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CONNECTED SUMS OF SIMPLICIAL COMPLEXES AND EQUIVARIANT COHOMOLOGY

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Abstract

In this paper, we introduce the notion of a *connected sum* $K_1 \#^Z K_2$ of simplicial complexes K_1 and K_2, as well as define a *strong* connected sum. Geometrically, the connected sum is motivated by Lerman's symplectic cut applied to a toric orbifold, and algebraically, it is motivated by the connected sum of rings introduced by Ananthnarayan–Avramov–Moore [1]. We show that the Stanley–Reisner ring of a connected sum $K_1 \#^Z K_2$ is the connected sum of the Stanley–Reisner rings of K_1 and K_2 along the Stanley–Reisner ring of $K_1 \cap K_2$. The strong connected sum $K_1 \#^Z K_2$ is defined in such a way that when K_1, K_2 are Gorenstein, and Z is a suitable subset of $K_1 \cap K_2$, then the Stanley–Reisner ring of $K_1 \#^Z K_2$ is Gorenstein, by work appearing in [1]. We also show that cutting a simple polytope by a generic hyperplane produces strong connected sums. These algebraic computations can be interpreted in terms of the equivariant cohomology of moment angle complexes and toric orbifolds.

1. Introduction

In this paper, we introduce a notion of the *connected sum of simplicial complexes*, abstracting the combinatorial aspect of cutting a simple polytope by a generic hyperplane. Let K_1 and K_2 be simplicial complexes on $[m] := \{1, \dots, m\}$ and let $Z \subset W := K_1 \cap K_2$ be a subset that does not contain the empty set. Assume that the neighborhood $O_{K_1 \cup K_2}(Z)$ of Z in $K_1 \cup K_2$ is contained in W. In Section 2, we define the *connected sum* $K_1 \#^Z K_2$ of K_1 and K_2 by

$$K_1 \#^Z K_2 := \mathrm{Del}_Z(K_1 \cup K_2).$$

Furthermore, we introduce the *strong* connected sum of K_1 and K_2 by assuming

$$Z = K_1 \setminus (\overline{K_1 \setminus W}) = K_2 \setminus (\overline{K_2 \setminus W}). \tag{1.1}$$

We show that if Δ_+ and Δ_- are simple polytopes obtained by cutting a simple polytope Δ with a generic hyperplane H_o, then the simplicial complex K associated to Δ

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is a strong connected sum of the simplicial complexes $K_\pm$ associated to $\Delta_\pm$. Interestingly, it is also shown that K_- is a strong connected sum of K_+ and K.

We then turn to study the algebraic structures of the corresponding Stanley–Reisner rings in the framework of the *connected sum of rings* introduced by Ananthnarayan–Avramov–Moore [1] (Section 3). Let $\mathsf{A}_1, \mathsf{A}_2$ and C be rings and V a C-module. Consider the following diagram

$$\begin{tikzcd} \mathsf{V} \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \mathsf{A}_1 \arrow[d, "\epsilon_1"] \\ \mathsf{A}_2 \arrow[r, "\epsilon_2"'] & \mathsf{C} \end{tikzcd} \tag{1.2}$$

where ϵ_1 and ϵ_2 are ring homomorphisms and φ_1 and φ_2 are module homomorphisms. The *connected sum of rings* associated to the diagram (1.2) is defined by

$$\mathsf{A}_1 \#_\epsilon^\varphi \mathsf{A}_2 := \frac{\ker \epsilon}{\operatorname{Im} \varphi}$$

where

$$\epsilon := \epsilon_1 - \epsilon_2 \colon \mathsf{A}_1 \oplus \mathsf{A}_2 \to \mathsf{C}$$

and

$$\varphi := (\varphi_1, \varphi_2) \colon \mathsf{V} \to \mathsf{A}_1 \oplus \mathsf{A}_2.$$

We show that the Stanley–Reisner ring $\mathbb{Z}[K_1 \#^Z K_2]$ of a connected sum $K_1 \#^Z K_2$ is the connected sum of Stanley–Reisner rings $\mathbb{Z}[K_1]$ and $\mathbb{Z}[K_2]$ (Theorem 3.5). More precisely, let $\mathcal{I}_Z$ be the ideal in $\mathbb{Z}[K_1 \cup K_2]$ generated by the monomials corresponding to the faces in Z. Then

Theorem A (Theorem 3.5)**.** *$\mathbb{Z}[K_1 \#^Z K_2]$ is isomorphic to the connected sum of rings, $\mathbb{Z}[K_1] \#_{\mathsf{g}}^\theta \mathbb{Z}[K_2]$, associated to the diagram*

$$\begin{tikzcd} \mathcal{I}_Z \arrow[r, "\theta_1"'] \arrow[d, "\theta_2"'] & \mathbb{Z}[K_1] \arrow[d, "\mathsf{g}_1"] \\ \mathbb{Z}[K_2] \arrow[r, "\mathsf{g}_2"'] & \mathbb{Z}[W], \end{tikzcd} \tag{1.3}$$

where all maps are given by the obvious quotient maps of Stanley–Reisner rings corresponding to the inclusions of simplicial complexes.

The extra assumption (1.1) for the strong connected sum is motivated by the following algebraic fact. If K_1 and K_2 are Gorenstein and W is Cohen–Macaulay, then assumption (1.1) implies that the ideal $\mathcal{I}_Z$ is a canonical module of $\mathbb{Z}[W]$. As a consequence, we can show purely algebraically from the work of [1] that if K_1 and K_2 are Gorenstein, $K_1 \#^Z K_2$ is a strong connected sum, and W is Cohen–Macaulay, then $K_1 \#^Z K_2$ is Gorenstein (see Corollary 3.10).

We also discuss the Tor algebra of the Stanley–Reisner ring of a connected sum. Let $[m]$ be the common vertex set of K_1, K_2 and K so that the corresponding Stanley–Reisner rings are the quotients of $\mathbb{Z}[x_1, \dots, x_m]$ by the ideals generated by monomials of non-faces. Pick an $n \times m$ integral matrix $B = (B_{ij}) \in \mathrm{Mat}_{n,m}(\mathbb{Z})$ of rank n and denote the corresponding map for tori also by $B\colon \mathsf{T} \to \mathsf{R}$. We have a polynomial ring $\mathbb{Z}[\mathsf{R}^*] := \mathbb{Z}[u_1, \dots, u_n]$ sitting inside of $\mathbb{Z}[\mathsf{T}^*] := \mathbb{Z}[x_1, \dots, x_m]$ where $u_i = \sum_{j=1}^m B_{ij}$. In Section 4.3, we show

Theorem B. *If* $\mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[L], \mathbb{Z}) = 0$ *for* $L = K, K_1, K_2, W$, *then* $\mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_1 \#^Z K_2], \mathbb{Z})$ *is isomorphic as a ring to* $\mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_1], \mathbb{Z}) \#_{\mathsf{g}}^{\theta} \mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_2], \mathbb{Z})$ *defined by the diagram*

$$\begin{tikzcd}
\mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathcal{I}_Z, \mathbb{Z}) \arrow[r, "\theta_1"'] \arrow[d, "\theta_2"'] & \mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_1], \mathbb{Z}) \arrow[d, "\mathsf{g}_1"'] \\
\mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_2], \mathbb{Z}) \arrow[r, "\mathsf{g}_2"'] & \mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[W], \mathbb{Z}),
\end{tikzcd}$$

where all the maps are induced from Diagram (1.3).

This analysis bears fruit in Section 4, where we relate the above results to the cohomology of the moment angle complex of a connected sum of simplicial complexes. The *moment angle complex* $\mathcal{Z}_K$ associated to a simplicial complex K was introduced by Buchstaber and Panov in [4] as a disc-circle decomposition of the Davis–Januszkiewicz universal space. It has been actively studied in *toric topology* and its connections to symplectic and algebraic geometry, and combinatorics. Since the (equivariant) cohomology of moment angle complexes are naturally related to the Stanley–Reisner rings and their Tor algebras (cf. [3], [11]), we have the corresponding theorem. More precisely, we can replace the Stanley–Reisner rings in Theorem A by the T-equivariant cohomology of the corresponding moment angle complexes where T is the m-dimensional torus acting on the moment angle complexes canonically (Corollary 4.3). Moreover, we can replace $\mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\ , \mathbb{Z})$ in Theorem B by the G-equivariant cohomology of the corresponding moment angle complexes where G is the kernel of $B\colon \mathsf{T} \to \mathsf{R}$ (Proposition 4.4). The connected sum of simplicial complexes can be used to construct interesting spaces (cf. [7]) and the techniques developed in this paper can be used to compute the (equivariant) cohomological invariants of these spaces.

