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INFINITE MARKOV PARTICLE SYSTEMS ASSOCIATED WITH ABSORBING STABLE MOTION ON A HALF SPACE

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Abstract

In general, for a Markov process which does not have an invariant measure, it is possible to realize a stationary Markov process with the same transition probability by extending the probability space and by adding new paths which are born at random times. The distribution (which may not be a probability measure) is called a *Kuznetsov measure*. By using this measure we can construct a stationary Markov particle system, which is called an *equilibrium process with immigration*. This particle system can be decomposed as a sum of the original part and the immigration part (see [2]). In the present paper, we consider an *absorbing stable motion* on a half space H, i.e., a time-changed absorbing Brownian motion on H by an increasing strictly stable process. We first give the martingale characterization of the particle system. Secondly, we discuss the finiteness of the number of particles near the boundary of the immigration part. (cf. [2], [3], [4].)

1. Introduction

Let $0 < \alpha < 2$, $d \geq 1$ and $H = \mathbf{R}^{d-1} \times (0, \infty)$ be the half space. Let $(w^-(t), P_x^-) = (w^{-,\alpha}(t), P_x^{-,\alpha})$ be an absorbing α-stable motion on H; $w^-(0) = x \in H$, i.e., $w^{-,\alpha}(t) = B^0(y^{\alpha/2}(t))$, where $B^0(t)$ is an absorbing Brownian motion on H; $B^0(0) = x$ and $y^{\alpha/2}(t)$ is an increasing strictly $\alpha/2$-stable process on $[0, \infty)$ such that $y^{\alpha/2}(0) = 0$ and $(y^{\alpha/2}(t))$ is independent of $(B^0(t))$. The Laplace transform of $y^{\alpha/2}(t)$ is given by $E[\exp(-uy^{\alpha/2}(t))] = \exp[-ctu^{\alpha/2}]$ $(c > 0)$ and the life time $\zeta(w^-) = \zeta(w^{-,\alpha})$ of $w^-(t)$ is given as $\zeta(w^-) = \inf\{t > 0;\ y^{\alpha/2}(t) \geq \zeta(B^0)\}$, where $\zeta(B^0)$ is the life time of $B^0(t)$.

Let $H_\Delta = H \cup \Delta$ with a fixed extra point $\Delta \notin H$, Define a path space W as $w \in W \overset{\text{def}}{\Longleftrightarrow} w\colon \mathbf{R} \to H_\Delta$; there exists $(\beta(w), \gamma(w)) \neq \emptyset$ such that w is H-valued càdlàg path on $(\beta(w), \gamma(w))$, $w(t) = \Delta$ for $t \notin (\beta(w), \gamma(w))$. Moreover set $W_0 = W \cap \{\beta = 0\}$ be a family of all càdlàg excursions on H.

In order to define a Kuznetsov measure associated with $(w^-(t), P_x^-)$, we need an entrance law $(\nu_t)_{t>0}$, that is, a family of σ-finite measures such that $\nu_s P_{t-s}^- = \nu_t$ for $s < t$, where (P_t^-) is the transition semi-group of $(w^-(t), P_x^-)$. (ν_t) is defined as for

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$x=(\tilde{x},x_d)\in H$ $(\tilde{x}\in\partial H=\mathbf{R}^{d-1},\ x_d>0)$,

$$\nu_t(dy):=\int_{\mathbf{R}^{d-1}} d\tilde{x}\,\partial_d P_t^-(x,dy)|_{x_d=0+},$$

where $\partial_d=\partial/\partial x_d$. Then we also have

$$\nu_t(dy)=\nu_t(y_d)\,dy \quad \text{with} \quad \nu_t(y_d)=-2\int_{\mathbf{R}^{d-1}} \partial_d p_t^\alpha((\tilde{x},y_d))\,d\tilde{x},$$

where $p_t^\alpha(x)$ is a density of the rotation invariant α-stable process $(w^\alpha(t),P_0^\alpha)$ on $\mathbf{R}^d$. In fact, the transition density of $P_t^-(x,dy)=p_t^-(x,y)\,dy$ is given by the following.

$$\begin{aligned} p_t^-(x,y)\equiv p_t^{\alpha,-}(x,y) &= p_t^\alpha(y-x)-p_t^\alpha((\tilde{y},-y_d)-x)\\ &= p_t^\alpha(y-x)-p_t^\alpha(y-(\tilde{x},-x_d))\\ &= -\int_{-x_d}^{x_d} \partial_d p_t^\alpha((\tilde{y}-\tilde{x},y_d+v))\,dv. \end{aligned}$$

Hence the density of an entrance law at a boundary point $\tilde{x}\in\partial H$ is defined as

$$\nu_t^{\tilde{x}}(y)=\partial_d p_t^-(x,y)|_{x_d=0+}=-2\partial_d p_t^\alpha((\tilde{y}-\tilde{x},y_d)).$$

We define the density of an entrance law by

$$\nu_t(y):=\int_{\mathbf{R}^{d-1}} \nu_t^{\tilde{x}}(y)\,d\tilde{x}=-2\int_{\mathbf{R}^{d-1}} \partial_d p_t^\alpha((\tilde{x},y_d))\,d\tilde{x}.$$

Since the last term is independent of $\tilde{y}$, we can write $\nu_t(y)=\nu_t(y_d)$.

By $p_t^\alpha(x)=t^{-d/\alpha}p_1^\alpha(t^{-1/\alpha}x)$, we have $\partial_d p_t^\alpha(x)=t^{-(d+1)/\alpha}\partial_d p_1^\alpha(t^{-1/\alpha}x)$ and

$$\nu_t(y_d)=t^{-2/\alpha}\nu_1(t^{-1/\alpha}y_d).$$

The density of the invariant measure is given as

$$m(y_d)=m^\alpha(y_d)=\int_0^\infty dt\,\nu_t(y_d)=c_\alpha y_d^{\alpha-2}$$

with a positive constant

$$c_\alpha=\alpha\int_0^\infty u^{1-\alpha}\nu_1(u)\,du=-2\alpha\int_0^\infty du\,u^{1-\alpha}\int_{\mathbf{R}^{d-1}} \partial_d p_1^\alpha(\tilde{v},u)\,d\tilde{v}.$$

Note that this integral is finite because it holds that with some constant $C>0$ (see [3]),

$$|\partial_d p_1^\alpha(x)|\le C(1\wedge x_d\wedge x_d|x|^{-2-d-\alpha}) \quad \text{for} \quad x\in H.$$

Let

$$m(dx) = m^\alpha(dx) := \int_0^\infty dt\ \nu_t(dx) = m^\alpha(x_d)\,dx = c_\alpha x_d^{\alpha-2}\,dx.$$

The *excursion law* Q^0 on W_0 and the *Kuznetsov measure* Q_m on W are defined by the following:

$$Q^0 := \lim_{r\downarrow 0}\int_H \nu_r(dx)P_x^-, \quad Q_m := \int_{-\infty}^\infty \theta_{-s}(Q^0)\,ds,$$

where θ_{-s} is a time-shift operator such that $\theta_{-s}w(t) := w(t-s)$ for $w \in W_0$ and $\theta_{-s}(Q^0)(A) = Q^0(\{w \in W_0;\, \theta_{-s}w \in A\})$ for $A \subset W \cap \{\beta = -s\}$. Note that these measures are infinite. Then $(w(t), Q_m)$ is a stationary Markov process with the invariant measure $m = m^\alpha$ and the same transition probability as the absorbing α-stable motion $(w^-(t), P_x^-) = (w^{-,\alpha}(t), P_x^{-,\alpha})$.

Furthermore by using this process we can introduce an infinite stationary Markov particle system; the *equilibrium process* $(X_t, \mathbf{P})$ as follows.

$$\omega \in \Omega \overset{\text{def}}{\iff} \omega = \sum \delta_{w_n}, \quad w_n \in W,$$
$$X_t(\omega) = \omega(t)|_H \quad \text{for} \quad \omega \in \Omega,$$
$$\mathbf{P} = \Pi_{Q_m} \quad \text{is } Q_m\text{-Poisson measure,}$$

i.e., the distribution of the Poisson random measure with intensity Q_m. Then $(X_t, \mathbf{P})$ is a stationary Markov particle system such that

$$\mathbf{E}\exp[-\langle X_t, f\rangle] = \exp[-\langle m, 1 - e^{-f}\rangle].$$

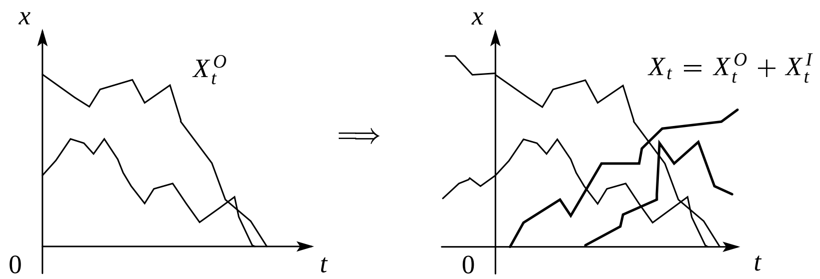
for nonnegative measurable function f.

