Title	THE DYNKIN INDEX AND CONFORMALLY INVARIANT SYSTEMS ASSOCIATED TO PARABOLIC SUBALGEBRAS OF HEISENBERG TYPE
Author(s)	Kubo, Toshihisa
Citation	Osaka Journal of Mathematics. 51(2) P.359-P.373
Issue Date	2014-04
Text Version	publisher
URL	https://doi.org/10.18910/29205
DOI	10.18910/29205
rights	
Note	

Kubo, T.
Osaka J. Math.
51 (2014), 359–373

THE DYNKIN INDEX AND CONFORMALLY INVARIANT SYSTEMS ASSOCIATED TO PARABOLIC SUBALGEBRAS OF HEISENBERG TYPE

TOSHIHISA KUBO

(Received August 31, 2012)

Abstract

Barchini, Kable, and Zierau constructed a number of conformally invariant systems of differential operators associated to parabolic subalgebras of Heisenberg type. When they constructed such systems of operators, two constants, which play a role for the construction, were defined as the constants of proportionality between two expressions. In this paper we give concrete and uniform expressions for these constants. To do so we introduce a new constant inspired by a formula on the Dynkin index of a finite dimensional representation of a complex simple Lie algebra.

1. Introduction

Let $D_1, \ldots, D_m$ be a system of differential operators on a smooth manifold M on which a Lie algebra $\mathfrak{g}$ acts by first order differential operators. We say that the system $D_1, \ldots, D_m$ of operators is *conformally invariant* if for all $X \in \mathfrak{g}$, it satisfies the bracket identity

$$[X, D_j] = \sum_i C_{ij}^X D_i,$$

where C_{ij}^X are smooth functions on M. (See for example p. 791 of [2] for the precise definition.) In [1], a number of examples of such systems of operators were constructed. On the construction two constants $c(\mathfrak{l}_j)$ and $p(\mathfrak{l}_j)$ were introduced. The main results of this paper concern the two constants $c(\mathfrak{l}_j)$ and $p(\mathfrak{l}_j)$.

To describe our results we start with briefly reviewing the work of [1]. Let $\mathfrak{g}$ be a complex simple Lie algebra with highest root γ. For each root α, if a triple $\{X_{-\alpha}, H_\alpha, X_\alpha\}$ is the corresponding $\mathfrak{sl}(2)$-triple in the Chevalley basis then $\mathrm{ad}(H_\gamma)$ on $\mathfrak{g}$ has eigenvalues -2, -1, 0, 1, and 2. Write the eigenspace decomposition as $\mathfrak{g} = \bigoplus_{j=-2}^{2} \mathfrak{g}(j)$ with $\mathfrak{g}(k)$ the k eigenspace of $\mathrm{ad}(H_\gamma)$. The parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ with $\mathfrak{l} = \mathfrak{g}(0)$ the Levi factor and $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ the nilpotent radical is of Heisenberg type, that is, $\mathfrak{n}$ is a two-step nilpotent algebra with one-dimensional center. Since $\mathfrak{g}(2) = \mathfrak{g}_\gamma$ with $\mathfrak{g}_\gamma$ the root space for γ, the parabolic

2010 Mathematics Subject Classification. Primary 17B10; Secondary 22E46.

subalgebra $\mathfrak{q}$ of Heisenberg type may be expressed as[1]

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{g}_\gamma. \tag{1.1}$$

Let G_0 be a real Lie group with complexified Lie algebra $\mathfrak{g}$ and $\bar{Q}_0$ be a parabolic subgroup of G_0 with complexified Lie algebra $\bar{\mathfrak{q}}$ opposite to $\mathfrak{q}$. Then, in [1], from each $2-k$ eigenspace $\mathfrak{g}(2-k)$ for $k = 1, \ldots, 4$, systems of k-th order differential operators acting on smooth sections on the line bundles $\mathcal{L}_{-s} \to G_0/\bar{Q}_0$ over a homogeneous manifold $G_0/\bar{Q}_0$ are constructed. Here s is a complex parameter, which indexes the line bundles $\mathcal{L}_{-s}$. The systems of operators are called Ω_k *systems*. It is not necessary that Ω_k systems are conformally invariant; their conformal invariance depends on the complex parameter s for the line bundle $\mathcal{L}_{-s}$. Then, in [1], a complex parameter s_0 is called a *special value* for an Ω_k system if the system is conformally invariant on the line bundle $\mathcal{L}_{-s_0}$.

In Section 2 of [1], given simple ideal $\mathfrak{l}_j$ of $\mathfrak{l}$ for the parabolic subalgebra $\mathfrak{q}$, two constants $c(\mathfrak{l}_j)$ and $p(\mathfrak{l}_j)$ were defined as proportionality of two expressions.[2] (See Definitions 2.1 and 2.2.) These two constants play a critical role for the construction of conformally invariant Ω_k systems. For instance, the special value s_0 can be expressed in terms of $c(\mathfrak{l}_j)$ and $p(\mathfrak{l}_j)$. Nonetheless, their true mathematical significance or any concrete expressions were not shown. Then, in this paper, we give explicit and uniform expressions for these constants. To do so we introduce a new constant $K(\mathfrak{l}_j; W)$ (See Definition 4.1.) associated to a simple ideal $\mathfrak{l}_j$ of $\mathfrak{l}$ and a finite dimensional $\mathfrak{l}$-module W. The formulation of the constant $K(\mathfrak{l}_j; W)$ is inspired by a formula on the Dynkin index of a finite dimensional representation of a complex simple Lie algebra. With the constants $K(\mathfrak{l}_j; W)$, we show that $c(\mathfrak{l}_j) = K(\mathfrak{l}_j; \mathfrak{g}(1))$ and $p(\mathfrak{l}_j) = K(\mathfrak{l}_j; \mathfrak{l}_j)$.

We now outline the remainder of this paper. In Section 2 we recall the definitions of $c(\mathfrak{l}_j)$ and $p(\mathfrak{l}_j)$ from [1]. The notation and normalizations that are in force for the rest of this paper are also introduced in this section. In Section 3 we review the Dynkin index of a simple Lie algebra $\mathfrak{g}$. In Section 4 we define the constant $K(\mathfrak{l}_j; W)$ and show its several properties. In Section 5, for the parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{g}_\gamma$ of Heisenberg type, we show that $c(\mathfrak{l}_j) = K(\mathfrak{l}_j; \mathfrak{g}(1))$ and $p(\mathfrak{l}_j) = K(\mathfrak{l}_j; \mathfrak{l}_j)$ as our main results. These are done in Theorems 5.1 and 5.4, respectively.