Finally, we come back to our original motivation to study the cohomology of a symplectic cut of a toric orbifold. Since a toric orbifold is topologically nothing but the quotient stack of a moment angle complex by a torus action, the above results can be applied. For example, we have

Theorem C (Proposition 4.7)**.** *Let $\mathcal{X}$ be a toric orbifold and $\mathcal{X}_+$ and $\mathcal{X}_-$ are symplectic cuts of of $\mathcal{X}$. Let $\mathcal{X}_o$ be the toric suborbifold of $\mathcal{X}_\pm$ that corresponds to the section of the cut. Let $\mathsf{f}_\pm \colon \mathcal{X}_o \hookrightarrow \mathcal{X}_\pm$ be the inclusion. Then $H^*(\mathcal{X};\mathbb{Q})$ is isomorphic to $H^*(\mathcal{X}_+;\mathbb{Q}) \#^{\mathsf{f}_*}_{\mathsf{f}^*} H^*(\mathcal{X}_-;\mathbb{Q})$ where the connected sum of rings is defined by the diagram*

$$\begin{tikzcd} H^*(\mathcal{X}_o;\mathbb{Q}) \arrow[r, "\mathsf{f}_{+*}"'] \arrow[d, "\mathsf{f}_{-*}"'] & H^*(\mathcal{X}_+;\mathbb{Q}) \arrow[d, "\mathsf{f}_+^*"'] \\ H^*(\mathcal{X}_-;\mathbb{Q}) \arrow[r, "\mathsf{f}_-^*"'] & H^*(\mathcal{X}_o;\mathbb{Q}). \end{tikzcd}$$

If $H^*(\mathcal{X}_o)$ and $H^*(\mathcal{X})$ are concentrated in even degree, then the statement holds over $\mathbb{Z}$-coefficients.

2. Connected sum of simplicial complexes

In this section, we define the (strong) connected sum $K_1 \#^Z K_2$ of simplicial complexes K_1 and K_2 on a vertex set $[m] := \{1, \ldots, m\}$. We show that cutting a simple polytopes produces strong connected sums of simplicial complexes.

2.1. Connected sums of simplicial complexes. A *simplicial complex* on the vertex set $\mathcal{S}$ is a collection K of subsets (called *faces*) of $\mathcal{S}$ such that if $\sigma \in K$, then all subsets including the empty $\varnothing$ of σ are in K. A simplicial complex K is called *pure* if all its maximal faces have the same dimension where the dimension of a face $\sigma \in K$ is $|\sigma| - 1$. A maximal face is called a *facet*. The set of all facets is denoted by $\mathcal{F}(K)$. A vertex x is called a *ghost vertex* if $\{x\} \notin K$. Let Z be a subset of a simplicial complex K. The *closure* $\overline{Z}$ of Z in K is the smallest subcomplex containing Z. The *open neighborhood* $O_K(Z)$ of Z in K is the set of all $\sigma \in K$ such that σ contains some $\tau \in Z$. Note that $O_K(Z) = Z$ if and only if $K \setminus Z$ is a subcomplex of K. The *star* of Z in K and the *deletion* of Z from K are the subcomplexes defined by $\mathrm{star}_K(Z) := \overline{O_K(Z)}$ and $\mathrm{Del}_Z(K) := K \setminus O_K(Z)$ respectively. If K_1 and K_2 are simplicial complexes on the same vertex set $\mathcal{S}$, then we can naturally take the intersection $K_1 \cap K_2$ and the union $K_1 \cup K_2$ that are also simplicial complexes on $\mathcal{S}$.

DEFINITION 2.1 (Connected sum). Let K_1 and K_2 be simplicial complexes on $[m] := \{1, \ldots, m\}$ and $W := K_1 \cap K_2$. Let $Z \subset W$ be a subset such that $\varnothing \notin Z$ and $O_{K_1 \cup K_2}(Z) \subset W$. We define the *connected sum* $K_1 \#^Z K_2$ of K_1 and K_2 along Z by

$$K_1 \#^Z K_2 := \mathrm{Del}_Z(K_1 \cup K_2).$$

EXAMPLE 2.2 (Connected sum along a facet p. 24 [3]). Let $\sigma_i \in \mathcal{F}(K_i)$, $i = 1, 2$, be facets of the same cardinaliry. If we identify the vertices of σ_1 and σ_2 and $\sigma :=$

$\sigma_1 = \sigma_2$, we have $W = \overline{\{\sigma\}}$. Let $Z := \{\sigma\}$ and then $O_{K_1 \cup K_2}(Z) = \{\sigma\} \subset K_1 \cap K_2$. The connected sum $K_1 \#^\sigma K_2 := K_1 \cup K_2 \setminus \{\sigma\}$ is exactly the "connected sum" defined in [3].

EXAMPLE 2.3. Let $v(K_1) = \{a, b, c, d\}$ and $v(K_2) = \{a, b, c, e\}$. Let $\mathcal{F}(K_1) = \{abc, bcd\}$ and $\mathcal{F}(K_2) = \{abc, ace\}$. Then $\mathcal{F}(W) = \{abc\}$ and let $Z = \{abc\} = O_K(Z)$. This is a connected sum in the sense of [3]. The result is not pure.

DEFINITION 2.4 (Strong connected sum). A connected sum $K_1 \#^Z K_2$ is called *strong* if K_1, K_2 and $W = K_1 \cap K_2$ are pure with the same dimension and

$$Z = W \setminus (\overline{K_1 \setminus W}) = W \setminus (\overline{K_2 \setminus W}).$$

The algebraic justification of Definition 2.4 comes in Section 3.2. Here we only show the following lemma that will be used later.

Lemma 2.5. *Let K be a simplicial complex and W a subcomplex of K. Let*

$$Z := \{\tau \in K \mid \tau \cup \sigma \notin K,\ \forall \sigma \in K \setminus W\}. \tag{2.1}$$

Then $O_K(Z) = Z$ and $Z = W \setminus (\overline{K \setminus W})$.

Proof. Let $\tau \in O_K(Z)$ and let $\tau' \in Z$ such that $\tau' \subset \tau$. If there is $\sigma \in K \setminus W$ such that $\tau \cup \sigma \in K$, then $\tau' \cup \sigma \in K$ so $\tau' \notin Z$. Thus $\tau \cup \sigma \notin K$ for all $\sigma \in K \setminus W$, i.e. $O_K(Z) \subset Z$. Since obviously $O_K(Z) \supset Z$, we have $O_K(Z) = Z$.

We have $Z \subset W$ since, if $\tau \notin W$, then $\tau \in K \setminus W$ and $\tau \cup \tau \in K$ so $\tau \notin Z$. Furthermore if $\tau \in \overline{K \setminus W}$, then there is $\sigma \in K \setminus W$ such that $\tau \subset \sigma$. Therefore $\tau \cup \sigma \in K$ so that $\tau \notin Z$. Thus $Z \subset W \setminus (\overline{K \setminus W})$. On the other hand, let $\tau \in W \setminus \overline{K \setminus W}$. If $\tau \notin Z$, then there is $\sigma \in K \setminus W$ such that $\tau \cup \sigma \in W$. This means $\tau \in \mathrm{star}_K(K \setminus W)$. However, we have that $\mathrm{star}_K(K \setminus W) = \overline{O_K(K \setminus W)} = \overline{K \setminus W}$. Thus $\tau \in \overline{K \setminus W}$ which is a contradiction. Thus $\tau \in Z$ and so $W \setminus \overline{K \setminus W} \subset Z$. $\square$

2.2. Polytope cutting and connected sum. A *polytope* Δ is defined to be the convex hull of a finite set of points in $\mathbb{R}^n$. We can choose $\lambda_i \in (\mathbb{R}^n)^*$ and $\eta_i \in \mathbb{R}$, $i = 1, \ldots, m$ such that

$$\Delta = \{\vec{x} \in \mathbb{R}^n \mid \langle \vec{x}, \lambda_i \rangle + \eta_i \geq 0,\ i = 1, \ldots, m\}.$$

Let $\tilde{H}_i := \{\vec{x} \in \mathbb{R}^n \mid \langle \vec{x}, \lambda_i \rangle + \eta_i = 0\}$ be the defining hyperplanes and we call $H_i := \Delta \cap \tilde{H}_i$ a *facet* for each $i = 1, \ldots, m$. If H_i is empty, we call it a *ghost facet*. A polytope Δ is *simple* if $\tilde{H}_i$, $i = 1, \ldots m$, are in a general position, i.e. if there are exactly n hyperplanes meeting at each vertex of Δ. For a simple polytope Δ with facets H_i, $i = 1, \dots, m$, the associated simplicial complex K_Δ is a simplicial complex on $[m]$ defined by

$$\sigma \subset K_\Delta \Leftrightarrow \sigma = \varnothing \quad \text{or} \quad \bigcap_{i \in \sigma} H_i \neq \varnothing.$$

