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Efficient Collusion in Repeated Auctions with Communication

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Abstract

This paper studies collusion in repeated auctions when bidders communicate prior to each stage auction. For independent and correlated private signals and general interdependent values, the paper identifies conditions under which an equilibrium collusion scheme is fully efficient in the sense that the bidders' payoff is close to what they get when the object is allocated to the highest valuation bidder at the reserve price in every period.

Key words: collusion, auction, communication, folk theorem.

JEL classification numbers: C72, D82.

1 Introduction

Collusion is a wide-spread phenomenon in auctions.¹ For example, the Japan Fair Trade Commission (JFTC) issued warnings to 928 firms in thirty three collusion cases in government procurement auctions for year 2001 alone.² According to a report from JFTC, colluding firms “collaborated to predetermine a winning bidder” in each and every one of those cases.

Economic theory suggests that repeated auctions where the same set of bidders meet time and time again provide an ideal ground for collusion: In repeated auctions, not only is it easy to enforce the collusive agreement through the threat of reversion to competitive bidding in the event of a deviation, but it is also possible to transfer payoffs within a cartel without explicit exchange of money. For example, bidders in repeated auctions can employ a simple bid rotation collusion scheme which appoints the winning bidder of a stage auction in turn and hence transfers the continuation payoff from the current winner to other members of the cartel.

From the point of view of bidders, the optimal collusion scheme is one which is *fully efficient* in the sense that their equilibrium payoff is close to what they would get when the object is allocated at the reserve price to the highest valuation bidder in every stage auction. For example, the simple bid rotation collusion scheme as described above may be an improvement over the one-shot equilibrium, but is not fully efficient since the highest valuation bidder may not win just because it is not his turn. One important question then is if and when there exists a fully efficient equilibrium collusion scheme. This paper attempts to answer this question in a model of collusion with bidder communication.

Formally, the model of repeated auctions considered in this paper is a repeated game with private information in which the players’ private signals are drawn identically and independently across periods. Because of the presence of private signals, it is known that standard folk theorems for repeated games do not apply except for some special cases as noted below. This paper shows, however, that with communication among bidders, an appropriate modification of the enforceability technique does yield an analytical framework for this class of games.

We formulate a collusion scheme in which bidders communicate their private signals with one another prior to every stage auction. At the beginning of each period, the bidders report their private signals to the *center*, which then return

¹Documentation of collusion in auctions by economists includes Baldwin et al. (1997), Marshall and Meurer (1995), Pesendorfer (2000), Porter and Zona (1993), and others.

²Recent Activities of JFTC, by the Administrative Office of JFTC (October 2002, in Japanese). JFTC’s investigation is most often initiated by a notification from the public, and the number of investigated cases may only be a fraction of all cases.

instructions to them based on the reported signal profile. An *instruction rule* is a functional relationship between the reported signal profile and the resulting instructions. For efficient collusion, of course, it is desirable to use the *efficient instruction rule*, which, based on the report profile, instructs the highest valuation bidder to bid the reserve price in the stage auction and all other bidders to stay out. It is easy to see, however, that this instruction rule is not incentive compatible as it gives the bidders an incentive to overstate their signals. Appropriate adjustment of continuation payoffs is hence essential for the enforcement of such an instruction rule. In the problem with two bidders, we will show that this adjustment can be accomplished by transition to the instruction rule called an *exclusion rule*, which allocates the object to only one of the bidders regardless of their reported signals. With three or more bidders, we show that the adjustment can be done through a generalization of the exclusion rule which allocates the object efficiently within a subset of bidders.

In standard repeated games without private information, the enforceability conditions are expressed in terms of action profiles: They check whether taking a certain action is optimal given the discounted sum of today's stage payoff and the continuation payoff from tomorrow on. With private information, the enforceability conditions are instead expressed in terms of instruction rules. In other words, they check whether truth-telling is incentive compatible given the discounted sum of the current stage payoff implied by the instruction rule, and the continuation payoff.

By the standard argument, identifying the set of equilibrium payoff set reduces to finding a self-decomposable set of repeated game payoffs.³ Following Fudenberg *et al.* (1994), the present paper solves the latter problem by finding a profile of transfer rules that satisfy weighted budget balance conditions. It describes when such a transfer rule profile exists in problems with two bidders and those with three or more bidders separately. In both cases, it is shown that a fully efficient collusion scheme exists under fairly permissive conditions.

In repeated auctions with two bidders, we assume that the private signals are linearly ordered and affiliated across bidders. Under these conditions, we construct a *redistribution scheme* in which the bidder who has reported the higher signal becomes the winner and his surplus is redistributed to the other bidder in the form of continuation payoffs. With only two bidders, one bidder's gain in the continuation payoff is necessarily the other bidder's loss. As will be seen, this trade-off creates a bound on the enforceable payoffs. In actual problems, we can check the feasibility of efficient collusion by explicitly computing this bound. In fact, we derive a sufficient condition for a full folk theorem when the bidders have linear valuation functions, and show that efficient collusion is possible when each bidder's private signal has a

³See, for example, Abreu *et al.* (1990), and Fudenberg *et al.* (1994).

binary distribution.

When there are three or more bidders, we assume that signals are either (i) linearly ordered and independent across bidders, or (ii) correlated. In both cases, we show that any relevant instruction rule is enforceable, and hence that efficient collusion is achieved. With three or more bidders, the key is to dissociate the inducement of truth-telling from the budget balance considerations. In other words, we can choose a continuation payoff function that ignores some bidder's report while letting him "absorb" the surplus or deficit caused by the inducement of truth-telling from another bidder.

In the analysis of repeated games with imperfect public monitoring, Fudenberg *et al.* (1994) discuss repeated adverse selection with communication. Specifically, they show that when players publicly announce their private signals, a folk theorem holds for an adverse selection model with *independent private values* (IPV), where private signals have independent distributions across players and their values depend only on their own signals. Their theorem readily implies that under the IPV assumption, efficient collusion is possible in repeated auctions when the bidders are sufficiently patient. On the other hand, the IPV assumption is often difficult to justify in actual auction environments. It is well recognized in the mechanism design literature that the IPV assumption possesses a number of special properties that would fail in other problems. The analysis of Fudenberg *et al.* (1994) does not make clear whether efficient collusion is a unique phenomenon in the IPV environment, or a more general conclusion. The present paper takes a new approach to enforceability and shows that the conclusion does not hinge on the IPV assumption. Furthermore, it explicitly characterizes the allocations that appear in the course of collusion.

The paper that is most closely related to the present one is Aoyagi (2003), which studies a collusion scheme with communication in repeated auctions that improves on the one-shot Nash equilibrium of the stage auction as well as the simple bid rotation scheme as described above. It develops the idea of *dynamic bid rotation* whereby intertemporal transition between instruction rules takes place as a function of the current and past reported signals. Specifically, the winner of the stage t auction in this scheme is required to stay out of the next few auctions. It should be noted that the collusion scheme in the present paper is a natural extension of that in Aoyagi (2003): As mentioned above, the collusion scheme in this paper supports an efficient payoff vector by switching among simple instruction rules that allocate the object efficiently within some subset of bidders. The intuition is as follows. Suppose that the efficient instruction rule is used in stage t . The bidders are given incentive for truth-telling through the requirement that the stage t winner must stay out in stage $t + 1$ (and possibly more). In stage $t + 1$, then, the scheme switches

to the quasi-efficient instruction rule that allocates the object efficiently within the losers of the stage t auction. These bidders are given incentive for truth-telling through the requirement that the winner there must stay out in stage $t + 2$ (and possibly more), *etc.* In other words, the collusion scheme proceeds by excluding recent winners and appoints the winner efficiently from the pool of recent losers. In short, we demonstrate the exact form of bid rotation required for efficient collusion.

The main difference between the present paper and Aoyagi (2003) is the cardinality of the signal set. Aoyagi (2003) assumes that it is the unit interval $[0, 1]$, while the present paper assumes that it is a *finite* set. The stronger conclusion obtained in this paper derives from the self-decomposability techniques available for finite-action games. While it is more standard in the auction literature to use a continuous signal space, we know of no continuous counterpart to the above technique. For this technical reason, such a generalization is beyond the scope of this paper and is left as a topic of future research.

Skrzypacz and Hopenhayn (2002) and Blume and Heidhues (2002) both study tacit collusion in repeated auctions, where bidders do not communicate prior to each stage auction. They show that a certain degree of improvement over one-shot Nash equilibrium as well as simple bid rotation is possible in independent private values models. The difference in our modeling choice is based on the following considerations: First, while per se illegal, bidder communication is often an integral part of actual collusion practice, and it is important to understand its implications. Second, with communication or not, little is known about the full scope of collusion in repeated auctions. It is hence useful to present a simple framework in which full efficiency can be achieved. Combined with the study of collusion without communication, the present analysis would make it possible to discuss the effectiveness of a strict ban on communication.⁴

In a repeated model of oligopoly, Athey and Bagwell (2001) consider collusion with communication when each player's private signal is binary. In the IPV environment, they show that efficient collusion is sustained for a discount factor strictly smaller than unity. The construction takes advantage of the special feature of their model that the two players' valuations are identical to each other at least one half of the time. Such an event renders every allocation efficient, and thus providing numerous opportunities for efficient adjustment of continuation payoffs. It does not appear straightforward to generalize this conclusion to the case with more signals and/or asymmetric valuations.

Communication mediated by the center as assumed in this paper mimics the

⁴Of course, the plethora of repeated game equilibria suggests that coordination on a particular equilibrium requires some form of communication before the play of a repeated game. Blume and Heidhues (2002) present an anti-folk theorem when no such communication is allowed.

direct revelation mechanism applied to each stage and hence represents a natural mode of information transmission. It also allows for a clean presentation of the enforceability conditions through the use of an instruction rule. However, what is essential for the argument is the functional relationship between a report profile and bidding behavior in the stage auction as well as continuation play, and the result continues to hold if we instead assume public communication, where bidders publicly reveal their private signals.

Although the discussion in this paper is embedded in the repeated auctions framework, its analysis applies to other problems of repeated adverse selection. They include, for example, models of repeated oligopoly with private cost signals, and unemployment insurance.⁵ In the latter application, for example, each agent either finds a job or not in each period and reports their job status to a social planner. Upon receiving the agents' reports, the planner determines whom to award unemployment insurance for the current period. This problem has the same structure as the present one once the social planner is identified with the center.

The paper is organized as follows: A model of repeated auctions is formulated in the next section. The enforceability of an instruction rule is defined in Section 3. Section 4 describes the feasible as well as self-decomposable payoff sets. Sections 5 and 6 study collusion by two bidders and by three or more bidders, respectively.

2 Model

The set I of I risk-neutral bidders participate in an infinite sequence of auctions, where a single indivisible object is sold in every period through a fixed auction format.⁶ In each period, bidder i draws a private signal s_i from a *finite* set S_i . The signal profile $s = (s_1, \dots, s_I)$ of I bidders has the joint distribution p in every period and is independent across periods.

Bidder i 's valuation of the object sold in each period is a function of the signal profile $s = (s_1, \dots, s_I)$ in that period and denoted $v_i(s) \geq 0$. A *stage auction* is any transaction mechanism that determines the allocation of the good as well as monetary transfer based on a single sealed bid submitted by each bidder.⁷ Participation in the stage auction is voluntary so that the set of each bidder's *generalized bids* is expressed as $B_1 = \dots = B_I = \{N\} \cup \mathbb{R}_+$, where N represents "no participation." The rule of the auction is summarized by mappings ω_i and ξ_i ($i \in I$) on the set $B = B_1 \times \dots \times B_I$ of bid profiles $b = (b_1, \dots, b_I)$: $\omega_i(b)$ is the probability that

⁵See Atkeson and Lucas (1992, 1995), Athey and Bagwell (2001) and Athey et al. (1998).

