The Temperature Dependence of the Phonon velocity and the Roton minimum in Liquid He II

by Takeji KEBUKAWA

Contents

The temperature dependence of the Phonon velocity and the roton minimum in liquid He II

Takeji KEBUKAWA

Department of Physics, College of General Education
Osaka University, Toyonaka

Die Abhängigkeit der Phonongeschwindigkeit und des Rotonminimums von Temperatur wird aus dem mikroskopishen Gesichtspunkt untersucht. Durch die Überlegung des Beitrags aus der in der früherer Abhandlung gegebenen Wechselwirkung zwischen Quasiteilchen finden wir, dass die Phonon-Roton-wechselwirkung und die Roton-Rotonwechselwirkung spielen eine grosse Rolle in der Abhängigkeit der Phonongeschwindigkeit und des Rotonminimums von Temperatur. Das Resultat der Rechnung stimmt mit dem Experiment überein.

§1. Introduction

The elementary excitations in liquid helium II have been the object of extensive study for many years. The phonon-roton type excitation spectrum was proposed phenomenologically by Landau(1) to explain superfluid properties and thermodynamic bahavior of liquid helium II below the λ-transition temperature. The existence of such a spectrum was demonstrated by the direct observation of single excitations from inelastic scattering of slow neutrons.(2) The curve H.W. in Fig B-2 indicates the experimental results obtained by Henshaw and Woods at T=0°K. The experimental spectrum H.W. clearly shows the existence of the phonon-roton type excitations, as has been proposed by Landau.(1)

A quantum theory of the excitations was developed by Feynman and Cohen.(3) In their calculation, the excitation energy is expressed in terms of the structure factor of liquid helium. Inserting the observed results into the structure factor in the formula of the excitation energy, they obtained the excitation spectrum of which agreement with experimental one was good qualitatively but was not quantitatively.

On the other hand, progress has been made along the microscopic point of view by many authors. Bogoliubov(4) derived a phonon-like elementary excitation spectrum in the low momentum region on the basis of the assumption that the finite fraction of helium atom was condensed into the zero-momentum state at T = 0°K, and hence the creation and the annihilation operator of helium atom at zero momentum (a_0 and a_0^+) might be replaced by N_0. Parry and ter Haar(5), however, have shown that the depletion of the zero-momentum state amounts to 270% if we take the Brueckner-Sawada[(6)] excitation spectrum which was obtained on the basis of Bogoliubov formalism and the method of t-matrix.

Miller, Pines and Nozieres[(7)] have emphasized that the phonon-phonon interaction is indispensable to account for the observed excitation spectrum. In the Bogoliubov formalism, however, the assumption that $a_0 = a_0^+ = N_0$ = c-number ignores the quantum fluctuation of the number of condensed atoms and hence does not lead to a unique expression for the phonon-phonon interaction. This difficulty has been overcome by Sunakawa et al.[(8)] by describing a Bose system in terms of the colletive variables which are appropriate to represent the liquid system, and the theory gives a microscopic theoretical version of the Landau quantum-hydrodynamics. The introduction of the collective variables yields a definite expression of the phonon-phonon interactions.

By taking into account the contributions of the definite phonon-phonon interaction, Sunakawa, Yamasaki and the present author[9),10)] have developed the theory of the excitation energy of liquid helium II at T = 0°K on the basis of the method which is slightly different from that of the standard perturbation theory, and obtained the convergent results for the correction of the excitation energy. Their results of the phonon-roton type excitation energy which are shown by the curve $E_{\mathbf{k}}^{I}$ in Fig B-2 are in good harmony with experimental curve H.W. On the basis of the same convergent formula for the excitation energy at $T = 0°K$, recently, they[11] have derived the multi-branch structure of the excitation spectra which have been confirmed experimentally by Woods et al..[12]

The temperature dependence of the excitation spectrum in liquid helium II, especially the phonon velocity and the roton minimum, was observed experimentally by many authors. Recently Dietrich et al.[13] have made measurements of the temperature dependence of the roton minimum by inelastic neutron scattering. Fig. 5 shows their experimental results which indicate that the roton-minimum decreases as temperature increases. Very accurate measurements of the temperature dependence of the phonon velocity have been made by Whitney and Chase by making use of pulse techniques developed by Chase. Their results for $1 M_c/s$ sound waves in helium under the saturated vapour pressure are shown in Fig. 3, which is written in a greatly expanded scale for the phonon velocity. A small maximum in the phonon velocity is observed at about 0.65°K. Fig. 4 shows the whole behavior of the temperature dependence of the phonon velocity, and shows rapid decrease of the phonon velocity in the region of temperature from 1.5°K to 2.19°K. A theoretical interpretation of the existence of a small maximum in the phonon velocity has been given by Khalatonikov et al.[15] on the basis of the phonomeno-logical kinetic equations for the distribution functions of phonons and rotons which are regarded as classical gases, respectively. From microscopic point of view, ter Haar et al.[16]

have developed a theory for the temperature dependence of the phonon velocity on the basis of the perturbation theory at finite temperature, by assuming Landau[1] phenomenological phonon-phonon interactions. They, however, considered only a part of all diagrams which contribute to the self-energy, and the neglected diagrams give divergent results.

The purpose of this paper is to calculate the temperature dependence of the phonon velocity observed by Whitney and Chase and of the roton minimum revealed by Dietrich et al., by taking into account the effect of all diagrams which contribute to the second order self-energy. The difficulty of the divergence of the self-energy which arises from the diagrams ignored by ter Haar and others will be removed with the aid of the method proposed in the previous papers[9],[10] (refered to as I and II hereafter), and later by Nishiyama.[17]

Section 2 is devoted to the derivation of a divergence free formula for the excitation spectrum at finite temperature. In §3, the temperature dependence of the phonon velocity is calculated quantitatively on the basis of the formula derived in §2, and is compared with experimental results[14] to have a good agreement. In §4, the temperature dependence of the roton minimum is calculated to yield also a fairly good agreement with experiments.[13] In Appendix A, we shall present a résumé of the collective description. Some lemmas which are necessary for the collective description will be proved in Appendix C. The formula of the excitation energy and its results will be shown in Appendix B.

§2. The excitation energy at finite temperature

Consider a system of N interacting helium atoms of mass m enclosed in a cubic box of volume V. The Hamiltonian of this system is described in terms of the density fluctuation operator ρ_k and the velocity operator v_k [9],[10],[18] in the following form

$$H = E_0^B + \sum_k E_k^B B_k^\dagger B_k + \sum_{\substack{p,q \neq 0 \\ p+q \neq 0}} \frac{1}{3}\Gamma_a(p,q)\left(B_p^\dagger B_q^\dagger B_{-p-q}^\dagger + B_{-p-q} B_q B_p\right)$$

$$+ \sum_{\substack{p,q \neq 0 \\ p+q \neq 0}} \Gamma_b(p,q)\left(B_q^\dagger B_{p+q}^\dagger B_p + B_p^\dagger B_{p+q} B_q\right) \qquad (2.1)$$

up to the order of $N^{-\frac{1}{2}}$, where

$$E_0^B = \frac{N(N-1)}{2V}\nu(0) + \frac{1}{2}\sum_{k \neq 0}\left(E_k^B - \frac{k^2}{2m} - \frac{N}{V}\nu(k)\right)$$

gives the ground state energy in the lowest approximation. The Bogoliubov excitation energy E_k^B [4] is expressed by

$$E_k^B = \frac{k^2}{2m}\Big/\lambda_k = \frac{k}{2m}\sqrt{k^2 + c^2(k)} \qquad (2.2)$$

and the structure factor λ_k in the lowest approximation is

$$\lambda_k = k\Big/\sqrt{k^2 + c^2(k)} \qquad (2.3)$$

, where

$$C(\mathbf{k}) = \sqrt{\frac{4Nm}{V}\nu(\mathbf{k})}$$

and $\nu(\mathbf{k})$ indicates the Fourier transform of the interaction potential between two helium atoms. The functions $\Gamma_a(\mathbf{p},\mathbf{q})$ and $\Gamma_b(\mathbf{p},\mathbf{q})$ in (2.1) are given by

$$\Gamma_a(\mathbf{p},\mathbf{q}) = \frac{1}{8m\sqrt{N}}\sqrt{\lambda_p\lambda_q\lambda_{p+q}}\left\{(\mathbf{p}\cdot\mathbf{q})\left(1+\frac{1}{\lambda_p\lambda_q}\right)-(\mathbf{p}\cdot\mathbf{q+p})\left(1+\frac{1}{\lambda_p\lambda_{p+q}}\right)-(\mathbf{q}\cdot\mathbf{p+q})\left(1+\frac{1}{\lambda_q\lambda_{p+q}}\right)\right.$$

and

$$\Gamma_b(\mathbf{p},\mathbf{q}) = \frac{1}{8m\sqrt{N}}\sqrt{\lambda_p\lambda_q\lambda_{p+q}}\left\{(\mathbf{p}\cdot\mathbf{q})\left(1-\frac{1}{\lambda_p\lambda_q}\right)-(\mathbf{p}\cdot\mathbf{p+q})\left(1-\frac{1}{\lambda_p\lambda_{p+q}}\right)-(\mathbf{q}\cdot\mathbf{p+q})\left(1+\frac{1}{\lambda_q\lambda_{p+q}}\right)\right. . \tag{2.4}$$

The detailed derivation of the Hamiltonian (2.1) will be found in Appendix A.

The excitation energy in liquid helium II at finite temperature is calculated by making use of the method of a temperature Green-function[19] defined by

$$\tilde{G}(\mathbf{k},\tau-\tau') = -\mathrm{Tr}\left(e^{\beta(\Omega-H)}T_\tau\left(B_{\mathbf{k}}(\tau)B^{\dagger}_{\mathbf{k}}(\tau')\right)\right) \tag{2.5}$$

, where

$$B_{\mathbf{k}}(\tau) = e^{H\cdot\tau} B_{\mathbf{k}} e^{-H\cdot\tau}$$

The symbol T_τ indicates the chronological operator concerning a temperature τ . The second order self-energy is given by the following three diagrams shown in Fig.-1.

Fig. 1

From three diagrams (a), (b), (c) in Fig.-1, we can readily write down the expressions

$$\tilde{\Sigma}^{(a)}(\mathbf{k},\omega_n) = 18 \times \frac{-1}{\beta} \sum_{n'} \sum_{\mathbf{p}} \left(\frac{\Gamma_a(\mathbf{k},\mathbf{p})}{3} \right)^2 \tilde{G}_0(\mathbf{p},\omega_{n'}) \tilde{G}_0(-\mathbf{p}-\mathbf{k},-\omega_{n'}-\omega_n), \quad (2.6)$$

$$\tilde{\Sigma}^{(b)}(\mathbf{k},\omega_n) = 2 \times \frac{-1}{\beta} \sum_{n'} \sum_{\mathbf{p}} \Gamma_b(\mathbf{k},\mathbf{p})^2 \tilde{G}_0(\mathbf{p},\omega_{n'}) \tilde{G}_0(\mathbf{p}+\mathbf{k},-\omega_{n'}+\omega_n) \quad (2.7)$$

and

$$\tilde{\Sigma}^{(c)}(\mathbf{k},\omega_n) = 4 \times \frac{-1}{\beta} \sum_{n'} \sum_{\mathbf{p}} \Gamma_b(\mathbf{p},\mathbf{k})^2 \tilde{G}_0(\mathbf{p},\omega_{n'}) \tilde{G}_0(\mathbf{p}-\mathbf{k},\omega_{n'}-\omega_n) \quad (2.8)$$

for the proper self-energy. In the above equations, $\tilde{G}_0(\mathbf{p},\omega_n)$ is the Fourier transform of the free temperature Green-function given by

$$\tilde{G}_0(\mathbf{p},\omega_n)=\frac{1}{i\omega_n-E_{\mathbf{p}}^B} \tag{2.9}$$

, where $\omega_n = 2n\pi/\beta$ $(n = 0, \pm 1, \ldots.)$. After taking the summation concerning n and replacing $i\omega_n$ by $E_K^B+i\eta$, we get the second order excitation energy at a temperature $T = (K_B\beta)^{-1}$ in the following form,

$$\Sigma^{(a)}(\mathbf{k},E_{\mathbf{k}}^B)=-2\sum_{\mathbf{p}}\frac{\Gamma_a(\mathbf{k},\mathbf{p})^2}{E_{\mathbf{k}}^B+E_{\mathbf{p}}^B+E_{\mathbf{p+k}}^B}-4\sum_{\mathbf{p}}\frac{\Gamma_a(\mathbf{k},\mathbf{p})^2}{E_{\mathbf{k}}^B+E_{\mathbf{p}}^B+E_{\mathbf{p+k}}^B}n(E_{\mathbf{p}}^B), \tag{2.10}$$

$$\Sigma^{(b)}(\mathbf{k},E_{\mathbf{k}}^B)=2\sum_{\mathbf{p}}\frac{\Gamma_b(\mathbf{k},\mathbf{p})^2}{E_{\mathbf{k}}^B-E_{\mathbf{p}}^B-E_{\mathbf{p+k}}^B+i\eta}+4\sum_{\mathbf{p}}\frac{\Gamma_b(\mathbf{k},\mathbf{p})^2}{E_{\mathbf{k}}^B-E_{\mathbf{p}}^B-E_{\mathbf{p+k}}^B+i\eta}n(E_{\mathbf{p}}^B) \tag{2.11}$$

and

$$\Sigma^{(c)}(\mathbf{k},E_{\mathbf{k}}^B)=-4\sum_{\mathbf{p}}\frac{\Gamma_b(\mathbf{p},\mathbf{k})^2}{E_{\mathbf{k}}^B+E_{\mathbf{p+k}}^B-E_{\mathbf{p}}^B+i\eta}\left(n(E_{\mathbf{p}}^B)-n(E_{\mathbf{p+k}}^B)\right) \tag{2.13}$$

, where

$$n(E_{\mathbf{p}}^B)=\frac{1}{e^{\beta E_{\mathbf{p}}^B}-1}\;.$$

From (2.10), (2.11) and (2.12), the excitation energy $E_{\mathbf{k}}(T)$ at T is written by

$$E_{\mathbf{k}}(T) = E_{\mathbf{k}}^{B} + \Sigma^{(a)}(\mathbf{k}, E_{\mathbf{k}}^{B}) + \Sigma^{(b)}(\mathbf{k}, E_{\mathbf{k}}^{B}) + \Sigma^{(c)}(\mathbf{k}, E_{\mathbf{k}}^{B}) . \qquad (2.14)$$

D. ter Haar and others[16] derived a similar result to calculate the absorption coefficient of the first sound in liquid helium II on the basis of the Landau phenomenological Hamiltonian. They consider only the effect of $\Sigma^{(c)}(\mathbf{k}, E_{\mathbf{k}}^{B})$ in (2.14) and do not take the contributions from $\Sigma^{(a)}(\mathbf{k}, E_{\mathbf{k}}^{B})$ and $\Sigma^{(b)}(\mathbf{k}, E_{\mathbf{k}}^{B})$ into consideration. Since $\Sigma^{(a)}(\mathbf{k}, E_{\mathbf{k}}^{B})$ and $\Sigma^{(b)}(\mathbf{k}, E_{\mathbf{k}}^{B})$ play a very important role in the temperature dependence of the phonon velocity and the roton minimum, as will be seen later, we must consider the effect of the new terms $\Sigma^{(a)}(\mathbf{k}, E_{\mathbf{k}}^{B})$ and $\Sigma^{(b)}(\mathbf{k}, E_{\mathbf{k}}^{B})$ in (2.14). However the first terms on the right hand side of (2.10) and (2.11) are strongly divergent. In order to avoid this serious difficulty, we make use of the method proposed in the previous paper I[9]. In this paper, the excitation spectrum at zero temperature has been calculated to yield a convergent result which is in good agreement with the experiment.[2] Nishiyama[17] derived later essentially the same result by making use of the ordinary perturbation formalism. Since the method of Nishiyama is more convenient for the present approach than the original method in I, we use his method to overcome the difficulty.