2. Preliminaries

In this section we recall from [1] two constants $c(\mathfrak{l}_j)$ and $p(\mathfrak{l}_j)$ associated to a simple ideal $\mathfrak{l}_j$ of the Levi subalgebra $\mathfrak{l}$ of the parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ of Heisenberg type. To do so we need to introduce some notation and fix normalizations on the root vectors $X_{-\alpha}$, H_α, and X_α.

---

[1] In [1] the 1 eigenspace $\mathfrak{g}(1)$ is denoted by V^+.

[2] In [1] the constants $c(\mathfrak{l}_j)$ and $p(\mathfrak{l}_j)$ are denoted by $c(\mathfrak{g}, \mathcal{C})$ and $p(\mathfrak{g}, \mathcal{C})$, respectively.

Let $\mathfrak{g}$ be a complex simple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ and write $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, the set of roots $\mathfrak{g}$ with respect to $\mathfrak{h}$. Choose a Borel subalgebra $\mathfrak{b}$ and denote by Δ^+ the corresponding set of positive roots so that $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ with $\mathfrak{g}_\alpha$ the root space of α. We write Π for the set of simple roots and ρ for half the sum of the positive roots. We denote by γ the highest root of $\mathfrak{g}$. Let $B_\mathfrak{g}$ be a positive multiple of the Killing form on $\mathfrak{g}$ and denote by $\langle \cdot, \cdot \rangle$ the corresponding inner product on $\mathfrak{h}^*$. A normalization of $B_\mathfrak{g}$ will be specified below.

As in [1], for each $\alpha \in \Delta^+$, we choose $X_\alpha \in \mathfrak{g}_\alpha$ and $H_\alpha \in \mathfrak{h}$ in such a way that the following properties hold:

(C1) For each $\alpha \in \Delta$, $\{X_\alpha, H_\alpha, X_{-\alpha}\}$ is an $\mathfrak{sl}(2)$-triple; in particular, $[X_\alpha, X_{-\alpha}] = H_\alpha$.
(C2) For each $\alpha, \beta \in \Delta$, $[H_\alpha, X_\beta] = \beta(H_\alpha)X_\beta$.
(C3) The inner product $\langle \cdot, \cdot \rangle$ is positive-definite on the real span of $\{H_\alpha \mid \alpha \in \Delta\}$.
(C4) For $\alpha \in \Delta$ we have $B_\mathfrak{g}(X_\alpha, X_{-\alpha}) = 2/\langle \alpha, \alpha\rangle$.
(C5) For $\alpha, \beta \in \Delta$ we have $\beta(H_\alpha) = 2\langle \beta, \alpha\rangle/\langle \alpha, \alpha\rangle$.

We normalize $B_\mathfrak{g}$ so that $B_\mathfrak{g}(X_\gamma, X_{-\gamma}) = 1$. By the condition (C4), this is equivalent to requiring $\langle \gamma, \gamma\rangle = 2$. For $\alpha \in \Delta$, we write $\|\alpha\|^2 = \langle \alpha, \alpha\rangle$ and $\alpha^\vee = 2\alpha/\langle \alpha, \alpha\rangle$.

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{g}_\gamma$ be the standard parabolic subalgebra of Heisenberg type, as described in (1.1). Observe that if $\mathfrak{g}$ is of type A_2 then, as $\mathfrak{q} = \mathfrak{b}$ the Borel subalgebra, we have $[\mathfrak{l}, \mathfrak{l}] = 0$. Thus, for the rest of this section, we assume that $\mathfrak{g}$ is not of type A_2 so that $[\mathfrak{l}, \mathfrak{l}] \neq 0$.

Now we recall the definitions of the constants $c(\mathfrak{l}_j)$ and $p(\mathfrak{l}_j)$ from [1]. If W is a finite dimensional representation of a complex reductive Lie algebra then we denote by $\Delta(W)$ the set of weights for W. When $\Delta(W)\backslash\{0\} \subset \Delta$, we write $\Delta^+(W) = \Delta(W) \cap \Delta^+$ and $\Pi(W) = \Delta(W) \cap \Pi$.

DEFINITION 2.1 ([1, Proposition 2.1]). Let $\mathfrak{g}$ be a complex simple Lie algebra, not of type A_2, and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{g}_\gamma$ be the parabolic subalgebra of $\mathfrak{g}$ of Heisenberg type. If $\mathfrak{l}_j$ is a simple ideal of $\mathfrak{l}$ then there exists a constant $c(\mathfrak{l}_j)$ such that, for all $\alpha \in \Delta(\mathfrak{g}(1))$ and $\delta \in \Delta(\mathfrak{l}_j)$,

$$\sum_{\beta\in\Delta(\mathfrak{g}(1))} \langle \alpha, \beta\rangle\langle \beta, \delta\rangle = c(\mathfrak{l}_j)\langle \alpha, \delta\rangle. \tag{2.1}$$

DEFINITION 2.2 ([1, Proposition 2.2]). Let $\mathfrak{g}$ be a complex simple Lie algebra, not of type A_2, and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{g}_\gamma$ be the parabolic subalgebra of $\mathfrak{g}$ of Heisenberg type. If $[\mathfrak{l}, \mathfrak{l}] = \bigoplus_{j=1}^m \mathfrak{l}_j$ with $\mathfrak{l}_j$ the simple ideals of $[\mathfrak{l}, \mathfrak{l}]$ then there exist constants $p(\mathfrak{l}_j)$ such that, for all $X \in \mathfrak{g}(1)$ and $Y \in \mathfrak{g}(-1)$,

$$\sum_{\beta\in\Delta(\mathfrak{g}(1))} \|\beta\|^2[[X, X_{-\beta}], [X_\beta, Y]] = \sum_{j=1}^m p(\mathfrak{l}_j)\,\mathrm{pr}_j([X, Y]), \tag{2.2}$$

where $\mathrm{pr}_j : [\mathfrak{l}, \mathfrak{l}] \to \mathfrak{l}_j$ is the projection map.

The goal of this paper is to express the constants $c(\mathfrak{l}_j)$ and $p(\mathfrak{l}_j)$ explicitly. To this end we show two technical lemmas, which will make certain expositions simple in later proofs. Let $\mathcal{D}_\gamma(\mathfrak{g}, \mathfrak{h})$ denote the subdiagram of the Dynkin diagram of $\mathfrak{g}$ obtained by deleting the nodes that are connected to $-\gamma$ in the extended Dynkin diagram and the edges that touch them. (See Appendix A for the diagrams $\mathcal{D}_\gamma(\mathfrak{g}, \mathfrak{h})$ for each of the complex simple Lie algebras.) For a simple ideal $\mathfrak{l}_j$ of $[\mathfrak{l}, \mathfrak{l}]$, we denote by ξ_j the highest root of $\mathfrak{l}_j$ with respect to the positive system $\Delta^+(\mathfrak{l}_j)$.