⁶Note that the symbol I represents both the set of bidders and its cardinality.

⁷The restriction to a sealed bid auction is purely for simplicity.

bidder i is awarded the good, and $\xi_i(b)$ is his expected payment to the auctioneer. We assume that ω_i and ξ_i satisfy the following conditions.

1. A bidder makes no payment when he does not participate: $\xi_i(b) = 0$ if $b_i = N$.
2. A bidder may win the object only if he submits a bid at or above the reserve price $R \in [0, \max_{s,i} v_i(s)]$: $\omega_i(b) = 0$ if $b_i \in \{N\} \cup [0, R)$.
3. If bidder i bids R and all other bidders stay out, then i wins the object at price R : $\omega_i(b) = 1$ and $\xi_i(b) = R$ if $b_i = R$ and $b_j = N$ for all $j \neq i$.

Note that the above conditions hold for most standard auctions including the first- and second-price auctions. Consider the Bayesian game in which bidder i 's (pure) strategy is a mapping $\eta_i : S_i \rightarrow B_i$ and his (*ex ante*) payoff function is

$$\sum_{s \in S} p(s) \{ \omega_i(\eta(s)) v_i(s) - \xi_i(\eta(s)) \}.$$

Let ΔB_i denote the probability distribution over B_i , and $\tilde{\eta}_i : S_i \rightarrow \Delta B_i$ denote bidder i 's mixed strategy in this game. We assume that this game has a (mixed) Nash equilibrium $\tilde{\eta}^0 = (\tilde{\eta}_1^0, \dots, \tilde{\eta}_I^0)$, which describes the non-cooperative bidding behavior in the stage auction. Let g_i^0 be the corresponding (*ex ante*) Nash equilibrium payoff to bidder i .

Collusion in the repeated auction takes the following form: At the beginning of each period, all bidders report their private signals s_i to the center. Upon receiving the report profile $\hat{s} = (\hat{s}_1, \dots, \hat{s}_I) \in S$, the center chooses an instruction to each bidder i on what to do in the stage auction.

In general, the bidders may report a false signal, and/or disobey the instruction. Bidder i 's *reporting rule* λ_i chooses report \hat{s}_i as a function of his true signal s_i , and his *bidding rule* μ_i chooses bid b_i in the stage action as a function of his signal, report and instruction. The reporting rule is *honest* if it always reports the true signal, and the bidding rule is *obedient* if it always obeys the instruction. Denote by λ_i^* and μ_i^* bidder i 's honest reporting rule and obedient action rule, respectively.

We assume that the identity of the winner of each stage auction (if any) is publicly observable. For simplicity, we assume that the center is also capable of observing this information.⁸ It follows that if any bidder disobeys the center's instruction and becomes an unexpected winner, it is an observable deviation. On the other hand, misreporting of a private signal is in general an unobservable deviation.

⁸If the center does not have this capability, we can replicate the situation by letting the bidders report the winner's identity to the center.

The *center* is an institution whose role in each stage is to transform the report profile into instructions. We suppose that the center's instruction to each bidder is chosen from the set $\{R, N, E\}$: instruction R implies that the bidder should submit the reserve price, N implies that he should stay out, and E implies that he should play the one-shot NE strategy $\tilde{\eta}_i^0$. An *instruction rule* $d = (d_1, \dots, d_I)$ is a mapping from the set S of report profiles to the set $\{R, N, E\}^I$ of instruction profiles: $d_i(\hat{s}) \in \{R, N, E\}$ is the instruction given to bidder i when the report profile equals $\hat{s} \in S$. At the beginning of each period, the center publicly announces which instruction rule is used in that period. Let d^0 be the *one-shot NE instruction rule* such that $d_i^0(\hat{s}) = E$ for each $\hat{s} \in S$ and $i \in I$.

Bidder i 's *communication history* in period t in the repeated auction game is the sequence of his reports and instructions in periods $1, \dots, t-1$. On the other hand, bidder i 's *private history* in period t is the sequence of his private signals s_i in periods $1, \dots, t-1$. Furthermore, the *public history* in period t is a sequence of instruction rules used by the center in periods $1, \dots, t$ and the identities of winners in the stage auctions in periods $1, \dots, t-1$.

Bidder i 's (pure) *strategy* σ_i in the repeated auction chooses the pair (λ_i, μ_i) of reporting and bidding rules in each period t as a function of his communication and private histories in t , and the public history in t . Let σ_i^* be bidder i 's *honest and obedient* strategy which plays the pair (λ_i^*, μ_i^*) of the honest reporting rule and obedient bidding rule for all histories.

The *collusion scheme* τ is the center's (contingent) choice of an instruction rule in every period as a function of communication and public histories. Let $\delta < 1$ be the bidders' common discount factor, and $\Pi_i(\sigma, \tau, \delta)$ be bidder i 's average discounted payoff (normalized by $(1 - \delta)$) in the repeated game under the profile (σ, τ) . The collusion scheme τ is an *equilibrium* if the profile $\sigma^* = (\sigma_1^*, \dots, \sigma_I^*)$ of honest and obedient strategies constitutes a *perfect public equilibrium* of the repeated game.

Our analysis will focus on the following class of "grim-trigger" collusion schemes with two phases: The game starts in the *collusion phase*, and reverts to the *punishment phase* forever if and only if the identity of the winner (if any) is inconsistent with the center's instructions. In the punishment phase, the center chooses the one-shot NE instruction rule d^0 every period. Furthermore, the center's instruction in the collusion phase is such that all but one bidder is instructed to stay out (N).

In the following sections, we will be interested in supporting payoffs above the one-shot NE level g^0 , and consideration of a grim-trigger scheme allows us to separate disobedience from misreporting in the following sense. If bidder i 's continuation payoff is strictly above his one-shot NE level g_i^0 , and if his discount factor $\delta < 1$ is sufficiently close to one, then he has no incentive to disobey the center's instruction

regardless of his reporting strategy. In other words, if a bidder who is instructed to stay out (N) deviates and wins, the deviation is observable and hence triggers the punishment. On the other hand, if he disobeys and submits a losing bid, the deviation may be unobservable but is not profitable. Furthermore, a bidder who is instructed to win with bid R cannot gain by bidding above R or by losing (since that would be an observable deviation). In what follows, therefore, our analysis will deal exclusively with the prevention of misreporting.

3 Enforceability of Instruction Rules

Define

$$g_i^d(\lambda) = \sum_{s \in S} p(s) \{ \omega_i(d(\lambda(s))) v_i(s) - \xi_i(d(\lambda(s))) \}.$$

to be bidder i 's *ex ante* expected stage payoff under the instruction rule d when the bidders use the reporting rule profile $\lambda = (\lambda_1, \dots, \lambda_I) \in \Lambda$ and *obey* the center's instruction. Denote by $q(\hat{s} \mid \lambda)$ the probability of the report profile \hat{s} under the reporting rule profile $\lambda \in \Lambda$:

$$q(\hat{s} \mid \lambda) = \sum_{\{s: \lambda(s) = \hat{s}\}} p(s).$$

The construction below is an adaptation of Fudenberg *et al.* (1994) to the repeated adverse selection framework.

Let $W \subset \mathbb{R}^I$ be a set of payoff vectors. The instruction rule d is (*truthfully*) *enforceable with respect to* δ and W if there exists $y = (y_1, \dots, y_I) : S \rightarrow W$, a profile of continuation payoff functions taking values in W , such that for every $i \in I$ and $\lambda_i \in \Lambda_i$,

$$(1 - \delta) g_i^d(\lambda^*) + \delta \sum_{\hat{s} \in S} q(\hat{s} \mid \lambda^*) y_i(\hat{s}) \geq (1 - \delta) g_i^d(\lambda_i, \lambda_{-i}^*) + \delta \sum_{\hat{s} \in S} q(\hat{s} \mid \lambda_i, \lambda_{-i}^*) y_i(\hat{s}).$$

In other words, truth-telling maximizes the (discounted) sum of today's stage payoff and the continuation payoff from tomorrow on among all possible reporting rules. Given a collection D of instruction rules, the set W is *locally self-decomposable with respect to* D if for each $w \in W$, there exist a discount factor $\delta < 1$ and an open neighborhood U of w such that for any $u = (u_1, \dots, u_I) \in U$, there exists an instruction rule $d \in D$ such that d is enforceable with respect to δ and W through some continuation payoff function profile $y = (y_1, \dots, y_I) : S \rightarrow W$, and

$$u_i = (1 - \delta) g_i^d(\lambda^*) + \delta \sum_{\hat{s} \in S} q(\hat{s} \mid \lambda^*) y_i(\hat{s})$$

for every $i \in I$.

As in Fudenberg *et al.* (1994), the case where the set W is a hyperplane is of particular importance in our analysis. For any $\alpha \in \mathbb{R}^I$ such that $\alpha \neq 0$, note that if d is enforceable with respect to $\delta < 1$ and the hyperplane $W = \{u \in \mathbb{R}^I : \alpha \cdot u = 0\}$, then for any $\delta' < 1$ and $\zeta \in \mathbb{R}^I$, d is enforceable with respect to δ' and a parallel hyperplane $W' = \{u \in \mathbb{R}^I : \alpha \cdot (u - \zeta) = 0\}$.⁹ For this reason, we say that an instruction rule d is *enforceable with respect to α* without reference to δ and the particular (α -normal) hyperplane. Note that enforceability with respect to the α -normal hyperplane through the origin involves the incentive compatibility conditions and the weighted budget balance condition $\alpha \cdot y(\hat{s}) = 0$ ($\hat{s} \in S$).

The set $W \subset \mathbb{R}^I$ is *smooth* if it is closed and convex, and if its interior is non-empty and its boundary is a C^2 -manifold. Following Fudenberg *et al.* (1994), we associate the local self-decomposability of a smooth set W with enforceability with respect to its supporting hyperplanes. Formally, a smooth set W is *decomposable on tangent hyperplanes* (given the set D of instruction rules) if for every point w on the boundary of W , there exists an instruction rule $d \in D$ such that (i) $g^d(\lambda^*)$ and W are separated by the supporting hyperplane H of W at w , and (ii) d is enforceable with respect to H .

Let $\alpha \neq 0$ be any I -dimensional vector, and

$$A(\alpha) = \{d : d \text{ is enforceable with respect to } \alpha\}.$$

Given any set D of instruction rules, define

$$V^*(D) = \bigcap_{\alpha \neq 0} \{u \in \mathbb{R}^I : u \geq g^0, \alpha \cdot u \leq \max_{d \in D \cap A(\alpha)} \alpha \cdot g^d(\lambda^*)\}.$$

By Theorem 4.1 of Fudenberg *et al.* (1994), if a smooth set W is decomposable on tangent hyperplanes given the set D of instruction rules, then W is locally self-decomposable with respect to D . Finally, Lemma 4.2 of Fudenberg *et al.* (1994) states that if W is compact, convex, and locally self-decomposable with respect to some D , then there exists a discount factor $\underline{\delta} < 1$ such that for any $\delta > \underline{\delta}$, any point $w \in W$ is sustained as a payoff vector of an equilibrium collusion scheme whose instruction rules are chosen from D . The following lemma summarizes the argument.

