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On realization of the discrete series for semisimple Lie groups

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§ 0. Introduction.

The main purpose of this paper is to show that most of the discrete series for a semisimple Lie group are realized on certain eigenspaces of the Casimir operator over the symmetric space. In more detail, let G be a connected non-compact semisimple Lie group with a finite dimensional faithful representation and K a maximal compact subgroup of G. Assume that $\operatorname{rank} G = \operatorname{rank} K$ (according to [**6**, Theorem 13], G has a discrete series if and only if G satisfies this condition). Let V_λ be an irreducible unitary K-module with lowest weight $\lambda+2\rho_k$, where ρ_k is the half sum of positive compact roots. We denote by $C^\infty(\mathcal{V}_\lambda)$ (resp. $L_2(\mathcal{V}_\lambda)$) the space consisting of all V_λ-valued C^∞ (resp. square-integrable) functions f on G such that $f(gk)=k^{-1}f(g)$ for $g \in G$, $k \in K$. Denoting by Ω the Casimir operator of G, let Ω act on $C^\infty(\mathcal{V}_\lambda)$ in the usual manner and denote by $\nu(\Omega)$ the differential operator given by the action of Ω on $C^\infty(\mathcal{V}_\lambda)$ in this sense (for a precise definition, see § 1). Put

$$\mathfrak{H}_\lambda = \{f \in C^\infty(\mathcal{V}_\lambda) \cap L_2(\mathcal{V}_\lambda)\,;\ \nu(\Omega)f = \langle \lambda+2\rho, \lambda \rangle f\}\,,$$

where ρ denotes the half sum of all positive roots and $\langle\ ,\ \rangle$ denotes the usual inner product on the set of weights induced by the Killing form. Since $\nu(\Omega)$ is elliptic on $C^\infty(\mathcal{V}_\lambda)$, $\mathfrak{H}_\lambda$ is then a Hilbert space and gives a unitary representation of G through the left translation. Assume that $\langle \lambda+\rho, \alpha \rangle < 0$ for all positive roots α. Then, there exists a constant a such that if $|\langle \lambda+\rho, \beta \rangle| > a$ for all non-compact positive roots β, $\mathfrak{H}_\lambda$ gives an irreducible unitary representation belonging to the discrete series for G, which is equivalent to the discrete class $\omega(\lambda+\rho)$ in the sense of [**6**] (§3, Corollary to Theorem 2). In view of Harish-Chandra's result [**5**], [**6**], the above result gives a procedure in order to realize most of the discrete series for G.

For our proof, we make use of the method established by M. S. Narasimhan and K. Okamoto in [**11**]. That is, the above result is deduced from Theorem 1 in § 2 and Lemma 9 in § 3, which amount to generalizations of the alternating sum formula and the vanishing theorem [**11**, Theorem 1 and Theorem 2] respectively.

We shall make some historical remarks. It is known by V. Bargmann [**2**] that all the discrete series for $SL(2, \boldsymbol{R})$ are constructed on certain spaces of square-integrable holomorphic functions on the unit disk. Generalizing this method, Harish-Chandra [**4**] constructed part of the discrete series for a group of motions of a bounded symmetric domain. Recently, M. S. Narasimhan and K. Okamoto [**11**] realized most of the discrete series for the above types of groups on the square-integrable cohomology spaces associated to holomorphic vector bundles over hermitian symmetric spaces. In these cases, considering the Laplace-Beltrami operators determined by the Cauchy-Riemann operators, one can see that these spaces of square-integrable holomorphic functions (or "cohomology") coincide with the previous eigenspaces of the Casimir operators (see § 3, Remark 2 to Theorem 2). Hence, for this type of groups, our result is a variation of the results cited above.

When G is not of the above type, G/K admits no invariant complex structures and so "holomorphic objects" as above cannot be considered over G/K. R. Takahashi [**16**], however, constructed all the discrete series for the universal covering group of the de Sitter group on eigenspaces of the Casimir operator. The generalization of this method is proposed in [**12**] and our result may be regarded as an answer to it.

As for different methods of realization of most of the discrete series for general semisimple Lie groups, there are the recent works due to W. Schmid; one is [**15**] and another is the method by means of certain invariant first order differential operators over G/K constructed in [**14**]. Our techniques used here also give a proof of the latter method[1] and one can see that this method is equivalent to that by means of the Casimir operator in a certain sense (see § 3, Theorem 3 and Remark 2 to it).

This paper is divided to three sections, each of which is again divided to several subsections. In § 1, we treat invariant differential operators on spaces of square-integrable sections of homogeneous vector bundles over G/K, which will be in need later. The aim of § 2 is to prove Theorem 1, and in § 3, (3.1), Lemma 9 is proved. With these preparations, the other subsections in § 3 are devoted to realization of the discrete series. That is, we shall obtain Theorem 2 and its Corollary in (3.2), and Theorem 3 in (3.4). We note that Lemmas 4, 5, 6 in § 1 are needless for a proof of Theorem 2. Besides these, in (3.3) we refer to a non-vanishing theorem for the elliptic complexes over G/K constructed in [**8**], which amounts to a generalization of the result in [**13**]. Throughout the paper, $\boldsymbol{Z}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the set of all integers, that of all real numbers and that of all complex numbers respectively. A summary of the present paper is found in [**10**].

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1) This fact is communicated by Prof. Schmid without a proof.

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§ 1. Invariant differential operators and induced representations.

This section is divided to two subsections. In (1.1), we notice some generalities for invariant differential operators on homogeneous vector bundles. From (1.2) throughout this paper, (G, K) is assumed to be a symmetric pair and we shall treat the relation of the discrete parts of induced representations with invariant differential operators.

1.1. Let G be a connected Lie group and K a closed subgroup of G. For a finite dimensional K-module V, let us introduce such an equivalence relation among the elements of $G\times V$ as $(gk, v)\sim(g, kv)$ for $g\in G$, $k\in K$, $v\in V$. Then the quotient space $\mathcal{V}$ of $G\times V$ by this relation is regarded as a vector bundle over G/K on which G acts as a vector bundle automorphism group. We shall call $\mathcal{V}$ the homogeneous vector bundle associated to a K-module V. Throughout this paper, when a K-module is given, the associated homogeneous vector bundle will be denoted by the corresponding script letter. We denote by $C^\infty(\mathcal{V})$ the space of C^∞ sections of $\mathcal{V}$, which can be identified with the following subspace of V-valued C^∞ functions on G, i. e.,

$$\{f\in C^\infty(G)\otimes V\,;\ f(gk)=k^{-1}f(g),\ g\in G,\ k\in K\}\,,$$

where $C^\infty(G)$ denotes the space of all C^∞ functions on G.

Let $\mathfrak{G}$ be the universal enveloping algebra of the complexification of the Lie algebra of G. For $X\in\mathfrak{G}$, $\nu(X)$ denotes the left invariant differential operator on $C^\infty(G)$ given by X. Through the right translation, $\mathfrak{G}$ can be provided with a G-module structure and therefore a K-module structure by restriction. Hence, for K-modules V, W, $\mathfrak{G}\otimes\mathrm{Hom}(V, W)$ has a K-module structure where $\mathrm{Hom}(V, W)$ denotes the K-module consisting of all linear mappings from V into W. Let $D(V, W)$ be the set of K-fixed elements in $\mathfrak{G}\otimes\mathrm{Hom}(V, W)$. An element $\sum_i X_i\otimes A_i$ ($X_i\in\mathfrak{G}$, $A_i\in\mathrm{Hom}(V, W)$) in $D(V, W)$ defines a G-invariant differential operator

$$D:\ C^\infty(\mathcal{V})\longrightarrow C^\infty(\mathcal{W})$$

which is given by

$$Df=\sum_i(\nu(X_i)\otimes A_i)f\qquad\text{for}\quad f\in C^\infty(\mathcal{V})\,.$$

Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebra of G, K respectively. We shall then say that the homogeneous space G/K is reductive if there exists an $\mathrm{Ad}\,K$-invariant subspace $\mathfrak{p}$ in $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ (direct sum).

LEMMA 1. *Suppose that G/K is reductive. Then, every G-invariant differ-*

ential operator from $C^\infty(\mathcal{V})$ *into* $C^\infty(\mathcal{W})$ *is given by an element of* $D(V, W)$ *under the above correspondence.*

PROOF. The lemma can be proved quite similarly to [7, Chap. X, Lemma 2.2].

If we denote by $\mathfrak{G}_0$ the subalgebra consisting of K-fixed elements in $\mathfrak{G}$, then $\mathfrak{G}_0$ is embedded in $D(V, V)$ by means of the injection $\mathfrak{G}_0 \ni X \to X \otimes 1 \in D(V, V)$. For a simplification, we write $\nu(X)$ for $\nu(X)\otimes 1$ in this case. In particular, denoting by $\mathfrak{Z}$ the center of $\mathfrak{G}$, we always have a well defined invariant differential operator $\nu(Z)$ on $C^\infty(\mathcal{V})$ for $Z \in \mathfrak{Z}$.

1.2. Henceforth, let us assume that G is a connected non-compact semisimple Lie group with a compact Cartan subgroup and K is a maximal compact subgroup of G. Let V be a finite dimensional unitary K-module. Then the associated homogeneous vector bundle $\mathcal{V}$ over G/K has a G-invariant hermitian inner product on each fibre and we obtain the space of all square-integrable sections $L_2(\mathcal{V})$. This is identified with the following subspace of V-valued square-integrable functions on G, i. e.,

$$L_2(\mathcal{V}) = \{f \in L_2(G) \otimes V\,;\ f(gk) = k^{-1}f(g),\ g \in G,\ k \in K\}\,,$$

where $L_2(G)$ denotes the Hilbert space of all square-integrable functions on G. The left regular representation on $L_2(G)$ then induces a unitary representation of G on $L_2(\mathcal{V})$.

For a unitary representation π of G on a space $\mathfrak{H}$, we denote by $(\pi_d, \mathfrak{H}_d)$ the discrete part of $(\pi, \mathfrak{H})$, i. e., $\mathfrak{H}_d$ is the smallest closed invariant subspace which contains every irreducible closed invariant subspace of $\mathfrak{H}$ and π_d is the restriction of π to $\mathfrak{H}_d$. We also denote by $[\mathfrak{H}]$ or $[\pi]$ (resp. $[V]$) the equivalence class to which a representation $(\pi, \mathfrak{H})$ (resp. a K-module V) belongs. By $\mathcal{E}_d$ we mean the discrete series for G, i. e., $\mathcal{E}_d$ is the set of all equivalence classes which are realized on an irreducible closed invariant subspace in $L_2(G)$ (under the left or right regular representation). An element in $\mathcal{E}_d$ will be called a discrete class of G. When σ, τ are equivalence classes of K-modules, we denote by $(\sigma : \tau)$ the intertwining number between σ and τ.

