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On homogeneous Kähler manifolds of solvable Lie groups

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Introduction

Let M be a connected homogeneous complex manifold on which a connected Lie group G acts transitively as a group of holomorphic transformations. We assume that M admits a G -invariant volume element v . If v has an expression

$$v = i^n F(z, \bar{z}) dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$$

in a local coordinate system $\{z_1, \dots, z_n\}$, then the G -invariant hermitian form

$$h = \sum_{i,j} \frac{\partial^2 \log F(z, \bar{z})}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

is called the canonical hermitian form of M . If M carries a G -invariant Kähler metric and if v is the volume element determined by this metric, the Ricci tensor of the Kähler manifold is equal to $-h$. From now on, M is assumed to be a homogeneous Kähler manifold unless otherwise specified. The canonical hermitian form h plays an important role in the investigation of homogeneous Kähler manifolds, and results in this direction are the following:

(i) If G is a semi-simple Lie group, then h is non-degenerate and the number of negative squares of h is equal to the difference between the dimension of a maximal compact subgroup of G and the dimension of the isotropy subgroup of G at a point of M [8].

(ii) If G is a unimodular Lie group and if h is non-degenerate, then G is a semi-simple Lie group [2].

(iii) h is negative definite if and only if G is a compact semi-simple Lie group [8], [11].

In [13], E.B. Vinberg, S.G. Gindikin, I.I. Pjateckii-Šapiro studied the structure of J -algebras. The J -algebra of a homogeneous bounded domain is proper in their sense. They proved the following:

(iv) Every proper J -algebra is isomorphic to the J -algebra of a homogeneous Siegel domain of the second kind. Since this domain is holomor-

phically isomorphic to a homogeneous bounded domain, the canonical hermitian form h of a proper J -algebra is positive definite. Moreover there exists a solvable Lie group which acts simply transitively on the homogeneous bounded domain.

Further they developed the theory of Kähler algebras and they proved [1], [14].

(v) If a connected simply connected Kähler manifold M admits a simply transitive solvable and splittable Lie group G , then M is a holomorphic fibre bundle whose base space is a homogeneous bounded domain and whose fibre is a locally flat homogeneous Kähler manifold. (A Lie group G is said to be splittable if the adjoint operator $\text{ad}(X)$ has only real eigenvalues for any element X in the Lie algebra of G .)

We assume that the canonical hermitian form of M is positive definite and that M admits a transitive solvable Lie group. Then the corresponding Kähler algebra of M is a proper J -algebra and hence the universal covering manifold of M is holomorphically isomorphic to a homogeneous bounded domain.

Now, we shall denote by $\hat{G}$ the identity component of the group of all holomorphic transformations of M leaving h invariant. If h is non-degenerate, the group $\hat{G}$ is a Lie group acting on M as a Lie transformation group. In [3], Hano proved

(vi) Let M be a homogeneous complex (not necessarily Kähler) manifold with non-degenerate canonical hermitian form. Then the adjoint group $\text{Ad}_{\hat{G}}(\hat{G})$ of $\hat{G}$ is the identity component of a real algebraic group in $GL(\hat{\mathfrak{g}}, \mathbf{R})$, where $\hat{\mathfrak{g}}$ is the Lie algebra of $\hat{G}$.

In the present paper, we show that the positive definiteness of h follows from its non-degeneracy provided that a (not necessarily splittable) solvable Lie group acts on M simply transitively. Precisely speaking, we prove the following theorem:

THEOREM. *Let M be a connected simply connected homogeneous Kähler manifold with non-degenerate canonical hermitian form h . If M admits a simply transitive solvable Lie group, then the canonical hermitian form h is positive definite and hence M is holomorphically isomorphic to a homogeneous bounded domain.*

We denote by $I^0(M)$ the identity component of the group of all isometries of M . If the canonical hermitian form is non-degenerate, then $I^0(M) \subset \hat{G}$ [6], and the center of a transitive subgroup of $I^0(M)$ is discrete. Moreover, if M is a homogeneous bounded domain, then we have $\hat{G} = I^0(M)$ [4] and the isotropy subgroup of $\hat{G}$ at a point $o \in M$ is a maximal compact subgroup of $\hat{G}$, and the center of the group $\hat{G}$ is reduced to the identity [5].

As immediate applications of our theorem and (vi), we have

COROLLARY 1. *Let M be a connected homogeneous Kähler manifold with non-degenerate canonical hermitian form. We assume that M admits a transitive Lie group G whose isotropy subgroup K at a point o of M is a maximal compact subgroup of G and that the group $\hat{G}$ coincides with $I^0(M)$ and of finite center. Then M is holomorphically isomorphic to a homogeneous bounded domain.*

COROLLARY 2. *Let M be a connected simply connected homogeneous Kähler manifold with non-degenerate canonical hermitian form. We assume that a point $o \in M$ has no conjugate point and that the group $\hat{G}$ coincides with $I^0(M)$ and of finite center. Then M is holomorphically isomorphic to a homogeneous bounded domain.*

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§ 1. Preliminaries

A. A $2m$ -dimensional real vector space V is called a symplectic space if there exists a skew symmetric bilinear form σ on V and a linear endomorphism J of V satisfying the following conditions; for $u, v \in V$

$$\begin{aligned} J^2 u &= -u, \\ \sigma(Ju, Jv) &= \sigma(u, v), \\ \sigma(Ju, u) &> 0, \quad u \neq 0. \end{aligned}$$

In this case V may be regarded as a complex vector space with the complex structure J , which we shall denote by $\tilde{V}$. Moreover,

$$(u, v) = \sigma(Ju, v) + i\sigma(u, v) \quad (u, v \in \tilde{V})$$

is a positive definite hermitian form on this complex vector space $\tilde{V}$.

For a real linear transformation f of V , we put

$$\begin{aligned} f^+(u) &= (1/2)(f(u) - Jf(Ju)), \\ f^-(u) &= (1/2)(f(u) + Jf(Ju)), \quad u \in V. \end{aligned}$$

Then, we have $f = f^+ + f^-$, $f^+ J = J f^+$, and $f^- J = -J f^-$. Let $u_1, u_2, \dots, u_m$ be an orthonormal basis of $\tilde{V}$ with respect to the hermitian form (u, v) . Put

$$\begin{aligned} f^+(u_j) &= \sum_{k=1}^m a_{kj} u_k, \quad a_{kj} \in \mathbb{C}, \\ f^-(u_j) &= \sum_{k=1}^m b_{kj} u_k, \quad b_{kj} \in \mathbb{C}. \end{aligned}$$

Identifying $\tilde{V}$ and C^m by means of the map $u = \sum_{j=1}^m z_j u_j \rightarrow z = {}^t(z_1, z_2, \dots, z_m)$, f may be considered as a map $z \rightarrow Az + B\bar{z}$, where $A = (a_{kj})$, $B = (b_{kj})$ and $\bar{z} = {}^t(\bar{z}_1, \dots, \bar{z}_m)$. We denote this map f by $f = (A, B)$. If we have $f = (A', B')$ with respect to another orthonormal basis of $\tilde{V}$, then there exists a unitary matrix U such that

$$A' = UA^t\bar{U}, \quad B' = UB^tU.$$

A real linear transformation f of V is said to be symplectic if

$$\sigma(f(u), v) + \sigma(u, f(v)) = 0$$

for $u, v \in V$. It is easy to see that $f = (A, B)$ is symplectic if and only if A is a skew-hermitian matrix and B is a symmetric matrix. By a simple calculation, we see that if f is symplectic, then $\text{Tr} f = 0$.

Now, let f be a real linear transformation of V which commutes with J . For real numbers α, β , we put

$$V_{(\alpha+i\beta)} = \{u \in V; (f - (\alpha + \beta J))^m u = 0 \text{ for some } m\}$$

and

$$V_{[\alpha]} = \sum_{\beta} V_{(\alpha+i\beta)}.$$

Then, we have $V = \sum_{\alpha} V_{[\alpha]}$ and $V_{[\alpha]}$ is the largest subspace of V on which the real parts of the eigenvalues of f are equal to α .

B. We denote by M a connected Kähler manifold on which a Lie group G acts simply transitively as a group of holomorphic isometries. Let (I, g) be the Kähler structure on M , i.e. I is a G -invariant complex structure tensor on M and g is a G -invariant Kähler metric on M .

