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ON THE GENERAL TRANSFORMATION OF
THE WIRTINGER INTEGRAL

HUMIHIKO WATANABE

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Abstract
The Wirtinger integral is the uniformization to the upper half plane $H$ of the hypergeometric function defined on the complex projective line $\mathbb{P}^1$. In [5] we established the transformation formulas of the Wirtinger integral for the linear fractional transformations $\tau \to \tau + 2$ and $\tau \to -1/\tau$ with the aide of the theory of theta functions. As a corollary we obtain the transformation formulas of the Wirtinger integral for the linear fractional transformations $\tau \to \tau + 2$ and $\tau \to \tau/(-2\tau + 1)$ which are identified with generators of the principal congruence subgroup $\Gamma(2)$ modulo center. These formulas correspond to the monodromy matrices of the hypergeometric function for generators of the fundamental group of $\mathbb{P}^1$ minus three points. The purpose of this paper is to generalize this result, that is, we establish the transformation formula of the Wirtinger integral for a general element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\Gamma(2)$, which corresponds to a general monodromy matrix of the hypergeometric function.

1. Introduction
Following the notation of Chandrasekharan [1], we introduce the four theta functions $\theta(v, \tau), \theta_i(v, \tau) (i = 1, 2, 3)$ by

$$\theta(v, \tau) = \frac{1}{i} \sum_{n=-\infty}^{+\infty} (-1)^n e^{(n+1/2)i\pi \tau} e^{(2n+1)i\pi v},$$

$$\theta_1(v, \tau) = \sum_{n=-\infty}^{+\infty} e^{(n+1/2)i\pi \tau} e^{(2n+1)i\pi v},$$

$$\theta_2(v, \tau) = \sum_{n=-\infty}^{+\infty} (-1)^n e^{n^2 \pi i \tau} e^{2n\pi i v},$$

$$\theta_3(v, \tau) = \sum_{n=-\infty}^{+\infty} e^{n^2 \pi i \tau} e^{2n\pi i v},$$

which are defined for all $(v, \tau) \in \mathbb{C} \times H$, where $\mathbb{C}$ denotes the complex plane. Mumford [2] adopts the symbols $\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}$ to denote the theta functions above. The relations

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between the two notations are as follows: \( \theta(v, \tau) = -\theta_{11}(v, \tau), \theta_1(v, \tau) = \theta_{10}(v, \tau), \theta_2(v, \tau) = \theta_{01}(v, \tau), \theta_3(v, \tau) = \theta_{00}(v, \tau) \). Note that \( \theta(v, \tau) \) has a simple zero at \( v = 0 \), \( \theta_1(v, \tau) \) at \( v = 1/2 \), \( \theta_2(v, \tau) \) at \( v = \tau/2 \), and \( \theta_3(v, \tau) \) at \( v = (1 + \tau)/2 \). In this paper we also use the following abbreviations: \( \theta_i(v) = \theta_i(v, \tau), \theta_i = \theta_i(0, \tau) \), etc. Moreover we set \( T(v, \tau)^{p,q,0,1} = T(v, \tau)^{p,q,0,1} = \theta(v, \tau)^p \theta_1(v, \tau)^q \theta_2(v, \tau)^r \theta_3(v, \tau)^s \).

We define two functions \( z_1(\tau), z_2(\tau) \), which we called Wirtinger integrals in our papers [5], [6] (see also [7], [8], [9]), by

\[
z_1(\tau) = \frac{\theta_3^2}{(1 - e^{4\pi i \alpha})(1 - e^{4\pi i (\gamma - \alpha)})} \times \int_{v_0}^{v_1} T(v, \tau)\sqrt{-2\gamma + 3.2\beta - 2\gamma + 3.2\beta - 1} \, dv,
\]

\[
z_2(\tau) = \frac{\theta_3^2}{(1 - e^{-4\pi i \beta})(1 - e^{4\pi i (\beta - \gamma)})} \times \int_{v_0}^{v_1} T(v, \tau)\sqrt{-2\gamma + 3.2\beta - 2\gamma + 3.2\beta - 1} \, dv,
\]

where we assume that the parameters \( \alpha, \beta, \gamma \) satisfy the conditions \( \alpha, \gamma - \alpha, \gamma - \beta, \beta \notin (1/2)\mathbb{Z} \), and \( ((1/2), 0+, (1/2)-, 0-) \) denotes a Pochhammer cycle with base point \( v = v_0 \), where \( \arg v_0 = \arg((1/2) - v_0) = 0 \), turning first around \( v = 1/2 \) once anticlockwisely, second around \( v = 0 \) once anticlockwisely, third around \( v = 1/2 \) once clockwise, and lastly around \( v = 0 \) once clockwise. These functions are the lifts of Gauss’ hypergeometric functions of \( SL \) type to the upper half plane, and form a fundamental system of solutions for the lift of Gauss’ hypergeometric differential equation of \( SL \) type to the upper half plane. Note that \( z_1(\tau) \) and \( z_2(\tau) \) are transformed to each other by the involution \( \iota \) defined by \( \iota(\alpha, \beta, \gamma) = (\beta - \gamma + 2, \alpha - \gamma, -\gamma + 2) \). Let \( \Gamma(2) \) be the principal congruence subgroup of level 2, and let \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) be an element of \( \Gamma(2) \).