Note that the equilibrium process $(X_t, \mathbf{P})$ is a stationary independent Markov particle system such that the initial distribution is a Poisson measure. More precisely, let m be a σ-finite measure on a state space S. We consider many independent identically distributed Markov processes $\{w_n(t)\}$ which have an invariant measure m. For the particle system $X_t = \sum \delta_{w_n(t)}$, if the initial state X_0 is a Poisson random measure with intensity m, then $\{X_t\}$ is a stationary Markov process. This particle system $\{X_t\}$ is called the *equilibrium process* (cf. [6]). This definition is equivalent to the above one (see below). The typical example is the independent Brownian particle system in $\mathbf{R}^d$ with the Lebesgue measure $m(dx) = dx$ on $\mathbf{R}^d$. In this case Q_m is identified as $P_m := \int_{\mathbf{R}^d} P_x\,dx$, where P_x is a distribution of a Brownian motion starting from $x \in \mathbf{R}^d$.

If we restrict to $t \geq 0$, then this process can be decomposed as $X_t = X_t^O + X_t^I$ such that

$$X_t^O = \int_{(-\infty,0)} \int_{W_0} \delta_{w(t-s)}|_H N^0(ds\,dw),$$
$$X_t^I = \int_{[0,t]} \int_{W_0} \delta_{w(t-s)}|_H N^0(ds\,dw),$$

where $N^0(ds\,dw)$ is the Poisson random measure with intensity $ds\,Q^0(dw)$ on $\mathbf{R} \times W_0$. X_t^O is the original part of X_t, X_t^I is the immigration part of X_t and they are independent. This process is called the *infinite Markov particle system with singular immigration associated with absorbing stable motions on* H.



Moreover for a fixed suitable measure $\mu = \sum \delta_{x_n}$ ($\in \mathcal{M}_{p,0}$, see the next section), it is possible to construct a Markov process $(X_t, \mathbf{P}_\mu)$ starting from μ such that $X_t = X_t^O + X_t^I$, where X_t^I is the same as above, X_t^O is the independent Markov particle system starting from μ, i.e., $X_0 = X_0^O = \mu$ and $\{X_t^O\}$, $\{X_t^I\}$ are independent. Furthermore $\mathbf{P}_\mu = \mathbf{P}_\mu^O \otimes \mathbf{P}^I$ where $\mathbf{P}_\mu^O = \bigotimes P_{x_n}^-$ and $\mathbf{P}^I$ is the distribution of $\{X_t^I\}$, that is, the $\int_0^\infty \theta_{-s}(Q^0)\,ds$-Poisson measure. Let $\{\mathcal{F}_t\}$ be the filtration generated by $\{X_t\}$. Then it satisfies that if $s < t$, then

$$\mathbf{E}_\mu[\exp(-\langle X_t,\ f\rangle) \mid \mathcal{F}_s] = \exp\left[-\langle X_s,\ V_{t-s} f\rangle - \int_0^{t-s} \langle \nu_r,\ 1 - e^{-f}\rangle\, dr\right]$$

for some nonnegative suitable functions f on H ($f \in D_p$, see also the next section), where

$$V_t f(x) = -\log E_x^-[\exp(-f(w^-(t))] = -\log\{1 - P_t^-(1 - e^{-f})(x)\}.$$

Note that if we let Π_m be the m-Poisson measure, then the distribution of the equilibrium process can be also defined as $\mathbf{P} = \int \Pi_m(d\mu)\mathbf{P}_\mu$ and we also have

$$\begin{aligned}\mathbf{E}\exp[-\langle X_t,\ f\rangle] &= \exp\left[-\langle m,\ 1 - e^{-V_t f}\rangle - \int_0^t \langle \nu_r,\ 1 - e^{-f}\rangle\, dr\right]\\ &= \exp[-\langle m,\ 1 - e^{-f}\rangle].\end{aligned}$$

by $1 - e^{-V_t f} = P_t^-(1 - e^{-f})$ and $mP_t^- = \int_0^\infty \nu_r P_t^- \, dr = \int_0^\infty \nu_{r+t} \, dr = \int_t^\infty \nu_r \, dr$.

In §2 we consider the martingale problem for $(X_t, \mathbf{P}_\mu)$. In order to give the martingale characterization we have to investigate the properties of the transition probabilities of absorbing stable motions in H (which are given in [4]) and the entrance laws.

In §3 we discuss the finiteness of the number of particles near the boundary of immigration. The obtained results and the proofs are similar to the case of absorbing the Brownian motion in [2]. However we have to extend the case by using the scaling property of the absorbing stable motion.

2. Martingale Problem for $(X_t, \mathbf{P}_\mu)$

The martingale problem can be shown by the same way as in case of the absorbing Brownian motion on H in [2] and the independent Markov particle system with no immigration in [4].

Let $h_0(v)$ be a C_b^∞-function on $(0, \infty)$ such that $0 < h_0 \le 1$ on $(0, \infty)$, $h_0(v) = v$ for $v \in (0, 1/2]$, $h_0(v) = 1$ for $v \ge 1$ and $\|h_0'\|_\infty = 1$.

Let $d < p < d + \alpha$. Set $g_p(x) := (1 + |x|^2)^{-p/2}$, $g_{p,0}(x) := g_p(x) h_0(x_d)$ for $x \in H$ and

$$f \in C_p \equiv C_p(H) \overset{\text{def}}{\iff} f \in C(\mathbf{R}^d)|_H; \quad \|f/g_p\|_\infty < \infty,$$

$$f \in C_{p,0} \equiv C_{p,0}(H) \overset{\text{def}}{\iff} f \in C(\mathbf{R}^d)|_H; \quad \|f/g_{p,0}\| < \infty.$$

Moreover set

$$f \in C_{p,0}^3 \equiv C_{p,0}^3(H) \overset{\text{def}}{\iff} f \in C_b^3(\mathbf{R}^d)|_H;$$
$$\text{for} \quad i, j \ne d, \ f, \partial_d^2 f, \partial_i f, \partial_{ij}^2 f \in C_{p,0} \quad \text{and} \quad \partial_d f, \partial_{id}^2 f \in C_p.$$

We define a space of counting measures $\mathcal{M}_{p,0}$ by

$$\mu \in \mathcal{M}_{p,0} \equiv \mathcal{M}_{p,0}(H) \overset{\text{def}}{\iff} \mu = \sum \delta_{x_n} \quad \text{on } H \text{ such that} \quad \langle \mu, g_{p,0} \rangle < \infty.$$

$\mathcal{M}_{p,0}$ is furnished with the vague topology, i.e.,

$$\mu_n \to \mu \text{ in } \mathcal{M}_{g_{p,0}} \overset{\text{def}}{\iff} \sup \langle \mu_n, g_{p,0} \rangle < \infty, \ \langle \mu_n, f \rangle \to \langle \mu, f \rangle \quad \text{for all} \quad f \in C_c,$$

where $C_c \equiv C_c(H)$ denotes the space of all continuous functions with compact supports in H. Then it holds that $\langle \mu, g_{p,0} \rangle \le \liminf \langle \mu_n, g_{p,0} \rangle < \infty$, and thus, $\mu \in \mathcal{M}_{g_{p,0}}$. Note that for each $1 \le K < \infty$, we define

$$\begin{cases} \mu \in \mathcal{M}_{g_{p,0},K} \equiv \mathcal{M}_{g_{p,0},K}(H) \overset{\text{def}}{\iff} \mu \in \mathcal{M}_{g_{p,0}}, \langle \mu, g_{p,0} \rangle \le K, \\ \mu_n \to \mu \quad \text{in } \mathcal{M}_{g_{p,0},K} \overset{\text{def}}{\iff} \langle \mu_n, f \rangle \to \langle \mu, f \rangle \quad \text{for all} \quad f \in C_c. \end{cases}$$

Then $\mathcal{M}_{g_{p,0},K}$ is a Polish space and $\mu_n \to \mu$ in $\mathcal{M}_{g_{p,0}}$ is equivalent to $\mu_n \to \mu$ in $\mathcal{M}_{g_{p,0},K}$ for some $K \geq 1$. Hence $\mathcal{M}_{g_{p,0}}$ is a metrizable and separable space (for the metric ρ in $\mathcal{M}_{p,0}$, see [4] in which $g_0 = g_{p,0}$).

For $\mu \in \mathcal{M}_{p,0}$, let $(X_t, \mathbf{P}_\mu)$ be the independent Markov particle system with singular immigration associated with $(w^-(t), P_x^-) = (w^{-,\alpha}(t), P_x^{-,\alpha})$ and $m = m^\alpha$ on H. We denote the generator of $(w^{-,\alpha}(t), P_x^{-,\alpha})$ as $A = A^{-,\alpha}$ (more concrete formula is given latter).