Let us now introduce the Feynman energy $E_{\mathbf{k}}^{F}$[3] defined by

$$E_{\mathbf{k}}^{F} = \frac{k^2}{2m} / S(\mathbf{k}) . \qquad (2.15)$$

$S(\mathbf{k})$ is the structure factor of liquid helium II at 0°K:

$$S(\mathbf{k}) = \langle G|\rho_{\mathbf{k}}\rho_{\mathbf{k}}^{\dagger}|G\rangle / \langle G|G\rangle = \lambda_{\mathbf{k}} \langle G|(B_{\mathbf{k}}+B_{-\mathbf{k}}^{\dagger})(B_{-\mathbf{k}}+B_{\mathbf{k}}^{\dagger})|G\rangle / \langle G|G\rangle \quad (2.16)$$

, where $|G\rangle$ denotes the ground state of the total Hamiltonian (2.1). After the straightforward calculation, we have

$$S(\mathbf{k}) = \lambda_{\mathbf{k}} + 4\lambda_{\mathbf{k}} \sum_{\mathbf{p}} \left\{ \frac{\Gamma_a(\mathbf{k},\mathbf{p})^2}{(E_{\mathbf{k}}^B + E_{\mathbf{p}}^B + E_{\mathbf{p}+\mathbf{k}}^B)^2} + \frac{\Gamma_a(\mathbf{k},\mathbf{p})\Gamma_b(\mathbf{k},\mathbf{p})}{E_{\mathbf{k}}^B(E_{\mathbf{k}}^B + E_{\mathbf{p}}^B + E_{\mathbf{p}+\mathbf{k}}^B)} \right\}. \quad (2.17)$$

Fig. 2

Substitution of (2.17) into (2.15) gives

$$E_{\mathbf{k}}^F = E_{\mathbf{k}}^B - 4E_{\mathbf{k}}^B \sum_{\mathbf{p}} \left\{ \frac{\Gamma_a(\mathbf{k},\mathbf{p})^2}{(E_{\mathbf{k}}^B + E_{\mathbf{p}}^B + E_{\mathbf{p}+\mathbf{k}}^B)^2} + \frac{\Gamma_a(\mathbf{k},\mathbf{p})\Gamma_b(\mathbf{k},\mathbf{p})}{E_{\mathbf{k}}^B(E_{\mathbf{k}}^B + E_{\mathbf{p}}^B + E_{\mathbf{p}+\mathbf{k}}^B)} \right\}$$

in the second-order approximation. From (2.14) and (2.18) we get

$$E_{\mathbf{k}}(T) = E_{\mathbf{k}}^F + 2\sum_{\mathbf{p}} \frac{\left\{ \Gamma_a(\mathbf{k},\mathbf{p})(E_{\mathbf{k}}^B - E_{\mathbf{p}}^B - E_{\mathbf{p}+\mathbf{k}}^B) + \Gamma_b(\mathbf{k},\mathbf{p})(E_{\mathbf{k}}^B + E_{\mathbf{p}}^B + E_{\mathbf{p}+\mathbf{k}}^B) \right\}^2}{(E_{\mathbf{k}}^B + E_{\mathbf{p}}^B + E_{\mathbf{p}+\mathbf{k}}^B)^2 (E_{\mathbf{k}}^B - E_{\mathbf{p}}^B - E_{\mathbf{p}+\mathbf{k}}^B + i\eta)}$$

$$- 4\sum_{\mathbf{p}} \frac{\Gamma_a(\mathbf{k},\mathbf{p})^2}{E_{\mathbf{k}}^B + E_{\mathbf{p}}^B + E_{\mathbf{p}+\mathbf{k}}^B} n(E_{\mathbf{p}}^B) + 4\sum_{\mathbf{p}} \frac{\Gamma_b(\mathbf{k},\mathbf{p})^2}{E_{\mathbf{k}}^B - E_{\mathbf{p}}^B - E_{\mathbf{p}+\mathbf{k}}^B + i\eta} n(E_{\mathbf{p}}^B)$$

$$- 4\sum_{\mathbf{p}} \frac{\Gamma_b(\mathbf{p},\mathbf{k})^2}{E_{\mathbf{k}}^B + E_{\mathbf{p}+\mathbf{k}}^B - E_{\mathbf{p}}^B + i\eta} \left\{ n(E_{\mathbf{p}}^B) - n(E_{\mathbf{p}+\mathbf{k}}^B) \right\}. \quad (2.19)$$

The second term on the right hand side of (2.19) is independent of temperature T and gives the correction term for the Feynman energy $E_{\mathbf{k}}^{F}$ at zero temperature. This correction term is strongly convergent in contrast to the first terms in $\sum^{(a)}(\mathbf{k}, E_{\mathbf{k}}^{B})$ and $\sum^{(b)}(\mathbf{k}, E_{\mathbf{k}}^{B})$. On the other hand, the third, fourth and fifth terms on the right hand side of (2.19) give the temperature dependence of the excitation spectrum.

§3. The temperature dependence of the phonon velocity

In this section we shall investigate the temperature dependence of the phonon velocity on the basis of the formula (2.19) derived in the preceding section. The temperature dependent part $\delta E_K(T)$ in (2.19) is given by

$$\delta E_{\mathbf{k}}(T) = \delta E_{\mathbf{k}}^{(a)}(T) + \delta E_{\mathbf{k}}^{(b)}(T) + \delta E_{\mathbf{k}}^{(c)}(T) \tag{3.1}$$

, where

$$\delta E_{\mathbf{k}}^{(a)}(T) = -4\sum_{\mathbf{p}} \frac{\Gamma_a(\mathbf{k},\mathbf{p})^2}{E_{\mathbf{k}}^B + E_{\mathbf{p}}^B + E_{\mathbf{p}+\mathbf{k}}^B}\, n(E_{\mathbf{p}}^B) \,, \tag{3.2}$$

$$\delta E_{\mathbf{k}}^{(b)}(T) = 4\sum_{\mathbf{p}} \mathcal{P} \frac{\Gamma_b(\mathbf{k},\mathbf{p})^2}{E_{\mathbf{k}}^B - E_{\mathbf{p}}^B - E_{\mathbf{p}+\mathbf{k}}^B}\, n(E_{\mathbf{p}}^B)$$

and

$$\delta E_{\mathbf{k}}^{(c)}(T) = -4\sum_{\mathbf{p}} \mathcal{P} \frac{\Gamma_b(\mathbf{p},\mathbf{k})^2}{E_{\mathbf{k}}^B + E_{\mathbf{p}+\mathbf{k}}^B - E_{\mathbf{p}}^B} \left\{ n(E_{\mathbf{p}}^B) - n(E_{\mathbf{p}+\mathbf{k}}^B) \right\} \,, \tag{3.4}$$

on taking the real part of (2.19). The symbol $\mathcal{P}$ indicates to take the principal values for the integrals. In order to proceed further, we replace $E_{\mathbf{k}}^B$ involved on the right hand side of (3.2), (3.3) and (3.4) by the experimental excitation spectrum $E_{\mathbf{k}}$ at zero temperature. This replacement may be plausible if we take the higher order corrections into consideration.

The deviation of the phonon velocity from that of zero temperature is given by

$$\delta C(T) = \delta C^{(a)}(T) + \delta C^{(b)}(T) + \delta C^{(c)}(T)$$

$$= \lim_{k\to 0} \frac{1}{k}\left\{\delta E_k^{(a)}(T) + \delta E_k^{(b)}(T) + \delta E_k^{(c)}(T)\right\}. \qquad (3.5)$$

We now first calculate the contribution from $\delta c^{(a)}(T)$ and $\delta c^{(B)}(T)$. Since we can put

$$E_k \to \frac{ck}{2m}$$

and

$$\lambda_k \longrightarrow k/c \qquad (3.6)$$

in the limit of $k \to 0$, we have

$$\delta C^{(a+b)}(T) = \delta C^{(a)}(T) + \delta C^{(b)}(T)$$

$$= -2\left\{\frac{1}{2}\frac{4}{(8m)^2}\frac{1}{N}\frac{1}{c}\sum_{P} f(P)\right\}, \qquad (3.7)$$

$$f(P) = \frac{\lambda_P^2\left(1+\frac{1}{\lambda_P^2}\right)^2 P^4}{E_P\left(e^{\beta E_P}-1\right)} \qquad (3.8)$$

, where $C = 3Å^{-1}$ is the phonon velocity at zero temperature. In order to carry out the integration in (3.7), we separate the region of the momentum integration by an appropriate momentum P_c into two parts, that is, the phonon region and the roton region. Then we have

$$\delta C^{(a+b)}(T) = -\frac{4}{(8m)^2}\frac{V}{N}\frac{1}{C}\frac{2}{(2\pi)^2}\left\{\int_0^{P_c} f(p)p^2dp + \int_{P_c}^{\infty} f(p)p^2dp\right\}. \qquad (3.9)$$

In the phonon region, we can replace $E_{\mathbf{p}}$ and $\lambda_{\mathbf{p}}$ by $cp/2m$ and p/c, resepctively, and we can extend the finite range of the integration to infinity without introducing any appreciate error in virtue of the existence of the statistical factor $n(E_{\mathbf{p}})$, and we have the contribution $\delta c^{(a+b)}_{(ph)}$ from the phonon region as

$$\delta C^{(a+b)}_{(ph)}(T) = -\frac{\pi^2}{120}\frac{V}{N}C^2\frac{(k_BT)^4}{(C^2/2m)^3} \qquad (3.10)$$

, where we have used the fact that $1 \gg \lambda_{\mathbf{p}}$. In the roton region, we can put

$$\lambda_{\mathbf{p}} \simeq 1$$

and

$$E_{\mathbf{p}} \simeq \Delta + \frac{1}{2\mu}(p-k_0)^2 \qquad (3.11)$$

, where

$$\Delta = 12\times10^{-16} \ \text{erg},$$

$$\mu = 10^{-24} \ \text{g}$$

and

$$k_0 = 1.9\,\text{Å}^{-1}.$$

By making use of (3.11) and by retaining the dominant terms, we have

$$\delta C^{(a+b)}_{(rot)}(T) \simeq 2\left\{ -\frac{1}{2}\frac{4}{(8m)^2}\frac{V}{N}\frac{1}{C}\frac{4k_0^6}{\Delta}e^{-\beta\Delta}\frac{2}{(2\pi)^2}\int_{p_c}^{\infty}dp\, e^{-\frac{\beta(p-k_0)^2}{2\mu}} \right\}$$

$$\simeq -\frac{1}{2\pi^{3/2}}\frac{V}{N}\frac{k_0^7}{C^5}\frac{\left(\frac{C^2}{2m}\right)^2}{\Delta}\sqrt{\frac{k_BT}{k_0^2/2\mu}}\; e^{-\frac{\Delta}{k_BT}} \qquad (3.12)$$

, where we have used the fact that

$$\int_{p_c}^{\infty}dp\, e^{-\beta\frac{(p-k_0)^2}{2\mu}} \simeq \int_{-\infty}^{+\infty}dp\, e^{-\beta\frac{p^2}{2\mu}} = \sqrt{2\pi\mu k_BT}\,. \qquad (3.13)$$

Collecting (3.10) and (3.12), we can write

$$\delta C^{(a+b)}(T) = -\frac{\pi^2}{120}\frac{V}{N}C^2\frac{(k_BT)^4}{(C^2/2m)^3} - \frac{1}{2\pi^{3/2}}\frac{V}{N}\frac{k_0^7}{C^5}\frac{\left(\frac{C^2}{2m}\right)^2}{\Delta}\sqrt{\frac{k_BT}{k_0^2/2\mu}}\; e^{-\frac{\Delta}{k_BT}}. \qquad (3.14)$$

We next calculate $\delta C^{(c)}(T)$ in (3.5). In the limit of $k \to 0$, the energy denominator and the statistical factor in (3.4) vanish simultaneously. Therefore we must expand them around $|\vec{k}| = 0$, and retain only nonvanishing lowest terms. In the phonon region $|p| \sim 0$, we can write

$$\delta C^{(c)}(T) = -\frac{1}{(2\pi)^2(8m)^2}\frac{4}{N}\frac{V}{N}C\int dp\int_{-1}^{1}d\mu\,\frac{p^4(1-2\mu)^2}{\frac{C}{2m}+|\vec{\nabla}_pE_p|\mu}\,|\vec{\nabla}_pE_p|\mu\,\beta n(E_p)(n(E_p)+1)$$

(3.15)

, where use is made of (3.6) in the integrand and the vertex function $\Gamma_b(\mathbf{p},\mathbf{k})$ is approximated only by the term which have lowest power in p. The angular integration gives

$$\delta C^{(c)}(T) = -\frac{1}{(2\pi)^2}\frac{4}{(9m)^2}\frac{V}{N}\beta c\int_0^\infty dp\, p^4\left\{20.6 - 9\ln\left(\frac{2}{3\gamma}\frac{1}{p^2}\right)\right\} n(E_p)(n(E_p)+1) \tag{3.16}$$

, where we have used the relation

$$|\vec{\nabla}_p E_p| = \frac{c}{2m}(1-3\gamma p^2)$$

and the expansion coefficient $\gamma = 2.2 \times 10^{37} g^{-2}\ cm^{-2}\ sec^{2}$ is obtained with the aid of the energy spectrum calculated in the previous papers I and II. Carrying out the integration in (3.16), we find

$$\delta C^{(c)}(T) = -\frac{\pi^2}{60}\frac{V}{N}c^2\frac{(k_BT)^4}{(c/2m)^3}\left\{20.6 - 18\ln\left(\sqrt{\frac{2}{27\gamma}}\frac{1}{c}\frac{\frac{c^2}{2m}}{k_BT}\right)\right\} \tag{3.17}$$

, where we have replaced p by $3k_BT/C$ in the logarithmic term of the integrand of (3.16), and the other factor in the integrand of (3.16) has a maximum value for this value of p. On the other hand, the contribution of the integral from the roton region is negligibly small. This comes from the fact that the leading term of the integrand in (3.4) is propotional to $(p-k_o)$, since the order of magnitude of the statistical factor in the roton region $p \simeq k_o$ is given by $\frac{\beta}{2\mu}\cdot(p-k_o)$ and the energy denominator is ck/2m.