Problem 1.1. Determine the constants \( A \) and \( B \) with respect to \( \tau \) such that

\[
z_1((a \tau + b)/(c \tau + d)) = A z_1((\tau) + B z_2(\tau)).
\]

Once Problem 1 is established, we have an analogous formula for \( z_2((a \tau + b)/(c \tau + d)) \) by applying the involution \( \iota \) to the formula for \( z_1((a \tau + b)/(c \tau + d)) \). Since the group \( \Gamma(2) \) modulo center is isomorphic to the fundamental group of the Riemann sphere minus three points (which is the defining region of Gauss’ hypergeometric differential equation), the constants \( A, B \) are identified with entries of the monodromy matrix for the element of the fundamental group corresponding to the matrix \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \).
Let $Z(\tau)$ be a vector-valued function defined by

$$Z(\tau) = \left( \frac{2\pi \Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} z_1(\tau), \frac{2\pi \alpha \Gamma(2 - \gamma)}{\Gamma(-\beta) \Gamma(1 + \beta - \gamma)} z_2(\tau) \right),$$

where $\Gamma(z)$ denotes the Gamma function. According to the results of [5], for any even integer $m$, the matrices $M_0(m)$ and $M_1(m)$ defined by $Z(\tau + m) = Z(\tau)M_0(m)$ and $Z(\tau/(m\tau + 1)) = Z(\tau)M_1(m)$ are given by

$$M_0(m) = \begin{bmatrix} e^{m\pi i(y-1)/2} & 0 \\ 0 & e^{m\pi i(1-\gamma)/2} \end{bmatrix}$$

and

$$M_1(m) = \begin{bmatrix} \sin \pi(\gamma - \alpha) \sin \pi(\gamma - \beta) e^{m\pi i(\alpha + \beta - \gamma + 1)/2} - \sin \pi \alpha \sin \pi \beta e^{m\pi i(y-\alpha-\beta-1)/2} \\ \sin \pi \gamma \sin \pi(\gamma - \alpha - \beta) \\ 2\pi i \Gamma(\gamma - 1) \Gamma(\gamma) \sin(m\pi(\alpha + \beta - \gamma + 1)/2) \\ \Gamma(\alpha + 1) \Gamma(\beta + 1) \Gamma(\gamma - \alpha) \Gamma(\gamma - \beta) \sin(\pi(\gamma - \alpha - \beta)) \\ \sin \pi(\gamma - \alpha) \sin \pi(\gamma - \beta) e^{m\pi i(y-\alpha-\beta-1)/2} - \sin \pi \alpha \sin \pi \beta e^{m\pi i(\alpha + \beta - 1)/2} \\ \sin \pi \gamma \sin \pi(\gamma - \alpha - \beta) \end{bmatrix}.$$

Let \((a \ b \ c \ d)\) be an element of the principal congruence subgroup $\Gamma(2)$. We assume that $d > 1$. It is well-known (e.g. [4]) that there exist $2n + 1$ even integers $m_0, m_1, m_2, \ldots, m_{2n}$ ($n \geq 1$) such that the linear fractional transformation \((a\tau + b)/(c\tau + d)\) is written in the form:

$$\frac{a\tau + b}{c\tau + d} = m_0 - \frac{1}{m_1 - \frac{1}{m_2 - \cdots - \frac{1}{m_{2n} + \tau}}}.$$

This is also written in the form

$$\frac{a\tau + b}{c\tau + d} = (\varphi_{m_0} \circ \psi_{m_1} \circ \varphi_{m_2} \circ \psi_{m_3} \circ \cdots \circ \psi_{m_{2n-1}} \circ \varphi_{m_{2n}})(\tau),$$

where $\varphi_m(\tau) = \tau + m$ and $\psi_m(\tau) = \tau/(m\tau + 1)$. Setting $Z((a\tau + b)/(c\tau + d)) = Z(\tau)M$ with a two-by-two matrix $M$, we have

$$M = M_0(m_0)M_1(m_1)M_0(m_2)M_1(m_3) \cdots M_1(m_{2n-1})M_0(m_{2n}).$$
The coefficients $A$ and $B$ in Problem 1.1 are given as entries of this matrix $M$. It seems, however, to be difficult to derive the explicit formulas of the entries of $M$ directly from calculating the product formula (1.1). In the next section we formulate our idea for giving the explicit formulas of the coefficients $A$ and $B$.

2. How to determine the coefficients $A$ and $B$

Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $\Gamma(2)$ such that $c > 0$. The following formulas are well-known (e.g. [3], [4]):

