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ON "EASY" ZETA FUNCTIONS

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The aim of this article is to explain that among the various types of zeta functions there are far more that have easy expressions than is commonly believed. Specifically, we would like to explain that in an unexpectedly large number of cases the zeta functions of prehomogeneous vector spaces and the zeta functions of automorphic forms can be calculated in terms of known functions if we bring to bear all our arithmetical knowledge, even though this "easiness" is often not apparent from the definitions. In the first half of the article more emphasis has been placed on the development of the ideas than on mathematical correctness.

1. Two kinds of zeta functions

This may sound a bit like a joke, but we have come to feel that there are two kinds of zeta functions: "easy zeta functions" and "difficult zeta functions." Even though we cannot define these precisely, we want to gradually explain this feeling.

The series $\sum_{n=1}^{\infty} a_n n^{-s}$ associated to a sequence $\{a_n\}$ and a complex number s is called a Dirichlet series. Depending on how the a_n are taken, this series may have very good properties. For instance, if we set $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, then we have the following:

(1) $\zeta(s)$ converges absolutely for $\Re(s) > 1$ and can be continued analytically to the whole s-plane as a meromorphic function.

(2) $\zeta(s)$ has a functional equation. Namely, if we set $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, then $\xi(1-s) = \xi(s)$.

(3) $\zeta(s)$ has an Euler product. Namely, $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ (p runs over prime numbers).

This $\zeta(s)$ is called the Riemann zeta function. Taking $\zeta(s)$ as a model, one can think up many other Dirichlet series which satisfy properties (1)–(3) or some part of them. These functions are labelled with a suitable adjective and are called zeta functions or L-functions. Here $\{a_n\}$ will often have a good definition with some natural arithmetical meaning, but there is no reason that the individual a_n themselves should be described by any especially concrete formula. Let us agree to speak somewhat vaguely of an "easy zeta function" whenever there is an easy explicit formula for the a_n. From this point of view, the Riemann zeta function is the model of an easy zeta function, while at the opposite extreme the L-function $L(s,\Delta) = \sum_{n=1}^{\infty} \tau(n) n^{-s}$ associated to Ramanujan's Delta function $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$ can be taken as the model of a difficult zeta function.

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What other kinds of easy zeta functions might there be, then? For example, for any ring A let us denote by $\mathrm{Sym}_n(A)$ the vector space of symmetric matrices of size m with coefficients in A. In particular, we write $L_m = \mathrm{Sym}_m(\mathbb{Z})$. Let $S \in L_m$, and for the sake of simplicity let us suppose that $S > 0$, i.e., S is positive definite. For $T \in L_n$ we set $A(S,T) = \#\{X \in M_{mn}(\mathbb{Z}) \mid {}^tXSX = T\}$. Then, for instance, if we set $\zeta(s,S) = \sum_{l=1}^{\infty} A(S,l)\, l^{-s}$, will $\zeta(s,S)$ be an easy zeta function? In other words, will there be a simple formula for the $A(S,l)$? Let us take as an example the Epstein zeta function $\zeta_m(s) = \sum_{x \in \mathbb{Z}^m - \{0\}} N(x)^{-s}$, where we have written $N(x) = {}^txx$. It has an analytic continuation to the whole plane and a functional equation. We cite a few concrete instances from the *Iwanami Mathematical Dictionary.*[1] Setting $L(s) = \sum_{n=1}^{\infty}(-1)^{n-1}(2n-1)^{-s}$ for simplicity, we have, e.g.,

$$\begin{aligned}
\zeta_6(s) &= -4\,\zeta(s)\,L(s-2) + 16\,\zeta(s-2)\,L(s)\,, \\
\zeta_8(s) &= 16\,(1 - 2^{1-s} + 2^{4-2s})\,\zeta(s)\,\zeta(s-3)\,, \\
\zeta_{12}(s) &= 8\,(1 - 2^{6-2s})\,\zeta(s)\,\zeta(s-5) + 16\,L(s,\phi)\,.
\end{aligned}$$

Here $L(s,\phi)$ is the L-function coming from a weight 6 cusp form (namely, $\phi = \sqrt{\Delta}$); so it is a difficult zeta function. To explain the complicated nature of $\zeta_{12}(s)$ in comparison with $\zeta_6(s)$ and $\zeta_8(s)$, we review Siegel's formula in a simple case. Let us say that S_1, $S_2 \in L_m$ belong to the same *class* if ${}^tgS_1g = S_2$ for a suitable $g \in GL_m(\mathbb{Z})$. The numbers of solutions $A(S,T)$ and other quantities described above depend only on the class. On the other hand, we say that S_1 and S_2 belong to the same *genus* if for an arbitrary prime number p there is a $g_p \in GL_m(\mathbb{Z}_p)$ ($\mathbb{Z}_p$ the ring of p-adic integers) with ${}^tg_pS_1g_p = S_2$ and if also S_1 and S_2 are equivalent after a change of basis over the field of real numbers. The number of classes within a single genus (that is, the subset of elements of L_n that are locally equivalent to a given element) is called the *class number* of the genus. This class number is in general larger than 1. If it is bigger than one, then it is not to be expected that global quantities like the number of solutions considered above can be found just by evaluating the number of local solutions. In fact, the class number for the rank 12 identity matrix 1_{12} is not 1, and this is the reason for the complexity of $\zeta_{12}(s)$. On the other hand, it might be that if we take the average over all the classes in a single genus, then this can be calculated in terms of local quantities. The first formulation of Siegel's formula states precisely this. Let $\mathcal{L} \subset L_m$ be a genus, which for simplicity we take to be that of a positive definite matrix. Let $S_1, \dots, S_h$ be the representatives of the classes in $\mathcal{L}$. Fix some $T \in L_n$. Also set $E(S) = A(S,S)$ and write $M(\mathcal{L}) = \sum_{i=1}^{h} E(S_i)^{-1}$. $M(\mathcal{L})$ is called the *mass* of $\mathcal{L}$.