DEFINITION 2.6 (Generic cut). Let $\Delta \subset \mathbb{R}^n$ be a n-dimensional simple polytope. Suppose that the facets are all non-ghost facets H_i, $i = 1, \dots, m$. Consider a new hyperplane

$$\tilde{H}_o := \{\vec{x} \in \mathbb{R}^n \mid \langle \vec{x},\ \lambda_0 \rangle + \xi = 0\}$$

and the corresponding closed half spaces $\tilde{H}_{\geq o} = \{\langle \vec{x}, \lambda_0 \rangle + \xi \geq 0\}$ and $\tilde{H}_{\leq o} = \{\langle \vec{x}, \lambda_0 \rangle + \xi \leq 0\}$. A *generic cut* of Δ is given by the pair (Δ, H_o) such that $\tilde{H}_o, \tilde{H}_1, \dots, \tilde{H}_m$ are in general position and $H_o := \tilde{H}_o \cap \Delta \neq \varnothing$. In this case, $\Delta_+ := \Delta \cap \tilde{H}_{\geq o}$ and $\Delta_- := \Delta \cap \tilde{H}_{\leq o}$ are non-empty simple polytopes.

We regard the vertex sets of the simplicial complexes K_Δ, K_+, K_- associated to Δ, Δ_+, Δ_- to be $\widetilde{[m]} := [m] \cup \{o\}$. More precisely, let

$$K_\Delta := \left\{ \sigma \subset \widetilde{[m]} \;\middle|\; o \notin \sigma \text{ and } \bigcap_{i \in \sigma} H_i \neq \varnothing \right\} \cup \{\varnothing\},$$

$$K_+ := \left\{ \sigma \subset \widetilde{[m]} \;\middle|\; \bigcap_{i \in \sigma} (H_i \cap \Delta_+) \neq \varnothing \right\} \cup \{\varnothing\},$$

$$K_- := \left\{ \sigma \subset \widetilde{[m]} \;\middle|\; \bigcap_{i \in \sigma} (H_i \cap \Delta_-) \neq \varnothing \right\} \cup \{\varnothing\}.$$

Let (Δ, H_o) be a generic cut of a simple polytope. For $\sigma \subset \widetilde{[m]}$, let $F_\sigma := \bigcap_{i \in \sigma} H_i$. First we show that K_Δ is a strong connected sum of K_+ and K_-.

Lemma 2.7.

$$(K_+ \cup K_-) \setminus K_\Delta = O_{K_+ \cup K_-}(o) = O_{K_+}(o) = O_{K_-}(o), \tag{2.2}$$

$$K_+ \cap K_- = \mathrm{star}_{K_+ \cup K_-}(o) = \mathrm{star}_{K_+}(o) = \mathrm{star}_{K_-}(o). \tag{2.3}$$

Proof. From the definition, it is clear that $\sigma \in (K_+ \cup K_-) \setminus K_\Delta$ if and only if $\sigma \in K_+ \cup K_-$ and $o \in \sigma$, i.e.

$$(K_+ \cup K_-) \setminus K_\Delta = \{\sigma \subset \widetilde{[m]} \mid o \in \sigma \text{ and } \sigma \in K_+ \cup K_-\} = O_{K_+ \cup K_-}(o).$$

On the other hand, $\bigcap_{i \in \sigma} (H_i \cap \Delta_+) = \left(\bigcap_{i \in \sigma} H_i\right) \cap H_o = \bigcap_{i \in \sigma} (H_i \cap \Delta_-)$ if $o \in \sigma$. Therefore for all σ that contains o, $\sigma \in K_+$ if and only if $\sigma \in K_-$. Thus $O_{K_+ \cup K_-}(o) =$

$O_{K_+}(o) = O_{K_-}(o)$. This proves (2.2) and also

$$\mathrm{star}_{K_+ \cup K_-}(o) = \mathrm{star}_{K_+}(o) = \mathrm{star}_{K_-}(o).$$

Since $\Delta_+ \cap \Delta_-$, it is clear that $\sigma \in K_+ \cap K_-$ if and only if $\sigma = \varnothing$ or $F_\sigma \cap H_o \neq \varnothing$. Thus

$$K_+ \cap K_- = \underbrace{\{\sigma \in K_+ \mid \sigma \cup \{o\} \in K_+\}}_{\mathrm{star}_{K_+}(o)} = \underbrace{\{\sigma \in K_- \mid \sigma \cup \{o\} \in K_-\}}_{\mathrm{star}_{K_-}(o)}. \qquad \square$$

An immediate corollary is that K_Δ is a connected sum of K_+ and K_- along $Z := \{o\}$. In fact, it is a strong connected sum, as is shown below.

Theorem 2.8. *If (Δ, H_o) is a generic cut, then K_Δ is the strong connected sum $K_+ \#^Z K_-$ where $Z = \{o\}$.*

Proof. To show it is a strong connected sum, we must prove $O_{K_\pm}(o) = W \setminus (\overline{K_\pm \setminus W})$ where $W := K_+ \cap K_-$. Suppose $\tau \in O_{K_+}(o)$. By (2.3), we have $\{o\} \cup \sigma \notin K_+$ for all $\sigma \in K_+ \setminus W$. Thus we have $\tau \cup \sigma \notin K_+$ for all $\sigma \in K_+ \setminus W$. Then Lemma 2.5 implies that $\tau \in W \setminus (\overline{K_+ \setminus W})$. To prove $W \setminus (\overline{K_+ \setminus W}) \subset O_{K_+}(o)$, we show that $W \setminus O_{K_+}(o) \subset \overline{K_+ \setminus W}$. Since $W = \mathrm{star}_{K_+}(o)$ by (2.3), we need to show that $\tau \in \mathrm{star}_{K_+}(o) \setminus O_{K_+}(o)$ implies $\tau \in \overline{K_+ \setminus \mathrm{star}_{K_+}(o)}$. Let $\tau \in \mathrm{star}_{K_+}(o) \setminus O_{K_+}(o)$, i.e. $o \notin \tau$ and $F_\tau \cap H_o \neq \varnothing$. Since the cutting is generic, F_τ has a vertex contained in Δ_+ but not contained in H_0. Let F_σ be such a vertex. Then $\sigma \in K_+ \setminus \mathrm{star}_{K_+}(o)$. Since $\tau \subset \sigma$, $\tau \in \overline{K_+ \setminus \mathrm{star}_{K_+}(o)}$. The same argument may be used to prove $O_{K_-}(o) = W \setminus (\overline{K_- \setminus W})$. $\square$

Now we show that K_- is a strong connected sum of K_Δ and K_+. Let

$$Z := \{\sigma \subset \widetilde{[m]} \mid F_\sigma \neq \varnothing \text{ and } F_\sigma \subset \Delta_+ \setminus H_o\}.$$

Lemma 2.9.

$$(K_+ \cup K_\Delta) \setminus K_- = Z, \tag{2.4}$$

$$K_+ \cap K_\Delta = \overline{Z}, \tag{2.5}$$

$$K_+ \setminus \overline{Z} = O_{K_+}(o), \tag{2.6}$$

$$K_\Delta \setminus \overline{Z} = \{\sigma \subset \widetilde{[m]} \mid F_\sigma \neq \varnothing \textit{ and } F_\sigma \subset \Delta_- \setminus H_o\}. \tag{2.7}$$

Proof. Equation (2.4) is obvious from the fact that $F_\sigma \subset \Delta_+ \setminus H_o$ if and only if $F_\sigma \cap \Delta_- = \varnothing$. Now observe that $K_+ \cap K_\Delta$ consists of $\varnothing$ and $\sigma \subset [m]$ such that $F_\sigma \cap \Delta_+ \neq \varnothing$. It is clear that $Z \subset K_+ \cap K_\Delta$ and hence $\overline{Z} \subset K_+ \cap K_\Delta$. Let $\sigma \in K_+ \cap K_\Delta$. If $\sigma \notin Z$, then $F_\sigma \cap H_o \neq \varnothing$. Since $\sigma \in K_\Delta$ so that $o \notin \sigma$, there is a vertex

