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# On the stability of minimal surfaces in $\boldsymbol{R}^3$

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## § 0. Introduction.

Minimal surfaces are exactly the critical points of area functional for all variations which keep their boundary values fixed. But they do not necessarily provide relative minima of area. When a minimal surface corresponds to a relative minimum of area for all such variations, we say it is stable, otherwise unstable.

In this paper we shall give sufficient conditions for the stability and the instability of minimal surfaces in the Euclidean space $\boldsymbol{R}^3$.

Let $D$ be a plane domain with compact closure $\bar{D}$, whose boundary $\partial D$ is a finite union of piecewise $C^\infty$ curves. Let $\mathfrak{X} : \bar{D} \to \boldsymbol{R}^3$ be a regular (i.e. immersed) minimal surface. And denote by $\mathfrak{G} : \bar{D} \to S = \{(x^1, x^2, x^3) \in \boldsymbol{R}^3 ; (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$ the Gauss map of $\mathfrak{X}$. Barbosa and do Carmo [1] gave a sufficient condition for $\mathfrak{X}$ to be stable:

THEOREM (Barbosa and do Carmo [1]). *If the area of $\mathfrak{G}(\bar{D})$ (as a point set on $S$) is smaller than $2\pi$, then $\mathfrak{X}$ is stable.*

This estimate is sharp in the following sense: There are examples of unstable minimal surfaces whose Gaussian image has area larger than $2\pi$ and as close to $2\pi$ as one pleases.

Now what can we say about the stability of minimal surfaces satisfying the condition that the area of $\mathfrak{G}(\bar{D})$ is exactly $2\pi$? Our purpose in this paper is to answer this question.

Except the case that $\mathfrak{G}$ is a branched covering of a hemisphere $H$ of $S$ (i.e. $\mathfrak{G}(\bar{D})$ coincides with $H$ and $\mathfrak{G}(\partial D) = \partial H$), minimal surface $\mathfrak{X}$ is always stable (Theorem 1 and Theorem 2).

In the case excepted above, let $f$ and $g$ be the factors of the Weierstrass representation of $\mathfrak{X}$ (cf. § 3). By a suitable rotation of the surface in $\boldsymbol{R}^3$, we may assume that $\mathfrak{G}(\bar{D})$ coincides with the lower hemisphere of $S$ : $H^- = \{(x^1, x^2, x^3) \in S ; x^3 \leqq 0\}$. In this situation, $g$ is a holomorphic function of $D$ onto $D_0 = \{w \in \boldsymbol{C} ; |w| < 1\}$. Here we define a function $F$ in $D_0$ as follows:

$$F(w)=\sum_{\{\zeta\in D;\ g(\zeta)=w\}}\frac{g'(\zeta)}{f(\zeta)}.$$

Then $F$ is seen to be holomorphic in $D_0$ and we can prove the following fact:

MAIN THEOREM. *Let $\mathfrak{X}:\bar{D}\to\boldsymbol{R}^3$ be a regular minimal surface and let $\mathfrak{G}:\bar{D}\to S$ be the Gauss map of $\mathfrak{X}$. Suppose that $\mathfrak{G}(\bar{D})$ coincides with the lower hemisphere $H^-$ of $S$ and that $\mathfrak{G}(\partial D)=\partial H^-$. If*

$$\operatorname{Re}F''(0)\neq 0,$$

*then $\mathfrak{X}$ is unstable.*

Therefore every minimal surface satisfying the assumption of the above theorem is not physically realized as soap film. It must be interesting that the instability of $\mathfrak{X}$ is decided only by the values of derivatives of $\mathfrak{X}$ at a finite number of points. This result is proved by calculating the third variation of area functional.

## §1. Notations and terminology.

A minimal surface $\mathfrak{X}$ in $\boldsymbol{R}^3$ is a $C^2$ mapping $\mathfrak{X}$ from some domain $D$ in the plane into $\boldsymbol{R}^3$ which is harmonic in $D$, extends continuously to the closure $\bar{D}$, and satisfies $\mathfrak{X}_\xi\cdot\mathfrak{X}_\eta=0$, $|\mathfrak{X}_\xi|=|\mathfrak{X}_\eta|$ in $D$ (where $\zeta=\xi+\sqrt{-1}\,\eta$ is the variable in the parameter domain). A branch point of $\mathfrak{X}$ is some point $\zeta\in\bar{D}$ where $\mathfrak{X}_\xi=\mathfrak{X}_\eta=0$. Branch points are the only possible singularities of minimal surfaces.

In the following the parameter domain $D$ is supposed to be a relatively compact domain whose boundary is a finite union of piecewise $C^\infty$ curves. And we shall be concerned only with regular minimal surfaces which can be extended as minimal surfaces across $\partial D$, where "regular" means that the surface considered has no branch points on $\bar{D}$.

If $\mathfrak{Y}:\bar{D}\to\boldsymbol{R}^3$ is of piecewise $C^1$-class on $\bar{D}$, then the area functional is defined as

$$A(\mathfrak{Y})=\iint_D|\mathfrak{Y}_\xi\times\mathfrak{Y}_\eta|\,d\xi d\eta=\iint_D\sqrt{E_{\mathfrak{Y}}G_{\mathfrak{Y}}-F_{\mathfrak{Y}}^2}\,d\xi d\eta,$$

where $E_{\mathfrak{Y}}=\mathfrak{Y}_\xi^2$, $F_{\mathfrak{Y}}=\mathfrak{Y}_\xi\cdot\mathfrak{Y}_\eta$, and $G_{\mathfrak{Y}}=\mathfrak{Y}_\eta^2$.

We give here the rigorous definitions for the stability and the instability of minimal surfaces in terms of the Gauss map and real-valued functions with vanishing boundary values:

$$\mathfrak{G}:\bar{D}\to S\qquad\text{the Gauss map of}\quad\mathfrak{X}$$

$$C_0^{2\prime}(\bar{D})=\{u:\bar{D}\to\boldsymbol{R}\ ;\ u\text{ is a piecewise }C^2\text{ function with }u|_{\partial D}=0\}.$$

For each smooth family $v(\varepsilon)\in C_0^{2\prime}(\bar{D})$ ($\varepsilon$ runs in an interval containing zero and "smooth" means that $v(\varepsilon)$ is smooth with respect to $\varepsilon$ and the derivatives are contained in $C_0^{2'}(\bar{D})$) with $v(0)=0$ and $[\partial v(\varepsilon)/\partial\varepsilon]_{\varepsilon=0}\neq 0$, we consider the normal variation of $\mathfrak{X}$ : $\mathfrak{X}+v(\varepsilon)\mathfrak{G}$, where $\mathfrak{G}$ is identified with the unit normal vector field of $\mathfrak{X}$.

DEFINITION. (i) A minimal surface $\mathfrak{X}$ is said to be *stable* if for each smooth family $v(\varepsilon)\in C_0^{2'}(\bar{D})$ with $v(0)=0$ and $[\partial v(\varepsilon)/\partial\varepsilon]_{\varepsilon=0}\neq 0$, there exists some $\varepsilon_0>0$ such that

$$A(\mathfrak{X})\leqq A(\mathfrak{X}+v(\varepsilon)\mathfrak{G})$$

holds for every $\varepsilon$, $|\varepsilon|<\varepsilon_0$.

(ii) A minimal surface $\mathfrak{X}$ is said to be *unstable* if $\mathfrak{X}$ is not stable, that is, there exists some smooth family $v(\varepsilon)\in C_0^{2'}(\bar{D})$ with $v(0)=0$ and $[\partial v(\varepsilon)/\partial\varepsilon]_{\varepsilon=0}\neq 0$, such that for each $\varepsilon_0>0$,

$$A(\mathfrak{X})>A(\mathfrak{X}+v(\varepsilon)\mathfrak{G})$$

holds for some $\varepsilon$, $|\varepsilon|<\varepsilon_0$.