Let $\mathfrak{g}$ be the Lie algebra of all left invariant vector fields on G , and let π be the canonical projection from G onto M defined by $\pi(a) = a \cdot o$, for $a \in G$, where o is a fixed point of M . Let π_e denote the differential of π at the identity e of G , and let X_e, I_o and g_o be the values of $X \in \mathfrak{g}$, I and g at e and o respectively. Then there exist a linear endomorphism J of $\mathfrak{g}$ and a skew symmetric bilinear form ρ on $\mathfrak{g}$ such that

$$\pi_e(JX)_e = I_o(\pi_e X_e),$$

$$\rho(X, Y) = g_o(\pi_e X_e, I_o \pi_e Y_e),$$

for $X, Y \in \mathfrak{g}$. Then $(\mathfrak{g}, J, \rho)$ satisfies the following properties ([1], [8]):

$$(K.1) \quad J^2 X = -X;$$

$$(K.2) \quad [JX, JY] = J[JX, Y] + J[X, JY] + [X, Y];$$

$$(K.3) \quad \rho(JX, JY) = \rho(X, Y);$$

$$(K.4) \quad \rho(JX, X) > 0, \quad X \neq 0;$$

$$(K.5) \quad \rho([X, Y], Z) + \rho([Y, Z], X) + \rho([Z, X], Y) = 0;$$

where $X, Y, Z \in \mathfrak{g}$.

$(\mathfrak{g}, J, \rho)$ will be called the normal Kähler algebra of M .

It is known that the canonical hermitian form h of a homogeneous Kähler manifold M has the following expression due to J.L. Koszul [8].

Putting

$$(1.1) \quad \eta(X, Y) = h_0(\pi_e X_e, \pi_e Y_e),$$

and

$$(1.2) \quad \phi(X) = \text{Tr}_{\mathfrak{g}}(\text{ad}(JX) - J \text{ad}(X)),$$

we have

$$(1.3) \quad \eta(X, Y) = (1/2)\phi([JX, Y]),$$

for $X, Y \in \mathfrak{g}$. The form η satisfies the following properties:

$$(1.4) \quad \eta(X, Y) = \eta(Y, X),$$

$$(1.5) \quad \eta(JX, JY) = \eta(X, Y),$$

for $X, Y \in \mathfrak{g}$.

Now, the following lemma is due to [1].

LEMMA 1. For $E, X, Y \in \mathfrak{g}$,

$$\begin{aligned} & \frac{d}{dt} \rho(\exp t \text{ad}(JE)X, \exp t \text{ad}(JE)Y) \\ &= \rho(JE, \exp t \text{ad}(JE)[X, Y]). \end{aligned}$$

PROOF. By the property (K.5) of the Kähler algebra,

$$\begin{aligned} & \frac{d}{dt} \rho(\exp t \text{ad}(JE)X, \exp t \text{ad}(JE)Y) \\ &= \rho([JE, \exp t \text{ad}(JE)X], \exp t \text{ad}(JE)Y) \\ & \quad + \rho(\exp t \text{ad}(JE)X, [JE, \exp t \text{ad}(JE)Y]) \\ &= \rho(JE, [\exp t \text{ad}(JE)X, \exp t \text{ad}(JE)Y]) \\ &= \rho(JE, \exp t \text{ad}(JE)[X, Y]). \end{aligned} \quad \text{Q. E. D.}$$

Now, let $\mathfrak{g}$ be a real Lie algebra and let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$. Suppose that there exist a linear endomorphism J of $\mathfrak{g}$ and a 1-form ω on $\mathfrak{g}$, which satisfy the following conditions:

$$(J.1) \quad J^2 \subset \mathfrak{k}, \quad J^2 X \equiv -X \pmod{\mathfrak{k}};$$

$$(J.2) \quad [W, JX] \equiv J[W, X] \pmod{\mathfrak{k}};$$

$$(J.3) \quad [JX, JY] \equiv J[JX, Y] + J[X, JY] + [X, Y] \pmod{\mathfrak{f}};$$

$$(J.4) \quad \omega([W, X]) = 0;$$

$$(J.5) \quad \omega([JX, JY]) = \omega([X, Y]);$$

$$(J.6) \quad \omega([JX, X]) > 0, \quad X \in \mathfrak{f};$$

where $X, Y \in \mathfrak{g}$, $W \in \mathfrak{f}$.

Then, $(\mathfrak{g}, \mathfrak{f}, J, \omega)$ will be called a J -algebra.

Let $(\mathfrak{g}, \mathfrak{f}, J, \omega)$ be a J -algebra and let $\mathfrak{g}'$ be a subalgebra such that

$$J\mathfrak{g}' \subset \mathfrak{g}' + \mathfrak{f}.$$

We can then define a linear endomorphism J' of $\mathfrak{g}'$ so that $JX' \equiv J'X' \pmod{\mathfrak{f}}$, for $X' \in \mathfrak{g}'$. We define a 1-form ω' on $\mathfrak{g}'$ as the restriction of ω on $\mathfrak{g}'$ and we put $\mathfrak{f}' = \mathfrak{f} \cap \mathfrak{g}'$. It is easy to see that $(\mathfrak{g}', \mathfrak{f}', J', \omega')$ is a J -algebra. It is called a J -subalgebra of $(\mathfrak{g}, \mathfrak{f}, J, \omega)$.

A J -algebra $(\mathfrak{g}, \mathfrak{f}, J, \omega)$ is said to be proper, if it satisfies the following condition (P):

(P) Every compact semi-simple J -subalgebra of $(\mathfrak{g}, \mathfrak{f}, J, \omega)$ is contained in $\mathfrak{f}$.

A J -algebra $(\mathfrak{g}, \mathfrak{f}, J, \omega)$ will be called normal if $\mathfrak{f} = 0$, and we denote it by $(\mathfrak{g}, J, \omega)$.

We know then the following theorem (cf. Introduction (iv)):

Let $(\mathfrak{g}, J, \omega)$ be a normal J -algebra and let $\mathfrak{g}$ be a solvable Lie algebra. Then this J -algebra is proper, and is isomorphic to the J -algebra of a homogeneous bounded domain.

§ 2. Statement of Theorem

In this section, we shall state our theorem and sketch the proof.

THEOREM. *Let M be a connected Kähler manifold on which a connected solvable Lie group G acts simply transitively as a group of holomorphic isometries. Let $(\mathfrak{g}, J, \rho)$ be the normal Kähler algebra of M . If the canonical hermitian form h of M is non-degenerate, then we get the decomposition*

$$(2.1) \quad \mathfrak{g} = \sum_{k=1}^m \mathfrak{g}_k$$

of $\mathfrak{g}$ into direct sum of vector spaces with the following properties:

1) $\mathfrak{g}_k$ is a J -invariant subalgebra in which there exist an element $E_k \in \mathfrak{g}_k$ and a subspace $\mathfrak{p}_k$ with the following properties:

$$(2.2) \quad \mathfrak{g}_k = \{JE_k\} + \{E_k\} + \mathfrak{p}_k;$$

$$(2.3) \quad J\mathfrak{p}_k \subset \mathfrak{p}_k;$$

$$(2.4) \quad [JE_k, E_k] = E_k;$$

$$(2.5) \quad [JE_k, \mathfrak{p}_k] \subset \mathfrak{p}_k.$$

Moreover the real parts of the eigenvalues of $\text{ad}(JE_k)$ on $\mathfrak{p}_k$ are equal to $1/2$, and

$$(2.6) \quad [E_k, \mathfrak{p}_k] = \{0\};$$

$$(2.7) \quad [\mathfrak{p}_k, \mathfrak{p}_k] \subset \{E_k\}.$$

2) Put

$$(2.8) \quad \mathfrak{g}^{k+1} = \mathfrak{g}_{k+1} + \mathfrak{g}_{k+2} + \cdots + \mathfrak{g}_m,$$

then, we have

$$(2.9) \quad [JE_k, \mathfrak{g}^{k+1}] \subset \mathfrak{g}^{k+1}.$$

Moreover the real parts of the eigenvalues of $\text{ad}(JE_k)$ on $\mathfrak{g}^{k+1}$ are equal to 0, and

$$(2.10) \quad [E_k, \mathfrak{g}^{k+1}] = \{0\};$$

$$(2.11) \quad [\mathfrak{p}_k, \mathfrak{g}^{k+1}] \subset \mathfrak{p}_k.$$

3) The form η defined by (1.2) and (1.3) is positive definite on $\mathfrak{g}$ and the factors of the decomposition $\mathfrak{g} = \sum_{k=1}^m (\{JE_k\} + \{E_k\} + \mathfrak{p}_k)$ are mutually orthogonal with respect to this form η .

Under the same assumption of Theorem, $(\mathfrak{g}, J, \phi)$ becomes a solvable normal J -algebra, since η is positive definite on $\mathfrak{g}$ (cf. § 1. B). Therefore it is isomorphic to a J -algebra of a homogeneous bounded domain (cf. Introduction (iv) and § 1. B). Hence we have our theorem in the introduction.

