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High-accuracy optical computing based on interval arithmetic and the fixed-point theorem

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A method for high-accuracy analog optical computing based on interval arithmetic and the fixed-point theorem is considered. Two-variable simultaneous equations are studied to investigate the proposed method. An optical implementation is considered by the use of spatial coding of intervals, affine transformation, and image magnification. Computational simulation verifies the principle of the method. © 1996 Optical Society of America

1. Introduction

In the course of exploring the capabilities of optical computing, we found the analog optical scheme attractive because of data capacity and processing capability provided by the physical characteristics of light. Various optical processes, such as the Fourier transform and optical matched filters, are good examples of the scheme.[1] However, inherent disadvantages also exist in analog optical computing, such as computational accuracy, dynamic range of data, and difficulty of implementation. Although the digital optical scheme[2,3] is a practical solution to the disadvantages, the sampling nature of the digital scheme greatly reduces the inherent capabilities of optical computing.

The objective of our research is to develop a new concept of analog computing with high accuracy and a large dynamic range of data representation. To this end, we consider a method for analog optical computing based on interval arithmetic[4] and the fixed-point theorem.[5] As an example, two-variable simultaneous equations are studied to investigate the proposed method. In Section 2 the mathematical basis of the method is explained with an example. In Section 3 an optical implementation for a two-variable problem is described with a simulation experiment. In Section 4 the features and future issues of the proposed method are discussed.

2. Mathematical Basis

One effective way to accomplish high accuracy in analog optical computing is to utilize accumulated resources in computer science. An enormous amount of effort has been made to improve the accuracy and efficiency of computation on digital computers. Among them, the authors found that interval arithmetic[4] and the fixed-point theorem[5] provide effective solutions for high-accuracy analog optical computing.

A. Interval Arithmetic

Interval arithmetic[4] is a computational scheme in which a number is represented by an interval that includes the object number. The four fundamental rules are defined as operations on the intervals to grasp computational error associated with implementation, e.g., limited number of bits in accumulators, rounding error, etc.

In interval arithmetic, a real number x is represented by a close interval $[a, b]$ where $[x | a \leq x \leq b]$. The four fundamental rules are defined as follows:

$$[a, b] + [c, d] = [a + c, b + d], \tag{1}$$

$$[a, b] - [c, d] = [a - d, b - d], \tag{2}$$

$$[a, b] \cdot [c, d] = [\min(ac, ad, bc, bd), \\ \max(ac, ad, bc, bd)], \tag{3}$$

$$[a, b]/[c, d] = [a, b] \cdot [1/d, 1/c]. \tag{4}$$

By using these rules, we can exactly grasp the behavior of the calculated numbers, which guarantees correctness of the computation.

B. Fixed-Point Theorem

The fixed-point theorem[5] indicates the existence of fixed point $\mathbf{x}^*$ for mapping f: $X \rightarrow X$, if $f(X) \subset X$, where X is a space $\mathbf{R}$ and $f(\mathbf{x}^*) = \mathbf{x}^*$. Although interval arithmetic guarantees computational correctness, the intervals must be shrunk to increase computational accuracy. For this purpose, reduction mapping around a fixed point can be utilized.

Figure 1 shows a conceptual diagram of the technique. The initial plane is converted into a shrunken (and rotated or translated) plane after a mapping, in which the fixed point is transformed into the same location. At the same time, an interval located on the initial plane shrinks and closes to the fixed point. Therefore, by repeating the same mapping sequentially, we can shrink the area of the interval, which results in increasing computational accuracy.

C. Computational Algorithm

Various kinds of computational algorithm have been developed with the above technique.[6] To illustrate the technique, a solution for a simultaneous linear equation is explained. The target equation is expressed as

$$\mathscr{A}\mathbf{x} = \mathbf{b}, \qquad (5)$$

where $\mathscr{A}$ is an $n \times n$ matrix and $\mathbf{x}$ and $\mathbf{b}$ are unknown and known n vectors, respectively. First, an approximate inverse matrix of $\mathscr{A}$ (denoted by $\mathscr{R}$) and an approximate solution (denoted by $\tilde{\mathbf{x}}$) are calculated. It seems strange to assume the calculation of these approximate values; this assumption might be a target of criticism, but abstract computing and accurate computing should be treated as different techniques. Note that most optical methods are categorized into the former, but few into the latter.