The notations and symbols below will be used throughout this paper without particular mentions:

$\zeta=\xi+\sqrt{-1}\,\eta$ ($\xi$, $\eta\in\boldsymbol{R}$) : the complex variable in the respective parameter domain;
$\Delta=\partial^2/\partial\xi^2+\partial^2/\partial\eta^2$ : the Laplacian on the parameter domain;
$\partial D$ : the boundary of $D$;
$\bar{D}$ : the closure of $D$;
$D_0$ : the unit open disk;
$S=\{(x^1, x^2, x^3)\in\boldsymbol{R}^3\ ;\ (x^1)^2+(x^2)^2+(x^3)^2=1\}$ : the unit sphere in $\boldsymbol{R}^3$;
$H$ : *a* (closed) hemisphere of $S$;
$H^-=\{(x^1, x^2, x^3)\in S\ ;\ x^3\leqq 0\}$ : the lower (closed) hemisphere of $S$;
$\mathfrak{X}$ : $\bar{D}\to\boldsymbol{R}^3$ : a regular minimal surface which can be extended as a minimal surface across $\partial D$;
$Ed\xi^2+2Fd\xi d\eta+Gd\eta^2$ : the first fundamental form of $\mathfrak{X}$;
$Ld\xi^2+2Md\xi d\eta+Nd\eta^2$ : the second fundamental form of $\mathfrak{X}$;
$W=\sqrt{EG-F^2}$ : the area element of $\mathfrak{X}$;
$\mathfrak{G}=\mathfrak{X}_\xi\times\mathfrak{X}_\eta/|\mathfrak{X}_\xi\times\mathfrak{X}_\eta|$ : $\bar{D}\to S$ : the Gauss map (sometimes identified with the unit normal vector field) of $\mathfrak{X}$;
$K$ : the Gaussian curvature of $\mathfrak{X}$;

Since our surface $\mathfrak{X}$ is minimal, it follows that

$$E=G,\qquad F=0,$$

and

$$L+N=\mathfrak{G}\cdot(\mathfrak{X}_{\xi\xi}+\mathfrak{X}_{\eta\eta})=0.$$

Therefore

$$W=\sqrt{EG-F^2}=E=G$$

and

(1) $$K=\frac{LN-M^2}{EG-F^2}=-\frac{L^2+M^2}{W^2}\leqq 0.$$

Moreover, $K$ can have only isolated zeros unless the locus of $\mathfrak{X}$ is contained in a plane (Lemma 3 in § 3).

REMARK 1. For each subdomain $\tilde{D}$ of the parameter domain, we denote the image of $\tilde{D}$ under $\mathfrak{G}$ by $\mathfrak{G}(\tilde{D})=\{\mathfrak{G}(\zeta)\ ;\ \zeta\in\tilde{D}\}$. Although $\mathfrak{G}$ is a complex analytic mapping of $\tilde{D}$, we regard $\mathfrak{G}(\tilde{D})$ as a mere subset of $S$, ignoring its number of sheets.

## § 2. The variations of area and the eigenvalue problem associated with area.

Let $v(\varepsilon)$ be a smooth family in $C_0^{2\prime}(\bar{D})$ with $v(0)=0$ and $[\partial v(\varepsilon)/\partial\varepsilon]_{\varepsilon=0}=u\neq 0$. With the aid of the minimality of $\mathfrak{X}$, the first and the second variations of area functional for a normal variation $\mathfrak{X}+v(\varepsilon)\mathfrak{G}$ are given by the following formulae (cf. Beeson [2]):

$$\frac{d}{d\varepsilon}A(\mathfrak{X}+v(\varepsilon)\mathfrak{G})\Big|_{\varepsilon=0}=0,$$

$$\frac{d^2}{d\varepsilon^2}A(\mathfrak{X}+v(\varepsilon)\mathfrak{G})\Big|_{\varepsilon=0}=\iint_D u(-\Delta u+2KWu)d\xi d\eta.$$

Therefore, as for the first and the second variations of area, it is sufficient to consider only variations formed as $\mathfrak{X}+\varepsilon u\mathfrak{G}$ ($\varepsilon\in\boldsymbol{R}$, $u\in C_0^{2\prime}(\bar{D})$ and $u\neq 0$) which we shall call "*variation* $u$". Then

(2) $$I^{(1)}(u)=\frac{d}{d\varepsilon}A(\mathfrak{X}+\varepsilon u\mathfrak{G})\Big|_{\varepsilon=0}=\frac{d}{d\varepsilon}A(\mathfrak{X}+v(\varepsilon)\mathfrak{G})\Big|_{\varepsilon=0}=0,$$

(3) $$\begin{aligned}I^{(2)}(u)&=\frac{d^2}{d\varepsilon^2}A(\mathfrak{X}+\varepsilon u\mathfrak{G})\Big|_{\varepsilon=0}\\&=\frac{d^2}{d\varepsilon^2}A(\mathfrak{X}+v(\varepsilon)\mathfrak{G})\Big|_{\varepsilon=0}=\iint_D u(-\Delta u+2KWu)d\xi d\eta.\end{aligned}$$

Moreover the third variation of area functional for variation $u$ is given by

(4) $$I^{(3)}(u)=\frac{d^3}{d\varepsilon^3}A(\mathfrak{X}+\varepsilon u\mathfrak{G})\Big|_{\varepsilon=0}=\iint_D\frac{6u}{W}\{L(u_\xi^2-u_\eta^2)+2Mu_\xi u_\eta\}\,d\xi d\eta$$

(cf. Nitsche [4, p. 93]).

When the locus of $\mathfrak{X}$ does not lie in a plane, we consider an eigenvalue problem related to the second variation above. Let $\tilde{D} \subsetneqq D$ be a subdomain of $D$ such that $\partial\tilde{D}$ is piecewise $C^\infty$. Then we pose the eigenvalue problem.

$$
(5) \qquad \begin{cases} \Delta u - \lambda KWu = 0 & \text{in } \tilde{D}, \\ u = 0 & \text{on } \partial\tilde{D}. \end{cases}
$$

If we denote by $\lambda_1(\tilde{D})$ the least eigenvalue of the problem (5), then we see almost immediately from Beeson [2] the following

LEMMA 1. (i) *If $D_1 \subsetneqq D_2$, then $\lambda_1(D_1) \geqq \lambda_1(D_2)$, where the equality holds if and only if $D_1 = D_2$.*

(ii) *According as $\partial\tilde{D}$ varies smoothly, $\lambda_1(\tilde{D})$ varies continuously.*

(iii) *$\lambda_1(\tilde{D})$ is equal to the minimum of the Rayleigh quotient:*

$$
R(u) = \frac{\iint_{\tilde{D}} (-u\Delta u)\, d\xi d\eta}{\iint_{\tilde{D}} (-KW) u^2 d\xi d\eta}, \qquad u \in C_0^{2'}(\tilde{D}),
$$

*and the equality "$R(u) = \lambda_1(\tilde{D})$" holds if and only if $u$ is a least eigenfunction (i.e. the one associated with the least eigenvalue $\lambda_1(\tilde{D})$).*

(iv) *Each least eigenfunction has the definite sign. But except them, every eigenfunction changes its sign.*

(v) *The eigenspace corresponding to the least eigenvalue is* 1*-dimensional.*

Now set $\tilde{D} = D$ in (5). By using the fact that $\lambda_1(D)$ minimizes $R(u)$ ((iii) of the above lemma), we can derive the relationship between the least eigenvalue of the problem (5) and the stability of the minimal surface:

LEMMA 2. (i) *If $\lambda_1(D) > 2$, then $I^{(2)}(u) > 0$ for all the variations $u$. Therefore $\mathfrak{X}$ is stable.*

(ii) *If $\lambda_1(D) = 2$, then $I^{(2)}(u) \geqq 0$ for all such variations $u$, and $I^{(2)}(u) = 0$ holds if and only if $u$ is a least eigenfunction of* (5).