As for the proof of Theorem, put $\mathfrak{g}^1 = \mathfrak{g}$. We shall show by induction on n that there exists a decomposition of $\mathfrak{g}$

$$(2.12) \quad \mathfrak{g} = \sum_{k=1}^{n-1} \mathfrak{g}_k + \mathfrak{g}^n$$

with the following properties:

1) For each $k = 1, 2, \dots, n-1$, there exist an element $E_k \in \mathfrak{g}_k$ and a subspace $\mathfrak{p}_k \subset \mathfrak{g}_k$ with the properties as stated in Theorem 1).

2) Put $\mathfrak{g}^{k+1} = \sum_{l=k+1}^{n-1} \mathfrak{g}_l + \mathfrak{g}^n$. Then $\mathfrak{g}^{k+1}$ has the properties as stated in Theorem 2).

3) The form η which is non-degenerate on $\mathfrak{g}$ is positive definite on $\sum_{k=1}^{n-1} \mathfrak{g}_k$ and the factors of the decomposition $\mathfrak{g} = \sum_{k=1}^{n-1} (\{JE_k\} + \{E_k\} + \mathfrak{p}_k) + \mathfrak{g}^n$ are mutually orthogonal with respect to this form η .

Now, our theorem will follow by this inductive process, since we shall

have $\mathfrak{g}^{m+1} = \{0\}$ for a certain m .

Suppose we have a decomposition (2.12) for an integer n . Then $\mathfrak{g}_k$ ($1 \leq k \leq n-1$) and $\mathfrak{g}^n$ are clearly J -invariant solvable subalgebras of $\mathfrak{g}$.

Define

$$(2.13) \quad \sigma_k(X, Y) = \phi([X, Y]),$$

for $X, Y \in \mathfrak{p}_k$ ($1 \leq k \leq n-1$). Then $(\mathfrak{p}_k, J, \sigma_k)$ becomes a symplectic space (cf. §1. A). A representation f_k of $\mathfrak{g}^n$ in $\mathfrak{p}_k$ is defined by

$$(2.14) \quad f_k(X)U = [X, V]$$

where $X \in \mathfrak{g}^n$, $U \in \mathfrak{p}_k$, and f_k is symplectic, i. e. $f_k(X)$ is a symplectic transformation of $\mathfrak{p}_k$ for $X \in \mathfrak{g}^n$. Indeed, for $X \in \mathfrak{g}^n$, $U, V \in \mathfrak{p}_k$, we have

$$[X, [U, V]] = [[X, U], V] + [U, [X, V]].$$

By (2.7) and (2.10), the left side of the equation is equal to 0, and hence

$$\phi([X, U], V) + \phi([U, [X, V]]) = 0.$$

Therefore $f_k(X)$ is symplectic.

§ 3. Proof of Theorem: Existence of E_n in $\mathfrak{g}^n$

We shall prove the following.

PROPOSITION 1. Suppose we have a decomposition (2.12) with the properties given there for an integer $n \geq 1$. Then, there exists a non-zero element E_n in $\mathfrak{g}^n$ such that

$$[X, E_n] = \lambda(X)E_n, \quad \lambda(X) \in \mathbf{R}, \quad \text{for } X \in \mathfrak{g}^n;$$

$$[JE_n, E_n] = E_n.$$

In the first place, we show

LEMMA 2. A real solvable Lie algebra $\mathfrak{g}$ contains a commutative ideal of dimension 1 or 2 spanned by the elements E, F such that for $X \in \mathfrak{g}$

$$(3.1) \quad \begin{aligned} [X, E] &= \lambda(X)E + \mu(X)F, \\ [X, F] &= -\mu(X)E + \lambda(X)F, \end{aligned}$$

where λ, μ are linear functions on $\mathfrak{g}$.

PROOF. Let $\mathfrak{g}$ be a real solvable Lie algebra and let $\mathfrak{g}^c = \{X + iY; X, Y \in \mathfrak{g}\}$ be its complexification. By Lie's theorem, there exists a non-zero element Z in $\mathfrak{g}^c$ such that $[W, Z] = k(W)Z$ holds for all $W \in \mathfrak{g}^c$ with $k(W) \in \mathbf{C}$.

Let E (resp. F) denote the real part (resp. imaginary part) of Z . Then for any $X \in \mathfrak{g}$,

$$(3.2) \quad \begin{aligned} [X, E] &= \lambda(X)E + \mu(X)F, \\ [X, F] &= -\mu(X)E + \lambda(X)F, \end{aligned}$$

hold, where λ, μ are linear functions on g .

Let $r = \{E, F\}$ be the real vector subspace spanned by the elements E, F . If E, F are linearly dependent, r is a one-dimensional ideal of g . If E, F are linearly independent, r is a two-dimensional ideal of g satisfying the above conditions. For, since $[E, E] = 0$, we get $\lambda(E) = \mu(E) = 0$ by the first relation in (3.2), which implies $[E, F] = 0$ by the second relation in (3.2) and then r is a commutative ideal. Q. E. D.

We shall now prove that g^n contains a one-dimensional ideal. In view of Lemma 2, it is sufficient to prove that g^n contains no two-dimensional ideal $r = \{E, F\}$ as in Lemma 2.

Let $r = \{E, F\}$ be such an ideal of g^n . By a simple calculation, we have

$$(3.3) \quad [[g^n, g^n], r] = \{0\}.$$

LEMMA 3. *Let X be an element of g^n . If $\phi([X, C]) = 0$ for all $C \in r$, then $[X, C] = 0$ for all $C \in r$.*

PROOF. First we note that $\phi \neq 0$ on r . In fact, if $\phi = 0$ on r , $\eta(Y, D) = \phi([JY, D]) = 0$ for $Y \in g^n, D \in r$. Since η is non-degenerate on g^n , it follows $r = \{0\}$, which is a contradiction. Therefore $\phi \neq 0$ on r . Since $[X, E] = \lambda(X)E + \mu(X)F$, $[X, F] = -\mu(X)E + \lambda(X)F$, we have $\lambda(X)\phi(E) + \mu(X)\phi(F) = 0$, $-\mu(X)\phi(E) + \lambda(X)\phi(F) = 0$. As $\phi(E) \neq 0$ or $\phi(F) \neq 0$, $\lambda(X)^2 + \mu(X)^2 = 0$ holds, and hence we have $\lambda(X) = \mu(X) = 0$, which implies $[X, C] = 0$ for all $C \in r$. Q. E. D.

We put

$$r^0 = \{C \in r; \phi([JC, E]) = \phi([JC, F]) = 0\}.$$

Then, the following three cases are possible:

$$\dim r^0 = 0, \quad \dim r^0 = 1 \quad \text{or} \quad \dim r^0 = 2.$$

We shall show that $\dim r^0 \neq 0$. Suppose $\dim r^0 = 0$. Then η is non-degenerate on r and hence there exists a unique non-zero element $A \in r$ such that $\phi([JA, C]) = \phi(C)$, for all $C \in r$. When $A = \alpha E + \beta F$ ($\alpha, \beta \in \mathbf{R}$), we put $B = -\beta E + \alpha F$. Then A, B is a base of r such that for $X \in g^n$

$$(3.4) \quad \begin{aligned} [X, A] &= \lambda'(X)A + \mu'(X)B, \\ [X, B] &= -\mu'(X)A + \lambda'(X)B, \end{aligned}$$

where λ' and μ' are linear functions on g^n . Now, for $C \in r$, we have by (1.5), (3.3),

$$\begin{aligned}
\phi([J[JA, A], C]) &= -\phi([[JA, A], JC]) \\
&= \phi([[A, JC], JA]) + \phi([[JC, JA], A]) \\
&= -\phi([A, JC]) \\
&= \phi([JA, C]) \\
&= \phi(C).
\end{aligned}$$

Thus we get $[JA, A] = A$, and so $[JA, B] = B$ by (3.4). When we put $[JB, A] = \lambda_0 A + \mu_0 B$, $\lambda_0, \mu_0 \in \mathbf{R}$, then $[JB, B] = -\mu_0 A + \lambda_0 B$ by (3.4). Hence by (K.2)

$$\begin{aligned}
[[JA, JB], A] &= [J[JA, B] + J[A, JB], A] \\
&= [JB - \lambda_0 JA - \mu_0 JB, A] \\
&= -\lambda_0 [JA, A] + (1 - \mu_0) [JB, A] \\
&= -\lambda_0 \mu_0 A + \mu_0 (1 - \mu_0) B.
\end{aligned}$$