Lemma 2.3. *If α_0 is a simple root whose node is deleted in $\mathcal{D}_\gamma(\mathfrak{g}, \mathfrak{h})$ then $\xi_j + \alpha_0 \in \Delta$. In particular, any simple ideal $\mathfrak{l}_j$ acts on $\mathfrak{g}(1)$ non-trivially.*

Proof. As both ξ_j and α_0 are roots, to prove this lemma, it suffices to show that $\langle \xi_j, \alpha_0 \rangle < 0$. For $\alpha \in \Pi$ observe that $\langle \alpha, \alpha_0 \rangle < 0$ if α is adjacent to α_0 in the Dynkin diagram and $\langle \alpha, \alpha_0 \rangle = 0$ otherwise. A direct observation on $\mathcal{D}_\gamma(\mathfrak{g}, \mathfrak{h})$ in Appendix A shows that, for any simple ideal $\mathfrak{l}_j$ of $\mathfrak{l}$, there exists a simple root α' in $\Pi(\mathfrak{l}_j)$ so that α' is adjacent to α_0. Write ξ_j as $\xi_j = \sum_{\alpha \in \Pi(\mathfrak{l}_j)} n_\alpha \alpha$ in terms of the simple roots $\alpha \in \Pi(\mathfrak{l}_j)$. Observe that since ξ_j is the highest root for the simple algebra $\mathfrak{l}_j$, we have $n_\alpha \in \mathbb{Z}_{>0}$ for all $\alpha \in \Pi(\mathfrak{l}_j)$; in particular, $n_{\alpha'} > 0$ for the simple root α'. Therefore, $\langle \xi_j, \alpha_0 \rangle \leq n_{\alpha'} \langle \alpha', \alpha_0 \rangle < 0$. $\square$

Observe that the Cartan subalgebra $\mathfrak{h}$ can be decomposed as $\mathfrak{h} = \mathfrak{z}(\mathfrak{l}) \oplus \mathfrak{h}_{ss}$, where $\mathfrak{h}_{ss}$ is the Cartan subalgebra of $[\mathfrak{l}, \mathfrak{l}]$. Then, for a finite dimensional $\mathfrak{l}$-module W, we say that a weight $\eta \in \Delta(W)$ is a *highest weight* for W if the restriction $\eta|_{\mathfrak{h}_{ss}}$ onto $\mathfrak{h}_{ss}$ is a highest weight for W as an $[\mathfrak{l}, \mathfrak{l}]$-module. We denote by μ a highest weight for $\mathfrak{g}(1)$.

Lemma 2.4. *If μ is a highest weight for $\mathfrak{g}(1)$ then $\mu - \xi_j \in \Delta$.*

Proof. As the proof for Lemma 2.3, since both μ and ξ_j are roots of $\mathfrak{g}$, to prove this lemma, it suffices to show that $\langle \mu, \xi_j \rangle > 0$. If the semisimple part $[\mathfrak{l}, \mathfrak{l}]$ of $\mathfrak{l}$ is $[\mathfrak{l}, \mathfrak{l}] = \bigoplus_{k=1}^m \mathfrak{l}_k$ with $\mathfrak{l}_k$ the simple ideals of $[\mathfrak{l}, \mathfrak{l}]$ then we write μ as $\mu = \sum_{k=1}^m \sum_{\alpha \in \Pi(\mathfrak{l}_k)} n_\alpha \varpi_\alpha$ with $n_\alpha \in \mathbb{Z}_{\geq 0}$, where ϖ_α are the fundamental weights of $\alpha \in \Pi(\mathfrak{l}_k)$. Observe that since ξ_j is the highest root for $\mathfrak{l}_j$, the weight ξ_j can be written as $\xi_j = \sum_{\beta \in \Pi(\mathfrak{l}_j)} m_\beta \beta$ with $m_\beta \in \mathbb{Z}_{>0}$, a linear combination of the simple roots $\beta \in \Pi(\mathfrak{l}_j)$ with coefficient $m_\beta \in \mathbb{Z}_{>0}$. Therefore, for each $\alpha \in \Pi(\mathfrak{l}_j)$,

$$\langle \varpi_\alpha, \xi_j \rangle = \sum_{\beta \in \Pi(\mathfrak{l}_j)} m_\beta \langle \varpi_\alpha, \beta \rangle = m_\alpha > 0$$

and, for all $\alpha \in \Pi(\mathfrak{l}_k)$ with $k \neq j$, we have $\langle \varpi_\alpha, \xi_j \rangle = 0$. It follows from Lemma 2.3 that $\mathfrak{l}_j$ acts on $\mathfrak{g}(1)$ non-trivially. In particular, there exists $\alpha'' \in \Pi(\mathfrak{l}_j)$ so that $n_{\alpha''} > 0$

for $\mu = \sum_{k=1}^{m} \sum_{\alpha \in \Pi(\mathfrak{l}_k)} n_\alpha \varpi_\alpha$. Therefore we obtain

$$\langle \mu, \xi_j \rangle = \sum_{k=1}^{m} \sum_{\alpha \in \Pi(\mathfrak{l}_k)} n_\alpha \langle \varpi_\alpha, \xi_j \rangle = \sum_{\alpha \in \Pi(\mathfrak{l}_j)} n_\alpha \langle \varpi_\alpha, \xi_j \rangle \geq n_{\alpha''} m_{\alpha''} > 0. \qquad \square$$

3. The Dynkin index

To give concrete expressions for $c(\mathfrak{l}_j)$ and $p(\mathfrak{l}_j)$, in the next section, we shall introduce a new constant inspired by a formula on the Dynkin index. Then, in this section, we recall the definition of the Dynkin index and its several properties. We keep the notation and normalizations on $\mathfrak{g}$ from Section 2, unless otherwise specified.

DEFINITION 3.1 ([5, Section 2]). Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two complex simple Lie algebras. If $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism then there exists a unique number $m_\phi \in \mathbb{C}$ such that, for all $X, Y \in \mathfrak{g}_1$,

$$B_{\mathfrak{g}_2}(\phi(X), \phi(Y)) = m_\phi B_{\mathfrak{g}_1}(X, Y),$$

where $B_{\mathfrak{g}_i}(\,\cdot\,,\,\cdot\,)$ is the positive multiple of the Killing form on $\mathfrak{g}_i$ normalized as in Section 2. The unique number m_ϕ is called the *Dynkin index* of ϕ.