⁹If d is enforceable with respect to δ and W through y , then it is enforceable with respect to δ' and W' through $y' : S \rightarrow W' \equiv H(\alpha, \zeta')$ defined by

$$y'(\hat{s}) = \zeta' + \frac{\delta(1 - \delta')}{\delta'(1 - \delta)} \{y(\hat{s}) - \zeta\}.$$

Lemma 1 *Suppose that $V^*(D)$ has a non-empty interior for some set D of instruction rules. Then any smooth subset W of the interior of $V^*(D)$ is locally self-decomposable with respect to D . Hence, for any $u \in V^*(D)$ and $\epsilon > 0$, there exists $\underline{\delta} < 1$ such that the following holds if $\delta > \underline{\delta}$: There exists an equilibrium collusion scheme τ which chooses instruction rules from D and yields the payoff vector $\Pi(\tau, \sigma^*, \delta)$ satisfying $\|\Pi(\tau, \sigma^*, \delta) - u\| < \epsilon$.*

Proof Take any smooth set $W \subset \text{int } V^*(D)$. Let w be a point on the boundary of W , and $\alpha \neq 0$ be the normal vector of the supporting hyperplane of W at w so that $\alpha \cdot u \leq \alpha \cdot w$ for any $u \in W$. Since $w \in V^*(D)$, there exists $d \in D$ such that $d \in A(\alpha)$ and $\alpha \cdot w \leq \alpha \cdot g^d(\lambda^*)$. Hence, W is decomposable on the tangent hyperplane at w using d . Therefore, W is decomposable on tangent hyperplanes, and hence is locally self-decomposable with respect to D by Theorem 4.1 of Fudenberg *et al.* (1994). The desired conclusion then follows from their Lemma 4.2. //

4 Instruction Rules for Bid Rotation

For efficient collusion, the bidder with the highest valuation should be instructed to bid the reserve price in the stage auction while other bidders are instructed to stay out. We begin with the description of such an instruction rule.

Since the signal space S_i is finite, more than one bidder may share the same highest valuation with positive probability. This suggests the possible multiplicity of efficient allocations according to different tie-breaking rules. To capture this possibility, we introduce a *permutation* on I that describes each player's rank in tie-breaking. Let Φ_I be the set of permutations on I : each $\phi \in \Phi_I$ is a one-to-one mapping from I to itself. For any $\phi \in \Phi_I$, let $d^{\phi*}$ denote the *efficient* instruction rule defined as follows: Given the report profile $\hat{s} \in S$, $d^{\phi*}$ instructs the bidder with the highest valuation (based on \hat{s}) to bid R if his valuation is higher than the reserve price R . If there exist two or more bidders with the highest valuation, then bidder i becomes the winner if and only if his index $\phi(i)$ according to ϕ is the smallest among all such bidders. Any other bidder is instructed to stay out. If we let $I^*(s) = \arg \max_{j \in I} v_j(s)$ be the set of bidders with the highest valuation under the signal profile s , then $d^{\phi*}$ can formally be described as:

$$d_i^{\phi*}(\hat{s}) = \begin{cases} R & \text{if } i \in I^*(\hat{s}), \phi(i) \leq \phi(j) \text{ for every } j \in I^*(\hat{s}), \text{ and } v_i(\hat{s}) \geq R, \\ N & \text{otherwise.} \end{cases}$$

For each $i \in I$, denote by $g_i^{\phi*}$ bidder i 's (ex ante) stage payoff $g_i^{d^{\phi*}}(\lambda^*)$ associated with $d^{\phi*}$. If we let

$$F = \text{Co} \{g^{\phi*} : \phi \in \Phi_I\},$$

then F is the set of (first-best) efficient payoff vectors. In the two-bidder problem, for example, the set F is an interval on the negative 45-degree line given the transferable utilities. Our analysis will assume that collusion is potentially profitable, *i.e.*, the one-shot Nash equilibrium is (strictly) Pareto dominated by any point on F : $g_i^0 < g_i^{\phi^*}$ for every $i \in I$ and $\phi \in \Phi_I$.

We will next describe instruction rules that are used for the adjustment of continuation payoffs. For each $i \in I$, let d^i be an asymmetric instruction rule defined as follows: Given the report profile $\hat{s} \in S$, d^i instructs (i) bidder i to bid R if his valuation $v_i(\hat{s})$ exceeds R , and to stay out otherwise, and (ii) bidder j ($j \neq i$) to stay out:

$$d_i^i(\hat{s}) = \begin{cases} R & \text{if } v_i(\hat{s}) > R \\ N & \text{otherwise,} \end{cases} \quad \text{and} \quad d_j^i(\hat{s}) = N \text{ for any } \hat{s}.$$

In other words, bidder i is the only potential winner under d^i . Let $g_j^i = g_j^{d^i}(\lambda^*)$ be bidder j 's (ex ante) stage payoff under d^i . We have $g_j^i = 0$ if $j \neq i$. For any i , we call d^i an *exclusion rule*.

The following lemma states that the exclusion rule d^i and the one-shot NE instruction rule d^0 are both enforceable with respect to any vector α .

Lemma 2 *For any $\alpha \neq 0$, d^i ($i \in I$) and d^0 are truthfully enforceable with respect to α : $\alpha \neq 0 \Rightarrow d^i, d^0 \in A(\alpha)$.*

Proof Whether $d = d^i$ or $d = d^0$, $g_i^d(\lambda^*) \geq g_i^d(\lambda_i, \lambda_{-i}^*)$ for every $i \in I$ so that d is enforceable with respect to α through $y(\cdot) \equiv 0$.

We next consider instruction rules that entail an efficient allocation for some subset of bidders. Formally, given the signal profile $s \in S$ and a subset $J \subset I$ of bidders, denote by $I^*(s, J) = \arg \max_{i \in J} v_i(s)$ the set of bidders with the highest valuation in set J . For each permutation $\phi \in \Phi_J$ on the subset J , define $d(\cdot \mid \phi, J)$ to be the *J-efficient instruction rule* such that

$$d_i(\hat{s} \mid \phi, J) = \begin{cases} R & \text{if } i \in I^*(\hat{s}, J) \text{ and } \phi(i) \leq \phi(j) \text{ for any } j \in I^*(\hat{s}, J), \\ N & \text{otherwise.} \end{cases}$$

In other words, each J -efficient instruction rule allocates the good efficiently *within* the set J while excluding all other bidders. Note that the I -efficient rule $d(\cdot \mid \phi, I)$ is equivalent to the efficient rule d^{ϕ^*} , and that the $\{i\}$ -efficient rule is equivalent to the exclusion rule d^i .

We now define

$$D^* = \{d^0\} \cup \{d(\cdot \mid \phi, J) : J \subset I, J \neq \emptyset, \phi \in \Phi_J\},$$

and

$$U^* = \text{Co}\{g(d) : d \in D^*\}.$$

In other words, D^* is a collection of all the J -efficient instruction rules for every non-empty subset J of I and the one-shot Nash equilibrium instruction rule, and U^* is the set of payoff vectors that can be expressed as a convex combination of the payoffs associated with those instruction rules. Let $V \subset \mathbb{R}^I$ be defined by

$$V = U^* \cap \{u \in \mathbb{R}^I : u \geq g^0\}.$$

We will be interested in supporting payoff vectors in the set V by equilibrium collusion schemes.

It can be seen that the efficiency frontier of V equals F introduced earlier.

Given any vector $\alpha \in \mathbb{R}^I$, let $J_\alpha = \{j : \alpha_j > 0\}$ be the set of bidders given positive weights by α . Theorem 3 below shows that if, for each $\alpha \neq 0$, the J -efficient rule $d(\cdot \mid \phi, J)$ is enforceable with respect to α for every $J \subset J_\alpha$ and $\phi \in \Phi_J$, then every point in set V can be supported as an equilibrium payoff vector when the bidders are patient.

Theorem 3 *Suppose that $d(\cdot \mid \phi, J) \in A(\alpha)$ for any vector $\alpha \neq 0$, $J \subset J_\alpha$ with $J \neq \emptyset$, and $\phi \in \Phi_J$. Then $V = V^*(D^*)$.*

Proof For any $J \subset I$ ($J \neq \emptyset$), $\phi \in \Phi_J$, and $i \in I$, write $g(\phi, J) = g(d(\cdot \mid \phi, J))$. Note first that U^* equals the intersection of all the closed half-spaces containing $\{g(d) : d \in D^*\}$ (Rockafellar (1970, Corollary 11.5.1):

$$U^* = \bigcap_{\alpha \neq 0} \{u \in \mathbb{R}^I : \alpha \cdot u \leq \max_{d \in D^*} \alpha \cdot g(d)\}.$$

It then follows from the definition of $V^*(D^*)$ that $V \supset V^*(D^*)$.

We next show that $V \subset V^*(D^*)$. For this, take any $u \in V$ and let $\alpha \neq 0$ be given. Since $u \in U^*$, there exists $d \in D^*$ such that $\alpha \cdot u \leq \alpha \cdot g(d)$. The proof is complete if we show that there exists $\bar{d} \in D^* \cap A(\alpha)$ such that $\alpha \cdot g(d) \leq \alpha \cdot g(\bar{d})$.

If $d = d^0 \in D^*$, then we can let $\bar{d} = d^0$ since $d^0 \in A(\alpha)$ by Lemma 2.

Otherwise, $d = d(\cdot \mid \phi, J') \in D^*$ for some $J' \subset I$ ($J' \neq \emptyset$) and $\phi \in \Phi_{J'}$. Let $J = J' \cap J_\alpha$.

If $J = \emptyset$, then take any $\phi' \in \Phi_{J_\alpha}$ and let $\bar{d} = g(\phi', J_\alpha)$. We then have $\bar{d} \in A(\alpha)$ by assumption, and also $\alpha \cdot g(d) \leq 0 \leq \alpha \cdot g(\bar{d})$.

If $J \neq \emptyset$, We let $\bar{d} = d(\cdot \mid \bar{\phi}, J)$, where $\bar{\phi}$ is the permutation on J obtained by restricting ϕ (on J') to J : For any $i, j \in J$,

$$\bar{\phi}(i) \leq \bar{\phi}(j) \Leftrightarrow \phi(i) \leq \phi(j).$$

Since $\alpha_i \leq 0$ for $i \in J' \setminus J_\alpha$, and $\alpha_i > 0$ and $g_i(\bar{\phi}, J) \geq g_i(\phi, J')$ for $i \in J$, we have

$$\alpha \cdot g(d) = \sum_{i \in J} \alpha_i g_i(\phi, J') + \sum_{i \in J' \setminus J_\alpha} \alpha_i g_i(\phi, J') \leq \sum_{i \in J} \alpha_i g_i(\phi, J') \leq \alpha \cdot g(\bar{d}).$$

Since $\bar{d} = d(\cdot \mid \bar{\phi}, J) \in A(\alpha)$ by assumption, we obtain the desired conclusion. //

5 Two Bidders: Redistribution Scheme

This section analyzes the problem with two bidders. We compute an explicit formula for continuation payoff functions that enforce the efficient instruction rules. These continuation payoff functions have the property that the winner's surplus in the present stage auction is transferred to the loser. We first make the following assumptions.

Assumption 1 For $i = 1, 2$, the set of signal $S_i = \{s^0, s^1, \dots, s^K\}$ is linearly ordered in the sense that $S_i \subset \mathbb{R}_+$ with $s^0 < s^1 < \dots < s^K$. Furthermore, the probability distribution p of signal profile s has full support over S , and satisfies the *monotone likelihood ratio property*:

$$(1) \quad \frac{p_j(s'_j \mid s_i)}{p_j(s_j \mid s_i)} \leq \frac{p_j(s'_j \mid s'_i)}{p_j(s_j \mid s'_i)} \text{ if } s'_i \geq s_i \text{ and } s'_j \geq s_j \text{ (} i = 1, 2, j \neq i \text{)}.$$

Assumption 2 The valuation functions v_1 and v_2 are *monotone*: $v_i(s') \geq v_i(s)$ if $s' \geq s$, and satisfies the *single crossing property*: $v_i(s) \geq v_j(s) \Leftrightarrow s_i \geq s_j$.