The generator $\mathcal{L}$ of this particle system is given as $\mathcal{L} = \mathcal{L}^O + \mathcal{L}^I$, where $\mathcal{L}^O$, $\mathcal{L}^I$ are given by the following: for $f \in C_c^\infty$,

$$\begin{aligned}\mathcal{L}^O e^{-\langle \cdot, f\rangle}(\mu) &= -\langle \mu, e^f A(1-e^{-f})\rangle e^{-\langle \mu, f\rangle}\\ &= -\langle \mu, Af - \Gamma f\rangle e^{-\langle \mu, f\rangle},\end{aligned}$$

where $\Gamma f := Af - e^f A(1-e^{-f})$, and

$$\begin{aligned}\mathcal{L}^I e^{-\langle \cdot, f\rangle}(\mu) &= -\lim_{r\downarrow 0}\langle \nu_r, 1-e^{-f}\rangle e^{-\langle \mu, f\rangle}\\ &= -c_0\langle \tilde{m}, \partial_d f(\,\cdot\,, 0+)\rangle e^{-\langle \mu, f\rangle},\end{aligned}$$

where $c_0 = \int_0^\infty y\nu_1(y)\,dy$ and $\tilde{m}$ is the Lebesgue measure on $\mathbf{R}^{d-1}$. Note that c_0 depends on α and it is finite. In fact, by $|\partial_d p_1^\alpha(x)| \leq C(1 \wedge x_d \wedge x_d|x|^{-2-d-\alpha})$ for $x \in H$,

$$c_0 = \int_0^\infty y\nu_1(y)\,dy = -2\int_0^\infty \left(y\int_{\mathbf{R}^{d-1}} \partial_d p_1^\alpha(\tilde{x}, y)\,d\tilde{x}\right)dy < \infty.$$

Hence

$$\mathcal{L}e^{-\langle \cdot, f\rangle}(\mu) = -\{\langle \mu, Af - \Gamma f\rangle + c_0\langle \tilde{m}, \partial_d f(\,\cdot\,, 0+)\rangle\}e^{-\langle \mu, f\rangle}.$$

Set $D_p := C_{p,0}^3$ (this is a core for A, see (B1) in §3 of [4]). Moreover for a function $f \in D_p$, let $\bar{f}$ be an extension of f to on $\mathbf{R}^d$ defined as

$$\bar{f}(x) = \begin{cases} f(x) & (x_d > 0),\\ f(\tilde{x}, 0+) = 0 & (x_d = 0),\\ -f(\tilde{x}, -x_d) & (x_d < 0).\end{cases}$$

Note that if $x \in H$, then $\bar{f}(x) = f(x)$. The generator $A \equiv A^- \equiv A^{-,\alpha}$ is given as (A is the same as L^- in §4 of [3], however in which we have some miss-prints)

$$\begin{aligned}A^{-,\alpha} f(x) &= c\int_{\mathbf{R}^d\setminus\{0\}}[\bar{f}(x+y) - \bar{f}(x) - \nabla\bar{f}(x)\cdot yI(|y|<1)]\frac{dy}{|y|^{d+\alpha}}\\ &= c\int_{\mathbf{R}^{d-1}} d\tilde{y}\int_{-x_d}^{x_d}[f(x+y) - f(x) - \nabla f(x)\cdot yI(|y|<1)]\frac{dy_d}{|y|^{d+\alpha}}\\ &\quad + c\int_{\mathbf{R}^{d-1}} d\tilde{y}\int_{x_d}^\infty [f(x+y) - f(\tilde{y}+\tilde{x}, y_d - x_d) - 2f(x)]\frac{dy_d}{|y|^{d+\alpha}}\end{aligned}$$

with a positive constant c. We can also write that if $0 < \alpha < 1$, then

$$\begin{aligned} &A^{-,\alpha} f(x) \\ &= c \int_{\mathbf{R}^d \setminus \{x\}} [\bar{f}(y) - \bar{f}(x)] \frac{dy}{|y-x|^{d+\alpha}} \\ &= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \left\{ \int_0^\infty [f(y) - f(x)] K(x,y)\, dy_d - 2 f(x) \int_0^\infty \frac{dy_d}{|(\tilde{y}-\tilde{x}, y_d + x_d)|^{d+\alpha}} \right\}, \\ &= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_0^\infty [f(y) - f(x)] K(x,y)\, dy_d - f(x) k(x), \end{aligned}$$

and that if $1 \le \alpha < 2$, then

$$\begin{aligned} &A^{-,\alpha} f(x) \\ &= c \int_{\mathbf{R}^d \setminus \{x\}} [\bar{f}(y) - \bar{f}(x) - \nabla \bar{f}(x) \cdot (y-x) I(|y-x|<1)] \frac{dy}{|y-x|^{d+\alpha}} \\ &= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \left\{ \int_0^\infty [f(y) - f(x) - \nabla f(x) \cdot (y-x) I(|y-x|<1)] K(x,y)\, dy_d \right. \\ &\qquad + \int_0^\infty [-2 f(x) - \nabla f(x) \cdot (y-x) I(|y-x|<1) \\ &\qquad\qquad \left. - \nabla f(x) \cdot (\tilde{y}-\tilde{x}, -y_d - x_d) I(|(\tilde{y}-\tilde{x}, y_d + x_d)|<1)] \frac{dy_d}{|(\tilde{y}-\tilde{x}, y_d + x_d)|^{d+\alpha}} \right\} \\ &= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_0^\infty [f(y) - f(x) - \nabla f(x) \cdot (y-x) I(|y-x|<1)] K(x,y)\, dy_d \\ &\quad - f(x) k(x) + \nabla f(x) \cdot c(x), \end{aligned}$$

where

$$\begin{aligned} K(x,y) &= \frac{I(y \neq x)}{|y-x|^{d+\alpha}} - \frac{1}{|(\tilde{y}-\tilde{x}, y_d + x_d)|^{d+\alpha}}, \\ k(x) = k(x_d) &= 2c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{x_d}^\infty \frac{dy_d}{|y|^{d+\alpha}} \end{aligned}$$

and

$$\begin{aligned} c(x) &= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_0^\infty [-(\tilde{y}-\tilde{x}, -y_d - x_d) I(|(\tilde{y}-\tilde{x}, y_d + x_d)| < 1) \\ &\qquad - (y-x) I(|y-x| < 1)] \frac{dy_d}{|(\tilde{y}-\tilde{x}, y_d + x_d)|^{d+\alpha}} \\ &= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_0^\infty [(\tilde{y}, y_d + x_d) I(|(\tilde{y}, y_d + x_d)| < 1) \\ &\qquad - (\tilde{y}, y_d - x_d) I(|(\tilde{y}, y_d - x_d)| < 1)] \frac{dy_d}{|(\tilde{y}, y_d + x_d)|^{d+\alpha}}. \end{aligned}$$

For $\eta \in \mathcal{M}_{p,0}$, let $F(\eta) = \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle) \in \mathcal{D}_0 \overset{\text{def}}{\Longleftrightarrow} \Phi(x) \in C^\infty(\mathbf{R}^n)$ is a polynomial growth function with polynomial growth derivatives of all orders and $f_i \in D_p$, $i = 1,\dots,n$. For this functional $F(\eta)$, the generator $\mathcal{L} = \mathcal{L}^O + \mathcal{L}^I$ can be expressed by the following form:

$$\begin{aligned}
&\mathcal{L}^O F(\eta) \\
&= \sum_{i=1}^n \partial_i \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle)\langle \eta, Af_i\rangle \\
&\quad + \int_H \Biggl\{ \int_{H\setminus\{x\}} \nu(x, dy) \biggl[\Phi(\langle \eta, f_1\rangle + f_1(y) - f_1(x), \dots, \langle \eta, f_n\rangle + f_n(y) - f_n(x)) \\
&\qquad\qquad - \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle) \\
&\qquad\qquad - \sum_{i=1}^n \partial_i \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle)(f_i(y) - f_i(x)) \biggr] \\
&\qquad + k(x) \biggl[\Phi(\langle \eta, f_1\rangle - f_1(x), \dots, \langle \eta, f_n\rangle - f_n(x)) - \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle) \\
&\qquad\qquad + \sum_{i=1}^n \partial_i \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle) f_i(x) \biggr] \Biggr\} \eta(dx),
\end{aligned}$$

where $\nu(x, dy)$ is the Lévy kernel of A (i.e., $\nu(x, dy) = cK(x, y)\,dy$) and $k(x) = k(x_d)$ is the killing rate given in the above. Moreover

$$\mathcal{L}^I F(\eta) = c_0 \sum_{i=1}^n \partial_i \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle)\langle \tilde{m}, \partial_d f(\,\cdot\,, 0+)\rangle.$$

Note that Γ can be expressed as

$$\Gamma f(x) = \int_{H\setminus\{x\}} (e^{-[f(y)-f(x)]} - 1 + [f(y) - f(x)])\nu(x, dy) + k(x)(e^{f(x)} - 1 - f(x)).$$

Theorem 1 (Martingale problem for $(\mathcal{L}, \mathcal{D}_0, \mu)$). *Let* $\mu \in \mathcal{M}_{p,0}$.
(i) $\mathbf{P}_\mu(X_0 = \mu) = 1$ *and for* $F(\eta) = \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle) \in \mathcal{D}_0$,

$$M_t^F = F(X_t) - F(X_0) - \int_0^t \mathcal{L}F(X_s)\,ds \quad \textit{is a } \mathbf{P}_\mu\textit{-martingale.}$$

(ii) *If there is a probability measure* $\mathbf{Q}_\mu$ *on* $\mathbf{D} = \mathbf{D}([0, \infty) \to \mathcal{M}_{p,0})$ *such that the canonical process* $\tilde{X}_t(\omega) = \omega(t)$ $(\omega \in \mathbf{D})$ *satisfies the same conditions as* $(X_t, \mathbf{P}_\mu)$ *in* (i) *and*

$$\int_0^t \langle \tilde{X}_s, g_p\rangle\,ds < \infty \quad \mathbf{Q}_\mu\textit{-a.s. for all} \quad t \ge 0,$$

then $\mathbf{Q}_\mu = \mathbf{P}_\mu \circ X^{-1}$ *on* $\mathbf{D}$, *that is, martingale problem for* $(\mathcal{L}, \mathcal{D}_0, \mu)$ *on* $\mathbf{D}$ *is well-posed.*

Proof of (i) in Theorem 1. In order to show it we need several lemmas.