DEFINITION 3.2 ([5, Section 2]). If V is a finite dimensional representation (not necessarily irreducible) of a complex simple Lie algebra $\mathfrak{g}$ then the *Dynkin index m_V of the representation V* is defined by that of a Lie algebra homomorphism

$$\phi\colon \mathfrak{g} \to \mathfrak{sl}(V),$$

where $\mathfrak{sl}(V)$ is the Lie algebra of trace zero endomorphisms of V.

Observe that since the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ is normalized as $\langle \gamma, \gamma \rangle = 2$, we have $\gamma^\vee = \gamma$. Nonetheless, for later convenience, we use the notation $\gamma^\vee$ in Proposition 3.3 below. (See Lemma 4.2 and some comments after Definition 4.1 below.)

Proposition 3.3 ([8, Lemma 5.2]). *Let $\mathfrak{g}$ be a complex simple Lie algebra with highest root γ. If V is a finite dimensional representation of $\mathfrak{g}$ then the Dynkin index m_V of the representation V is given by*

$$m_V = \frac{1}{2} \sum_{\lambda \in \Delta(V)} \dim(V_\lambda) \langle \lambda, \gamma^\vee \rangle^2, \tag{3.1}$$

where V_λ is the weight space of V for weight λ. In particular, for the adjoint

Table 1. The dual Coxeter numbers.

Types of $\mathfrak{g}$	dual Coxeter numbers
A_n, $n \geq 1$	$n+1$
B_n, $n \geq 2$	$2n-1$
C_n, $n \geq 2$	$n+1$
D_n, $n \geq 4$	$2n-2$
E_6	12
E_7	18
E_8	30
F_4	9
G_2	4

representation ad *of* $\mathfrak{g}$, *we have*

$$m_{\mathrm{ad}} = \sum_{\alpha \in \Delta^+} \langle \alpha, \gamma^\vee \rangle^2 = 2(1 + \langle \rho, \gamma^\vee \rangle). \tag{3.2}$$

REMARK 3.4. The formula $m_{\mathrm{ad}} = 2(1 + \langle \rho, \gamma^\vee \rangle)$ in (3.2) was first found by Dynkin ([5, Theorem 2.5]) without using (3.1).

REMARK 3.5. The number $1 + \langle \rho, \gamma^\vee \rangle$ is called the *dual Coxeter number* of $\mathfrak{g}$. (See for instance [6, Section 6.1 and Exercise 6.2].) Table 1 summarizes the dual Coxeter number for each complex simple Lie algebra $\mathfrak{g}$.

We shall later use a modification of (3.2). To prove the modification we first recall the following lemma due to Braden. Let Λ denote the weight lattice of $\mathfrak{g}$ and $V(\lambda)$ denote the finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight $\lambda \in \Lambda$. We denote by rank($\mathfrak{g}$) the rank of $\mathfrak{g}$.

Lemma 3.6 ([4, Lemma 1.3])**.** *Let* $\mathfrak{g}$ *be a complex simple Lie algebra. Then, for any* $\nu, \epsilon \in \Lambda$, *we have*

$$\sum_{\eta \in \Delta(V(\lambda))} \langle \nu, \eta \rangle \langle \eta, \epsilon \rangle = \frac{\langle \nu, \epsilon \rangle}{\mathrm{rank}(\mathfrak{g})} \sum_{\eta \in \Delta(V(\lambda))} \langle \eta, \eta \rangle.$$

Here is a modification of (3.2).

Lemma 3.7. *Let* $\mathfrak{g}$ *be a complex simple Lie algebra with highest root* γ. *Then, for any* $\beta_1, \beta_2 \in \Delta$, *we have*

$$\|\beta_1\|^2 \sum_{\alpha \in \Delta^+} \langle \alpha, \beta_1^\vee \rangle^2 = \|\beta_2\|^2 \sum_{\alpha \in \Delta^+} \langle \alpha, \beta_2^\vee \rangle^2.$$

In particular, for any $\beta \in \Delta$,

$$\frac{\|\beta\|^2}{\|\gamma\|^2} \sum_{\alpha \in \Delta^+} \langle \alpha, \beta^\vee \rangle^2 = 2(1 + \langle \rho, \gamma^\vee \rangle).$$

Proof. Via the adjoint representation, the simple algebra $\mathfrak{g}$ is regarded as the finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight γ. Since the set of weights of $\mathfrak{g}$ is $\Delta \cup \{0\}$, it follows from Lemma 3.6 that, for any $\beta \in \Delta$,

$$\sum_{\alpha \in \Delta} \langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = \frac{\|\beta\|^2}{\operatorname{rank}(\mathfrak{g})} \sum_{\alpha \in \Delta} \|\alpha\|^2.$$

Thus if $\beta_1, \beta_2 \in \Delta$ then

$$\frac{1}{\|\beta_1\|^2} \sum_{\alpha \in \Delta} \langle \alpha, \beta_1 \rangle^2 = \frac{1}{\operatorname{rank}(\mathfrak{g})} \sum_{\alpha \in \Delta} \|\alpha\|^2 = \frac{1}{\|\beta_2\|^2} \sum_{\alpha \in \Delta} \langle \alpha, \beta_2 \rangle^2.$$

Since $\langle \alpha, \beta_i^\vee \rangle = 2\langle \alpha, \beta_i \rangle / \|\beta_i\|^2$ and $\langle \alpha, \beta_i^\vee \rangle^2 = \langle -\alpha, \beta_i^\vee \rangle^2$, this implies that

$$\|\beta_1\|^2 \sum_{\alpha \in \Delta^+} \langle \alpha, \beta_1^\vee \rangle^2 = \|\beta_2\|^2 \sum_{\alpha \in \Delta^+} \langle \alpha, \beta_2^\vee \rangle^2. \tag{3.3}$$

Now if $\beta_1 = \beta$ and $\beta_2 = \gamma$ in (3.3) then it follows from (3.2) that

$$\frac{\|\beta\|^2}{\|\gamma\|^2} \sum_{\alpha \in \Delta^+} \langle \alpha, \beta^\vee \rangle^2 = \sum_{\alpha \in \Delta^+} \langle \alpha, \gamma^\vee \rangle^2 = 2(1 + \langle \rho, \gamma^\vee \rangle).$$ □

To conclude this section we show an interesting relationship between the Dynkin index m_{ad} and minimal nilpotent orbit $\mathcal{O}_{\min}$, although the result is not used in this paper. In Theorem 1 in [11], Wang showed that

$$\dim \mathcal{O}_{\min} = 2(1 + \langle \rho, \gamma^\vee \rangle) - 2. \tag{3.4}$$

With (3.2) and (3.4) in hand, we obtain the following formula.