Collecting (3.14) and (3.17), we have the final result for the temperature dependent part of the phonon velocity;

$$\delta c(T) = \delta c^{(a+b)}(T) + \delta c^{(c)}(T)$$

$$= -\frac{\pi^2}{120}\frac{V}{N}c^2\frac{(k_BT)^4}{\left(\frac{c^2}{2m}\right)^3} - \frac{1}{2\pi^{3/2}}\frac{V}{N}\frac{k_0^7}{c^5}\frac{\left(\frac{c^2}{2m}\right)^2}{\Delta}\sqrt{\frac{k_BT}{k_0^2/2\mu}}\,e^{-\frac{\Delta}{k_BT}}$$

$$-\frac{\pi^2}{60}\frac{V}{N}c^2\frac{(k_BT)^4}{(c^2/2m)^3}\left\{20.6 - 18\ln\left(\sqrt{\frac{2}{27\gamma}}\frac{1}{c}\frac{c^2/2m}{k_BT}\right)\right\}. \qquad (3.18)$$

Introducing the numerical values

$$c = 3\,\mathring{A}^{-1},$$

$$\frac{V}{N} = 45.8\,\mathring{A}^3$$

and

$$\gamma = 2.2\times10^{37}\ g^{-2}\,cm^{-2}\,sec^{2},$$

we have the result

$$\delta c(T) = 10.4\,T^4\ln\left(\frac{3.1}{T}\right) - 5.7\times10^4\sqrt{T}\,e^{-\frac{8.6}{T}}\quad\left(\frac{cm}{sec}\right). \qquad (3.19)$$

In the expression (3.19), the first term on the right hand side is the contribution from the phonon region and the second term comes from the effect of the roton region. The first term gives increase of the phonon velocity as temperature rises, and the second term, on the other hand, decreases the phonon velocity. This behavior of the second term is indispensable for explaining the experimental results of the temperature dependence for the phonon velocity which show characteristic decrease. We should note here that this second term comes from (3.1) and (3.2) which are not considered in the theory of ter Haar et al.. The numerical result of (3.19) is shown in Table-I, and compared with the experimental result[14] in Fig.-3 and Fig.-4.

Table I

Fig. 3

Fig. 4

Although the position of the maximum point of the theoretical curve is slightly larger than the experimental one in Fig.-3, the whole behavior of the theoretical curve is very similar to the measured one.

In conclusion, we can see that the phonon velocity increases slightly due to the effect of the phonon-phonon interaction up to 0.6°K, and the phonon-roton interaction becomes effective above 0.6°K to decrease strongly the phonon velocity. We can further expect that the phonon-roton interaction also plays an important role in the temperature dependence of the absorption coefficient of the first sound in liquid helium II, and this problem is now under consideration.

§4. The temperature dependence of the roton minimum

In this section we shall study the temperature dependence of the roton minimum Δ. The deviation of the roton minimum $\delta\Delta(T)$ from that at zero temperature is described from (3.2), (3.3) and (3.4) by

$$\delta\Delta(T) = \delta\Delta^{(a)}(T) + \delta\Delta^{(b)}(T) + \delta\Delta^{(c)}(T) \quad , \tag{4.1}$$

$$\delta\Delta^{(a)}(T) = -4\sum_{\mathbf{p}} \frac{\Gamma_a(\mathbf{k}_0,\mathbf{p})^2}{\Delta + E_{\mathbf{p}} + E_{\mathbf{p}+\mathbf{k}_0}} n(E_{\mathbf{p}}) \quad , \tag{4.2}$$

$$\delta\Delta^{(b)}(T) = 4\sum_{\mathbf{p}} \mathcal{P} \frac{\Gamma_b(\mathbf{k}_0,\mathbf{p})^2}{\Delta - E_{\mathbf{p}} - E_{\mathbf{p}+\mathbf{k}_0}} n(E_{\mathbf{p}}) \tag{4.3}$$

and

$$\delta\Delta^{(c)}(T) = -4\sum_{\mathbf{p}} \mathcal{P} \frac{\Gamma_b(\mathbf{p},\mathbf{k}_0)^2}{\Delta + E_{\mathbf{p}+\mathbf{k}_0} - E_{\mathbf{p}}} \left\{ n(E_{\mathbf{p}}) - n(E_{\mathbf{p}+\mathbf{k}_0}) \right\} , \tag{4.4}$$

where

$$|\mathbf{k}_0| = k_0 \ ,$$
$$E_{\mathbf{k}_0} = \Delta$$

and

$$\lambda_{\mathbf{k}_0} \simeq 1 \quad . \tag{4.5}$$

As in the case of the phonon velocity, we separate the region of the momentum integrations of (4.2), (4.3) and (4.4) into two parts, that is, the phonon region and the roton region. We first consider the contributions from the phonon region ($p \sim 0$) by using the following approximated relations ;

$$\Gamma_a(k_0,p)^2 \simeq \frac{1}{(8m)^2}\frac{1}{N}\frac{4k_0^2}{C}p \quad ,$$

$$E_{k_0}+E_p+E_{p+k_0} \simeq 2\Delta \quad ,$$

$$\Gamma_b(k_0,p)^2 \simeq \frac{1}{(8m)^2}\frac{1}{N}4Ck_0^2\, p\cos^2\theta \quad , \tag{4.6}$$

$$E_{k_0}-E_p-E_{p+k_0} \simeq -\frac{cp}{2m} \quad ,$$

$$\Gamma_b(p,k_0)^2 \simeq \frac{1}{(8m)^2}\frac{1}{N}\frac{4k_0^4}{c}p$$

and

$$E_{k_0}+E_{p+k_0}-E_p \simeq 2\Delta \, .$$

The substitution of the relations in (4.6) into (4.2), (4.3) and (4.4) and the momentum integrations give

$$\delta\Delta^{(a)}_{(pk)}(T) = -\frac{\pi^2}{60}\frac{V}{N}\frac{k_0^4}{C}\frac{(k_BT)^4}{\left(\frac{C^2}{2m}\right)^2\Delta} \quad , \tag{4.7}$$

$$\delta\Delta^{(b)}_{(pk)}(T) = -\frac{A}{6\pi^2}\frac{V}{N}Ck_0^2\frac{(k_BT)^3}{\left(\frac{C^2}{2m}\right)^2} \tag{4.8}$$

and

$$\delta\Delta^{(c)}_{(pk)}(T) = -\frac{\pi^2}{60}\frac{V}{N}\frac{k_0^4}{C}\frac{(k_BT)^4}{\left(\frac{C^2}{2m}\right)^2\Delta} + \frac{9}{4\pi^2}\frac{V}{N}\frac{k_0^8}{C^5}\frac{\left(\frac{C^2}{2m}\right)^2}{\Delta}\frac{(k_BT)^2}{\left(\frac{k_0^2}{2\mu}\right)^2}e^{-\frac{\Delta}{k_BT}} \quad , \tag{4.9}$$

where in (4.8)

$$A = \int_0^\infty dx \frac{x^2}{e^x - 1} \simeq 2.4 \ ,$$

and in the second term of (4.9) we have used the following approximation;

$$e^{-\beta \frac{1}{2\mu}(|p+k_0|-k_0)^2} \simeq e^{-\beta \frac{p^2}{6\mu}} . \qquad (4.10)$$

Collecting (4.7), (4.8) and (4.9), we have the expression for the temperature dependence of the roton minimum

$$\delta\Delta_{(ph)}(T) = -\frac{2.4}{6\pi^2}\frac{V}{N} C k_0^2 \frac{(k_B T)^3}{\left(\frac{C^2}{2m}\right)^2} - \frac{\pi^2}{30}\frac{V}{N}\frac{k_0^4}{C}\frac{(k_B T)^4}{\left(\frac{C^2}{2m}\right)^2 \Delta}$$
$$+ \frac{9}{8\pi^2}\frac{V}{N}\frac{k_0^8}{C^5}\frac{\left(\frac{C^2}{2m}\right)^2}{\Delta}\frac{(k_B T)^2}{\left(\frac{k_0^2}{2\mu}\right)^2} e^{-\frac{\Delta}{k_B T}} \qquad (4.11)$$

from the phonon region.

We next calculate the contributions from the roton region ($|\mathbf{p}| \sim k_0$) by using (4.5) as the approximated relations for $E_{\mathbf{p}}$ and $\lambda_{\mathbf{p}}$ in the integrands of (4.2), (4.3) and (4.4). The contributions from the roton region are given by

$$\delta\Delta^{(a)}_{(rot)}(T) = -\frac{4}{(8m)^2}\frac{1}{N}\sum_{\mathbf{p}} \frac{\lambda_{\mathbf{p}+\mathbf{k}_0}\left(2k_0^2 + \frac{(\mathbf{p}+\mathbf{k}_0)^2}{\lambda_{\mathbf{p}+\mathbf{k}_0}}\right)^2}{2\Delta + E_{\mathbf{p}+\mathbf{k}_0}} n\left(\Delta + \frac{1}{2\mu}(p-k_0)^2\right), \quad (4.12)$$

$$\delta\Delta^{(b)}_{(rot)}(T) = -\frac{4}{(8m)^2}\frac{1}{N}\sum_{p}\frac{\lambda_{p+k_0}(p+k_0)^4}{E_{p+k_0}}n\left(\Delta+\frac{1}{2\mu}(p-k_0)^2\right) \qquad (4.13)$$

and

$$\delta\Delta^{(c)}_{(rot)}(T) = -\frac{4}{(8m)^2}\frac{1}{N}\sum_{p}\frac{\lambda_{p+k_0}(p+k_0)^4}{E_{p+k_0}}\left\{n\left(\Delta+\frac{1}{2\mu}(p-k_0)^2\right)-n(E_{p+k_0})\right\} \qquad (4.14)$$

, where we have used the fact that $|p| \sim k_o$. The integrations on the right hand side of (4.12), (4.13) and (4.14) are to be carried out for two regions when $|p+k_o|$ is in the phonon region and when it is in the roton region, respectively. After the integrations for these regions, we obtain

$$\delta\Delta^{(a)}_{(rot)}(T) = -\frac{3}{8\pi^{3/2}}\frac{V}{N}\frac{k_0^7}{C^4}\frac{\left(\frac{C^2}{2m}\right)^2}{\Delta}\sqrt{\frac{k_BT}{k_0^2/2\mu}}\, e^{-\frac{\Delta}{k_BT}} - \frac{9}{8\pi^2}\frac{V}{N}\frac{k_0^8}{C^5}\frac{\left(\frac{C^2}{2m}\right)^2}{\Delta}\frac{(k_BT)^2}{(k_0^2/2\mu)^2}\, e^{-\frac{\Delta}{k_BT}}, \qquad (4.15)$$

$$\delta\Delta^{(b)}_{(rot)}(T) = -\frac{1}{8\pi^{3/2}}\frac{V}{N}\frac{k_0^7}{C^4}\frac{\left(\frac{C^2}{2m}\right)^2}{\Delta}\sqrt{\frac{k_BT}{k_0^2/2\mu}}\, e^{-\frac{\Delta}{k_BT}} + (\text{small terms}) \qquad (4.16)$$

and

$$\delta\Delta^{(c)}_{(rot)}(T) \simeq (\text{small terms})$$

for the contributions from the roton region. In (4.15) and (4.16), we have written down explicitly only the leading terms, and the other small terms are negligible compared with the leading terms.

Collecting (4.11), (4.15) and (4.16), we have the final result for the deviation of the roton minimum from that at zero temperature ;

$$\delta\Delta(T) = -\frac{2.4}{6\pi^2}\frac{V}{N}ck_0^2\frac{(k_BT)^3}{(\frac{c^2}{2m})^2} - \frac{\pi^2}{30}\frac{V}{N}\frac{k_0^4}{c}\frac{(k_BT)^4}{(\frac{c^2}{2m})^2\Delta}$$
$$-\frac{1}{2\pi^2}\frac{V}{N}\frac{k_0^7}{c^4}\frac{(\frac{c^2}{2m})^2}{\Delta}\sqrt{\frac{k_BT}{k_0^2/2\mu}}\; e^{-\frac{\Delta}{k_BT}} .$$

By using the numerical values of the constants Δ , c, V/N and k_o given in the preceding section, we obtain the result

$$\delta\Delta(T) = -9.5\times10^{-19}T^3 - 3.5\times10^{-19}T^4 - 17.5\times10^{-15}\sqrt{T}\,e^{-8.6/T}(\text{erg}). \quad (4.18)$$

Fig. 5

Tabe. II

The first and second terms in (4.18) are the contributions from the phonon region to the roton minimum, and the third term in (4.18) is the correction term from the roton region. From (4.18), the numerical values for the roton minimum $\Delta(T) = \Delta + \delta\Delta(T)$ are shown in Table-2 and compared with experiments[13] in Fig.-5, from which we find that the theoretical result is in fairly good agreement with the experimental one.

In conclusion we can see that the roton minimum gradually decreases due to the effect of the roton-phonon interaction up to 0.7°K, and begins to decrease rapidly above 0.7°K by the effect of the roton-roton interaction.