Lemma 1. $\sup_{0<r\le T}\langle \nu_r, g_{p,0}\rangle < \infty$ *for each* $T > 0$.

Proof. We first note that $|\partial_d g_{p,0}(x)| \le (p+1)g_p(x) \le (p+1)g_p(\tilde{x})$ for any $x = (\tilde{x}, x_d) \in H$. Moreover note that $g_{p,0}(x) = x_d\, \partial_d g_{p,0}((\tilde{x}, \theta x_d))$ with some $0 < \theta < 1$. Now by using scaling property; $\nu_t(y) = t^{-2/\alpha}\nu_1(t^{-1/\alpha}y)$ and changing of variable $t^{-1/\alpha}x_d = y$ we can get

$$\begin{aligned}
\langle \nu_r, g_{p,0}\rangle &= \int_{\mathbf{R}^{d-1}} d\tilde{x} \int_0^\infty t^{-2/\alpha}\nu_1(t^{-1/\alpha}x_d) g_{p,0}(x)\, dx_d \\
&= \int_{\mathbf{R}^{d-1}} d\tilde{x} \int_0^\infty t^{-1/\alpha}\nu_1(y) g_{p,0}(\tilde{x}, t^{-1/\alpha}y)\, dy \\
&= \int_{\mathbf{R}^{d-1}} d\tilde{x} \int_0^\infty \nu_1(y) y\, \partial_d g_{p,0}(\tilde{x}, \theta t^{-1/\alpha}y)\, dy \\
&\le C \int_{\mathbf{R}^{d-1}} g_p(\tilde{x})\, d\tilde{x} \int_0^\infty y\nu_1(y)\, dy < \infty.
\end{aligned}$$
□

By this lemma the following holds: for every $f \in D_p$, noting that $f(\tilde{x}, 0+) = 0$,

$$\lim_{r\downarrow 0}\langle \nu_r, 1 - e^{-f}\rangle = \lim_{r\downarrow 0}\langle \nu_r, f\rangle = c_0\langle \tilde{m}, f(\,\cdot\,, 0+)\rangle$$

and if $n \ge 2$, then $\lim_{r\downarrow 0}\langle \nu_r, f\rangle^n = 0$.

The following is given in [4] as Lemma 4.1.

Lemma 2. *For each* $f \in C_c^\infty$ *and* $T > 0$, $\sup_{t\in[0,T]} \|g_{p,0}^{-1}\,\partial_t V_t f\|_\infty < \infty$.

Proof. Let $g_0 \equiv g_{p,0}$. In [4] we have $A^- C^3_{p,0} \subset C_{p,0}$ and $\sup_{t\le T}\|g_0^{-1}P_t^- g_0\|_\infty < \infty$. Hence by $\|V_t f\|_\infty \le \|f\|_\infty$ and $|A^-(1 - e^{-f})| \le C g_0$ we have

$$|\partial_t V_t f| = |e^{V_t f} P_t^- A^-(1 - e^{-f})| \le C e^{\|f\|_\infty} P_t^- g_0.$$

Thus the claim follows. □

From the above results we can show the martingale property (i) of Theorem 1. In fact, it is reduced to the following first result.

Theorem 2. *For* $f \in C_c^\infty$,

$$e^{-\langle X_t, f\rangle} - e^{-\langle X_0, f\rangle} - \int_0^t \mathcal{L}e^{-\langle \cdot, f\rangle}(X_s)\, ds$$

is a $\mathbf{P}_\mu$*-martingale. Moreover*

$$H_t(f) = \exp\left[-\langle X_t,\ f\rangle + \int_0^t \langle X_s,\ Af - \Gamma f\rangle\, ds + tc_0\langle \tilde m,\ \partial_d f(\,\cdot\,, 0+)\rangle\right]$$

is also a $\mathbf{P}_\mu$*-martingale.*

Proof. By the above lemma, if $s < t$, then

$$\begin{aligned}
\partial_t \mathbf{E}_\mu[e^{-\langle X_t, f\rangle} \mid \mathcal{F}_s] &= \partial_t \exp\left[-\langle X_s, V_{t-s} f\rangle - \int_0^{t-s} \langle \nu_r, 1 - e^{-f}\rangle\, dr\right] \\
&= \partial_{u=0+} \exp\left[-\langle X_s, V_{t-s+u} f\rangle - \int_0^{t-s+u} \langle \nu_r, 1 - e^{-f}\rangle dr\right] \\
&= \partial_{u=0+}\mathbf{E}_\mu\left[\exp\left\{-\langle X_t, V_u f\rangle - \int_0^u \langle \nu_r, 1 - e^{-f}\rangle\, dr\right\} \,\middle|\, \mathcal{F}_s\right] \\
&= \mathbf{E}_\mu\left[\partial_{u=0+} \exp\left[-\langle X_t, V_u f\rangle - \int_0^u \langle \nu_r, 1 - e^{-f}\rangle\, dr\right] \,\middle|\, \mathcal{F}_s\right] \\
&= \mathbf{E}_\mu[\mathcal{L} e^{-\langle\cdot, f\rangle}(X_t) \mid \mathcal{F}_s],
\end{aligned}$$

where $\partial_{u=0+}$ denotes the right differential operator at $u = 0$. Hence the first claim follows. The second claim follows from Corollary 3.3 of Chapter 2 in [1]. □

In order to show (ii) of Theorem 1, we need the semi-martingale representation of $(X_t, \mathbf{P}_\mu)$.

Theorem 3 (Semi-martingale representation of $(X_t, \mathbf{P}_\mu)$)**.** *Let* $\mu \in \mathcal{M}_{p,0}$. $(X_t, \mathbf{P}_\mu)$ *has the following semi-martingale representation*: *for* $f \in D_p$,

$$\langle X_t,\ f\rangle = \langle X_0,\ f\rangle + \int_0^t \langle X_s,\ Af\rangle\, ds + tc_0\langle \tilde m,\ f(\,\cdot\,, 0+)\rangle + M_t^d(f),$$

where

$$M_t^d(f) = \int_0^t \int_{\mathcal{M}_{p,0}^{\pm}} \langle \mu,\ f\rangle \tilde N(ds, d\mu) \quad \text{is a purely discontinuous } L^2\text{-martingale}$$

with $\tilde N = N - \hat N$ *is the martingale measure such that for* $\Delta X_u = X_u - X_{u-}$,

$$N(ds, d\mu) = \sum_{u;\Delta X_u \neq 0} \delta_{(u,\Delta X_u)}(ds, d\mu): \quad \text{the jump measure of } \{X_t\},$$

$$\hat N(ds, d\mu) = ds \int X_s(dx)\left(\int \nu(x, dy)\delta_{(\delta_y - \delta_x)} + k(x)\delta_{-\delta_x}\right)(d\mu):$$

the compensator of N,

where $\mathcal{M}^{\pm}_{p,0}$ is the family of signed-measures of $\mu^+ - \mu^-$; $\mu^+, \mu^- \in \mathcal{M}_{p,0}$.

Proof. For $f \in C_c^\infty$, let

$$G_t(f) = \exp\left[-\int_0^t \langle X_s, Af - \Gamma f\rangle\, ds - tc_0\langle \tilde{m}, \partial_d f(\,\cdot\,, 0+)\rangle\right]$$

be a continuous process of bounded variation. Since $H_t(f)$ in Theorem 2 is a martingale,

$$Z_t(f) := \exp[-\langle X_t, f\rangle] = H_t(f)G_t(f)$$

is a semi-martingale, more exactly, a special semi-martingale, i.e., a bounded variation part is (locally) integrable. In fact, by Proposition 3.2 of Chapter 2 in [1] we have

$$\begin{aligned} dZ_t(f) &= H_t(f)\, dG_t(f) + G_t(f)\, dH_t(f) \\ &= -(\langle X_t, Af - \Gamma f\rangle + c_0\langle \tilde{m}, \partial_d f(\,\cdot\,, 0+)\rangle)Z_t(f)\, dt + d(\text{martingale}). \end{aligned} \tag{2.1}$$