F_τ of F_σ contained in $\Delta_+ \setminus H_o$, which means $\tau \in Z$. Thus $\sigma \in \overline{Z}$. Thus $K_+ \cap K_\Delta \subset \overline{Z}$. This proves (2.5). Equation (2.6) follows from the fact that $\sigma \in K_+ \setminus \overline{Z}$ if and only if $o \in \sigma$ and $F_\sigma \neq \varnothing$. Equation (2.7) follows from (2.6) and that $\sigma \in K_\Delta \setminus \overline{Z}$ if and only if $F_\sigma \neq \varnothing$ and $F_\sigma \subset \Delta_- \setminus H_o$. □

Theorem 2.10. *Let (Δ, H_o) be a generic cut. Let $Z = \{\sigma \subset \widetilde{[m]} \mid F_\sigma \neq \varnothing$ and $F_\sigma \subset \Delta_+ \setminus H_o\}$ as above. Then K_- is the strong connected sum $K_+ \#^Z K_\Delta$ of K_+ and K_Δ along Z.*

Proof. K_- is a connected sum of K_+ and K_Δ along Z by (2.4) and (2.5). Let $W := K_+ \cap K_\Delta$. First we show that $Z = W \setminus (\overline{K_+ \setminus W})$. Since $\overline{K_+ \setminus W} = \mathrm{star}_{K_+}(o)$ by (2.6), we must show $Z = W \setminus \mathrm{star}_{K_+}(o)$. Suppose $\sigma \in Z$. If $\sigma \in \mathrm{star}_{K_+}(o)$, then there must be $\tau \in O_{K_+}(o)$ such that $\sigma \subset \tau$. Since $o \in \tau$, we have $F_\sigma \cap H_o \neq \varnothing$ which contradicts $F_\sigma \subset \Delta_+ \setminus H_o$. Thus $\sigma \in W \setminus \mathrm{star}_{K_+}(o)$. On the other hand, if $\sigma \in W \setminus \mathrm{star}_{K_+}(o)$, then $F_\sigma \cap \Delta_+ \neq \varnothing$ and there is no vertex of F_σ that lies on H_o. Therefore $F_\sigma \subset \Delta_+ \setminus H_o$, i.e. $\sigma \in Z$. Finally we show that $W \setminus (\overline{K_+ \setminus W}) = W \setminus (\overline{K_\Delta \setminus W})$. Let $\varnothing \neq \sigma \in W \cap \overline{K_+ \setminus W}$. Then $\sigma \subset [m]$ and $F_\sigma \cap H_o \neq \varnothing$. Thus $\dim F_\sigma \geq 1$ and there is a vertex F_τ of F_σ that lies in $\Delta_- \setminus H_o$. Since $\tau \in K_\Delta \setminus \overline{Z}$, we have $\sigma \in \overline{K_+ \setminus W}$. On the other hand, suppose that $\varnothing \neq \sigma \in W \cap \overline{K_\Delta \setminus W}$, then $F_\sigma \cap \Delta_+ \neq \varnothing$ and there is a vertex of F_σ that lies in $\Delta_- \setminus H_o$. Thus $F_\sigma \cap H_o \neq \varnothing$ which implies $\sigma \in \mathrm{star}_{K_+}(o)$. □

3. Stanley–Reisner rings and connected sum

We study the algebraic structure of the Stanley–Reisner ring of the connected sum $K_1 \#^Z K_2$ defined in the previous section. The algebraic model is the *connected sum of rings* introduced and studied by Ananthnarayan–Avramov–Moore [1]. In Section 3.1, we review the definitions and show that the Stanley–Reisner ring $\mathbb{Z}[K_1 \#^Z K_2]$ is the connected sum of the Stanley–Reisner ring of K_1 and K_2. In Section 3.2, we study the Gorenstein property of $\mathbb{Z}[K_1 \#^Z K_2]$ in terms of the same property of K_1, K_2 and $K_1 \cap K_2$ for strong connected sums. Here Corollary 3.10 is our motivation to define *strong* connected sums. In Section 3.3, we discuss how those properties descend to Tor algebras of Stanley–Reisner rings.

3.1. Connected sum of rings.

DEFINITION 3.1 (Fiber product and connected sum of rings). Let $\epsilon_i\colon \mathsf{A}_i \to \mathsf{C}$, $i = 1, 2$, be ring homomorphisms. Then the *fiber product* $\mathsf{A}_1 \times_\epsilon \mathsf{A}_2$ is the subring of $\mathsf{A}_1 \oplus \mathsf{A}_2$ defined as the kernel of $\epsilon := \epsilon_1 - \epsilon_2$, i.e.

$$\mathsf{A}_1 \times_\epsilon \mathsf{A}_2 := \{(x_1, x_2) \in \mathsf{A}_1 \oplus \mathsf{A}_2 \mid \epsilon_1(x_1) = \epsilon_2(x_2)\}.$$

Now take a C-module V and regard it as a A_i-module via ϵ_i for each $i = 1, 2$. Given a commutative diagram

$$\begin{tikzcd} \mathsf{V} \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \mathsf{A}_1 \arrow[d, "\epsilon_1"] \\ \mathsf{A}_2 \arrow[r, "\epsilon_2"'] & \mathsf{C} \end{tikzcd} \tag{3.1}$$

where φ_i is a homomorphism of A_i-modules for $i = 1, 2$, we set $\varphi := (\varphi_1, \varphi_2)\colon \mathsf{V} \to \mathsf{A}_1 \oplus \mathsf{A}_2$. The *connected sum* of the diagram (3.1) is given by

$$\mathsf{A}_1 \#^{\varphi}_{\epsilon} \mathsf{A}_2 := \frac{\ker \epsilon}{\operatorname{Im} \varphi} = \frac{\mathsf{A}_1 \times_{\epsilon} \mathsf{A}_2}{\{(\varphi_1(v), \varphi_2(v)) \in \mathsf{A}_1 \oplus \mathsf{A}_2 \mid v \in \mathsf{V}\}}.$$

REMARK 3.2. Equivalently, one may also view the definition of the connected sum of rings as arising via the following exact sequences:

$$0 \longrightarrow \mathsf{A}_1 \times_{\epsilon} \mathsf{A}_2 \longrightarrow \mathsf{A}_1 \oplus \mathsf{A}_2 \xrightarrow{\epsilon} \mathsf{C}, \tag{3.2}$$

$$\mathsf{V} \xrightarrow{\varphi} \mathsf{A}_1 \times_{\epsilon} \mathsf{A}_2 \longrightarrow \mathsf{A}_1 \#^{\varphi}_{\epsilon} \mathsf{A}_2 \longrightarrow 0. \tag{3.3}$$

DEFINITION 3.3. Let K be a simplicial complex on $[m]$. The *Stanley–Reisner ring* $\mathbb{Z}[K]$ is the quotient of the polynomial ring $\mathbb{Z}[x_1,\dots,x_m]$ by the ideal generated by $x_\sigma := \prod_{i\in\sigma} x_i$ for all non-face σ of K. For a monomial $p = \prod_{i=1}^m x_i^{a_i}$ in $\mathbb{Z}[x_1, \dots, x_m]$, let $\sigma := \{i \in [m] \mid a_i \neq 0\}$. Let M_K be the set of monomials p such that $\sigma(p)$ does not contain any non-face of K. We have the canonical choice of representatives of elements of $\mathbb{Z}[K]$:

$$\mathbb{Z}[K] \cong \bigoplus_{p\in M_K} \mathbb{Z} \cdot p. \tag{3.4}$$

Theorem 3.4. *Let K_1 and K_2 are simplicial complexes on $[m]$. Let $W := K_1 \cap K_2$. Let $\mathsf{g}_i\colon \mathbb{Z}[K_i] \to \mathbb{Z}[W]$ be the quotient map of Stanley–Reisner rings for the inclusion $W \hookrightarrow K_i$ for each $i = 1, 2$ and let $\mathsf{g} := \mathsf{g}_1 - \mathsf{g}_2$. Let $\theta_i\colon \mathbb{Z}[K_1 \cup K_2] \to \mathbb{Z}[K_i]$ also be the quotient map for the inclusion $K_i \hookrightarrow K_1 \cup K_2$. Then $\theta := (\theta_1, \theta_2)$ defines an isomorphism of rings over $\mathbb{Z}[x_1, \dots, x_m]$:*

$$\theta\colon \mathbb{Z}[K_1 \cup K_2] \to \mathbb{Z}[K_1] \times_{\mathsf{g}} \mathbb{Z}[K_2].$$

Proof. The following short exact sequence is obvious

$$0 \to \mathbb{Z}[K_1 \cup K_2] \xrightarrow{\theta} \mathbb{Z}[K_1] \oplus \mathbb{Z}[K_2] \xrightarrow{\mathsf{g}} \mathbb{Z}[W] \to 0.$$