LEMMA 2. *Let* V *be a finite dimensional unitary* K*-module. Then*

$$[L_2(\mathcal{V})_d] = \bigoplus_{\omega \in \mathcal{E}_d} (\omega | K : [V])\omega\,,$$

where $\omega | K$ *denotes the equivalence class containing the* K*-module which is obtained by the restriction to* K *of a representation contained in* ω. *Here the sum is finite.*

PROOF. The proof is similar to that of [**11**, Lemma 1.2] without any essential change.

When there is given an invariant differential operator $D : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{W})$, by the maximal extension of D we mean the densely defined closed linear operator $\tilde{D}: L_2(\mathcal{V}) \to L_2(\mathcal{W})$ where the domain of $\tilde{D}$ consists of all $f \in L_2(\mathcal{V})$ such that Df, formed in the sense of distributions, belongs to $L_2(\mathcal{W})$ and $\tilde{D}f = Df$ for f in the domain of D. Hereafter, if not otherwise stated, we shall always consider differential operators on spaces of square-integrable sections in this sense, and write D instead of $\tilde{D}$ for a notational simplicity.

LEMMA 3. *Let* $D: L_2(\mathcal{V}) \to L_2(\mathcal{W})$ *be an invariant differential operator (maximally extended). Then* $L_2(\mathcal{V})_d \subset C^\infty(\mathcal{V})$, $L_2(\mathcal{W})_d \subset C^\infty(\mathcal{W})$, *and the domain of* D *contains* $L_2(\mathcal{V})_d$. *If we denote by* D_d *the restriction of* D *to* $L_2(\mathcal{V})_d$, *then* D_d *is a bounded operator from* $L_2(\mathcal{V})_d$ *into* $L_2(\mathcal{W})_d$.

PROOF. Let $\mathfrak{H}_\pi$ be an irreducible closed invariant subspace of $L_2(\mathcal{V})_d$ and π the representation on $\mathfrak{H}_\pi$ ($[\pi] \in \mathcal{E}_d$). We denote by $\mathfrak{H}_\pi^0$ the dense subspace of all C^∞ vectors under π in $\mathfrak{H}_\pi$. Let Ω be the Casimir operator of G and χ_π the infinitesimal character of π. We then have $\pi(\Omega)f = \chi_\pi(\Omega)f$ and $\pi(\Omega)f = \nu(\Omega)f$ for $f \in \mathfrak{H}_\pi^0$, since $\Omega \in \mathfrak{Z}$ and $\mathfrak{H}_\pi^0 \subset C^\infty(\mathcal{V})$. Consider the differential operator $\nu(\Omega) - \chi_\pi(\Omega)1$ on $L_2(\mathcal{V})$ where $\nu(\Omega)$ is maximally extended and 1 denotes the identity operator. Put

$$H_\pi = \{f = L_2(\mathcal{V})\,;\ \nu(\Omega)f = \chi_\pi(\Omega)f\}\,.$$

Then H_π is a closed subspace of $L_2(\mathcal{V})$ and $H_\pi \subset C^\infty(\mathcal{V})$ because $\nu(\Omega)$ is elliptic on $C^\infty(\mathcal{V})$. On the other hand $\mathfrak{H}_\pi^0 \subset H_\pi$, hence the closure $\mathfrak{H}_\pi$ of $\mathfrak{H}_\pi^0$ is contained in H_π. Since $L_2(\mathcal{V})_d$ has only finitely many irreducible components by Lemma 2, this implies $L_2(\mathcal{V})_d \subset C^\infty(\mathcal{V})$. For $\mathcal{W}$, it holds quite similarly.

Next, for $f \in \mathfrak{H}_\pi$ we put

$$f_v(g) = (f(g), v)_V \qquad \text{for} \quad v \in V,\ g \in G\,,$$

where $(\ ,\ )_V$ denotes the inner product on V. We shall first show that f_v is a C^∞ vector in $L_2(G)$ under the right regular representation. In fact, since $f_v(gk) = f_{kv}(g)$ for $k \in K$, right K-translates of f_v span a finite dimensional subspace. Moreover, we have $\nu(Z)f_v = \chi_\pi(Z)f_v$ for every $Z \in \mathfrak{Z}$. Hence f_v is a C^∞ function on G whose right K-translates and $\mathfrak{Z}$-translates span a finite dimensional space. Therefore, from [**6**, Theorem 1], we can see that f_v is a C^∞ vector under the right regular representation. Secondly, we shall show $(Df)_w \in L_2(G)_d$ for every $w \in W$. Here Df is considered as an operation of D on a C^∞ section $f \in C^\infty(\mathcal{V})$. From Lemma 1, we may assume that D is given by $\sum_i X_i \otimes A_i \in D(V, W)$ ($X_i \in \mathfrak{G}$, $A_i \in \mathrm{Hom}(V, W)$). We then see that for $w \in W$

$$(Df)_w(g) = \sum_i ((\nu(X_i) \otimes A_i) f(g), w)_W$$
$$= \sum_i (\nu(X_i) f_{A_i^* w})(g)\,,$$

where A_i^* denotes the adjoint operator of A_i. From the above argument,

$f_{A_i^* w}$ is a C^∞ vector under the right regular representation, and therefore $\nu(X_i) f_{A_i^* w} \in C^\infty(G) \cap L_2(G)_d$. Hence $(Df)_w \in L_2(G)_d$, which implies that $Df \in L_2(\mathcal{W})_d$. Finally, since D_d is a closed operator whose domain is the whole space $L_2(\mathcal{V})_d$, the boundedness is clear from the closed graph theorem. q. e. d.

Let $C_c^\infty(\mathcal{V})$, $C_c^\infty(\mathcal{W})$ denote the spaces of all C^∞ sections of $\mathcal{V}$, $\mathcal{W}$ respectively which have compact supports. For an invariant differential operator D, we denote by D_0 the restriction of D to $C_c^\infty(\mathcal{V})$ and consider the densely defined linear operator

$$D_0 : L_2(\mathcal{V}) \longrightarrow L_2(\mathcal{W}) .$$

Then the adjoint operator D_0^* of D_0 coincides with the maximal extension of the formal adjoint operator

$$D^* : L_2(\mathcal{W}) \longrightarrow L_2(\mathcal{V}) .$$

By Lemma 3, the restriction $(D^*)_d$ of D^* to the discrete part $L_2(\mathcal{W})_d$ is a bounded operator. On the other hand, we have the adjoint operator $(D_d)^*$ of the bounded operator D_d in the sense of Hilbert space. We ask whether $(D^*)_d$ coincides with $(D_d)^*$. In case D is of at most first order, it will be answered in the affirmative making use of a technique analogous to the one in [**1**] as follows.

LEMMA 4. *Assume that D is of first order. Let f be a C^∞ section of $\mathcal{V}$ belonging to the domain of D in $L_2(\mathcal{V})$. There then exists a sequence $\{f_j\}$ in $C_c^\infty(\mathcal{V})$ such that f_j converges to f strongly in $L_2(\mathcal{V})$ and Df_j converges to Df strongly in $L_2(\mathcal{W})$.*

PROOF. Take a real valued C^∞ function μ on $\boldsymbol{R}$ such that $0 \leqq \mu(t) \leqq 1$, $\mu(t) = 1$ for $t \leqq 1$, and $\mu(t) = 0$ for $t \geqq 2$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. Every element g of G then has the unique decomposition $g = (\exp X)k$ where $X \in \mathfrak{p}$, $k \in K$. Denoting by $\|\cdot\|$ a K-invariant norm on $\mathfrak{p}$, we define

$$\sigma(g) = \|X\|$$

for $g = (\exp X)k$ $(X \in \mathfrak{p},\ k \in K)$. Since $\sigma(gk) = \sigma(g)$ for $k \in K$, σ is also regarded as a function on G/K. It is seen that $\sigma(g)$ is equal to the distance from eK to gK with respect to the G-invariant riemannian metric on G/K induced by $\|\cdot\|$. It holds that

$$\sigma(gh) \leqq \sigma(g) + \sigma(h) \qquad \text{for} \quad g, h \in G \tag{1}$$

(see, for example, [**6**, Lemma 10]). Put

$$w_j(g) = \mu(\sigma(g)/j) \qquad \text{for} \quad j = 1, 2, \cdots .$$

Then w_j may be regarded as a C^∞ function on G/K with a compact support contained in $\{x \in G/K;\ \sigma(x) \leqq 2j\}$.

Put $f_j = w_j f$ for f given in the lemma. We shall then show that $\{f_j\}$

satisfies the condition of the lemma. It is clear that f_j converges to f strongly in $L_2(\mathcal{V})$. Since D is of first order, we may assume that

$$D=\sum_i \nu(X_i)\otimes A_i+B$$

where $X_i\in\mathfrak{p}$, A_i, $B\in\mathrm{Hom}(V, W)$. Denote by $|\cdot|_V$, $|\cdot|_W$ the K-invariant norms on V, W respectively. We then have

$$\begin{aligned}|Df-Df_j|_W &= |Df-w_jDf-\sum_i(\nu(X_i)w_j)A_if|_W \\ &\leqq |(1-w_j)Df|_W+|\sum_i(\nu(X_i)w_j)A_if|_W \\ &\leqq |(1-w_j)Df|_W+\sum_i a_i|\nu(X_i)w_j|\,|f|_V \qquad (2)\end{aligned}$$

where w_j is considered as a function on G ($\nu(X_i)w_j$ is not a well defined function on G/K), and a_i is a constant depending only on A_i. Now, the support of $\nu(X_i)w_j$ is contained in $\{g\in G\,;\, j\leqq\sigma(g)\leqq 2j\}$ and we have

$$(\nu(X_i)w_j)(g)=\frac{1}{j}(\nu(X_i)\sigma)(g)\frac{d\mu}{dt}\Big(\frac{\sigma(g)}{j}\Big) \qquad (3)$$

for $j\leqq\sigma(g)\leqq 2j$. By (1), we have

$$\sigma(g(\exp tX_i))-\sigma(g)\leqq\sigma(\exp tX_i)=|t|\,\|X_i\|$$

$$\sigma(g)-\sigma(g(\exp tX_i))\leqq\sigma(\exp(-tX_i))=|t|\,\|X_i\|\,,$$

hence

$$|\nu(X_i)\sigma|\leqq\|X_i\|\,.$$

Therefore, if we put $M=\sup\left|\frac{d\mu}{dt}\right|$, then there exists a constant C such that

$$|Df-Df_j|_W\leqq|(1-w_j)Df|_W+\frac{C}{j}|f|_V$$

in view of (2), (3). Denoting by $\|\cdot\|_{\mathcal{V}}$, $\|\cdot\|_{\mathcal{W}}$ the norms on $L_2(\mathcal{V})$, $L_2(\mathcal{W})$ respectively, we have

$$\|Df-Df_j\|_{\mathcal{W}}\leqq\int_{G-B(j)}|Df|_W dg+\frac{C}{j}\|f\|_{\mathcal{V}}$$

where $B(j)=\{g\in G\,;\ \sigma(g)\leqq 2j\}$. Since $\|Df\|_{\mathcal{W}}<\infty$, Df_j converges to Df strongly in $L_2(\mathcal{W})$. q. e. d.