\[
\begin{align*}
\theta \left( \frac{v}{c \tau + d} \cdot \frac{a \tau + b}{c \tau + d} \right) &= \left( \frac{c}{d} \right) e^{\pi i (3d - 2b - c d)/4} \sqrt{\frac{c \tau + d}{i}} e^{\pi i v^2/(c \tau + d)} \theta (v, \tau), \\
\theta_1 \left( \frac{v}{c \tau + d} \cdot \frac{a \tau + b}{c \tau + d} \right) &= \left( \frac{c}{d} \right) e^{\pi i (d - 2c + b d - c d)/4} \sqrt{\frac{c \tau + d}{i}} e^{\pi i v^2/(c \tau + d)} \theta_1 (v, \tau), \\
\theta_2 \left( \frac{v}{c \tau + d} \cdot \frac{a \tau + b}{c \tau + d} \right) &= \left( \frac{c}{d} \right) e^{\pi i (3d - 2a - b d - c d)/4} \sqrt{\frac{c \tau + d}{i}} e^{\pi i v^2/(c \tau + d)} \theta_2 (v, \tau), \\
\theta_3 \left( \frac{v}{c \tau + d} \cdot \frac{a \tau + b}{c \tau + d} \right) &= \left( \frac{c}{d} \right) e^{\pi i (3d - 2a + 2c - b d - c d)/4} \sqrt{\frac{c \tau + d}{i}} e^{\pi i v^2/(c \tau + d)} \theta_3 (v, \tau),
\end{align*}
\]

where $\left( \frac{c}{d} \right)$ denotes Legendre–Jacobi’s symbol. Substitution $\tau \rightarrow (a \tau + b)/(c \tau + d)$ and $v \rightarrow v/(c \tau + d)$ makes the integral representation for $z_1(\tau)$ given in Section 1 into

\[
z_1 \left( \frac{a \tau + b}{c \tau + d} \right) = \frac{1}{(1 - e^{4 \pi i a})(1 - e^{4 \pi i (\gamma - a)})} \theta_1 \left( 0, \frac{a \tau + b}{c \tau + d} \right)^2 \\
\times \int_{\Omega} T \left( \frac{v}{c \tau + d}, \frac{a \tau + b}{c \tau + d} \right)^{2a - 1, 2, -2a - 1, 2, -2a - 1, 2, -2a - 1, 2} \frac{dv}{c \tau + d}.
\]

Applying the transformation formulas above for theta functions to the integral representation for $z_1((a \tau + b)/(c \tau + d))$, we have

\[
z_1 \left( \frac{a \tau + b}{c \tau + d} \right) = \frac{e^{\pi i d(b + c)/2} e^{\pi i a} e^{\pi i d(a - \gamma) - \gamma - a - \beta/2} e^{\pi i a(b - 2\gamma)/2} \theta_3^2}{(1 - e^{4 \pi i a})(1 - e^{4 \pi i (\gamma - a)})} \\
\times \int_{\Omega} T \left( v, \frac{a \tau + b}{c \tau + d} \right)^{2a - 1, 2, -2a - 1, 2, -2a - 1, 2, -2a - 1, 2} dv
\]

for $\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma(2)$ with $c > 0$. In the rest of this paper we concentrate our attention on the case $0 < c < d$. Applying the reasoning for this case to the other one $d < c$.
which we do not refer to in this paper, we would obtain a result similar to the main theorem stated later. Noticing that the integers $c$, $d$ are relatively prime to each other, we define integers $k_v$ by

$$k_0 = 0, \quad k_v = \left[ \frac{d_v}{c} \right] + 1 \quad (1 \leq v \leq c - 1), \quad k_c = d,$$

where for a real number $x$ the symbol $[x]$ denotes the maximal integer not exceeding to $x$. The branch cuts $L_v$ ($v = 0, \ldots, c$) of the integrand $T(v)^{2a-1.2y-2\alpha-1.2\beta-2\gamma+3,-2\beta-1}$ of the integral in (2.1) are given as follows. Namely, $L_0$ is the union of two rays on $\mathbb{C}$ defined by the equation $v = s$ for the real parameter $s$ such that $s \leq 0$, $1/2 \leq s$; if $v \geq 1$, $L_v$ is the union of two rays on $\mathbb{C}$ defined by $v = s + (v/2)\tau$ for the real parameter $s$ such that $s \leq (k_v - 1)/2$, $k_v/2 \leq s$. We set $l_{v+1} = k_{v+1} - k_v$. Let $l_{\mu \nu}$, $J_{\nu}$ ($0 \leq \mu \leq l_{v+1} - 1$, $0 \leq \nu \leq c - 1$) be given by

(2.2)

$$I_{\mu \nu} = \frac{1}{(1 - \varepsilon((k_v + \mu + 1 + \nu \tau)/2))(1 - \varepsilon((k_v + \mu + \nu \tau)/2))} \times \int \left( \frac{2a-1.2y-2\alpha-1.2\beta-2\gamma+3,-2\beta-1}{T(v + \frac{k_v + \mu}{2} + \frac{v}{2}\tau)} \right) dv,$$

and

(2.3)

$$J_{\nu} = \frac{1}{(1 - \varepsilon((k_v + 1 + \nu \tau)/2))(1 - \varepsilon((k_v + 1 + (\nu + 1)\tau)/2))} \times \int \left( \frac{2a-1.2y-2\alpha-1.2\beta-2\gamma+3,-2\beta-1}{T(v + \frac{k_v + 1 + \nu \tau}{2})} \right) dv,$$

where $\varepsilon((m + n\tau)/2)$, $m$ and $n$ being integers, denotes the local monodromy of the function $T(v)^{2a-1.2y-2\alpha-1.2\beta-2\gamma+3,-2\beta-1}$ along a small circle turning around the center $v = (m + n\tau)/2$ in the anticlockwise direction, and therefore it coincides with one of the four quantities $e^{4\pi i \alpha}$, $e^{4\pi i (y-\alpha)}$, $e^{4\pi i (\beta-\gamma)}$, $e^{-4\pi i \beta}$. In the integral representation (2.2) we take the base point $v = v_0$ of the cycle $((1/2)+, 0+, (1/2)-, 0-)$ in such a manner that arg $v_0 = 0$ and arg $(v_0 - 1/2) = \pi$; in the integral representation (2.3) we take the base point $v = v_0$ of the cycle $((\tau/2)+, 0+, (\tau/2)-, 0-)$ in such a manner that arg $v_0 = \arg \tau$ and arg $(v_0 - \tau/2) = \arg \tau + \pi$. Then the integral in (2.1) has the following decomposition:

(2.4)

$$\frac{1}{(1 - e^{4\pi i \alpha})(1 - e^{4\pi i (y-\alpha)})} \int \left( \frac{2a-1.2y-2\alpha-1.2\beta-2\gamma+3,-2\beta-1}{T(v)^2} \right) dv$$

$$= \sum_{\nu=0}^{c-1} (I_{\nu} + J_{\nu}),$$
In fact, the constants \( I \) of the integrals \( I \) are written by

\[
(2.5) \quad I = \sum_{\mu=0}^{l_{\mu}-1} I_{\mu}u.
\]

Thus, Problem 1.1 follows immediately from the following:

**PROBLEM 2.1.** Determine the constants \( P, Q, R, S \) with respect to \( \tau \) such that \( I = Pz_1(\tau)/\theta_3^2 + Qz_2(\tau)/\theta_3^2 \) and \( J = Rz_1(\tau)/\theta_3^2 + Sz_2(\tau)/\theta_3^2 \).

In fact, the constants \( A \) and \( B \) in Problem 1.1 are written by

\[
A = e^{i\pi(d+e)/2}e^{i\pi\alpha}e^{i\pi(c+\gamma)/2}e^{i\pi\beta/2} \sum_{\nu=0}^{c-1} (P + R),
\]

\[
B = e^{i\pi(d+e)/2}e^{i\pi\alpha}e^{i\pi(c+\gamma)/2}e^{i\pi\beta/2} \sum_{\nu=0}^{c-1} (Q + S).
\]

We will give the explicit formulas for \( I \) and \( J \) in Theorems 4.1 and 5.1.

3. **Auxiliary formulas**

In this section we introduce some auxiliary formulas which are applied to the study of the integrals \( I \) and \( J \).

**Lemma 3.1.** Let \( p, q, r, s \) be complex constants but not integers. Then we have

\[
\int_{(1/2)+,0+,(1/2)-,0-} T(v)^{p,q,r,s} dv = \int_{(1/2)+,0+,(1/2)-,0-} T(v)^{p,q,r,s} dv.
\]

We omit the proof.

**Lemma 3.2.** We have:

\[
(3.2) \quad \frac{1}{(1 - e^{4\pi i\alpha})(1 - e^{4\pi i\beta -\gamma})} \int_{((\tau/2)+,0+,((\tau/2)-,0-)} T(v)^{2\alpha -1,2\beta -1,2\gamma +3,-2\beta-1} dv
\]

\[
= \frac{1}{1 - e^{-2\pi i\gamma}} \frac{e^{\pi i(\alpha -\gamma)}(e^{\pi i\beta} - e^{-\pi i\beta})}{1 - e^{-2\pi i\gamma}} z_1(\tau)/\theta_3^2,
\]

\[
(3.3) \quad \frac{1}{(1 - e^{-4\pi i\beta})(1 - e^{4\pi i\gamma -\alpha})} \int_{((\tau/2)+,0+,((\tau/2)-,0-)} T(v)^{2\gamma -2\alpha -1,2\beta -1,2\gamma +3} dv
\]

\[
= \frac{1}{1 - e^{-2\pi i\alpha}} \frac{e^{\pi i(-\alpha +\beta +\gamma)} - e^{\pi i(-\alpha +\beta -\gamma)}}{1 - e^{-2\pi i\gamma}} z_2(\tau)/\theta_3^2.
\]
Proof. Applying Cauchy’s theorem on the contour integral to the integration of
\(T(v)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}\) along the parallelogram with verices 0, 1/2, (1 + \(\tau\))/2, \(\tau/2\), we have

\[(3.4)\]
\[
\frac{1}{(1 - e^{4\pi i\alpha})(1 - e^{4\pi i(y-\gamma)})} \int_{(0^+, (\tau/2)+, 0^-, (\tau/2)-)} T(v)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} dv
\]
\[+ \frac{1}{(1 - e^{4\pi i\alpha})(1 - e^{4\pi i(y-\alpha)})} \int_{((1/2)+, 0^+, (1/2)-, 0^-)} T(v)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} dv
\]
\[+ \frac{1}{(1 - e^{4\pi i(a-\beta)})} \int_{(0^+, (\tau/2)+, 0^-)} T(v)^{2\gamma-2\alpha=1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} dv
\]
\[= 0.\]

Substitution \(\alpha \rightarrow \gamma - \alpha\) and \(\beta \rightarrow \gamma - \beta - 2\) makes (3.4) into

\[(3.5)\]
\[
\frac{1}{(1 - e^{4\pi i\beta})(1 - e^{4\pi i(y-\gamma)})} \int_{(0^+, (\tau/2)+, 0^-, (\tau/2)-)} T(v)^{2\gamma-2\alpha=1,2\alpha-1,2\beta-2\gamma+3,-2\beta-1} dv
\]
\[+ \frac{1}{(1 - e^{4\pi i(y-\alpha)})} \int_{((1/2)+, 0^+, (1/2)-, 0^-)} T(v)^{2\alpha-1,2\beta-2\gamma+3,-2\beta-1} dv
\]
\[+ \frac{1}{(1 - e^{4\pi i(y-\alpha)})} \int_{(0^+, (\tau/2)+, 0^-)} T(v)^{2\gamma-2\alpha=1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} dv
\]
\[= 0.\]

Combining (3.1), (3.4), (3.5), we have (3.2) and (3.3).

