Title	Positivity of Eta Products : a Certain Case of K. Saito's Conjecture
Author(s)	Ibukiyama, Tomoyoshi
Citation	Publications of the Research Institute for Mathematical Sciences. 2005, 41(3), p. 683-693
Version Type	VoR
URL	https://hdl.handle.net/11094/3071
rights	
Note	

Positivity of Eta Products
—a Certain Case of K. Saito's Conjecture

By

Tomoyoshi Ibukiyama*

Abstract

We prove that for any prime p, the Fourier coefficients of $\eta(p\tau)^p/\eta(\tau)$ are all non-negative, where $\eta(\tau)$ is the Dedekind eta function. This is a proof of some parts of K. Saito's conjecture on such positivity of eta products associated with regular systems of weight.

§1.   Introduction

Let $\eta(\tau)$ be the Dedekind eta function defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

where $q = e^{2\pi i \tau}$ and $\tau \in \mathbb{C}$, $Im(\tau) > 0$. In his theory of extended affine root systems and other things, K. Saito treated eta products of the form

$$\prod_i \eta(i\tau)^{e(i)}$$

where $e(i)$ are integers which might be negative, and considered the condition that the coefficients of the q-expansion of this function are all non-negative. For example, in his paper [3], he defined a notion of elliptic eta product and he proved that an eta product of this kind has only non-negative coefficients if

and only if this is not a cusp form. There are exactly four such eta products. These cases are examples of his more general conjecture on the positivity of eta products defined by "regular systems of weight" ([3], [4]). Apparently irrelevant to this, he also gave a conjecture in his paper [5] that for any natural number h the eta product

$$\frac{\eta(h\tau)^{\phi(h)}}{\prod_{d|h} \eta(d\tau)^{\mu(d)}}$$

has only non-negative Fourier coefficients, where $\phi(h)$ is the Euler function and $\mu(d)$ is the Möbius function. He has proved this conjecture for $h = 2$, 3, 5, 6, 10. When h is a prime p or a product of two different primes p, q, we can see that the latter conjecture is contained in the former conjecture. (Put $h = p$, $a = (p-1)/2$, $b = c = 1$, or $h = pq$, $a = p$, $b = q$, $c = 1$ in [4]).

The aim of this paper is to prove that this conjecture is true when h is a power of any prime p.

Main Theorem

(1) *For any prime p, all the Fourier coefficients of*

$$\frac{\eta(p\tau)^p}{\eta(\tau)} = q^{(p^2-1)/12} \prod_{n=1}^{\infty} (1 - q^{pn})^p (1 - q^n)^{-1}$$

are non-negative.

(2) *For any prime p and any natural number a, all the Fourier coefficients of*

$$\frac{\eta(p^a\tau)^{p^a - p^{a-1}} \eta(p\tau)}{\eta(\tau)}$$

are non-negative.

The assertion (2) is an easy corollary of the assertion (1) as we see in §4. A key point of the proof of (1) is to express this function as a difference $\theta_{L_1}(\tau) - \theta_{L_2}(\tau)$ of theta functions associated with a lattice L_1 and a sublattice $L_2 \subset L_1$ up to constant. To find such lattices, the theory of cyclotomic fields is helpful. After giving a useful characterization of our eta products in §2, we explain lattices in the p-th cyclotomic fields in §3. In §4, using these lattices, we prove our Main Theorem. As soon as we discover the lattices, we can define lattices directly in down to earth fashion without theory of cyclotomic fields. We also explain this in §5. The case where h has at least two distinct prime factors is not clear at moment and we shall give a short comment at the end of this paper. The whole nature of the conjecture seems still conceptually unclear.

I would like to thank K. Saito for his clear talk on his conjectures in the Second Spring Conference on Modular Forms and Related Topics in 2003 at

Hamana Lake and useful discussions. The content of his talk was published as [5]. This mathematical exchange gave me a motivation to do the present work.