(iii) *If $\lambda_1(D) < 2$, then there exists some $u$ such that $I^{(2)}(u) < 0$. Therefore $\mathfrak{X}$ is unstable.*

PROOF. Since $\lambda_1$ $(= \lambda_1(D))$ minimizes $R(u)$,

$$
\lambda_1 \leqq \frac{\iint_D (-u\Delta u)\, d\xi d\eta}{\iint_D (-KW) u^2 d\xi d\eta}
$$

for every variation $u$. By using the fact that $-KW$ is non-negative (§ 1), we see

$$
\lambda_1 \iint_D (-KW) u^2 d\xi d\eta \leqq \iint_D (-u\Delta u)\, d\xi d\eta.
$$

Therefore

$$I^{(2)}(u)=\iint_D u(-\Delta u+2KWu)d\xi d\eta$$

$$\geqq(\lambda_1-2)\iint_D(-KW)u^2 d\xi d\eta .$$

Hence, the assumption $\lambda_1>2$ implies $I^{(2)}(u)>0$ for all $u$. Moreover, if $\lambda_1=2$, $I^{(2)}(u)\geqq 0$ for all $u$, and $I^{(2)}(u)=0$ if and only if $\lambda_1=R(u)$ which is just the case in which $u$ is a least eigenfunction by virtue of Lemma 1, (iii). Thus we have proved (i) and (ii).

Suppose that $\lambda_1<2$ and that $u$ is a least eigenfunction. Then

$$2>\lambda_1=\frac{\iint_D(-u\Delta u)d\xi d\eta}{\iint_D(-KW)u^2 d\xi d\eta}.$$

Therefore

$$I^{(2)}(u)=\iint_D u(-\Delta u+2KWu)d\xi d\eta$$

$$=(\lambda_1-2)\iint_D(-KW)u^2 d\xi d\eta$$

$$<0. \qquad \text{Q.E.D.}$$

REMARK 2. In the case (ii) of Lemma 2, we cannot so easily arrive at any conclusion about the stability of the minimal surface $\mathfrak{X}$. Calculating the third variation is one of the ways to obtain some conclusion. In fact, if, for some $u$, $I^{(2)}(u)=0$ and $I^{(3)}(u)\neq 0$, then $\mathfrak{X}$ is clearly unstable.

## § 3. The Weierstrass representation and the second fundamental form.

In this section we recall the Weierstrass representation of minimal surfaces and derive a certain important relation between the factors of the representation and the second fundamental form of the surface. The facts mentioned in this section will be used effectively in § 6 to investigate the stability of a certain kind of minimal surfaces.

Since $\mathfrak{X}=(\mathfrak{X}^1, \mathfrak{X}^2, \mathfrak{X}^3): \bar{D}\to \boldsymbol{R}^3$ is a minimal surface, each of the functions

$$\phi_j=\frac{\partial \mathfrak{X}^j}{\partial \xi}-\sqrt{-1}\frac{\partial \mathfrak{X}^j}{\partial \eta}, \qquad j=1, 2, 3, \tag{6}$$

is holomorphic in $D$. Let us introduce two functions with Enneper-Weierstrass:

$$f=\phi_1-\sqrt{-1}\,\phi_2, \qquad g=\frac{\phi_3}{\phi_1-\sqrt{-1}\,\phi_2} \tag{7}$$

Then $f$ is holomorphic and $g$ is meromorphic in $D$. Moreover, for any point $\zeta\in D$ which is not a pole of $g$, $f(\zeta)=0$ if and only if $\zeta$ is a branch point of $\mathfrak{X}$. Since $\zeta$ is an isothermal parameter of $\mathfrak{X}$,

$$\phi_1^2+\phi_2^2+\phi_3^2=0.$$

From this fact, it follows that

$$(8)\qquad \phi_1=\frac{1}{2}f(1-g^2), \qquad \phi_2=\frac{\sqrt{-1}}{2}f(1+g^2), \qquad \phi_3=fg.$$

Therefore if $\zeta_0\in D$,

$$\begin{pmatrix}\mathfrak{X}^1(\zeta)\\ \mathfrak{X}^2(\zeta)\\ \mathfrak{X}^3(\zeta)\end{pmatrix}=\begin{pmatrix}\mathrm{Re}\int_{\zeta_0}^{\zeta}\frac{1}{2}f(1-g^2)d\zeta\\ \mathrm{Re}\int_{\zeta_0}^{\zeta}\frac{\sqrt{-1}}{2}f(1+g^2)d\zeta\\ \mathrm{Re}\int_{\zeta_0}^{\zeta}fgd\zeta\end{pmatrix}+\begin{pmatrix}\mathfrak{X}^1(\zeta_0)\\ \mathfrak{X}^2(\zeta_0)\\ \mathfrak{X}^3(\zeta_0)\end{pmatrix}.$$

This representation is called the Weierstrass representation of the minimal surface $\mathfrak{X}$. Let us call $f$ and $g$ the first and the second factor of the Weierstrass representation of $\mathfrak{X}$ respectively (or, for short, the first W-factor and the second W-factor of $\mathfrak{X}$ respectively). And sometimes we call $f$ and $g$ the factors of the Weierstrass representation (or the W-factors) of $\mathfrak{X}$ in the lump.

From the equations (6) and (8) we derive

$$(9)\qquad W=|\mathfrak{X}_\xi|^2=|\mathfrak{X}_\eta|^2=\frac{1}{2}\sum_{j=1}^{3}|\phi_j|^2=\left\{\frac{|f|(1+|g|^2)}{2}\right\}^2.$$

Now, by some calculations, we observe

$$(10)\qquad \mathfrak{G}=\left(\frac{2\,\mathrm{Re}\,g}{|g|^2+1},\ \frac{2\,\mathrm{Im}\,g}{|g|^2+1},\ \frac{|g|^2-1}{|g|^2+1}\right).$$

Consequently $g$ coincides with the composition $P\circ\mathfrak{G}$ of the Gauss map $\mathfrak{G}$ with the stereographic projection $P$ from the point (0, 0, 1) onto the $(x^1, x^2)$-plane.

The following proposition will give some information about the geometrical meaning of the holomorphic function $f$.