Since $[[JA, JB], A] = 0$ by (3.3), we have $\mu_0 = 0$ or $\mu_0 = 1$ and $\lambda_0 = 0$. In the case $\mu_0 = 0$, put $\gamma = \phi(B)$, $\delta = -\phi(A)$. Then, since $\phi \neq 0$ on $\mathfrak{r}$, at least one of γ, δ is not zero and $\gamma\phi(A) + \delta\phi(B) = 0$. Let $X = \gamma A + \delta B$. Then $X \neq 0$ and we have

$$\begin{aligned}
\phi([JA, X]) &= \phi(\gamma A + \delta B) = \gamma\phi(A) + \delta\phi(B) = 0, \\
\phi([JB, X]) &= \phi(\gamma\lambda A + \delta\lambda B) = \lambda(\gamma\phi(A) + \delta\phi(B)) = 0.
\end{aligned}$$

Since η is non-degenerate on $\mathfrak{r}$, it follows that $X = 0$, which is a contradiction. Suppose $\mu_0 = 1$ and $\lambda_0 = 0$. Then we have by (K.2),

$$[JA, JB] = J[JA, B] + J[A, JB] = JB - JB = 0.$$

From this and (K.5), it follows

$$\begin{aligned}
\rho([JA, JB], B) + \rho([JB, B], JA) + \rho([B, JA], JB) &= 0, \\
\rho(JA, A) + \rho(JB, B) &= 0,
\end{aligned}$$

which is a contradiction since $\rho(JA, A) > 0$ and $\rho(JB, B) > 0$. Thus we have shown that $\dim \mathfrak{r}^0 = 0$ is impossible.

We show that $\dim \mathfrak{r}^0 \neq 1$. Suppose $\dim \mathfrak{r}^0 = 1$ and let A be a non-zero element in $\mathfrak{r}^0$. When $A = \alpha E + \beta F$ ($\alpha, \beta \in \mathbf{R}$), we put $B = -\beta E + \alpha F$. Then A, B form a basis of $\mathfrak{r}$ such that for $X \in \mathfrak{g}^n$

$$\begin{aligned}
(3.5) \quad [X, A] &= \lambda'(X)A + \mu'(X)B \\
[X, B] &= -\mu'(X)A + \lambda'(X)B,
\end{aligned}$$

where λ' and μ' are linear functions on $\mathfrak{g}^n$. Since $\phi([JA, A]) = \phi([JA, B]) = 0$, we have $\phi([JA, C]) = 0$ for all $C \in \mathfrak{r}$ and it follows by Lemma 3 that $[JA, A] = 0$ and $[JA, B] = 0$. From this and (3.3), we have

$$\begin{aligned}
0 &= \phi([\![JA, JB]\!], A) \\
&= \phi([J[JA, B] + J[A, JB], A) \\
&= \phi([J[A, JB], A]), \\
0 &= \phi([\![JA, JB]\!], B) \\
&= \phi([J[JA, B] + J[A, JB], B) \\
&= \phi([J[A, JB], B]).
\end{aligned}$$

Hence there exists a real number λ_0 such that $[JB, A] = \lambda_0 A$. Then $[JB, B] = \lambda_0 B$ by (3.5). Since $B \in \mathfrak{r}^0$, it follows that either $\phi([JB, B]) = \lambda_0 \phi(B)$ or $\phi([JB, A]) = \lambda_0 \phi(A)$ is not zero. Therefore $\lambda_0 \neq 0$. On the other hand, we have by (K.2) and (K.5)

$$\begin{aligned}
0 &= \rho([JA, JB], A) + \rho([JB, A], JA) + \rho([A, JA], JB) \\
&= \rho(J[JA, B], A) + \rho(J[A, JB], A) + \rho([JB, A], JA) + \rho([A, JA], JB) \\
&= -2\lambda_0 \rho(JA, A).
\end{aligned}$$

Since $\rho(JA, A) > 0$, we get $\lambda_0 = 0$, which is a contradiction. Thus $\dim \mathfrak{r}^0 = 1$ does not occur.

Suppose now $\dim \mathfrak{r}^0 = 2$. Since $\phi([JC, E]) = 0$, $\phi([JC, F]) = 0$ for any $C \in \mathfrak{r}$, it follows by Lemma 3 that $[JC, E] = [JC, F] = 0$ for all $C \in \mathfrak{r}$ and hence we have

$$(3.6) \quad [J\mathfrak{r}, \mathfrak{r}] = \{0\}.$$

Now, we put

$$(3.7) \quad \mathfrak{p} = \{P \in \mathfrak{g}^n; [P, E] = [JP, E] = 0\}.$$

Then, clearly

$$(3.8) \quad J\mathfrak{p} \subset \mathfrak{p}.$$

Moreover we see

$$(3.9) \quad \text{ad}(JE)\mathfrak{p} \subset \mathfrak{p}, \quad \text{and}$$

$$(3.10) \quad \text{ad}(JE)J = J \text{ad}(JE) \quad \text{on } \mathfrak{p}.$$

Indeed, since $[JE, JP] = J[JE, P] + J[E, JP] + [E, P] = J[JE, P]$ for $P \in \mathfrak{p}$, we have $\text{ad}(JE)J = J \text{ad}(JE)$ on $\mathfrak{p}$. Since $[[JE, P], E] = 0$ and $[J[JE, P], E] = [[JE, JP], E] = 0$ by (3.3), we have $\text{ad}(JE)\mathfrak{p} \subset \mathfrak{p}$.

Now, for $X \in \mathfrak{g}^n$, we get by (3.3), (3.6)

$$[[JE, X], E] = 0,$$

$$\begin{aligned}
[J[JE, X], E] &= [[JE, JX] - J[E, JX] - [E, X], E] \\
&= [[JE, JX], E] \\
&= 0,
\end{aligned}$$

which implies that $[JE, X] \in \mathfrak{p}$, and hence we have

$$(3.11) \quad \text{ad}(JE)\mathfrak{g}^n \subset \mathfrak{p}.$$

Let $P \in \mathfrak{p}$. We have

$$\begin{aligned}
\rho(JE, [JE, P]) &= -\rho(E, J[JE, P]) \\
&= -\rho(E, [JE, JP]) \\
&= \rho(JE, [JP, E]) + \rho(JP, [E, JE]) \\
&= 0,
\end{aligned}$$

and it follows that for $X \in \mathfrak{g}^n$

$$(3.12) \quad \rho(JE, \text{ad}(JE)^2 X) = 0.$$

Applying Lemma 1 and (3.12), we have for $X, Y \in \mathfrak{g}^n$

$$\begin{aligned}
&\frac{d^3}{dt^3} \rho(\exp t \text{ad}(JE)X, \exp t \text{ad}(JE)Y) \\
&= \frac{d^2}{dt^2} \rho(JE, \exp t \text{ad}(JE)[X, Y]) \\
&= \rho(JE, \text{ad}(JE)^2 \exp t \text{ad}(JE)[X, Y]) \\
&= 0.
\end{aligned}$$

Hence we may put

$$(3.13) \quad \rho(\exp t \text{ad}(JE)X, \exp t \text{ad}(JE)Y) = at^2 + bt + c,$$

where a, b and c are real numbers not depending on t .

Now, let $\alpha + i\beta$ ($\alpha, \beta \in \mathbf{R}$) be an eigenvalue of $\text{ad}(JE)$ on $\mathfrak{p}$. Since $\text{ad}(JE)J = J\text{ad}(JE)$ on $\mathfrak{p}$ by (3.10), there exists a non-zero element $P \in \mathfrak{p}$ such that $\text{ad}(JE)P = (\alpha + \beta J)P$, and hence $\exp t \text{ad}(JE)P = \exp t(\alpha + \beta J)P$. Therefore we have by (K.3) and (3.10),

$$\begin{aligned}
&\rho(\exp t \text{ad}(JE)JP, \exp t \text{ad}(JE)P) \\
&= \rho(J \exp t \text{ad}(JE)P, \exp t \text{ad}(JE)P) \\
&= \rho(J \exp t(\alpha + \beta J)P, \exp t(\alpha + \beta J)P) \\
&= \rho(\exp t(\alpha + \beta J)JP, \exp t(\alpha + \beta J)P) \\
&= e^{(\alpha + i\beta)t} \overline{e^{(\alpha + i\beta)t}} \rho(JP, P) \\
&= e^{2\alpha t} \rho(JP, P).
\end{aligned}$$

From this and (3.13), we have

$$at^2 + bt + c = e^{2\alpha t} \rho(JP, P).$$

Since $\rho(JP, P) > 0$, it follows $\alpha = 0$. Thus the real parts of the eigenvalues of $\text{ad}(JE)$ on $\mathfrak{p}$ are equal to 0.