Appendix A

Collective description and phonon-phonon interaction

We consider a system of N interacting helium atoms of mass m enclosed in a cubic box of volume V at T = 0°K. The Hamiltonian for our system is given by

$$H = \sum_{k} \frac{k^2}{2m} a_k^\dagger a_k + \frac{1}{2} \sum_{k,\ell,\ell'} \frac{\nu(k)}{V} a_\ell^\dagger a_{\ell'}^\dagger a_{\ell'+k} a_{\ell-k} \qquad \text{(A-1)}$$

, where $\nu(k)$ is the Fourier transform of the effective interatomic potential $V(x)$. The commutation relations are given by

$$[a_k, a_{k'}^\dagger] = \delta_{k,k'} \ , \ [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0 \ . \qquad \text{(A-2)}$$

In this appendix, we rewrite the original Hamiltonian (A-1) in terms of the collective variables which are appropriate to describe the collective behavior of the system. As the collective variables, we first introduce the density fluctuation operator and the current operator defined by

$$\rho_k = \sum_p a_{p+k}^\dagger a_p$$

and

$$\vec{g}_k = \sum_p p \, a_{p-k/2}^\dagger a_{p+k/2} \ . \qquad \text{(A-3)}$$

From (A-2) and (A-3), we can readily derive the following commutation relations,

$$[\rho_{\mathbf{k}}, \rho_{\mathbf{k}'}] = 0 , \qquad [\vec{g}_{\mathbf{k}}, \rho_{\mathbf{k}'}] = \mathbf{k}' \rho_{\mathbf{k}'-\mathbf{k}}$$

and

$$[g_{\mathbf{k}}^{(i)}, g_{\mathbf{k}'}^{(j)}] = \left(g^{(i)}_{\mathbf{k}+\mathbf{k}'} k_j - g^{(j)}_{\mathbf{k}+\mathbf{k}'} k_i \right) , \qquad \text{(A-4)}$$

where

$$\vec{g}_{\mathbf{k}} = \left(g_{\mathbf{k}}^{(1)}, g_{\mathbf{k}}^{(2)}, g_{\mathbf{k}}^{(3)} \right) , \quad \mathbf{k} = (k_1, k_2, k_3) .$$

One cannot treat $\rho_{\mathbf{k}}$ and $\vec{g}_{\mathbf{k}}$ as a canonical set of variables because of the fact that the commutator between $\vec{g}_{\mathbf{k}}$ and $\rho_{\mathbf{k}'}$ is not a c-number, and the third commutation relation in (A-4) does not vanish even for the case $\mathbf{k} = \mathbf{k}'$. As the canonical variables for $\rho_{\mathbf{k}}$, however, we can introduce the new collective variables, i.e. the velocity operator $\vec{v}_{\mathbf{k}}$, defined by the integral equation

$$\vec{v}_{\mathbf{k}} = \frac{1}{\hat{N}} \vec{g}_{\mathbf{k}} - \frac{1}{\hat{N}} \sum_{\mathbf{p} \neq \mathbf{k}} \rho_{\mathbf{p}-\mathbf{k}} \vec{v}_{\mathbf{p}} , \qquad \text{(A-5)}$$

where $\hat{N} = \rho_0 = \sum_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}$.

From (A-4) and (A-5), we can prove the commutation relations

$$[\vec{v}_{\mathbf{k}}, \rho_{\mathbf{k}'}] = \mathbf{k}' \delta_{\mathbf{k},\mathbf{k}'}$$

$$\text{(A-6)}$$

and

$$[v_{\mathbf{k}}^{(i)}, v_{\mathbf{k}'}^{(j)}] = 0 ,$$

where the last commutation relation in (A-6) will be derived in the appendix C. In contrast to the case of $\vec{g}_k$, the new variables $\vec{v}_k$ do commute exactly with $\rho_{k'}$, for $k \neq k'$, and we can rewrite (A-5) in the following form :

$$\vec{v}_k = \frac{1}{N}\vec{g}_k - \frac{1}{N}\sum_{p \neq k} \vec{v}_p \rho_{p-k} \quad . \tag{A-7}$$

It should be noted that the commutation relations (A-6) coincide precisely with those assumed by Landau[1], except the fact that the second commutation relation in (A-6) vanishes.

In order to express the Hamiltonian (A-1) in terms of the collective variables, we now introduce a projection operator defined by

$$P(k-k') = \sum_p a_{p+k}\frac{1}{N}a^{\dagger}_{p+k'} - \sum_{\substack{m \neq p \\ p}} a_{p+k}\frac{1}{N}\rho_{p-m}\frac{1}{N}a^{\dagger}_{m+k'} + \cdots \quad , \tag{A-8}$$

which will be proved in (C-8) to be

$$P(k-k') = \mathbb{1}\,\delta_{k,k'} \tag{A-9}$$

for the case of Bose system. Noting the relation (A-9), we insert (A-8) into (A-1) and hence the kinetic energy T is written as

$$T = \frac{1}{2m}\left\{\sum_{\substack{k,k' \\ p}} k a^{\dagger}_k a_{p+k}\frac{1}{N}k' a^{\dagger}_{p+k'}a_{k'} - \sum_{q \neq p}\sum_{k,k'} k a^{\dagger}_k a_{p+k}\frac{1}{N}\rho_{p-q}\frac{1}{N}k' a^{\dagger}_{q+k'}a_{k'} + \cdots\right\}. \tag{A-10}$$

Introducing the relations

$$\sum_k k a^{\dagger}_k a_{k+p} = \vec{g}_p - \frac{1}{2}p\rho_p$$

and

$$\sum_{k} k\, a^{\dagger}_{k+p} a_{k} = \vec{g}_{-p} - \frac{1}{2} p\, \rho_{p} \tag{A-11}$$

into (10), we obtain

$$\begin{aligned} T = & \frac{1}{2m}\left\{ \sum_{p} \vec{g}_{p} \frac{1}{N} \vec{g}_{-p} - \sum_{q \neq p} \vec{g}_{p} \frac{1}{N} \rho_{p-q} \frac{1}{N} \vec{g}_{-q} + \cdots \right\} \\ & + \frac{1}{2m}\left\{ \sum_{p} \frac{p^{2}}{4} \rho_{-p} \frac{1}{N} \rho_{p} - \sum_{q \neq p} \frac{(p \cdot q)}{4} \rho_{-p} \frac{1}{N} \rho_{p-q} \frac{1}{N} \rho_{q} + \cdots \right\} \\ & - \frac{1}{2m}\left\{ \sum_{p} \frac{p}{2} \cdot \left(\vec{g}_{p} \frac{1}{N} \rho_{p} - \rho_{p} \frac{1}{N} \vec{g}_{p} \right) - \sum_{q \neq p} \frac{p}{2} \cdot \left(\vec{g}_{q} \frac{1}{N} \rho_{q-p} \frac{1}{N} \rho_{p} - \rho_{p} \frac{1}{N} \rho_{q-p} \frac{1}{N} \vec{g}_{q} \right) + \cdots \right\} \end{aligned} \tag{A-12}$$

In order to express (A-12) in terms of the velocity operator defined by (A-5), let us consider in detail the first term on the right-hand side of (A-12). The first series can be transformed into the following form :

$$\frac{1}{2m}\left\{ \sum_{p} \vec{g}_{p} \frac{1}{N} \vec{g}_{-p} - \sum_{q \neq p} \vec{g}_{p} \frac{1}{N} \rho_{p-q} \frac{1}{N} \vec{g}_{-q} + \cdots \right\}$$

$$= \frac{1}{2m}\left\{ \sum_{p,p',p''} \vec{v}_{p'} \rho_{p'-p} \frac{1}{N} \rho_{p''+p} \vec{v}_{p''} - \sum_{q \neq p} \sum_{p',p''} \vec{v}_{p'} \rho_{p'-p} \frac{1}{N} \rho_{p-q} \frac{1}{N} \rho_{p''+q} \vec{v}_{p''} + \cdots \right\}.$$

Noting that $\rho_0 = \tilde{N}$, we divide each term of the last expression into two parts, i.e. the part of $\mathbf{p}' = \mathbf{p}$ and that of $\mathbf{p}' \neq \mathbf{p}$, and obtain

$$\frac{1}{2m}\left\{ \sum_{\mathbf{p}} \vec{g}_{\mathbf{p}} \frac{1}{\tilde{N}} \vec{g}_{-\mathbf{p}} - \sum_{\mathbf{q}\neq\mathbf{p}} \vec{g}_{\mathbf{p}} \frac{1}{\tilde{N}} \rho_{\mathbf{p}-\mathbf{q}} \frac{1}{\tilde{N}} \vec{g}_{-\mathbf{q}} + \cdots \right\}$$

$$= \frac{1}{2m}\Bigg\{ \sum_{\mathbf{p},\mathbf{p}''} \vec{v}_{\mathbf{p}}\, \rho_{\mathbf{p}''+\mathbf{p}}\, \vec{v}_{\mathbf{p}''} + \sum_{\substack{\mathbf{p},\mathbf{p}',\mathbf{p}'' \\ \mathbf{p}'\neq\mathbf{p}}} \vec{v}_{\mathbf{p}'}\, \rho_{\mathbf{p}'-\mathbf{p}} \frac{1}{\tilde{N}}\, \rho_{\mathbf{p}''+\mathbf{p}}\, \vec{v}_{\mathbf{p}''}$$

$$- \sum_{\mathbf{q}\neq\mathbf{p},\mathbf{p}''} \vec{v}_{\mathbf{p}}\, \rho_{\mathbf{p}-\mathbf{q}} \frac{1}{\tilde{N}}\, \rho_{\mathbf{p}''+\mathbf{q}}\, \vec{v}_{\mathbf{p}''} - \sum_{\substack{\mathbf{q}\neq\mathbf{p},\mathbf{p}' \\ \mathbf{p}'',\mathbf{p}'\neq\mathbf{p}}} \vec{v}_{\mathbf{p}'}\, \rho_{\mathbf{p}'-\mathbf{p}} \frac{1}{\tilde{N}}\, \rho_{\mathbf{p}-\mathbf{q}} \frac{1}{\tilde{N}}\, \rho_{\mathbf{p}''+\mathbf{q}}\, \vec{v}_{\mathbf{p}''}$$

$$+ \cdots \Bigg\}. \qquad \text{(A-13)}$$

All the terms cancel out except the first term on the right hand side of (A-13), and we have

$$\frac{1}{2m}\left\{ \sum_{\mathbf{p}} \vec{g}_{\mathbf{p}} \frac{1}{\tilde{N}} \vec{g}_{-\mathbf{p}} - \sum_{\mathbf{q}\neq\mathbf{p}} \vec{g}_{\mathbf{p}} \frac{1}{\tilde{N}}\, \rho_{\mathbf{p}-\mathbf{q}} \frac{1}{\tilde{N}} \vec{g}_{-\mathbf{q}} + \cdots \right\} = \frac{1}{2m} \sum_{\mathbf{p},\mathbf{q}} \vec{v}_{\mathbf{p}}\, \rho_{\mathbf{p}+\mathbf{q}}\, \vec{v}_{\mathbf{q}}. \qquad \text{(A-14)}$$

In the quite similar way, we obtain

$$-\frac{1}{2m}\left\{ \sum_{\mathbf{p}} \frac{\mathbf{p}}{2}\left(\vec{g}_{\mathbf{p}} \frac{1}{\tilde{N}} \rho_{\mathbf{p}} - \rho_{\mathbf{p}} \frac{1}{\tilde{N}} \vec{g}_{\mathbf{p}}\right) - \sum_{\mathbf{q}\neq\mathbf{p}} \frac{\vec{\mathbf{p}}}{2}\left(\vec{g}_{\mathbf{q}} \frac{1}{\tilde{N}}\, \rho_{\mathbf{q}-\mathbf{p}} \frac{1}{\tilde{N}}\, \rho_{\mathbf{p}} - \rho_{\mathbf{p}} \frac{1}{\tilde{N}}\, \rho_{\mathbf{q}-\mathbf{p}} \frac{1}{\tilde{N}} \vec{g}_{\mathbf{q}}\right) + \cdots \right\}$$

$$= -\frac{1}{4m} \sum_{\mathbf{p}} \mathbf{p}\cdot\left(\vec{v}_{\mathbf{p}}\, \rho_{\mathbf{p}} - \rho_{\mathbf{p}}\, \vec{v}_{\mathbf{p}}\right) = -\sum_{\mathbf{p}} \frac{\mathbf{p}^2}{4m} \qquad \text{(A-15)}$$

with the aid of the commutation relations (A-6). The interaction energy is expressed by

$$\frac{1}{2}\sum_{\substack{k,\ell\\ \ell'}}\frac{\nu(k)}{V}a_\ell^\dagger a_{\ell'}^\dagger a_{\ell'+k}a_{\ell-k}=\frac{N(N-1)}{2V}\nu(0)-\frac{N}{2V}\sum_{k\neq 0}\nu(k)+\frac{1}{2}\sum_{k\neq 0}\frac{\nu(k)}{V}\rho_k\rho_{-k}, \qquad \text{(A-16)}$$

where N is the total number of the particles in our system and the operator $\tilde{N}$ has been replaced by a c-number N by virtue of the fact that the total number of the particles is always conserved.

Collecting (A-14), (A-15) and (A-16), we find the Hamiltonian expressed in terms of the collective variables in the following form :

$$\begin{aligned} H=&\left\{\frac{N(N-1)}{2V}\nu(0)-\frac{N}{2V}\sum_{k\neq 0}\nu(k)-\frac{1}{4m}\sum_{k}k^2\right\}\\ &+\sum_{k}\left\{\frac{N}{2m}\vec{v}_k\cdot\vec{v}_{-k}+\left(\frac{k^2}{8mN}+\frac{\nu(k)}{2V}\right)\rho_k\rho_{-k}\right\}\\ &+\sum_{\substack{p,q\\ p+q\neq 0}}\left\{\frac{1}{2m}\vec{v}_p\,\rho_{p+q}\,\vec{v}_q+\frac{1}{8mN^2}(p\cdot q)\rho_p\,\rho_{-p-q}\,\rho_q\right\}-\cdots\quad . \end{aligned} \qquad \text{(A-17)}$$

The second term of the Hamiltonian (A-17) can be regarded as the free part, and the third and higher terms give the phonon-phonon interactions.

If we introduce the operators B_k and B_k^+ by

$$\rho_k=\sqrt{N}\lambda_k\left(B_k+B_{-k}^\dagger\right)$$

and $\qquad$ (A-18)

$$\vec{v}_k=\frac{k}{2\sqrt{N}\lambda_k}\left(B_{-k}-B_k^\dagger\right) ,$$

the new operators satisfy the commutation relations

$$[B_{\mathbf{k}}, B_{\mathbf{k}'}] = 0 \ , \quad [B^{\dagger}_{\mathbf{k}}, B^{\dagger}_{\mathbf{k}'}] = 0$$

and

$$[B_{\mathbf{k}}, B^{\dagger}_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'} \ , \tag{A-19}$$

where

$$\lambda_{\mathbf{k}} = k \Big/ \sqrt{\mathbf{k}^2 + C^2(\mathbf{k})} \ , \quad k = |\mathbf{k}| \tag{A-20}$$

and

$$C(\mathbf{k}) = \sqrt{\frac{4Nm}{V} \nu(\mathbf{k})} \ . \tag{A-21}$$

Inserting (A-18) into (A-17), we have the new representation of the Hamiltonian:

$$H = H_0 + H_I$$

$$H_0 = E^B_0 + \sum_{\mathbf{k}} E^B_{\mathbf{k}} B^{\dagger}_{\mathbf{k}} B_{\mathbf{k}}$$

and

$$H_I = \frac{1}{8m\sqrt{N}} \sum_{\substack{p,q \\ p+q \neq 0}} (p \cdot q) \sqrt{\frac{\lambda_{p+q}}{\lambda_p \lambda_q}} (1 + \lambda_p \lambda_q) \left\{ B^{\dagger}_p B^{\dagger}_{-p-q} B^{\dagger}_q + B_p B_{-p-q} B_q + B^{\dagger}_p B^{\dagger}_q B_{p+q} + B^{\dagger}_{p+q} B_p B_q \right\}$$

$$- \frac{1}{8m\sqrt{N}} \sum_{\substack{p,q \\ p+q \neq 0}} (p \cdot q) \sqrt{\frac{\lambda_{p+q}}{\lambda_p \lambda_q}} (1 - \lambda_p \lambda_q) \left\{ B^{\dagger}_q B^{\dagger}_{p+q} B_p + B^{\dagger}_p B_{p+q} B_{-q} \right\} \ , \tag{A-24}$$

where E^B_0 stands for the lowest order ground state energy given by

$$E_0^B = \frac{N(N-1)}{2V}\nu(0) + \frac{1}{2}\sum_{k\neq 0}\left(E_k^B - \frac{k^2}{2m} - \frac{N}{V}\nu(k)\right), \tag{A-25}$$

and

$$E_k^B = \frac{k}{2m}\sqrt{k^2 + c^2(k)} \tag{A-26}$$

gives the excitation energy in the lowest approximation. The phonon-phonon interactions of the order of $\sqrt{N}$ are given by (A-24).