PROPOSITION 1. *Let $\mathfrak{X}:\bar{D}\to\boldsymbol{R}^3$ be a minimal surface and let $f, g$ be the W-factors of $\mathfrak{X}$. Denote the second fundamental form of $\mathfrak{X}$ by $\alpha=Ld\xi^2+2Md\xi d\eta+Nd\eta^2$. Then*

$$(11)\qquad L-\sqrt{-1}M=-fg',$$

$$\alpha=-(1/2)fg'd\zeta^2-(1/2)\overline{fg'}d\bar{\zeta}^2,$$

*where $'$ means the derivative of a holomorphic function, $d\zeta=d\xi+\sqrt{-1}\,d\eta$, and $d\bar{\zeta}=d\xi-\sqrt{-1}\,d\eta$.*

Proof. Set $f^{(1)}=\mathrm{Re}\,f$, $f^{(2)}=\mathrm{Im}\,f$, $g^{(1)}=\mathrm{Re}\,g$, and $g^{(2)}=\mathrm{Im}\,g$. By the equation (10), we see

$$\begin{aligned}(12)\qquad \mathfrak{G}_\xi=(&2\{-g_\xi^{(1)}(g^{(1)})^2-g_\xi^{(1)}(g^{(2)})^2-2g_\xi^{(2)}g^{(1)}g^{(2)}+g_\xi^{(1)}\}/(|g|^2+1)^2,\\ &2\{g_\xi^{(2)}(g^{(1)})^2-g_\xi^{(2)}(g^{(2)})^2-2g_\xi^{(1)}g^{(1)}g^{(2)}+g_\xi^{(2)}\}/(|g|^2+1)^2,\\ &4(g_\xi^{(1)}g^{(1)}+g_\xi^{(2)}g^{(2)})/(|g|^2+1)^2).\end{aligned}$$

On the other hand,

$$\begin{aligned}L-\sqrt{-1}M&=-\mathfrak{G}_\xi\cdot\mathfrak{X}_\xi-\sqrt{-1}(-\mathfrak{G}_\xi\cdot\mathfrak{X}_\eta)\\ &=-\mathfrak{G}_\xi\cdot(\mathfrak{X}_\xi-\sqrt{-1}\mathfrak{X}_\eta)\\ &=-\mathfrak{G}_\xi\cdot(\phi_1,\ \phi_2,\ \phi_3)\\ (13)\qquad &=-\mathfrak{G}_\xi\cdot(f(1-g^2)/2,\ \sqrt{-1}f(1+g^2)/2,\ fg).\end{aligned}$$

By some calculation using (12) and (13), we obtain

$$L-\sqrt{-1}M=-fg'.$$

Since $\mathfrak{X}$ is minimal, $N=-L$ (§1). Therefore

$$\begin{aligned}\alpha&=L(d\xi^2-d\eta^2)+2Md\xi d\eta\\ &=\mathrm{Re}\{(L-\sqrt{-1}M)(d\xi+\sqrt{-1}\,d\eta)^2\}\\ &=\{(L-\sqrt{-1}M)(d\xi+\sqrt{-1}\,d\eta)^2+(L+\sqrt{-1}M)(d\xi-\sqrt{-1}\,d\eta)^2\}/2.\end{aligned}$$

Q. E. D.

Although the following lemma is well known, we contain a brief sketch of its proof by using the above proposition.

Lemma 3.

$$(14)\qquad K=-\left\{\frac{4|g'|}{|f|(1+|g|^2)^2}\right\}^2.$$

*Therefore $K$ is non-positive. Moreover, $K$ can have only isolated zeros unless the locus of $\mathfrak{X}$ lies entirely in a plane.*

Proof. By using (1), (9), and Proposition 1, we obtain the equation (14). Since $g$ is meromorphic, $g'$ can have only isolated zeros unless $g'$ is identically zero. Moreover, $g'\equiv 0$ if and only if $\mathfrak{G}$ is constant by virtue of the equation $g=P\circ\mathfrak{G}$. Q. E. D.

## §4. Case 1. $\mathfrak{G}(\bar{D})$ has area $2\pi$ but does not coincide with $H$.

From now on, we assume that the area of $\mathfrak{G}(\bar{D})$ equals exactly $2\pi$. At first, in this section, we are concerned with the case in which $\mathfrak{G}(\bar{D})$ does not coincide with any hemisphere $H$ of $S$.

When we regard $S$ as a Riemannian manifold with the Riemannian metric induced from the Euclidean space $\boldsymbol{R}^3$, we denote the Laplacian for functions in $S$ by $\Delta_S$ : Denote by $g_{jk}$ $(j, k=1, 2)$ the components of the Riemannian metric on $S$ with respect to a system of local coordinates $(y^1, y^2)$ in $S$. Set $G=\det(g_{jk})$, and set $(g^{jk})=(g_{jk})^{-1}$. Then the operator $\Delta_S$ is defined as follows:

$$\Delta_S h=\frac{1}{\sqrt{G}}\sum_{j,k=1,2}\frac{\partial}{\partial y^j}\left[\sqrt{G}\, g^{jk}\frac{\partial h}{\partial y^k}\right].$$

Let $\Omega$ be a domain in $S$. Consider the eigenvalue problem:

$$\left\{\begin{array}{ll}\Delta_S v+\lambda v=0 & \text{in } \Omega, \\ v=0 & \text{on } \partial\Omega.\end{array}\right. \tag{15}$$

We denote by $\tilde{\lambda}_1(\Omega)$ the least eigenvalue of this problem.

LEMMA 4 (Peetre [5]). *Among all spherical domains with the same area, only the spherical cap minimizes* $\tilde{\lambda}_1$.

LEMMA 5 (Barbosa and do Carmo [1]). $\tilde{\lambda}_1(\mathrm{Int}\,H)=2$, *where* $\mathrm{Int}\,H$ *stands for the interior of* $H$.

LEMMA 6 (Barbosa and do Carmo [1]). *If* $\tilde{\lambda}_1(\mathfrak{G}(D))>2$, *then* $I^{(2)}(u)>0$ *for all* $u$, *so* $\mathfrak{X}$ *is stable.*

Now that $\tilde{\lambda}_1(\mathfrak{G}(D))>2$ for our minimal surface $\mathfrak{X}$ by Lemmas 4 and 5, Lemma 6 is applicable. Thus we have proved

THEOREM 1. *Let the image of the Gauss map of a regular minimal surface* $\mathfrak{X}$ *have area* $2\pi$. *Assume that it does not coincide with any hemisphere of* $S$. *Then* $\mathfrak{X}$ *is stable.*

## §5. Case 2. $\mathfrak{G}$ maps $\bar{D}$ onto $H$ but $\mathfrak{G}(\partial D)\neq\partial H$.

THEOREM 2. *Let the image of the Gauss map* $\mathfrak{G}$ *of a regular minimal surface* $\mathfrak{X}$ *coincide with a hemisphere* $H$ *of* $S$. *Suppose that* $\mathfrak{G}(\partial D)\neq\partial H$. *Then the second variation of area is always positive, and hence* $\mathfrak{X}$ *is stable.*

PROOF. Assume that $I^{(2)}(u)\leqq 0$ for some $u$. Then $\lambda_1(D)\leqq 2$ (Lemma 2). Since $\mathfrak{G}(\partial D)\neq\partial H$, there exists some arc $\gamma\subset\partial D$ such that $\mathfrak{G}(\gamma)\subset\mathrm{Int}\,H$ (Fig. 1). By assumption, $\mathfrak{X}$ can be extended across $\gamma$ up to some domain $D_1\supsetneqq D$ such that $\mathfrak{G}(\bar{D}_1)=H$, too (Fig. 2). Because $\lambda_1$ is strictly decreasing (Lemma 1, (i)), $\lambda_1(D_1)$

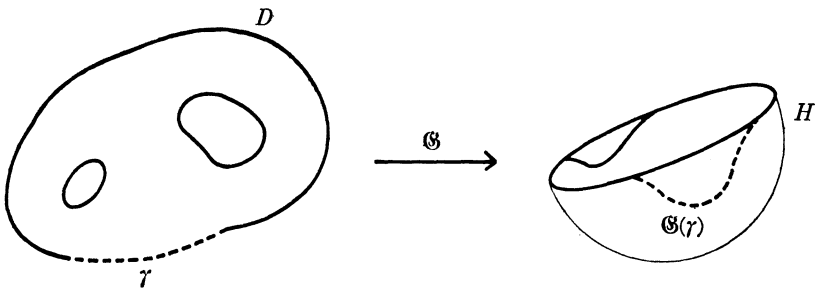


Figure 1.