Now, we put

$$(3.14) \quad \phi_n(X) = \text{Tr}_{\mathfrak{g}^n}(\text{ad}(JX) - J\text{ad}(X)) \quad \text{for } X \in \mathfrak{g}^n.$$

From the facts that $\text{ad}(JE)\mathfrak{g}^n \subset \mathfrak{p}$ (3.11) and the real parts of the eigenvalues of $\text{ad}(JE)$ on $\mathfrak{p}$ are equal to 0, it follows that $\text{Tr}_{\mathfrak{g}^n} \text{ad}(JE) = \text{Tr}_{\mathfrak{p}} \text{ad}(JE) = 0$. On the other hand, by (3.6) and $J\text{ad}(E)\mathfrak{g}^n \subset \mathfrak{r}$, we have $\text{Tr}_{\mathfrak{g}^n} J\text{ad}(E) = \text{Tr}_{\mathfrak{r}} J\text{ad}(E) = 0$. These imply that

$$\phi_n(E) = 0.$$

Taking F instead of E , we have

$$\phi_n(F) = 0.$$

Thus we know

$$(3.15) \quad \phi_n = 0 \quad \text{on } \mathfrak{r}.$$

Now, for $X \in \mathfrak{g}^n$, $P \in \mathfrak{p}_k$, we have $[X, JE_k] \in \mathfrak{g}^{k+1}$, $[X, E_k] = 0$, $[X, P] \in \mathfrak{p}_k$ and $\text{Tr}_{\mathfrak{p}_k} f_k(X) = 0$, where f_k is a symplectic representation of $\mathfrak{g}^n$ in $\mathfrak{p}_k$ (2.14). It follows that for $X \in \mathfrak{g}^n$

$$\begin{aligned} (3.16) \quad \phi(X) &= \text{Tr}_{\mathfrak{g}}(\text{ad}(JX) - J\text{ad}(X)) \\ &= \sum_{k=1}^{n-1} \text{Tr}_{\mathfrak{p}_k}(f_k(JX) - Jf_k(X)) + \phi_n(X) \\ &= - \sum_{k=1}^{n-1} \text{Tr}_{\mathfrak{p}_k} Jf_k(X) + \phi_n(X). \end{aligned}$$

Put $f_k(E) = (A, B)$, $f_k(JE) = (C, D)$ in the sense of § 1.A. Since $[f_k(JE), f_k(E)] = f_k([JE, E]) = 0$, it becomes that

$$(3.17) \quad CA - AC + D\bar{B} - B\bar{D} = 0.$$

By (K.2), we have

$$(3.18) \quad [J, f_k(JE) - (1/2)[J, f_k(E)]] = 0.$$

From this and $J = (i, 0)$, we know

$$(3.19) \quad D = iB.$$

Since B is a symmetric matrix, there exists a unitary matrix U such that

$$UB^tU = \begin{pmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & \lambda_{r+1} & \\ 0 & & & & \ddots \\ & & & & & \lambda_l \end{pmatrix}$$

where $\lambda_i > 0$ for $1 \leq i \leq r$, $\lambda_i = 0$ for $r+1 \leq i \leq l$. Hence we may assume that

$$(3.20) \quad B = \begin{pmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & \lambda_{r+1} & \\ 0 & & & & \ddots \\ & & & & & \lambda_l \end{pmatrix}$$

where $\lambda_i > 0$ for $1 \leq i \leq r$, $\lambda_i = 0$ for $r+1 \leq i \leq l$. Since $D = iB$, we have

$$(3.21) \quad CA - AC + 2iB^2 = 0.$$

Taking the trace of the both sides of the formula (3.21), it follows that

$$2i \sum_{k=1}^l \lambda_k^2 = 0,$$

which implies that $B = 0$, and hence $f_k(E) = (A, 0)$. From this we know that

$$(3.22) \quad f_k(E)J = Jf_k(E) \quad \text{on } \mathfrak{p}_k.$$

Using (3.15), (3.16) and (3.22), we have

$$\begin{aligned} (3.23) \quad \phi([E, X]) &= - \sum_{k=1}^{n-1} \text{Tr}_{\mathfrak{p}_k} Jf_k([E, X]) + \phi_n([E, X]) \\ &= - \sum_{k=1}^{n-1} \text{Tr}_{\mathfrak{p}_k} J[f_k(E), f_k(X)] \\ &= - \sum_{k=1}^{n-1} \text{Tr}_{\mathfrak{p}_k} (Jf_k(E)f_k(X) - Jf_k(X)f_k(E)) \\ &= - \sum_{k=1}^{n-1} \text{Tr}_{\mathfrak{p}_k} (f_k(E)Jf_k(X) - Jf_k(X)f_k(E)) \\ &= - \sum_{k=1}^{n-1} \text{Tr}_{\mathfrak{p}_k} [f_k(E), Jf_k(X)] \\ &= 0, \quad \text{for } X \in \mathfrak{g}^n. \end{aligned}$$

Since η is non-degenerate on $\mathfrak{g}^n$, it follows that $E = 0$, which is a contradiction. Thus we have known that $\mathfrak{g}^n$ contains a one-dimensional ideal $\mathfrak{r}$ of $\mathfrak{g}^n$.

Now, let E be a non-zero element of $\mathfrak{r}$ and suppose that $[JE, E] = 0$. Put

$$\mathfrak{p} = \{P \in \mathfrak{g}^n; [P, E] = [JP, E] = 0\}.$$

Then, by the argument as used above, we see the followings. First $\text{ad}(JE)\mathfrak{g}^n \subset \mathfrak{p}$, and the real parts of the eigenvalues of $\text{ad}(JE)$ on $\mathfrak{p}$ are equal to 0, and $\phi([E, X]) = 0$ for any $X \in \mathfrak{g}^n$. This is a contradiction, because η is non-degenerate on $\mathfrak{g}^n$. Therefore $[JE, E] \neq 0$.

Putting $E_n = \lambda E$ with a non-zero constant λ , we have

$$(3.24) \quad [JE_n, E_n] = E_n.$$

Thus Proposition 1 is proved.

§ 4. Proof of Theorem (continued): Decomposition of $\mathfrak{g}^n$

PROPOSITION 2. *Let E_n be an element in $\mathfrak{g}^n$ as in Proposition 1. Then, we get the decomposition*

$$(4.1) \quad \mathfrak{g}^n = \{JE_n\} + \{E_n\} + \mathfrak{p}_n + \mathfrak{g}^{n+1}$$

of $\mathfrak{g}^n$ into the direct sum of vector spaces with the following properties:

1) $\mathfrak{g}_n = \{JE_n\} + \{E_n\} + \mathfrak{p}_n$ is a J -invariant subalgebra such that

$$(4.2) \quad J\mathfrak{p}_n \subset \mathfrak{p}_n;$$

$$(4.3) \quad [JE_n, E_n] = E_n;$$

$$(4.4) \quad [JE_n, \mathfrak{p}_n] \subset \mathfrak{p}_n.$$

Moreover the real parts of the eigenvalues of $\text{ad}(JE_n)$ on $\mathfrak{p}_n$ are equal to $1/2$, and

$$(4.5) \quad [E_n, \mathfrak{p}_n] = 0;$$

$$(4.6) \quad [\mathfrak{p}_n, \mathfrak{p}_n] \subset \{E_n\}.$$

2) $\mathfrak{g}^{n+1}$ is a J -invariant subalgebra such that

$$(4.7) \quad [JE_n, \mathfrak{g}^{n+1}] \subset \mathfrak{g}^{n+1}.$$

Moreover the real parts of the eigenvalues of $\text{ad}(JE_n)$ on $\mathfrak{g}^{n+1}$ are equal to 0, and

$$(4.8) \quad [E_n, \mathfrak{g}^{n+1}] = 0;$$

$$(4.9) \quad [\mathfrak{p}_n, \mathfrak{g}^{n+1}] \subset \mathfrak{p}_n.$$

3) *The form η is positive definite on $\mathfrak{g}_n$ and the factors of the decomposition $\mathfrak{g}_n = \{JE_n\} + \{E_n\} + \mathfrak{p}_n$ are mutually orthogonal with respect to this form η . Further, the form η is non-degenerate on $\mathfrak{g}^{n+1}$.*

PROOF. For the convenience of notation, we denote the element E_n by E . We put

$$(4.10) \quad \mathfrak{p} = \{P \in \mathfrak{g}^n; [P, E] = [JP, E] = 0\}.$$

Then, we have

$$(4.11) \quad J\mathfrak{p} \subset \mathfrak{p}, \quad \text{ad}(JE)\mathfrak{p} \subset \mathfrak{p},$$

$$(4.12) \quad \text{ad}(JE)J = J \text{ad}(JE) \quad \text{on } \mathfrak{p},$$

$$(4.13) \quad \mathfrak{g}^n = \{JE\} + \{E\} + \mathfrak{p}.$$

Indeed, (4.11) and (4.12) can be shown in the same way as (3.8), (3.9) and (3.10). Since $\{E\}$ is a one-dimensional ideal of $\mathfrak{g}^n$, we get $[X, E] = \alpha(X)E$, $[JX, E] = \beta(X)E$ for $X \in \mathfrak{g}^n$, where α, β are linear functions on $\mathfrak{g}^n$. It is easily seen that $P = X - \alpha(X)JE - \beta(X)E$ belongs to $\mathfrak{p}$ for any $X \in \mathfrak{g}^n$.