On the other hand, $\langle X_t, f\rangle$ is also a special semi-martingale. Hence by (1.10) of Chapter 4 in [5], $\langle X_t, f\rangle$ has the following expression:

$$\langle X_t, f\rangle = \langle X_0, f\rangle + C_t(f) + M_t^c(f) + \tilde{N}_t(f) + N_t(f),$$

where $C_t(f)$ is a continuous process of locally bounded variation, $M_t^c(f)$ is a continuous L^2-martingale with quadratic variation $\langle\langle M^c(f)\rangle\rangle_t$, and

$$\begin{aligned} \tilde{N}_t(f) &= \int_0^t \int_{\mathcal{M}^\pm} \langle \mu, f\rangle I(\|\mu\| < 1)\tilde{N}(ds, d\mu), \\ N_t(f) &= \int_0^t \int_{\mathcal{M}^\pm} \langle \mu, f\rangle I(\|\mu\| \geq 1)N(ds, d\mu) \end{aligned}$$

with the jump measure N of X, its compensator $\hat{N}$ and $\tilde{N} = N - \hat{N}$. By using Ito's formula we have

$$\begin{aligned} dZ_t(f) = Z_{t-}(f)\Big\{ &-dC_t(f) + \frac{1}{2}\, d\langle\langle M^c(f)\rangle\rangle_t \\ &+ \int_{\mathcal{M}^\pm} [e^{-\langle \mu, f\rangle} - 1 + \langle \mu, f\rangle] I(\|\mu\| < 1)\hat{N}(dt, d\mu) \\ &+ \int_{\mathcal{M}^\pm} [e^{-\langle \mu, f\rangle} - 1] I(\|\mu\| \geq 1) N(dt, d\mu)\Big\} + d(\text{martingale}) \\ = Z_{t-}(f)\Big\{ &-\Big(dC_t(f) + \int_{\{\|\mu\|\geq 1\}} \langle \mu, f\rangle \hat{N}(dt, d\mu)\Big) + \frac{1}{2}\, d\langle\langle M^c(f)\rangle\rangle_t \\ &+ \int_{\mathcal{M}^\pm} [e^{-\langle \mu, f\rangle} - 1 + \langle \mu, f\rangle]\hat{N}(dt, d\mu)\Big\} + d(\text{martingale}). \end{aligned} \tag{2.2}$$

If we set

$$B_t(f) = C_t(f) + \int_0^t \int_{\{\|\mu\|\geq 1\}} \langle \mu,\ f\rangle \hat{N}(ds, d\mu),$$

then by the expressions (2.1), (2.2) and by the uniqueness of special semi-martingale with predictable locally bounded part (see Theorem 2.1.1 in [5]), we have

$$\begin{aligned} &-dB_t(f) + \frac{1}{2} d\langle\langle M^c(f)\rangle\rangle_t + \int [e^{-\langle \mu, f\rangle} - 1 + \langle \mu,\ f\rangle]\hat{N}(dt, d\mu) \\ &= [-(\langle X_t, Af\rangle + c_0\langle \tilde{m}, \partial_d f(\,\cdot\,, 0+)\rangle) + \langle X_t, \Gamma f\rangle]\, dt. \end{aligned}$$

Hence it is easy to see that $\langle\langle M^c(f)\rangle\rangle_t \equiv 0$, i.e., $M_t^c(f) \equiv 0$,

$$B_t(f) = \int_0^t \langle X_s, Af\rangle\, ds + tc_0\langle \tilde{m}, \partial_d f(\,\cdot\,, 0+)\rangle$$

and

$$\begin{aligned} &\int_0^t \int [e^{-\langle \mu, f\rangle} - 1 + \langle \mu,\ f\rangle]\hat{N}(ds, d\mu) \\ &= \int_0^t \langle X_s, \Gamma f\rangle\, ds \\ &= \int_0^t ds \int X_s(dx)\bigg\{\int (e^{-[f(y)-f(x)]} - 1 + [f(y) - f(x)])\nu(x, dy) \\ &\qquad\qquad + k(x)(e^{f(x)} - 1 - f(x))\bigg\}. \end{aligned}$$

That is,

$$\hat{N}(ds, d\mu) = ds \int X_s(dx)\bigg(\int \nu(x, dy)\delta_{(\delta_y - \delta_x)} + k(x)\delta_{-\delta_x}\bigg)(d\mu).$$

Therefore we have the desired representation of $\langle X_t,\ f\rangle$ for $f \in C_c^\infty$. Finally it is possible to extend $f \in C_c^\infty$ to $f \in D_g$. □

Moreover we can get the following (which is shown in the proof of Corollary 5.1 in [4]).

Lemma 3. *For each* $f \in (C_c^\infty)^+$, $t > 0$, $AV_t f = -\log(1 - P_t^-(1 - e^{-f}))$ *is well-defined and* $AV_t f$ *is continuous in* t *under the norm* $\|\cdot/g_p\|_\infty$, *i.e.,*

$$\|(AV_t f - AV_{t_0} f)/g_p\|_\infty \to 0 \quad (t \to t_0).$$

Proof of (ii) in Theorem 1. In order to show the uniqueness of martingale problem it is enough to show that

$$\exp\left[-\langle \tilde{X}_t, V_{T-t} f\rangle + \int_0^t \langle \nu_{T-r}, 1-e^{-f}\rangle\, dr\right]$$

is $\mathbf{Q}_\mu$-martingale. In fact, this implies

$$\tilde{\mathbf{Q}}_\mu[e^{-\langle \tilde{X}_t, f\rangle} \mid \mathcal{F}_s] = \exp\left[-\langle \tilde{X}_s, V_{t-s} f\rangle - \int_0^{t-s} \langle \nu_u, 1-e^{-f}\rangle\, du\right].$$

Hence this implies the uniqueness in the sense of finite dimensional distributions, and separability of $(\mathcal{M}_{p,0}, \rho)$ implies the uniqueness in the sense of distributions on **D**).

Now for a fixed $f \in (C_c^\infty)^+$, by Lemma 2 we have

$$\partial_t V_t f \quad \text{is continuous in } t \text{ under the norm } \|\cdot\|_{g_{p,0}} = \|\cdot/g_{p,0}\|_\infty.$$

Moreover by the above lemma we see that

$$\Gamma V_t f \in C_b \quad \text{is continuous in } t \text{ under the norm } \|\cdot/g_p\|_\infty$$

and $v_t \equiv v_t^T = V_{T-t} f$ $(0 \le t \le T)$ is the unique solution to the equation:

$$(\partial_t + A - \Gamma)v_t = 0 \quad \text{and} \quad v_T = f.$$

Let $\Phi(v) = e^{-v}$ and

$$\phi_t \equiv \phi_t^T = \exp\left[\int_0^t \langle \nu_{T-r}, 1-e^{-f}\rangle\, dr\right].$$

By Theorem 3 $(\tilde{X}_t, \mathbf{Q}_\mu)$ has the same semi-martingale representation as $(X_t, \mathbf{P}_\mu)$. Hence by using the above results and Ito's formula the following is a $\mathbf{Q}_\mu$-martingale:

$$\begin{aligned}
&\Phi(\langle \tilde{X}_t, v_t\rangle)\phi_t - \Phi(\langle \tilde{X}_0, v_0\rangle)\phi_0 - \int_0^t \Phi(\langle \tilde{X}_s, v_s\rangle)\phi_s \langle \nu_{T-s,1-e^{-f}}\rangle\, ds \\
&- \int_0^t \Phi'(\langle \tilde{X}_s, v_s\rangle)\phi_s(\langle \tilde{X}_s, \partial_s v_s + A v_s\rangle + c_0\langle \tilde{m}, \partial_d v_s(\cdot, 0+)\rangle)\, ds \\
&- \int_0^t \int_{\mathcal{M}_\pm} [\Phi(\langle \tilde{X}_s + \eta, v_s\rangle) - \Phi(\langle \tilde{X}_s, v_s\rangle) - \Phi'(\langle \tilde{X}_s, v_s\rangle)\langle \eta, v_s\rangle]\phi_s \hat{N}(ds\, d\eta) \\
&= \exp[-\langle \tilde{X}_t, v_t\rangle]\phi_t - \exp[-\langle \tilde{X}_0, v_0\rangle] + \int_0^t \langle \tilde{X}_s, \partial_s v_s + A v_s\rangle \exp[-\langle \tilde{X}_s, v_s\rangle]\phi_s\, ds \\
&\quad - \int_0^t \langle \tilde{X}_s, \Gamma v_s\rangle \exp[-\langle \tilde{X}_s, v_s\rangle]\phi_s\, ds
\end{aligned}$$

$$= \exp[-\langle \tilde{X}_t, v_t\rangle]\phi_t - \exp[-\langle \tilde{X}_0, v_0\rangle] + \int_0^t \langle \tilde{X}_s, (\partial_s + A - \Gamma)v_s\rangle \exp[-\langle \tilde{X}_s, v_s\rangle]\phi_s \, ds$$

$$= \exp[-\langle \tilde{X}_t, V_{T-t} f\rangle]\phi_t - \exp[-\langle \tilde{X}_0, V_T f\rangle].$$

In the first equality we use

$$\langle \nu_{T-s}, 1 - e^{-f}\rangle = \lim_{r\downarrow 0}\langle \nu_r, V_{T-s} f\rangle = c_0\langle \tilde{m}, \partial_d v_s(\,\cdot\,, 0+)\rangle.$$

Therefore we have the desired result. □

3. Immigration particles near the boundary

For $S = \mathbf{R}^d$ or H, we denote

$$\mu \in \mathcal{M}_p(S) \overset{\text{def}}{\iff} \mu = \sum \delta_{x_n} \quad \text{on } S \text{ such that} \quad \langle \mu, g_p\rangle < \infty.$$

In particular, if $S = H$, then we simply denote $\mathcal{M}_p = \mathcal{M}_p(H)$.