§2.   Preliminaries

For a sake of simplicity, for any prime p we write

$$f_p(\tau) = \frac{\eta(p\tau)^p}{\eta(\tau)}.$$

First we give some characterization of $f_p(\tau)$. When $p = 2$ or 3, it is better to take $f_2(8\tau)$ or $f_3(3\tau)$ instead. But since these cases are known already by Saito (cf. [3], [4]) and a correction we need in these cases is almost trivial, we assume $p \geq 5$ from now on. We put

$$\Gamma_0(p) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); c \equiv 0 \bmod p \right\}.$$

We define a character ψ of $\Gamma_0(p)$ by

$$\psi(\gamma) = \left(\frac{(-1)^{(p-1)/2}p}{d} \right)$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$ and the right hand side is the Kronecker symbol associated with the quadratic field $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$. A holomorphic function $f(\tau)$ on the upper half plane H is called a modular form of weight k of $\Gamma_0(p)$ with character ψ if

$$f(\gamma\tau) = \psi(\gamma)(c\tau + d)^k f(\tau)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$ and besides if f is bounded at each cusp. We denote by $A_k(\Gamma_0(p), \psi)$ the space of such modular forms. It is known by K. Saito (cf. [3] Lemma 1) that when $p \geq 5$ we have $f_p(\tau) \in A_{(p-1)/2}(\Gamma_0(p), \psi)$ and $f_p(\tau)$ does not vanish at the cusp 0. Directly from the definition we see that the order of zero of $f_p(\tau)$ at the cusp $i\infty$ is $(p^2 - 1)/24$. The following lemma is trivial but crucial.

Lemma 2.1.   *If a non-zero modular form $f(\tau) \in A_{(p-1)/2}(\Gamma_0(p), \psi)$ has a zero at $i\infty$ at least of order $(p^2 - 1)/24$, then $f_p(\tau)$ is a constant multiple of $f(\tau)$.*

Proof. Since $f_p(\tau)$ does not vanish on any point of H and at the cusp 0, the condition of the order of $f(\tau)$ at $i\infty$ implies that $f(\tau)/f_p(\tau)$ is a holomorphic function of the compact Riemann surface $\overline{\Gamma_0(p)\backslash H}$. Hence this is a constant. $\square$

We note that we cannot prove this kind of theorem by usual Riemann Roch theorem easily since we cannot conclude $H^1 = 0$ automatically.

We shall use theta functions to describe $f_p(\tau)$, so we review the theory of theta functions of lattices. We take a positive definite quadratic form $Q(x)$ on a vector space V of dimension n over $\mathbb{Q}$. We define the symmetric bilinear form $B(x, y)$ associated with $Q(x)$ by $B(x, y) = (Q(x+y) - Q(x) - Q(y))/2$. Let L be a lattice of V. When $L = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$, we define an $n \times n$ symmetric matrix S by $S = (B(\omega_i, \omega_j))$ and $\det(S)$ is called the discriminant of L. When $Q(x) \in 2\mathbb{Z}$ for all $x \in L$, we say that L or S is even integral. A lattice L is even if and only if all the components of S are integral and the diagonal components are even besides. We define a theta function associated with L or S by

$$\theta_L(\tau) = \theta_S(\tau) = \sum_{x \in L} e^{\pi i Q(x)\tau} = \sum_{x \in \mathbb{Z}^n} e^{\pi i \, (^t x S x)\tau}.$$

We define the dual L^* of L by

$$L^* = \{y \in V; B(x, y) \in \mathbb{Z} \text{ for all } x \in L\}.$$

The level of L is the least natural number N such that $\sqrt{N}L^*$ is even integral. This is also the least natural number such that NS^{-1} is even integral. The following proposition is classically well known (e.g. see [2] p. 63).