With $I = 2$, the monotone likelihood ratio property in Assumption 1 is equivalent to affiliation in Milgrom and Weber (1982). Note that Assumption 2 holds in a private values model where $v_i(s) = s_i$ for every $s \in S$ and $i \in I$, and more generally, in a symmetric linear valuation model discussed in the second half of this section. We also assume in this section that the reserve price R equals zero. For the permutation $\phi \in \Phi_I$ such that $\phi(i) = 1$ and $\phi(j) = 2$, we write d^{ij*} for the efficient instruction rule $d^{\phi*}$. Hence, the set D^* of relevant instruction rules can be expressed as

$$D^* = \{d^0, d^1, d^2, d^{12*}, d^{21*}\}.$$

In this section, we use the convention that the first argument of v_i is s_i (own signal) and the second is s_j (the other bidder's signal), *i.e.*, $v_i(s^k, s^l) = v_i(s_i = s^k, s_j = s^l)$.

As mentioned above, our focus in this section is on a *redistribution scheme* in which the winner's surplus in each stage auction is redistributed to the loser through

an adjustment in continuation payoffs. Such a transfer has a natural interpretation, and is most likely at the heart of many actual collusion schemes. The theoretical analysis of such a collusion scheme is first provided by McAfee and McMillan (1992) under the assumption that side transfer among bidders is possible. Aoyagi (2003) extends their analysis and shows that monetary transfer can partially be compensated by the adjustment in continuation payoffs in repeated auctions. The scheme analyzed in Aoyagi (2003) is described in more detail as follows: It begins with the efficient instruction rule d^* , and switches with positive probability to the exclusion rule d^j when bidder i wins the stage auction today.¹⁰ In this event, d^j is used for a fixed number of periods and then play returns to the original phase where d^* is used. The same process is repeated thereafter unless there is a disobedience, which would trigger reversion to the one-shot NE instruction rule d^0 . It can be seen that this collusion scheme embodies the idea of redistribution as described above, but entails inefficiency since the (inefficient) exclusion rules d^i are invoked with positive probability. The redistribution scheme of this section uses the same type of rotation over efficient and exclusion rules, but inefficiency is eliminated by the use of the self-decomposability argument.

Formally, let $x_i : S_i \rightarrow \mathbb{R}_+$ and $x_j : S_j \rightarrow \mathbb{R}_+$ be non-negative functions of i 's and j 's reports, respectively, and let continuation payoff functions $y = (y_1, y_2)$ be given by

$$(2) \quad y_i(\hat{s}) = \begin{cases} -x_i(\hat{s}_i) & \text{if } \hat{s}_i \geq \hat{s}_j \\ \frac{\alpha_j}{\alpha_i} x_j(\hat{s}_j) & \text{otherwise,} \end{cases} \quad \text{and} \quad y_j(\hat{s}) = \begin{cases} \frac{\alpha_i}{\alpha_j} x_i(\hat{s}_i) & \text{if } \hat{s}_i \geq \hat{s}_j \\ -x_j(\hat{s}_j) & \text{otherwise.} \end{cases}$$

It is clear that the budget balance condition $\alpha \cdot y(\hat{s}) = 0$ holds for any $\hat{s} \in S$. We can think of $x_i(\hat{s}_i)$ as a compensation payment from bidder i to bidder j when the former wins with report \hat{s}_i . We will choose x_i and x_j so that they satisfy the incentive compatibility conditions. For this, we will first consider local incentive conditions that guarantee that the bidders do not have incentive to report a signal one-step above or below its true value. We will then show that x_i and x_j chosen to satisfy these conditions are monotone increasing functions of the signals when the ratio α_j/α_i is not extreme. We will finally globalize the local incentive properties of x_i and x_j using the monotonicity and the monotone likelihood ratio property. The last step is standard in the mechanism design literature.

Formally, for $i = 1, 2$ and $j \neq i$, let

$$\rho_j^{k,l} = \frac{p_j(s^l \mid s_i = s^k)}{\sum_{s_j \leq s^l} p_j(s_j \mid s_i = s^k)} \quad (k, l = 0, 1, \dots, K)$$

¹⁰Since Aoyagi (2003) considers a continuous signal set, the efficient instruction rule does not need to specify a tie-breaking rule.

be a *hazard rate* of s_j conditional on $s_i = s^k$. By the monotone likelihood property (1), we have $\rho_j^{k,l} \leq \rho_j^{k',l}$ if $k \leq k'$. For each $k = 1, \dots, K$, define $\theta_{ij}^k \leq 1$ by

$$\theta_{ij}^k = \frac{\rho_j^{k-1,k} v_i(s^{k-1}, s^k)}{v_j(s^k, s^k) - \{1 - \rho_j^{k-1,k}\} v_i(s^{k-1}, s^{k-1})},$$

and let

$$\theta_{ij} = \max_k \theta_{ij}^k.$$

It should be noted that $\theta_{ij} \leq 1$ since $v_j(s^k, s^k) = v_i(s^k, s^k) \geq v_i(s^{k-1}, s^k)$ by Assumption 2. The following lemma proves the existence of incentive compatibility transfer functions when the exchange rate θ for transfer between the two bidders is not extreme.

Lemma 4 *Suppose that Assumptions 1 and 2 hold. If $\alpha = (\alpha_1, \alpha_2)$ is such that $\alpha_1, \alpha_2 > 0$ and $\theta = \alpha_j/\alpha_i \in [\theta_{ij}, 1]$, then there exist functions x_i and x_j such that the efficient instruction rule d^{ij*} is enforceable with respect to α through y in (2).*

Proof See the Appendix.

The exact forms of the functions x_i and x_j are presented in the Appendix. To see why we need a bound on the exchange rate θ for transfer between the two bidders, suppose for example that money worth one dollar to bidder j is worth very little to bidder i . It will then be the case that any monetary transfer to/from bidder j designed to induce truth-telling from him will be insufficient to induce truth-telling from bidder i .

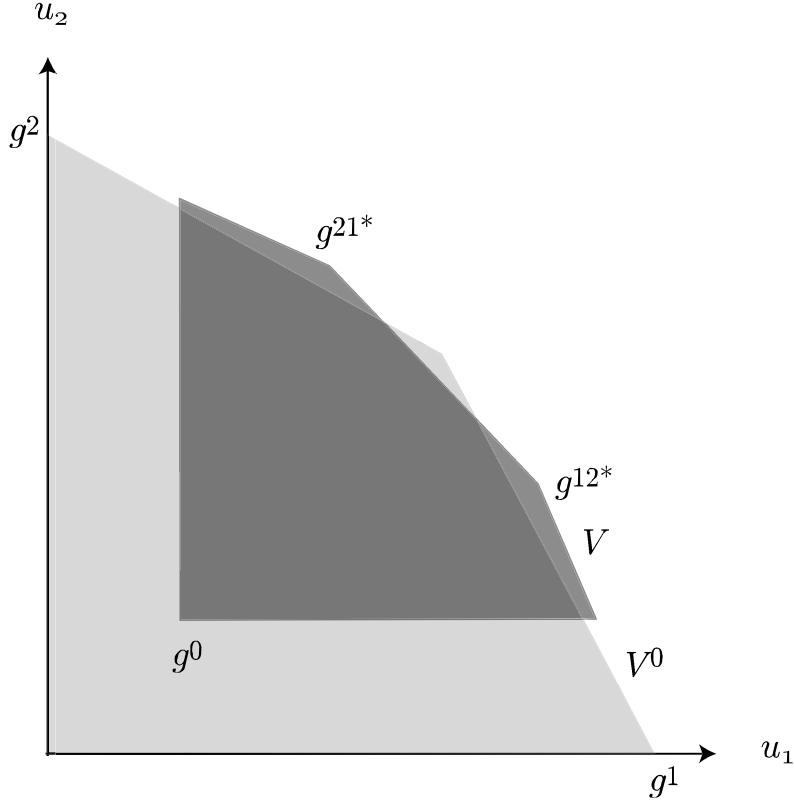
Let

$$V^0 = \{u \in \mathbb{R}_+^2 : u_i + \theta_{ij} u_j \leq g_i^i \text{ for } i = 1, 2, j \neq i\}.$$

In the two-dimensional plane (u_1, u_2) , the set V^0 is the area below the two straight lines $u_2 = -(u_1 - g_1^1)/\theta_{12}$ and $u_2 = -\theta_{21} u_1 + g_2^2$ (Figure 5).

The next theorem asserts any payoff vector in the intersection $V^0 \cap V$ can be supported in equilibrium when the instruction rules are chosen from D^* .

Theorem 5 *Suppose that $I = 2$ and that Assumptions 1 and 2 hold. Then $V^0 \cap V \subset V^*(D^*)$. That is, if $V^0 \cap V$ has a non-empty interior, then for any $u \in V^0 \cap V$ and $\epsilon > 0$, there exists $\underline{\delta} < 1$ such that the following holds for $\delta > \underline{\delta}$: There exists an equilibrium collusion scheme τ which chooses instruction rules from $D^* = \{d^0, d^1, d^2, d^{12*}, d^{21*}\}$ and yields the payoff vector $\Pi(\tau, \sigma^*, \delta)$ satisfying $\|\Pi(\tau, \sigma^*, \delta) - u\| < \epsilon$.*



Proof See the Appendix.

It follows that *every* payoff vector in V can be supported in equilibrium when $V \subset V^0$. Note that the latter condition is equivalent to $g^{12*}, g^{21*} \in V^0$ given our assumption that the efficient payoff vectors Pareto dominate the one-shot NE point g^0 . Hence, we have the following corollary to Theorem 5.

Corollary 6 *Under the conditions of Theorem 5, $V = V^*(D^*)$ if $g^{12*}, g^{21*} \in V^0$.*

On the other hand, even when $V \not\subset V^0$, *some* efficient payoff vectors can still be supported in equilibrium if V^0 has a non-empty intersection with the efficiency frontier F . For the rest of this section, we will focus attention on a linear valuation model and derive a sufficient condition for efficient collusion.

Assumption 3 The set $S_i = \{s^0, \dots, s^K\}$ of signals equals $\{0, 1/K, \dots, 1-1/K, 1\}$, and the valuation function is *linear* in the sense that $v_i(s) = cs_i + (1-c)s_j$ for $i = 1, 2$, $j \neq i$, and $c \in [1/2, 1]$.

The linear valuation function encompasses the private values model ($c = 1$) as well as the pure common values model ($c = 1/2$). The requirement $c \geq 1/2$ is equivalent to the single crossing condition in Assumption 2, which expresses the idea that one's own signal is at least as important as the other bidder's signal in determining valuation. It is clear that Assumption 3 implies Assumption 2.

The following theorem identifies the conditions on the distribution of private signals under which every point on the efficiency frontier can be supported as an equilibrium payoff.

Theorem 7 *Suppose that $I = 2$ and that Assumptions 1 and 3 hold. Then $V \subset V^0$ if and only if*

$$(3) \quad \frac{\rho_j^{k-1,k}(k-c)}{1 + \rho_j^{k-1,k}(k-1)} \leq \frac{1-c + c\nu_{ij}}{(1-c)\nu_{ij} + c},$$

$$\text{where } \nu_{ij} = \frac{\sum_{s_j > s_i} p(s) s_i}{\sum_{s_j > s_i} p(s) s_j} < 1.$$

Proof See the Appendix.

Since the left-hand side of (3) is increasing in $\rho_j^{k-1,k}$ (≤ 1) and in k ($\leq K$), and the right-hand side is increasing in ν_{ij} (≥ 0), a sufficient condition for (3) can be obtained by replacing these parameters by their bounds.

Corollary 8 *Suppose that $I = 2$ and that Assumptions 1 and 3 hold. Then $V \subset V^0$ if $1 - \frac{c}{K} \leq \frac{1-c}{c}$.*

By Corollary 8, we have $V \subset V^0$ under Assumptions 1 and 3 if either

1. c is sufficiently close to $1/2$ (i.e., values are almost common), or
2. $K = 1$ (i.e., the private signal of each bidder is binary $S_i = \{0, 1\}$).