Indeed, the injectivity of θ and the surjectivity of g are obvious. Also it is obvious that $\operatorname{Im}\theta \subset \ker \mathsf{g}$. We define the inverse $\theta^{-1}\colon \mathbb{Z}[K_1] \times_{\mathsf{g}} \mathbb{Z}[K_2] \to \mathbb{Z}[K_1 \cup K_2]$. In the notation in Definition 3.3, $M_{K_1} \cap M_{K_2} = M_W$. Therefore for each $(r_1, r_2) \in \ker \mathsf{g}$, we have the unique representatives

$$r_1 = \sum_{p \in M_{K_1} \setminus M_W} a_p \cdot p + \sum_{p \in M_W} a_p \cdot p \quad \text{and} \quad r_2 = \sum_{p \in M_{K_2} \setminus M_W} a_p \cdot p + \sum_{p \in M_W} a_p \cdot p$$

and we can associate

$$\theta(r_1, r_2) := \sum_{p \in M_{K_1} \setminus M_W} a_p \cdot p + \sum_{p \in M_{K_2} \setminus M_W} a_p \cdot p + \sum_{p \in M_W} a_p \cdot p.$$

Here we note that $M_{K_1 \cup K_2} = (M_{K_1} \setminus M_W) \sqcup (M_{K_2} \setminus M_W) \sqcup M_W$ and hence this clearly defines the inverse of θ. $\square$

Theorem 3.5. *Let $K_1 \#^Z K_2$ be a connected sum introduced at* Definition 2.1. *Let $\mathcal{I}_Z$ be the ideal in $\mathbb{Z}[K_1 \cup K_2]$ generated by x_σ, $\sigma \in Z$. Then as an algebra over $\mathbb{Z}[x_1, \ldots, x_m]$, $\mathbb{Z}[K_1 \#^Z K_2]$ is isomophic to the connected sum of rings, $\mathbb{Z}[K_1] \#^{\theta}_{\mathsf{g}} \mathbb{Z}[K_2]$, associated to the diagram*

$$\begin{tikzcd} \mathcal{I}_Z \arrow[r, "\theta_1"'] \arrow[d, "\theta_2"'] & \mathbb{Z}[K_1] \arrow[d, "\mathsf{g}_1"'] \\ \mathbb{Z}[K_2] \arrow[r, "\mathsf{g}_2"'] & \mathbb{Z}[W]. \end{tikzcd}$$

Proof. Since $K_1 \#^Z K_2 = (K_1 \cup K_2) \setminus O_{K_1 \cup K_2}(Z)$, we have the following short exact sequence of $\mathbb{Z}[x_1, \ldots, x_m]$-modules

$$0 \to \mathcal{I}_Z \to \mathbb{Z}[K_1 \cup K_2] \to \mathbb{Z}[K_1 \#^Z K_2] \to 0.$$

By Theorem 3.4, we have the isomorphism of rings over $\mathbb{Z}[x_1, \ldots, x_m]$

$$\mathbb{Z}[K_1 \#^Z K_2] \cong \frac{\ker(\mathsf{g}\colon \mathbb{Z}[K_1] \oplus \mathbb{Z}[K_2] \to \mathbb{Z}[W])}{\operatorname{Im}(\theta\colon \mathcal{I}_Z \to \mathbb{Z}[K_1] \oplus \mathbb{Z}[K_2])}.$$

To complete the proof, we need to show that $\mathcal{I}_Z$ is a $\mathbb{Z}[W]$-module and that $\theta_i\colon \mathcal{I}_Z \to \mathbb{Z}[K_i]$ is a $\mathbb{Z}[K_i]$-module homomorphism with respect to g_i for each $i = 1, 2$. But this is clear since $O_{K_1 \cup K_2}(Z) \subset W$ implies that the natural quotient map $\mathbb{Z}[K_1 \cup K_2] \to \mathbb{Z}[W]$ sends $\mathcal{I}_Z$ to the ideal in $\mathbb{Z}[W]$ which is isomorphic to $\mathcal{I}_Z$ as a $\mathbb{Z}[x_1, \ldots, x_m]$-module. $\square$

From Theorems 2.8 and 2.10, we have

Corollary 3.6. *Let (Δ, H_o) be a generic cut of a simple polytope. Then $\mathbb{Z}[K_\Delta]$ is isomorphic to the connected sum of $\mathbb{Z}[K_+]$ and $\mathbb{Z}[K_-]$ associated to the corresponding diagram*

$$\begin{array}{ccc} \mathcal{I}_{\{o\}} & \longrightarrow & \mathbb{Z}[K_+] \\ \downarrow & & \downarrow \\ \mathbb{Z}[K_-] & \longrightarrow & \mathbb{Z}[K_+ \cap K_-]. \end{array}$$

Moreover $\mathbb{Z}[K_-]$ is isomorphic to the connected sum of $\mathbb{Z}[K_\Delta]$ and $\mathbb{Z}[K_+]$ associated to the corresponding diagram

$$\begin{array}{ccc} \mathcal{I}_{Z} & \longrightarrow & \mathbb{Z}[K_\Delta] \\ \downarrow & & \downarrow \\ \mathbb{Z}[K_+] & \longrightarrow & \mathbb{Z}[K_\Delta \cap K_+] \end{array}$$

where $Z = (K_\Delta \cap K_+) \setminus K_-$.

3.2. Connected sum of Gorenstein rings. This section explains our algebraic motivation for Definition 3.1 of the strong connected sum. Let W be a subcomplex of a simplicial complex K on $[m]$. Let $\mathcal{I}_{K\setminus W}$ be a kernel of the quotient map $\mathbb{Z}[K] \to \mathbb{Z}[W]$.

Lemma 3.7. *The annihilator $(0 :_{\mathbb{Z}[K]} \mathcal{I}_{K\setminus W})$ is generated by x_σ, $\sigma \in W \setminus (\overline{K \setminus W})$.*

Proof. The annihilator is generated by x_σ where $\sigma \in K$ s.t. $\sigma \cup \tau \notin K$, $\forall \tau \in K \setminus W$. The claim is a corollary of Lemma 2.5. $\square$

The following is a basic fact about the canonical module of a Cohen–Macaulay ring [2, Theorem 3.3.7]:

Lemma 3.8. *Suppose that W and K are pure with the same dimension. If K is Gorenstein and W is Cohen–Macaulay, then $(0 :_{\mathbb{Z}[K]} \mathcal{I}_{K\setminus W})$ is a canonical module of $\mathbb{Z}[W]$.*

From [1], we have the following theorem.

Theorem 3.9. *In the* Definition 3.1, $\mathsf{A}_1 \#_\epsilon^\varphi \mathsf{A}_2$ *is Gorenstein if* A_i *is Gorenstein for each* $i = 1, 2$, C *is Cohen–Macaulay and* V *is a canonical module of* C.

As a corollary, together with Lemmas 3.7 and 3.8, we have

Corollary 3.10. *Let K_1 and K_2 are simplicial complexes on $[m]$ such that K_1, K_2 and $W := K_1 \cup K_2$ are pure with the same dimension. Assume that K_1, K_2 are Gorenstein and W is Cohen–Macaulay. If $K_1 \#^Z K_2$ is a strong connected sum, then $K_1 \#^Z K_2$ is Gorenstein.*

3.3. Tor algebra of connected sums. Let K_1 and K_2 be simplicial complexes on $[m]$ and $K := K_1 \#^Z K_2$ a connected sum of K_1 and K_2 along Z. Let $\tilde{K} = K_1 \cup K_2$ and $W := K_1 \cap K_2$. In Theorems 3.4 and 3.5, we see that there are two short exact sequences of algebras and modules over $\mathbb{Z}[x_1, \dots, x_m]$:

$$0 \longrightarrow \mathbb{Z}[\tilde{K}] \xrightarrow[\theta]{} \mathbb{Z}[K_1] \oplus \mathbb{Z}[K_2] \xrightarrow[\mathsf{g}]{} \mathbb{Z}[W] \longrightarrow 0;$$

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathbb{Z}[\tilde{K}] \longrightarrow \mathbb{Z}[K] \longrightarrow 0.$$