Proposition 2.2. *If L is a positive definite even integral lattice of level N of dimension $2k$ with discriminant $\det(S)$, then $\theta_L(\tau) \in A_k(\Gamma_0(N), \chi)$ where we put $\chi(\gamma) = \left(\frac{(-1)^k \det(S)}{d} \right)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$*

It is clear that if $L_2 \subset L_1$ are lattices in the same vector space V, then Fourier coefficients of $\theta_{L_1}(\tau) - \theta_{L_2}(\tau)$ are all non-negative, since the coefficient of q^n are the number of vectors $x \in L_1 \backslash L_2$ such that $Q(x) = 2n$. In next section, we find such pair of lattices to express our $f_p(\tau)$.

§3.   Cyclotomic Fields

To discover the lattices we want, a general theory of cyclotomic field is helpful. Let p be a prime such that $p \geq 5$ and put $\zeta = e^{2\pi i/p}$. The cyclotomic

field $V = \mathbb{Q}(\zeta)$ is an abelian extension of degree $p - 1$ over $\mathbb{Q}$. The ring $\mathfrak{O}$ of algebraic integers in V is given by $\mathfrak{O} = \mathbb{Z}[\zeta]$. The unique prime ideal of $\mathfrak{O}$ over p is given by $\mathfrak{P} = (1 - \zeta)\mathfrak{O}$ and we have $\mathfrak{P}^{p-1} = p\mathfrak{O}$. The discriminant of V over $\mathbb{Q}$ is known to be p^{p-2}, and this implies that $\mathrm{Tr}_{V/\mathbb{Q}}(xy) \in \mathbb{Z}$ for all $y \in \mathfrak{O}$ if and only if $x \in \mathfrak{P}^{-p+2}$, where $\mathrm{Tr}_{V/\mathbb{Q}}$ is the usual trace from V to $\mathbb{Q}$. We regard V as a $(p-1)$-dimensional vector space over $\mathbb{Q}$ and denote by $Q(x)$ the positive definite quadratic form on V defined by

$$Q(x) = \frac{1}{p}\mathrm{Tr}_{V/\mathbb{Q}}(x\overline{x})$$

for any $x \in V$. If we regard $\mathfrak{O}$ as a lattice in V, then since the usual discriminant of V is p^{p-2}, the discriminant of $\mathfrak{O}$ with respect to $Q(x)$ is $p^{p-2}/p^{p-1} = p^{-1}$. Since $\#(\mathfrak{O}/\mathfrak{P}^k) = p^k$, the discriminant of $\mathfrak{P}^k$ is $p^{-1}(p^k)^2 = p^{2k-1}$. It is easy to see that the prime ideal $\mathfrak{P}$ is an even lattice with respect to Q which is nothing but the root lattice A_{p-1} of rank $p - 1$ as shown in [1]. For any natural number k, we have $\mathfrak{P} \supset \mathfrak{P}^k$, so the lattice of the ideal $\mathfrak{P}^k$ is even. The dual of $\mathfrak{P}^k$ with respect to $Q(x)$ is $p\mathfrak{P}^{-p+2-k} = \mathfrak{P}^{-k+1}$ and $\sqrt{p}\mathfrak{P}^{-k+1}$ is even integral if and only if $(p-1)/2 + (1-k) \geq 1$ namely $k \leq (p-1)/2$. Hence, if we put $L_1 = \mathfrak{P}^{(p-3)/2}$ and $L_2 = \mathfrak{P}^{(p-1)/2}$, then the level of L_1 or L_2 is p and the discriminant is p^{p-4} or p^{p-2}. This means that $\theta_{L_1}(\tau)$, $\theta_{L_2}(\tau) \in A_{(p-1)/2}(\Gamma_0(p), \psi)$. The Fourier coefficients of $\theta_{L_1}(\tau) - \theta_{L_2}(\tau)$ are of course all non-negative. Now by virtue of Lemma 2.1, all we should show is that this has zero at $i\infty$ of order $(p^2 - 1)/24$.