Proposition 3.8. *If* $\mathcal{O}_{\min}$ *is the minimal nilpotent orbit of* $\mathfrak{g}$ *then*

$$\dim \mathcal{O}_{\min} = m_{\mathrm{ad}} - 2.$$

Proof. This lemma directly follows from (3.2) and (3.4). □

For more properties of the Dynkin index, see for example [4], [5], [7], [8], [9], and [10]. Some topological interpretation of the Dynkin index is discussed in [9] and Section 4 of [7].

4. The constant $K(\mathfrak{l}_j; W)$

In this section we introduce constants $K(\mathfrak{l}_j; W)$ associated to simple ideals $\mathfrak{l}_j$ of a Levi subalgebra $\mathfrak{l}$ and finite dimensional $\mathfrak{l}$-modules W. These constants will play an essential role for concrete expressions for $c(\mathfrak{l}_j)$ and $p(\mathfrak{l}_j)$. We continue to use the notation from the previous sections.

DEFINITION 4.1. Let $\mathfrak{g}$ be a complex simple Lie algebra with highest root γ, and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ be any standard parabolic subalgebra of $\mathfrak{g}$. If W is a finite dimensional $\mathfrak{l}$-module and if $\mathfrak{l}_j$ is a simple ideal of $[\mathfrak{l}, \mathfrak{l}]$ then we define $K(\mathfrak{l}_j; W)$ by

$$K(\mathfrak{l}_j; W) := \frac{1}{2} \cdot \frac{\|\xi_j\|^2}{\|\gamma\|^2} \sum_{\lambda \in \Delta(W)} \dim(W_\lambda) \langle \lambda, \xi^\vee \rangle_j^2,$$

where W_λ is the weight space of W for weight λ.

Remark that, since the values of $\|\xi_j\|^2/\|\gamma\|^2$ and $\langle \lambda, \xi_j^\vee \rangle$ are independent from the normalizations of the inner product $\langle \cdot, \cdot \rangle$, the value of $K(\mathfrak{l}_j; W)$ is defined intrinsically.

It is immediate from the definition that $K(\mathfrak{l}_j; \mathfrak{l}_k) = 0$, unless $j = k$. In the case that $j = k$, the following formula holds.

Lemma 4.2. *Given simple ideal $\mathfrak{l}_j$ of $\mathfrak{l}$ with highest root ξ_j, we have*

$$K(\mathfrak{l}_j; \mathfrak{l}_j) = \frac{2\|\xi_j\|^2}{\|\gamma\|^2}(1 + \langle \rho(\mathfrak{l}_j), \xi_j^\vee \rangle), \tag{4.1}$$

where $\rho(\mathfrak{l}_j)$ is half the sum of the positive roots in $\Delta^+(\mathfrak{l}_j)$.

Proof. This directly follows from Proposition 3.3 and Definition 4.1. □

EXAMPLE 4.3. Let $\mathfrak{g}$ be the complex simple Lie algebra of type B_n with $n \geq 4$, and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{g}_\gamma$ be the parabolic subalgebra of $\mathfrak{g}$ of Heisenberg type described as in (1.1). If we use the standard realization of the root system Δ with ε_j the dual basis of the standard orthonormal basis for $\mathbb{R}^n$ then the highest root γ is $\gamma = \varepsilon_1 + \varepsilon_2$. As $\mathfrak{g}(1)$ is the 1 eigenspace of $\mathrm{ad}(H_\gamma)$, the set $\Delta(\mathfrak{g}(1)) = \{\beta \in \Delta^+ \mid \langle \beta, \gamma \rangle = 1\}$ is given by

$$\Delta(\mathfrak{g}(1)) = \{\varepsilon_j \pm \varepsilon_k \colon j = 1, 2 \text{ and } 3 \leq k \leq n\} \cup \{\varepsilon_1, \varepsilon_2\}.$$

There are two simple ideals $\mathfrak{l}_1$ and $\mathfrak{l}_2$ in $[\mathfrak{l}, \mathfrak{l}]$, where $\mathfrak{l}_1 \cong \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{l}_2 \cong \mathfrak{so}(2(n-2)+1, \mathbb{C})$ with highest roots $\xi_1 = \varepsilon_1 - \varepsilon_2$ and $\xi_2 = \varepsilon_3 + \varepsilon_4$, respectively. (See the diagram $\mathcal{D}_\gamma(\mathfrak{g}, \mathfrak{h})$ in Appendix A.) In this case, for $j = 1, 2$, $K(\mathfrak{l}_j : \mathfrak{g}(1))$ and $K(\mathfrak{l}_j : \mathfrak{l}_j)$ are computed as follows.

(1) $K(\mathfrak{l}_j;\mathfrak{g}(1))$: Observe that we have $\|\gamma\|^2 = \|\xi_j\|^2$ for $j = 1, 2$. Also, observe that since any weights β for $\mathfrak{g}(1)$ are roots of $\mathfrak{g}$, each weight space $\mathfrak{g}(1)_\beta$ of $\mathfrak{g}(1)$ is one-dimensional. Thus $K(\mathfrak{l}_1;\mathfrak{g}(1))$ is given by

$$\begin{aligned} K(\mathfrak{l}_1;\mathfrak{g}(1)) &= \frac{1}{2}\cdot\frac{\|\xi_j\|^2}{\|\gamma\|^2}\sum_{\beta\in\Delta(\mathfrak{g}(1))}\dim(\mathfrak{g}(1)_\beta)\langle\beta,\xi^\vee\rangle_1^2 \\ &= \frac{1}{2}\sum_{\beta\in\Delta(\mathfrak{g}(1))}\langle\beta,(\varepsilon_1-\varepsilon_2)^\vee\rangle^2 \\ &= \frac{1}{2}\left(\sum_{\substack{j=1,2\\k=3,\dots,n}}\langle\varepsilon_j\pm\varepsilon_k,(\varepsilon_1-\varepsilon_2)^\vee\rangle^2+\langle\varepsilon_1,(\varepsilon_1-\varepsilon_2)^\vee\rangle^2+\langle\varepsilon_2,(\varepsilon_1-\varepsilon_2)^\vee\rangle^2\right) \\ &= 2n-3. \end{aligned}$$

Similarly, we have $K(\mathfrak{l}_2;\mathfrak{g}(1)) = 4$.