Equation (5) can be rewritten as the following form:

$$\mathscr{R}(\mathbf{b} - \mathscr{A}\tilde{\mathbf{x}}) + (\mathscr{T} - \mathscr{R}\mathscr{A})\mathbf{x}^* = \mathbf{x}^*, \qquad (6)$$

where $\mathscr{T}$ is a unit matrix and $\tilde{\mathbf{x}} + \mathbf{x}^*$ gives the correct solution of Eq. (5).

Referring to Eq. (6), if the right-hand side is considered as a mapping of a space $\mathbf{R}^n$, $\mathbf{x}^*$ can be regarded as the fixed point of the mapping, namely,

$$g(\mathbf{x}^*) = \mathscr{R}(\mathbf{b} - \mathscr{A}\tilde{\mathbf{x}}) + (\mathscr{T} - \mathscr{R}\mathscr{A})\mathbf{x}^* = \mathbf{x}^* \quad (7)$$

indicates that $\mathbf{x}^*$ is the fixed point of the mapping g.

Although Eq. (7) describes mapping of the conventional number representation system, the mapping can also be applied to interval arithmetic:

$$g(\mathbf{X}) = \mathscr{R}(\mathbf{b} - \mathscr{A}\tilde{\mathbf{x}}) + (\mathscr{T} - \mathscr{R}\mathscr{A})\mathbf{X}, \qquad (8)$$

where $\mathbf{X}$ is an n vector whose elements are intervals that represent the range of individual elements of the vector $\mathbf{x}^*$. According to the fixed-point theorem, if $g(\mathbf{X}) \subset \mathbf{X}$, the existence of the fixed point $\mathbf{x}^* \in \mathbf{X}$ is guaranteed.

To ensure convergence of the mapping, the following iterative calculation is applied:

$$\mathbf{X}^{(k)} = g[\mathbf{X}^{(k-1)}] \cap \mathbf{X}^{(k-1)},$$

$$\mathbf{X}^{(0)} = \mathbf{X}, \qquad (10)$$

where the superscript indicates the iteration number. As k is increased, $\mathbf{X}^{(k)}$ converges to the fixed point $\mathbf{x}^*$. Consequently the solution of Eq. (5) is given by $\tilde{\mathbf{x}} + \mathbf{X}^{(k)}$ for sufficiently large k.

D. Numerical Simulation

To illustrate the above algorithm, a simple example is given. We consider the following two-variable simultaneous linear equation:

$$\begin{bmatrix} 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \qquad (11)$$

For Eq. (11), the correct answer is $(x_0, x_1) = (5/6, 1/3)$, which are infinite decimals. As the first step, approximate inverse matrix $\mathscr{R}$ and approximate solution $\tilde{\mathbf{x}}$ are assumed (or calculated by an appropriate method) as follows:

$$\mathscr{R} = \begin{bmatrix} -0.2 & 0.6 \\ 0.3 & -0.3 \end{bmatrix},$$

$$\tilde{\mathbf{x}} = \begin{pmatrix} 0.8 \\ 0.3 \end{pmatrix}.$$

According to Eq. (8), the mapping for the given problem is obtained by $\mathscr{R}$ and $\tilde{\mathbf{x}}$:

$$g\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{bmatrix} 0.2 & 0.2 \\ 0 & 0.1 \end{bmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + \begin{pmatrix} 0.02 \\ 0.03 \end{pmatrix}. \qquad (12)$$

Table 1 shows the time development of the intervals transformed by Eq. (12), in which the initial
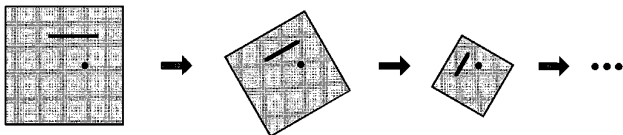

Fig. 1. Conceptual diagram of interval shrinking with reduction mapping around the fixed point. The filled circles and the lines indicate the fixed point and the intervals, respectively.