§4. Order of Zero at Infinity

We shall show that for any $n < (p^2 - 1)/12$ we have $\{x \in L_1; Q(x) = 2n\} = \{x \in L_2; Q(x) = 2n\}$. We take a suitable $\mathbb{Z}$ basis of L_1 and L_2 and write down the quadratic form $Q(x)$ more explicitly. Since the element $1 - \zeta^k$ is a generator of $\mathfrak{P}$ for any integer k with $p \nmid k$, we have $L_2 = \mathfrak{P}^{(p-1)/2} = \prod_{t=1}^{(p-1)/2}(1 - \zeta^t)\mathfrak{O}$, and we can take $\omega_k = \zeta^k \prod_{t=1}^{(p-1)/2}(1 - \zeta^t)$ $(1 \leq k \leq p - 1)$ as a $\mathbb{Z}$-basis of L_2. We have $\prod_{t=1}^{(p-1)/2}(1 - \zeta^t)(1 - \zeta^{-t}) = p$ and

$$\mathrm{Tr}(\zeta^k) = \begin{cases} -1 & \text{if } p \nmid k \\ p - 1 & \text{if } p \mid k. \end{cases}$$

So $B(\omega_k, \omega_j) = -1 + \delta_{kj}p$ where δ_{kj} is Kronecker's delta. For $x = \sum_{k=1}^{p-1} x_k\omega_k \in L_2$ $(x_k \in \mathbb{Z})$, we see

$$Q(x) = \sum_{k=1}^{p-1}(p-1)x_k^2 - 2\sum_{1 \leq j < k \leq p-1} x_j x_k = \sum_{k=1}^{p-1} x_k^2 + \sum_{1 \leq j < k \leq p-1}(x_k - x_j)^2.$$

It is easily observed that $Q(x) \geq p-1$ for all $x \in L_2 \backslash \{0\}$, though we do not need this later. Next, we describe elements in L_1. Since $L_1 = \prod_{t=2}^{(p-1)/2}(1 - \zeta^t)\mathfrak{O} = (1 - \zeta)^{-1}L_2$ and $\mathfrak{O}/\mathfrak{P}$ is represented by $a = 0, 1, \ldots, p-1$, any element y in L_1 is written as

$$y = (a + (1 - \zeta)x) \prod_{t=2}^{(p-1)/2} (1 - \zeta^t)$$

where $a \in \mathbb{Z}$ and $x = \sum_{k=1}^{p-1} x_k \zeta^k \in \mathfrak{O}$ with $x_k \in \mathbb{Z}$. Since

$$\frac{1}{1 - \zeta} = -\frac{1}{p}\sum_{k=1}^{p-1} k\zeta^k$$

we have

$$y = \left(\sum_{k=1}^{p-1}(x_k - \frac{ak}{p})\zeta^k\right) \prod_{t=1}^{(p-1)/2} (1 - \zeta^t).$$

This implies that

$$Q(y) = \sum_{k=1}^{p-1} \left(x_k - \frac{ak}{p}\right)^2 + \sum_{1 \leq j < k \leq p-1} \left(x_k - \frac{ak}{p} - x_j + \frac{aj}{p}\right)^2.$$

For $a \in \mathbb{Z}$ and $x \in \mathfrak{O}$, we have $a + (1 - \zeta)x \in \mathfrak{P}$ if and only if $a \in p\mathbb{Z}$. Hence if $y \in L_1$ and $y \notin L_2$, we see $a \not\equiv 0 \bmod p$. In other words, if $y \in L_1$ and $y \notin L_2$, then by taking $y_k = px_k - ak$, we see that there are $y_k \in \mathbb{Z}$ with $1 \leq k \leq p-1$ which satisfy the following two conditions (1) and (2).

(1) $\{y_1, \ldots, y_{p-1}\}$ is a complete set of representatives of non-zero elements of $\mathbb{Z}/p\mathbb{Z}$.