LEMMA 5. *Let $D: L_2(\mathcal{V})\to L_2(\mathcal{W})$ be an invariant first order differential operator, and $D^*: L_2(\mathcal{W})\to L_2(\mathcal{V})$ the formal adjoint operator of D. Then $(D_d)^*=(D^*)_d$.*

PROOF. Take $f\in L_2(\mathcal{W})_d$. Then $D^*f\in L_2(\mathcal{V})_d$ by Lemma 3, and

$$(D^*f, \varphi)_{\mathcal{V}}=(f, D\varphi)_{\mathcal{W}} \qquad \text{for} \quad \varphi\in C_c^\infty(\mathcal{V})\,,$$

where $(\ ,\ )_{\mathcal{V}}$, $(\ ,\ )_{\mathcal{W}}$ denote the inner products on $L_2(\mathcal{V})$, $L_2(\mathcal{W})$ respectively.

For $\varphi \in L_2(\mathcal{V})_d$, choose a sequence $\{\varphi_j\}$ in $C_c^\infty(\mathcal{V})$ such as in Lemma 4. Since $(D^*f, \varphi_j)_{\mathcal{V}} = (f, D\varphi_j)_{\mathcal{W}}$ for $\varphi_j \in C_c^\infty(\mathcal{V})$, we have $(D^*f, \varphi)_{\mathcal{V}} = (f, D\varphi)_{\mathcal{W}}$ for $\varphi \in L_2(\mathcal{V})_d$. Hence it holds

$$((D^* - (D_d)^*) f, \varphi)_{\mathcal{V}} = 0 \qquad \text{for every} \quad \varphi \in L_2(\mathcal{V})_d$$

because $(f, D\varphi)_{\mathcal{W}} = ((D_d)^* f, \varphi)_{\mathcal{V}}$. Therefore we have $D^*f = (D_d)^* f$ for every $f \in L_2(\mathcal{W})_d$. q. e. d.

LEMMA 6. *In the same notation as in Lemma* 5, *let* Ker D (*resp.* Ker D^*) *be the null space of* D (*resp.* D^*). *Then we have* $(\mathrm{Ker}\, D^*)_d = \mathrm{Ker}\,(D_d^*)$.

PROOF. Clear from Lemmas 3 and 5.

§ 2. The difference formula for the discrete parts.

The aim of this section is to prove the difference formula for the characters of the discrete parts of certain two induced representations from K (Theorem 1). The section is divided to three subsections. In (2.1), we shall recall Harish-Chandra's fundamental result on the discrete series for semisimple Lie groups in [5], [6]. We shall state Theorem 1 in (2.2) and give a proof of this theorem in (2.3). Our proof is based upon the same lines as those of Narasimhan and Okamoto in [11, § 6]. We, however, treat a somewhat general situation and so some important points in [11] will be repeated here for the sake of completeness.

2.1. In addition to the assumption in (1.2), we shall assume, for covenience, that G has a finite dimensional faithful representation and that its complexification G^C is simply connected. Let T be a maximal torus contained in K. Then T is seen to be a Cartan subgroup of G from the assumption and we fix it once for all. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}$ be the Lie algebras of G, K, T respectively and denote by $\mathfrak{g}^C, \mathfrak{k}^C, \mathfrak{h}^C$ their complexifications. Let Δ be the root system for $(\mathfrak{g}^C, \mathfrak{h}^C)$ and W_G the Weyl group for $(\mathfrak{k}^C, \mathfrak{h}^C)$. We denote by $\mathcal{F}$ the space of all real valued linear forms on $\sqrt{-1}\,\mathfrak{h}$ and by $\langle\ ,\ \rangle$ the inner product on $\mathcal{F}$ induced by the Killing form of $\mathfrak{g}^C$ in the usual way. The character group L of T is then identified with the lattice in $\mathcal{F}$ consisting of all integral linear forms, i. e.,

$$L = \{\lambda \in \mathcal{F}\,;\ 2\langle\lambda, \alpha\rangle / \langle\alpha, \alpha\rangle \in \boldsymbol{Z} \text{ for every } \alpha \in \Delta\}\,.$$

Put

$$L' = \{\lambda \in L\,;\ \langle\lambda, \alpha\rangle \neq 0 \text{ for every } \alpha \in \Delta\}$$

and

$$T' = T - \exp \mathfrak{h}_s\,,$$

where $\mathfrak{h}_s = \{H \in \mathfrak{h}\,;\ \alpha(H) = 0 \text{ for some } \alpha \in \Delta\}$. Choose a positive root system P in Δ and fix it once for all. If we need a linear order on Δ, we shall always consider the one given by P fixed above. We put

$$\varepsilon(\lambda)=\operatorname{sign}\prod_{\alpha\in P}\langle\lambda,\alpha\rangle \qquad \text{for} \quad \lambda\in L'$$

and denote by $\varepsilon(s)$ the determinant of $s\in W_G$ as an orthogonal transformation on $\mathcal{F}$.

Then, Harish-Chandra's fundamental result on the discrete series is stated as follows ([5, Theorem 3], [6, Theorem 16]). *For $\lambda\in L'$, there exists a unique tempered invariant distribution Θ_λ such that*

$$\Delta(\exp H)\Theta_\lambda(\exp H)=\sum_{s\in W_G}\varepsilon(s)e^{s\lambda(H)}$$

for $\exp H\in T'$, *where*

$$\Delta(\exp H)=\prod_{\alpha\in P}(e^{\alpha(H)/2}-e^{-\alpha(H)/2}).$$

For $\lambda\in L'$, there exists a unique discrete class $\omega(\lambda)$ in $\mathcal{E}_d$ such that the character $\Theta_{\omega(\lambda)}$ of $\omega(\lambda)$ coincides with $(-1)^n\varepsilon(\lambda)\Theta_\lambda$ where $n=\frac{1}{2}\dim G/K$. The map $L'\ni\lambda\mapsto\omega(\lambda)\in\mathcal{E}_d$ is surjective and $\omega(\lambda)=\omega(\lambda')$ if and only if there exists an $s\in W_G$ such that $s\lambda=\lambda'$.

2.2. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition as in (1.2) and $\mathfrak{p}^C$ the complexification of the subspace $\mathfrak{p}$ in $\mathfrak{g}^C$. Denoting by E_α a non-zero eigenvector in $\mathfrak{g}^C$ for a root $\alpha\in\Delta$, we shall say that a root α is compact (resp. non-compact) if $E_\alpha\in\mathfrak{k}^C$ (resp. $E_\alpha\in\mathfrak{p}^C$). We denote by P_k (resp. P_n) the set of all compact (resp. non-compact) positive roots. Put $\rho=\frac{1}{2}\sum_{\alpha\in P}\alpha$ and $\rho_k=\frac{1}{2}\sum_{\alpha\in P_k}\alpha$. For $\lambda\in L$, put $\varepsilon_k(\lambda)=\operatorname{sign}\prod_{\alpha\in P_k}\langle\lambda,\alpha\rangle$ if $\prod_{\alpha\in P_k}\langle\lambda,\alpha\rangle\neq 0$ and $\varepsilon_k(\lambda)=0$ otherwise. By $\mathcal{E}_K$, we denote the set of all equivalence classes of irreducible unitary K-modules.

We shall define the map

$$L\ni\lambda\longmapsto[\lambda]\in\mathcal{E}_K$$

as follows. When $\varepsilon_k(\lambda+\rho_k)=0$, we then put $[\lambda]=0$ (the equivalence class of the zero K-module). When $\varepsilon_k(\lambda+\rho_k)\neq 0$, there then exists the unique element $s\in W_G$ such that $\langle s(\lambda+\rho_k),\alpha\rangle>0$ for all $\alpha\in P_k$. In this case we denote by $[\lambda]$ the equivalence class containing an irreducible K-module with highest weight $s(\lambda+\rho_k)-\rho_k$. We notice that $[\lambda]\in\mathcal{E}_K$ can be realized from $\lambda\in L$ by means of Borel-Weil theorem for the pair (K, T). For $\lambda\in L$, define a class function $\chi(\lambda)$ on K such as

$$\chi(\lambda)(h)=\Delta_k(h)^{-1}\sum_{s\in W_G}\varepsilon(s)e^{s(\lambda+\rho_k)(H)}$$

for $h=\exp H\in T$, where

$$\Delta_k(h)=\prod_{\alpha\in P_k}(e^{\alpha(H)/2}-e^{-\alpha(H)/2}).$$

In the above, notice that Δ_k by itself is not generally a well defined function on T. We easily see that the character of $[\lambda]\in\mathcal{E}_K$ is $\varepsilon_k(\lambda+\rho_k)\chi(\lambda)$, from Weyl's character formula.

Let V, W be finite dimensional unitary K-modules and $\mathcal{V}$, $\mathcal{W}$ the homogeneous vector bundles over G/K associated to V, W respectively. Defining $L_2(\mathcal{V})$, $L_2(\mathcal{W})$ as in (1.2), let us suppose that there is given an invariant first order differential operator

$$D: L_2(\mathcal{V}) \longrightarrow L_2(\mathcal{W}).$$

Let D^* denote the formal adjoint operator of D. Considering D, D^* as the closed linear operators extended maximally as in (1.2), we obtain two unitary representations of G on the spaces $\operatorname{Ker} D$, $\operatorname{Ker} D^*$. Taking up the discrete parts respectively, we have the unitary representations $(\pi_V, (\operatorname{Ker} D)_d)$, $(\pi_W, (\operatorname{Ker} D^*)_d)$ as in Lemma 6. Let $C_c^\infty(G)$ be the space of all C^∞ functions with compact supports on G. It follows from Lemma 2 that the bounded operator

$$\pi_V(\varphi)=\int_G \varphi(g)\pi_V(g)dg$$

is of trace class for $\varphi\in C_c^\infty(G)$ (the same holds also for π_W). Hence we have the invariant distributions $\operatorname{Trace}\pi_V$, $\operatorname{Trace}\pi_W$ on G.