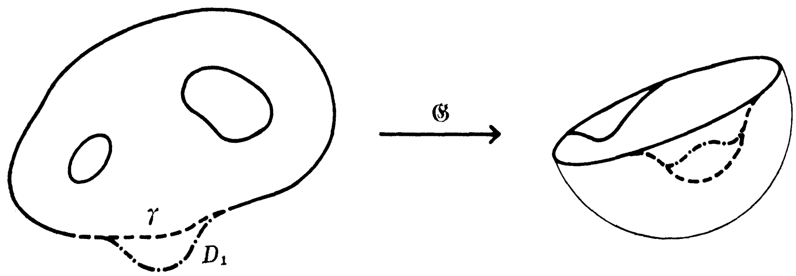


Figure 2.

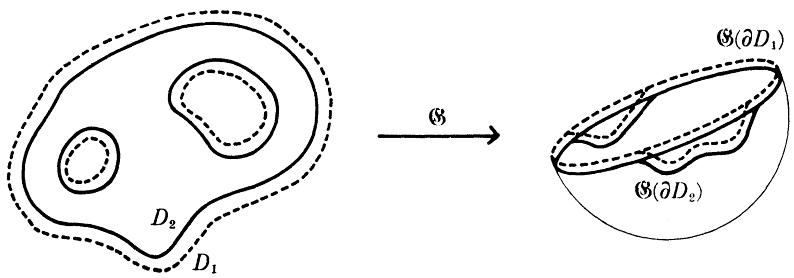


Figure 3.

$<\lambda_1(D)\leqq 2$. Owing to this fact and the continuity of $\lambda_1$ (Lemma 1,(ii)), we can take some domain $D_2$ whose closure is contained in Int$D_1$, such that $\lambda_1(D_2)<2$ (Fig. 3).

On the other hand, the second W-factor $g$ of $\mathfrak{X} : \bar{D}_1 \to \boldsymbol{R}^3$ is holomorphic in $D_1$ and $g(\bar{D}_1)=\bar{D}_0=\{w\,;\ |w|\leqq 1\}$. Therefore $g(D_2)\subsetneqq D_0$ by the maximum principle, which implies that $\mathfrak{G}(D_2)\subsetneqq \mathrm{Int}\,H$ (Fig. 3). Since $\tilde{\lambda}_1$ is decreasing, $\tilde{\lambda}_1(\mathfrak{G}(D_2))>2$ (Lemma 5). Thus, by applying Lemmas 2 and 6 to the minimal surface

$\mathfrak{X}|_{D_2}$, we obtain $\lambda_1(D_2)>2$. This is a contradiction. Q.E.D.

REMARK 3. From the proof above, it is obvious that the assumption of the extensibility of $\mathfrak{X}$ across $\partial D$ can be replaced by the following weaker assumption $(*)$:

$(*)$ There exists some arc $\gamma\subset\partial D$, $\mathfrak{G}(\gamma)\subset\operatorname{Int}H$, such that $\mathfrak{X}$ can be extended across $\gamma$ as a minimal surface.

## §6. Case 3. $\mathfrak{G}$ maps $\bar{D}$ onto $H$ and $\mathfrak{G}(\partial D)=\partial H$.

Finally we consider the case in which $\mathfrak{G}(\bar{D})$ coincides with a hemisphere $H$ of $S$ and $\mathfrak{G}(\partial D)=\partial H$. By a suitable rotation of the surface in $\boldsymbol{R}^3$, we may assume that $H$ coincides with the lower hemisphere $H^-=\{(x^1, x^2, x^3)\in S\ ;\ x^3\leqq 0\}$. Then $g=P\circ\mathfrak{G}$ (see §3) maps $D$ onto the unit open disk $D_0=\{w\ ;\ |w|<1\}$.

LEMMA 7. *Suppose that $\mathfrak{G}(\bar{D})=H^-$ and that $\mathfrak{G}(\partial D)=\partial H^-$. Then $g|_D : D\to D_0$ is a finite-sheeted branched covering (in other words, $(D, D_0\ ;\ g)$ is a finite-sheeted unlimited covering surface), and the number of branch points of $g$ is finite.*

(For the notion of (global) branched covering, cf. Gunning [**3**, pp. 220-221].)

PROOF. Because $\mathfrak{X}$ can be extended as a minimal surface across $\partial D$, $g$ is holomorphic on $\bar{D}$. From this fact and the fact that $g(\partial D)=\partial D_0$, the lemma holds immediately.

LEMMA 8. *It holds that $\lambda_1(D)=2$. And the eigenspace corresponding to the least eigenvalue of the problem* (5) *is given by*

$$E_1=\{av_0\circ\mathfrak{G}\ ;\ v_0((x^1, x^2, x^3))\equiv x^3,\ a\in\boldsymbol{R}\}\,.$$

PROOF. Put $u_0=v_0\circ\mathfrak{G}$ with $v_0((x^1, x^2, x^3))\equiv x^3$. Then by (10) (§3)

$$u_0(\zeta)=\frac{|g(\zeta)|^2-1}{|g(\zeta)|^2+1}\,. \tag{16}$$

From some easy calculations, we derive

$$\Delta u_0=\frac{8|g'|^2(1-|g|^2)}{(|g|^2+1)^3}\,. \tag{17}$$

On the other hand, by virtue of Proposition 1 (§3), we observe

$$K=\frac{LN-M^2}{EG-F^2}=-\frac{L^2+M^2}{W^2}=-\frac{|fg'|^2}{W^2}\,. \tag{18}$$

From (9) (§3), (16), (17), and (18), we conclude that

$$\Delta u_0-2KWu_0=0\,.$$

Moreover, $u_0<0$ on $D$ and $u_0|_{\partial D}=0$. Therefore $u_0$ is one of the least eigenfunctions of the problem (5) (Lemma 1,(iv)), and the least eigenvalue $\lambda_1(D)$ equals 2. Since the eigenspace corresponding to $\lambda_1$ is 1-dimensional (Lemma 1,(v)), the proof is completed.

From Lemmas 2 and 8, we observe

LEMMA 9. *$I^{(2)}(u)\geqq 0$ for all $u$. The equality holds if and only if $u\in E_1$.*

In view of Remark 2 and the above lemma we shall investigate the third variation of area for $u\in E_1$.

LEMMA 10. *If $u$ belongs to $E_1$, then*

$$I^{(3)}(u)=\mathrm{Re}\iint_D \frac{384a^3(1-|g|^2)|g|^4|g'|^2}{(1+|g|^2)^7}\cdot\frac{g'}{fg^2}\,d\xi d\eta, \tag{19}$$

*where $a$ is a real constant determined by the choice of $u$, i.e. $u=av_0$ (cf. Lemma 8).*

PROOF. By virtue of (4) (§2), (9), and (11) (§3), we see

$$I^{(3)}(u)=\iint_D \frac{6u}{W}\mathrm{Re}\,\{(L-\sqrt{-1}\,M)(u_\xi+\sqrt{-1}\,u_\eta)^2\}\,d\xi d\eta$$

$$=\iint_D \frac{-24u}{|f|^2(1+|g|^2)^2}\cdot\mathrm{Re}\,\{fg'(u_\xi+\sqrt{-1}\,u_\eta)^2\}\,d\xi d\eta. \tag{20}$$

Since $u\in E_1$, $u$ can be written in the form:

$$u=a\cdot\frac{|g|^2-1}{|g|^2+1},\qquad a\in\boldsymbol{R}$$

((16) in the proof of Lemma 8). Therefore, by some calculation, we observe

$$(u_\xi+\sqrt{-1}\,u_\eta)^2=\frac{16a^2g^2\overline{g'}^2}{(|g|^2+1)^4}. \tag{21}$$

Using (20) and (21), we see

$$I^{(3)}(u)=\iint_D \frac{384a^3(1-|g|^2)}{|f|^2(1+|g|^2)^7}\cdot\mathrm{Re}\{fg^2g'\overline{g'^2}\}\,d\xi d\eta$$

$$=\iint_D \frac{384a^3(1-|g|^2)|g'|^2|g|^4}{(1+|g|^2)^7}\cdot\mathrm{Re}\left\{\frac{\overline{g'}}{\bar{f}\bar{g}^2}\right\}d\xi d\eta$$

$$=\iint_D \frac{384a^3(1-|g|^2)|g'|^2|g|^4}{(1+|g|^2)^7}\cdot\mathrm{Re}\left\{\frac{g'}{fg^2}\right\}d\xi d\eta.$$

Q.E.D.