(2) We have

$$p^2 Q(y) = \sum_{k=1}^{p-1} y_k^2 + \sum_{1 \leq j < k \leq p-1} (y_k - y_j)^2.$$

The following lemma is a key to our result.

Lemma 4.1. *For any integers y_k $(1 \leq k \leq p-1)$, put*

$$P(y) = \sum_{k=1}^{p-1} y_k^2 + \sum_{1 \leq j < k \leq p-1} (y_j - y_k)^2.$$

If $p \nmid y_k$ for any k and $y_j \not\equiv y_k \bmod p$ for any $j \neq k$, then we have

$$P(y) \geq \frac{p^2(p^2 - 1)}{12}.$$

Proof. To prove this in a smart way, we consider the quadratic form Q^* of p variables z_k $(1 \leq k \leq p)$ defined by

$$Q^*(z_1, \ldots, z_{p-1}, z_p) = \sum_{1 \leq j < k \leq p} (z_k - z_j)^2.$$

We assume that z_k $(1 \leq k \leq p)$ are integers and that $\{z_1, \ldots, z_p\}$ is a complete set of representatives of $\mathbb{Z}/p\mathbb{Z}$. Renumbering if necessary, we may assume that $z_1 \leq \cdots \leq z_p$. Since $z_j \neq z_k$ for any $k \neq j$, we have $z_k + 1 \leq z_{k+1}$ for $1 \leq k \leq p - 1$ and hence for $j < k$ we have $0 < k - j \leq z_k - z_j$. This implies

$$\begin{aligned}
Q^*(z_1, \ldots, z_p) &\geq \sum_{1 \leq j < k \leq p} (k - j)^2 \\
&= \frac{1}{2} \sum_{j,k=1}^{p} (k^2 + j^2 - 2kj) \\
&= \frac{p^2(p+1)(2p+1)}{6} - \left(\frac{p(p+1)}{2}\right)^2 \\
&= \frac{p^2(p^2 - 1)}{12}.
\end{aligned}$$

Now, let y_k be as in the lemma. Then $\{y_1, \ldots, y_{p-1}, 0\}$ is a complete set of representatives of $\mathbb{Z}/p\mathbb{Z}$ and

$$P(y) = Q^*(y_1, \ldots, y_{p-1}, 0) \geq \frac{p^2(p^2 - 1)}{12}.$$

Thus we prove the lemma. By the way, in the above proof, we did not use the assumption that p is a prime.

By this lemma, we see that

$$f_p(\tau) = \frac{1}{p(p-1)} \left(\theta_{L_1}(\tau) - \theta_{L_2}(\tau)\right)$$

which has only non-negative Fourier coefficients. So we prove (1) of our main theorem. (The coefficient $1/p(p-1)$ comes from $p-1$ numbers of $a \not\equiv 0 \bmod p$ and the p numbers of choice of a continuous sequence of p numbers including 0.)

Now we show (2) of Main Theorem. Namely we show that for any natural number a, the modular form

$$\frac{\eta(p^a\tau)^{\phi(p^a)}\eta(p\tau)}{\eta(\tau)} = \frac{\eta(p^a\tau)^{p^a - p^{a-1}}\eta(p\tau)}{\eta(\tau)}$$

has only non-negative Fourier coefficients. We prove this by induction on a. If $a = 1$ then the above modular form is $\eta(p\tau)^p/\eta(\tau)$, so the claim was already proved. We assume $a \geq 2$ and suppose that the claim is true for $a - 1$. The above modular form is equal to

$$\left(\frac{\eta(p^a\tau)^p}{\eta(p^{a-1}\tau)}\right)^{\phi(p^{a-1})} \times \frac{\eta(p^{a-1}\tau)^{\phi(p^{a-1})}\eta(p\tau)}{\eta(\tau)}.$$