Appendix B

Phonon-Phonon interactions and the excitation energy

In A, we have obtained the definite expression (A-24) of the phonon-phonon interactions by introducing the collective variables $\rho_{\mathbf{k}}$ and $\vec{\mathcal{V}}_{\mathbf{k}}$. In this section, let us derive a formula for the excitation spectrum in which the effects of the phonon-phonon interactions are taken into account. However, the traditional perturbation treatment of the interactions will produce a strongly divergent result, because the phonon-phonon interactions (A-24) involve the products of momenta $(\mathbf{p}\cdot\mathbf{q})$.. If we introduce a cutoff parameter to yield a finite result, the result will be fortuitous even if it is in good agreement with experiment for some value of the parameter, because the result is very sensitive to the magnitude of the parameter.

In order to overcome the difficulty mentioned above, we now adopt a slightly different method from the standard perturbation theory. The Schrödinger equation for the state of a single excitation $|\Psi_k\rangle$ is given by

$$(H - E_0 - E_{\mathbf{k}}^F)|\Psi_{\mathbf{k}}\rangle = \delta E_{\mathbf{k}}|\Psi_{\mathbf{k}}\rangle \qquad \text{(B-1)}$$

and

$$E_{\mathbf{k}} = E_{\mathbf{k}}^F + \delta E_{\mathbf{k}} \quad ,$$

where E_0 is the eigenvalue for the exact ground state $|G\rangle$ of the total system and E_k denotes the excitation energy of a single physical phonon. The Feynman excitation energy E_k^F is defined by

$$E_k^F = \left(\frac{k^2}{2m}\right) \Big/ \left(\frac{\langle G|\rho_k \rho_k^\dagger|G\rangle}{\langle G|G\rangle}\right) \quad . \tag{B-2}$$

Along the line of reasoning of the Feynman-Cohen theory[3], we assume that the single excitation state is approximately expressed by

$$|\Psi_k\rangle = \rho_k^\dagger|G\rangle + \frac{1}{2}\sum_{\ell \neq k} C_{k-\ell,\ell}\, \rho_{k-\ell}^\dagger \rho_\ell^\dagger |G\rangle \quad , \tag{B-3}$$

where we should note that the first and the second terms on the right-hand side of (B-3) are not orthogonal to each other. Introducing (B-3) into (B-1), we may obtain a set of equations :

$$\delta E_k \langle G|\rho_k \rho_k^\dagger|G\rangle = \frac{1}{2}\sum_{\ell' \neq k} C_{k-\ell',\ell'} \langle G|\rho_k (H-E_0-E_k^F)\rho_{k-\ell'}^\dagger \rho_{\ell'}^\dagger|G\rangle$$

$$+ \frac{1}{2}\sum_{\ell' \neq k} C_{k-\ell',\ell'} \langle G|\rho_k \rho_{k-\ell'}^\dagger \rho_{\ell'}^\dagger|G\rangle \delta E_k \tag{B-4}$$

and

$$\frac{1}{2}\sum_{\ell' \neq k} C_{k-\ell',\ell'} \langle G|\rho_{k-\ell}\rho_\ell (H-E_0-E_k)\rho_{k-\ell}^\dagger \rho_\ell^\dagger|G\rangle$$

$$= -\langle G|\rho_{k-\ell}\rho_\ell (H-E_0-E_k^F)\rho_k^\dagger|G\rangle + \delta E_k \langle G|\rho_k \rho_{k-\ell}\rho_k^\dagger|G\rangle , \tag{B-5}$$

where use is made of the sum rule

$$\langle G|\rho_k (H-E_0-E_k^F)\rho_k^\dagger|G\rangle = 0 \quad .$$

The prescription of our method of calculation is to retain only the lowest nonvanishing matrix elements. Hence, the exact structure factor $S(\mathbf{k})$ is approximated by $\lambda_{\mathbf{k}}$ in the following way,

$$\langle G|\rho_{\mathbf{k}}\rho_{\mathbf{k}}^{\dagger}|G\rangle \simeq (G|\rho_{\mathbf{k}}\rho_{\mathbf{k}}^{\dagger}|G) = N\lambda_{\mathbf{k}} \;, \tag{B-7}$$

where the state $|G)$ denotes the ground state of the free Hamiltonian H_0 ;

$$(H_0 - E_0^B)|G) = 0 \;. \tag{B-8}$$

Because of the approximation adopted above, Feynman's excitation energy $E_{\mathbf{k}}^F$ in (B-4) and (B-5) is replaced by Bogoliubov's one $E_{\mathbf{k}}^B$, i.e.

$$E_{\mathbf{k}}^F = \left(\frac{\mathbf{k}^2}{2m}\right)\Big/\left(\langle G|\rho_{\mathbf{k}}\rho_{\mathbf{k}}^{\dagger}|G\rangle/N\right) \rightarrow E_{\mathbf{k}}^B = \frac{\mathbf{k}^2}{2m}\Big/\lambda_{\mathbf{k}} \;. \tag{B-9}$$

If we use the same prescription as above for the ground state energy, the energy E_0 which includes the effect of the phonon-phonon interactions should be replaced by the ground state energy E_0^B in the nonvanishing lowest approximation. In the same way, we have

$$\begin{aligned}
&\langle G|\rho_{\mathbf{k}-\boldsymbol{\ell}}\rho_{\boldsymbol{\ell}}(H-E_0-E_{\mathbf{k}})\rho_{\mathbf{k}-\boldsymbol{\ell}'}^{\dagger}\rho_{\boldsymbol{\ell}'}^{\dagger}|G\rangle \\
&\simeq (G|\rho_{\mathbf{k}-\boldsymbol{\ell}}\rho_{\boldsymbol{\ell}}(H-E_0^B-E_{\mathbf{k}})\rho_{\mathbf{k}-\boldsymbol{\ell}'}^{\dagger}\rho_{\boldsymbol{\ell}'}^{\dagger}|G) \\
&= (E_{\mathbf{k}-\boldsymbol{\ell}}^B + E_{\boldsymbol{\ell}}^B - E_{\mathbf{k}})\lambda_{\mathbf{k}-\boldsymbol{\ell}}\lambda_{\boldsymbol{\ell}}(\delta_{\boldsymbol{\ell},\boldsymbol{\ell}'} + \delta_{\boldsymbol{\ell}',\mathbf{k}-\boldsymbol{\ell}})N^2 \;,
\end{aligned}$$

where the excitation energy $E_{\mathbf{k}}$ remains as it is. Introducing (B-9) and (B-10) into (B-4) and (B-5), we may readily obtain

$$\delta E_{\mathbf{k}} = -\sum_{\boldsymbol{\ell} \neq \mathbf{k}} \frac{1}{2N^3 \lambda_{\boldsymbol{\ell}} \lambda_{\boldsymbol{\ell}-\mathbf{k}} \lambda_{\mathbf{k}}} \frac{|\langle G | \rho_{\mathbf{k}-\boldsymbol{\ell}} \rho_{\boldsymbol{\ell}} (H - E_0^B - E_{\mathbf{k}}^B) \rho_{\mathbf{k}}^{\dagger} | G \rangle|^2}{E_{\boldsymbol{\ell}}^B + E_{\mathbf{k}-\boldsymbol{\ell}}^B - E_{\mathbf{k}}} \tag{B-11}$$

and

$$|\Psi_{\mathbf{k}}\rangle = \rho_{\mathbf{k}}^{\dagger} |G\rangle - \frac{1}{2} \sum_{\boldsymbol{\ell} \neq \mathbf{k}} \frac{\langle G | \rho_{\mathbf{k}-\boldsymbol{\ell}} \rho_{\boldsymbol{\ell}} (H - E_0^B - E_{\mathbf{k}}^B) \rho_{\mathbf{k}}^{\dagger} | G \rangle}{N^2 \lambda_{\boldsymbol{\ell}} \lambda_{\mathbf{k}-\boldsymbol{\ell}} (E_{\boldsymbol{\ell}}^B + E_{\mathbf{k}-\boldsymbol{\ell}}^B - E_{\mathbf{k}})} \rho_{\boldsymbol{\ell}}^{\dagger} \rho_{\mathbf{k}-\boldsymbol{\ell}}^{\dagger} |G\rangle . \tag{B-12}$$

In the course of derivation of (B-11) and (B-12), the second terms on the right hand sides of (B-4) and (B-5) have been neglected, since they can be regarded as higher order terms in comparison with other retained terms.

The calculation of the matrix elements in (B-11) and (B-12) should be carried out in the same spirit as the calculations of (B-9) and (B-10) ; that is, we take only the lowest nonvanishing value of the matrix element. Since the matrix elements in the numerators of (B-11) and (B-12) have odd density operators, it is required for us to take the phonon -phonon interaction H_I into account, and the physical ground state $|G\rangle$ is written as

$$|G\rangle \simeq |G) + \frac{1}{E_0^B - H_0} H_I |G)$$

$$= |G) - \frac{1}{8m\sqrt{N}} \sum_{\substack{p,q \\ p+q \neq 0}} \frac{(p \cdot q)}{E_p^B + E_q^B + E_{p+q}^B} \sqrt{\frac{\lambda_{p+q}}{\lambda_p \lambda_q}} (1 + \lambda_p \lambda_q) B_{-p}^{\dagger} B_{p+q}^{\dagger} B_q^{\dagger} |G) . \tag{B-13}$$

By making use of (A-24) and (B-13), we have

$$\langle G|\rho_{\ell}\,\rho_{k-\ell}\,\rho_{k}^{\dagger}|G\rangle$$

$$\simeq -\frac{(\sqrt{N})^3}{8m\sqrt{N}}\sum_{\substack{p,q\\ p+q\neq 0}}\frac{(p\cdot q)}{E_p^B+E_q^B+E_{p+q}^B}\sqrt{\frac{\lambda_{p+q}}{\lambda_p\lambda_q}}\,(1+\lambda_p\lambda_q)\sqrt{\lambda_\ell\lambda_{k-\ell}\lambda_k}\times\Big\{(G|B_q B_{p+q} B_p B_\ell^{\dagger} B_{\ell-k}^{\dagger} B_k^{\dagger}|G)$$
$$+\,(G|B_\ell B_{k-\ell} B_k B_p^{\dagger} B_{p+q}^{\dagger} B_q^{\dagger}|G)\Big\}$$

$$=\frac{(\sqrt{N})^3}{2m\sqrt{N}}\Big\{(k\cdot\ell)\lambda_{k-\ell}(1+\lambda_k\lambda_\ell)+(k\cdot k-\ell)\lambda_\ell(1+\lambda_{k-\ell}\lambda_\ell)-(k\cdot k-\ell)\lambda_k(1+\lambda_\ell\lambda_{\ell-k})\Big\}\Big/(E_\ell^B+E_{k-\ell}^B+E_k^B)$$

(B-14)

and

$$\langle G|\rho_\ell\,\rho_{k-\ell}(H-E_0^B)\rho_k^{\dagger}|G\rangle$$

$$\simeq (G|\rho_\ell\,\rho_{k-\ell}H_I\rho_k^{\dagger}|G)-\frac{1}{8m\sqrt{N}}\sum_{\substack{p,q\\ p+q\neq 0}}\frac{(p\cdot q)}{E_p^B+E_q^B+E_{p+q}^B}\sqrt{\frac{\lambda_{p+q}}{\lambda_p\lambda_q}}\sqrt{\lambda_k\lambda_{k-\ell}\lambda_\ell}\,(1+\lambda_p\lambda_q)$$
$$\times\Big\{(G|B_p B_{p+q} B_q B_\ell^{\dagger} B_{\ell-k}^{\dagger}(H_0-E_0^B)B_k^{\dagger}|G)+(G|B_\ell B_{k-\ell}(H_0-E_0^B)B_k B_p^{\dagger} B_{p-q}^{\dagger} B_q^{\dagger}|G)\Big\}$$

$$=\frac{(\sqrt{N})^3}{8m\sqrt{N}}\sum_{\substack{p,q\\ p+q\neq 0}}(p\cdot q)\sqrt{\frac{\lambda_{p+q}}{\lambda_p\lambda_q}}\,(1+\lambda_p\lambda_q)\sqrt{\lambda_k\lambda_{k-\ell}\lambda_\ell}\,(G|B_\ell B_{k-\ell} B_p^{\dagger} B_q^{\dagger} B_{p+q} B_k^{\dagger}|G)$$

$$-\frac{2(\sqrt{N})^3}{8m\sqrt{N}}\sum_{\substack{p,q\\ p+q\neq 0}}(p\cdot q)\sqrt{\frac{\lambda_{p+q}}{\lambda_p\lambda_q}}\,(1-\lambda_p\lambda_q)\sqrt{\lambda_k\lambda_{k-\ell}\lambda_\ell}\,(G|B_\ell B_{k-\ell} B_q^{\dagger} B_{-p-q}^{\dagger} B_{-p} B_k^{\dagger}|G)$$

$$-\frac{(\sqrt{N})^3}{8m\sqrt{N}}\sum_{\substack{p,q\\ p+q\neq 0}}(p\cdot q)\sqrt{\frac{\lambda_{p+q}}{\lambda_p\lambda_q}}\,(1+\lambda_p\lambda_q)\sqrt{\lambda_k\lambda_{k-\ell}\lambda_\ell}\,(G|B_q B_{p+q} B_p B_\ell^{\dagger} B_{k-\ell}^{\dagger} B_k^{\dagger}|G)$$

$$=\frac{(\sqrt{N})^3}{2m\sqrt{N}}\Big\{(k\cdot\ell)\lambda_{k-\ell}+(k\cdot k-\ell)\lambda_\ell\Big\}\,.$$

(B-15)