For a set A, we shall denote by $|A|$ the number of elements in A and for a finite subset Q contained in $\mathcal{F}$, put

$$\langle Q\rangle=\sum_{\mu\in Q}\mu \qquad (\langle Q\rangle=0 \text{ if } Q=\phi).$$

THEOREM 1. *Suppose that there are given two finite dimensional unitary K-modules V, W and an invariant first order differential operator $D: L_2(\mathcal{V})\to L_2(\mathcal{W})$. Let χ_V (resp. χ_W) be the characters of V (resp. W). Assume that*

$$\chi_V-\chi_W=\varepsilon_k(\lambda+\rho)\sum_{Q\subset P_n}(-1)^{|Q|}\chi(\lambda+\langle Q\rangle)$$

for some $\lambda\in L$ such that $\lambda+\rho\in L'$. We then have

$$\operatorname{Trace}\pi_V-\operatorname{Trace}\pi_W=(-1)^{q_\lambda}\Theta_{\omega(\lambda+\rho)},$$

where $q_\lambda=|\{\beta\in P_n;\ \langle\lambda+\rho,\beta\rangle>0\}|$.

REMARK. According to [3], we say that a non-compact positive root $\beta\in P_n$ is totally positive if $\beta-\alpha\in P_n$ for every compact positive root $\alpha\in P_k$ such that $\beta-\alpha$ is also a root. When all the elements in P_n are totally positive, the symmetric pair (G, K) must then be a hermitian symmetric pair and so in this case the above theorem amounts to [**11**, Theorem 1].

2.3. We begin with a few lemmas essentially due to [**11**]. For the time being, we denote by τ_V the representation $\tau_V: K\to\operatorname{End} V$ on a K-module V given. The orthogonal projection $E_0: L_2(G)\otimes V\to L_2(\mathcal{V})$ is then given by

$$E_0=\int_K r(k)\otimes\tau_V(k)dk$$

where r denotes the right regular representation and dk the normalized Haar measure on K. Let E be the orthogonal projection of $L_2(G)$ onto $L_2(G)_d$. We notice that E is given explicitly, as follows, on the Schwartz space $\mathcal{C}(G)$ of G (for definition, see [**6**, §9]). For a discrete class $\omega\in\mathcal{E}_d$, $d(\omega)$ denotes the formal degree of ω, and Θ_ω the character of ω. For $\varphi\in\mathcal{C}(G)$, put

$${}^0\varphi(g)=\sum_{\omega\in\mathcal{E}_d}d(\omega)\Theta_{\omega^*}(r(g)\varphi)\qquad\text{for}\quad g\in G\,,$$

where ω^* denotes the equivalence class of the representation contragredient to one contained in ω. It is then known that ${}^0\varphi$ is a continuous function on G and ${}^0\varphi=E\varphi$ ([**6**, Corollary 3 to Lemma 69], or [**11**, Lemma 2.3]). We have the representation T^V on $L_2(G)\otimes V$ defined by $T^V=l\otimes 1$, where l denotes the left regular representation of G. For $\varphi\in C_c^\infty(G)$, we put

$$T_\varphi^V=\int_G\varphi(g)T_g^V dg\,.$$

We shall say that $\varphi\in C_c^\infty(G)$ is K-finite if both $l(K)\varphi$ and $r(K)\varphi$ span finite dimensional vector spaces.

LEMMA 7. *Fix a K-finite function $\varphi\in C_c^\infty(G)$. Then the operator $\tilde{K}_\varphi^V=T_\varphi^V\circ E_0\circ(E\otimes 1)$ on $L_2(G)\otimes V$ is of finite rank and coincides with an integral operator with an* End *V-valued C^∞ kernel function K_φ^V which is given by*

$$K_\varphi^V(x,y)=\int_K{}^0\varphi(xky^{-1})\tau_V(k)dk$$

for $(x,y)\in G\times G$. The integral $\int_G\mathrm{Trace}\,K_\varphi^V(x,x)dx$ exists and we have

$$\mathrm{Trace}\,\tilde{K}_\varphi^V=\int_G\mathrm{Trace}\,K_\varphi^V(x,x)dx\,.$$

For a proof, see [**11**, Proposition 6.1]. Notice that in [**11**] the above fact is stated for certain special K-modules, but it is easy to see that it holds for any finite dimensional K-module V and under our general situation.

By means of this lemma, we have the following lemma which is implicitly included also in [**11**].

LEMMA 8. *For a K-finite function $\varphi\in C_c^\infty(G)$, we have*

$$\mathrm{Trace}\,\pi_V(\varphi)-\mathrm{Trace}\,\pi_W(\varphi)=\mathrm{Trace}\,\tilde{K}_\varphi^V-\mathrm{Trace}\,\tilde{K}_\varphi^W\,,$$

where $\mathrm{Trace}\,\pi_V$, $\mathrm{Trace}\,\pi_W$ *are as in Theorem* 1.

PROOF. We denote by $\tilde{\pi}_V$ (resp. $\tilde{\pi}_W$) the unitary representation on $L_2(\mathcal{V})_d$ (resp. $L_2(\mathcal{W})_d$). Then $\tilde{\pi}_V$ is obtained by the restriction of T^V to $L_2(\mathcal{V})_d$, and we easily see that $\mathrm{Trace}\,\tilde{\pi}_V(\varphi)=\mathrm{Trace}\,\tilde{K}_\varphi^V$. In fact, the range of $\tilde{K}_\varphi^V$ is contained in $L_2(\mathcal{V})_d$ and the restriction of $\tilde{K}_\varphi^V$ to $L_2(\mathcal{V})_d$ coincides with $\tilde{\pi}_V(\varphi)$ by definition. On the other hand, we have a bounded operator $D_d: L_2(\mathcal{V})_d \to L_2(\mathcal{W})_d$ in (1.2) and $\{\tilde{\pi}_V(\varphi), \tilde{\pi}_W(\varphi)\}$ is a system of endomorphisms commuting with D_d. By Lemma 7, $\tilde{\pi}_V(\varphi)$ and $\tilde{\pi}_W(\varphi)$ are of finite rank, and by Lemma 6, π_W coincides with the representation on $\operatorname{Ker}(D_d)^*$. Hence by the lemma of Atiyah and Bott (see [**11**, Lemma 6.1]), we have

$$\operatorname{Trace}\tilde{\pi}_V(\varphi)-\operatorname{Trace}\tilde{\pi}_W(\varphi)=\operatorname{Trace}\pi_V(\varphi)-\operatorname{Trace}\pi_W(\varphi)\,.$$

Since the left hand side is equal to $\operatorname{Trace}\tilde{K}_\varphi^V-\operatorname{Trace}\tilde{K}_\varphi^W$, we can complete the proof. q. e. d.

PROOF OF THEOREM 1. Under the assumption of Theorem 1, we first show the equality

$$\chi_V(h)-\chi_W(h)=(-1)^{q_\lambda}|\Delta_n(h)|^2\Theta_{\omega(\lambda+\rho)}(h) \tag{1}$$

for $h\in T'$, where

$$\Delta_n(h)=\prod_{\beta\in P_n}(e^{\beta(H)/2}-e^{-\beta(H)/2})$$

for $h=\exp H\in T$ (note that $|\Delta_n(h)|^2$ is well defined for $h\in T$). By the assumption, we have

$$\chi_V(h)-\chi_W(h)=\varepsilon_k(\lambda+\rho)\Delta_k(h)^{-1}\sum_{s\in W_G}\sum_{Q\subset P_n}(-1)^{|Q|}\varepsilon(s)e^{s(\lambda+\rho_k+\langle Q\rangle)(H)}$$

for $h=\exp H\in T$. Put $\rho_n=\rho-\rho_k$. Then it holds

$$\begin{aligned}&\sum_{Q\subset P_n}(-1)^{|Q|}e^{s(\lambda+\rho_k+\langle Q\rangle)(H)}\\ &=\sum_{Q\subset P_n}(-1)^{|Q|}e^{s(\lambda+\rho)(H)}e^{-s\rho_n(H)}e^{s\langle Q\rangle(H)}\\ &=e^{s(\lambda+\rho)(H)}e^{-s\rho_n(H)}\prod_{\beta\in P_n}(1-e^{s\beta(H)})\\ &=e^{s(\lambda+\rho)(H)}\prod_{\beta\in P_n}(e^{-s\beta(H)/2}-e^{s\beta(H)/2})\end{aligned}$$

for $H\in\mathfrak{h}$, $s\in W_G$. Since

$$\prod_{\beta\in P_n}(e^{\beta(H)/2}-e^{-\beta(H)/2})=\prod_{\beta\in P_n}(e^{s\beta(H)/2}-e^{-s\beta(H)/2})$$

for $s\in W_G$, we have

$$\begin{aligned}\chi_V(h)-\chi_W(h)&=\varepsilon_k(\lambda+\rho)\Delta_k(h)^{-1}\overline{\Delta_n(h)}\sum_{s\in W_G}\varepsilon(s)e^{s(\lambda+\rho)(H)}\\ &=\varepsilon_k(\lambda+\rho)|\Delta_n(h)|^2\Theta_{\lambda+\rho}(h)\end{aligned}$$

for $h=\exp H\in T'$, where $\Theta_{\lambda+\rho}$ is as in (2.1). Because

$$(-1)^n\varepsilon(\lambda+\rho)=\varepsilon_k(\lambda+\rho)(-1)^{q_\lambda}\qquad\left(n=\frac{1}{2}\dim G/K=|P_n|\right),$$

we have the formula (1).