In Lemma 7, we observed that $g: D\to D_0$ is a finite-sheeted branched covering. Therefore, for each point $\zeta\in D$, there exists some open neighborhood $U\subset D$ of $\zeta$, such that $g(U)$ is open in $D_0$ and the restriction of $g$ to $U-\{\zeta\}$ is a finite-sheeted covering of $g(U)-\{g(\zeta)\}$. The number of such sheets is called the multiplicity of the point $\zeta$ for $g$. (Of course, if $\zeta\in D$ is not a branch point of

$g$, namely if $g'(\zeta)\neq 0$, the multiplicity of $\zeta$ is one.)

Let us introduce a single-valued function $F$ in $D_0$ as

$$F(w)=\sum_{\{\zeta\in D;\ g(\zeta)=w\}}\frac{g'(\zeta)}{f(\zeta)}. \tag{22}$$

Then we have the following lemma which plays an important rôle in subsequent arguments.

LEMMA 11. *$F$ is holomorphic in $D_0$.*

PROOF. Note that $F(w)$ coincides with $\sum_{\{\zeta\in D;\ g(\zeta)=w\}}\nu(\zeta)\cdot g'(\zeta)/f(\zeta)$, where $\nu(\zeta)\geqq 1$ is the multiplicity of the point $\zeta$ for the function $g$. Indeed $\nu(\zeta)>1$ if and only if $g'(\zeta)=0$.

Denote the branch points of $g$ by $\zeta_1, \cdots, \zeta_m$ (Lemma 7). Set $D_1=D-\{\zeta_1, \cdots, \zeta_m\}$ and $D_2=D_0-\{g(\zeta_1), \cdots, g(\zeta_m)\}$.

Let $q\in D_0$, let $g^{-1}(q)=\{p_1, \cdots, p_n\}$ ($p_j\neq p_k$, if $j\neq k$), and let $\nu_j$ be the multiplicity of $p_j$, $j=1, \cdots, n$. Then there exists a neighborhood $V\subset D_2\cup\{q\}$ of $q$ such that the set formed by the connected components of $g^{-1}(V)$ is the disjoint union of neighborhoods $U_1, \cdots, U_n\subset D$ of the points $p_1, \cdots, p_n$, respectively, and the restriction of $g$ to $U_j-\{p_j\}$ is a $\nu_j$-sheeted covering of $V-\{q\}$, $\nu_j\geqq 1$. Define functions $F_j$ in $V$ as follows:

$$F_j(q)=\nu_j\cdot\frac{g'(p_j)}{f(p_j)},$$

and if $\tilde{q}\in V-\{q\}$,

$$F_j(\tilde{q})=\sum_{k=1}^{\nu_j}\frac{g'(\tilde{p}^j_k)}{f(\tilde{p}^j_k)},$$

where $g^{-1}(\tilde{q})\cap U_j=\{\tilde{p}^j_1, \cdots, \tilde{p}^j_{\nu_j}\}$. Then the restriction $F|_V=\sum_{j=1}^{n}F_j$.

When $q\in D_2$, then each restriction $g_j=g|_{U_j}$ is injective and has an inverse $g_j^{-1}: V\to U_j$ which is holomorphic in $V$. Moreover, $F_j=(g'/f)\circ g_j^{-1}$ is holomorphic in $V$. Therefore $F=\sum_{j=1}^{n}F_j$ is holomorphic in $V$. Because of the arbitrariness of $q$, $F$ is holomorphic in $D_2$. In particular, $F$ is continuous in $D_2$.

Now let us prove the continuity of $F$ at $q=g(\zeta_j)$, $j=1, \cdots, m$. We may assume $V$ to be a small disk around $q$. Let $l$ be a segment joining $q$ to $\partial V$. Then $g$ has inverses:

$$g_{jk}^{-1}: V-l\longrightarrow U_j$$

which are holomorphic in $V-l$, and

$$F_j|_{V-l}=\sum_{k=1}^{\nu_j}\frac{g'}{f}\circ g_{jk}^{-1}.$$

So each $F_j$ is continuous on $V-l$. Since $l$ is arbitrary, $F_j$ is continuous in $V-\{q\}$. But, for any choice of $l$, we have that

$$\lim_{\tilde{q}\to q}\sum_{k=1}^{\nu_j}\frac{g'}{f}\circ g_{jk}^{-1}(\tilde{q})=\nu_j\cdot\frac{g'(p_j)}{f(p_j)}$$

Consequently, $F_j$ is continuous in $V$, hence $F$ is continuous in $V$.

Now we have proved that (i) $F$ is holomorphic in $D_2=D_0-\{g(\zeta_1), \cdots, g(\zeta_m)\}$ and that (ii) $F$ is continuous up to $D_0$. Therefore, $g(\zeta_1), \cdots, g(\zeta_m)$ are removable singularities of $F|_{D_2}$, hence $F$ is holomorphic in the whole $D_0$.

REMARK 4. $F$ remains invariant under any parameter change of the "minimal surface" $\mathfrak{X}$. Namely, let $\tau: \tilde{D}\to D$ be a conformal mapping from some domain $\tilde{D}$ in the plane onto $D$, then $\tilde{F}$ which is constructed from the minimal surface $\tilde{\mathfrak{X}}=\mathfrak{X}\circ\tau$ by (22) coincides exactly with $F$.

PROOF. Let $\tilde{f}$, $\tilde{g}$ be the W-factors of $\tilde{\mathfrak{X}}=(\tilde{\mathfrak{X}}^1, \tilde{\mathfrak{X}}^2, \tilde{\mathfrak{X}}^3)$. And set

$$\tilde{\phi}_j(z)=2\cdot\frac{\partial\tilde{\mathfrak{X}}^j(z)}{\partial z}, \qquad j=1, 2, 3.$$

Then

$$\tilde{\phi}_j=2\cdot\frac{\partial\mathfrak{X}^j}{\partial\zeta}\circ\tau\cdot\tau'.$$

Therefore, by definition (see (6) and (7) in § 3),

$$\tilde{f}=\tilde{\phi}_1-\sqrt{-1}\,\tilde{\phi}_2=2\left(\frac{\partial\mathfrak{X}^1}{\partial\zeta}-\sqrt{-1}\,\frac{\partial\mathfrak{X}^2}{\partial\zeta}\right)\circ\tau\cdot\tau'=f\circ\tau\cdot\tau',$$

$$\tilde{g}=\frac{\tilde{\phi}_3}{\tilde{\phi}_1-\sqrt{-1}\,\tilde{\phi}_2}=\frac{2\cdot\dfrac{\partial\mathfrak{X}^3}{\partial\zeta}\circ\tau\cdot\tau'}{2\left(\dfrac{\partial\mathfrak{X}^1}{\partial\zeta}-\sqrt{-1}\,\dfrac{\partial\mathfrak{X}^2}{\partial\zeta}\right)\circ\tau\cdot\tau'}=g\circ\tau.$$

Thus we obtain

$$\begin{aligned}\tilde{F}(w)&=\sum_{\{z\in\tilde{D};\ \tilde{g}(z)=w\}}\frac{\tilde{g}'(z)}{\tilde{f}(z)}\\&=\sum_{\{\zeta\in D;\ g(\zeta)=w\}}\frac{\tilde{g}'(\tau^{-1}(\zeta))}{\tilde{f}(\tau^{-1}(\zeta))}\\&=\sum_{\{\zeta\in D;\ g(\zeta)=w\}}\frac{g'(\zeta)\cdot\tau'(\tau^{-1}(\zeta))}{f(\zeta)\cdot\tau'(\tau^{-1}(\zeta))}\\&=F(w).\end{aligned}$$

Q.E.D.