By using (B-14) and (B-15), the straightforward calculation yields

$$\langle G|\rho_{\mathbf{k}-\boldsymbol{\ell}}\rho_{\boldsymbol{\ell}}(H-E_0^B-E_{\mathbf{k}}^B)\rho_{\mathbf{k}}^{\dagger}|G\rangle$$

$$\simeq \frac{(\sqrt{N})^3}{4m\sqrt{N}}\lambda_{\mathbf{k}}\frac{\left\{(\mathbf{k}^2+\boldsymbol{\ell}^2+(\mathbf{k}-\boldsymbol{\ell})^2)\left((\mathbf{k}\cdot\mathbf{k}-\boldsymbol{\ell})\lambda_{\boldsymbol{\ell}}^2(1-\lambda_{\mathbf{k}-\boldsymbol{\ell}}^2)+(\mathbf{k}\cdot\boldsymbol{\ell})\lambda_{\mathbf{k}-\boldsymbol{\ell}}^2(1-\lambda_{\boldsymbol{\ell}}^2)-2(\mathbf{k}\cdot\boldsymbol{\ell})(\mathbf{k}\cdot\mathbf{k}-\boldsymbol{\ell})(\lambda_{\boldsymbol{\ell}}-\lambda_{\mathbf{k}-\boldsymbol{\ell}})^2\right)\right\}}{\boldsymbol{\ell}^2\lambda_{\mathbf{k}}\lambda_{\mathbf{k}-\boldsymbol{\ell}}+(\mathbf{k}-\boldsymbol{\ell})^2\lambda_{\mathbf{k}}\lambda_{\boldsymbol{\ell}}+\mathbf{k}^2\lambda_{\boldsymbol{\ell}}\lambda_{\mathbf{k}-\boldsymbol{\ell}}}. \qquad \text{(B-16)}$$

Inserting (B-16) into (B-11), we have a fundamental formula for the excitation energy in the following form ;

$$E_{\mathbf{k}} = E_{\mathbf{k}}^B + \delta E_{\mathbf{k}} = E_{\mathbf{k}}^B(1-\eta_{\mathbf{k}}), \qquad \text{(B-17)}$$

where $\eta_{\mathbf{k}}$ indicates the ratio of the correction due to the phonon-phonon interaction to the lowest Bogoliubov excitation energy E_k^B and is determined by the equation

$$\eta_{\mathbf{k}} = \frac{\lambda_{\mathbf{k}}^3}{8N\mathbf{k}^2}\sum_{\boldsymbol{\ell}\neq\mathbf{k}}\frac{\left[\{\mathbf{k}^2+\boldsymbol{\ell}^2+(\mathbf{k}-\boldsymbol{\ell})^2\}\{(\mathbf{k}\cdot\mathbf{k}-\boldsymbol{\ell})\lambda_{\boldsymbol{\ell}}^2(1-\lambda_{\mathbf{k}-\boldsymbol{\ell}}^2)+(\mathbf{k}\cdot\boldsymbol{\ell})\lambda_{\mathbf{k}-\boldsymbol{\ell}}^2(1-\lambda_{\boldsymbol{\ell}}^2)-2(\mathbf{k}\cdot\boldsymbol{\ell})(\mathbf{k}\cdot\mathbf{k}-\boldsymbol{\ell})(\lambda_{\boldsymbol{\ell}}-\lambda_{\mathbf{k}-\boldsymbol{\ell}})^2\}\right]^2}{(\boldsymbol{\ell}^2\lambda_{\mathbf{k}}\lambda_{\mathbf{k}-\boldsymbol{\ell}}+(\mathbf{k}-\boldsymbol{\ell})^2\lambda_{\mathbf{k}}\lambda_{\boldsymbol{\ell}}+\mathbf{k}^2\lambda_{\boldsymbol{\ell}}\lambda_{\mathbf{k}-\boldsymbol{\ell}})^2(\boldsymbol{\ell}^2\lambda_{\mathbf{k}}\lambda_{\mathbf{k}-\boldsymbol{\ell}}+(\mathbf{k}-\boldsymbol{\ell})^2\lambda_{\mathbf{k}}\lambda_{\boldsymbol{\ell}}-\mathbf{k}^2\lambda_{\boldsymbol{\ell}}\lambda_{\mathbf{k}-\boldsymbol{\ell}}(1-\eta_{\mathbf{k}}))}. \qquad \text{(B-18)}$$

It should be noted here that the correction term $\eta_{\mathbf{k}}$ in (B-18) vanishes for the case of the non-interacting system ($\lambda_{\mathbf{k}} = 1$) as they should, and that the integral in (B-18) does strongly converge irrespective of the form of the potential function between helium atoms in contrast to that based on the standard perturbation theory. By virtue of this reason, we can use a simple square-well potential (see Fig. B-1)

$$V(r) = \begin{cases} V_a & \text{for } a > r \\ -V_b & \text{for } b > r > a \\ 0 & \text{for } r > b \end{cases} \qquad \text{(B-19)}$$

Fig. B-1

as an idealization of the realistic potential, where Va indicates the height of the repulsive potential and V_b denotes the depth of the attractive potential. The parameters a and b represent the range of the repulsive potential and that of the attractive one, respectively. In the case of the realistic potential, however, the repulsive core is not soft like (B-19), but is regarded as a hard core. In this respect, the potential (B-19) seems to be very far from reality. For the hard sphere potential, however, the Fourier transform of the potential function produces divergence, because the plane wave does not vanish in the hard sphere region. In reality, the amplitude of the wave function vanishes in the region of the hard sphere potential, and the product of the hard sphere potential and the wave function remains to be finite. So long as one uses the Fourier transform, therefore, use of the hard sphere potential is not adequate and it should be replaced by a finite effective potential. From the arguments mentioned above, we should realize that the height Va of the repulsive potential in (B-19) does not literally represent the real height of the repulsive potential, but gives the effective magnitude of the hard sphere interaction.

The Fourier transform of the potential (B-19) is given by

$$\nu(k) = \frac{4}{3}\pi a^3 V_b (n+1)(1-\gamma) H(k) ,$$

$$\gamma = \frac{1}{n+1}\left(\frac{b}{a}\right)^3 , \quad n = V_a / V_b \tag{B-20}$$

and

$$H(k) = \frac{3}{1-\gamma}\left\{ \frac{j_1(ka)}{ka} - \frac{j_1(kb)}{kb} \right\} , \tag{B-21}$$

where j_1 is the first order spherical Bessel function. The function $c(k)$ defined by (A-21) is written as

$$c^2(k) = c^2 H(k) \tag{B-22}$$

and

$$c^2 = \frac{16}{3}\pi \frac{N}{V} V_b \frac{ma^3}{\hbar^2}(n+1)(1-\gamma) . \tag{B-23}$$

We now fix the mean density N/V and the constant c with the observed values

$$N/V = 1/45.8 \ \text{Å}^{-3}$$

and

$$c = \frac{2ms}{\hbar} = 2.99 \ \text{Å}^{-1} ,$$

where $s = 2.37 \times 10^4$ cm/sec is the observed phonon velocity. For the parameters a and b, we set a = 2.8Å and b = 4.4Å, which are reasonable values as the ranges of the interaction potential between two helium atoms. The remaining parameter should be determined so that we have a reasonable value for the depth of the square-well attractive potential V_b from (B-23) and (B-20). If we choose $\gamma = 0.60$ which corresponds to take $n = 5.5$, we have a reasonable value $V_b = 7.2 \times 10^{-16}$ erg .

By using the parameters chosen above, the equation (B-18) for η_k is solved numerically and the solutions give the multi-branch excitation spectra shown in Fig. B-2.

Fig. B-2

Appendix C

Projection operator and the velocity operator

In A, we have introduced the projection operator $P(\mathbf{k})$ and the velocity operator to rewrite the original Hamiltonian (A-1) in terms of the collective variables. This appendix is devoted to the proof of the property of the projection operator, i.e. $P(\mathbf{k}) = \mathbb{1}\delta_{\mathbf{k},0}$ and the derivation of the second commutation relation of (A-6).

The projection operator $P(\mathbf{k})$ is given by

$$P(\mathbf{k}) = \sum_{\ell} a_{\ell} \frac{1}{N} a^{\dagger}_{\ell+\mathbf{k}} - \sum_{m \neq 0} \sum_{\ell} a_{\ell} \frac{1}{N} \rho_m \frac{1}{N} a^{\dagger}_{\ell-m+\mathbf{k}} + \cdots \quad . \tag{C-1}$$

We transform each term of the right hand side of (C-1) into

$$\sum_{\ell} a_{\ell} \frac{1}{N} a^{\dagger}_{\ell+\mathbf{k}} = \sum_{\ell} \frac{1}{N} a_{\ell} a^{\dagger}_{\ell+\mathbf{k}} - \sum_{\ell} \frac{1}{N} a_{\ell} \frac{1}{N} a^{\dagger}_{\ell+\mathbf{k}} \ , \tag{C-2}$$

$$\sum_{\substack{m \neq 0 \\ \ell}} a_{\ell} \frac{1}{N} \rho_m \frac{1}{N} a^{\dagger}_{\ell-m+\mathbf{k}} = \sum_{m \neq 0} \frac{1}{N} \rho_m a_{\ell} \frac{1}{N} a^{\dagger}_{\ell-m+\mathbf{k}} + \sum_{\substack{m \neq 0 \\ \ell}} \frac{1}{N} a_{\ell-m} \frac{1}{N} a^{\dagger}_{\ell-m+\mathbf{k}}$$
$$- \sum_{\substack{m \neq 0 \\ \ell}} \frac{1}{N} a_{\ell} \frac{1}{N} \rho_m \frac{1}{N} a^{\dagger}_{\ell-m+\mathbf{k}} \tag{C-3}$$

and so on. Combining the second terms on the right hand side of (C-2) and (C-3), we have

$$- \sum_{\ell} \frac{1}{N} a_{\ell} \frac{1}{N} a^{\dagger}_{\ell+\mathbf{k}} - \sum_{\substack{m \neq 0 \\ \ell}} \frac{1}{N} a_{\ell} \frac{1}{N} \rho_m \frac{1}{N} a^{\dagger}_{\ell-m+\mathbf{k}}$$

$$= - \sum_{m,\ell} \frac{1}{N} a_{\ell} \frac{1}{N} \rho_m \frac{1}{N} a^{\dagger}_{\ell-m+\mathbf{k}} \quad .$$

By repeating this procedure for all terms of (C-1), we rewrite (C-1) in the following form;

$$\mathcal{P}(k) = \sum_{\ell} \frac{1}{\tilde{N}} a_\ell a^{\dagger}_{\ell+k} - \sum_{m\neq 0} \frac{1}{\tilde{N}} \rho_m \Big\{ \sum_{\ell} a_\ell \frac{1}{\tilde{N}} a^{\dagger}_{\ell-m+k} - \sum_{\substack{n\neq 0\\ \ell}} a_\ell \frac{1}{\tilde{N}} \rho_n \frac{1}{\tilde{N}} a^{\dagger}_{\ell-m-n+k}$$

$$+ \cdots \Big]$$

$$- \frac{1}{\tilde{N}} \sum_{m} \Big\{ \sum_{\ell} a_\ell \frac{1}{\tilde{N}} a^{\dagger}_{\ell-m+k} - \sum_{\substack{n\neq 0\\ \ell}} a_{\ell-m} \frac{1}{\tilde{N}} \rho_n \frac{1}{\tilde{N}} a^{\dagger}_{\ell-m-n+k} \cdots \Big] \qquad (C-4)$$

Using the equation (C-1) in (C-4), we obtain an integral equation

$$\mathcal{P}(k) = \sum_{\ell} \frac{1}{\tilde{N}} a_\ell a^{\dagger}_{\ell+k} - \sum_{m\neq 0} \frac{1}{\tilde{N}} \rho_m \mathcal{P}(k-m) - \frac{1}{\tilde{N}} \Delta \mathcal{P}(k), \qquad (C-5)$$

where Δ indicates $\sum_{\ell} 1$. The integral equation for $\mathcal{P}(k)$ is rewritten in the following form

$$\mathcal{P}(k) = \frac{\Delta \delta_{k,0} + \rho_k}{\tilde{N} + \Delta} - \frac{1}{\tilde{N}+\Delta} \sum_{\ell\neq 0} \rho_\ell \mathcal{P}(k-\ell) \qquad (C-6)$$

for the case of the Bose system. In the case of Fermi system, on the other hand, we get

$$\mathcal{P}(k) = \frac{\Delta \delta_{k,0} - \rho_k}{\tilde{N} + \Delta} - \frac{1}{\tilde{N}+\Delta} \sum_{\ell\neq 0} \rho_\ell \mathcal{P}(k-\ell), \qquad (C-7)$$

where we should note the difference of the sign in the inhomogenous term in (C-6) and (C-7). Iterating the equation (C-6), we have

$$\mathcal{P}(k) = \frac{\Delta \delta_{k,0} + \rho_k}{\tilde{N} + \Delta} - \frac{1}{(\tilde{N}+\Delta)^2} \sum_{\ell\neq 0} \rho_\ell \left(\Delta \delta_{k-\ell,0} + \rho_{k-\ell} \right)$$

$$+ \frac{1}{(\tilde{N}+\Delta)^3} \sum_{\substack{\ell\neq 0\\ m\neq 0}} \rho_\ell \rho_m \left(\delta_{k-\ell-m,0} + \rho_{k-\ell-m} \right) - \cdots \qquad (C-8)$$

In the case of $\mathbb{k}=0$, the equation (c-8) becomes

$$
\begin{aligned}
P(0) &= 1 - \frac{1}{\tilde{N}+\Delta}\sum_{\ell\neq 0}\rho_\ell\rho_{-\ell} + \frac{1}{(\tilde{N}+\Delta)^3}\sum_{\substack{\ell\neq 0\\ m\neq 0}}\rho_\ell\rho_m\left(\delta(0)\delta_{\ell+m,0}+\rho_{-\ell-m}\right) \\
&\quad - \cdots \\
&= 1 - \frac{1}{(\tilde{N}+\Delta)^2}\sum_{\ell\neq 0}\rho_\ell\rho_{-\ell} + \frac{1}{(\tilde{N}+\Delta)^2}\sum_{\ell\neq 0}\rho_\ell\rho_{-\ell} + \frac{1}{(\tilde{N}+\Delta)^3}\sum_{\substack{\ell\neq 0\\ m\neq 0\\ \ell+m\neq 0}}\rho_\ell\rho_m\rho_{-\ell-m} \\
&\quad - \cdots \\
&= 1 .
\end{aligned}
\tag{C-9}
$$

In the case of $\mathbb{k}\neq 0$, on the other hand, one can see that

$$
\begin{aligned}
P(\mathbb{k}) &= \frac{\rho_{\mathbb{k}}}{\tilde{N}+\Delta} - \frac{1}{(\tilde{N}+\Delta)^2}\sum_{\ell\neq 0}\rho_\ell\left(\Delta\delta_{\mathbb{k}-\ell,0}+\rho_{\mathbb{k}-\ell}\right) \\
&\quad + \frac{1}{(\tilde{N}+\Delta)^3}\sum_{\substack{\ell\neq 0\\ m\neq 0}}\rho_\ell\rho_m\left(\Delta\delta_{\mathbb{k}-\ell-m,0}+\rho_{\mathbb{k}-\ell-m}\right) - \cdots \\
&= \frac{\rho_{\mathbb{k}}}{\tilde{N}+\Delta} - \frac{\rho_{\mathbb{k}}}{\tilde{N}+\Delta} - \frac{1}{(\tilde{N}+\Delta)^2}\sum_{\ell\neq 0,\mathbb{k}}\rho_\ell\rho_{\mathbb{k}-\ell} \\
&\qquad + \frac{1}{(\tilde{N}+\Delta)^2}\sum_{\ell\neq 0,\mathbb{k}}\rho_\ell\rho_{\mathbb{k}-\ell} - \cdots \\
&= 0 .
\end{aligned}
\tag{C-10}
$$

Hence we can conclude that

$$
P(\mathbb{k}) = 1\,\delta_{\mathbb{k},0} \tag{C-11}
$$

for the case of Bose system.