Secondly, for a K-finite function $\varphi\in C_c^\infty(G)$, we have by Lemma 7,

$$\text{Trace}\,\tilde{K}_\varphi^V - \text{Trace}\,\tilde{K}_\varphi^W$$

$$= \int_G (\text{Trace}\,K_\varphi^V(x, x) - \text{Trace}\,K_\varphi^W(x, x))dx$$

$$= \int_G dx \int_K {}^0\varphi(xkx^{-1})(\chi_V(k) - \chi_W(k))dk$$

$$= |W_G|^{-1} \int_G dx \int_{T\times K} {}^0\varphi(xkhk^{-1}x^{-1})\,|\Delta_k(h)|^2(\chi_V(h) - \chi_W(h))dhdk\,, \tag{2}$$

where the last equality follows from Weyl's integral formula. When we put $\Phi_\omega(h) = \Delta(h)\Theta_\omega(h)$ for $h \in T'$, Φ_ω then extends to T as a C^∞ function and we see from (1) that the integral (2) is equal to

$$(-1)^{q_\lambda}\,|W_G|^{-1} \int_G dx \int_{T\times K} (-1)^{n+k} \Phi_{\omega(\lambda+\rho)}(h)\Delta(h)^0\varphi(xkhk^{-1}x^{-1})dhdk\,, \tag{3}$$

where $k = |P_k|$. Since φ is K-finite in $C_c^\infty(G)$ (hence in $\mathcal{C}(G)$), we know that ${}^0\varphi$ is in $\mathcal{C}(G)$ (see [**11**, Lemma 3.2]). Therefore, in view of [**6**, Theorem 5], the function defined by $|\Delta(h)| \int_G |{}^0\varphi(xkhk^{-1}x^{-1})|\,dx$ for $k \in K$, $h \in T'$ is bounded in T', which allows us to make use of Fubini's theorem in (3). Hence if we put

$$F_f(h) = \Delta(h) \int_G f(xhx^{-1})dx \qquad \text{for} \quad f \in \mathcal{C}(G)\,,$$

we see that the integral (3) is equal to

$$(-1)^{q_\lambda+n+k}\,|W_G|^{-1} \int_T \Phi_{\omega(\lambda+\rho)}(h) F_{{}^0\varphi}(h)dh\,. \tag{4}$$

Here we note that F_f is a C^∞ function on T' and the restriction of F_f to each connected component of T' extends to its closure as a continuous function (see [**11**, § 2]). Moreover, if $\nu(Z)f$ spans a finite dimensional space when Z runs over all elements in $\mathfrak{Z}$ (i. e. $\mathfrak{Z}$-finite), then

$$\Theta_\omega(f) = (-1)^{n+k}\,|W_G|^{-1} \int_T F_f(h)\Phi_\omega(h)dh$$

for $\omega \in \mathcal{E}_d$ (see [**6**, Lemma 79]). Utilizing this fact, we have by (4)

$$\text{Trace}\,\tilde{K}_\varphi^V - \text{Trace}\,\tilde{K}_\varphi^W = (-1)^{q_\lambda}\Theta_{\omega(\lambda+\rho)}({}^0\varphi)\,.$$

By [**11**, Lemma 3.2], it holds $\Theta_{\omega(\lambda+\rho)}({}^0\varphi) = \Theta_{\omega(\lambda+\rho)}(\varphi)$. Hence by Lemma 8, we have

$$\text{Trace}\,\pi_V(\varphi) - \text{Trace}\,\pi_W(\varphi) = (-1)^{q_\lambda}\Theta_{\omega(\lambda+\rho)}(\varphi)$$

for every K-finite function $\varphi \in C_c^\infty(G)$. Since K-finite functions are dense in $C_c^\infty(G)$, we have

$$\text{Trace}\,\pi_V - \text{Trace}\,\pi_W = (-1)^{q_\lambda}\Theta_{\omega(\lambda+\rho)}\,. \qquad \text{q. e. d.}$$

§ 3. Realization of the discrete series.

This section is divided to four subsections. In (3.1), we give Lemma 9 which will be indispensable for our purpose. Though this lemma is proved similarly to [**11**, Theorem 2], we must mind our general situation (see Remark at the end of (3.1)). Subsection (3.2) is devoted to our main purpose, Theorem 2 and its Corollary. In (3.3), we show the non-vanishing theorem for the elliptic complexes over G/K constructed in [8] and in (3.4), we refer to the realization of the discrete series by means of Schmid's operator (Theorem 3). We retain the situation and notation as in the previous sections.

3.1. For $\lambda \in L$, $Q \subset P_n$, let V_λ^Q be an irreducible unitary K-module such that

$$[V_\lambda^Q] = [\lambda + \langle Q \rangle] .$$

Denoting by Ω the Casimir operator of G, we shall consider the differential operator $\nu(\Omega)$ on the homogeneous vector bundle $\mathcal{V}_\lambda^Q$ associated to V_λ^Q as in § 1. Put

$$H_\lambda^Q = \{ f \in C^\infty(\mathcal{V}_\lambda^Q) \cap L_2(\mathcal{V}_\lambda^Q) ; \nu(\Omega) f = \langle \lambda + 2\rho, \lambda \rangle f \} .$$

Thanks to the ellipticity of $\nu(\Omega)$, H_λ^Q is a closed invariant subspace of $L_2(\mathcal{V}_\lambda^Q)$. It is known that $H_\lambda^Q \subset L_2(\mathcal{V}_\lambda^Q)_d$ from the recent result of Harish-Chandra (see [**11**, the proof of Proposition 4.1]). For $\lambda \in L$, put

$$Q_\lambda = \{ \beta \in P_n ; \langle \lambda + \rho, \beta \rangle > 0 \} .$$

For $\lambda \in L$, $s \in W_G$, $Q \subset P_n$, put

$$A_\lambda(s, Q) = \langle \langle Q \rangle - \rho, \langle Q \rangle + \rho - 2 \langle Q_\lambda \rangle \rangle + \langle \rho, \rho \rangle + 2 \langle \rho_k, \rho_n - \langle Q_\lambda \rangle - s(\rho_n - \langle Q_\lambda \rangle) \rangle .$$

LEMMA 9. *If $\varepsilon_k(\lambda + \rho_k + \langle Q \rangle) = 0$, then $H_\lambda^Q = 0$. If $\varepsilon_k(\lambda + \rho_k + \langle Q \rangle) \neq 0$, then there exists the unique element $s_\lambda^Q \in W_G$ such that $\langle s_\lambda^Q(\lambda + \rho_k + \langle Q \rangle), \alpha \rangle > 0$ for every $\alpha \in P_k$. In this case, suppose that*

$$|\langle \lambda + \rho, \beta \rangle| > \frac{1}{2} A_\lambda(s_\lambda^Q, Q) \qquad \textit{for all} \quad \beta \in P_n .$$

Then $H_\lambda^Q = 0$, if $Q \neq Q_\lambda$.

PROOF. The first half is clear since $V_\lambda^Q = 0$ under the given assumption. Under the assumption of the second part, suppose that $H_\lambda^Q \neq 0$. Let $\mathfrak{H}_\pi$ be an irreducible closed invariant subspace of H_λ^Q and π the representation on $\mathfrak{H}_\pi$ ($[\pi] \in \mathcal{E}_d$). From Lemma 2, we then see that

$$([\pi | K] : [\lambda + \langle Q \rangle]) \neq 0 .$$

When we denote by χ_π the infinitesimal character of π, we have

$$\chi_\pi(\Omega) = \langle \lambda + 2\rho, \lambda \rangle$$

by the same argument as utilized in the proof of Lemma 3.

Now, let $\mathfrak{g}_u=\mathfrak{k}\oplus\sqrt{-1}\mathfrak{p}$ be the compact real form of $\mathfrak{g}^C$ dual to $\mathfrak{g}$ and θ the conjugation with respect to $\mathfrak{g}_u$. Let $\{E_\alpha, H_i;\ \alpha\in\Delta,\ i=1,\cdots,l\}$ be a base of $\mathfrak{g}^C$ such that E_α is an eigenvector for a root $\alpha\in\Delta$ and

$$B(E_\alpha, E_{-\alpha})=1,$$

$$\theta E_\alpha=-E_{-\alpha}$$

for every $\alpha\in\Delta$, and that $\{H_i;\ i=1,\cdots,l\}$ is an orthonormal base of $\mathfrak{h}^C$ with respect to the Killing form B. We then have $\pi(E_\alpha)^*=\pi(E_{-\alpha})$ for $\alpha\in P_k$ and $\pi(E_\beta)^*=-\pi(E_{-\beta})$ for $\beta\in P_n$, where $\pi(X)^*$ denotes the adjoint operator of $\pi(X)$. This follows from the fact that π is a unitary representation of G. Under this base, we have

$$\Omega=\sum_i H_i^2+\sum_{\alpha\in\Delta}E_{-\alpha}E_\alpha.$$

Let $\mathfrak{H}_{\pi,[\lambda+\langle Q\rangle]}$ denote the subspace of all the elements in $\mathfrak{H}_\pi$ which transform according to $[\lambda+\langle Q\rangle]$ under $\pi|K$. Take a unit vector ψ which is a highest weight vector of some irreducible component in $\mathfrak{H}_{\pi,[\lambda+\langle Q\rangle]}$. Then ψ is a C^∞ vector under π and we have

$$\pi(\Omega)\psi=\chi_\pi(\Omega)\psi$$

and

$$\pi(\Omega_k)\psi=\langle\mu+2\rho_k, \mu\rangle\psi,$$

where $\Omega_k=\sum_{i=1}^{l}H_i^2+\sum_{\alpha\in P_k\cup(-P_k)}E_{-\alpha}E_\alpha$ and μ is the highest weight of $[\lambda+\langle Q\rangle]$. For a notational simplification, we write s for s_λ^Q in the lemma. Hence, $\mu=s(\lambda+\rho_k+\langle Q\rangle)-\rho_k$. Put $P_n'=(-sQ_\lambda)\cup s(P_n-Q_\lambda)$. Then we have $P_n'\cup(-P_n')=P_n\cup(-P_n)$ and $P_n'\cap(-P_n')=\phi$. For $\gamma\in\mathcal{F}$, denoting by $H_\gamma\in\mathfrak{h}^C$ the unique element of $\mathfrak{h}^C$ such that $B(H_\gamma, H)=\gamma(H)$ for all $H\in\mathfrak{h}^C$, we easily see that

$$\begin{aligned}\Omega-\Omega_k&=\sum_{\beta\in P_n\cup(-P_n)}E_{-\beta}E_\beta\\&=H_{2\rho_n'}+2\sum_{\beta\in P_n'}E_{-\beta}E_\beta,\end{aligned}$$

where $2\rho_n'=\langle P_n'\rangle$, since $[E_\beta, E_{-\beta}]=H_\beta$.