Theorem (Siegel). *With the above notation, the following holds:*

$$\begin{aligned}
&\left(\sum_{i=1}^{h} A(S_i,T)\,E(S_i)^{-1}\right) M(\mathcal{L})^{-1} \\
&\qquad = c_{m,n}\,\det(S)^{-n/2}\,\det(T)^{(m-n-1)/2}\prod_p \alpha_p(\mathcal{L},T)\,.
\end{aligned}$$

[1] Translated into English as *Encyclopedic Dictionary of Mathematics*, MIT Press, Cambridge, MA, 1977, 1987.

Here $c_{m,n}$ is a constant depending on m and n whose detailed description we omit. $\alpha_p(\mathcal{L},T)$ is a quantity called the *local density*: If for $S \in \mathrm{Sym}_m(\mathbb{Z}_p)$ we set $A_{p^\nu}(S,T) = \#\{X \in M_{mn}(\mathbb{Z}/p^\nu\mathbb{Z}) \mid {}^tXSX \equiv T \pmod{p^\nu}\}$, and if $\epsilon = 1$ for $m > n$ and $\epsilon = 1/2$ for $m = n$, then we define

$$\alpha_p(S,T) = \epsilon \lim_{\nu\to\infty} p^{\nu(m(m+1)/2-mn)} A_{p^\nu}(S,T).$$

The quantity in the limit on the right-hand side stabilizes for ν sufficiently large; so the definition is consistent. Moreover, since it also does not depend on how we choose $S \in \mathcal{L}$, we wrote it above as $\alpha_p(\mathcal{L},T)$.

Now Siegel's formula has another aspect. In order to keep the explanation simple we consider the case $n = 1$. Write $\mathfrak{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ for the complex upper half plane. The *theta function* on $\mathfrak{H}$ associated to $S \in L_n$,

$$\vartheta(S,z) = \sum_{l=0}^{\infty} A(S,l)\exp(2\pi i l z),$$

is a modular form on $\mathfrak{H}$ with respect to a suitable discrete group.

Theorem (Siegel). *With the notation taken as above,*

$$M(\mathcal{L})^{-1}\sum_{i=1}^{h}\vartheta(S_i,z)\,E(S_i)^{-1}$$

is equal to a modular form on $\mathfrak{H}$ called an Eisenstein series.

That is, the "average" of the theta functions as we run through the classes is a relatively standard modular form. The important point is that not only are the "averages" over the classes of the $A(S_i,l)$ for the individual l expressible in terms of local invariants, but the average over the whole set of zeta functions $\zeta(s,S_i)$ is also a standard object. (In case the dimension n of the matrix T is bigger than 1, one must consider theta functions and Eisenstein series on the Siegel upper half-plane, but we omit the explanation of this.) The local densities are fairly well understood. Actually, no completely general formula for the local densities is known, but still it is not too unreasonable to call the above "average" of the zeta functions over the classes in a genus an "easy" zeta function, and it may also be permissible to call Eisenstein series a "source" of easy zeta functions. (We will touch upon this point again later on.)

Now, the identity matrices 1_6 and 1_8 both have class number 1. From this point of view one would not expect $\zeta_6(s)$ and $\zeta_8(s)$ to differ, but the former does not have an Euler product, while the latter one does. Yet, seen from the point of view of Siegel's theorem, we have no grounds to consider the latter more natural. So we cannot avoid calling $\zeta_6(s)$ a completely standard zeta function. Based on these thoughts, we would like to make the following assertion:

Observation 1. *It is a misconception that good zeta functions must have Euler products.*

As prototypical zeta functions which do not have an Euler product but which do have a functional equation we can take the zeta functions of prehomogeneous vector spaces. These will be discussed in the next section.