(2) $K(\mathfrak{l}_j;\mathfrak{l}_j)$: By using the dual Coxeter numbers in Table 1, it follows from Lemma 4.2 that

$$K(\mathfrak{l}_1;\mathfrak{l}_1) = \frac{2\|\xi_1\|^2}{\|\gamma\|^2}(1+\langle\rho(\mathfrak{l}_1),\xi_1^\vee\rangle) = 2\cdot 2 = 4,$$

and

$$K(\mathfrak{l}_2;\mathfrak{l}_2) = \frac{2\|\xi_2\|^2}{\|\gamma\|^2}(1+\langle\rho(\mathfrak{l}_2),\xi_2^\vee\rangle) = 2(2(n-2)-1) = 2(2n-5).$$

By Proposition 3.3, we have

$$\frac{1}{2}m_{\mathrm{ad}} = 1+\langle\rho,\gamma^\vee\rangle,$$

where m_{ad} is the Dynkin index of the adjoint representation. To finish this section we show an analogous formula for $K(\mathfrak{l}_j;\mathfrak{g}(k))$.

Proposition 4.4. *Let $\mathfrak{g} = \bigoplus_{k=-r}^{r}\mathfrak{g}(k)$ be a complex simple graded Lie algebra with parabolic subalgebra $\mathfrak{q} = \mathfrak{l}\oplus\mathfrak{n} = \mathfrak{g}(0)\oplus\bigoplus_{k>0}\mathfrak{g}(k)$. If γ is the highest root for $\mathfrak{g}$ and if $\mathfrak{l}_j$ is a simple ideal of $\mathfrak{l} = \mathfrak{g}(0)$ then*

$$\frac{1}{2}K(\mathfrak{l}_j;\mathfrak{l}_j)+\sum_{k=1}^{r}K(\mathfrak{l}_j;\mathfrak{g}(k)) = 1+\langle\rho,\gamma^\vee\rangle.$$

Proof. First observe that since $\mathfrak{l} = \mathfrak{g}(0)$, each subspace $\mathfrak{g}(k)$ is an $\mathfrak{l}$-module. Then we have

$$\begin{aligned}
&\frac{1}{2}K(\mathfrak{l}_j;\mathfrak{l}_j) + \sum_{k=1}^{r} K(\mathfrak{l}_j;\mathfrak{g}(k)) \\
&= \frac{1}{2}\cdot\frac{\|\xi_j\|^2}{\|\gamma\|^2}\left(\frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{l}_j)}\langle\alpha,\xi^\vee\rangle_j^2 + \sum_{k=1}^{r}\sum_{\beta\in\Delta(\mathfrak{g}(k))}\langle\beta,\xi_j^\vee\rangle^2\right) \\
&= \frac{1}{2}\cdot\frac{\|\xi_j\|^2}{\|\gamma\|^2}\left(\sum_{\alpha\in\Delta^+(\mathfrak{l}_j)}\langle\alpha,\xi_j^\vee\rangle^2 + \sum_{k=1}^{r}\sum_{\beta\in\Delta(\mathfrak{g}(k))}\langle\beta,\xi_j^\vee\rangle^2\right).
\end{aligned}$$

If the semisimple part $[\mathfrak{l},\mathfrak{l}]$ of $\mathfrak{l}$ is $[\mathfrak{l},\mathfrak{l}] = \bigoplus_{t=1}^{m}\mathfrak{l}_t$ with $\mathfrak{l}_t$ simple ideals then $\Delta(\mathfrak{g}(0))$ is given by $\Delta(\mathfrak{g}(0)) = \Delta(\mathfrak{l}) = \bigcup_{t=1}^{m}\Delta(\mathfrak{l}_t)$. As $\alpha\perp\xi_j$ for any $\alpha\in\Delta(\mathfrak{l}_t)$ for $t\neq j$, we have

$$\sum_{\alpha\in\Delta^+(\mathfrak{l}_j)}\langle\alpha,\xi_j^\vee\rangle^2 = \sum_{t=1}^{m}\sum_{\alpha\in\Delta^+(\mathfrak{l}_t)}\langle\alpha,\xi_j^\vee\rangle^2 = \sum_{\alpha\in\Delta^+(\mathfrak{l})}\langle\alpha,\xi_j^\vee\rangle^2.$$

Therefore we obtain

$$\begin{aligned}
&\frac{1}{2}K(\mathfrak{l}_j;\mathfrak{l}_j) + \sum_{k=1}^{r} K(\mathfrak{l}_j;\mathfrak{g}(k)) \\
&= \frac{1}{2}\cdot\frac{\|\xi_j\|^2}{\|\gamma\|^2}\left(\sum_{\alpha\in\Delta^+(\mathfrak{l}_j)}\langle\alpha,\xi_j^\vee\rangle^2 + \sum_{k=1}^{r}\sum_{\beta\in\Delta(\mathfrak{g}(k))}\langle\beta,\xi_j^\vee\rangle^2\right) \\
&= \frac{1}{2}\cdot\frac{\|\xi_j\|^2}{\|\gamma\|^2}\left(\sum_{\alpha\in\Delta^+(\mathfrak{g}(0))}\langle\alpha,\xi_j^\vee\rangle^2 + \sum_{k=1}^{r}\sum_{\beta\in\Delta(\mathfrak{g}(k))}\langle\beta,\xi_j^\vee\rangle^2\right) \\
&= \frac{1}{2}\cdot\frac{\|\xi_j\|^2}{\|\gamma\|^2}\sum_{\alpha\in\Delta^+}\langle\alpha,\xi_j^\vee\rangle^2.
\end{aligned}$$

Note that it is applied from line three to line four that $\Delta^+ = \Delta^+(\mathfrak{g}(0))\cup\bigcup_{k=1}^{r}\Delta(\mathfrak{g}(k))$. Now Lemma 3.7 concludes the proposition. □

5. Main results

In this section, as our main results, we show that $c(\mathfrak{l}_j) = K(\mathfrak{l}_j;\mathfrak{g}(1))$ and $p(\mathfrak{l}_j) = K(\mathfrak{l}_j;\mathfrak{l}_j)$, where $c(\mathfrak{l}_j)$ and $p(\mathfrak{l}_j)$ are the two constants defined in Definitions 2.1 and 2.2, respectively.