Let $0 < \alpha < 2$ and $d < p < d + \alpha$. In [4] it is shown that if $(X_t^\alpha, \mathbf{P}_\mu)$ is an infinite independent Markov particle system associated with α-stable motions, i.e., rotation invariant α-stable processes on $\mathbf{R}^d$, starting from $X_0^\alpha = \mu \in \mathcal{M}_p(\mathbf{R}^d)$, then

$$\mathbf{P}_\mu(X_t^\alpha \in \mathcal{M}_p \text{ for all } t \geq 0) = 1.$$

This implies that for the original part $(X_t^O, \mathbf{P}_\mu)$ (which is an infinite independent Markov particle system associated with absorbing stable motions on H), by $X_t^O \leq X_t^\alpha|_H$, if $\mu \in \mathcal{M}_p$, then

$$\mathbf{P}_\mu(X_t^O \in \mathcal{M}_p \text{ for all } t \geq 0) = 1.$$

Moreover if $\mu \in \mathcal{M}_{p,0}$ (and even if $\mu \notin \mathcal{M}_p$), then

$$\mathbf{P}_\mu(X_t^O \in \mathcal{M}_p \text{ for all } t > 0) = 1.$$

On the other hand for the immigration part X_t^I,

$$\mathbf{E}[\langle X_t^I, g_p\rangle] \leq \langle m^\alpha, g_p\rangle$$

and the right hand side is finite at least if $1 < \alpha < 2$. This implies that at least if $1 < \alpha < 2$, then for each fixed $t > 0$, $\mathbf{P}(X_t^I \in \mathcal{M}_p) = 1$ and

$$\mathbf{P}(X_t^I \in \mathcal{M}_p \text{ for } dt\text{-a.a. } t > 0) = 1.$$

Here we have the following question.

QUESTION 1. For the immigration part X_t^I, which does it hold that $X_t^I \in \mathcal{M}_p$ or $X_t^I \in \mathcal{M}_{p,0} \setminus \mathcal{M}_p$ for all $t > 0$, **P**-a.s.?

We can obtain the following answer.

Theorem 4. *For every* $0 < \alpha < 2$ *and for all* $T > 0$, *it holds that*

$$\mathbf{P}(X_t^I \in \mathcal{M}_{p,0} \setminus \mathcal{M}_p \textit{ for infinitely many } t\textit{'s in } (0, T]) = 1.$$

This result depends on the number of particles near the boundary ∂H.

From now on, we only consider the immigration part X_t^I. and for simplicity, we consider the one-dimensional case, i.e., $d = 1$, $H = (0, \infty)$ (the case of $d \geq 2$ is essentially the same).

For any $\varepsilon > 0$, by $\nu_t(x) = t^{-2/\alpha}\nu_1(t^{-1/\alpha}x)$, $\nu_1(y) = (p_1^\alpha)'(y)$ and $|(p_1^\alpha)'(y)| \leq C(1 \wedge y \wedge y^{-2-\alpha})$ $(y > 0)$, we have

$$\begin{aligned}\int_\varepsilon^\infty \nu_t(x)\,dx &\leq t^{-1/\alpha}\int_{t^{-1/\alpha}\varepsilon}^\infty \nu_1(y)\,dy \\ &\leq C/(1+\alpha)t^{-1/\alpha}(t^{-1/\alpha}\varepsilon)^{-1-\alpha} \\ &= C/(1+\alpha)\varepsilon^{-1-\alpha}t \to 0 \quad (t \downarrow 0).\end{aligned}$$

Moreover

$$\begin{aligned}\mathbf{E}X_t^I((0,\varepsilon)) &= \int_0^t \int_{W_0} 1_{(0,\varepsilon)}(w(t-s))\,ds\; Q^0(dw) \\ &= \int_0^t \nu_u((0,\varepsilon))\,du \\ &\leq m^\alpha((0,\varepsilon)) = c_\alpha \int_0^\varepsilon y^{\alpha-2}\,dy.\end{aligned}$$

This is finite at least for $1 < \alpha < 2$. Hence Question 1 is reduced to the following.

QUESTION 2. Let $\varepsilon > 0$. At least if $1 < \alpha < 2$, then for each fixed time $t > 0$, the number of particles near the boundary is finite with probability one: $\mathbf{P}(X_t^I((0,\varepsilon)) < \infty) = 1$. Moreover it also holds that

$$\mathbf{P}(X_t^I((0,\varepsilon)) < \infty \text{ for } dt\text{-a.a.}t) = 1.$$

Now does it hold that for each $0 < \alpha < 2$ and for any $0 \leq a < b$,

$$\mathbf{P}(X_t^I((0,\varepsilon)) < \infty, \text{ for all } t \in (a,b)) = 1?$$

For this question we have the following answer, which implies Theorem 4.

Theorem 5. *Let $\varepsilon > 0$. For each $0 < \alpha < 2$ and for any $0 \le a < b$,*

$$\mathbf{P}(X_t^I((0,\varepsilon)) = \infty \text{ for some } t \in (a,b)) = 1.$$

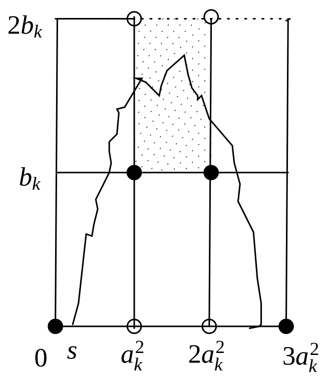
REMARK 1. It also holds that $\mathbf{P}\big(\sup_{t\in\mathbf{Q}\cap(a,b)} X_t^I((0,\varepsilon)) = \infty\big) = 1$.

Proof of Theorem 5. For the absorbing Brownian motion on the half space, we already gave the same result in [2]. So the proof is essentially the same as it. However the key is the Claim 1 of the following proposition.

Let $\varepsilon > 0$. $X_t^I((0,\varepsilon))$ can be expressed as $N^0(D_t)$ with

$$D_t = \{(s.w) \in [0,\infty) \times W_0 \colon w(t-s) \in (0,\varepsilon),\ 0 \le s < t\}.$$

We define a smaller process $S_{k,t} \le X_t^I((0,\varepsilon))$ as follows (k is determined by ε):



$$a_k = 1/2^k, \quad b_k = a_k^{2/\alpha},$$

$\xi^k := N^0(V^k)$: the number of excursions in V^k,

$$v^k = \{(s,w) \in [0,a_k^2) \times W_0;\ w(a_k^2 - s),\ w(2a_k^2 - s) \in [b_k, 2b_k),\ w(3a_k^2 - s) = \Delta,\ \gamma(w) < T_{[2b_k,\infty)}^{a_k^2 - s}(w)\},$$

where $T_{[a,\infty)}^{t_0}(w)$ is the hitting time after the time t_0 to $[a,\infty)$ of w, i.e.,

$$T_{[a,\infty)}^{t_0}(w) := \inf\{t > t_0;\ w(t) \in [a,\infty)\}.$$

For each $j \ge 1$, let $t_j^k = j/4^k$. If $t_j^k \le t < t_{j+1}^k$, then set

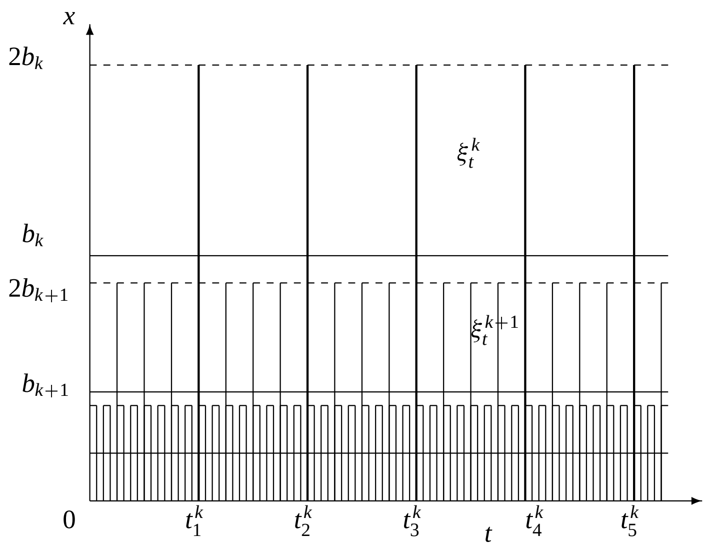
$$\xi_t^k \equiv \xi_{t_j^k}^k := N^0(V_j^k) \quad \text{with} \quad V_j^k = \theta_{-t_{j-1}^k}(V^k)$$

(note that ξ_t^k is undefined for $0 \le t < t_1^k$). It holds that $\xi_t^k \overset{(d)}{=} \xi^k$. In particular, if we denote $\xi_j^{(k)} = \xi_{t_j^k}^k$, then $\{\xi_j^{(k)} \colon j = 1, 2, \ldots,\ k = 1, 2, \ldots\}$ are independent. Because $b_k = a_k^{2/\alpha} > 2a_{k+1}^{2/\alpha} = 2b_{k+1}$ by $2/\alpha > 1$ (this is important).