Table 1. Time Development of Transformed Intervals

Iteration	$x_{0\text{bottom}}$	$x_{0\text{top}}$	$x_{1\text{bottom}}$	$x_{1\text{top}}$
0	−1.0000000	1.0000000	−1.0000000	1.0000000
1	−0.3800000	0.4200000	−0.0700000	0.1300000
2	−0.0700000	0.1300000	0.0230000	0.0430000
3	0.0106000	0.0546000	0.0323000	0.0343000
4	0.0285800	0.0377800	0.0332300	0.0334300
5	0.0323620	0.0342420	0.0333230	0.0333430
6	0.0331370	0.0335170	0.0333323	0.0333343

intervals for x_0 and x_1 are both $[-1, 1]$. As seen from Table 1, both intervals are shrunk as the transform is repeated. Adding the approximate solution $\tilde{x}$ to the result of the sixth transformation, we find that the solution (x_0, x_1) exists in the intervals $([0.8331370, 0.8335170], [0.333323, 0.333343])$.

3. Optical Implementation

The computational algorithm described in Section 2 can be applied to analog optical computing. The authors found that a rectangle on an image plane, as shown in Fig. 2, can be used as a representation of intervals for two variables. For this case, optical implementation becomes quite simple and elegant. We consider a method for two-variable simultaneous linear equations.

With the spatial interval representation, the mapping that appeared in the algorithm of Section 2 is regarded as image transformation. Note that for two-variable case the transformation is identical to affine transformation:

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}. \tag{13}$$

This fact indicates that the transformation can be implemented optically with the same optical system as that used in the optical fractal synthesizer[7] shown in Fig. 3. The optical fractal synthesizer is composed of two branches in which image rotation, reduction, and translation are executed. The input plane is imaged onto the output plane through the two branches. The amounts of rotation angle and translation of the transformation are specified by the rotation angle of the Dove prisms (DP's) and the mirrors (M's), respectively. Reduction is accomplished by zoom lenses (not shown in the figure). The output image obtained at the output plane is fed back to the input plane for the following iteration. The feedback line is operated either in the clocked latch-and-transfer mode or in the continuous delay-transfer mode. Although the feedback is achieved by a CCD camera directly connected to a CRT display in the experimental system,[7] a parallel feed-
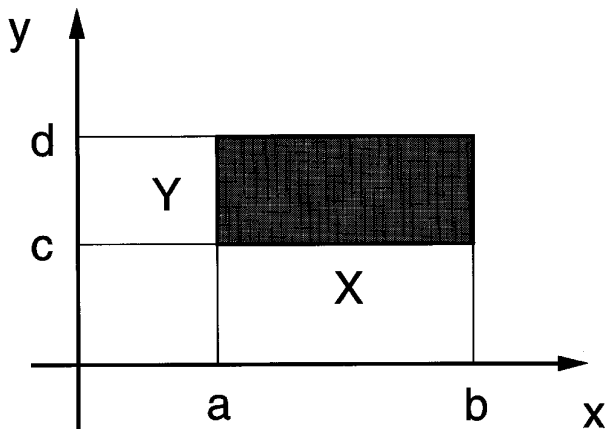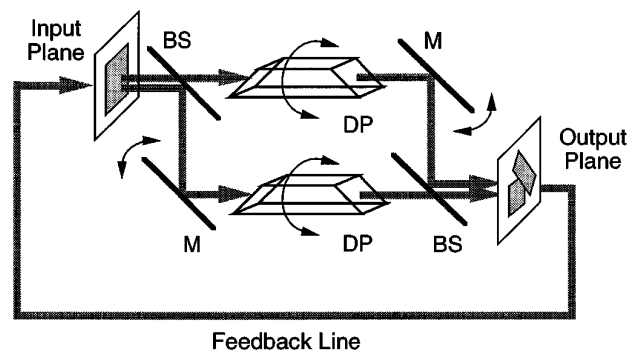