Theorem 5.1. *Let $\mathfrak{g}$ be a complex simple Lie algebra, not of type A_2, and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{g}_\gamma$ be the parabolic subalgebra of $\mathfrak{g}$ of Heisenberg type. If $\mathfrak{l}_j$ is a simple ideal of $\mathfrak{l}$ then*

$$c(\mathfrak{l}_j) = K(\mathfrak{l}_j; \mathfrak{g}(1)).$$

Proof. Observe that, as $\beta \in \Delta(\mathfrak{g}(1))$ are roots for $\mathfrak{g}$, we have $\dim(\mathfrak{g}(1)_\beta) = 1$ for all $\beta \in \Delta(\mathfrak{g}(1))$. Since $\|\gamma\|^2$ is normalized as $\|\gamma\|^2 = 2$, it then follows that

$$K(\mathfrak{l}_j; \mathfrak{g}(1)) = \frac{1}{2} \cdot \frac{\|\xi_j\|^2}{\|\gamma\|^2} \sum_{\beta \in \Delta(\mathfrak{g}(1))} \dim(\mathfrak{g}(1)_\beta) \langle \beta, \xi_j^\vee \rangle^2 = \frac{\|\xi_j\|^2}{4} \sum_{\beta \in \Delta(\mathfrak{g}(1))} \langle \beta, \xi_j^\vee \rangle^2.$$

Thus, to prove this theorem, it suffices to show that

$$c(\mathfrak{l}_j) = \frac{\|\xi_j\|^2}{4} \sum_{\beta \in \Delta(\mathfrak{g}(1))} \langle \beta, \xi_j^\vee \rangle^2. \tag{5.1}$$

Since $\langle \mu, \xi_j \rangle = (\|\xi_j\|^2/2) \langle \mu, \xi_j^\vee \rangle$, if $\alpha = \mu$ and $\lambda = \xi_j$ in (2.1) then

$$\sum_{\beta \in \Delta(\mathfrak{g}(1))} \langle \mu, \beta \rangle \langle \beta, \xi_j \rangle = c(\mathfrak{l}_j) \frac{\|\xi_j\|^2}{2} \langle \mu, \xi_j^\vee \rangle. \tag{5.2}$$

On the other hand, by Lemma 2.4, we have $\mu - \xi_j \in \Delta(\mathfrak{g}(1))$. So, if $\alpha = \mu - \xi_j$ and $\lambda = \xi_j$ in (2.1) then, as $\langle \mu - \xi_j, \xi_j \rangle = \langle \mu, \xi_j \rangle - \|\xi_j\|^2 = (\|\xi_j\|^2/2)(\langle \mu, \xi_j^\vee \rangle - 2)$, we have

$$\sum_{\beta \in \Delta(\mathfrak{g}(1))} \langle \mu - \xi_j, \beta \rangle \langle \beta, \xi_j \rangle = c(\mathfrak{l}_j) \frac{\|\xi_j\|^2}{2} (\langle \mu, \xi_j^\vee \rangle - 2). \tag{5.3}$$

The difference of the left hand sides of (5.2) and (5.3) is

$$\begin{aligned} \sum_{\beta \in \Delta(\mathfrak{g}(1))} \langle \mu, \beta \rangle \langle \beta, \xi_j \rangle - \langle \mu - \xi_j, \beta \rangle \langle \beta, \xi_j \rangle &= \sum_{\beta \in \Delta(\mathfrak{g}(1))} \langle \xi_j, \beta \rangle \langle \beta, \xi_j \rangle \\ &= \frac{\|\xi_j\|^4}{4} \sum_{\beta \in \Delta(\mathfrak{g}(1))} \langle \beta, \xi_j^\vee \rangle^2, \end{aligned}$$

and that of the right hand sides is

$$c(\mathfrak{l}_j) \frac{\|\xi_j\|^2}{2} \langle \mu, \xi_j^\vee \rangle - c(\mathfrak{l}_j) \frac{\|\xi_j\|^2}{2} (\langle \mu, \xi_j^\vee \rangle - 2) = \|\xi_j\|^2 c(\mathfrak{l}_j).$$

Therefore we have

$$\frac{\|\xi_j\|^4}{4} \sum_{\beta \in \Delta(\mathfrak{g}(1))} \langle \beta, \xi_j^\vee \rangle^2 = \|\xi_j\|^2 c(\mathfrak{l}_j). \tag{5.4}$$

Now the equation (5.1) follows from dividing both sides of (5.4) by $\|\xi_j\|^2$. □

Next we show that $p(\mathfrak{l}_j) = K(\mathfrak{l}_j; \mathfrak{l}_j)$. To this end we show one simple result on the parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{g}_\gamma$ of Heisenberg type.

Lemma 5.2. *If $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{g}_\gamma$ is the parabolic subalgebra of $\mathfrak{g}$ of Heisenberg type then*

$$1 + \langle \rho, \gamma^\vee \rangle = \frac{\dim(\mathfrak{g}(1)) + 4}{2}.$$

Proof. By the equation (3.2), we have

$$\begin{aligned} 1 + \langle \rho, \gamma^\vee \rangle &= \frac{1}{2} \sum_{\alpha \in \Delta^+} \langle \alpha, \gamma^\vee \rangle^2 \\ &= \frac{1}{2} \left(\sum_{\alpha \in \Delta^+(\mathfrak{l})} \langle \alpha, \gamma^\vee \rangle + \sum_{\alpha \in \Delta(\mathfrak{g}(1))} \langle \alpha, \gamma^\vee \rangle^2 + \langle \gamma, \gamma \rangle^2 \right). \end{aligned}$$

As $\mathfrak{l}$ and $\mathfrak{g}(1)$ are the 0 eigenspace and 1 eigenspace of $\mathrm{ad}(H_\gamma)$, respectively, it follows from the normalization that $\langle \gamma, \gamma \rangle = 2$ that

$$\langle \alpha, \gamma^\vee \rangle = \langle \alpha, \gamma \rangle = \begin{cases} 0 & \text{if} \quad \alpha \in \Delta^+(\mathfrak{l}), \\ 1 & \text{if} \quad \alpha \in \Delta(\mathfrak{g}(1)). \end{cases}$$

Therefore, we have $\sum_{\alpha \in \Delta^+(\mathfrak{l})} \langle \alpha, \gamma^\vee \rangle = 0$ and $\sum_{\alpha \in \Delta(\mathfrak{g}(1))} \langle \alpha, \gamma^\vee \rangle = \dim(\mathfrak{g}(1))$. Hence,

$$\begin{aligned} 1 + \langle \rho, \gamma^\vee \rangle &= \frac{1}{2} \left(\sum_{\alpha \in \Delta^+(\mathfrak{l})} \langle \alpha, \gamma^\vee \rangle + \sum_{\alpha \in \Delta(\mathfrak{g}(1))} \langle \alpha, \gamma^\vee \rangle^2 + \langle \gamma, \gamma \rangle^2 \right) \\ &= \frac{\dim(\mathfrak{g}(1)) + 4}{2}. \end{aligned}$$

□

Proposition 5.3 ([1, Proposition 3.1])**.** *We have*

$$\frac{1}{2} p(\mathfrak{l}_j) + c(\mathfrak{l}_j) = \frac{\dim(\mathfrak{g}(1)) + 4}{2}.$$

Now we are ready to show that $p(\mathfrak{l}_j) = K(\mathfrak{l}_j; \mathfrak{l}_j)$.