2. Zeta functions of prehomogeneous vector spaces

We will illustrate our motivation for studying these objects through an example. For any ring A and any natural number n we set $V(A) = \mathrm{Sym}_n(A)$. Then $GL_n(\mathbb{C})$ acts on $V(\mathbb{C})$ by $x \to {}^t gxg$ ($x \in V(\mathbb{C})$, $g \in GL_n(\mathbb{C})$). In fact (GL_n, V) is an example of a prehomogeneous vector space defined over $\mathbb{Q}$. Now let L_n^* be the lattice

$$L_n^* = \{x = (x_{ij}) \in V \mid x_{ii} \in \mathbb{Z}, \quad 2x_{ij} \in \mathbb{Z} \quad (1 \leq i, j \leq n)\}$$

in V. The elements of L_n^* are the so-called semi-integral symmetric matrices. For an element x of L_n^* with $x > 0$ (positive definite) we set $\Gamma_x = \{\gamma \in SL_n(\mathbb{Z}) \mid {}^t\gamma x \gamma = x\}$ and write $\mu(x) = \#(\Gamma_x)^{-1}$. We define the zeta function of L_n^* by

$$\zeta(s, L_n^*) = \sum_{x \in L_n^*/\sim,\, x>0} \frac{\mu(x)}{\det(x)^s},$$

where $L_n^*/\sim$ denotes a set of representatives of the orbits of the action of $SL_n(\mathbb{Z})$ on L_n^*. (Note that this is totally different from the zeta functions $\zeta(s, S)$ associated to the various symmetric matrices themselves which we considered in the previous section.) There are already works by Siegel concerning the convergence of these series, but it was T. Shintani who later studied them deeply and proved their analytic continuation and functional equation (cf. [16]). One of the reasons that great attention has been paid to these functions is their special values; specifically, their values at non-negative integers s appear as one ingredient in the dimension formula for Siegel modular forms. This is due to Y. Morita [12] for $n = 2$ and to Shintani [16] for general n. It has been possible to actually calculate this special value if $n = 2$. This is because of the special circumstance that in that case $\det(x)$ can be considered as a ternary quadratic form and can be understood from Siegel's results on zeta functions of quadratic forms. For general n everybody assumed that, because of their role in the dimension formula (which is after all a formula for an integral quantity), the special values should be rational numbers, but for $n \geq 3$ nothing was known. The sole exception was the result $\zeta(0, L_3^*) = 1/3456$ obtained by Hashimoto and Tsushima, but the proof of this was terribly complicated. Namely, they calculated the dimension formula for modular forms in two ways, using both the Selberg trace formula and the Lefschetz fixed point theorem, and then by comparing the non-understood terms which were left over were able to determine the value in question. On the other hand, $\zeta(s, L_n^*)$ itself was given by a complicated expression even for $n = 2$; so it was vaguely thought to be a difficult object. Thus one could imagine that perhaps $\zeta(s, L_n^*)$ itself was a complicated mystery and that just its special values were nice because of some accidental circumstances, so that one might try to calculate them by some kind of contour integral techniques, with the amount of calculation needed probably being too great to carry out in practice. But the truth is:

Observation 2. *It is a misconception to think that $\zeta(s, L_n^*)$ is complicated. In fact it can be written in terms of known zeta functions and is "easy."*

It is in fact the principal aim of this article to amplify and explain this observation as fully as possible.

Before stating Observation 2 as a precise theorem, we will give a simple review of prehomogeneous vector spaces (under suitable conditions). This should make it easier for the reader to understand the overall picture. Let V be a finite-dimensional vector space over $\mathbb{Q}$ and G a connected reductive algebraic group over $\mathbb{Q}$ acting on V. In general, for an algebraic variety W and a ring A we write $W(A)$ for the *A-valued points* (points with coordinates in A). If there is a proper algebraic subset S of $V(\mathbb{C})$ such that $G(\mathbb{C})$ acts transitively on $V(\mathbb{C}) - S(\mathbb{C})$, then (G, V) is called a prehomogeneous vector space. For the sake of simplicity we shall assume in what follows that $S(\mathbb{C})$ is an irreducible hypersurface $S(\mathbb{C}) = \{x \in V(\mathbb{C}) \mid P(x) = 0\}$. Here $P(x)$ is taken to be a generator of the relative invariants of G, i.e., of the polynomials satisfying $f(gx) = \chi(g)f(x)$ ($\chi(g) \in \mathbb{C}$, $g \in G(\mathbb{C})$). Next, let $\Gamma \subset G(\mathbb{R})$ be a discrete subgroup of finite covolume and L a Γ-invariant lattice in $V(\mathbb{Q})$. We would like to assign a weight to each $x \in L$, and for this purpose we fix an invariant measure dg on G and a G-invariant measure ω on V. For $x \in L$, $x \notin S$, we write $G_x = \{g \in G(\mathbb{R}) \mid gx = x\}$ and $\Gamma_x = \Gamma \cap G_x$. Because of the prehomogeneous property, an invariant measure on G_x can be defined as the "ratio" of dg and ω. The effect of this definition is that when looking at all x simultaneously we can eliminate the ambiguity of the invariant measure up to a constant factor. Then, if the volume $\mu(x) = \mathrm{vol}(G_x/\Gamma_x)$ with respect to the measure just defined is finite for every $x \in L$, we can define zeta functions in the following way. We first decompose $V(\mathbb{R}) - S(\mathbb{R})$ into $G(\mathbb{R})$-orbits, say $V(\mathbb{R}) - S(\mathbb{R}) = \coprod_i V^i$ (disjoint). Then for each V^i we define

$$\zeta_i(s, L) = \sum_{x \in L \cap V^i/\Gamma} \frac{\mu(x)}{|P(x)|^s}\,.$$

Thus *a priori* there is one such partial zeta function corresponding to each $G(\mathbb{R})$-orbit, but in fact it can happen that the zeta functions for different i's coincide. The functional equation of the zeta function is then actually a relationship between the vectors $(\zeta_i(s, L))$ and $(\zeta_i(s, L^*))$ formed from these, and the individual functions do not have to have functional equations when taken separately. Here L^* is the dual lattice of L.