REMARK 2. ξ^k denotes the number of particles which are born during the time interval $[0,a_k^2)$, stay in $[b_k, 2b_k)$ at each time point a_k^2, $2a_k^2$ and die during the time interval $(2a_k^2, 3a_k^3]$, and also which never hit $2b_k$ after the time a_k^2. Also ξ_t^k is the shifted ξ^k by the time t_{j-1}^k if $t_j^k \le t < t_{j+1}^k$. Hence we may regard that the box $[t_j^k, t_{j+1}^k) \times [b_k, 2b_k)$ has the random number ξ_t^k.

Now for each $k \ge 1$, set

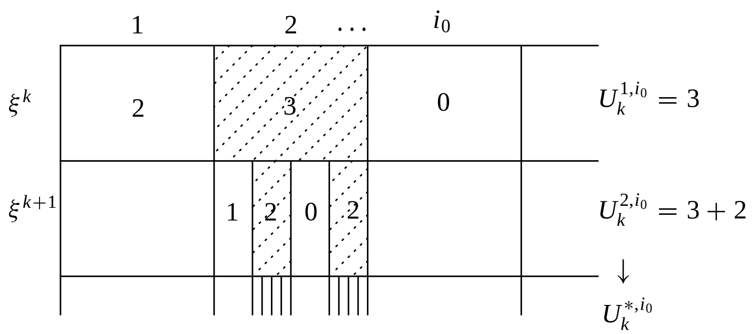
$$S_{k,t} = \sum_{n=k}^{\infty} \xi_t^n.$$



Clearly for every k; $2b_k \le \varepsilon$, if $t \ge t_1^k$ $(= a_k^2 = 1/4^k)$, then $S_{k,t} \le X_t^I((0,\varepsilon))$. Hence in order to prove Theorem 5 it is enough to show the following proposition:

Proposition 1. *For each* $k, i \ge 1$, $\mathbf{P}(S_{k,t} = \infty$ *for some* $t_1^k \le t < t_{i+1}^k) = 1$.

Proof. We define a random variable U_k^{*,i_0} for each $k, i_0 \ge 1$ as follows:



We first start from $\xi_j^{(k)}$'s which take the maximum of $\xi_j^{(k)}$, $j = 1, 2, \ldots, i_0$, and look at $\xi_j^{(k+1)}$'s under the above $\xi_j^{(k)}$'s. Next we start from $\xi_j^{(k+1)}$'s which take the maximum of the above $\{\xi_j^{(k+1)}\}$. We continue these operations and define U_k^{*,i_0} by adding each maximum number. That is, we set

$$U_k^{1,i_0} = \max_{1\le j\le i_0} \xi_j^{(k)}$$

and

$$I_1 = \left\{ i = 1, 2, \ldots, i_0 : \xi_i^{(k)} = \max_{1\le j\le i_0} \xi_j^{(k)} \right\},$$
$$J_1 = 4^k + [(4I_1 - 3) \cup (4I_1 - 2) \cup (4I_1 - 1) \cup 4I_1].$$

Also we define

$$U_k^{2,i_0} = U_k^{1,i_0} + \max_{j\in J_1} \xi_j^{(k+1)}.$$

If we have I_n, J_n, U_k^{n+1,i_0}, then set

$$I_{n+1} = \left\{ i \in J_n : \xi_i^{(k+n)} = \max_{j\in J_n} \xi_j^{(k+n)} \right\},$$
$$J_{n+1} = 4^{k+n} + [(4I_{n+1} - 3) \cup (4I_{n+1} - 2) \cup (4I_{n+1} - 1) \cup 4I_{n+1}]$$

and

$$U_k^{n+2,i_0} = U_k^{n+1,i_0} + \max_{j\in J_{n+1}} \xi_j^{(k+n+1)}.$$

So we define

$$U_k^{*,i_0} = \lim_{n\to\infty} U_k^{n,i_0}.$$

We can show the following two claims:

Claim 1. $\lambda_k := \mathbf{E}\xi^k = \lambda_0/2^{k(2-2/\alpha)} = 2^{2/\alpha-2}\lambda_{k-1}$ *for all* $k \ge 1$ $(\lambda_0 = \mathbf{E}\xi^0 > 0)$.

Claim 2. *For each* $k, i_0 \ge 1$, $\mathbf{P}(U_k^{*,i_0} = \infty) = 1$.

Obviously Claim 2 implies Proposition 1.

Proof of Claim 1. Let $(w^\alpha(t), P_x^\alpha)$ be the rotation invariant α-stable process on $\mathbf{R}$. Then for $a > 0$, $w^\alpha(a^2 t) \overset{(d)}{=} a^{2/\alpha} w^\alpha(t)$ and recall $\nu_r(a^{2/\alpha}x) = a^{-4/\alpha}\nu_{r/a^2}(x)$ $(r, x > 0)$.

Let V_s^k denote the s-section of V^k.

$$\begin{aligned}
\lambda_k &= \mathbf{E}N^0(V^k) = \int_0^{a_k^2} ds\ \ Q^0(V_s^k) \\
&= \int_0^{a_k^2} ds\ \ Q^0\big(w(a_k^2-s)\in[b_k,2b_k), \\
&\qquad\qquad w(2a_k^2-s)\in[b_k,2b_k),\ w(3a_k^2-s)=\Delta,\ \gamma(w)<T^{a_k^2-s}_{[2b_k,\infty)}(w)\big) \\
&= \int_0^{a_k^2} du\ \ Q^0\big(w(u)\in[b_k,2b_k), \\
&\qquad\qquad w(a_k^2+u)\in[b_k,2b_k),\ w(2a_k^2+u)=\Delta,\ \gamma(w)<T^{u}_{[2b_k,\infty)}(w)\big) \\
&= \int_0^{a_k^2} du \int_{b_k}^{2b_k} dx\ \ \nu_u(x)P_x^-\big(w^-(a_k^2)\in[b_k,2b_k),\ \zeta(w^-)\le(2a_k^2)\wedge T_{[2b_k,\infty)}(w^-)\big) \\
&= \int_0^{a_k^2} du \int_{b_k}^{2b_k} dx\ \ \nu_u(x)P_x^\alpha\big(w^\alpha(a_k^2)\in[b_k,2b_k),\ a_k^2<\zeta(w^\alpha)\le(2a_k^2)\wedge T_{[2b_k,\infty)}(w^\alpha)\big),
\end{aligned}$$

where $T_{[a,\infty)}=T^0_{[a,\infty)}$ is a hitting time to $[a,\infty)$ of $w^-(t)$ or $w^\alpha(t)$, and $\zeta(w^\alpha)$ is essentially the same as $\zeta(w^-)$. That is, by $w^\alpha(t)=B(y^{\alpha/2}(t))$

$$\zeta(w^\alpha):=\inf\{t>0;\ y^{\alpha/2}(t)\ge T_0(B)\},$$

where $\{B(t)\}$ is the Brownian motion on $\mathbf{R}$ which is independent of $\{y^{\alpha/2}(t)\}$ such that $B(0)=x$, and $T_0(B)$ is the hitting time to 0 of $B(t)$. By changing variables $u/a_k^2=v$, $x/b_k=y$ and by using scaling properties; $w^\alpha(a_k^2t)\overset{(\mathrm{d})}{=}a_k^{2/\alpha}w^\alpha(t)=b_kw^\alpha(t)$, $\nu_{a_k^2v}(b_ky)=b_k^{-2}\nu_v(y)$, we have

$$\begin{aligned}
\lambda_k &= \int_0^1 a_k^2\,dv\int_1^2 b_k\,dy\ \ \nu_{a_k^2v}(b_ky)P^\alpha_{b_ky}(w^\alpha(a_k^2)\in[b_k,2b_k),a_k^2<\zeta(w^\alpha)\le(2a_k^2)\wedge T_{[2b_k,\infty)}(w^\alpha)) \\
&= \int_0^1 a_k^2\,dv\int_1^2 b_k\,dy\ \ b_k^{-2}\nu_v(y)P^\alpha_y(b_kw^\alpha(1)\in[b_k,2b_k),1<\zeta(w^\alpha)\le 2\wedge T_{[2,\infty)}(w^\alpha)) \\
&= a_k^2b_k^{-1}\int_0^1 dv\int_1^2 dy\ \ \nu_v(y)P^\alpha_y(w^\alpha(1)\in[1,2),1<\zeta(w^\alpha)\le 2\wedge T_{[2,\infty)}(w^\alpha)) \\
&= a_k^{2-2/\alpha}\lambda_0,
\end{aligned}$$

where we use the following result. For $a>0$, by $y^{\alpha/2}(at)=a^{4/\alpha}y^{\alpha/2}(t)$, $\zeta(w^\alpha(a^2\,\cdot\,))$ under $P^\alpha_{a^{2/\alpha}y}$ has the same distribution as $\zeta(a^{2/\alpha}w^\alpha(\,\cdot\,))=\zeta(w^\alpha)$ under P^α_y. In fact, under P^α_{by}, $[\zeta(w^\alpha)>a^2t\iff T_0(B)>y^{\alpha/2}(a^2t)\iff T_0(B)>a^{4/\alpha}y^{\alpha/2}(t)\iff\exists t_0>y^{\alpha/2}(t);\ B(a^{4/\alpha}t_0)=0]\iff$ under P^α_y, $[\exists t_0>y^{\alpha/2}(t);\ a^{2/\alpha}B(t_0)=0$, i.e., $B(t_0)=0\iff T_0(B)>y^{\alpha/2}(t)\iff\zeta(w^\alpha)>t]$.