The first observation above on the almost common value environment is intuitively clear: As $c \rightarrow 1/2$, g_i^i approaches $g_i^{ij*} + g_j^{ij*}$ (from below) since the allocation induced by the exclusion rule d^i becomes almost efficient. Geometrically, the line connecting points g^{ij*} and g^i (as well as g^{ji*} and g^j) approximates the negative 45-degree line. It follows that the inequality $g_i^{ij*} + \theta_{ij} g_j^{ij*} \leq g_i^i$ holds for any $\theta_{ij} < 1$.

Example 1 Suppose that the signals s_i and s_j are *independent* and have the uniform distribution: $p_i(s_i) = 1/(K+1)$ for any $s_i \in S_i$ ($i = 1, 2$). In this example, it can be checked that

$$\nu_{ij} = \frac{\sum_{s_j > s_i} p(s) s_i}{\sum_{s_j > s_i} p(s) s_j} = \frac{K-1}{2K+1},$$

and that

$$\rho_j^{k-1,k} = \frac{1}{k+1}.$$

By Theorem 7, hence, $V \subset V^0$ if

$$c \leq \frac{3K^2 + 5K + 1 - \sqrt{9K^4 + 18K^3 - 5K^2 - 14K + 1}}{2(K+2)}.$$

We can verify that the right-hand side approaches 1 as $K \rightarrow \infty$. In other words, for any $c < 1$, there exists \underline{K} such that efficient collusion is possible if the number of each bidder's private signals is at least \underline{K} . We also see that the inequality fails to hold (*i.e.*, $g^{ij*} \notin V^0$) if, for example, $c = 1$ (private values) and $K \geq 2$. However, it can be verified that F has a non-empty intersection with V^0 for any value of K and c . To see this, note that the payoffs corresponding to the efficient instruction rule d^{12*} can be computed as:

$$\begin{aligned} g_1^{12*} &= \sum_{k=0}^K \sum_{l=0}^k \frac{ck + (1-c)l}{K} \frac{1}{(K+1)^2} \\ &= \frac{1+c}{2K(K+1)^2} \sum_{k=0}^K k(1+k) = \frac{1+c}{2} \frac{K+2}{3(K+1)}, \end{aligned}$$

and

$$\begin{aligned} g_2^{12*} &= \sum_{k=0}^K \sum_{l=0}^{k-1} \frac{ck + (1-c)l}{K} \frac{1}{(K+1)^2} \\ &= \frac{1+c}{2K(K+1)^2} \sum_{k=0}^K k^2 = \frac{1+c}{2} \frac{2K+1}{6(K+1)}. \end{aligned}$$

It follows that the *symmetric* point $g^* = (g_1^{12*} + g_2^{12*})/2$ on the efficiency frontier is given by

$$g_1^* = g_2^* = \frac{1}{2} (g_1^{12*} + g_2^{12*}) = \frac{1+c}{2} \frac{4K+5}{12(K+1)}.$$

It is also seen that bidder i 's exclusion payoff equals $g_i^i = 1/2$. This implies that the bound on the exchange rate is given by

$$\theta_{ij} = \theta_{ij}^K = \frac{K-c}{K(1+c)}.$$

It then follows that for any $K \geq 1$,

$$g_i^* + \theta_{ij} g_j^* = \frac{(4K+5) \{(2+c)K - c\}}{24K(K+1)} < \frac{1}{2} = g_i^i.$$

As depicted in Figure 1, hence, the symmetric efficiency point $g^* = (g_1^*, g_2^*) \in V^0$. By Theorem 5, therefore, there exists an equilibrium collusion scheme whose payoff vector approximates g^* provided that the discount factor is close to one.

6 Three or More Bidders

With three or more bidders, efficient collusion is obtained under very weak assumptions. As mentioned in the Introduction, the key advantage of having more than two bidders is that inducement of truth-telling can be considered separately from the budget balance requirement. The particular construction below is based on that in Aoyagi (1998).¹¹

Recall from Theorem 3 that any payoff vector in V can be sustained in equilibrium if the J -efficient instruction rule $d(\cdot \mid \phi, J)$ is enforceable with respect to α for every $\alpha \neq 0$, $J \subset J_\alpha$ ($J \neq \emptyset$), and $\phi \in \Phi_J$. We first derive some general properties that can be used for the analysis of specific cases in Subsections 6.1 and 6.2.

Since $d(\cdot \mid \phi, \{i\}) = d^i$ is enforceable with respect to any $\alpha \neq 0$ by Lemma 2, suppose that $J_\alpha = \{1, \dots, n\}$ for some $n \geq 2$, and let $\phi \in \Phi_{J_\alpha}$ be given. By the definition of $d(\cdot \mid \phi, J_\alpha)$, the enforceability conditions for bidders $n+1, \dots, I$ are satisfied if we take $y_{n+1}(\cdot) = \dots = y_I(\cdot) \equiv 0$. For bidders $1, \dots, n$, we express their enforceability conditions in matrix form. For each $i = 1, \dots, n$, let $\Lambda_i^0 = \Lambda_i \setminus \{\lambda_i^*\}$ be the set of “untruthful” reporting rules, $m_i = |\Lambda_i^0|$ and $m = \sum_{i=1}^n m_i$. For any reporting rule profile λ , let $q(\cdot \mid \lambda) = (q(\hat{s} \mid \lambda))_{\hat{s} \in S}$ represent the $|S|$ -dimensional vector of probability distributions of report profiles under λ . Denote by B_i the $m_i \times |S|$ matrix whose row equals

$$b_i(\lambda_i) = q(\cdot \mid \lambda^*) - q(\cdot \mid \lambda_i, \lambda_{-i}^*) \quad (\lambda_i \in \Lambda_i^0).$$

In other words, each row of B_i corresponds to the difference in probability distributions of report profiles between λ_i^* and λ_i ($\neq \lambda_i^*$). Write $d = d(\cdot \mid \phi, J_\alpha)$ for simplicity, and let $\hat{v}_i(\lambda_i) = g_i^d(\lambda_i, \lambda_{-i}^*) - g_i^d(\lambda^*)$, and \hat{v}_i be the m_i -dimensional vector $\hat{v}_i = (\hat{v}_i(\lambda_i))_{\lambda_i \in \Lambda_i^0}$. It follows that d is enforceable with respect to α if there exist $y_1, \dots, y_n \in \mathbb{R}^S$ such that

$$(4) \quad B_i y_i \geq \hat{v}_i \text{ for } i = 1, \dots, n, \text{ and } \sum_{i=1}^n \alpha_i y_i = 0.$$

Note that we have set $\delta = 1/2$ in the expression of enforceability above based on the remark in Section 3. Eliminating y_n using the second condition and writing $\beta_i = \alpha_i / \alpha_n > 0$ ($i = 1, \dots, n-1$), we can further rewrite (4) as:

$$(5) \quad B \hat{y} \geq \hat{v},$$

¹¹See Kandori and Matsushima (1998) for a related idea.

where

$$B = \begin{bmatrix} B_1 & & O \\ & \ddots & \\ O & & B_{n-1} \\ -\beta_1 B_n & \cdots & -\beta_{n-1} B_n \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}, \quad \text{and} \quad \hat{v} = \begin{bmatrix} \hat{v}_1 \\ \vdots \\ \hat{v}_n \end{bmatrix}.$$

Therefore, d is enforceable with respect to α if the inequality (5) has a solution \hat{y} . The following lemma is a simple application of the theorem of the alternatives.

Lemma 9 *Inequality (5) has a solution \hat{y} if and only if for any $(\gamma_1, \dots, \gamma_n)$ such that $\gamma_i \in \mathbb{R}_+^{m_i}$ ($i = 1, \dots, n$),*

$$\gamma_i B_i - \beta_i \gamma_n B_n = 0 \text{ for every } i = 1, \dots, n-1 \quad \Rightarrow \quad \sum_{i=1}^n \gamma_i \cdot \hat{v}_i \leq 0.$$

Proof See the Appendix.

The following lemma states the conditions under which the budget balance requirement $\sum_i \alpha_i y_i = 0$ can be ignored.

Lemma 10 *For every $i \in J = \{1, \dots, n\}$ and $j \neq i$, if there exists $z : S \rightarrow \mathbb{R}$ such that $B_i z \geq \hat{v}_i$ and $B_j z = 0$, then (4) has a solution $y = (y_1, \dots, y_n) : S \rightarrow \mathbb{R}^n$.*

Proof In view of Lemma 9, suppose that $\gamma_i B_i - \beta_i \gamma_n B_n = 0$ ($i = 1, \dots, n-1$) for some $(\gamma_1, \dots, \gamma_n) \geq 0$. For any $i < n$, if we multiply from the right $z_i \in \mathbb{R}^S$ such that $B_i z_i \geq \hat{v}_i$ and $B_n z_i = 0$, then

$$0 = \gamma_i B_i z_i - \beta_i \gamma_n B_n z_i = \gamma_i B_i z_i \geq \gamma_i \cdot \hat{v}_i.$$

Likewise, if we multiply $z_n \in \mathbb{R}^S$ such that $B_i z_n = 0$ and $B_n z_n \geq \hat{v}_n$, then

$$0 = \gamma_i B_i z_n - \beta_i \gamma_n B_n z_n = -\beta_i \gamma_n B_n z_n \leq -\beta_i \gamma_n \cdot \hat{v}_n,$$

which implies $\gamma_n \cdot \hat{v}_n \leq 0$. It hence follows that $\sum_{i=1}^n \gamma_i \cdot \hat{v}_i \leq 0$. //

The simplest way to have $B_j z = 0$ is to make z independent of j 's report as shown in the lemma below. For $j \in I$, let $G_j \subset \mathbb{R}^S$ be the set of continuation payoff functions that do not depend on j 's report:

$$G_j = \{z \in \mathbb{R}^S : z(\hat{s}) = z(\hat{s}') \text{ if } \hat{s}_{-j} = \hat{s}'_{-j}\}.$$

The following lemma is immediate.

Lemma 11 *If $z \in G_j$, then $B_j z = 0$.*

Proof See the Appendix.

6.1 Independent Signals

In this section, we make the following assumptions about the signal distribution and the valuation functions.

Assumption 4 For every $i \in I$, the signal set $S_i = \{s^0, s^1, \dots, s^K\} \subset \mathbb{R}_+$ for $s^0 < s^1 < \dots < s^K$. Furthermore, the probability distribution p of signal profile s has full support over S , and is *independent*: $p(s) = \prod_i p_i(s_i)$.

Assumption 5 The valuation functions v_1, \dots, v_I are *monotone in the own signal*: $v_i(s'_i, s_{-i}) \geq v_i(s_i, s_{-i})$ for $s'_i \geq s_i$.