In the previously discussed case $V = \mathrm{Sym}_n$ and $G = GL_n$, we take V^i $(0 \le i \le n)$ to be the set of real symmetric matrices of signature $(i, n-i)$, corresponding to the decomposition into real orbits. Then there are *a priori* $n+1$ zeta functions (of course $\zeta(s, L_n^*) = \zeta_n(s, L_n^*)$), but from the definitions one immediately sees that $\zeta_i(s, L) = \zeta_{n-i}(s, L)$. From now on, we consider the case $L_n = \mathrm{Sym}_n(\mathbb{Z})$, $L = L_n$ or L_n^*, and $\Gamma = SL_n(\mathbb{Z})$. In fact, for some reason the following fact has escaped attention, namely, the arithmetical theory of quadratic forms shows that $\zeta_i(s, L)$ depends only on $\delta = (-1)^{n-i}$ and $\epsilon = (-1)^{(n-i)(n-i+1)/2}$. Therefore for $L = L_n$ or L_n^* let us write $\zeta_i(s, L) = \zeta(s, L, \delta, \epsilon)$. We also suppose from now on that $n \ge 3$. (This is because the case $n = 2$ is already known and also presents various peculiarities. We will touch on this again later.) The concrete formulas for the zeta functions are completely different for odd and even n. In order to state the theorem in a simple form, let us prepare some notation. For an integer $i \ge 0$, the *Bernoulli number* B_i is defined by $te^t/(e^t - 1) = \sum_{i=0}^{\infty} B_i t^i/i!$. Furthermore, set

$$b_n = \frac{|\prod_{i=1}^{[(n-1)/2]} B_{2i}|}{2^{n-1}\,([\frac{n-1}{2}])!}\,, \qquad B''_{n/2} = 2\,(\frac{n}{2})!\,(2\pi)^{-n/2}\,\zeta(\frac{n}{2})\,.$$

Finally, for the case when n is odd, we set

$$Q_n(s) = \zeta(s - \frac{n-1}{2}) \prod_{i=1}^{(n-1)/2} \zeta(2s-(2i-1)), \qquad R_n(s) = \zeta(s) \prod_{i=1}^{(n-1)/2} \zeta(2s-2i).$$

Theorem (cf. [7]).

$$\zeta(s, L_n^*, \delta, \epsilon) = b_n\, 2^{(n-1)s} \left(Q_n(s) + \epsilon\, \delta^{(n+1)/2}(-1)^{(n^2-1)/8} R_n(s)\right),$$
$$\zeta(s, L_n, \delta, \epsilon) = b_n \left(2^{(n-1)/2} Q_n(s) + \epsilon\, \delta^{(n+1)/2}(-1)^{(n^2-1)/8} R_n(s)\right).$$

We see from this that these zeta functions do not have Euler products but that they are the sums of two pieces which do have Euler products.

Now let us suppose that n is even. We first set

$$A_n(s) = \prod_{i=1}^{n/2-1} \zeta(2s-2i), \qquad B_n(s) = \prod_{i=1}^{n/2} \zeta(2s-(2i-1)).$$

But the case of even n is more complicated, and we need more notation.

If K is a quadratic extension of $\mathbb{Q}$, we denote by d_K the discriminant and by χ_K the quadratic character corresponding to K. We also consider the case $K = \mathbb{Q} \oplus \mathbb{Q}$ and in that case set $d_K = 1$ and take χ_K to be the unit character. Then for $\delta = \pm 1$ we set

$$D_n^*(s, \delta) = (-1)^{[n/4]} \sum_{(-1)^{n/2}\delta d_K > 0} 2\,(2\pi)^{-n/2}\, \Gamma(n/2)\, |d_K|^{(n-1)/2}\, L(n/2, \chi_K)$$
$$\times \frac{\zeta(2s)\zeta(2s-n+1)}{L(2s-n/2+1, \chi_K)}\, |d_K|^{-s}.$$

Here the sum runs over all quadratic fields K or $K = \mathbb{Q} \oplus \mathbb{Q}$ with $(-1)^{n/2}\delta d_K > 0$.[2] Then setting

$$D_n^*(s, \delta) = \sum_{d=1}^{\infty} H(n/2, d, \delta)\, d^{-s}$$

defines the notation $H(n/2, d, \delta)$, after which we define

$$D_n(s, \delta) = \sum_{d=1}^{\infty} H(n/2, 4d, \delta)\, d^{-s}.$$

These functions D_n^* and D_n have a close relation with Eisenstein series, as will be discussed in the following section.

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[2] *Translator's remark.* We can write $D_n^*(s, \delta)$ more simply as $c_n \sum |\Delta|^{-s+(n-1)/2} L(n/2, \Delta)$, where $c_n = (-1)^{[n/4]} 2 (2\pi)^{-n/2} \Gamma(n/2)$, the sum is over all integers $\Delta \equiv 0$ or $1 \pmod 4$ with $(-1)^{n/2}\delta\Delta > 0$, and $L_\Delta(\cdot)$ is a certain standard L-function defined in terms of binary quadratic forms of discriminant Δ. (Cf. *Modular Functions of One Variable VI*, Springer Lecture Notes 627, pp. 109–110 and 130.) In this notation, $H(n/2, d, \delta)$ equals $c_n\, d^{(n-1)/2}\, L(n/2, (-1)^{n/2}\delta d)$.