Under these preparations, we have

$$\pi(\Omega)\psi-\pi(\Omega_k)\psi=\pi(H_{2\rho_n'})\psi-2\sum_{\beta\in P_n'}\pi(E_\beta)^*\pi(E_\beta)\psi,$$

hence

$$-2\sum_{\beta\in P_n'}\pi(E_\beta)^*\pi(E_\beta)\psi=(\langle\lambda+2\rho, \lambda\rangle-\langle\mu+2\rho_k+2\rho_n', \mu\rangle)\psi.$$

Since $\|\psi\|=1$ where $\|\cdot\|$ denotes the norm of $\mathfrak{H}_\pi$, we have

$$-2\sum_{\beta\in P_n'}\|\pi(E_\beta)\psi\|^2$$

$$=\langle\lambda+2\rho,\lambda\rangle-\langle\mu+2\rho_k+2\rho_n',\mu\rangle$$

$$=\langle\lambda+2\rho,\lambda\rangle-\langle s(\lambda+\rho_k+\langle Q\rangle)+\rho_k+2\rho_n',\ s(\lambda+\rho_k+\langle Q\rangle)-\rho_k\rangle\,.$$

Noticing that $\langle Q_\lambda\rangle=\rho_n-s^{-1}\rho_n'$, we see, after some computations, that

$$-2\sum_{\beta\in P_n'}\|\pi(E_\beta)\psi\|^2=2\langle\lambda+\rho,\langle Q_\lambda-Q\rangle-\langle Q-Q_\lambda\rangle\rangle-A_\lambda(s,Q)\,.$$

If we assume that $Q_\lambda\neq Q$, then $(Q_\lambda-Q)\cup(Q-Q_\lambda)\neq\phi$. Hence, there exists $\beta\in P_n$ such that

$$\langle\lambda+\rho,\langle Q_\lambda-Q\rangle-\langle Q-Q_\lambda\rangle\rangle\geqq|\langle\lambda+\rho,\beta\rangle|\,.$$

On the other hand, we have

$$\frac{1}{2}A_\lambda(s,Q)\geqq\langle\lambda+\rho,\langle Q_\lambda-Q\rangle-\langle Q-Q_\lambda\rangle\rangle$$

because $-2\sum\|\pi(E_\beta)\psi\|^2\leqq 0$. Therefore it must hold

$$\frac{1}{2}A_\lambda(s,Q)\geqq|\langle\lambda+\rho,\beta\rangle|$$

for some $\beta\in P_n$, which contradicts the assumption. q. e. d.

REMARK. In [**11**], it is assumed that all the roots in P_n are totally positive, so that $s\rho_n=\rho_n$ and $sQ\subset P_n$ for $s\in W_G$. Hence the regularity condition of λ for the vanishing is obtained in a slightly simpler form in this case (see [**11**, Theorem 2]).

3.2. We put for $\lambda\in L$

$$a_\lambda=\frac{1}{2}\max_{Q\subset P_n}A(s_\lambda^Q,Q)\,,$$

where Q runs over subsets in P_n such that $\varepsilon_k(\lambda+\rho_k+\langle Q\rangle)\neq 0$. We then have

THEOREM 2. *Let λ be a character of T such that $\lambda+\rho\in L'$ and assume that*

$$|\langle\lambda+\rho,\beta\rangle|>a_\lambda\qquad \textit{for all}\quad \beta\in P_n\,.$$

Then the Hilbert space $H_\lambda^{Q_\lambda}$ gives an irreducible unitary representation belonging to the discrete series for G and its character is $\Theta_{\omega(\lambda+\rho)}$.

PROOF. Put

$$V=\bigoplus_{Q\subset P_n,\ \delta_\lambda(Q)=1}V_\lambda^Q\,,$$

$$W=\bigoplus_{Q\subset P_n,\ \delta_\lambda(Q)=-1}V_\lambda^Q\,,$$

where $\delta_\lambda(Q)=\varepsilon_k(\lambda+\rho)\varepsilon_k(\lambda+\rho_k+\langle Q\rangle)(-1)^{|Q|}$ and V_λ^Q is as in (3.1). Then it is obvious that V, W satisfy the condition of Theorem 1. As an invariant differential operator $D: L_2(\mathcal{V}) \to L_2(\mathcal{W})$, we consider $D=0$. In this case, the representation spaces of π_V, π_W are $L_2(\mathcal{V})_d$, $L_2(\mathcal{W})_d$ respectively. Since $\mathrm{Trace}\,\pi_V - \mathrm{Trace}\,\pi_W = (-1)^{q_\lambda}\Theta_{\omega(\lambda+\rho)}$ by Theorem 1, there exists a $Q \subset P_n$ such that $L_2(\mathcal{V}_\lambda^Q)_d$ contains an irreducible closed invariant subspace $\mathfrak{H}_{\omega(\lambda+\rho)}$ which gives a representation equivalent to $\omega(\lambda+\rho) \in \mathcal{E}_d$. For the infinitesimal character $\chi_{\omega(\lambda+\rho)}$ of $\omega(\lambda+\rho)$, we know that

$$\chi_{\omega(\lambda+\rho)}(\Omega) = \langle \lambda+2\rho, \lambda \rangle .$$

Hence it follows that $\mathfrak{H}_{\omega(\lambda+\rho)} \subset H_\lambda^Q$. In fact, for a C^∞ vector f in $\mathfrak{H}_{\omega(\lambda+\rho)}$ we have $\nu(\Omega)f = \langle \lambda+2\rho, \lambda \rangle f$, which implies $f \in H_\lambda^Q$. Since C^∞ vectors are dense in $\mathfrak{H}_{\omega(\lambda+\rho)}$ and H_λ^Q is closed, the above assertion follows. On the other hand, under the assumption for λ, we see that

$$H_\lambda^Q = 0 \qquad \text{for every} \quad Q \neq Q_\lambda$$

from Lemma 9. Therefore it must hold that $\mathfrak{H}_{\omega(\lambda+\rho)} \subset H_\lambda^{Q_\lambda}$.

Assume next that $\mathfrak{H}_{\omega(\lambda+\rho)} \neq H_\lambda^{Q_\lambda}$. If we take an irreducible closed invariant subspace $\mathfrak{H}_\pi$ in $H_\lambda^{Q_\lambda}$ contained in the orthogonal complement of $\mathfrak{H}_{\omega(\lambda+\rho)}$, it holds

$$\chi_\pi(\Omega) = \langle \lambda+2\rho, \lambda \rangle ,$$

where χ_π is the infinitesimal character of $\mathfrak{H}_\pi$. Noting that $\mathfrak{H}_\pi \subset L_2(\mathcal{V}_\lambda^{Q_\lambda})_d$, it follows from Theorem 1 that there exists $Q \neq Q_\lambda$ in P_n such that $L_2(\mathcal{V}_\lambda^Q)_d$ contains an irreducible closed invariant subspace $\mathfrak{H}'_\pi$ which gives a representation equivalent to the one given by $\mathfrak{H}_\pi$. Hence, by the same argument as above, $\mathfrak{H}'_\pi \subset H_\lambda^Q$, which contradicts that $H_\lambda^Q = 0$. Therefore, $H_\lambda^{Q_\lambda}$ is irreducible and gives a unitary representation equivalent to $\omega(\lambda+\rho)$ in $\mathcal{E}_d$. q. e. d.

Making use of Theorem 2, we shall state a procedure in order to realize most of the discrete classes on the eigenspaces of the Casimir operator. For a positive root system P in the root system Δ, we put

$$a_P = \frac{1}{2} \max_{Q \subset P_n,\, s \in W_G} (\langle\langle Q \rangle, \langle Q \rangle\rangle + 2\langle \rho_k, \rho_n - s\rho_n \rangle) .$$

Here notice that P determines P_n, ρ_k, ρ_n. Choose a positive number a such as

$$a \geqq a_P \qquad \text{for all positive root systems } P \text{ in } \Delta .$$

COROLLARY. *Take $\Lambda \in L'$ such that $|\langle \Lambda, \beta \rangle| > a$ for all non-compact roots β. Choose a positive root system P such as $P = \{\alpha \in \Delta\,;\, \langle \Lambda, \alpha \rangle < 0\}$. Under the linear order given by P, define ρ, P_k, ρ_k as before and put $\lambda = \Lambda - \rho$. Then $\langle \lambda + 2\rho_k, \alpha \rangle \leqq 0$ for all compact positive roots $\alpha \in P_k$. Let V_λ be the irreducible unitary K-module with lowest weight $\lambda + 2\rho_k$, and $C^\infty(\mathcal{V}_\lambda)$ (resp. $L_2(\mathcal{V}_\lambda)$) the space of all V_λ-valued C^∞ (resp. square-integrable) functions f on G such that $f(gk) = k^{-1}f(g)$ for $g \in G$, $k \in K$. Put*

$$\mathfrak{H}_\lambda=\{f\in C^\infty(\mathcal{V}_\lambda)\cap L_2(\mathcal{V}_\lambda)\,;\ \nu(\Omega)f=\langle\lambda+2\rho,\lambda\rangle f\}\,.$$

Then the Hilbert space $\mathfrak{H}_\lambda$ gives an irreducible unitary representation belonging to the discrete series for G, and its character is $\Theta_{\omega(\lambda+\rho)}=\Theta_{\omega(\Lambda)}$.

PROOF. We first notice the following. Let μ be an integral linear form for P_k, i. e. $2\langle\mu,\alpha\rangle/\langle\alpha,\alpha\rangle\in \boldsymbol{Z}$ for every $\alpha\in P_k$. Then $\langle\mu-\rho_k,\alpha\rangle\geqq 0$ for all $\alpha\in P_k$ if $\langle\mu,\alpha\rangle>0$ for all $\alpha\in P_k$. From this fact, it follows that $\langle\rho_n,\alpha\rangle\geqq 0$ for all $\alpha\in P_k$, because $\langle\rho,\alpha\rangle>0$ for all $\alpha\in P_k$. Now, by the definition, $\langle\lambda+\rho,\alpha\rangle<0$ for all $\alpha\in P_k$. Therefore $\langle\lambda+\rho_k,\alpha\rangle=\langle\lambda+\rho,\alpha\rangle-\langle\rho_n,\alpha\rangle<0$ for all $\alpha\in P_k$. Since $\lambda+\rho_k$ is an integral linear form for P_k, we have $\langle\lambda+2\rho_k,\alpha\rangle\leqq 0$ for all $\alpha\in P_k$. Hence one can consider the irreducible K-module V_λ with lowest weight $\lambda+2\rho_k$.

Secondly, under this outset, it is easy to see that $[V_\lambda]=[\lambda]$ and $Q_\lambda=\phi$. Hence $\mathfrak{H}_\lambda$ coincides with $H_\lambda^{\rho_\lambda}$ in Theorem 2. For λ such that $Q_\lambda=\phi$,

$$A_\lambda(s,Q)=\langle\langle Q\rangle,\langle Q\rangle\rangle+2\langle\rho_k,\rho_n-s\rho_n\rangle\,.$$

Hence, in our case, we have $a_P\geqq a_\lambda$, where a_λ is as in Theorem 2. Therefore, if $|\langle\Lambda,\beta\rangle|>a$ for all non-compact roots β, we have $|\langle\lambda+\rho,\beta\rangle|>a_\lambda$ for all $\beta\in P_n$, which implies the requirement by Theorem 2. q. e. d.