Fix any $J \subset I$ such that $|J| \geq 2$ and $\phi \in \Phi_J$, and consider the instruction rule $d = d(\cdot \mid \phi, J)$. For $i \in J$ and $l = 0, 1, \dots, K$, let $\bar{Z}_i(s^l \mid \phi, J) \subset S_{-i}$ be the set of signal profiles $s_{-i} = (s_j)_{j \neq i}$ of bidders other than i such that according to $d(\cdot \mid \phi, J)$, i becomes the winner and instructed to bid R under $(s_i = s^l, s_{-i})$:

$$\bar{Z}_i(s^l \mid \phi, J) = \left\{ s_{-i} \in S_{-i} : d_i(s_i = s^l, s_{-i} \mid \phi, J) = R \right\}.$$

Let $Z_i(s^l \mid \phi, J) = \bar{Z}_i(s^l \mid \phi, J) \setminus \bar{Z}_i(s^{l-1} \mid \phi, J)$. In other words, $Z_i(s^l \mid \phi, J)$ is the set of s_{-i} 's against which bidder i wins when reporting $\hat{s}_i = s^l$ but loses when reporting $\hat{s}_i = s^{l-1}$. For any $i \in J$, $j \neq i$, $k = 0, 1, \dots, K$, and $l = 1, \dots, K$, define

$$w_i^j(s^k, s^l \mid \phi, J) = \frac{\sum_{s_{-i} \in Z_i(s^l \mid \phi, J)} \{v_i(s_i = s^k, s_{-i}) - R\} p_{-i}(s_{-i})}{p_j(s_j = s^l)}.$$

It then follows from Assumption 5 that for any $k, k' = 0, 1, \dots, K$ and $l = 1, \dots, K$,

$$(6) \quad w_i^j(s^{k'}, s^l \mid \phi, J) \geq w_i^j(s^k, s^l \mid \phi, J) \text{ if } k' \geq k.$$

For any $j \neq i$, let

$$\rho_j^l = \frac{p_j(s_j = s^l)}{\sum_{s_j \leq s^l} p_j(s_j)} \quad (l = 0, 1, \dots, K),$$

and consider a continuation payoff function $z : S \rightarrow \mathbb{R}$ such that

$$(7) \quad z(\hat{s}) = \begin{cases} -x_i(\hat{s}_i) & \text{if } \hat{s}_i \geq \hat{s}_j \\ 0 & \text{otherwise,} \end{cases}$$

where $x_i : S_i \rightarrow \mathbb{R}_+$ is a non-negative function of i 's report defined recursively by

$$(8) \quad \begin{aligned} x_i(s^0) &= 0, \\ x_i(s^k) &= \rho_j^k w_i^j(s^{k-1}, s^k \mid \phi, J) + (1 - \rho_j^k) x_i(s^{k-1}) \quad (k = 1, \dots, K). \end{aligned}$$

According to z , i 's continuation payoff depends only on his and another bidder j 's reports. Just as in the redistribution scheme for two bidders, i 's continuation payoff is reduced when he reports a higher signal than j . The lemma below shows that z defined above makes truth-telling incentive compatible. Unlike in the two-bidder case, however, the required adjustment in bidder i 's transfer does not need to come from bidder j , but can come from yet another bidder. This slackness is what permits stronger conclusions with more than two bidders.

Lemma 12 *Under Assumptions 4 and 5, $B_i z \geq \hat{v}_i$ for z given in (7).*

Proof See the Appendix.

Since z is a function of the reports of only two bidders, there is at least one other bidder h for whom $z \in G_h$ and hence $B_h z = 0$ by Lemma 11. This along with Lemmas 10 and 12 as well as Theorem 3 yields the following theorem.

Theorem 13 *Suppose that $I \geq 3$ and that Assumptions 4 and 5 hold. For any $u \in V$ and $\epsilon > 0$, there exists $\underline{\delta} < 1$ such that the following holds if $\delta > \underline{\delta}$: There exists an equilibrium collusion scheme τ which chooses instruction rules from D^* and yields the payoff vector $\Pi(\tau, \sigma^*, \delta)$ satisfying $\|\Pi(\tau, \sigma^*, \delta) - u\| < \epsilon$.*

6.2 Correlated Signals

When private signals are correlated, continuation payoffs can be determined using the functional relationship between a bidder's private signal and the probability distribution of other bidders' signal profiles. When there exist three or more bidders, use of such a mechanism yields an extremely powerful conclusion that does not depend on the detailed specification of the valuation functions or the signal distribution. The analysis in this subsection draws heavily on Aoyagi (1998).

For each $s_i \in S_i$, let $p_{-i}(\cdot | s_i)$ and $p_{-i-j}(\cdot | s_i)$ denote the following vectors of conditional probabilities:

$$\begin{aligned} p_{-i}(\cdot | s_i) &= (p_{-i}(s_{-i} | s_i))_{s_{-i} \in S_{-i}}, \\ p_{-i-j}(\cdot | s_i) &= (p_{-i-j}(s_{-i-j} | s_i))_{s_{-i-j} \in S_{-i-j}}. \end{aligned}$$

Assumption 6 The probability distribution p of the private signals satisfy $p_{-i-j}(\cdot | s_i) \neq p_{-i-j}(\cdot | s'_i)$ for any $s_i \neq s'_i$ and $i \neq j$.

When the set of probability distributions p of $s \in S$ is identified with the $(|S|-1)$ -dimensional simplex $\Delta^{|S|-1}$, Assumption 6 holds generically in this set as long as $|S_i| \geq 2$ for each $i \in I$. In particular, it holds when the distribution satisfies

affiliation with strict inequality. Fix any $J \subset I$ such that $|J| \geq 2$ and $\phi \in \Phi_J$ and consider the instruction rule $d = d(\cdot \mid \phi, J)$.

Lemma 14 *Suppose that $I \geq 3$ and that Assumption 6 holds. For any $i, j \in I$ ($i \neq j$), there exists a continuation payoff function $z \in G_j$ such that $B_i z \geq \hat{v}_i$.*

Proof See the Appendix.

Combining Lemmas 10 and 14, and Theorem 3, we obtain the following theorem.

Theorem 15 *Suppose that $I \geq 3$ and that Assumption 6 holds. Then for any $u \in V$ and $\epsilon > 0$, there exists $\underline{\delta} < 1$ such that the following holds if $\delta > \underline{\delta}$: There exists an equilibrium collusion scheme τ which chooses instruction rules from D^* and yields the payoff vector $\Pi(\tau, \sigma, \delta)$ satisfying $\|\Pi(\tau, \sigma, \delta) - u\| < \epsilon$.*

Appendix

Proof of Lemma 4 For $\rho_i^{k,l}$ defined in the text, let x_i and x_j be defined recursively by

$$\begin{aligned}
 (9) \quad & x_i(s^0) = x_j(s^0) = 0, \\
 & x_i(s^k) = x_i(s^{k-1}) + \rho_j^{k-1,k} \left[v_i(s^{k-1}, s^k) - t^{k-1} \right. \\
 & \quad \left. - \rho_i^{k-1,k-1} \{ \theta v_j(s^{k-1}, s^{k-1}) - t^{k-1} \} \right], \\
 & x_j(s^k) = x_j(s^{k-1}) + \rho_i^{k-1,k-1} \left[v_j(s^{k-1}, s^{k-1}) - \frac{1}{\theta} t^{k-1} \right] \\
 & \quad (k = 1, \dots, K),
 \end{aligned}$$

where

$$t^k = x_i(s^k) + \theta x_j(s^k).$$

The functions in (9) are derived from the “one-step upward” incentive conditions for truth-telling, which ensure that the bidders do not report a signal one-step above its true value. The proof proceeds in three steps. The first step shows that x_i and x_j thus defined are monotone. The second step shows that they do satisfy the local (both upward and downward) incentive conditions. The third step concludes by demonstrating the global incentive conditions.

Step 1 For $\theta = \frac{\alpha_j}{\alpha_i} \in [\theta_{ij}, 1]$, both x_i and x_j are increasing: $x_i(s^{k+1}) \geq x_i(s^k)$ and $x_j(s^{k+1}) \geq x_j(s^k)$ for $k = 0, 1, \dots, K-1$.

Write $\Delta_i^k = x_i(s^k) - x_i(s^{k-1})$ and $\Delta_j^k = x_j(s^k) - x_j(s^{k-1})$ ($k = 1, \dots, K$). It suffices to show that $\Delta_i^k, \Delta_j^k \geq 0$ for each k . We first show by induction that

$$t^k \leq \theta v_j(s^k, s^k) \quad \text{and} \quad (1 - \rho_i^{k,k}) t^k \leq v_i(s^k, s^{k+1}) - \rho_i^{k,k} \theta v_j(s^k, s^k).$$

for $k = 0, 1, \dots, K$. These clearly hold when $k = 0$ since $v_i(s^0, s^1) \geq v_j(s^0, s^0)$ by Assumption 2. For $k \geq 1$, (9) implies that t^k satisfies

$$(10) \quad \begin{aligned} t^k &= \rho_j^{k-1,k} v_i(s^{k-1}, s^k) + \rho_i^{k-1,k-1} \{1 - \rho_j^{k-1,k}\} \theta v_j(s^{k-1}, s^{k-1}) \\ &\quad + \{1 - \rho_i^{k-1,k-1}\} \{1 - \rho_j^{k-1,k}\} t^{k-1}. \end{aligned}$$

Suppose that $t^{k-1} \leq \theta v_j(s^{k-1}, s^{k-1})$. Then (10) implies that

$$\begin{aligned} t^k &\leq \rho_j^{k-1,k} v_i(s^{k-1}, s^k) + \rho_i^{k-1,k-1} \{1 - \rho_j^{k-1,k}\} \theta v_j(s^{k-1}, s^{k-1}) \\ &\quad + \{1 - \rho_i^{k-1,k-1}\} \{1 - \rho_j^{k-1,k}\} \theta v_j(s^{k-1}, s^{k-1}) \\ &= \rho_j^{k-1,k} v_i(s^{k-1}, s^k) + \{1 - \rho_j^{k-1,k}\} \theta v_j(s^{k-1}, s^{k-1}). \end{aligned}$$

It can be verified that the RHS is $\leq \theta v_j(s^k, s^k)$ if and only if $\theta \geq \theta_{ij}$. Suppose next that

$$(1 - \rho_i^{k-1,k-1}) t^{k-1} \leq v_i(s^{k-1}, s^k) - \rho_i^{k-1,k-1} \theta v_j(s^{k-1}, s^{k-1}).$$

It then follows from (10) that

$$\begin{aligned} t^k &\leq \rho_j^{k-1,k} v_i(s^{k-1}, s^k) \\ &\quad + \rho_i^{k-1,k-1} \{1 - \rho_j^{k-1,k}\} \theta v_j(s^{k-1}, s^{k-1}) \\ &\quad + \{1 - \rho_j^{k-1,k}\} \{v_i(s^{k-1}, s^k) - \rho_i^{k-1,k-1} \theta v_j(s^{k-1}, s^{k-1})\} \\ &= v_i(s^{k-1}, s^k). \end{aligned}$$

Since $\theta \leq 1$, Assumption 2 implies that

$$(1 - \rho_i^{k,k}) v_i(s^{k-1}, s^k) \leq v_i(s^k, s^{k+1}) - \rho_i^{k,k} \theta v_j(s^k, s^k).$$

We hence obtain the desired conclusion. Note now that from (9), $\Delta_i^k = x_i(s^k) - x_i(s^{k-1})$ and $\Delta_j^k = x_j(s^k) - x_j(s^{k-1})$ can be expressed in terms of t^{k-1} as

$$(11) \quad \Delta_i^k = \rho_j^{k-1,k} \left[v_i(s^{k-1}, s^k) - t^{k-1} - \rho_i^{k-1,k-1} \{ \theta v_j(s^{k-1}, s^{k-1}) - t^{k-1} \} \right],$$

$$(12) \quad \Delta_j^k = \rho_i^{k-1,k-1} \left[v_j(s^{k-1}, s^{k-1}) - \frac{1}{\theta} t^{k-1} \right]$$

for $k = 1, \dots, K$. The above conclusions then imply that $\Delta_i^k, \Delta_j^k \geq 0$ for $k = 1, \dots, K$.

Step 2 y_i and y_j satisfy the local incentive constraints.