Theorem (cf. [7]). *Let $n \geq 4$ be an even number. Then the following formulas hold:*

$$\begin{aligned}
&\zeta(s, L_n^*, \delta, \epsilon) \\
&\quad = b_n\, 2^{ns} \left((-1)^{[n/4]}\, D_n^*(s,\delta)\, A_n(s) \,+\, \epsilon\, \delta_n\, (-1)^{n(n+2)/8}\, \frac{2B''_{n/2}}{n}\, B_n(s)\right), \\
&\zeta(s, L_n, \delta, \epsilon) \\
&\quad = b_n \left((-1)^{[n/4]}\, D_n(s,\delta)\, A_n(s) \,+\, \epsilon\, \delta_n\, (-1)^{n(n+2)/8}\, \frac{2^{(n+2)/2} B''_{n/2}}{n}\, B_n(s)\right),
\end{aligned}$$

where we have set $\delta_n = 1$ if $(-1)^{n/2} = \delta$ and $\delta_n = 0$ otherwise.

Here the Dirichlet series $D_n(s,\delta)$ and $D_n^*(s,\delta)$ do not have Euler products, but if one looks closely one sees that they are sums of infinitely many terms which do have Euler products. In fact it has been shown that this can be said in general about the zeta functions of prehomogeneous vector spaces. (Writing down a concrete formula, however, is a different matter.)

The proofs of these two theorems do not use the general theory of prehomogeneous vector spaces at all. Instead, one carries out an arithmetical calculation directly from the definitions. If we try to obtain other concrete theorems in the same spirit, then a number of results which were previously unclear emerge as simple corollaries. For example—although we will not enlarge on this here—one can give a direct proof of the functional equation without appealing to the general theory. Moreover, our results turn out to be much simpler than those obtained by Shintani. In particular, the special values $\zeta(1-m, L_n^*)$ ($m = 1, 2, \dots$) for arbitrary n and m can all finally be expressed in terms of Bernoulli numbers. The dimension formula for automorphic forms comes out as one would expect from the general theory, and the residues can also be written down fairly concretely.

If $i = n$, then the zeta functions we have been discussing are zeta functions of *self-dual homogeneous cones* (also called *symmetric cones*). For the general orbits, it would be more correct to speak of the zeta functions of formally real Jordan algebras. These types of functions were investigated by Satake, Ogata and others in order to get a better understanding of the geometric invariants of quotient spaces of bounded symmetric domains by discrete groups (cf. [13]). For them, too, one can use basically the same computational techniques, and, at least for typical $\mathbb{Q}$-forms, the zeta functions can be shown to be "easy" ones. These are only a subclass of prehomogeneous vector spaces, but can be extended in large part to the general case. And also for zeta functions like those of binary cubic forms, which were previously thought to be exceedingly complicated, a totally unexpected relation has been found (work of Ohno and Nakagawa). At this point we will risk proposing the following question:

Problem. Are the "averages" of the zeta functions of prehomogeneous vector spaces all "easy zetas"?

Here the word "average" is like the corresponding notion in the Siegel formula, where we took an average over the genus of a lattice L. For many (G, V) and L the class number is one, as happened for example for the two theorems stated above, and in this case there is no need for any averaging. The notion of "easy zeta" still remains imprecise, but we are now thinking, for instance, of the zeta functions of a wider class of Eisenstein series (e.g. for metaplectic groups). The meaning of this will be explained more fully below.

3. Real-analytic Eisenstein series

This section has two aims. The first is to give an explanation of $D_n(s,\delta)$ and related quantities in terms of automorphic forms; the second, to make some comments about the particular automorphic aspects of the functional equation, etc., in the special (and pathological) case $n=2$. Our results will extend the results obtained by Cohen [3], Sturm [18] and others for the case $\delta=1$. We begin by giving the definition of one-variable Eisenstein series of half-integral weight following Shimura [15] and others. For a positive or negative odd number k, a parameter $\sigma\in\mathbb{C}$ with $\Re(\sigma)\gg 0$, and a variable $z\in\mathfrak{H}$ ($\mathfrak{H}$ = complex upper half-plane), we define the Eisenstein series $E(k,\sigma,z)$ and $E^*(k,\sigma,z)$ by

$$\begin{aligned} E(k,\sigma,z) &= y^{\sigma/2}\sum_{\substack{d=1\\ d\text{ odd}}}^{\infty}\sum_{c=-\infty}^{\infty}\left(\frac{4c}{d}\right)\epsilon_d^{-k}\,(4cz+d)^{k/2}\,|4cz+d|^{-\sigma}\,,\\ E^*(k,\sigma,z) &= E\big(-\frac{1}{4z}\big)\,(-2iz)^{k/2}\\ &= y^{\sigma/2}\,2^{k/2-\sigma}\,e\big(-\frac{k}{8}\big)\sum_{\substack{d=1\\ d\text{ odd}}}^{\infty}\sum_{b=-\infty}^{\infty}\left(\frac{-b}{d}\right)\epsilon_d^{-k}\,(dz+b)^{k/2}\,|dz+b|^{-\sigma}\,. \end{aligned}$$