On the other hand, we cannot derive the relation (C-11) for the case of the Fermi system, because of the difference of the sign in front of ρ_k in the first term on the right hand side of (c-7).

The proof of the commutation relation $[V_k^{(i)}, V_{k'}^{(j)}] = 0$ is achieved by using the property of the projection operator mentioned above. If we introduce the integral equations defined by

$$R_k = \frac{1}{N} a_k^{\dagger} - \sum_{l \neq 0} \frac{1}{N} \rho_l R_{k-l} \tag{C-12}$$

and by

$$S_k = \frac{1}{N} \delta_{k,0} - \sum_{l \neq 0} \frac{1}{N} \rho_l S_{k-l} \quad , \tag{C-13}$$

the velocity operator given by (A-5) is rewritten as

$$V_k = \sum_m R_{m-k} m a_m + \frac{1}{2} \sum_m S_{k-m} m a_m \ . \tag{C-14}$$

First we shall show that R_k commutes exactly with a_p . We can see from (c-12) that the commutation relation of these operators is readily given by

$$\begin{aligned}[a_p, R_k] = \frac{1}{N} \delta_{p,k} - \frac{1}{N} a_p \Big[\frac{1}{N} a_k^{\dagger} - \sum_{l \neq 0} \frac{1}{N} \rho_l R_{k-l} \Big] \\ - \frac{1}{N} \sum_{l \neq 0} a_{p-l} R_{k-l} - \sum_{l \neq 0} \frac{1}{N} \rho_l [a_p, R_{k-l}] \ .\end{aligned} \tag{C-15}$$

By using (c-12), the second term on the right hand side of (c-15) becomes

$$-\frac{1}{N} a_p R_k$$

and the integral equation (C-15) is expressed as

$$[a_p, R_k] = \frac{1}{N}\delta_{p,k} - \sum_{\ell} \frac{1}{N} a_{p-\ell} R_{k-\ell} - \sum_{\ell \neq 0} \frac{1}{N} \rho_\ell [a_p, R_{k-\ell}]. \qquad \text{(C-16)}$$

Noting (C-11), we find that the first term on the right hand side of (C-16) cancels out the second term, and hence we can conclude that

$$[a_p, R_k] = 0 . \qquad \text{(C-17)}$$

We now turn our attention to the proof of the relation $k \times V_k = 0$. By introducing two operators $\vec{Q}_k$ and $\vec{T}_k$ defined by

$$\vec{Q}_k = \sum_m (m-k) R_{m-k} \times m\, a_m \qquad \text{(C-18)}$$

and by

$$\vec{T}_k = \sum_m (m+k) S_{k-m} \times m\, \rho_m \qquad \text{(C-19)}$$

,

respectively, $k \times V_k$ is expressed as

$$k \times V_k = -\vec{Q}_k + \frac{1}{2}\vec{T}_k . \qquad \text{(C-20)}$$

From (C-12), $\vec{Q}_k$ is rewritten as

$$\vec{Q}_k = \sum_m (m-k)\frac{1}{N}a^\dagger_{m-k}\times m a_m - \sum_m \sum_l \frac{1}{N} l\, P_l R_{m-k-l}\times m a_m$$
$$-\sum_m \sum_{l\neq 0}\frac{1}{N}P_l (m-k-l) R_{m-k-l}\times m a_m \;, \qquad \text{(C-21)}$$

where the restriction concerning the sum over momenta l on the right hand side of (C-21) is taken off, because the contribution from $l=0$ vanishes. The second term on the right hand side of (C-21) is transformed into

$$-\sum_{l,m,p}\frac{1}{N} l\, a^\dagger_p a_{p-l} R_{m-k-l}\times m a_m$$
$$= -\sum_{l,m,p}\frac{1}{N}(p-l)a^\dagger_p a_l R_{m+l-k-p}\times m a_m$$
$$= -\sum_{l,m,p}\frac{1}{N} p\, a^\dagger_p a_l R_{l+m-k-p}\times m a_m$$
$$= -\sum_m \frac{1}{N}(m-k)\, a^\dagger_{m-k}\times m a_m \;, \qquad \text{(C-22)}$$

where the last equality is due to (c-11), and the second one comes from the following relation

$$\sum_{m,l} a^\dagger_p R_{l+m-k-p}\, l\, a_l \times m a_m = 0 \;. \qquad \text{(C-23)}$$

From (C-22) and (C-18), we can conclude that

$$\vec{Q}_k = 0 \;. \qquad \text{(C-24)}$$

In the same way, we find

$$\vec{T}_{k} = 0 - \sum_{l \neq k} \frac{1}{N} \rho_{l} \vec{T}_{k+l} \tag{C-25}$$

and hence

$$\vec{T}_{k} = 0 \ . \tag{C-26}$$

With the aid of (C-24) and (C-26), one can conclude that

$$k \times V_{k} = 0 \ . \tag{C-27}$$

The commutation relation between the velocity operators is given by

$$[V_{k}^{(i)}, V_{k'}^{(j)}] = -\frac{1}{N^2} \sum_{p} \rho_{p-k-k'} (p_i V_{p}^{(j)} - p_j V_{p}^{(i)}) - \frac{1}{N^2} \sum_{p,q}{}' \rho_{p-k} \rho_{q-k'} [V_{p}^{(i)}, V_{q}^{(j)}] , \tag{C-28}$$

where the symbol $\sum_{p,q}{}'$ indicates that the term with $p = k$ and $q = k'$ at the same time should be omitted from the sum. Noting (c-27), we find the second commutation relation in (A-6) ;

$$[V_{k}^{(i)}, V_{k'}^{(j)}] = 0 \tag{C-29}$$

Closing this section, we should give the remarks that the relations (c-11), (c-27) and (c-29) are valid only in the case of the Bose system.

Acknowledgements

The author would like to express his sincere thanks to Professor S. Sunakawa for his quidance and encouragement. Thanks are also due to Professor T. Nishiyama and Professor S. Yamasaki for their useful discussions.

Reference

(1) L. D. Landau, J. Phys. 5 (1941), 71; 11 (1947), 91.

(2) D. G. Henshaw and A. D. B. Woods, Phys. Rev. 121 (1961), 1266.

(3) R. P. Feynman and M. Cohen, Phys. Rev. 102 (1956), 1183.

(4) N. N. Bogoliubov, J. Phys. 11 (1947), 23.

(5) W. A. Parry and D. ter Haar, Ann. of phys. 19 (1962), 496.

(6) K. A. Brueckner and K. Sawada, Phys. Rev. 106 (1957), 1117.

(7) A. Miller, D. Pines and P. Nozieres, Phys. Rev. 127 (1962), 1452.

(8) S. Sunakawa, Y. Yoko-o and H. Nakatani, Prog. Theor. Phys. 27 (1962), 589, 600 ; 28 (1962), 127.
Y. Fukui and S. Sunakawa, Prog. Theor. Phys. 34 (1965), 499.

(9) S. Sunakawa, S. Yamasaki and T. Kebukawa, Prog. Theor. Phys. 41 (1969), 919.

(10) T. Kebukawa, S. Yamasaki and S. Sunakawa, Prog. Theor. Phys. 44 (1970), 565.

(11) T. Kebukawa, S. Yamasaki and S. Sunakawa, preprint.

(12) R. A. Cowley and A. D. N. Woods, Can. J. Phys. 49 (1971), 177. A. D. B. Woods, E. C. Svensson and P. Martel, preprint.

(13) O. W Dietrich, E. H. Graf, C. H. Huang and L. Passell, Phys. Rev. A 5 (1972), 1377.

(14) W. M. Whitney and C. E. Chase, Phys. Rev. Lett. 9 (1962), 243.

(15) I. M. Khalatnikov and D. M. Chernikova, JETP 22 (1966), 1336 ; 23 (1966), 274.
A. Andreev and I. Khalatnikov, JETP 17 (1963), 1384.

(16) A. J. Legett and D. Ter Haar, Phys. Rev. 139 A 779.
C. J. Pethick and D. ter Haar, Physica 32 (1966), 1905.

(17) T. Nishiyama, Prog. Theor. Phys. 45 (1971), 730.

(18) S. Sunakawa, S. Yamasaki and T. Kebukawa, Prog. Theor. Phys. 44 (1970), 819.

(19) A. A. Abrikosov, L. P. Gorkov and I. E. Dzyaloshinskii, Quantum Field Theoretical Methods in Statistical Physics, (Tokyo tosho).
A. L. Fetter and J. D. Walecka, Quantum Theory of Many Particle System, (M_cGraw-hill book company).

Figure Captions

Fig - 1 Lowest order diagrams for the self energy.

Fig - 2 Lowest order diagrams for the structure factor.

Fig - 3 The deviation of the phonon velocity from that at zero temperature in liquid helium II : the continuous curve gives the theoretical values and the points are the experimental data of Whitney and Chase.[14]

Fig - 4 The phonon velocity in liquid helium II : the continuous curve gives the theoretical values and the points are the experimental data of Whitney and Chase.[14]

Fig - 5 The roton minimum in liquid helium II : the continuous curve gives the theoretical values and the points are the experimental data of Dietrich et al..[13]

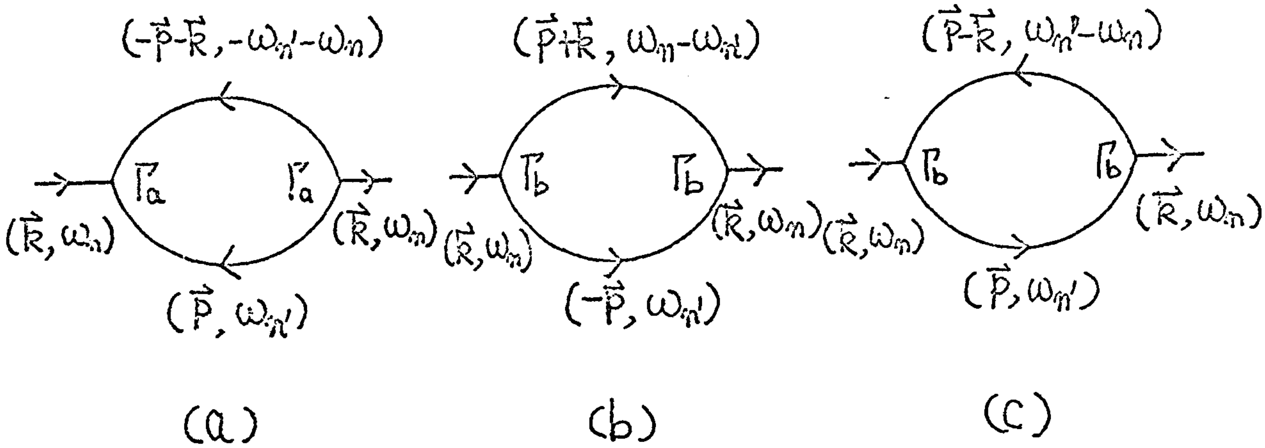
Table - 1 The calculated numerical values of phonon velocity versus temperature in liquid helium II.

Table - 2 The calculated numerical values of roton minimum versus temperature in liquid helium II.

Fig. B-1 The effective interaction potential (B-19) between helium atoms.

Fig, B-2 The theoretical and experimental curves of excitation spectra in liquid HeII. The experimental curve H. W. has been observed by Henshaw and Woods.[2] The experimental curve C. W. are found Cowley and Woods.[12] The curve $E_{\mathbf{k}}^{B}$ is the Bogoliubov excitation spectrum in the present theory. The curves $E_{\mathbf{k}}^{I}$, $E_{\mathbf{k}}^{III,A}$, $E_{\mathbf{k}}^{III,B}$, $E_{\mathbf{k}}^{III,2}$ and $E_{\mathbf{k}}^{III,1}$ are the theoretical excitation spectra.[11]

(Fig - 1)



$(-\vec{p}-\vec{k}, -\omega_{n'}-\omega_n)$
Γ_a
Γ_a
$(\vec{k}, \omega_n)$
$(\vec{k}, \omega_n)$
$(\vec{p}, \omega_{n'})$
$(\vec{p}+\vec{k}, \omega_n-\omega_{n'})$
Γ_b
Γ_b
$(\vec{k}, \omega_n)$
$(\vec{k}, \omega_n)$
$(-\vec{p}, \omega_{n'})$
$(\vec{p}-\vec{k}, \omega_{n'}-\omega_n)$
Γ_b
Γ_b
$(\vec{k}, \omega_n)$
$(\vec{k}, \omega_n)$
$(\vec{p}, \omega_{n'})$
(a)
(b)
(c)