Consider an integer $n \times m$ matrix B of rank n. The choice of such B corresponds uniquely to a choice of a surjective map $\mathsf{T} := U(1)^m \to \mathsf{R} := U(1)^n$. Denote $\mathbb{Z}[\mathsf{T}^*] := \mathbb{Z}[x_1, \dots, x_m]$. Let $u_i := \sum_{j=1}^m B_{ij} x_j$, and denote $\mathbb{Z}[\mathsf{R}^*] := \mathbb{Z}[u_1, \dots, u_n] \subset \mathbb{Z}[\mathsf{T}^*]$. Recall that the Koszul complex $\mathcal{K}^{\mathbb{Z}[\mathsf{R}^*]}$ is a $\mathbb{Z}[\mathsf{R}^*]$-free resolution of $\mathbb{Z}$. Therefore, tensoring the above short exact sequences with $\mathcal{K}^{\mathbb{Z}[\mathsf{R}^*]}$ and taking homology, we get the following long exact sequences:

$$\begin{aligned} &\cdots \to \mathrm{Tor}_i^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[\tilde{K}], \mathbb{Z}) \to \mathrm{Tor}_i^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_1], \mathbb{Z}) \oplus \mathrm{Tor}_i^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_2], \mathbb{Z}) \\ &\quad \to \mathrm{Tor}_i^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[W], \mathbb{Z}) \to \cdots, \end{aligned} \tag{3.5}$$

$$\cdots \to \mathrm{Tor}_i^{\mathbb{Z}[\mathsf{R}^*]}(\mathcal{I}_Z, \mathbb{Z}) \to \mathrm{Tor}_i^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[\tilde{K}], \mathbb{Z}) \to \mathrm{Tor}_i^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) \to \cdots. \tag{3.6}$$

Let

$$\bar{\mathsf{g}} := \bar{\mathsf{g}}_1 - \bar{\mathsf{g}}_2 \colon \mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_1], \mathbb{Z}) \oplus \mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_2], \mathbb{Z}) \to \mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[W], \mathbb{Z});$$

$$\bar{\theta} \colon (\bar{\theta}_1, \bar{\theta}_2) \colon \mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathcal{I}_Z, \mathbb{Z}) \to \mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_1], \mathbb{Z}) \oplus \mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_2], \mathbb{Z})$$

be the induced maps on Tor. The following claims can be easily observed:

Lemma 3.11. *Suppose that* $\mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[W], \mathbb{Z}) = 0$. *Then one has* $\mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[\tilde{K}], \mathbb{Z}) = 0$ *if and only if* $\mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_1], \mathbb{Z}) = \mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_2], \mathbb{Z}) = 0$. *In this case,*

$$\mathrm{Tor}_0^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[\tilde{K}], \mathbb{Z}) = \mathrm{Tor}_0^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_1], \mathbb{Z}) \times_{\bar{\mathsf{g}}} \mathrm{Tor}_0^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_2], \mathbb{Z}).$$

Proposition 3.12. *If* $\mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_1], \mathbb{Z}) = \mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_2], \mathbb{Z}) = \mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) = \mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[W], \mathbb{Z}) = 0$, *then*

$$\mathrm{Tor}_0^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) = \mathrm{Tor}_0^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_1], \mathbb{Z}) \#_{\bar{\mathsf{g}}}^{\bar{\theta}} \mathrm{Tor}_0^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_2], \mathbb{Z}).$$

REMARK 3.13. By Proposition 2.3 [8], $\mathrm{Tor}_1 = 0$ implies $\mathrm{Tor}_i = 0$ for all $i > 0$. Therefore, in the above lemmata, we actually have $\mathrm{Tor}_0^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[\tilde{K}],\mathbb{Z}) = \mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[\tilde{K}],\mathbb{Z})$ and $\mathrm{Tor}_*^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K],\mathbb{Z}) = \mathrm{Tor}_0^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K],\mathbb{Z})$.

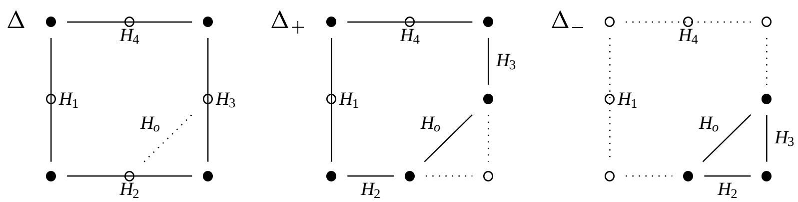
Lemma 3.14. *Let (Δ, H_o) be a generic cut of a simple polytope as in* Definition 2.6. *Let $W := K_+ \cap K_-$. Regard K_Δ as the connected sum of K_+ and K_- along $Z := \{o\}$. If $\mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[W],\mathbb{Z}) = \mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_\Delta],\mathbb{Z}) = 0$, then $\mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_+],\mathbb{Z}) = \mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_-],\mathbb{Z}) = 0$. In this case, we have*

$$\mathrm{Tor}_0^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_\Delta],\mathbb{Z}) = \mathrm{Tor}_0^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_+],\mathbb{Z}) \,\#_{\bar{\mathfrak{g}}}^{\bar{\theta}}\, \mathrm{Tor}_0^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_-],\mathbb{Z}).$$

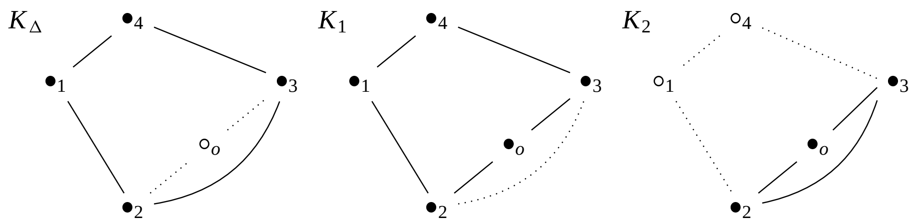
Proof. In this case, observe that $\mathcal{I}_Z \cong \mathbb{Z}[W]$ as $\mathbb{Z}[\mathsf{T}^*]$-modules. Thus $\mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[W],\mathbb{Z}) = \mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_\Delta],\mathbb{Z}) = 0$ implies $\mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_+ \cup K_-],\mathbb{Z}) = 0$ and hence $\mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_+],\mathbb{Z}) = \mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_-],\mathbb{Z}) = 0$. □

REMARK 3.15. The converse of Lemma 3.14 is not true; we give an example such that $\mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[W],\mathbb{Z}) = \mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_+],\mathbb{Z}) = \mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_-],\mathbb{Z}) = 0$ but $\mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_\Delta],\mathbb{Z}) \neq 0$.

Consider the following cutting of a cube: Δ is the cube with the facets $H_1, \ldots, H_4$ and we cut it by the facet H_o to obtain Δ_+ and Δ_- as shown below.



The following are the corresponding simplicial complexes.



K_Δ is a strong connected sum of K_+ and K_- along $Z = \{o\}$.

Consider the following 2×5 matrix B:

$$B = \begin{pmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 2 & 0 & -1 & 1 \end{pmatrix}.$$

By direct computation (we used *Macaulay*2), we find that

$$\mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[W], \mathbb{Z}) = \mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_1], \mathbb{Z}) = \mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K_2], \mathbb{Z}) = 0$$

but $\mathrm{Tor}_1^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) \neq 0$.

Note that this example comes from cutting the labeled polytope (Δ, b) that corresponds to the direct product of weighted projective space $\mathbb{CP}^1_{12} \times \mathbb{CP}^1_{12}$.

4. Moment angle complexes and toric orbifolds

4.1. Cohomology of moment angle complexes. We use the following notation for convenience. Let X be a set and Y, Z subsets of X. Let $\sigma \subset [m]$ be a subset. Then $Y^\sigma \times Z^{[m]\setminus\sigma} \subset X^m$ denotes the direct product of Y and Z's where i-th component is Y if $i \in \sigma$ and Z if $i \in [m] \setminus \sigma$.

DEFINITION 4.1 (Moment angle complexes). Let K be a simplicial complex on the vertex set $[m] := \{1, \ldots, m\}$ (with possible ghost vertices). The *moment angle complex* $\mathcal{Z}_{K,[m]} \subset \mathbb{C}^m$ is defined by

$$\mathcal{Z}_K := \bigcup_{\sigma \in K} \mathsf{D}^\sigma \times \partial\mathsf{D}^{[m]\setminus\sigma} = \bigcup_{\sigma \in \mathcal{F}(K)} \mathsf{D}^\sigma \times \partial\mathsf{D}^{[m]\setminus\sigma}$$

where $\mathsf{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and $\partial\mathsf{D} = \{z \in \mathbb{C} \mid |z| = 1\}$. The standard action of $\mathsf{T} := \mathrm{U}(1)^m$ on $\mathbb{C}^m$ can be restricted to the one on $\mathcal{Z}_K$.