Here $\left(\frac{*}{*}\right)$ is the quadratic residue symbol, while ϵ_d is 1 or $\sqrt{-1}$ according as $d\equiv 1$ or $3 \bmod 4$. To get the zeta function we want, it is convenient to define the combination

$$F(k,\sigma,z) \;=\; E(k,\sigma,z)+2^{k/2-\sigma}\,\big(e(k/8)+e(-k/8)\big)\,E^*(k,\sigma,z)\,.$$

Since F is invariant under translation by integers, if we write $z=x+iy$ it has a Fourier expansion of the form

$$F(k,\sigma,z) \;=\; \sum_{d=-\infty}^{\infty} c_d(y)\,e^{2\pi i dx}\,.$$

An explicit formula for $c_d(y)=c_d(y,\sigma)$ can be obtained easily from, e.g., Shimura [15]. We further set $G(y)=\sum_{d=-\infty}^{\infty}c_d(y)$ and $H(y)=\sum_{d=-\infty}^{\infty}c_{4d}(y/4)$. Then from the modularity we get

$$G(1/4y) \;=\; (-1)^{(k^2-1)/8}\,2^{1/2}\,y^{\sigma-k/2}\,H(y)\,.$$

We consider the two Mellin transforms

$$\begin{aligned} \Psi^*_{\sigma,k}(s) &= \int_0^{\infty}\big(G(y)-c_0(y)\big)\,y^{s-1}\,dy,\\ \Psi_{\sigma,k}(s) &= \int_0^{\infty}\big(H(y)-c_0(y/4)\big)\,y^{s-1}\,dy\,. \end{aligned}$$

Then for n an even number, $n > 2$, it can be shown that

$$\begin{aligned}
&\Psi^*_{0,-n-1}(s)\\
&\quad= (-1)^{n(n+2)/8+[n/4]}\pi^{n+1/2}\Gamma\big((n+1)/2\big)^{-1}\Gamma(n/2)^{-1}\zeta(n)^{-1}(2\pi)^{-s}\Gamma(s)D^*_n(s,1)\,,\\
&\Psi^*_{2,-n+3}(s)\\
&\quad= (-1)^{n(n+2)/8+1+[n/4]}\,\pi^{n+1/2}\,\Gamma(n/2)^{-1}\Gamma\big((n-1)/2\big)^{-1}\zeta(n)^{-1}(2\pi)^{-s}\Gamma(s)\\
&\qquad\times\big(D^*_n(s,1)\,I\big(s,\tfrac{n-1}{2},1\big)+D^*_n(s,-1)\,I\big(s,1,\tfrac{n-1}{2}\big)\big)\,,
\end{aligned}$$

for $\Re(s) \gg 0$, and similar formulas for Ψ with all asterisks removed. Here we have set

$$I(s,\alpha,\beta) \;=\; \int_0^\infty \frac{(1+u)^{\alpha-1}u^{\beta-1}}{(1+2u)^s}\,du\,.$$

Then by the usual Hecke argument, the functions $\Psi^*_{\sigma,k}$ and $\Psi_{\sigma,k}$ extend meromorphically to all of $\mathbb{C}$ and satisfy

$$\Psi^*_{\sigma,k}(\sigma-k/2-s) \;=\; (-1)^{(k^2-1)/8}\,2^{2s-2\sigma+k+1}\,\Psi_{\sigma,k}(s),$$

from which the functional equation and position of the poles follow. We have thus interpreted the Dirichlet series which we are interested in in terms of Eisenstein series. It is noteworthy that here, rather than using the zeta functions of automorphic forms of integral weight, as is done in the usual Shimura correspondence, we directly take the Mellin transform of the form of half-integral weight.

The case $n=2$, on the other hand, is less simple. The reason is that if $x \in L_2 \cap V^1$ and $-\det x$ is a perfect square, then the quantity $\mu(x)$ becomes infinite, so that even the definition of $\zeta_1(s, L_2)$ becomes problematic in this case. One could try to proceed by summing only over the x for which $\mu(x)$ is finite, but in fact if we do this then the functional equation is not satisfied. The question therefore arises of how to modify the definition to get a good zeta function. It does not seem to be known yet how to make this modification for prehomogeneous vector spaces in general. In our case it was done by Shintani [16] and F. Sato [14], but our methods give an alternative solution. Namely, if we put $k=1$ in the above Ψ^* and Ψ and think of them as functions of σ and s, then the situation is roughly speaking as follows. After a suitable normalization, these functions have the shape $A(\sigma, s) + \zeta(\sigma-1)B(\sigma, s)$ for suitable holomorphic functions A and B, with a pole at $\sigma = 2$. The residue of this pole is expressible in terms of the Riemann zeta function and is not especially interesting. However, the part of the sum with $\mu(x)$ finite appears naturally in $A(2, s)$, so if we look at the constant term of the Laurent expansion around $\sigma = 2$ then a correction term arises from the $B(\sigma, s)$ part, and the above functional equation implies a functional equation between the corrected zeta functions (viz., the constant terms). The calculation is somewhat tricky, but all known results can be recovered using this method. That this method succeeds in explaining everything starting from automorphic forms, and in particular Eisenstein series, is especially attractive from our point of view.