One can prove the following similarly.

**Lemma 3.3.** We have:

\[(3.6)\]
\[
\frac{1}{(1 - e^{4\pi i(y-\gamma)})} \int_{((\tau/2)+, 0^+, (\tau/2)-, 0^-)} T(v)^{2\gamma-2\alpha=1,2\gamma-2\alpha-1,2\gamma-2\alpha-1} dv
\]
\[= \frac{1}{1 - e^{2\pi i y}} z_1(\tau)/\theta_3^2 + \frac{1}{1 - e^{2\pi i y}} z_2(\tau)/\theta_3^2,
\]

\[(3.7)\]
\[
\frac{1}{(1 - e^{4\pi i(y-\alpha)})} \int_{((\tau/2)+, 0^+, (\tau/2)-, 0^-)} T(v)^{-2\beta-1,2\beta-2\gamma+3,2\gamma-2\alpha-1,2\alpha-1} dv
\]
\[= \frac{1}{1 - e^{2\pi i y}} z_1(\tau)/\theta_3^2 + \frac{1}{1 - e^{2\pi i y}} z_2(\tau)/\theta_3^2.
\]
4. The integrals $I_v$

The explicit formulas for the integrals $I_v$ are as follows:

**Theorem 4.1.** (i) We have

$$I_0 = e^{i(2\alpha - \gamma)/2} \left\{ \cos \frac{\pi}{2} (\gamma - 2\alpha) \cdot \frac{1 - e^{i(1-\gamma)k_1}}{1 - e^{i(1-\gamma)}} + i \sin \frac{\pi}{2} (\gamma - 2\alpha) \cdot \frac{1 - (-1)^{k_1} e^{i(1-\gamma)k_1}}{1 + e^{i(1-\gamma)}} \right\} z_1(\tau)/\theta_3^2.$$  

(ii) We have

$$I_1 = -e^{i(2\alpha - \gamma)/2} e^{2\pi i(\alpha + \beta - \gamma)} e^{i(1-\gamma)k_1}$$

$$\times \left\{ \cos \Theta_1 \cdot \frac{1 - e^{i(1-\gamma)l_2}}{1 - e^{i(1-\gamma)}} + i \sin \Theta_1 \cdot \frac{1 - (-1)^{k_2} e^{i(1-\gamma)l_2}}{1 + e^{i(1-\gamma)}} \right\} z_2(\tau)/\theta_3^2,$$

where

$$\Theta_1 = \pi \left\{ 2\alpha - \gamma - \frac{(-1)^{k_1}}{2}(2\beta - \gamma) \right\}.$$  

(iii) If $v$ is a positive even integer, then we have

$$I_v = e^{i(2\alpha - \gamma)/2} e^{2\pi i(l_1 + l_2 + \cdots + l_k)} e^{2\pi i(\alpha + \beta - \gamma)} e^{i(1-\gamma)k},$$

$$\times \left\{ \cos \Theta_v \cdot \frac{1 - e^{i(1-\gamma)l_{v+1}}}{1 - e^{i(1-\gamma)}} + i \sin \Theta_v \cdot \frac{1 - (-1)^{k_{v+1}} e^{i(1-\gamma)l_{v+1}}}{1 + e^{i(1-\gamma)}} \right\} z_1(\tau)/\theta_3^2,$$

where

$$\Theta_v = \pi \left\{ (-1)^{\nu} (2\beta - \gamma) + (-1)^{l_1 + \cdots + l_k} (2\alpha - \gamma) + \cdots + (-1)^{l_1 + \cdots + l_k} (2\beta - \gamma) \right\} + \frac{2 - (-1)^{k_v}}{2}(2\alpha - \gamma).$$

(iv) If $v$ is an odd integer and $v > 1$, then we have

$$I_v = -e^{i(2\alpha - \gamma)/2} e^{2\pi i(l_1 + l_2 + \cdots + l_k)} e^{2\pi i(\alpha + \beta - \gamma)} e^{i(1-\gamma)k_v}$$

$$\times \left\{ \cos \Theta_v \cdot \frac{1 - e^{i(1-\gamma)l_{v+1}}}{1 - e^{i(1-\gamma)}} + i \sin \Theta_v \cdot \frac{1 - (-1)^{k_{v+1}} e^{i(1-\gamma)l_{v+1}}}{1 + e^{i(1-\gamma)}} \right\} z_2(\tau)/\theta_3^2,$$