Fig-1

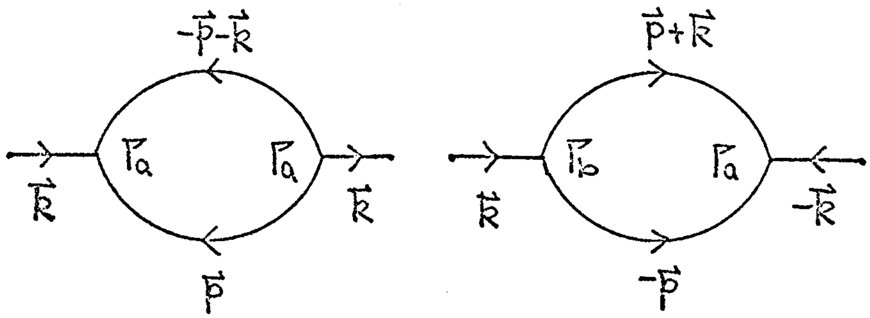


Fig-2

Fig-3

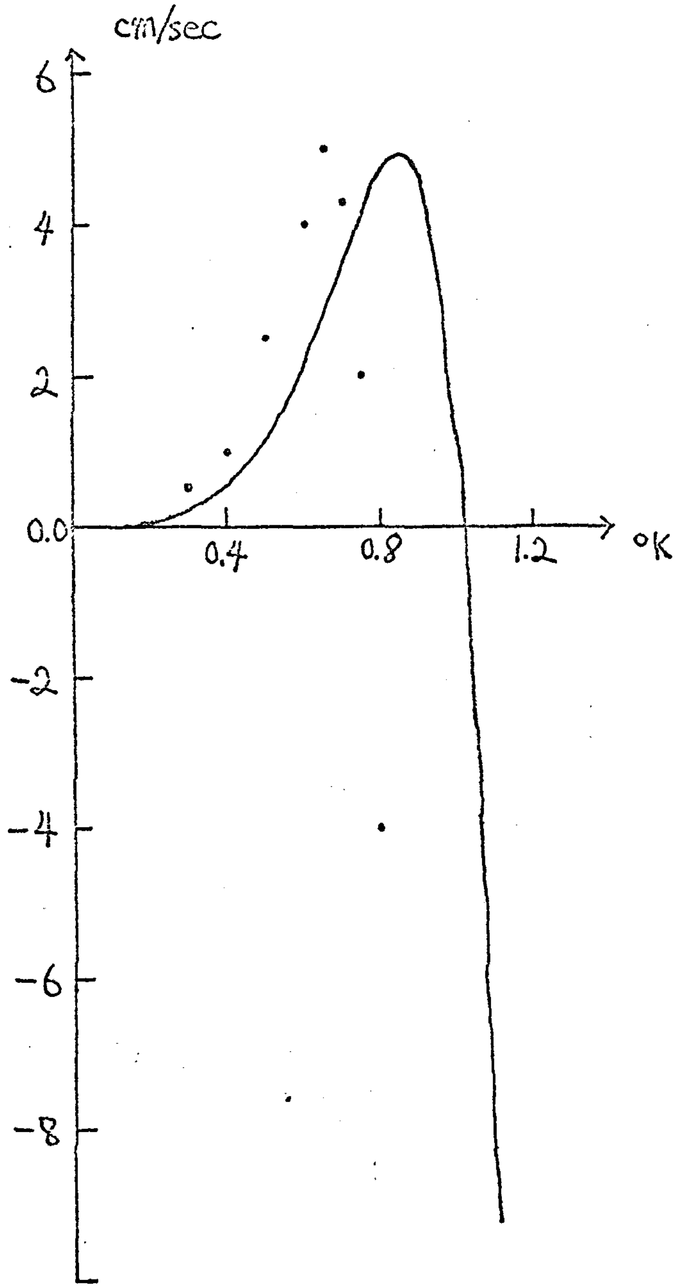


Fig-4

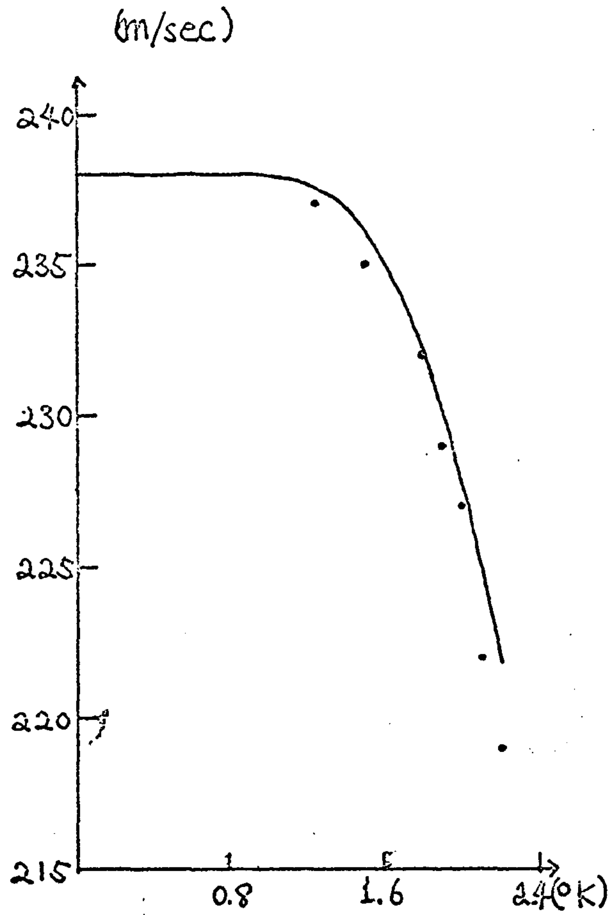
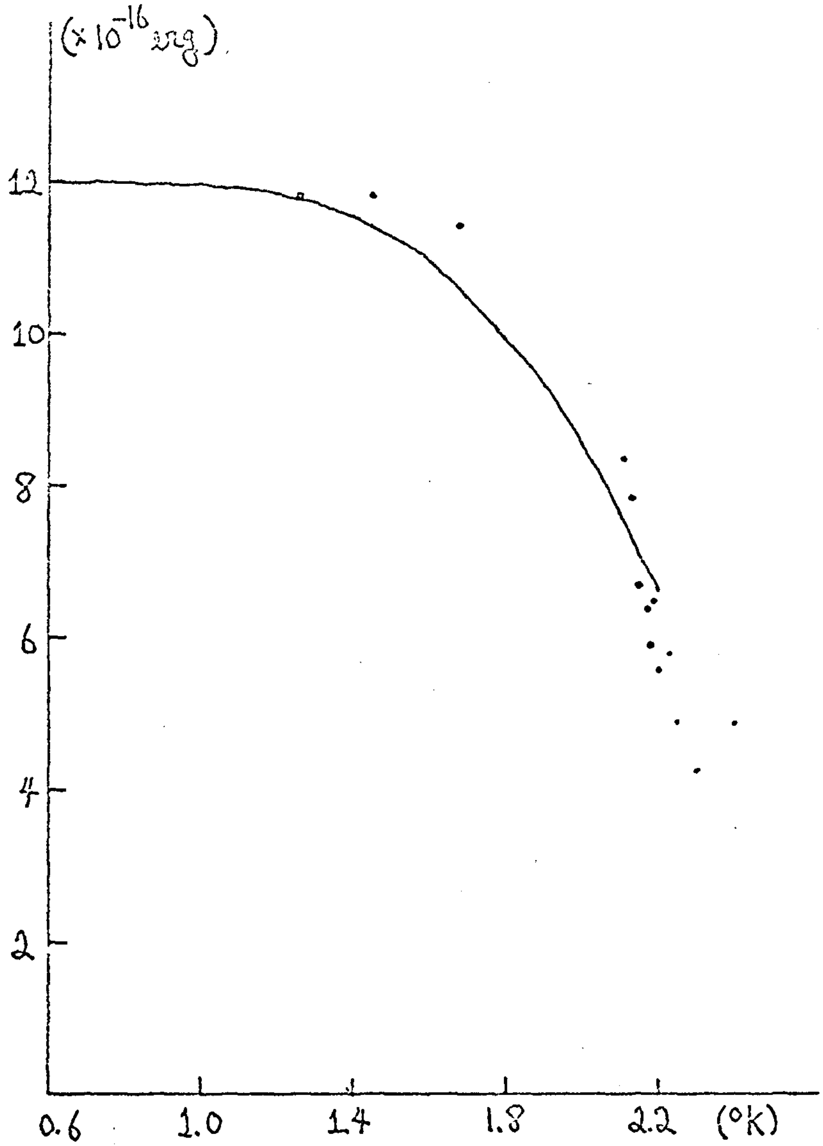


Fig -5



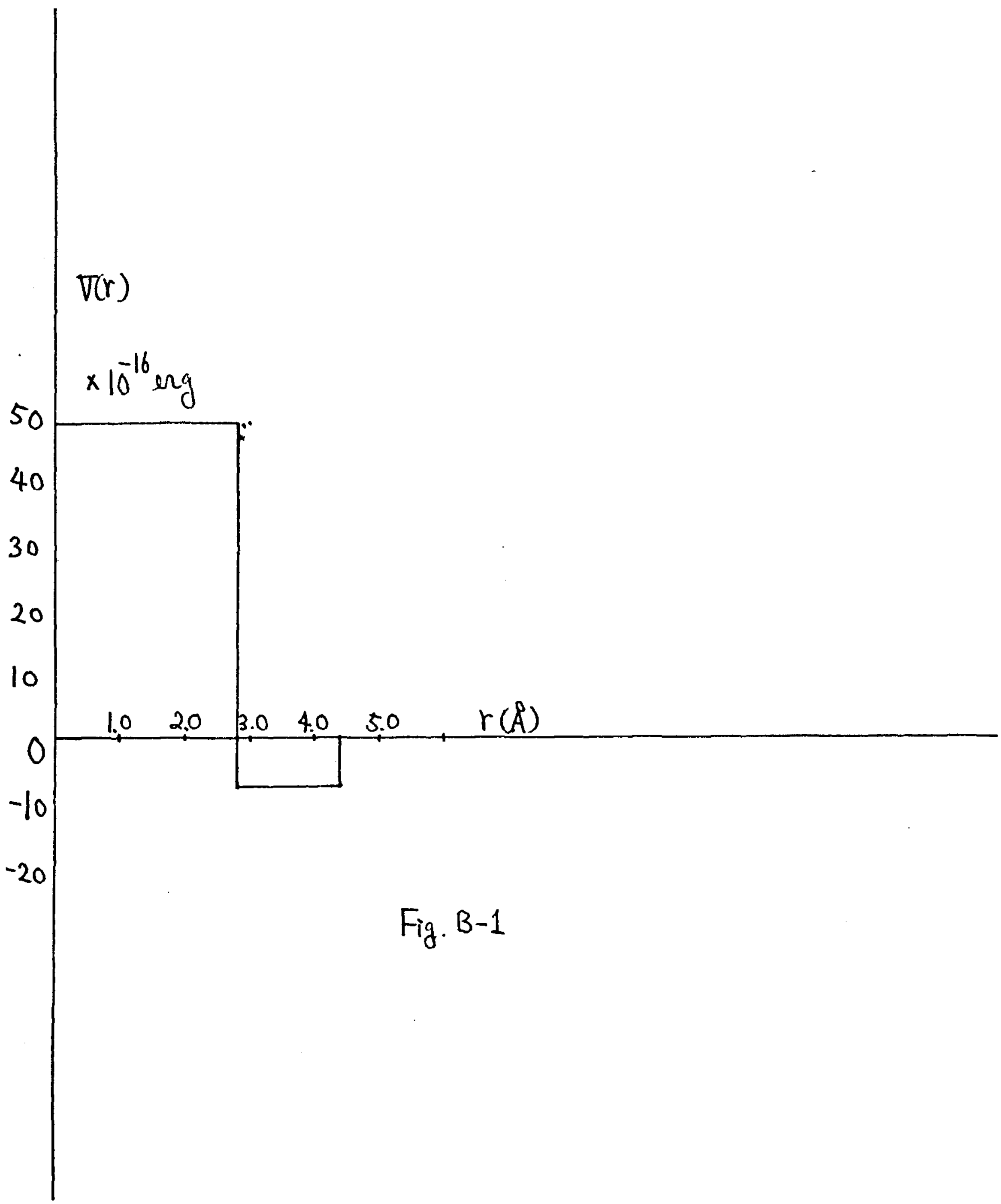


Fig. B-1

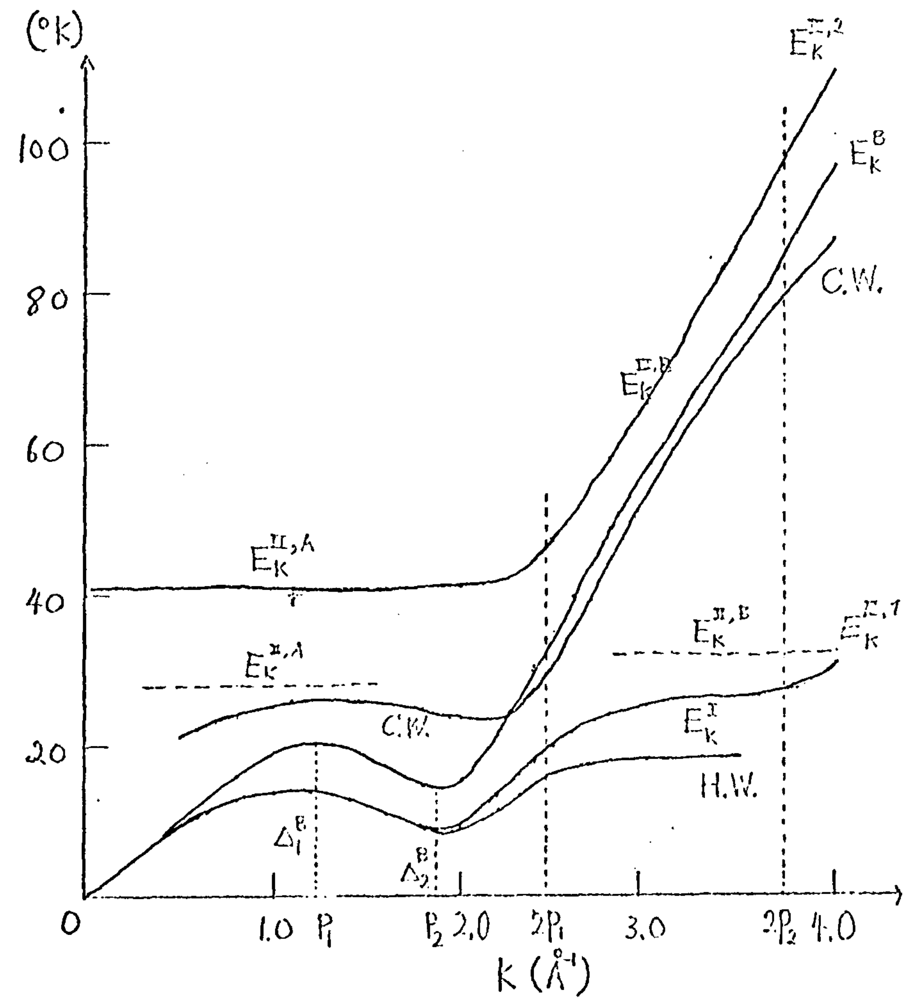


Fig. B-2

Table - 1

T(°k)	C(T) ($\frac{m}{sec}$)
0.1	238
0.4	238
0.6	238
0.8	238
1.0	238
1.1	237.9
1.2	237.7
1.3	237.4
1.4	236.9
1.5	236.1
1.6	235.1
1.7	235.8
1.8	233.2
1.9	230.2
2.0	227.8
2.1	225.0
2.2	221.9

Table - 2

T(°k)	Δ(T) ($\times 10^{-16}$ erg)
0.1	12
0.4	12
0.6	11.99
0.8	11.98
1.0	11.96
1.1	11.91
1.2	11.83
1.3	11.70
1.4	11.52
1.5	11.26
1.6	10.91
1.7	10.47
1.8	9.93
1.9	9.28
2.0	8.51
2.1	7.62
2.2	6.61