PROPOSITION 2. *If $u$ belongs to $E_1$, then*

$$I^{(3)}(u)=\pi a^3\cdot\mathrm{Re}\{F''(0)\}, \tag{23}$$

*where a is a real constant determined by the choice of u, i.e.* $u=av_0$ *(cf. Lemma 8).*

PROOF. Let us first prove that

$$I^{(3)}(u)=\mathrm{Re}\iint_{D_0}\frac{384a^3(1-|w|^2)|w|^4}{(1+|w|^2)^7}\cdot\frac{F(w)}{w^2}dx^1dx^2, \tag{24}$$

where $w=x^1+\sqrt{-1}\,x^2$. Set

$$A(\zeta)=\frac{384a^3(1-|g(\zeta)|^2)|g(\zeta)|^4}{(1+|g(\zeta)|^2)^7}\cdot\frac{g'(\zeta)}{f(\zeta)\{g(\zeta)\}^2},$$

then

$$I^{(3)}(u)=\mathrm{Re}\iint_D A(\zeta)|g'(\zeta)|^2 d\xi d\eta$$

by virtue of Lemma 10.

Let $D_1$ and $D_2$ be what were defined in the proof of Lemma 11. Let $q\in D_2$ and let $g^{-1}(q)=\{p_1,\cdots,p_n\}$. Then similarly to the proof of Lemma 11, there exists a neighborhood $V\subset D_2$ of $q$ such that $g^{-1}(V)$ is the disjoint union of neighborhoods $U_1,\cdots,U_n\subset D_1$ of the points $p_1,\cdots,p_n$, respectively, and each restriction $g|_{U_j}$ is injective and has a holomorphic inverse $g_j^{-1}: V\to U_j$, $j=1,\cdots,n$. Therefore

$$\sum_{j=1}^n\iint_{U_j}A(\zeta)|g'(\zeta)|^2d\xi d\eta=\sum_{j=1}^n\iint_V A\circ g_j^{-1}(w)dx^1dx^2$$

$$=\iint_V\frac{384a^3(1-|w|^2)|w|^4}{(1+|w|^2)^7}\cdot\frac{F(w)}{w^2}dx^1dx^2.$$

Consequently

$$\iint_{D_1}A(\zeta)|g'(\zeta)|^2d\xi d\eta=\iint_{D_2}\frac{384a^3(1-|w|^2)|w|^4}{(1+|w|^2)^7}\cdot\frac{F(w)}{w^2}dx^1dx^2. \tag{25}$$

Now we assert that each integrand of the both sides of (25) is bounded. In fact, because of the regularity and the extensibility of $\mathfrak{X}$, $1/f$ and $g'$ are both bounded on $\bar{D}$ (cf. § 3). Moreover, not only $D-D_1$, but also $D_0-D_2$ is of measure zero. Therefore (25) implies (24).

Let us introduce the polar coordinates $(r,\theta)$ in $w$-plane: $w=x^1+\sqrt{-1}x^2=re^{\sqrt{-1}\theta}$. By using (24) and the residue theorem, we obtain

$$I^{(3)}(u)=\mathrm{Re}\int_{r=0}^1\frac{384a^3(1-r^2)r^5}{(1+r^2)^7}\cdot\frac{1}{\sqrt{-1}}\left(\int_{\theta=0}^{2\pi}\frac{F(re^{\sqrt{-1}\theta})}{(re^{\sqrt{-1}\theta})^3}\cdot\sqrt{-1}re^{\sqrt{-1}\theta}d\theta\right)dr$$

$$=\mathrm{Re}\int_{r=0}^1\frac{384a^3(1-r^2)r^5}{(1+r^2)^7}\cdot\frac{1}{\sqrt{-1}}\left(\int_{|w|=r}\frac{F(w)}{w^3}dw\right)dr$$

$$=\mathrm{Re}\int_{r=0}^1\frac{384a^3(1-r^2)r^5}{(1+r^2)^7}\cdot\frac{1}{\sqrt{-1}}\cdot 2\pi\sqrt{-1}\cdot\operatorname*{Res}_{w=0}\left\{\frac{F(w)}{w^3}\right\}dr$$

$$=\pi a^3\cdot \mathrm{Re}\,\{F''(0)\}\,,$$

which was to be proved.

Therefore, by taking account of Remark 2 (§2) and Lemma 9, we obtain

MAIN THEOREM. *Assume that the Gauss map $\mathfrak{G}$ of a regular minimal surface $\mathfrak{X}:\bar{D}\to\boldsymbol{R}^3$ is a mapping from $\bar{D}$ onto the lower hemisphere $H^-$ of $S$ and that $\mathfrak{G}$ maps $\partial D$ onto $\partial H^-$. If*

$$\mathrm{Re}\,\{F''(0)\}\neq 0\,, \tag{26}$$

*then $\mathfrak{X}$ is unstable, where $F$ is the holomorphic function defined by* (22).

Especially, for the case in which $\mathfrak{G}$ is injective, we get

COROLLARY 1. *Assume that the Gauss map $\mathfrak{G}$ of a regular minimal surface $\mathfrak{X}:\bar{D}\to\boldsymbol{R}^3$ is injective and maps $\bar{D}$ onto the lower hemisphere of $S$. Let $f$, $g$ be the first and the second W-factor of $\mathfrak{X}$, respectively. If*

$$\mathrm{Re}\left\{\left(\frac{g'}{f}\circ g^{-1}\right)''(0)\right\}\neq 0\,,$$

*then $\mathfrak{X}$ is unstable.*

When $\mathfrak{G}$ is not injective, it is difficult in general to express the left hand side of (26) using $f$ and $g$ explicitly. However, whenever $g^{-1}(0)$ contains no singularity of $g$, (that is, $\mathfrak{G}^{-1}((0, 0, -1))$ contains no singularity of $\mathfrak{G}$,) we obtain

COROLLARY 2. *Assume that the Gauss map $\mathfrak{G}$ of a regular minimal surface $\mathfrak{X}:\bar{D}\to\boldsymbol{R}^3$ is a mapping from $\bar{D}$ onto the lower hemisphere $H^-$ of $S$ and maps $\partial D$ onto $\partial H^-$, and moreover that $\mathfrak{G}^{-1}((0, 0, -1))$ contains no singularity of $\mathfrak{G}$. Let $f$, $g$ be the W-factors of $\mathfrak{X}$. If*

$$\mathrm{Re}\sum_{\zeta\in g^{-1}(0)}\left[\frac{2g'''(\zeta)}{g'(\zeta)}\cdot\frac{1}{f(\zeta)g'(\zeta)}+\frac{3g''(\zeta)}{g'(\zeta)}\left(\frac{1}{f(\zeta)g'(\zeta)}\right)'+\left(\frac{1}{f(\zeta)g'(\zeta)}\right)''\right]\neq 0\,,$$

*then $\mathfrak{X}$ is unstable.*

Of course, the last result implies Corollary 1.