Fig. 3. Schematic diagram of the optical fractal synthesizer. DP's, dove prisms for rotation; M's, mirrors for translation; BS's, beam splitters.

back line based on imaging can be used for higher performance. Consequently, in optical transformation, all points on an image are mapped at one time without quantizing error. Therefore parallelism and continuity of the data plane, which are attractive features of optics, can be effectively utilized in this method.

In addition, to extend the dynamic range and the precision of data representation, magnification of the image plane is introduced. Because dynamic range and precision are determined by the spatial resolution of the image plane, some techniques should be used to enlarge the image plane effectively. For this purpose, magnification is an effective and suitable solution for optical implementation. To clarify the procedure, we introduce a plane of attention (POA) that indicates the area of observation. The POA is defined as a rectangle $\{(x, y)|x_{\text{bottom}} \leq x \leq x_{\text{top}}, y_{\text{bottom}} \leq y \leq y_{\text{top}}\}$ that has local coordinate system $\{(i, j)|0 \leq i \leq s, 0 \leq j \leq t\}$. Figure 4 shows the relationship between the global and the local coordinate systems. For the local coordinate system, the affine

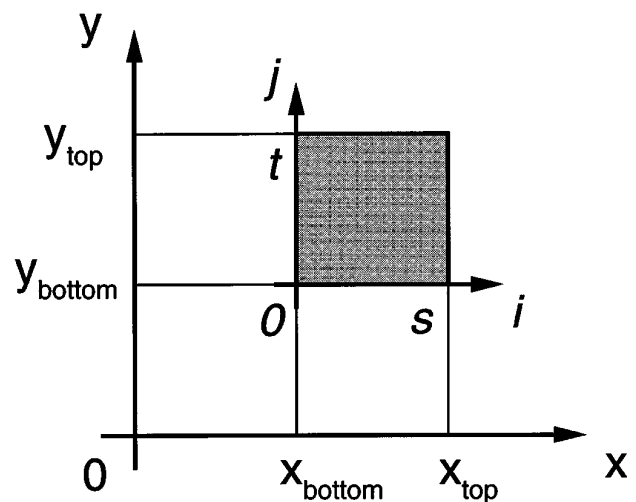Fig. 2. Spatial coding for two-dimensional data.

Fig. 4. Relationship between global and local coordinate systems. Block (italic) letters indicate the number in a global (local) coordinate system.
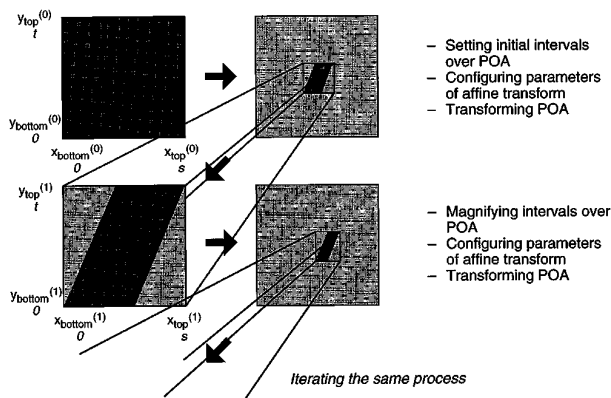
Fig. 5. Processing procedure of the optical implementation of high-accuracy computing algorithm. Block (italic) letters indicate boundaries of the POA in a global (local) coordinate system.