In this section, all cohomology rings are taken with integer coefficients unless otherwise specified. The basic fact about the T-equivariant cohomology ring of $\mathcal{Z}_K$ is

Theorem 4.2 (Davis–Januszkiewicz [5]). *There is an isomorphism of graded rings $\mathbb{Z}[K] \cong H^*_{\mathsf{T}}(\mathcal{Z}_K; \mathbb{Z})$. This isomorphism is natural in the sense that, for a subcomplex $W \subset K$, we have the commutative diagram of short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}_{K\setminus W} \arrow[r] \arrow[d, "\cong"'] & \mathbb{Z}[K] \arrow[r] \arrow[d, "\cong"'] & \mathbb{Z}[W] \arrow[r] \arrow[d, "\cong"'] & 0 \\ 0 \arrow[r] & H^*_{\mathsf{T}}(\mathcal{Z}_K, \mathcal{Z}_W) \arrow[r] & H^*_{\mathsf{T}}(\mathcal{Z}_K) \arrow[r] & H^*_{\mathsf{T}}(\mathcal{Z}_W) \arrow[r] & 0 \end{tikzcd}$$

where $\mathcal{I}_{K\setminus W}$ is the ideal in $\mathbb{Z}[K]$ generated by monomials x_σ, $\sigma \in K \setminus W$ and $H^*_{\mathsf{T}}(\mathcal{Z}_K, \mathcal{Z}_W; \mathbb{Z})$ is the relative equivariant cohomology for $\mathcal{Z}_W \subset \mathcal{Z}_K$. The vertical isomorphism on the left is induced from the other two isomorphisms and the short exactness of rows.

Theorems 3.5 and 4.2 has an immediate corollary.

Corollary 4.3. *Let K_1 and K_2 be simplicial complexes on $[m]$ and let $K = K_1 \#^Z K_2$ be a connected sum as in* Definition 2.1. *Let $W := K_1 \cap K_2$ and $\tilde{K} := K_1 \cup K_2$. As rings over $H^*(B\mathsf{T})$, $H^*_{\mathsf{T}}(\mathcal{Z}_K)$ is isomorphic to $H^*_{\mathsf{T}}(\mathcal{Z}_{K_1}) \#^{\theta^*}_{\mathsf{g}^*} H^*_{\mathsf{T}}(\mathcal{Z}_{K_2})$ defined by the diagram*

$$\begin{array}{ccc} H^*_{\mathsf{T}}(\mathcal{Z}_{\tilde{K}}, \mathcal{Z}_W) & \xrightarrow{\theta_1^*} & H^*_{\mathsf{T}}(\mathcal{Z}_{K_1}) \\ \Big\downarrow{\scriptstyle \theta_2^*} & & \Big\downarrow{\scriptstyle \mathsf{g}_1^*} \\ H^*_{\mathsf{T}}(\mathcal{Z}_{K_2}) & \xrightarrow{\mathsf{g}_2^*} & H^*_{\mathsf{T}}(\mathcal{Z}_W), \end{array}$$

where θ_i^* and g_i^*'s are the obvious pullback maps and $\theta^* := (\theta_1^*, \theta_2^*)$ and $\mathsf{g}^* := \mathsf{g}_1^* - \mathsf{g}_2^*$.

Let B be a $n \times m$ integer matrix of rank n where $n < m$. Let G be the kernel of the corresponding map $\mathsf{T} \to \mathsf{R}$. Note that every subgroup of T can be obtained this way. To obtain what corresponds to Proposition 3.12 for G-equivariant cohomology, we use the two long exact sequences, the Mayer–Vietoris and the relative cohomology sequence:

$$\cdots \longrightarrow H^i_{\mathsf{G}}(\mathcal{Z}_{\tilde{K}}) \longrightarrow H^i_{\mathsf{G}}(\mathcal{Z}_{K_1}) \oplus H^i_{\mathsf{G}}(\mathcal{Z}_{K_2}) \xrightarrow[\mathsf{g}^*]{} H^*_{\mathsf{G}}(\mathcal{Z}_W) \longrightarrow \cdots,$$

$$\cdots \longrightarrow H^i_{\mathsf{G}}(\mathcal{Z}_{\tilde{K}}, \mathcal{Z}_K) \longrightarrow H^i_{\mathsf{G}}(\mathcal{Z}_{\tilde{K}}) \longrightarrow H^i_{\mathsf{G}}(\mathcal{Z}_K) \longrightarrow \cdots.$$

When these sequences split into short exact sequences, we can write the equivariant cohomology of $\mathcal{Z}_K$ in terms of the connected sum of rings.

Proposition 4.4. *If $H^*_{\mathsf{G}}(\mathcal{Z}_K), H^*_{\mathsf{G}}(\mathcal{Z}_{K_1}), H^*_{\mathsf{G}}(\mathcal{Z}_{K_2})$ and $H^*_{\mathsf{G}}(\mathcal{Z}_W)$ are concentrated in even degree, then $H^*_{\mathsf{G}}(\mathcal{Z}_K)$ is isomorphic as a ring to the connected sum $H^*_{\mathsf{G}}(\mathcal{Z}_{K_1}) \#^{\theta^*}_{\mathsf{g}^*} H^*_{\mathsf{G}}(\mathcal{Z}_{K_2})$ defined by the diagram*

$$\begin{array}{ccc} H^*_{\mathsf{G}}(\mathcal{Z}_{\tilde{K}}, \mathcal{Z}_K) & \xrightarrow{\theta_1^*} & H^*_{\mathsf{G}}(\mathcal{Z}_{K_1}) \\ \Big\downarrow{\scriptstyle \theta_2^*} & & \Big\downarrow{\scriptstyle \mathsf{g}_1^*} \\ H^*_{\mathsf{G}}(\mathcal{Z}_{K_2}) & \xrightarrow{\mathsf{g}_2^*} & H^*_{\mathsf{G}}(\mathcal{Z}_W), \end{array}$$

where θ_i^* and g_i^*'s are the obvious pullback maps and $\theta^* := (\theta_1^*, \theta_2^*)$ and $\mathsf{g}^* := \mathsf{g}_1^* - \mathsf{g}_2^*$.

REMARK 4.5. The assumption in Proposition 3.12 is equivalent to the one in Proposition 4.4 by [11]. Moreover, it is also true that, for any simplicial complex K on $[m]$ and for any subgroup G of T, if $H^{\mathrm{odd}}_{\mathsf{G}}(\mathcal{Z}_K) = 0$, then there is a natural isomorphism of rings

$$H^{\mathrm{even}}_{\mathsf{G}}(\mathcal{Z}_K) \cong \mathrm{Tor}_0^{\mathbb{Z}[\mathsf{R}^*]}(\mathbb{Z}[K], \mathbb{Z}).$$

Therefore Proposition 4.4 is a direct consequence of Proposition 3.12.

Let (Δ, H_o) be a generic cut of a simple polytope Δ and regard K_Δ as the connected sum of K_+ and K_- as in Theorem 2.8. In this case, the relative cohomology of the pair $(\mathcal{Z}_{K_+\cup K_-}, \mathcal{Z}_{K_\Delta})$ can be replaced by the cohomology of $\mathcal{Z}_W$. Namely, for any subgroup G of $\mathsf{T} := (\mathrm{U}(1))^{\widetilde{[m]}}$, consider the isomorphism

$$\begin{aligned}\mathscr{T}\colon H^{*-2}_{\mathsf{G}}(\mathcal{Z}_W) \cong H^{*-2}_{\mathsf{G}}(\mathcal{Z}^\circ_W) &\overset{\text{Thom}}{\cong} H^*_{\mathsf{G}}(\mathcal{Z}_W, \mathcal{Z}_W \setminus \mathcal{Z}^\circ_W)\\ &\cong H^*_{\mathsf{G}}(\mathcal{Z}_W, \mathcal{Z}_{\mathrm{Del}_{\{o\}} W}) \cong H^*_{\mathsf{G}}(\mathcal{Z}_{K_+\cup K_-}, \mathcal{Z}_{K_\Delta}),\end{aligned}$$

where

$$\mathcal{Z}^\circ_W := \bigcup_{\substack{\sigma\in\mathcal{F}(W)\\ o\in\sigma}} \{0\}^{\{o\}} \times \mathsf{D}^{\sigma\setminus\{o\}} \times (\partial\mathsf{D})^{\widetilde{[m]}\setminus\sigma}$$

and all maps except the second one are pullback maps and the second one is the Thom isomorphism. Composing $\mathscr{T}$ with the pullback, we have the pushforward map

$$\theta_{\pm *}\colon H^*_{\mathsf{G}}(\mathcal{Z}_W) \to H^*_{\mathsf{G}}(\mathcal{Z}_{K_\pm}).$$

Let $\theta^*_\pm\colon H^*_{\mathsf{G}}(\mathcal{Z}_{K_\pm}) \to H^*_{\mathsf{G}}(\mathcal{Z}_W)$ be the pullback maps for the inclusion $W \hookrightarrow K_\pm$. As a corollary of Lemma 3.14, we have