The positivity of $\lambda_0 = \mathbf{E}\xi^0$ would be intuitively obvious. However for the completeness of the proof we shall show it. Recall

$$\lambda_0 = \int_0^1 dt \int_1^2 dx \ \nu_t(x) P_x^\alpha(w^\alpha(1) \in [1,2),\, 1 < \zeta(w^\alpha) \le 2 \wedge T_{[2,\infty)}(w^\alpha)).$$

By $w^\alpha(t) \overset{(d)}{=} B(y^{\alpha/2}(t))$ and independence of $\{B(t)\}$, $\{y^{\alpha/2}\}$, we have

$$\begin{aligned} &P_x^\alpha(w^\alpha(1) \in [1,2),\, 1 < \zeta(w^\alpha) \le 2 \wedge T_{[2,\infty)}(w^\alpha)) \\ &\ge P_0(y^{\alpha/2}(1) \in [1, 1+\varepsilon),\, y^{\alpha/2}(2) \ge 2) \\ &\quad \times P_x(B(t) \in (1,2) \text{ for all } 0 \le t \le 1+\varepsilon,\, T_0 \le 2,\, T_0 < T_2). \end{aligned}$$

Moreover for $0 < \varepsilon < 1/2$,

$$\begin{aligned} &P_0(y^{\alpha/2}(1) \in [1, 1+\varepsilon),\, y^{\alpha/2}(2) \ge 2) \\ &\ge P_0(y^{\alpha/2}(1) \in [1, 1+\varepsilon)) P_0(y^{\alpha/2}(1) \ge 1) =: C_\varepsilon > 0. \end{aligned}$$

If $1 < x < 2$, then

$$\begin{aligned} &P_x(B(t) \in (1,2) \text{ for all } 0 \le t \le 1+\varepsilon,\, T_0 \le 2,\, T_0 < T_2) \\ &\ge \int_1^2 dy \ p_{1+\varepsilon}^{1,2}(x,y) P_y(T_0 \le 2,\, T_0 < T_2), \end{aligned}$$

where $p_t^{1,2}(x,y) = p_t^{0,1}(x-1, y-1)$ and for $0 < u, v < b$

$$p_t^{0,b}(u,v) = P_u(B(t) = v;\, t < T_0 \wedge T_b) = \sum_{n=-\infty}^{\infty} p_t^0(u, v+2nb)$$

with

$$p_t^0(u,v) = p_t(u,v) - p_t(u,-v) \quad \text{and} \quad p_t(u,v) = \frac{1}{\sqrt{2\pi t}} e^{-(v-u)^2/(2t)},$$

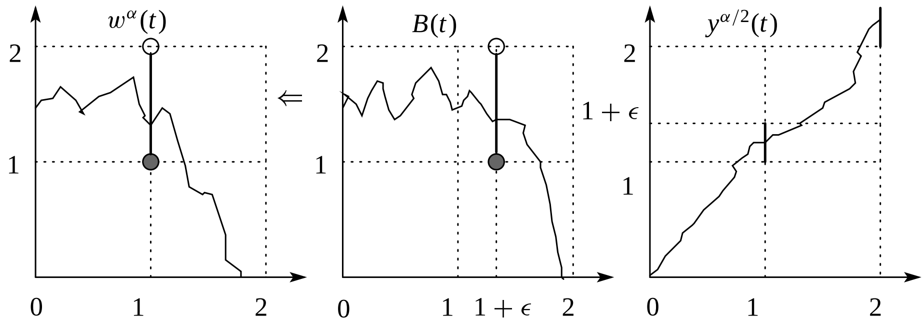
and for $0 < y < b$

$$P_y(T_0 \in dt;\, T_0 < T_b) = \frac{dt}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} (y + 2nb) \exp\left[-\frac{(y+2nb)^2}{2t}\right].$$

Hence

$$\lambda_0 \ge C_\varepsilon \int_0^1 dt \int_1^2 dx \ \nu_t(x) \int_1^2 dy \ p_{1+\varepsilon}^{1,2}(x,y) P_y(T_0 \le 2,\, T_0 < T_2).$$

By the continuities and the positivities of $p_{1+\varepsilon}^{1,2}(x,y)$, $P_y(T_0 \le 2,\, T_0 < T_2)$, we have $\lambda_0 > 0$. (Note that the positivities follows directly from the positivities of each sum of terms for $n = k, -k-1$ $(k \ge 0)$ in the above summations in $n \in \mathbf{Z}$.)



Proof of Claim 2. We shall show that for each k, $i_0 \geq 1$ and $m \geq 0$, $\mathbf{P}(U_k^{*,i_0} \leq m) = 0$ by the mathematical induction.

(1) For each k, $i_0 \geq 1$, $\mathbf{P}(U_k^{*,i_0} = 0) = 0$.

In fact, if $U_k^{*,i_0} = 0$, then $\xi_j^{(k+n)} = 0$ for all $n \geq 0$, $4^n \leq j < 4^n(i_0+1)$. However the sum of these expectations is given as $i_0(\lambda_k + 4\lambda_{k+1} + 4^2\lambda_{k+2} + \cdots)$ and this is infinite by Claim 1 (the rate is $4(a_{k+1}/a_k)^{2-2/\alpha} = 2^2 \cdot 2^{2/\alpha-2} = 2^{2/\alpha} > 1$). Hence the probability of this event is 0.

$$\begin{aligned}\mathbf{P}(U_k^{*,i_0} = 0) &\leq \lim_{n\to\infty} \mathbf{P}(\xi^k = 0)^{i_0}\mathbf{P}(\xi^{k+1} = 0)^{4i_0} \cdots \mathbf{P}(\xi^{k+n} = 0)^{4^n i_0} \\ &= \lim_{n\to\infty} \exp[-i_0(\lambda_k + 4\lambda_{k+1} + \cdots + 4^n\lambda_{k+n})] = 0.\end{aligned}$$

(2) If we assume that $\mathbf{P}(U_k^{*,i_0} \leq m-1) = 0$ for all k, $i_0 \geq 1$, then

$$\begin{aligned}\mathbf{P}(U_k^{*,i_0} \leq m) &= \sum_{m_k=0}^{m} \mathbf{P}(\xi^k = m_k)^{i_0}\mathbf{P}\big(U_{k+1}^{*,4i_0} \leq m - m_k\big) \\ &\quad + \sum_{j=1}^{i_0-1}\binom{i_0}{j}\sum_{m_k=1}^{m}\mathbf{P}(\xi^k = m_k)^j\mathbf{P}(\xi^k \leq m_k - 1)^{i_0-j}\mathbf{P}\big(U_{k+1}^{*,4j} \leq m - m_k\big) \\ &= \mathbf{P}(\xi^k = 0)^{i_0}\mathbf{P}\big(U_{k+1}^{*,4i_0} \leq m\big) \\ &= \mathbf{P}(\xi^k = 0)^{i_0}\mathbf{P}(\xi^{k+1} = 0)^{4i_0}\mathbf{P}\big(U_{k+2}^{*,4^2 i_0} \leq m\big) \\ &\leq \mathbf{P}(\xi^k = 0)^{i_0}\mathbf{P}(\xi^{k+1} = 0)^{4i_0} \cdots \mathbf{P}(\xi^{k+n} = 0)^{4^n i_0} \\ &= \exp[-i_0(\lambda_k + 4\lambda_{k+1} + \cdots + 4^n\lambda_{k+n})] \to 0 \quad (n \to \infty)\end{aligned}$$

by Claim 1. This implies $\mathbf{P}(U_k^{*,i_0} \leq m) = 0$ for all k, $i_0 \geq 1$.

(3) From the above results (1) and (2) we have Claim 2. □

Proof of Remark 1. For each $k \geq 1$, we set

$$S_k^* \equiv \lim_{n\to\infty} \max_{t_1^k \leq t < t_2^k} \sum_{j=k}^{k+n} \xi_t^j.$$

It holds that for all $m \geq 0$,

$$\mathbf{P}(S_k^* \leq m) = \sum_{m_k=0}^{m} \mathbf{P}(\xi^k = m_k)\mathbf{P}(S_{k+1}^* \leq m - m_k)^4.$$

Hence by the same way as in the case of $U_k^{*,i}$, we can show $\mathbf{P}(S_k^* \leq m) = 0$ for all $m \geq 0$. This implies for all $\varepsilon > 0$ and $k; 2b_k \leq \varepsilon$, $\mathbf{P}\bigl(\sup_{t\in\mathbf{Q}\cap[t_1^k, t_2^k)} X_t^I((0,\varepsilon)) = \infty\bigr) = 1$. Furthermore if we change t_1^k, t_2^k to any t_j^k, t_{j+1}^k $(j \geq 1)$, then the same result holds. $\square$

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