where

$$\Theta_v = \pi \left\{ 2\alpha - \gamma + (-1)^{\nu} (2\beta - \gamma) + (-1)^{l_1 + \cdots + l_k} (2\alpha - \gamma) + \cdots + (-1)^{l_1 + \cdots + l_k} (2\beta - \gamma) \right\} + \frac{(-1)^{k_v}}{2}(2\beta - \gamma).$$
We prove (4.3) only because the other formulas are proved similarly. Let \( \nu \) be a positive even integer. Let \( T_\pm(v + (k_v + \mu)/2 + (v/2)\tau)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} \) be the branches of \( T(v + (k_v + \mu)/2 + (v/2)\tau)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} \) analytically continued on the real ray \( v \geq -\mu/2 \) minus half periods from the upper side for the plus sign and from the lower side for the minus sign, respectively, such that they coincide with each other on the real interval \( -(\mu + 1)/2 < v < -\mu/2 \). Then we have

\[
T_+ \left( v + \frac{k_v + \mu}{2} + \frac{v}{2} \tau \right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} = 1 + \frac{(-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} e^{\pi i(2\alpha-2\gamma+1)} T \left( v + \frac{k_v - 1}{2} + \frac{v}{2} \tau \right)^{2\gamma-2\alpha+1,2\alpha-1,-2\beta-1,2\beta-2\gamma+3} + 1 + \frac{(-1)^{\mu+1}}{2} e^{\pi i(1-\gamma)(\mu+1)} T \left( v + \frac{k_v - 1}{2} + \frac{v}{2} \tau \right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}
\]

and

\[
T_- \left( v + \frac{k_v + \mu}{2} + \frac{v}{2} \tau \right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} = 1 + \frac{(-1)^\mu}{2} e^{\pi i(1-\gamma-1)\mu} e^{\pi i(2\alpha-2\gamma-1)} T \left( v + \frac{k_v - 1}{2} + \frac{v}{2} \tau \right)^{2\gamma-2\alpha+1,2\alpha-1,-2\beta-1,2\beta-2\gamma+3} + 1 + \frac{(-1)^{\mu+1}}{2} e^{\pi i(1-\gamma-1)(\mu+1)} T \left( v + \frac{k_v - 1}{2} + \frac{v}{2} \tau \right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}
\]

Combining these relations, we have

\[
T_+ \left( v + \frac{k_v + \mu}{2} + \frac{v}{2} \tau \right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} = e^{\pi i(-2\gamma\mu+2\alpha-3\gamma+(1)^\mu(2\alpha-\gamma))} T_- \left( v + \frac{k_v + \mu}{2} + \frac{v}{2} \tau \right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1}
\]

Now we have

\[
T_- \left( v + \frac{k_v + \mu}{2} + \frac{v}{2} \tau \right)^{2\alpha-1,2\gamma-2\alpha-1,2\beta-2\gamma+3,-2\beta-1} = e^{\pi i(\alpha+\beta-\gamma+1)} T_+ \left( v + \frac{k_v + \mu}{2} + \frac{v - 1}{2} \tau \right)^{2\beta-2\gamma+3,-2\beta-1,2\alpha-1,2\gamma-2\alpha-1}
\]
Combining the preceding two relations, we have

\[
T_+ \left( v + \frac{k_v + \mu}{2} + \frac{v}{2} \right)^{2\alpha - 1, 2\gamma - 2\alpha - 1, 2\beta - 2\gamma + 3, -2\beta - 1}
\]

(4.5)

\[
e^{\pi i (-2\gamma u + 2\alpha - 3\gamma + (-1)^\gamma (2\alpha - \gamma))} e^{\pi i (\alpha + \beta - \gamma + 1)}
\times T_+ \left( v + \frac{k_v + \mu}{2} + \frac{v - 1}{2} \right)^{2\beta - 2\gamma + 3, -2\beta - 1, 2\alpha - 1, 2\gamma - 3\alpha - 1}
\]

By the same argument, we have

\[
T_+ \left( v + \frac{k_v + \mu}{2} + \frac{v - 1}{2} \right)^{2\beta - 2\gamma + 3, -2\beta - 1, 2\alpha - 1, 2\gamma - 2\alpha - 1}
\]

(4.6)

\[
e^{\pi i (2\gamma (\mu + l_v) + 2\beta + \gamma + (-1)^{\gamma} (2\beta - \gamma))} e^{\pi i (\alpha + \beta - \gamma + 1)}
\times T_+ \left( v + \frac{k_v + \mu}{2} + \frac{v - 2}{2} \right)^{2\alpha - 1, 2\gamma - 2\alpha - 1, 2\beta - 2\gamma + 3, -2\beta - 1}
\]

From (4.5) and (4.6) we have

\[
T_+ \left( v + \frac{k_v + \mu}{2} + \frac{v}{2} \right)^{2\alpha - 1, 2\gamma - 2\alpha - 1, 2\beta - 2\gamma + 3, -2\beta - 1}
\]

\[
e^{\pi i (2\gamma (\mu + l_v) + (-1)^{\gamma} (2\alpha - \gamma)) + (-1)^{\gamma} (2\beta - \gamma))} e^{4\pi i (\alpha + \beta - \gamma)}
\times T_+ \left( v + \frac{k_v + \mu}{2} + \frac{v - 2}{2} \right)^{2\alpha - 1, 2\gamma - 2\alpha - 1, 2\beta - 2\gamma + 3, -2\beta - 1}
\]