Here $\eta(p^a\tau)^p/\eta(p^{a-1}\tau) = f_p(p^{a-1}\tau)$ has of course only non-negative Fourier coefficients. So does the second function by inductive assumption. Hence inductively we see that the Fourier coefficients are non-negative. $\square$

§5. Direct Approach

It would be more convincing if we give lattices more directly. We explain this in this section. We assume that p is any natural number with $p \geq 2$ not necessarily a prime. We consider the following quadratic form $P(x)$ of $x = (x_1, \ldots, x_{p-1}) \in \mathbb{Q}^{p-1}$.

$$P(x) = \sum_{k=1}^{p-1} x_k^2 + \sum_{1 \leq j < k \leq p-1} (x_k - x_j)^2.$$

We consider two lattices M_1 and M_2 defined by

$$M_1 = \text{ the lattice generated by } p^{-1}(1, 2, \ldots, p - 1) \in \mathbb{Q}^{p-1}$$
$$\text{and } e_i \text{ with } 2 \leq i \leq p - 1,$$
$$M_2 = \mathbb{Z}^{p-1},$$

where e_i is the vector in $\mathbb{Q}^{p-1}$ such that the i-th component is one and the other components are zero. Then the $(p - 1) \times (p - 1)$ symmetric matrix S_2 associated with M_2 with respect to P is given by

$$S_2 = \begin{pmatrix} p-1 & -1 & -1 & \cdots & -1 \\ -1 & p-1 & -1 & \ldots & -1 \\ \vdots & -1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \ldots & \ldots & p-1 \end{pmatrix}.$$

Then it is obvious that the symmetric matrix S_1 associated with M_1 is given by $RS_2\,{}^t R$ where we put

$$
R = \begin{pmatrix}
p^{-1} & 2p^{-1} & 3p^{-1} & \cdots & (p-1)p^{-1} \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & 0 & 1 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}.
$$

Since

$$
S_1 = RS_2\,{}^t R = \begin{pmatrix}
(p^2-1)/12 & 2-(p-1)/2 & 3-(p-1)/2 & \cdots & (p-1)/2 \\
2-(p-1)/2 & p-1 & -1 & \cdots & -1 \\
3-(p-1)/2 & -1 & p-1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(p-1)/2 & -1 & \cdots & \cdots & p-1
\end{pmatrix}
$$

we see that S_1 is also even integral if $p^2 \equiv 1 \bmod 24$, namely if p is odd and $3 \nmid p$. We see directly that

$$
pS_2^{-1} = \begin{pmatrix}
2 & 1 & 1 & \cdots & 1 \\
1 & 2 & 1 & \cdots & 1 \\
1 & 1 & 2 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 2
\end{pmatrix}
$$

Hence S_2 is of level p. By calculating $pS_1^{-1} = {}^t R^{-1}(pS_2^{-1})R^{-1}$ we see that the level of S_1 is also p. Besides, we have $\det(S_2) = p^{p-2}$ and $\det(S_1) = p^{p-4}$. Indeed we can prove it by induction on p. We write $S_2 = S_2(p)$ to make its dependence on p clearer. We have $\det S_2(2) = 1$ so the claim is true for $p = 2$. Assume that $\det S_2(p-1) = (p-1)^{p-3}$. If we put

$$
T = \begin{pmatrix}
1 & 0 \\
(p-1)^{-1}b & 1_{p-2}
\end{pmatrix}
$$

where $b = {}^t(1,1,\ldots,1) \in \mathbb{Z}^{p-2}$, then we have

$$
TS_2(p)\,{}^t T = \begin{pmatrix}
p-1 & 0 \\
0 & \frac{p}{p-1}S_2(p-1)
\end{pmatrix}
$$

Hence $\det(S_2(p)) = (p - 1) \times p^{p-2}(p-1)^{-p+2} \det(S_2(p-1)) = p^{p-2}$. Since $\det(R) = p^{-1}$ we get $\det(S_1) = p^{p-4}$. As a conclusion, M_1 and M_2 are positive definite even integral lattices of level p having odd power of p as their discriminants if $p^2 \equiv 1 \bmod 24$. So the result in the last section is now written as

$$\frac{\eta(p\tau)^p}{\eta(\tau)} = \frac{1}{p(p-1)}(\theta_{S_2}(\tau) - \theta_{S_1}(\tau))$$

for any prime $p \geq 5$.