Let $\pi_i(s_i, \hat{s}_i \mid d, y)$ denote bidder i 's (*interim*) expected payoff under the instruction rule d and the continuation payoff function profile y when he observes signal s_i and reports \hat{s}_i , and bidder j reports his signal truthfully. The global incentive conditions can be expressed as:

$$(13) \quad \pi_i(s_i, s_i \mid d^{ij*}, y) \geq \pi_i(s_i, \hat{s}_i \mid d^{ij*}, y)$$

for any $s_i, \hat{s}_i \in S_i$, and

$$(14) \quad \pi_j(s_j, s_j \mid d^{ij*}, y) \geq \pi_j(s_j, \hat{s}_j \mid d^{ij*}, y)$$

for any $s_j, \hat{s}_j \in S_j$. This step shows that x_i and x_j defined in (9) satisfy the “local” or “one-step” incentive compatibility constraints.

We first note that x_i and x_j defined in (9) satisfy the inequalities (15)-(18) below.

$$(15) \quad \frac{1}{\rho_j^{k-1,k}} \Delta_i^k + \theta \Delta_j^k \geq v_i(s^{k-1}, s^k) - t^{k-1},$$

$$(16) \quad \frac{1}{\rho_j^{k,k}} \Delta_i^k + \theta \Delta_j^k \leq v_i(s^k, s^k) - t^{k-1},$$

$$(17) \quad \frac{1}{\rho_i^{k-1,k-1}} \Delta_j^k \geq v_j(s^{k-1}, s^{k-1}) - \frac{1}{\theta} t^{k-1},$$

$$(18) \quad \frac{1}{\rho_i^{k,k-1}} \Delta_j^k \leq v_j(s^k, s^{k-1}) - \frac{1}{\theta} t^{k-1}.$$

It can be readily verified that x_i and x_j satisfy (15) and (17) with equality. Since $\Delta_i^k, \Delta_j^k \geq 0$ by Step 1, (16) holds since $\rho_j^{k,k} \geq \rho_j^{k-1,k}$ and $v_i(s^k, s^k) \geq v_i(s^{k-1}, s^k)$, and (18) holds since $\rho_i^{k,k-1} \geq \rho_i^{k-1,k-1}$ and $v_j(s^k, s^{k-1}) \geq v_j(s^{k-1}, s^{k-1})$.

We now show that (15)-(18) above correspond to the local incentive constraints. For y given in (2), $\pi_i(s_i, \hat{s}_i \mid d^{ij*}, y)$ can be written as

$$\begin{aligned} \pi_i(s_i, \hat{s}_i \mid d^{ij*}, y) &= \sum_{s_j \leq \hat{s}_i} v_i(s_i, s_j) p_j(s_j \mid s_i) \\ &\quad - x_i(\hat{s}_i) \sum_{s_j \leq \hat{s}_i} p_j(s_j \mid s_i) + \sum_{s_j > \hat{s}_i} \theta x_j(s_j) p_j(s_j \mid s_i). \end{aligned}$$

Hence, (13) for $s_i = s^{k-1}$ and $\hat{s}_i = s^k$ (*i.e.*, the “one-step upward” incentive compatibility condition) is equivalent to ($k = 1, \dots, K$):

$$\begin{aligned} (19) \quad &\pi_i(s^{k-1}, s^{k-1} \mid d^{ij*}, y) - \pi_i(s^{k-1}, s^k \mid d^{ij*}, y) \\ &= -v_i(s^{k-1}, s^k) p_j(s^k \mid s_i = s^{k-1}) + x_i(s^{k-1}) p_j(s^k \mid s_i = s^{k-1}) \\ &\quad + \theta x_j(s^k) p_j(s^k \mid s_i = s^{k-1}) + \Delta_i^k \sum_{s_j \leq s^k} p_j(s_j \mid s_i = s^{k-1}) \geq 0, \end{aligned}$$

and (13) for $s_i = s^k$ and $\hat{s}_i = s^{k-1}$ (*i.e.*, the “one-step downward” incentive compatibility condition) is equivalent to:

$$\begin{aligned}
(20) \quad & \pi_i(s^k, s^k \mid d^{ij*}, y) - \pi_i(s^k, s^{k-1} \mid d^{ij*}, y) \\
& = v_i(s^k, s^k) p_j(s^k \mid s_i = s^k) - x_i(s^{k-1}) p_j(s^k \mid s_i = s^k) \\
& \quad - \theta x_j(s^k) p_j(s^k \mid s_i = s^k) - \Delta_i^k \sum_{s_j \leq s^k} p_j(s_j \mid s_i = s^k) \geq 0.
\end{aligned}$$

Rearranging, we see that (19) and (20) are equivalent to (15) and (16), respectively. For bidder j , $\pi_j(s_j, \hat{s}_j \mid d^{ij*}, y)$ can be written as

$$\begin{aligned}
\pi_j(s_j, \hat{s}_j \mid d^{ij*}, y) & = \sum_{s_i < \hat{s}_j} v_j(s_j, s_i) p_i(s_i \mid s_j) \\
& \quad - x_j(\hat{s}_j) \sum_{s_i < \hat{s}_j} p_i(s_i \mid s_j) + \sum_{s_i \geq \hat{s}_j} \theta^{-1} x_i(s_i) p_i(s_i \mid s_j).
\end{aligned}$$

Hence, (14) for $s_j = s^{k-1}$ and $\hat{s}_j = s^k$ is equivalent to ($k = 1, \dots, K$):

$$\begin{aligned}
(21) \quad & \pi_j(s^{k-1}, s^{k-1} \mid d^{ij*}, y) - \pi_j(s^{k-1}, s^k \mid d^{ij*}, y) \\
& = -v_j(s^{k-1}, s^{k-1}) p_i(s^{k-1} \mid s_j = s^{k-1}) + x_j(s^{k-1}) p_i(s^{k-1} \mid s_j = s^{k-1}) \\
& \quad + \theta^{-1} x_i(s^{k-1}) p_i(s^{k-1} \mid s_j = s^{k-1}) + \Delta_j^k \sum_{s_i \leq s^{k-1}} p_i(s_i \mid s_j = s^{k-1}) \geq 0,
\end{aligned}$$

and (14) for $s_i = s^k$ and $\hat{s}_i = s^{k-1}$ is equivalent to:

$$\begin{aligned}
(22) \quad & \pi_j(s^k, s^k \mid d^{ij*}, y) - \pi_j(s^k, s^{k-1} \mid d^{ij*}, y) \\
& = v_j(s^k, s^{k-1}) p_i(s^{k-1} \mid s_j = s^k) - x_j(s^{k-1}) p_i(s^{k-1} \mid s_j = s^k) \\
& \quad - \theta^{-1} x_i(s^{k-1}) p_i(s^{k-1} \mid s_j = s^k) - \Delta_j^k \sum_{s_i \leq s^{k-1}} p_i(s_i \mid s_j = s^k) \geq 0.
\end{aligned}$$

Rearrangement shows that (21) and (22) are equivalent to (17) and (18), respectively.

Step 3 y_i and y_j satisfy the global incentive constraints (13) and (14).

As an induction hypothesis, suppose that (13) holds for $s_i = s^{k-1}$ and $\hat{s}_i = s^{k+l-1}$ ($l = 1, \dots, K - k$). When $s_i = s^{k-1}$ and $\hat{s}_i = s^{k+l}$, we have

$$\begin{aligned}
& \pi_i(s^{k-1}, s^{k-1} \mid d^{ij*}, y) - \pi_i(s^{k-1}, s^{k+l} \mid d^{ij*}, y) \\
& = \pi_i(s^{k-1}, s^{k-1} \mid d^{ij*}, y) - \pi_i(s^{k-1}, s^{k+l-1} \mid d^{ij*}, y) \\
& \quad + \{-v_i(s^{k-1}, s^{k+l}) + \theta x_j(s^{k+l}) + x_i(s^{k+l-1})\} p_j(s^{k+l} \mid s_i = s^{k-1}) \\
& \quad + \Delta_i^{k+l} \sum_{s_j \leq s^{k+l}} p_j(s_j \mid s_i = s^{k-1}).
\end{aligned}$$

By the induction hypothesis, the RHS is ≥ 0 if

$$\begin{aligned} & -v_i(s^{k-1}, s^{k+l}) + \theta x_j(s^{k+l}) + x_i(s^{k+l-1}) + \Delta_i^{k+l} \frac{\sum_{s_j \leq s^{k+l}} p_j(s_j \mid s_i = s^{k-1})}{p_j(s^{k+l} \mid s_i = s^{k-1})} \\ & = -v_i(s^{k-1}, s^{k+l}) + \theta \Delta_j^{k+l} + t^{k+l-1} + \frac{1}{\rho_j^{k-1, k+l}} \Delta_i^{k+l} \geq 0, \end{aligned}$$

or equivalently,

$$\frac{1}{\rho_j^{k-1, k+l}} \Delta_i^{k+l} + \theta \Delta_j^{k+l} \geq v_i(s^{k-1}, s^{k+l}) - t^{k+l-1}.$$

Since $\rho_j^{k-1, k+l} \leq \rho_j^{k+l-1, k+l}$ by (1) and $v_i(s^{k-1}, s^{k+l}) \leq v_i(s^{k+l-1}, s^{k+l})$, (15) for $k+l$ implies the above inequality. Therefore, (13) holds for $s_i = s^{k-1}$ and $\hat{s}_i = s^{k+l}$ ($k = 1, \dots, K, l = 0, \dots, K-k-1$). We can show by an analogous argument that (14) holds for $s_i = s^k$ and $\hat{s}_i = s^{k-l}$ ($k = 1, \dots, K, l = 1, \dots, k$). The argument for j is similar and is omitted. //

Proof of Theorem 5 Take any $u \in V^0 \cap V$, and let $\alpha \neq 0$ be given. If $\alpha_1, \alpha_2 \leq 0$, then we have $\alpha \cdot u \leq \alpha \cdot g^0$. If $\alpha_i > 0$ and $\alpha_j \leq 0$, then we have $\alpha \cdot u \leq \alpha \cdot g^i$ ($i = 1, 2, j \neq i$). In both cases, we are done since $d^0, d^i \in A(\alpha)$ by Lemma 2. Suppose then that $\alpha_i \geq \alpha_j > 0$. If $\alpha \cdot u \leq \alpha \cdot g^i$ for $i = 1$ or 2 , then the proof is complete again by Lemma 2. Otherwise, since $u \in V$, we must have $\alpha \cdot u < \alpha \cdot g^{\phi*}$ for either $\phi = ij$ or $\phi = ji$. Since $g_i^{ij*} + g_j^{ij*} = g_i^{ji*} + g_j^{ji*}$ and $g_j^{ij*} \leq g_j^{ji*}$, we also have

$$\alpha \cdot (g^{ij*} - g^{ji*}) = (\alpha_i - \alpha_j)(g_j^{ji*} - g_j^{ij*}) \geq 0.$$

Therefore, $\alpha \cdot u \leq \alpha \cdot g^{ij*}$ must hold. If $d^{ij*} \in A(\alpha)$, then the proof is complete. Otherwise, $\alpha_j/\alpha_i < \theta_{ij}$ must hold by Lemma 4 and hence $u \in V^0$ implies that

$$\alpha \cdot u = \alpha_i \left(u_i + \frac{\alpha_j}{\alpha_i} u_j \right) \leq \alpha_i (u_i + \theta_{ij} u_j) \leq \alpha_i g_i^i = \alpha \cdot g^i.$$

We hence obtain the desired conclusion. //

Proof of Theorem 7 The desired conclusion follows if $g^{ij*} = (g_i^{ij*}, g_j^{ij*}) \in V^0$ for $i \neq j$, where g^{ij*} is the payoff vector corresponding to the efficient instruction rule d^{ij*} . For this, it suffices to show that $g_i^{ij*} + \theta_{ij} g_j^{ij*} \leq g_i^i$ ($i = 1, 2, j \neq i$): If this holds, then we also have $g_j^{ij*} + \theta_{ji} g_i^{ij*} \leq g_j^j$ since $g_j^{ij*} + \theta_{ji} g_i^{ij*} \leq g_j^{ji*} + \theta_{ji} g_i^{ji*}$ by