4. Koecher-Maass series

Up to the last section we have been speaking somewhat vaguely of the zeta functions associated with Eisenstein series as "easy". Now we want to state a little more correctly what this means, for instance, for the zeta functions associated to automorphic forms on the Siegel modular group $Sp(n,\mathbb{Z})$. In the interest of brevity we omit the precise definition of Siegel automorphic forms, but in any case they are holomorphic functions $F(Z)$ in the Siegel upper half-space $\mathfrak{H}_n = \{Z = X+iY \in \mathrm{Sym}_n(\mathbb{C}) \mid X,\, Y \in \mathrm{Sym}_n(\mathbb{R}),\ Y>0\}$ that satisfy a suitable invariance property with respect to the operation of $Sp(n,\mathbb{Z})$ (and a growth condition). These automorphic forms have Fourier expansions of the form

$$F(Z) = \sum_{T\in L_n^*} a(T)\,\exp\bigl(2\pi i\,\mathrm{tr}(TZ)\bigr)\,,$$

with $a(T)$ equal to 0 unless T is half-integral. We then set

$$\zeta(s,F) = \sum_{T\in L_n^*/\sim,\ T>0} \frac{\mu(T)\,a(T)}{\det(T)^s}\,.$$

This is called the *Koecher-Maass series* associated to F. For $n=1$ it is just the zeta function defined by Hecke, and in particular has an Euler product if F is a simultaneous eigenfunction of all Hecke operators, but the Koecher-Maass series for $n\geq 2$ do not in fact belong to the mainstream of the zeta functions that have been studied up to now. From the point of view of modern representation theory, it is more natural to define zeta functions by local decompositions of representations (Euler products!), so the zeta functions whose definitions use the eigenvalues with respect to Hecke operators have been studied the most. Such zeta functions have Euler products from the very beginning, but the proof of their functional equations, etc., is all the more difficult. The Koecher-Maass series for $n\geq 2$, on the other hand, behave in a completely different way: the proof of the functional equation is comparatively easy, but they do not in general possess an Euler product.

However, the function $\zeta(s,F)$ defined above at least visually resembles $\zeta(s,L_n^*)$. Of course, since one cannot in general even hope for a formula for the Fourier coefficients when F is a cusp form, it is not possible always to call $\zeta(s,F)$ an easy zeta function. But if F is related to some sort of lifting, then the zeta function may be easy. And indeed, in the case $n=2$ there are results of Böcherer [2] for the Maass space and for Klingen-Eisenstein series. For general n the following theorem results from the joint work of Ibukiyama and Katsurada.

Theorem. *If F is a Siegel Eisenstein series, then $\zeta(s,F)$ can be written down by an explicit formula involving the Riemann zeta function and zeta functions associated to Eisenstein series of half-integral weight in one variable.*

This has been extended by Saito from the ground field $\mathbb{Q}$ to the case of general ground fields. For the details we would like to refer to the papers now in preparation. Because the weight enters as a parameter, the results are more complicated than for $\zeta(s,L_n^*)$, but their general nature is surprisingly similar. If we repeat the above theorem in the form of a catchword, we get

Observation 3. *The Koecher-Maass series of Siegel Eisenstein series are "easy zetas".*

Now, since Eisenstein series are averages of theta series, and $\zeta(s,F)$ is an average of Fourier coefficients, can one speak of $\zeta(s,F)$ as being an average of averages?

Problem. Compute the Koecher-Maass series of elements belonging to the orthogonal complement of the space of cuspidal automorphic forms on a Siegel domain of the first kind (tube domain), e.g., of Klingen-Eisenstein series. Are they "easy zetas"?

Of course, since these automorphic forms become cusp forms on the boundary, we cannot avoid the data coming from this part. The question is whether what remains after we remove this part is "easy".

In any case, it seems that among the zeta functions which we normally meet the family of "easy" zetas is rather extensive.

5. Dimension formula and exponential sums

We now discuss two other important related topics. The first is the dimension formula for holomorphic automorphic forms on a bounded symmetric domain D. This is provided by the Selberg trace formula with a suitable integral kernel, but the actual calculation of the dimension is extremely troublesome and in fact the number of formulas that are known is quite small. Since the integral kernel has the form of a sum of translates of a function by the action of the discrete group Γ with respect to which the automorphic forms are defined, it makes sense to speak of the contribution coming from an element or (if it converges) a subset of Γ. Let us suppose for instance that the algebraic group G used in writing $D = G/K$ is defined over $\mathbb{Q}$ and that Γ is an arithmetic subgroup of $G(\mathbb{Q})$. Then it is generally believed that:

(1) If an element $\gamma \in \Gamma$ has no eigenvalue of absolute value 1, then its contribution vanishes.

(2) Suppose that a suitable power of γ is a unipotent element of Γ. If the unipotent part of its Jordan decomposition is contained in the center U of the unipotent radical of a maximal parabolic $\mathbb{Q}$-subgroup, we call γ central. Then the contribution of non-central elements vanishes.