LEMMA 4. *The real parts of the eigenvalues of $\text{ad}(JE)$ on $\mathfrak{p}$ are equal to 0 or $1/2$.*

PROOF. By Lemma 1, we have for $P \in \mathfrak{p}$,

$$\begin{aligned} & \frac{d}{dt} \rho(\exp t \text{ad}(JE)E, \exp t \text{ad}(JE)P) \\ &= \rho(JE, \exp t \text{ad}(JE)[E, P]) \\ &= 0. \end{aligned}$$

Since $\exp t \text{ad}(JE)E = e^t E$, this implies that

$$\rho(E, \exp t \text{ad}(JE)P) = a' e^{-t}$$

where a' is a constant determined by P and independent of t . We have then

$$\begin{aligned} \rho(JE, \exp t \text{ad}(JE)P) &= -\rho(E, J \exp t \text{ad}(JE)P) \\ &= -\rho(E, \exp t \text{ad}(JE)JP) \\ &= a e^{-t} \end{aligned}$$

where a is the constant determined by JP . Since any element X in $\mathfrak{g}^n$ is expressed in the form $X = \lambda JE + \mu E + P$, where $\lambda, \mu \in \mathbf{R}$ and $P \in \mathfrak{p}$ (4.13), we have

$$\begin{aligned} \rho(JE, \exp t \text{ad}(JE)X) &= \rho(JE, \lambda JE + \mu e^t E + \exp t \text{ad}(JE)P) \\ &= \mu \rho(JE, E) e^t + \rho(JE, \exp t \text{ad}(JE)P) \\ &= a e^{-t} + b e^t \end{aligned}$$

where a, b are constant independent of t . This fact and Lemma 1 imply that for $X, Y \in \mathfrak{g}^n$

$$\begin{aligned} & \frac{d}{dt} \rho(\exp t \text{ad}(JE)X, \exp t \text{ad}(JE)Y) \\ &= \rho(JE, \exp t \text{ad}(JE)[X, Y]) \\ &= a e^{-t} + b e^t. \end{aligned}$$

Hence we obtain

$$(4.14) \quad \rho(\exp t \operatorname{ad}(JE)X, \exp t \operatorname{ad}(JE)Y) = ae^{-t} + be^t + c,$$

where a , b and c are constant independent of t . Let $\alpha + i\beta$ ($\alpha, \beta \in \mathbf{R}$) be an eigenvalue of $\operatorname{ad}(JE)$ on $\mathfrak{p}$. Since $\operatorname{ad}(JE)J = J\operatorname{ad}(JE)$ on $\mathfrak{p}$, there exists a non-zero element P in $\mathfrak{p}$ such that $\operatorname{ad}(JE)P = (\alpha + \beta J)P$. Hence we have

$$\begin{aligned} & \rho(\exp t \operatorname{ad}(JE)JP, \exp t \operatorname{ad}(JE)P) \\ &= \rho(J \exp t \operatorname{ad}(JE)P, \exp t \operatorname{ad}(JE)P) \\ &= \rho(J \exp t(\alpha + \beta J)P, \exp t(\alpha + \beta J)P) \\ &= \rho(\exp t(\alpha + \beta J)JP, \exp t(\alpha + \beta J)P) \\ &= e^{(\alpha + i\beta)t} \overline{e^{(\alpha + i\beta)t}} \rho(JP, P) \\ &= e^{2\alpha t} \rho(JP, P). \end{aligned}$$

Therefore

$$e^{2\alpha t} \rho(JP, P) = ae^{-t} + be^t + c.$$

This implies that $\alpha = 0$ or $1/2$ or $-1/2$, since $\rho(JP, P) > 0$. We put

$$(4.15) \quad \mathfrak{p}_{(\alpha + i\beta)} = \{P \in \mathfrak{p}; (\operatorname{ad}(JE) - (\alpha + \beta J))^m P = 0 \text{ for some integer } m > 0\}.$$

Then we have

$$(4.16) \quad \mathfrak{p} = \sum_{\alpha + i\beta} \mathfrak{p}_{(\alpha + i\beta)},$$

where α is equal to 0 or $1/2$ or $-1/2$. Let P be a non-zero element in $\mathfrak{p}_{(\alpha + i\beta)}$. Then there exists a positive integer m such that $(\operatorname{ad}(JE) - (\alpha + \beta J))^m P = 0$. Hence we have

$$\begin{aligned} \exp t \operatorname{ad}(JE)P &= \exp t(\alpha + \beta J) \sum_{l=0}^{m-1} \frac{t^l}{l!} (\operatorname{ad}(JE) - (\alpha + \beta J))^l P \\ &= e^{\alpha t} \left\{ \cos \beta t \sum_{l=0}^{m-1} \frac{t^l}{l!} (\operatorname{ad}(JE) - (\alpha + \beta J))^l P \right. \\ &\quad \left. + \sin \beta t \sum_{l=0}^{m-1} \frac{t^l}{l!} (\operatorname{ad}(JE) - (\alpha + \beta J))^l JP \right\}. \end{aligned}$$

Therefore

$$\begin{aligned} & \rho(JE, \exp t \operatorname{ad}(JE)P) \\ &= e^{\alpha t} \left\{ \cos \beta t \sum_{l=0}^{m-1} \frac{1}{l!} \rho(JE, (\operatorname{ad}(JE) - (\alpha + \beta J))^l P) t^l \right. \\ &\quad \left. + \sin \beta t \sum_{l=0}^{m-1} \frac{1}{l!} \rho(JE, (\operatorname{ad}(JE) - (\alpha + \beta J))^l JP) t^l \right\}. \end{aligned}$$

We put

$$h(t) = \sum_{l=0}^{m-1} \frac{1}{l!} \rho(JE, (\operatorname{ad}(JE) - (\alpha + \beta J))^l P) t^l,$$

$$k(t) = \sum_{l=0}^{m-1} \frac{1}{l!} \rho(JE, (\operatorname{ad}(JE) - (\alpha + \beta J))^l JP) t^l.$$

Then, $h(t)$ and $k(t)$ are polynomials whose degrees are $m-1$ at most. We have then

$$(4.17) \quad \begin{aligned} h(t) \cos \beta t + k(t) \sin \beta t &= a e^{-(1+\alpha)t}, \\ \left| \frac{h(t)}{t^m} \cos \beta t + \frac{k(t)}{t^m} \sin \beta t \right| &= \left| a \frac{e^{-(1+\alpha)t}}{t^m} \right|. \end{aligned}$$

We assume that $a \neq 0$. Since $1+\alpha > 0$ and since $h(t)$ and $k(t)$ are polynomials of degree $\leq m-1$, the left side of the above formula (4.17) approaches to 0 and the right side to ∞ , when $t \rightarrow -\infty$. This is a contradiction, and we must have $a=0$. This implies that

$$\rho(JE, \exp t \operatorname{ad}(JE)P) = 0, \quad \text{for } P \in \mathfrak{p}_{(\alpha+i\beta)}.$$

Hence we have

$$\rho(JE, \exp t \operatorname{ad}(JE)P) = 0, \quad \text{for } P \in \mathfrak{p}.$$

Therefore

$$e^{2\alpha t} \rho(JP, P) = b e^t + c.$$

This implies that $\alpha = 0$ or $1/2$, which proves Lemma 4.