$g_i^{ij*} + g_j^{ij*} = g_i^{ji*} + g_j^{ji*}$ and $g_i^{ij*} \geq g_i^{ji*}$. We have

$$\begin{aligned}\theta_{ij}^k &= \frac{\rho_j^{k-1,k} v_i(s^{k-1}, s^k)}{v_j(s^k, s^k) - v_i(s^{k-1}, s^{k-1}) + \rho_j^{k-1,k} v_i(s^{k-1}, s^{k-1})} \\ &= \frac{\rho_j^{k-1,k} (k - c)}{1 + \rho_j^{k-1,k} (k - 1)}\end{aligned}$$

for any $k = 1, \dots, K$, and

$$\begin{aligned}\frac{g_i^i - g_i^{ij*}}{g_j^{ij*}} &= \frac{\sum_{s_j > s_i} p(s) v_i(s)}{\sum_{s_j > s_i} p(s) v_j(s)} \\ &= \frac{c \sum_{s_j > s_i} p(s) s_i + (1 - c) \sum_{s_j > s_i} p(s) s_j}{(1 - c) \sum_{s_j > s_i} p(s) s_i + c \sum_{s_j > s_i} p(s) s_j} \\ &= \frac{1 - c + c\nu_{ij}}{(1 - c)\nu_{ij} + c}.\end{aligned}$$

It can be seen from these that the inequality in the theorem is equivalent to $\theta_{ij} \leq \frac{g_i^i - g_i^{ij*}}{g_j^{ij*}}$, which in turn implies $g_i^{ij*} + \theta_{ij} g_j^{ij*} \leq g_i^i$. //

Proof of Lemma 9 By the theorem of the alternatives (Rockafellar (1970, Theorem 22.1)), (5) has a solution if and only if for any $\gamma \in \mathbb{R}_+^s$,

$$\gamma B = 0 \Rightarrow \sum_{i=1}^n \gamma_i \cdot \hat{v}_i \leq 0.$$

Take any $\gamma \in \mathbb{R}_+^s$ and write $\gamma = (\gamma_1, \dots, \gamma_n)$, where γ_i is m_i -dimensional. Simple algebra shows that $\gamma B = 0$ is equivalent to

$$\gamma_i B_i - \beta_i \gamma_n B_n = 0 \quad \text{for } i = 1, \dots, n - 1,$$

and that $\gamma \cdot \hat{v} \leq 0$ is equivalent to

$$\sum_{i=1}^n \gamma_i \cdot \hat{v}_i \leq 0.$$

Hence, the desired conclusion follows. //

Proof of Lemma 11 Fix any $\hat{s}_j \in S_j$. For any $\lambda_j \in \Lambda_j$,

$$\begin{aligned} b_j(\lambda_j) \cdot z &= \sum_{\hat{s} \in S} \{p(\hat{s} \mid \lambda^*) - p(\hat{s} \mid \lambda_j, \lambda_{-j}^*)\} z(\hat{s}) \\ &= \sum_{\hat{s}_{-j} \in S_{-j}} z(\hat{s}_j, \hat{s}_{-j}) \sum_{\hat{s}_j \in S_j} \{p(\hat{s}_j, \hat{s}_{-j} \mid \lambda^*) - p(\hat{s}_j, \hat{s}_{-j} \mid \lambda_j, \lambda_{-j}^*)\} \\ &= \sum_{\hat{s}_{-j} \in S_{-j}} z(\hat{s}_j, \hat{s}_{-j}) \{p(\hat{s}_{-j} \mid \lambda_{-j}^*) - p(\hat{s}_{-j} \mid \lambda_{-j}^*)\} = 0, \end{aligned}$$

where the second equality follows since $z(\hat{s}'_j, \hat{s}_{-j}) = z(\hat{s}_j, \hat{s}_{-j})$ for any $\hat{s}'_j \in S_j$. //

Proof of Lemma 12 As in the proof of Lemma 4, let $\pi_i(s_i, \hat{s}_i \mid d, y)$ denote bidder i 's (*interim*) expected payoff under the instruction rule d and the continuation payoff function y when he has signal s_i and reports \hat{s}_i , and other bidders report their signals truthfully. Write $d(\cdot \mid \phi, J) = d$, $w_i^j(\cdot \mid \phi, J) = w_i^j(\cdot)$ and $\bar{Z}_i(\cdot \mid \phi, J) = \bar{Z}_i(\cdot)$ for simplicity. The conclusion follows if

$$(23) \quad \pi_i(s_i, s_i \mid d, y) \geq \pi_i(s_i, \hat{s}_i \mid d, y)$$

for any $s_i \neq \hat{s}_i$. Note first that x_i defined in (8) satisfies (24) and (25) below for $k = 1, \dots, K$:

$$(24) \quad x_i(s^k) \geq \rho_j^k w_i^j(s^{k-1}, s^k) + (1 - \rho_j^k) x_i(s^{k-1}),$$

$$(25) \quad x_i(s^k) \leq \rho_j^k w_i^j(s^k, s^k) + (1 - \rho_j^k) x_i(s^{k-1}),$$

x_i satisfies (24) with equality, and hence also satisfies (25) since $w_i^j(s^{k-1}, s^k) \leq w_i^j(s^k, s^k)$ by (6).

We next show that x_i satisfies the local incentive conditions. In other words, (23) holds for $s_i = s^k$ when $\hat{s}_i = s^{k-1}$ or $\hat{s}_i = s^{k+1}$. Note that when $y_i = z$ for z given in (7), $\pi_i(s_i, \hat{s}_i \mid d, y)$ can be written as

$$\pi_i(s_i, \hat{s}_i \mid d, y) = \sum_{s_{-i} \in \bar{Z}_i(\hat{s}_i)} \{v_i(s_i, s_{-i}) - R\} p_{-i}(s_{-i}) - x_i(\hat{s}_i) \sum_{s_j \leq \hat{s}_i} p_j(s_j).$$

Hence, (23) for $s_i = s^{k-1}$ and $\hat{s}_i = s^k$ is equivalent to

$$\begin{aligned} &\pi_i(s^{k-1}, s^{k-1} \mid d, y) - \pi_i(s^{k-1}, s^k \mid d, y) \\ (26) \quad &= -w_i^j(s^{k-1}, s^k) p_j(s^k) + x_i(s^k) \sum_{s_j \leq s^k} p_j(s_j) - x_i(s^{k-1}) \sum_{s_j \leq s^{k-1}} p_j(s_j) \geq 0. \end{aligned}$$

Since $\sum_{s_j \leq s^{k-1}} p_j(s_j) = (1 - \rho_j^k) \sum_{s_j \leq s^k} p_j(s_j)$, (26) simplifies to (24). Likewise, (23) for $s_i = s^k$ and $\hat{s}_i = s^{k-1}$ is equivalent to (25).

We now show that x_i satisfies the global incentive conditions: (23) holds for any s_i and \hat{s}_i .

As an induction hypothesis, suppose that (23) holds for $s_i = s^{k-1}$ and $\hat{s}_i = s^{k+l-1}$ ($l = 1, \dots, K - k$). When $s_i = s^{k-1}$ and $\hat{s}_i = s^{k+l}$, we have

$$\begin{aligned} & \pi_i(s^{k-1}, s^{k-1} \mid d, y) - \pi_i(s^{k-1}, s^{k+l} \mid d, y) \\ &= \pi_i(s^{k-1}, s^{k-1} \mid d, y) - \pi_i(s^{k-1}, s^{k+l-1} \mid d, y) \\ &= w_i^j(s^{k-1}, s^{k+l}) p_j(s^{k+l}) + x_i(s^{k+l}) \sum_{s_j \leq s^{k+l}} p_j(s_j) - x_i(s^{k+l-1}) \sum_{s_j \leq s^{k+l-1}} p_j(s_j), \end{aligned}$$

By the induction hypothesis, the RHS is ≥ 0 if

$$\begin{aligned} & -w_i^j(s^{k-1}, s^{k+l}) p_j(s^{k+l}) + x_i(s^{k+l}) \sum_{s_j \leq s^{k+l}} p_j(s_j) - x_i(s^{k+l-1}) \sum_{s_j \leq s^{k+l-1}} p_j(s_j) \\ &= \left\{ x_i(s^{k+l}) - \rho_j^{k+l} w_i^j(s^{k-1}, s^{k+l}) - (1 - \rho_j^{k+l}) x_i(s^{k+l-1}) \right\} \sum_{s_j \leq s^{k+l}} p_j(s_j) \geq 0. \end{aligned}$$

Since $w_i^j(s^{k-1}, s^{k+l}) \leq w_i^j(s^{k+l-1}, s^{k+l})$ by (6), (24) for $k+l$ implies the above inequality. Therefore, (23) holds for $s = s^{k-1}$ and $\hat{s} = s^{k+l}$ ($k = 1, \dots, K$, $l = 0, \dots, K - k - 1$). An analogous argument shows that (23) holds for $s_i = s^k$ and $\hat{s}_i = s^{k-l}$ ($k = 1, \dots, K$, $l = 1, \dots, k$). This completes the proof. //

Proof of Lemma 14 Write $\|\cdot\|$ for the square norm, and let $z \in G_j$ be defined by $z(s_i, s_j, s_{-i-j}) = \bar{z}(s_i, s_{-i-j})$ for any $(s_i, s_j, s_{-i-j}) \in S$, where

$$\bar{z}(s_i, \cdot) = \frac{p_{-i-j}(\cdot \mid s_i)}{\|p_{-i-j}(\cdot \mid s_i)\|}.$$

Then for any $s_i, s'_i \in S_i$,

$$\begin{aligned} \bar{z}(s_i, \cdot) \cdot p_{-i-j}(\cdot \mid s_i) &= \frac{p_{-i-j}(\cdot \mid s_i) \cdot p_{-i-j}(\cdot \mid s_i)}{\|p_{-i-j}(\cdot \mid s_i)\|} = \|p_{-i-j}(\cdot \mid s_i)\|, \quad \text{and} \\ \bar{z}(s'_i, \cdot) \cdot p_{-i-j}(\cdot \mid s_i) &= \frac{p_{-i-j}(\cdot \mid s'_i) \cdot p_{-i-j}(\cdot \mid s_i)}{\|p_{-i-j}(\cdot \mid s'_i)\|}. \end{aligned}$$

If $s_i \neq s'_i$, then $p_{-i-j}(\cdot \mid s_i) \neq p_{-i-j}(\cdot \mid s'_i)$ by Assumption 6 and hence by the Cauchy-Schwartz inequality,

$$p_{-i-j}(\cdot \mid s_i) \cdot p_{-i-j}(\cdot \mid s'_i) < \|p_{-i-j}(\cdot \mid s_i)\| \|p_{-i-j}(\cdot \mid s'_i)\|.$$

It follows that

$$\bar{z}(s_i, \cdot) \cdot p_{-i-j}(\cdot \mid s_i) > \bar{z}(s'_i, \cdot) \cdot p_{-i-j}(\cdot \mid s_i).$$

This further implies that for any $\lambda_i \in \Lambda_i^0$, we have

$$b_i(\lambda_i) \cdot z = \sum_{s_i \in S_i} p(s_i) \{ \bar{z}(s_i, \cdot) \cdot p_{-i-j}(\cdot | s_i) - \bar{z}(\lambda_i(s_i), \cdot) \cdot p_{-i-j}(\cdot | s_i) \} > 0,$$

or equivalently, $B_i z > 0$. Therefore, we have $B_i z \geq \hat{v}_i$ if we redefine z to be kz for $k > 0$ sufficiently large. //

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