transformation of Eq. (13) is rewritten as

$$g\begin{pmatrix} i \\ j \end{pmatrix} = \begin{bmatrix} a & \dfrac{\alpha}{\beta} b \\ \dfrac{\beta}{\alpha} c & d \end{bmatrix} \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} \alpha e \\ \beta f \end{pmatrix}$$

$$+ \begin{bmatrix} \alpha(a-1) & \alpha b \\ \beta c & \beta(d-1) \end{bmatrix} \begin{pmatrix} x_{\text{bottom}} \\ y_{\text{bottom}} \end{pmatrix}, \quad (14)$$

$$\alpha = \frac{s}{x_{\text{top}} - x_{\text{bottom}}}, \quad (15)$$

$$\beta = \frac{t}{y_{\text{top}} - y_{\text{bottom}}}. \quad (16)$$

By the use of spatial interval representation and image magnification, high-accuracy analog optical computing is accomplished. The processing procedure is shown in Fig. 5. First the initial interval area is magnified to cover the POA and transformed with Eq. (14). Then the transformed area, which is shrunk, is magnified to cover the POA, and the same process is repeated. Note that the set operation in Eq. (9) is nothing but a clipping operation with the POA. In a usual optical system, the POA corresponds to the plane of observation, e.g., the detector plane of a CCD, so that the local coordinate system does not change during iteration. On the other hand, the global coordinates of POA will vary along with the iteration. Consequently, by tracing the

global coordinates of the POA, or more exactly the boundaries of the POA, we can obtain fine intervals, including the solution of the given equation.

To verify the above principle, an experimental simulation is attempted. As the target equation, Eq. (5) is used. $[-1, 1]$ is assumed for both initial intervals. Table 2 shows the time development of the affine parameters, a, b, c, d, e, and f, in the local coordinate system and the boundaries of the POA, x_{bottom}, x_{top}, y_{bottom}, and y_{top}, in the global coordinate system. Figure 6 shows POA's before and after transformation at the first–sixth iterations. The POA is assumed to have discrete local addresses 0–127 for i and j, which corresponds to the addresses of an array of detectors such as a CCD. As seen from columns 8–11 of Table 2, the POA, or the area of the intervals, is shrunk along with the iteration. After the sixth iteration, it is found that the solution of Eq. (5) is located inside intervals $[0.833101, 0.833452]$ and $[0.333332, 0.333334]$ for x_0 and x_1, respectively.

4. Discussion

The number representation system used in the proposed method is different from conventional ones, namely, a number is represented by a position on the POA specified by the lower and the upper boundaries. Although this method seems complicated, the boundaries of the POA have enough information to represent a number, after the concept of the interval arithmetic. Actually, for a specific range of the POA, important information on a number is not the position in the POA but whether the number is located inside the POA. If more accuracy is required, the POA will be magnified. As a result, both a large dynamic range and high precision are achieved with this technique.

Convergence characteristics of the mapping that appeared in our method depend on the quality of the approximate inverse matrix $\mathcal{R}$ and the approximate solution $\tilde{\mathbf{x}}$. In our simulation, when those values are far from the correct ones, we could not obtain the answer, that is, if the mapping provided by Eq. (8) is not an appropriate one, its transformed intervals go out of the POA, and the iteration will be terminated. Therefore, for obtaining a solution with high accuracy, finding good approximate values is important.

In terms of accuracy, the spatial resolution of the POA, i.e., the element numbers of the local coordinate on the POA, is an important parameter. We

Table 2. Time Development of Parameters of Affine Transformation and Transformed Intervals