REMARK 1. In view of Harish-Chandra's result cited in (2.1), the above procedure in Corollary makes it possible to realize most of the discrete classes, i. e. $\{\omega(\Lambda)\in\mathcal{E}_d\,;\ |\langle\Lambda,\beta\rangle|>a$ for all non-compact roots $\beta\}$.

REMARK 2. When the symmetric space G/K admits an invariant complex structure and all the roots in P_n are totally positive, Theorem 2 nearly amounts to Proposition 9.1 in [**11**]. In more detail, under such an assumption, one can introduce an invariant complex structure on G/K such as the anti-holomorphic cotangent space at the origin eK is identified with the K-invariant space $\mathfrak{p}_+=\sum_{\beta\in P_n}\boldsymbol{C}E_\beta$ where E_β is as in (3.1). Let V_λ be the irreducible unitary K-module with lowest weight $\lambda+2\rho_k$. Then the associated homogeneous vector bundle $\mathcal{V}_\lambda$ over G/K has a structure of a holomorphic vector bundle. When we consider the Laplace-Beltrami operator $\square$ for the Dolbeault complex associated to the holomorphic vector bundle $\mathcal{V}_\lambda$, we know that

$$\square=-\frac{1}{2}(\nu(\Omega)-\langle\lambda+2\rho,\lambda\rangle 1)$$

as a differential operator on the homogeneous vector bundle associated to the K-module $V_\lambda\otimes\wedge^q\mathfrak{p}_+$ which is regarded as the tensor bundle of $(0,q)$ forms (see [**13**], also [**9**]). Consequently, in this case, our realization using the Casimir operator is equivalent to that in the framework of the square-integrable cohomology space, which is established by Narasimhan and Okamoto in [**11**]. In particular, the Hilbert space $\mathfrak{H}_\lambda$ in Corollary coincides with the space of all square-integrable holomorphic sections of $\mathcal{V}_\lambda$ in the above case; hence the corollary is included in Harish-Chandra's realization in [4]. It is known in [4] that the constant a_λ may be chosen as $a_\lambda=0$ in this case.

REMARK 3. When G/K admits no invariant complex structures (or an element of P_n is not totally positive), Theorem 2 and Corollary is a generalization of Takahashi's realization for the universal covering group of the de Sitter group in [16]. Further, this method (the realization by means of the Casimir operator) is indicated by Okamoto in [12]. We notice that, according to [16], when G is the universal covering group of the de Sitter group, the constant a in the corollary may be chosen as $a=0$, and in this case all the discrete classes are realized by this method.

3.3. We shall here refer to a relation of the discrete series with the elliptic complexes constructed in [8] over the symmetric space G/K. When $\lambda \in L$ satisfies the condition that $\langle \lambda+\rho_k+\langle Q\rangle, \alpha\rangle \leqq 0$ for all $\alpha \in P_k$ and all $Q \subset P_n$, we shall say that λ satisfies the condition (♯). We choose such a special positive root system as follows. We say that a positive root system P is admissible if

$$\beta_1+\cdots+\beta_q \neq \gamma_1+\cdots+\gamma_r$$

for any $\beta_1, \cdots, \beta_q, \gamma_1, \cdots, \gamma_r \in P_n$ such that $q \neq r$. It is known that there exists an admissible positive root system in Δ (see [8, Lemma 3.2]). We keep an admissible positive root system P and the linear order on Δ given by P fixed in this subsection.

Under this situation, let V_λ be the irreducible K-module with lowest weight $\lambda+2\rho_k$. Assume that λ satisfies the condition (♯) with respect to the admissible positive root system P fixed above. Then for $Q \subset P_n$, $[\lambda+\langle Q\rangle] \in \mathcal{E}_K$ is the equivalence class of an irreducible K-module with lowest weight $\lambda+2\rho_k+\langle Q\rangle$. For $q=1, \cdots, n$ $\left(n=\frac{1}{2}\dim G/K\right)$, put

$$V_\lambda^q = \bigoplus_{|Q|=q,\, Q\subset P_n} V_\lambda^Q,$$

where V_λ^Q is as in (3.1). There then exist invariant first order differential operators $\mathcal{D}^q: C^\infty(\mathcal{V}_\lambda^q) \to C^\infty(\mathcal{V}_\lambda^{q+1})$ for $q=0, 1, \cdots, n-1$ ($V_\lambda^0=V_\lambda$) such that the sequence

$$0 \longrightarrow C^\infty(\mathcal{V}_\lambda) \xrightarrow{\mathcal{D}^0} C^\infty(\mathcal{V}_\lambda^1) \xrightarrow{\mathcal{D}^1} \cdots \xrightarrow{\mathcal{D}^{n-1}} C^\infty(\mathcal{V}_\lambda^n) \longrightarrow 0$$

is an elliptic complex over G/K (see [8, Theorem 3.1], where this complex is called the (♯)-complex associated to $\mathcal{V}_\lambda$). We note that when G/K is a hermitian symmetric space, P is admissible if all roots in P_n are totally positive. Further, in this case the above elliptic complex is the Dolbeault complex for the holomorphic vector bundle $\mathcal{V}_\lambda$ (cf. Remark 2 in (3.2)).

Denoting by $(\mathcal{D}^q)^*$ the formal adjoint of $\mathcal{D}^q$, we define the "square-integrable cohomology space" of this elliptic complex $\mathcal{V}_\lambda^*$ as

$$H_2^q(\mathcal{V}_\lambda^*)=\{f\in L_2(\mathcal{V}_\lambda^q)\,;\,(\mathcal{D}^q+(\mathcal{D}^{q-1})^*)f=0\}\,,$$

where $\mathcal{D}^q+(\mathcal{D}^{q-1})^*$ is extended maximally on $L_2(\mathcal{V}_\lambda^q)$ as in (1.2). We notice that this definition coincides with that by means of the space of square-integrable harmonic sections, i. e., the null space of the laplacian $\square=\mathcal{D}^{q-1}(\mathcal{D}^{q-1})^*+(\mathcal{D}^q)^*\mathcal{D}^q$. This is shown by a technique similar to the one in [1] because of the completeness of the invariant riemannian metric on G/K (cf. Lemma 4 in § 1).

Let s_0 be the unique element in W_G such that $s_0P_k=-P_k$. Put

$$b_\lambda=\frac{1}{2}\max_{Q\subset P_n}A_\lambda(s_0,\,Q)$$

where A_λ is as in (3.1).

As for the above square-integrable cohomology space of the elliptic complex $\mathcal{V}_\lambda^*$, the following proposition is a generalization of [**13**], in which the non-vanishing theorem for the Dolbeault complex over a hermitian symmetric space is treated.

PROPOSITION. *Assume that $\lambda\in L$ satisfies the condition (♯) and let $\mathcal{V}_\lambda^*$ be the above elliptic complex for λ. If $|\langle\lambda+\rho,\beta\rangle|>b_\lambda$ for all $\beta\in P_n$, then $H_2^{q_\lambda}(\mathcal{V}_\lambda^*)\neq 0$ and an irreducible unitary representation of G equivalent to $\omega(\lambda+\rho)\in\mathcal{E}_d$ is realized on a closed invariant subspace of $H_2^{q_\lambda}(\mathcal{V}_\lambda^*)$.*

PROOF. Put $V=\bigoplus_{q:\,\mathrm{even}}V_\lambda^q$ and $W=\bigoplus_{q:\,\mathrm{odd}}V_\lambda^q$. Consider an invariant differential operator $\mathcal{L}: C^\infty(\mathcal{V})\to C^\infty(\mathcal{W})$ defined so that $\mathcal{L}$ coincides with $\mathcal{D}^q+(\mathcal{D}^{q-1})^*$ on $C^\infty(\mathcal{V}_\lambda^q)$ for each even q. Since

$$\varepsilon_k(\lambda+\rho)\varepsilon_k(\lambda+\rho_k+\langle Q\rangle)=1$$

under the condition (♯) for λ, the differential operator

$$\mathcal{L}:\ L_2(\mathcal{V})\longrightarrow L_2(\mathcal{W})$$

satisfies the condition of Theorem 1. So there exists q such that $H_2^q(\mathcal{V}_\lambda^*)_d$ contains an irreducible unitary representation equivalent to $\omega(\lambda+\rho)$, because $\operatorname{Ker}\mathcal{L}=\bigoplus_{q:\,\mathrm{even}}H_2^q(\mathcal{V}_\lambda^*)$ and $\operatorname{Ker}\mathcal{L}^*=\bigoplus_{q:\,\mathrm{odd}}H_2^q(\mathcal{V}_\lambda^*)$. On the other hand, since λ satisfies the condition (♯), the element s_λ^Q in Lemma 9 for our $\lambda+\langle Q\rangle$ is s_0. Hence if $|\langle\lambda+\rho,\beta\rangle|>b_\lambda$ for all $\beta\in P_n$, $H_2^q(\mathcal{V}_\lambda^*)_d$ contains no closed invariant subspaces which give representations equivalent to $\omega(\lambda+\rho)$ when $q\neq q_\lambda$ in view of Lemma 9. The proposition follows immediately from these facts. q. e. d.

3.4. We shall now see that another realization of the discrete series is obtained in a similar way. That is, most of the discrete classes are also realized by means of a certain invariant first order differential operator introduced by W. Schmid in [**14**]. We start with recalling the definition of this differential operator. Fix a positive root system P and the linear order on the root system Δ given by P (P is not assumed to be admissible in the sense of (3.3)). Define $\mathfrak{p}^C$ as in (2.1) and regard $\mathfrak{p}^C$ as a K-module. We choose an eigenvector E_β for a root β such that $B(E_\beta, E_{-\beta})=1$ where B is the Killing form of $\mathfrak{g}^C$. By the definition, we know $\mathfrak{p}^C=\sum_{\beta\in P_n\cup(-P_n)} CE_\beta$.