Theorem 5.4. *Let $\mathfrak{g}$ be a complex simple Lie algebra, not of type A_2, and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{g}_\gamma$ be the parabolic subalgebra of $\mathfrak{g}$ of Heisenberg type. If $[\mathfrak{l}, \mathfrak{l}] = \bigoplus_{j=1}^m \mathfrak{l}_j$ with $\mathfrak{l}_j$ the simple ideals of $[\mathfrak{l}, \mathfrak{l}]$ then*

$$p(\mathfrak{l}_j) = K(\mathfrak{l}_j; \mathfrak{l}_j).$$

Proof. As in the proof for Lemma 5.2, observe that $\mathfrak{l} = \mathfrak{g}(0)$ the 0 eigenspace of $\mathrm{ad}(H_\gamma)$. Thus, $\alpha \perp \gamma$ for all $\alpha \in \Delta(\mathfrak{l})$; in particular, $\xi_j \perp \gamma$. Therefore, as $\mathfrak{g}_\gamma = \mathfrak{g}(2)$, we have $K(\mathfrak{l}_j; \mathfrak{g}(2)) = K(\mathfrak{l}_j; \mathfrak{g}_\gamma) = 0$. It then follows from Proposition 4.4 that

$$\frac{1}{2}K(\mathfrak{l}_j; \mathfrak{l}_j) + K(\mathfrak{l}_j; \mathfrak{g}(1)) = 1 + \langle \rho, \gamma^\vee \rangle. \tag{5.5}$$

On the other hand, by Proposition 5.3, we have

$$\frac{1}{2}p(\mathfrak{l}_j) + c(\mathfrak{l}_j) = \frac{\dim(\mathfrak{g}(1)) + 4}{2}.$$

Thus it follows from Lemma 5.2 and (5.5) that

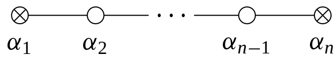
$$\frac{1}{2}K(\mathfrak{l}_j; \mathfrak{l}_j) + K(\mathfrak{l}_j; \mathfrak{g}(1)) = \frac{1}{2}p(\mathfrak{l}_j) + c(\mathfrak{l}_j).$$

Now the theorem is concluded by Theorem 5.1. □

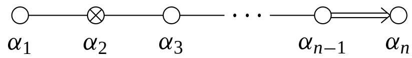
A. Appendix

In this appendix we give the diagram $\mathcal{D}_\gamma(\mathfrak{g}, \mathfrak{h})$ for each complex simple Lie algebra. (See Section 2.) For simplicity we depict the diagrams by crossing out the deleted nodes. We use the Bourbaki conventions [3] for the labeling of the simple roots for exceptional algebras.

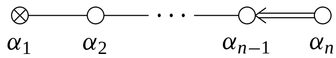
- A_n, $n \geq 2$:



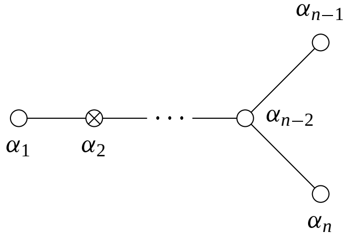
- B_n, $n \geq 3$:



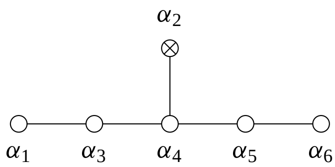
- C_n, $n \geq 2$:



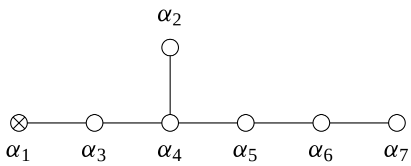
- D_n, $n \geq 4$:



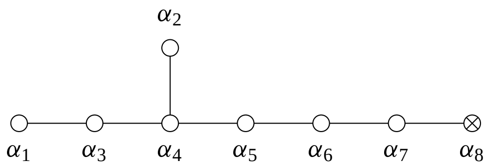
- E_6:



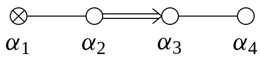
- E_7:



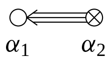
- E_8:



- F_4:



- G_2:



References

[1] L. Barchini, A.C. Kable and R. Zierau: *Conformally invariant systems of differential equations and prehomogeneous vector spaces of Heisenberg parabolic type*, Publ. Res. Inst. Math. Sci. **44** (2008), 749–835.

[2] L. Barchini, A.C. Kable and R. Zierau: *Conformally invariant systems of differential operators*, Adv. Math. **221** (2009), 788–811.

[3] N. Bourbaki: Groupes et Algébres de Lie, Masson, Paris, 1981, Chapters 4–6.

[4] H.W. Braden: *Integral pairings and Dynkin indices*, J. London Math. Soc. (2) **43** (1991), 313–323.

[5] E.B. Dynkin: *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl., Ser. II, **6** (1957), 111–244.

[6] V.G. Kac: Infinite-Dimensional Lie Algebras, third edition, Cambridge Univ. Press, Cambridge, 1990.

[7] S. Kumar and M.S. Narasimhan: *Picard group of the moduli spaces of G-bundles*, Math. Ann. **308** (1997), 155–173.

[8] S. Kumar, M.S. Narasimhan and A. Ramanathan: *Infinite Grassmannians and moduli spaces of G-bundles*, Math. Ann. **300** (1994), 41–75.

[9] A.L. Onishchik: *Torsion of special Lie groups*, Amer. Math. Soc. Transl., Ser. II, **50** (1966), 1–4.

[10] D.I. Panyushev: *On the Dynkin index of a principal* $\mathfrak{sl}_2$*-subalgebra*, Adv. Math. **221** (2009), 1115–1121.

[11] W. Wang: *Dimension of a minimal nilpotent orbit*, Proc. Amer. Math. Soc. **127** (1999), 935–936.

Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
Japan
e-mail: toskubo@ms.u-tokyo.ac.jp