Repeating the same procedure, we arrive at the following

**Lemma 4.2.** For a positive even integer \( v \) we have

(4.7)

\[
T_+ \left( v + \frac{k_v + \mu}{2} + \frac{v}{2} \right)^{2\alpha - 1, 2\gamma - 2\alpha - 1, 2\beta - 2\gamma + 3, -2\beta - 1}
\]

\[
e^{\pi i ((-1)^{\gamma} (2\alpha - \gamma) + (-1)^{\gamma} (2\beta - \gamma) + (-1)^{\gamma} (2\alpha - \gamma) + (-1)^{\gamma} (2\beta - \gamma) + \cdots + (-1)^{\gamma} + l_v + l_v + (-1)^{\gamma} (2\alpha - \gamma) + (-1)^{\gamma} (2\beta - \gamma)) + \cdots}
\times e^{2\pi i (\alpha + \beta - \gamma) T_+ \left( v + \frac{k_v + \mu}{2} \right)^{2\alpha - 1, 2\gamma - 2\alpha - 1, 2\beta - 2\gamma + 3, -2\beta - 1}}
\]
Let us compute the integral $I_v$ for a positive even integer $v$. Substituting (4.7) into (2.2) we have

$$I_{\mu v} = e^{\pi i((-1)^{v/2}(2\alpha - \gamma')) + (-1)^{v+1/2}(2\beta - \gamma) + \cdots + (-1)^{v+(v-1)/2}(2\gamma-\gamma') + \cdots}$$

$$+ e^{2\pi i(l_0 + l_1 + \cdots + l_{v-1})} e^{2\pi i(\alpha + \beta - \gamma)}$$

$$\times \frac{1}{1 - \varepsilon((k_v + \mu)/2))} (1 - \varepsilon((k_v + \mu + 1)/2))$$

$$\times \int_{(1/2)^+,0^+,0^-} T(v + \frac{k_v + \mu}{2})^{2\alpha - 2\alpha - 2\beta - 2\gamma + 3, -2\beta - 1} dv.$$  

Note that

$$\frac{1}{1 - \varepsilon((k_v + \mu)/2))} (1 - \varepsilon((k_v + \mu + 1)/2))$$

$$\times \int_{(1/2)^+,0^+,0^-} T(v + \frac{k_v + \mu}{2})^{2\alpha - 2\alpha - 2\beta - 2\gamma + 3, -2\beta - 1} dv$$

$$= \frac{1}{2} \pi i(1-\gamma)\mu e^{\pi i(1-\gamma)\mu}$$

$$\times \int_{(1/2)^+,0^+,0^-} T(v + \frac{k_v + \mu}{2})^{2\alpha - 2\alpha - 2\beta - 2\gamma + 3, -2\beta - 1} dv$$

$$+ \frac{1}{2} \pi i(1-\gamma)\mu e^{\pi i(2\alpha - \gamma)}$$

$$\times \int_{(1/2)^+,0^+,0^+} T(v + \frac{k_v + \mu}{2})^{2\gamma - 2\alpha - 2\alpha - 1, -2\beta - 2\gamma + 3} dv.$$  

Combining (4.8) with (4.9) and substituting the resulting equality into (2.5), we have after some calculation

$$I_v = e^{\pi i(2\alpha - \gamma)} e^{\pi i((-1)^{v/2}(2\beta - \gamma) + (-1)^{v+1/2}(2\alpha - \gamma) + \cdots + (-1)^{v+(v-1)/2}(2\gamma - \gamma'))}$$

$$\times e^{2\pi i(l_0 + l_1 + \cdots + l_{v-1})} e^{2\pi i(\alpha + \beta - \gamma)}$$

$$\times \int_{(1/2)^+,0^+,0^-} T(v + \frac{k_v + \mu}{2})^{2\alpha - 2\alpha - 2\beta - 2\gamma + 3, -2\beta - 1} dv$$

$$+ e^{\pi i((-1)^{v/2}(2\beta - \gamma) + (-1)^{v+1/2}(2\alpha - \gamma) + \cdots + (-1)^{v+(v-1)/2}(2\gamma - \gamma'))}$$

$$\times e^{2\pi i(l_0 + l_1 + \cdots + l_{v-1})} e^{2\pi i(\alpha + \beta - \gamma)}$$

$$\times \int_{(1/2)^+,0^+,0^+} T(v + \frac{k_v + \mu}{2})^{2\gamma - 2\alpha - 2\alpha - 1, -2\beta - 2\gamma + 3} dv.$$
Now we have

\[(4.11)\]

\[
\frac{1}{(1 - e(k_v/2))(1 - e((k_v + 1)/2))} \times \int_{((1/2)+,0+,(1/2)-,0-)} (v + \frac{k_v}{2})^{2\alpha-1.2y-2\alpha-1.2\beta-2\gamma+3,-2\beta-1} d\nu
\]

\[
= e^{\pi i(1-\gamma)k_v} e^{(1-(-1)^{y+}1)^{\pi i(2\alpha-2\gamma)/2}} \int_{((1/2)+,0+,(1/2)-,0-)} T(v)^{2\alpha-1.2y-2\alpha-1.2\beta-2\gamma+3,-2\beta-1} d\nu
\]

and

\[(4.12)\]

\[
\frac{1}{(1 - e(k_v/2))(1 - e((k_v + 1)/2))} \times \int_{((1/2)+,0+,(1/2)-,0-)} T(v)^{2\gamma-2\alpha-1.2\beta-2\gamma+3} d\nu
\]

\[
= e^{\pi i(1-\gamma)k_v} e^{(1-(-1)^{\gamma+}1)^{\pi i(2\gamma-2\gamma)/2}} \int_{((1/2)+,0+,(1/2)-,0-)} T(v)^{2\alpha-1.2y-2\alpha-1.2\beta-2\gamma+3,-2\beta-1} d\nu.
\]