If we believe these statements, then the only contributions to the dimension formula for a congruence subgroup of sufficiently high level will come from ± 1 (unit element) and the central unipotent elements. On the other hand, for any maximal parabolic subgroup, U has the structure of a Jordan algebra. Therefore we should expect to be able to write its contribution as a special value of the zeta function of a cone. In order to prove this, one has to prove new functional equations, etc., which are not contained in the general theory of prehomogeneous vector spaces, but under suitable arithmetical assumptions these can actually be obtained, and because one can interpret them as special values of easy zeta functions it is possible to write down explicitly a conjectural formula for the dimension of the space of automorphic forms. For this we refer the reader, for instance, to [6] and [4]. It used to be assumed that the complexity would become unmanageable as the dimension of the domain increased, but the above considerations suggest that it may really be possible in future to write down the dimension formula for automorphic forms on bounded symmetric domains.

Next we say something about exponential sums. In connection with the dimension formula for Siegel modular forms of degree 2, Lee and Weintraub formulated a conjecture saying that a certain sum of roots of unity of a previously unknown type could be expressed in terms of Bernoulli numbers. Several people considered the challenge of finding an elementary proof of this conjecture, but no one succeeded. By the work of Hashimoto and Arakawa [1], it could be shown that the conjecture reduced to the evaluation of special values of the L-functions of a prehomogeneous vector space of a certain type. But in fact, this L-function can be written down in explicit form by using our theory, and one also obtains the Lee-Weintraub conjecture ([11], [10], [9]). But even now there is no elementary proof. Going even further, it may be that there are exponential sums of a similar type for more general prehomogeneous vector spaces, but this is not yet known. We think that there may be the possibility of a general theory here.

6. Concerning the proofs

Unfortunately, we are gradually reaching the end of our allotted space. Also, to be honest, the proofs are very complicated—in contrast to the easiness of the results. But, even if we can't explain the proof of the formula for $\zeta(s, L_n)$ in detail, we would still like to discuss it briefly. Looking carefully, one sees that $\zeta(s, L_n)$ describes the distribution of masses of genera in L_n. Each mass is given by Siegel's formula as a product of reciprocals of local densities. The form of the local densities $\alpha_p(S, S)$ occurring here is extremely well understood. In the case $p = 2$ it is quite complicated, but still there is a complete formula. Using it, one can therefore in principle carry out the calculation. Specifically, we should first calculate the sum over equivalence classes of $S \in \mathrm{Sym}_n(\mathbb{Z}_p)$ of the quantity $\alpha_p(S, S)^{-1}|\det(S)|_p^s$ and then take the product over p. In practice, however, there are several complications.

(1) First of all, even if the determinants of a collection of local (i.e., over $\mathbb{Z}_p$) symmetric matrices are equal, this does not imply the existence of a global (i.e., over $\mathbb{Z}$) symmetric matrix. The necessary condition for this is that the product of the Hasse invariants is 1. But since verifying this condition each time would be difficult, one instead considers the local Dirichlet series both including and not including the Hasse invariant, forms the products over all p and then adds them, so that only the part where the product of the Hasse invariants is 1 survives. This is the same technique as is used, for example, for the "relative trace formula."

(2) The next problem is that the classification of the local isomorphism classes is extremely complicated for $p = 2$. But in fact the zeta function only involves cruder data than the mass of a single genus. By lumping together all the genera with the same determinant we obtain a quantity which is much simpler than in Siegel's formula, and in fact, if we collect together all of the ones with the same Jordan decomposition (= class of symmetric matrices with the same normal form), then we can calculate the average of the reciprocals of the local densities without using the classification of the equivalence classes. This part is one of the key lemmas.

(3) The quantities calculated as above for each Jordan decomposition must then be put together into a local Dirichlet series which is as simple as possible. We do this in two steps, first collecting the contributions corresponding to Jordan decompositions of a fixed size and then letting the size vary, using the q-analogue of the Taylor expansion for the latter.

(4) The coefficient a_{p^ν} of the Dirichlet series occurring in (3) is obtained as the sum (or as the sum weighted by the Hasse invariants) of the reciprocals of the local densities taken over all local equivalence classes with $\det(S) = d_0 p^\nu$ for a fixed number d_0 prime to p. This involves studying a certain decomposition of the so-called *Igusa local zeta function.* If the result were independent of the value of d_0,

then we could obtain the global Dirichlet series simply by multiplying together the local ones corresponding to all p, but in fact it does depend strongly on d_0. We therefore have to study the individual local Dirichlet series carefully and rewrite them before we can assemble them into a global object.

Unfortunately, we have no idea why in the above proof the final formula should become so simple, and in particular, why quantities like $D_n(s,\delta)$ appear. We can only say that when one does the calculation this is what happens, and that it seems to be an extremely attractive problem to explain the essential reason for such an outcome and to give a different proof.

Our results saying that there are not as many difficult zeta functions in the world as one might have expected could perhaps be seen as not very positive. But on the other hand, it may be that some new insight has been gained and that some new questions have arisen whose meaning is only beginning to emerge.

We have restricted ourselves here to quoting only the most basic references. For more details, we refer to the bibliographies at the end of [7]–[9].

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