| Iteration	a	b	c	d	e	f	x_{bottom}	x_{top}	y_{bottom}	y_{top}
0	0.2	0.200000	0	0.1	0.020000	0.030000	-1.000000	1.000000	-1.000000	1.000000
1	0.2	0.200000	0	0.1	39.680000	59.520000	-0.390625	0.40625	-0.078125	0.125000
2	0.2	0.050980	0	0.1	50.898824	63.212308	-0.079346	0.119873	0.021851	0.040894
3	0.2	0.019118	0	0.1	56.442353	69.464615	0.007812	0.051392	0.032116	0.034050
4	0.2	0.008876	0	0.1	59.252437	72.513136	0.027900	0.036752	0.0033204	0.033400
5	0.2	0.004438	0	0.1	62.481293	75.944725	0.032187	0.033985	0.033319	0.033339
6	0.2	0.002219	0	0.1	65.054251	83.148058	0.033101	0.033452	0.033332	0.033334
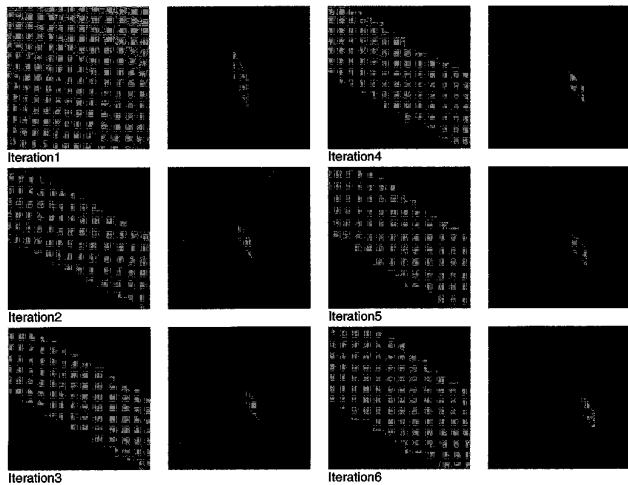
Fig. 6. Simulation result of computing of Eq. (5). POA's before and after transformation at first–sixth iterations are shown.

examine the same simulation as described in Section 3 with three different spatial resolutions, 32, 64, and 128, for each dimension. Usually the iteration is terminated when the transformed intervals go out of the POA. According to the simulation result, the iteration numbers before termination for individual spatial resolutions are 3, 6, and 13, respectively. In general, as the iteration number is increased, the accuracy of the final result will be improved. Therefore a sufficient number of spatial resolutions are required for obtaining high accuracy, while the quantitative relationship between spatial resolution and accuracy remains the feature study.

There are many practical issues for implementing the proposed method: accuracy in the controlling optical system, processing speed, flexibility of processing, etc. Although most of them are future works, we point out two significant views on the proposed method.

The first point is that fabrication and controlling issues on the proposed method have something in common with those of optical fine instruments and apparatus. In both cases, fine mechanics plays an important role, and a lot of resources and techniques accumulated in the field are expected to be utilized. The method of computation proposed in this paper is regarded as a technique that converts a computing scheme into a suitable form for conventional technologies in optics.

The second point is the analogy of the proposed method with the gazing mechanism of the human visual system. Finding and concentrating a target is an example of sophisticated mechanisms to accomplish flexibility and a large dynamic range for sensing information by human visual system. The procedure executed in the proposed method is nothing but the same operation as gazing. Therefore the proposed scheme is expected to be a hint of optical computing associated with human visual processing.

Finally, we comment on the limitation of the proposed technique on the matrix size. As described in the beginning of Section 3, the proposed method treats two variables as a rectangle on an image plane. This spatial encoding is an essential idea of the proposed method and also imposes a limitation on the variable number. Namely, according to the dimensionality of image plane, the variable number is fixed as two. This fact prevents us from extending the size of matrices $\mathcal{A}$ and $\mathcal{R}$ to more than two. Thus the discussion in this paper is valid only for a two-variable problem, as shown in Table 1. Although the authors recognize that this is a serious drawback of our technique, we also believe the proposed method shows a capability of analog optical computing. A technique for extending the matrix size is an important issue for future study.

5. Conclusion

A method for high-accuracy analog optical computing based on interval arithmetic and the fixed-point theorem has been considered. Two-variable simultaneous equations have been studied to investigate the proposed method. An optical implementation is considered by the use of spatial coding of intervals, a fine transformation, and image magnification. Computational simulation verifies the principle of the method.

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