Now, put

$$\mathfrak{p}_{[\alpha]} = \sum_{\beta} \mathfrak{p}_{(\alpha+i\beta)}.$$

Then, we have

$$\begin{aligned} J\mathfrak{p}_{[\alpha]} &\subset \mathfrak{p}_{[\alpha]}, \quad \operatorname{ad}(JE)\mathfrak{p}_{[\alpha]} \subset \mathfrak{p}_{[\alpha]}, \\ \mathfrak{p} &= \mathfrak{p}_{[0]} + \mathfrak{p}_{[1]}. \end{aligned}$$

Moreover the real parts of the eigenvalues of $\operatorname{ad}(JE)$ on $\mathfrak{p}_{[\alpha]}$ are equal to α . Hence we get the following decomposition;

$$(4.18) \quad \mathfrak{g}^n = \mathfrak{g}_{[0]} + \mathfrak{g}_{[1/2]} + \mathfrak{g}_{[1]},$$

where $\mathfrak{g}_{[0]} = \{JE\} + \mathfrak{p}_{[0]}$, $\mathfrak{g}_{[1/2]} = \mathfrak{p}_{[1/2]}$ and $\mathfrak{g}_{[1]} = \{E\}$. Moreover, $\operatorname{ad}(JE)\mathfrak{g}_{[\alpha]} \subset \mathfrak{g}_{[\alpha]}$ and the real parts of the eigenvalues of $\operatorname{ad}(JE)$ on $\mathfrak{g}_{[\alpha]}$ are equal to α . We put

$$(4.19) \quad \mathfrak{p}_n = \mathfrak{p}_{[1/2]},$$

$$(4.20) \quad \mathfrak{g}_n = \{JE\} + \{E\} + \mathfrak{p}_{[1/2]},$$

$$(4.21) \quad \mathfrak{g}^{n+1} = \mathfrak{p}_{[0]}.$$

Then, $\mathfrak{g}_n$ is clearly a J -invariant subalgebra. We prove that $\mathfrak{g}^{n+1}$ is also a J -invariant subalgebra. First we have $J\mathfrak{g}^{n+1} \subset \mathfrak{g}^{n+1}$. Now, since $[\mathfrak{g}_{[0]}, \mathfrak{g}_{[0]}] \subset \mathfrak{g}_{[0]}$,

if $P, Q \in \mathfrak{g}^{n+1}$, we have

$$[P, Q] = \lambda JE + P',$$

where $\lambda \in \mathbf{R}$ and $P' \in \mathfrak{p}_{[0]}$, and therefore

$$[[P, Q], E] = \lambda E.$$

On the other hand, we have

$$[[P, Q], E] = [[P, E], Q] + [P, [Q, E]] = 0.$$

This implies that $\lambda = 0$. Thus we have $[P, Q] \in \mathfrak{g}^{n+1}$, which shows that $\mathfrak{g}^{n+1}$ is a subalgebra.

Now, we shall show that the factors of the decomposition

$$\mathfrak{g}^n = \{JE\} + \{E\} + \mathfrak{p}_{[1/2]} + \mathfrak{p}_{[0]}$$

are mutually orthogonal with respect to the non-degenerate form η . Indeed, for $P \in \mathfrak{p}_{[1/2]}$, $Q \in \mathfrak{p}_{[0]}$, put $P' = [JP, Q]$. Then $P' \in \mathfrak{p}_{[1/2]}$. Since the real parts of the eigenvalues of $\text{ad}(JE)$ on $\mathfrak{p}_{[1/2]}$ are equal to $1/2$, $\text{ad}(JE)$ is non-singular on $\mathfrak{p}_{[1/2]}$ and hence there exists an element $P'' \in \mathfrak{p}_{[1/2]}$ such that $[JE, P''] = P'$. We have then $2\eta(P, Q) = \phi([JP, Q]) = \phi([JE, P'']) = -\phi([E, JP'']) = 0$. This shows that $\mathfrak{p}_{[1/2]}$ and $\mathfrak{p}_{[0]}$ are orthogonal with respect to η . It is clear that the other pairs of factors are mutually orthogonal with respect to η .

LEMMA 5.

$$\phi(E) > 0.$$

PROOF. Recall that $(\mathfrak{p}_k, J, \sigma_k)$ is a symplectic space where σ_k is defined in (2.13) and that f_k is a symplectic representation of $\mathfrak{g}^n$ in $\mathfrak{p}_k$ defined by (2.14). Since

$$[f_k(JE), f_k(E)] = f_k(E),$$

$$[J, f_k(JE) - (1/2)[J, f_k(E)]] = 0,$$

we have by [10]

- 1) $\mathfrak{p}_k = \mathfrak{p}_k^+ + \mathfrak{p}_k^- + \mathfrak{p}_k^0$ direct sum;
- 2) $\mathfrak{p}_k^+$, $\mathfrak{p}_k^-$ and $\mathfrak{p}_k^0$ are invariant by $f_k(JE)$;
- 3) the real parts of the eigenvalues of $f_k(JE)$ on $\mathfrak{p}_k^+$, $\mathfrak{p}_k^-$ and $\mathfrak{p}_k^0$ are $1/2$, $-1/2$ and 0 respectively;
- 4) $J\mathfrak{p}_k^- = \mathfrak{p}_k^+$, $J\mathfrak{p}_k^0 = \mathfrak{p}_k^0$, in particular $\dim \mathfrak{p}_k^- = \dim \mathfrak{p}_k^+$, $\text{Tr}_{\mathfrak{p}_k} f_k(JE) = 0$;
- 5)

$$f_k(E) = \begin{cases} J & \text{on } \mathfrak{p}_k^-, \\ 0 & \text{on } \mathfrak{p}_k^+ + \mathfrak{p}_k^0. \end{cases}$$

These show that $\text{Tr}_{\mathfrak{p}_k} Jf_k(E) = -\dim \mathfrak{p}_k^-$. On the other hand $\text{Tr}_{\mathfrak{g}^n}(\text{ad}(JE) - J\text{ad}(E)) > 0$. Indeed, $\text{Tr}_{\mathfrak{g}^n} J\text{ad}(E) = -1$ because $J\text{ad}(E)\mathfrak{g}^n \subset \{JE\}$ and $J\text{ad}(E)JE = -JE$. Moreover, the real parts of the eigenvalues of $\text{ad}(JE)$ on $\mathfrak{g}^n$ are equal to 0 , $1/2$ or 1 , and $\text{ad}(JE)E = E$. Therefore we have $\text{Tr}_{\mathfrak{g}^n} \text{ad}(JE) > 0$.

These imply that $\text{Tr}_{\mathfrak{g}^n}(\text{ad}(JE) - J \text{ad}(E)) > 0$. Therefore we have

$$\begin{aligned}\phi(E) &= -\sum_{k=1}^{n-1} \text{Tr}_{\mathfrak{p}_k} J f_k(E) + \text{Tr}_{\mathfrak{g}^n}(\text{ad}(JE) - J \text{ad}(E)) \\ &= \sum_{k=1}^{n-1} \dim \mathfrak{p}_k^- + \text{Tr}_{\mathfrak{g}^n}(\text{ad}(JE) - J \text{ad}(E)) > 0.\end{aligned}$$

Q. E. D.

LEMMA 6. η is positive definite on $\mathfrak{p}_{[1/2]}$.

PROOF. We shall first prove that the decomposition $\mathfrak{p}_{[1/2]} = \sum_{\beta} \mathfrak{p}_{(1/2+i\beta)}$ is an orthogonal decomposition with respect to η . Let P and Q be non-zero elements in $\mathfrak{p}_{(1/2+i\beta)}$, $\mathfrak{p}_{(1/2+i\beta')}$ respectively and assume $\beta \neq \beta'$. Then there exist positive integers m, n such that

$$\begin{aligned}(\text{ad}(JE) - (1/2 + \beta J))^m P &= 0, \\ (\text{ad}(JE) - (1/2 + \beta' J))^n Q &= 0.\end{aligned}$$

Hence we have

$$\begin{aligned}\exp t \text{ad}(JE) P &= \exp t(1/2 + \beta J) \sum_{l=0}^{m-1} \frac{t^l}{l!} (\text{ad}(JE) - (1/2 + \beta J))^l P, \\ \exp t \text{ad}(JE) Q &= \exp t(1/2 + \beta' J) \sum_{l=0}^{n-1} \frac{t^l}{l!} (\text{ad}(JE) - (1/2 + \beta' J))^l Q.\end{aligned}$$

Since $[\mathfrak{p}_{[1/2]}, \mathfrak{p}_{[1/2]}] \subset \{E\}$, it becomes that $[JP, Q] = \lambda E$, where $\lambda \in \mathbf{R}$. By Lemma 1, we have

$$\begin{aligned}(4.22) \quad & \frac{d}{dt} \rho(\exp t \text{ad}(JE) JP, \exp t \text{ad}(JE) Q) \\ &= \rho(JE, \exp t \text{ad}(JE) [JP, Q]).\end{aligned}$$