Proposition 4.6. *For a generic cut (Δ, H_o) and any subgroup of $\mathsf{G} \subset \mathsf{T}$, if $H^*_{\mathsf{G}}(\mathcal{Z}_W)$ and $H^*_{\mathsf{G}}(\mathcal{Z}_{K_\Delta})$ are concentrated in even degree, then as rings*

$$H^*_{\mathsf{G}}(\mathcal{Z}_{K_\Delta}) \cong H^*_{\mathsf{G}}(\mathcal{Z}_{K_+}) \#^{\theta_*}_{\theta^*} H^*_{\mathsf{G}}(\mathcal{Z}_{K_-})$$

where the connected sum of rings is defined for the diagram

$$\begin{array}{ccc} H^*_{\mathsf{G}}(\mathcal{Z}_W) & \xrightarrow{\theta_{+*}} & H^*_{\mathsf{G}}(\mathcal{Z}_{K_+}) \\ \downarrow{\scriptstyle\theta_{-*}} & & \downarrow{\scriptstyle\theta^*_+} \\ H^*_{\mathsf{G}}(\mathcal{Z}_{K_-}) & \xrightarrow[\theta^*_-]{} & H^*_{\mathsf{G}}(\mathcal{Z}_W). \end{array}$$

4.2. Application to toric orbifolds. A *labeled polytope* (Δ, b) is an n-dimensional rational simple polytope Δ in $\mathbb{R}^n$ where each facet H_i, $i = 1, \ldots, m$ is labeled by a positive integer b_i. Let $\rho_1, \ldots, \rho_m$ be the inward primitive normal vectors to the facets. Let B be the $n \times m$ integer matrix $[b_1\rho_1, \ldots, b_m\rho_m]$ and also denote the corresponding surjective homomorphism of the tori by $B\colon \mathsf{T} \to \mathsf{R}$ where $\mathsf{T} = \mathrm{U}(1)^m$ and $\mathsf{R} = \mathrm{U}(1)^n$. From a labeled polytope (Δ, b), a symplectic toric orbifold $\mathcal{X}$ is constructed by the symplectic reduction of the complex plane $\mathbb{C}^m$ by $\mathsf{G} := \ker B$. See [10] for the detail. Topologically $\mathcal{X}$ is nothing but the quotient stack given by

$$\mathcal{X} = [\mathcal{Z}_{K_\Delta}/\mathsf{G}]$$

together with the residual R-action.

The cohomology of a quotient stack can be defined as the equivariant cohomology $H^*(\mathcal{X}) := H^*_{\mathsf{G}}(\mathcal{Z}_{K_\Delta})$ (cf. [6]). For a labeled polytope (Δ, b), consider a generic cut of a rational polytope Δ by a rational hyperplane $\tilde{H}_o$. The resulting polytopes $\Delta_\pm$ are endowed with labeling where the new facet H_o is labeled by 1. The corresponding toric orbifolds $\mathcal{X}_\pm$ are the results of the symplectic cut by the one dimensional subgroup of R defined by the rational hyperplane $\tilde{H}_o$. Proposition 4.6 can be rewritten in terms of the cohomology of $\mathcal{X}_\pm$ and the toric suborbifold $\mathcal{X}_o$ corresponding to the facet H_o.

Proposition 4.7. *Let $\mathsf{f}_\pm \colon \mathcal{X}_o \hookrightarrow \mathcal{X}_\pm$ be the inclusion. As graded rings,*

$$H^*(\mathcal{X}; \mathbb{Q}) \cong H^*(\mathcal{X}_+; \mathbb{Q}) \#^{\mathsf{f}_*}_{\mathsf{f}^*} H^*(\mathcal{X}_-; \mathbb{Q})$$

where the connected sum of rings is defined by the diagram

$$\begin{tikzcd} H^*(\mathcal{X}_o; \mathbb{Q}) \arrow[r, "\mathsf{f}_{+*}"'] \arrow[d, "\mathsf{f}_{-*}"'] & H^*(\mathcal{X}_+; \mathbb{Q}) \arrow[d, "\mathsf{f}_+^*"'] \\ H^*(\mathcal{X}_-; \mathbb{Q}) \arrow[r, "\mathsf{f}_-^*"'] & H^*(\mathcal{X}_o; \mathbb{Q}). \end{tikzcd}$$

If $H^*(\mathcal{X}_o)$ and $H^*(\mathcal{X})$ are concentrated in even degree, then the statement holds over $\mathbb{Z}$-coefficients.

Furthermore, Proposition 4.4 can be also applied to write the cohomology of $\mathcal{X}_-$ in terms of $\mathcal{X}$ and $\mathcal{X}_+$ as follows. Let U_o be a small neighborhood of H_o in Δ_+ and let $\Delta'_+ := \Delta_+ \setminus U_o$. Let $\mathcal{Y}$ be the suborbifold of $\mathcal{X}$ defined by the preimage of $\Delta'_+ \subset \Delta$ under the projection (or the moment map) $\mathcal{X} \to \Delta$. Also let $\mathcal{Y}_o$ be the preimage of $H'_o \subset \Delta$ where $H'_o := \Delta'_+ \cap \overline{U_o}$. It is clear that $\mathcal{Y}$ and $\mathcal{Y}_o$ are also naturally suborbifolds of $\mathcal{X}_+$. Let $\mathsf{f} \colon \mathcal{Y} \hookrightarrow \mathcal{X}$ and $\mathsf{f}_+ \colon \mathcal{Y} \hookrightarrow \mathcal{X}_+$ be the inclusions. Consider the maps

$$\theta_1 \colon H^*(\mathcal{Y}, \mathcal{Y}_o) \cong H^*(\mathcal{X}_+; \mathcal{X}_o) \to H^*(\mathcal{X}_+)$$

and

$$\theta_2 \colon \mathsf{g}_1 \colon H^*(\mathcal{Y}, \mathcal{Y}_o) \cong H^*(\mathcal{X}; \mathcal{X}_-) \to H^*(\mathcal{X})$$

where the first isomorphisms are excisions and the second maps are the pullback maps. Then we have the following statement that is actually a special case of what is proved by Hausmann–Knutson [9] for more general symplectic cuts.

Proposition 4.8. *If* f^* *and* f_+^* *are surjective with* $\mathbb{Z}$*-coefficients, then as graded rings,*

$$H^*(\mathcal{X}_-) \cong H^*(\mathcal{X}) \#_{\mathsf{f}^*}^{\theta} H^*(\mathcal{X}_+)$$

where the connected sum of rings is defined by the diagram

$$\begin{tikzcd} H^*(\mathcal{Y}, \mathcal{Y}_o) \arrow[r, "\theta_1"'] \arrow[d, "\theta_2"'] & H^*(\mathcal{X}) \arrow[d, "\mathsf{f}^*"'] \\ H^*(\mathcal{X}_+) \arrow[r, "\mathsf{f}_+^*"'] & H^*(\mathcal{Y}). \end{tikzcd}$$

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References

[1] H. Ananthnarayan, L.L. Avramov and W.F. Moore: *Connected sums of Gorenstein local rings*, J. Reine Angew. Math. **667** (2012), 149–176.

[2] W. Bruns and J. Herzog: Cohen–Macaulay Rings, Cambridge Studies in Advanced Mathematics **39**, Cambridge Univ. Press, Cambridge, 1993.

[3] V.M. Buchstaber and T.E. Panov: Torus Actions and Their Applications in Topology and Combinatorics, University Lecture Series **24**, Amer. Math. Soc., Providence, RI, 2002.

[4] V.M. Bukhshtaber and T.E. Panov: *Tours actions and the combinatorics of polytopes*, Tr. Mat. Inst. Steklova **225** (1999), 96–131.

[5] M.W. Davis and T. Januszkiewicz: *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), 417–451.

[6] D. Edidin: *Equivariant geometry and the cohomology of the moduli space of curves*, arXiv:1006.2364.

[7] G. Ewald and C. Schulz: *Nonstarshaped spheres*, Arch. Math. (Basel) **59** (1992), 412–416.

[8] M. Franz and V. Puppe: *Exact cohomology sequences with integral coefficients for torus actions*, Transform. Groups **12** (2007), 65–76.

[9] J.-C. Hausmann and A. Knutson: *Cohomology rings of symplectic cuts*, Differential Geom. Appl. **11** (1999), 197–203.

[10] E. Lerman and S. Tolman: *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, Trans. Amer. Math. Soc. **349** (1997), 4201–4230.

[11] S. Luo, T. Matsumura and W.F. Moore: *Moment angle complexes and big Cohen–Macaulayness*, arXiv:1205.1566.

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