Let V_λ be the irreducible K-module with lowest weight $\lambda+2\rho_k$ and $\mathcal{V}_\lambda$ the homogeneous vector bundle over G/K associated to V_λ. Denote by $\mathcal{P}$ the homogeneous vector bundle associated to the K-module $\mathfrak{p}^C$. We define an invariant differential operator $D: C^\infty(\mathcal{V}_\lambda)\to C^\infty(\mathcal{V}_\lambda\otimes\mathcal{P})$ by

$$D=\sum_{\beta\in P_n\cup(-P_n)}\nu(E_\beta)\otimes\varepsilon(E_{-\beta}),$$

where $\varepsilon(E_{-\beta})\in \mathrm{Hom}(V_\lambda, V_\lambda\otimes\mathfrak{p}^C)$ is a tensoring operator by $E_{-\beta}\in\mathfrak{p}^C$ (for the interpretation as a differential operator, see (1.1)). In view of (1.1), it is easily seen that the above definition of D is well defined and independent of choice of a base $\{E_\beta\}$ under the condition $B(E_\beta, E_{-\beta})=1$. Let V_λ^1 be the minimal K-submodule of $V_\lambda\otimes\mathfrak{p}^C$ which contains every irreducible component of $V_\lambda\otimes\mathfrak{p}^C$ with lowest weight $\lambda+2\rho_k+\beta$ ($\beta\in P_n$) and U_λ^1 that which contains every irreducible component with lowest weight $\lambda+2\rho_k-\beta$ ($\beta\in P_n$). We then have the unique direct sum decomposition

$$V_\lambda\otimes\mathfrak{p}^C=V_\lambda^1\oplus U_\lambda^1$$

(see [14, § 5]). Let $p: C^\infty(\mathcal{V}_\lambda\otimes\mathcal{P})\to C^\infty(\mathcal{V}_\lambda^1)$ be the projection induced from the projection $V_\lambda\otimes\mathfrak{p}^C\to V_\lambda^1$ for the above decomposition. The composition $\mathcal{D}=p\circ D$ is then an invariant first order differential operator from $C^\infty(\mathcal{V}_\lambda)$ into $C^\infty(\mathcal{V}_\lambda^1)$ and we call $\mathcal{D}$ Schmid's operator. Notice that the definition of $\mathcal{D}$ depends on choice of the positive root system P. As for this operator we know

LEMMA 10. *If $\lambda\in L$ satisfies the condition (#) in the sense of (3.3), then $\mathcal{D}$ is elliptic. Moreover, for $q=1,\cdots,n$ $\left(n=\frac{1}{2}\dim G/K\right)$, let V_λ^q be a K-module such that $[V_\lambda^q]=\bigoplus_{|Q|=q,\ Q\subset P_n}[\lambda+\langle Q\rangle]$ and put $W_\lambda^e=\bigoplus_{q:\text{ even}}V_\lambda^q$ $(V_\lambda^0=V_\lambda)$, $W_\lambda^o=\bigoplus_{q:\text{ odd}}V_\lambda^q$. Then there exists an elliptic operator*

$$\mathcal{L}:\ C^\infty(\mathcal{W}_\lambda^e)\longrightarrow C^\infty(\mathcal{W}_\lambda^o)$$

such that $\mathcal{L}|C^\infty(\mathcal{V}_\lambda)=\mathcal{D}$.

For a proof, see [8, § 3 and Lemma 6.1], where $\mathcal{L}: C^\infty(\mathcal{W}_\lambda^e)\to C^\infty(\mathcal{W}_\lambda^o)$ is called the (#)-complex for an arbitrary linear order on Δ.

REMARK. When P is admissible, Schmid's operator $\mathcal{D}$ coincides with $\mathcal{D}^0$ in (3.3) and W_λ^e, W_λ^o respectively coincide with V, W in the proof of Proposition in (3.3).

Take $\Lambda \in L'$ and choose a positive root system P such that $P=\{\alpha \in \Delta;\ \langle \Lambda, \alpha \rangle < 0\}$. Put $\lambda = \Lambda - \rho$. Then $\langle \lambda + 2\rho_k, \alpha \rangle \leqq 0$ for all $\alpha \in P_k$ as is shown in (3.2). We may therefore consider the irreducible unitary K-module V_λ with lowest weight $\lambda + 2\rho_k$. For V_λ, we have Schmid's operator $\mathcal{D}: C^\infty(\mathcal{V}_\lambda) \to C^\infty(\mathcal{V}_\lambda^1)$ as above. Assume that λ satisfies the condition (♯). Because of the ellipticity of $\mathcal{D}$, the space

$$\mathcal{H}_\lambda = \{ f \in C^\infty(\mathcal{V}_\lambda) \cap L_2(\mathcal{V}_\lambda);\ \mathcal{D}f = 0\}$$

is then a Hilbert space and gives a unitary representation π_λ of G. As for $(\pi_\lambda, \mathcal{H}_\lambda)$, we see

LEMMA 11. *Assume that λ satisfies the condition (♯) and that $\langle \lambda + 2\rho_n + \beta, \alpha \rangle < 0$ for all $\alpha \in P_k$, $\beta \in P_n$. Then we have*

$$([\pi_\lambda | K] : [V_\lambda]) \leqq 1 .$$

PROOF. Consider the G-module $\mathcal{F}_\lambda = \{ f \in C^\infty(\mathcal{V}_\lambda);\ \mathcal{D}f = 0\}$ through the left translation. Let Y_λ be the subspace of $\mathcal{F}_\lambda$ spanned by all the elements in $\mathcal{F}_\lambda$ which transform according to $[V_\lambda]$. When λ satisfies the condition (♯), we see that $\langle \lambda + 2\rho_n, \alpha \rangle < 0$ for all $\alpha \in P_k$ since $\langle \rho_k, \alpha \rangle > 0$ for all $\alpha \in P_k$. Under these conditions combining with the last condition in the lemma, we know that Y_λ is finite dimensional and irreducible as a K-module (see [14] and [8, Theorem 6.2]). Since $\mathcal{H}_\lambda \subset \mathcal{F}_\lambda$, we see that $([\pi_\lambda | K] : [V_\lambda]) \leqq 1$. q. e. d.

Now, we put

$$c_\lambda = \frac{1}{2} \max_{Q \subset P_n} A_\lambda(s_0, Q) ,$$

where s_0 is as in (3.3). We notice that

$$A_\lambda(s_0, Q) = \langle\langle Q \rangle, \langle Q \rangle\rangle + 2\langle \rho_k, \rho_n - s_0\rho_n \rangle$$

in this case, where the choice of P depends on λ (or Λ) and so does c_λ.

THEOREM 3[2]. *Let λ be in L such that $\langle \lambda + \rho, \alpha \rangle < 0$. Assume that λ satisfies the condition in Lemma 11 for the compact positive roots and, moreover, that $|\langle \lambda + \rho, \beta \rangle| > c_\lambda$ for all $\beta \in P_n$. Then $\mathcal{H}_\lambda$ gives an irreducible unitary representation belonging to the discrete series for G and its character is $\Theta_{\omega(\lambda+\rho)}$.*

PROOF. Consider the elliptic operator $\mathcal{L}: L_2(\mathcal{W}_\lambda^e) \to L_2(\mathcal{W}_\lambda^o)$ in Lemma 10, where $\mathcal{L}$ is extended maximally as in (1.2). By the definition, the K-modules W_λ^e, W_λ^o satisfy the condition in Theorem 1. By the choice of λ, we have $q_\lambda = 0$ and therefore there exists an irreducible closed invariant subspace $\mathfrak{H}_\pi$ in Ker $\mathcal{L}$ such that $[\pi] = \omega(\lambda + \rho)$ in $\mathcal{E}_d$, where π denotes the representation on $\mathfrak{H}_\pi$. Since $|\langle \lambda + \rho, \beta \rangle| > c_\lambda$ for all $\beta \in P_n$ and $Q_\lambda = \phi$, tnere are no closed

2) The result in Theorem 3 is communicated by Prof. Schmid. The proof stated here is due to the author.

invariant subspaces in $L_2(\mathcal{V}_\lambda^q)_d$ for $q \geqq 1$ which give representations equivalent to $\omega(\lambda+\rho)$ in view of Lemma 9. Hence we have $\mathfrak{H}_\pi \subset L_2(\mathcal{V}_\lambda)_d$. Since $\mathcal{L} | L_2(\mathcal{V}_\lambda) = \mathcal{D}$ and $\mathcal{D}$ is elliptic, we see that $\operatorname{Ker} \mathcal{L} \cap L_2(\mathcal{V}_\lambda) = \mathcal{H}_\lambda$, which implies $\mathfrak{H}_\pi \subset \mathcal{H}_\lambda$. By Lemma 2, we have $([\pi | K] : [V_\lambda]) \geqq 1$ and so $([\pi_\lambda | K] : [V_\lambda]) \geqq 1$ by the above result. On the other hand, under the condition of the theorem we have $([\pi_\lambda | K] : [V_\lambda]) \leqq 1$ by Lemma 11. Hence $([\pi_\lambda | K] : [V_\lambda]) = 1$. Now, since the representation space $\mathcal{H}_\lambda$ consists of C^∞ (and actually analytic) functions on G thanks to the ellipticity of $\mathcal{D}$, one can conclude from $([\pi_\lambda | K] : [V_\lambda]) = 1$ that π_λ is irreducible. This is shown quite similarly to [**12**, Proposition 2]. Therefore we have $\mathfrak{H}_\pi = \mathcal{H}_\lambda$. q. e. d.

REMARK 1. There exists a positive number c_k such that λ satisfies the condition of Lemma 11 if $|\langle \Lambda, \alpha \rangle| > c_k$ for all $\alpha \in P_k$. As in (3.2), one can therefore choose a positive number c such that λ satisfies the condition of the theorem if $|\langle \Lambda, \alpha \rangle| > c$ for all roots $\alpha \in \Delta$. For this reason, we may say that Theorem 3 also gives a procedure in order to realize most of the discrete classes.

REMARK 2. For a relation with the realization by means of the Casimir operator, we remark the following. If λ satisfies the condition of Theorem 3, we then see that $\mathcal{H}_\lambda \subset \mathfrak{H}_\lambda$ where $\mathfrak{H}_\lambda$ is as in Corollary in (3.2). As is shown in the proof of Theorem 3, Theorem 1 and Lemma 9 are valid under this condition. From this fact, it is easily seen that $\mathfrak{H}_\lambda$ is irreducible in the same way as in Theorem 2. Hence $\mathcal{H}_\lambda = \mathfrak{H}_\lambda$ and so the two procedures are mutually equivalent.

REMARK 3. For the connected component of the identity of the generalized Lorentz group $SO(2n, 1)$ (or its universal covering group), the above procedure realizes all the discrete classes, i.e. the constant c in Remark 1 may be chosen as $c = 0$ (see [**8**, Corollary to Theorem 6.3]). This is, however, shown in a different way from here.

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Added in proof.

R. Parthasarathy has recently realized most of the discrete series by means of Dirac operators. He has sharpened the vanishing theorem and thereby constructed discrete classes under the less restrictive "regularity" condition.