The left side of this equation is equal to

$$\begin{aligned}& \frac{d}{dt} \rho(J \exp t \text{ad}(JE) P, \exp t \text{ad}(JE) Q) \\ &= \frac{d}{dt} \rho(J \exp t(1/2 + \beta J) \sum_{l=0}^{m-1} \frac{t^l}{l!} (\text{ad}(JE) - (1/2 + \beta J))^l P, \\ & \quad \exp t(1/2 + \beta' J) \sum_{l=0}^{n-1} \frac{t^l}{l!} (\text{ad}(JE) - (1/2 + \beta' J))^l Q) \\ &= \frac{d}{dt} e^t \rho(\exp \beta t J \sum_{l=0}^{m-1} \frac{t^l}{l!} (\text{ad}(JE) - (1/2 + \beta J))^l JP, \\ & \quad \exp \beta' t J \sum_{l=0}^{n-1} \frac{t^l}{l!} (\text{ad}(JE) - (1/2 + \beta' J))^l Q) \\ &= \frac{d}{dt} e^t \rho(\{\cos \beta t + (\sin \beta t) J\} u(t), \{\cos \beta' t + (\sin \beta' t) J\} v(t))\end{aligned}$$

$$\begin{aligned}
&= -\frac{d}{dt} e^t (\cos \beta t \cos \beta' t + \sin \beta t \sin \beta' t) \rho(u(t), v(t)) \\
&\quad + (\sin \beta t \cos \beta' t - \cos \beta t \sin \beta' t) \rho(Ju(t), v(t)) \\
&= -\frac{d}{dt} e^t \{h(t) \cos (\beta - \beta')t + k(t) \sin (\beta - \beta')t\} \\
&= e^t \{a(t) \cos (\beta - \beta')t + b(t) \sin (\beta - \beta')t\},
\end{aligned}$$

where

$$\begin{aligned}
u(t) &= \sum_{l=0}^{m-1} \frac{t^l}{l!} (\operatorname{ad}(JE) - (1/2 + \beta J))^l JP, \\
v(t) &= \sum_{l=0}^{n-1} \frac{t^l}{l!} (\operatorname{ad}(JE) - (1/2 + \beta' J))^l Q, \\
h(t) &= \rho(u(t), v(t)), \\
k(t) &= \rho(Ju(t), v(t)), \\
a(t) &= h(t) + h'(t) + (\beta - \beta')k(t), \\
b(t) &= k(t) + k'(t) - (\beta - \beta')h(t).
\end{aligned}$$

Hence $a(t)$ and $b(t)$ are polynomials. On the other hand, the right side of the equation (4.22) is equal to

$$\begin{aligned}
\rho(JE, \exp t \operatorname{ad}(JE)[JP, Q]) &= \rho(JE, \lambda e^t E) \\
&= e^t \lambda \rho(JE, E).
\end{aligned}$$

Therefore we have

$$a(t) \cos (\beta - \beta')t + b(t) \sin (\beta - \beta')t = \lambda \rho(JE, E).$$

Since $a(t) - \lambda \rho(JE, E)$ is a polynomial and since $a(t_n) - \lambda \rho(JE, E) = 0$ for $t_n = 2n\pi/(\beta - \beta')$, where n integer, it follows that $a(t)$ is a constant a . Similarly $b(t)$ is a constant b . Hence we have

$$a \cos (\beta - \beta')t + b \sin (\beta - \beta')t = \lambda \rho(JE, E).$$

By this formula, we have $(\beta - \beta')^2 \lambda \rho(JE, E) = 0$. Since $\beta - \beta' \neq 0$ and $\rho(JE, E) > 0$, λ must be 0. Thus we have

$$\eta(P, Q) = \phi([JP, Q]) = \lambda \phi(E) = 0.$$

This implies that $\mathfrak{p}_{(1/2+i\beta)}$ and $\mathfrak{p}_{(1/2+i\beta')}$ are mutually orthogonal with respect to η .

Now, let P be a non-zero element in $\mathfrak{p}_{(1/2+i\beta)}$. Then, there exists a positive integer m such that

$$(\operatorname{ad}(JE) - (1/2 + \beta J))^m P = 0,$$

and hence

$$\exp t \operatorname{ad} (JE)P = \exp t (1/2 + \beta J)u(t),$$

where $u(t) = \sum_{l=0}^{m-1} \frac{t^l}{l!} (\operatorname{ad} (JE) - (1/2 + \beta J))^l P$. On the other hand, we have by Lemma 1

$$(4.23) \quad \begin{aligned} & -\frac{d}{dt} \rho(\exp t \operatorname{ad} (JE)JP, \exp t \operatorname{ad} (JE)P) \\ & = \rho(JE, \exp t \operatorname{ad} (JE)[JP, P]). \end{aligned}$$

The left side of this equation is equal to

$$\begin{aligned} & -\frac{d}{dt} \rho(J \exp t \operatorname{ad} (JE)P, \exp t \operatorname{ad} (JE)P) \\ & = -\frac{d}{dt} \rho(J \exp t (1/2 + \beta J)u(t), \exp t (1/2 + \beta J)u(t)) \\ & = -\frac{d}{dt} \rho(\exp t (1/2 + \beta J)Ju(t), \exp t (1/2 + \beta J)u(t)) \\ & = -\frac{d}{dt} e^{(1/2 + i\beta)t} \overline{e^{(1/2 + i\beta)t}} \rho(Ju(t), u(t)) \\ & = -\frac{d}{dt} e^t \rho(Ju(t), u(t)) \\ & = e^t (h'(t) + h(t)), \end{aligned}$$

where $h(t) = \rho(Ju(t), u(t))$, and $h(t)$ is a polynomial of degree $\leq 2m-2$. Because $[JP, P] = \lambda E$ where $\lambda \in \mathbf{R}$, the right side of the equation (4.23) is equal to

$$\rho(JE, \lambda e^t E) = e^t \lambda \rho(JE, E).$$

Hence we have

$$h'(t) + h(t) = \lambda \rho(JE, E).$$

The solution of this equation is $h(t) = ce^{-t} + \lambda \rho(JE, E)$, where c is an arbitrary constant. However, $h(t)$ is a polynomial, and so c must be 0. Hence we have

$$h(t) = \lambda \rho(JE, E),$$

and hence it follows that

$$(4.24) \quad \lambda = \frac{h(t)}{\rho(JE, E)} = \frac{h(0)}{\rho(JE, E)} = \frac{\rho(JP, P)}{\rho(JE, E)} > 0.$$

Therefore we have by Lemma 5 and (4.24)

$$\eta(P, P) = \phi([JP, P]) = \lambda \phi(E) > 0.$$

This shows that η is positive definite on $\mathfrak{p}_{(1/2+i\beta)}$, and hence on $\mathfrak{p}_{[1/2]} = \sum_{\beta} \mathfrak{p}_{(1/2+i\beta)}$.
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This completes the proof of Proposition 2.

As explained in § 2 our theorem follows then by induction on n , applying Propositions 1 and 2 successively.

§ 5. Proof of corollaries

PROOF OF COROLLARY 1. Let $\hat{K}$ be the isotropy subgroup of $\hat{G}$ at the point o . Since $\hat{G} = I^0(M)$, $\hat{K}$ is a compact subgroup of $\hat{G}$. By a theorem of Iwasawa, $\hat{G}/\hat{K} = G/K$ is homeomorphic to a Euclidean space. Hence $\hat{K}$ is a maximal compact subgroup of $\hat{G}$. Let $\hat{\mathfrak{k}}$ be the Lie subalgebra of $\hat{\mathfrak{g}}$ corresponding to $\hat{K}$. By a theorem of Hano (Introduction (vi)), the adjoint group $\tilde{G} = \text{Ad}_{\hat{G}}(\hat{G})$ of $\hat{G}$ is the identity component of a real algebraic group in $GL(\hat{\mathfrak{g}}, \mathbf{R})$. Since the center of $\hat{G}$ is finite and since $\tilde{K} = \text{Ad}_{\hat{G}}(\hat{K})$ is a compact subgroup of $\tilde{G}$, $\tilde{K}$ is a maximal compact subgroup of $\tilde{G}$. Hence there exists a connected triangular subgroup $\tilde{T}$ of $\tilde{G}$ such that $\tilde{G} = \tilde{T}\tilde{K}$, where $\tilde{T} \cap \tilde{K}$ consists of the identity only [12]. Thus we have $\hat{G} = \hat{T}\hat{K}$, where $\hat{T} = \text{Ad}_{\hat{G}}^{-1}(\tilde{T})$, and $\hat{T}$ is a solvable Lie group which acts transitively on M . The Kähler algebra corresponding to $\hat{T}$ is normal, solvable and the canonical hermitian form is non-degenerate. By our theorem, M is holomorphically isomorphic to a homogeneous bounded domain.

PROOF OF COROLLARY 2. Since M is complete, simply connected, and since o has no conjugate point, the exponential map $\exp_o: T_o(M) \rightarrow M$ is a homeomorphism. Therefore the isotropy subgroup of $I^0(M)$ at o is a maximal compact subgroup of $I^0(M)$. Therefore, by Corollary 1 the proof is completed.

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