PROOF. Let us recall the proof of the holomorphy of $F$ (Lemma 11). Let $g^{-1}(0)=\{p_1, \cdots, p_n\}$. Then each $p_j$ is not a singularity of $g$ by assumption. Therefore there exist a neighborhood $V$ of $w=0$ and neighborhoods $U_j$ of $p_j$ such that each $g_j=g|_{U_j}$ has a holomorphic inverse $g_j^{-1}:V\to U_j$. Moreover

$$F|_V=\sum_{j=1}^{n}\frac{g'}{f}\circ g_j^{-1}\,.$$

Set $F_j=\dfrac{g'}{f}\circ g_j^{-1}$. Then

$$F|_V=\sum_{j=1}^{n} F_j. \tag{27}$$

Obviously,

$$\frac{d}{dw}g_j^{-1}(w)=\left[\frac{1}{g_j'(\zeta)}\right]_{\zeta=g_j^{-1}(w)}=\left[\frac{1}{g'(\zeta)}\right]_{\zeta=g_j^{-1}(w)}.$$

Therefore

$$F_j'(w)=\left[2g''\cdot\frac{1}{fg'}+g'\cdot\left(\frac{1}{fg'}\right)'\right]_{\zeta=g_j^{-1}(w)},$$

$$F_j''(w)=\left[\frac{2g'''}{g'}\cdot\frac{1}{fg'}+\frac{3g''}{g'}\cdot\left(\frac{1}{fg'}\right)'+\left(\frac{1}{fg'}\right)''\right]_{\zeta=g_j^{-1}(w)}.$$

By using this equation and (27), we obtain the desired result by virtue of Main Theorem.

Finally we give an interesting example of the case that $\mathfrak{G}^{-1}((0, 0, -1))$ contains singularities of $\mathfrak{G}$.

EXAMPLE. Suppose that $D=D_0$ and that $g(\zeta)\equiv\zeta^n$ $(n\geqq 1)$. Then, for $u\in E_1$,

$$I^{(3)}(u)=\frac{2\pi a^3 n^2}{(n+1)!}\cdot\mathrm{Re}\left\{\left[\frac{d^{n+1}}{d\zeta^{n+1}}\frac{1}{f(\zeta)}\right]_{\zeta=0}\right\}. \tag{28}$$

Therefore, if $\mathrm{Re}\{[(d^{n+1}/d\zeta^{n+1})(1/f(\zeta))]_{\zeta=0}\}\neq 0$, $\mathfrak{X}$ is unstable.

PROOF. Let $w_0$ be an arbitrary point of $D_0-\{0\}$, and let $g^{-1}(w_0)=\{\zeta_0, \cdots, \zeta_{n-1}\}$. Then there exists some neighborhood $V\subset D_0-\{0\}$ of $w_0$ such that $g^{-1}(V)$ is the disjoint union of neighborhoods $U_0, \cdots, U_{n-1}\subset D_0-\{0\}$ of the points $\zeta_0, \cdots, \zeta_{n-1}$, respectively, and each restriction $g_j=g|_{U_j}$ is injective and has a holomorphic inverse $\phi_j: V\to U_j$. Moreover we may assume that $\zeta_j=e^{2\pi\sqrt{-1}j/n}\zeta_0$ and that $\phi_j=e^{2\pi\sqrt{-1}j/n}\phi_0$, $j=0, 1, \cdots, n-1$. Since

$$\phi_0'(w)=\frac{1}{g'(\phi_0(w))}=\frac{1}{n(\phi_0(w))^{n-1}},$$

we see

$$\phi_j'(w)=e^{2\pi\sqrt{-1}j/n}\phi_0'(w)=\frac{e^{2\pi\sqrt{-1}j/n}}{n(\phi_0(w))^{n-1}}.$$

Define a holomorphic function $h$ in $D_0$ as $h=1/f$. In $V$ we observe

$$F(w)=\sum_{j=0}^{n-1} g'(\phi_j(w))h(\phi_j(w)). \tag{29}$$

Now we introduce the Taylor expansion of $h$ around $\zeta=0$:

$$h(\zeta)=\sum_{k=0}^{\infty}\alpha_k\zeta^k.$$

Then

$$h(\psi_j(w))=\sum_{k=0}^{\infty}\alpha_k(\psi_j(w))^k=\sum_{k=0}^{\infty}\alpha_k e^{2\pi\sqrt{-1}jk/n}(\psi_0(w))^k. \tag{30}$$

On the other hand,

$$\begin{aligned} g'(\psi_j(w)) &= n(\psi_j(w))^{n-1}=ne^{2\pi\sqrt{-1}j(n-1)/n}(\psi_0(w))^{n-1} \\ &= ne^{-2\pi\sqrt{-1}j/n}(\psi_0(w))^{n-1}. \end{aligned} \tag{31}$$

From (29), (30), and (31), we obtain

$$\begin{aligned} F(w) &= \sum_{j=0}^{n-1}\sum_{k=0}^{\infty} n\alpha_k e^{2\pi\sqrt{-1}j(k-1)/n}(\psi_0(w))^{n+k-1} \\ &= nw\sum_{k=0}^{\infty}\alpha_k(\psi_0(w))^{k-1}\sum_{j=0}^{n-1}(e^{2\pi\sqrt{-1}(k-1)/n})^j. \end{aligned}$$

Set $k-1=pn+q$ ($p$, $q\in\boldsymbol{Z}$, $0\leqq q\leqq n-1$). We claim

$$\begin{aligned} \sum_{j=0}^{n-1}(e^{2\pi\sqrt{-1}(k-1)/n})^j &= \sum_{j=0}^{n-1}(e^{2\pi\sqrt{-1}q/n})^j \\ &= \begin{cases} n & \text{for } q=0. \\ 0 & \text{for } q\neq 0. \end{cases} \end{aligned}$$

Therefore

$$\begin{aligned} F(w) &= n^2 w\sum_{k=pn+1}\alpha_k(\psi_0(w))^{k-1} \\ &= n^2 w\sum_{p=0}^{\infty}\alpha_{pn+1}(\psi_0(w))^{pn} \\ &= n^2 w\sum_{p=0}^{\infty}\alpha_{pn+1}w^p=\sum_{p=0}^{\infty}n^2\alpha_{pn+1}w^{p+1} \end{aligned}$$

$$= \sum_{l=1}^{\infty}n^2\alpha_{(l-1)n+1}w^l \tag{32}$$

in $V$. Since $w_0$ is an arbitrary point in $D_0-\{0\}$ and $F$ is holomorphic in $D_0$, (32) is valid in the whole $D_0$. Thus, in $D_0$,

$$F'(w)=\sum_{l=1}^{\infty}ln^2\alpha_{(l-1)n+1}w^{l-1}.$$

Consequently,

$$F''(w)=\sum_{l=2}^{\infty}l(l-1)n^2\alpha_{(l-1)n+1}w^{l-2}$$

and

$$F''(0)=2n^2\alpha_{n+1}=2n^2\cdot\frac{h^{(n+1)}(0)}{(n+1)!}.$$

Therefore, by using Proposition 2, we obtain

$$I^{(3)}(u)=\frac{2\pi a^3 n^2}{(n+1)!}\cdot \mathrm{Re}\,\{h^{(n+1)}(0)\}\,,$$

which means (28).

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