Substituting (4.11) and (4.12) into (4.10), we have

\[I_v = e^{\pi i(2\alpha-2\gamma)} e^{\pi i((-1)^{y+1}(2\beta-2\gamma)+(-1)^{y+1}1(2\alpha-2\gamma)+\cdots+(-1)^{y+1}1+\cdots+1)(2\beta-2\gamma))}
\]

\[
\times e^{2\pi i((l_1+l_2)+\cdots+l_k+l_2)} e^{2\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_v} e^{(1-(-1)^{y+}1)^{\pi i(2\alpha-2\gamma)/2}}
\]

\[
\times \left\{ \frac{1}{2} \left[ \frac{1 - e^{\pi i(1-\gamma)y_{l+1}}}{1 - e^{\pi i(1-\gamma)}} + \frac{1 - (-1)^{y+1}e^{\pi i((1-\gamma)y_{l+1}}}{1 + e^{\pi i(1-\gamma)}} \right] z_1(\tau) / \theta_3^2 \right. + e^{\pi i((-1)^{y+1}(2\beta-2\gamma)+(-1)^{y+1}1(2\alpha-2\gamma)+\cdots+(-1)^{y+1}1+\cdots+1)(2\beta-2\gamma))}
\]

\[
\times e^{2\pi i((l_1+l_2)+\cdots+l_k+l_2)} e^{2\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_v} e^{(1-(-1)^{y+}1)^{\pi i(1-\gamma)/2}}
\]

\[
\times \left\{ \frac{1}{2} \left[ \frac{1 - e^{\pi i((1-\gamma)y_{l+1}}}{1 - e^{\pi i(1-\gamma)}} + \frac{1 - (-1)^{y+1}e^{\pi i((1-\gamma)y_{l+1}}}{1 + e^{\pi i(1-\gamma)}} \right] z_1(\tau) / \theta_3^2 \right. + e^{\pi i(\alpha-\gamma)} e^{\pi i(1-\gamma)k_v} e^{(1-(-1)^{y+}1)^{\pi i(1-\gamma)/2}}
\]

from which (4.3) follows immediately.

5. The integrals \(J_v\)

The explicit formulas for integrals \(J_v\) are as follows:
Lemma 5.2. For a positive even integer $v$ we have

$$T_+(v + \frac{k_{v+1}}{2}) 2^{\alpha-1,2} 2^{n-1,2} 2^{n-2} 2^{3,2}$$

(5.4)

$$= e^{\pi i ((-1)^{v+1} (2\alpha - \gamma) + (-1)^{v+1} i (2\beta - \gamma) + \cdots + (-1)^{v+1} i (2\alpha - \gamma) + (-1)^{v+1} i (2\beta - \gamma))} e^{2\pi i (\alpha + \beta - \gamma)} T_+ \left( v + \frac{k_{v+1}}{2} \right) 2^{\alpha-1,2} 2^{n-1,2} 2^{n-2} 2^{3,2}$$
Substitution of (5.4) into (2.3) and some calculation makes \( J \) into

\[
J_y = e^{\pi i ((-1)^y + i (2a - y) + \ldots + \ldots + i (2a - y) + \ldots + (-1)^{y+1} + i (2a - y)) + \ldots + i (2a - y)) + \ldots + \ldots + i (2a - y))}
\]

\[
\times e^{2\pi i (l_0 + l_1 + \ldots + l_y + l_z)} e^{2\nu i (\alpha + \beta - y)}
\]

\[
\times \left\{ \frac{1 + (-1)^{k_y+1}}{2} \frac{e^{\pi i (1 - y)k_y+1}}{1 - e^{4\pi i (\beta - y)}} \right. \\
\int_{((\tau/2) + 0 + (\tau/2) - 0)}^{((\tau/2) + 0 + (\tau/2) - 0)} T(v)^{2a - 1, 2a - 2a - 2a - 2 \beta - 2 \gamma + 3, -2 \beta - 1} dv \\
+ \frac{1 - (-1)^{k_y+1}}{2} \frac{e^{\pi i (1 - y)k_y+1} e^{\pi i (2a - y)}}{1 - e^{4\pi i (\gamma - a)}} \\
\left. \times \int_{((\tau/2) + 0 + (\tau/2) - 0)}^{((\tau/2) + 0 + (\tau/2) - 0)} T(v)^{2a - 2a - 1, 2a - 1, -2 \beta - 2 \gamma + 3} dv \right\}.
\]

Applying the formulas of Lemma 3.2 to (5.5), we have (5.2) immediately. One can derive the other formulas (5.1) and (5.3) similarly.

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References