Since Lemma 4.1 is valid for any natural number p, we get the following non trivial claim by the same argument. Let N be a positive odd number such that $N^2 \equiv 1 \bmod 24$ (namely $3 \nmid N$), and ψ be a character of $\Gamma_0(N)$ defined by $\psi(\gamma) = \left(\frac{(-1)^{(N-1)/2} N}{d} \right)$. Then there is a modular form $f \in A_{(N-1)/2}(\Gamma_0(N), \psi)$ such that all the Fourier coefficients of the q-expansion of f at $i\infty$ is non-negative and the order of zero of f at $i\infty$ is at least $(N^2 - 1)/24$.

§6.   Concluding Remarks

K. Saito's general conjecture claims that for a positive integer $h \geq 2$, the eta product

$$\frac{\eta(h\tau)^{\phi(h)}}{\prod_{d|h} \eta(d\tau)^{\mu(d)}}$$

has only non-negative Fourier coefficients.

When h is a prime or a power of prime, we have already proved this conjecture. The simplest case where h has two distinct prime factors is the case $h = 6$ which has already been proved by K. Saito by using Euler factors. We can give an alternative proof for this case by using lattices as follows. Consider the quadratic form on $\mathbb{Q}^2$ defined by $2(x^2 + y^2)$. Define four lattices by

$$L_1 = \{(x, y) \in \mathbb{Z}^2 ; x + y \equiv 0 \bmod 3\}$$
$$L_2 = \{(x, y) \in L_1 ; x \equiv y \equiv 0 \bmod 3\}$$
$$L_3 = \{(x, y) \in L_1 ; x + y \equiv 0 \bmod 2\}$$
$$L_4 = L_2 \cap L_3.$$

Then we have

$$4 \times \frac{\eta(72\tau)\eta(36\tau)\eta(24\tau)}{\eta(12\tau)} = \theta_{L_1}(\tau) - \theta_{L_2}(\tau) - \theta_{L_3}(\tau) + \theta_{L_4}(\tau).$$

The positivity is proved easily since the coefficient of q^n of the right hand side appears as the number of vectors $x \in L_1$ not in $L_2 \cup L_3$ of length $2n$. This

is essentially the same description by the root lattice G_2 given by V. Kac (cf. [5]). We do not know at moment if we can expect the same kind of expression for more general case.

References

[1] Ebeling, W., *Lattices and Codes, A course partially based on Lectures by F. Hirzebruch*, Adv. Lectures Math., Vieweg, Braunschweig, Wiesbaden, 1994.

[2] Eichler, M., *Einführung in die Theorie der algebraischen Zahlen und Funkitionen*, Birkhäuser Verlag, Basel, Stuttgart, 1963.

[3] Saito, K., Extended affine root systems V. Elliptic eta-products and their Dirichlet series, *Proceedings on Moonshine and related topics* (Montréal, QC, 1999)*, 185-222, CRM Proc. Lecture Notes*, **30**, Amer. Math. Soc. Providence, RI, 2001.

[4] ———, Duality for regular systems of weights. Mikio Sato: a great Japanese mathematician of the twentieth century, *Asian J. Math.*, **2** (1998), 983-1047.

[5] ———, Non-negativity of Fourier Coefficients of Eta-products (in Japanese), *Proceedings of the Second Spring Conference on Modular Forms and Related Topics*, 2003 at Hamana Lake, ed. T. Ibukiyama, (2004), 95-142.