Bid Rotation and Collusion in Repeated Auctions†

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Abstract

This paper studies bidder collusion with communication in repeated auctions when no side transfer is possible. It presents a simple *dynamic bid rotation scheme* which coordinates bids based on communication history and enables intertemporal transfer of bidders' payoffs. The paper derives a sufficient condition for such a dynamic scheme to be an equilibrium and characterizes the equilibrium payoffs in a general environment with affiliated signals and private or interdependent values. With IPV, it is shown that this dynamic scheme yields a strictly higher payoff to the bidders than any static collusion scheme which coordinates bids based only on the current reported signals.

Key words: collusion, auction, repeated games.

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1. Introduction

It is well recognized that bidder collusion is a serious problem in many auctions: Collusion is documented in auctions for used machinery, timbers, frequency spectrums, Treasury securities, the procurement of construction work, *etc.* (Marshall and Meurer [13], Porter and Zona [17], Baldwin *et al.* [3]). Despite its significance as an empirical phenomenon, relatively little is understood about the theory of collusion in auctions, which is distinguished from the standard collusion theory by the presence of asymmetric information across bidders about their valuations of the object.

Most of the existing analysis of collusion in auctions is conducted in the *one-shot* framework. One important contribution in this case is made by McAfee and McMillan [14], who analyze bidder collusion with communication in first-price auctions under the independent private values (IPV) assumption. Their key findings include the identification of the most efficient collusion schemes with and without side transfer. In particular, they show that full collusion is possible with side transfer, but that the scope of bidder collusion is severely limited without it.

If collusion is a product of frequent interaction, however, a more appropriate framework for analysis is that of repeated games, where the same set of bidders participate in a series of auctions held sequentially over time.[1] The purpose of this paper is to show that in infinitely repeated auctions, collusion is possible through intertemporal payoff transfer even if there is no side payment of money. In other words, bidders in repeated auctions can collude through the adjustment of continuation payoffs in a way that partially compensates for the lack of monetary transfer. We derive a sufficient condition for such a collusion scheme to be an equilibrium and characterize the equilibrium payoffs in a general environment with affiliated signals and private or interdependent values. Specifically, the collusion scheme considered in this paper builds on communication between the bidders, which we think is an integral part of many collusion practices.[2] The analysis shows how communication coordinates bidding in a dynamic environment.

We consider a model of infinitely repeated auctions with two symmetric bidders. In

[1]See also Hendricks and Porter [10], who emphasize the need of a repeated model of collusion in auctions.

[2]The Japanese word for collusion in procurement auctions is "dangō," which literally means "to discuss."

every period, a single indivisible object is sold through the same auction format, and the bidders' private signals are drawn from the same distribution.

The bidders collude by coordinating their bids in each auction with the help of a communication device referred to here as a *center.* In each period, the center receives reports from the bidders about their private signals and then instructs them on what bid to submit in the stage auction. This stage mechanism, which chooses instructions as a function of reports, is called an *instruction rule* in this paper. A *collusion scheme* represents the center's choice of an instruction rule in every period contingent on history. A collusion scheme is an *equilibrium* if truth-telling is incentive compatible and obedience to the instructions is rational for each bidder. The paper's analysis focuses on a class of (grim-trigger) collusion schemes called *bid rotation* schemes. In these schemes, play begins with the collusion phase where no more than one bidder is instructed to bid in each stage auction, and any deviation from the center's instruction triggers the punishment phase where the one-shot Nash equilibrium of the stage auction is played.

The bid rotation scheme constructed in this paper is as follows: Play during the collusion phase rotates among three (sub-)phases S, A_1 and A_2. In the original symmetric phase S, the center uses the *efficient* instruction rule which instructs the bidder with the higher valuation (based on the reports) to bid the reserve price R if and only if his valuation exceeds R. It instructs the other bidder to stay out. In phase A_i ($i = 1, 2$), the center uses an *asymmetric* instruction rule which favors bidder $j \neq i$ in the sense that j's *ex ante* stage payoff is higher than that of bidder i. It can be seen that the efficient instruction rule used in phase S is not incentive compatible by itself since the bidders would overstate their signals in the hope of winning the object at the reserve price. The incentive for truth-telling in phase S is provided through the adjustment in continuation payoffs as follows: When bidder i's reported signal is higher than that of j, a transition to phase A_i takes place with positive probability so that bidder i's continuation payoff would be lower. The instruction rule used in phase A_i is chosen to be incentive compatible so that no further adjustment in continuation payoffs is necessary. After a fixed number of play in phase A_i, the game returns to phase S.

The above bid rotation scheme is *dynamic* in the sense that the instruction rule used during the collusion phase is chosen as a function of past reports. In contrast, the collusion

scheme proposed by McAfee and McMillan [14] is a *static* bid rotation scheme, which uses the same instruction rule throughout the collusion phase independent of the history.[3] It should be noted that in a static scheme, incentive compatibility of the instruction rule must hold period by period since no adjustment in continuation payoffs is possible. With this constraint relaxed, a dynamic bid rotation scheme can be strictly more efficient. Specifically, this paper shows that when the stage auction is first-price and when we have IPV, a dynamic bid rotation scheme as described above is an equilibrium for sufficiently patient bidders and yields a strictly higher payoff than the optimal static scheme.

As mentioned above, most existing models of collusion in auctions are one-shot. Robinson [18] and von Ungern-Sternberg [20] are among the first to point out the vulnerability of the English and second-price sealed-bid auctions to bidder collusion.[4] Graham and Marshall [8] and Graham *et al.* [9] present particular side transfer schemes for second-price and English auctions with pre-auction communication. Subsequently, Mailath and Zemsky [12] and McAfee and McMillan [14] identify the optimal schemes under the IPV assumption. The former examine collusion with side transfer in second-price auctions, while the latter look at that with and without side transfer in first-price auctions. Both papers conclude that efficient collusion is possible with side transfer.[5] More recently, Athey *et al.* [2], Johnson and Robert [11], and Skrzypacz and Hopenhayn [19] analyze collusion in repeated IPV auctions without side transfer.[6] Among them, Skrzypacz and Hopenhayn [19] prove the existence of a collusion scheme without communication that performs strictly better than the static scheme of McAfee and McMillan [14] when the stage auction is first-price and the reserve price equals zero. While the intuition behind their results is closely related to ours, their formal logic is specialized to the particular auction format as well as the IPV assumption. In contrast, this paper presents a simple collusion scheme which is robust with respect to these specifications, and characterizes the collusive payoff explicitly.

It should be noted that the collusion scheme considered in this paper does not extract full surplus from the auctioneer since the instruction rule used in the asymmetric phase

[3]Note that in both "static" and "dynamic" schemes, any deviation from the instruction is met by reversion to the one-shot Nash equilibrium.
[4]Brusco and Lopomo [4] and Engelbrecht-Wiggans and Kahn [5] analyze collusion in (one-shot) multi-object ascending price auctions.
[5]An earlier version of this paper contains a generalization of this result.
[6]The present paper was developed independently of them.

A_i is not efficient. In other words, the scheme is not first-best efficient from the point of view of the bidders. The question of first-best efficiency in collusion without side transfer under asymmetric information is indeed very difficult. Efficiency results are available only in IPV models with *finite* signals: Fudenberg *et al.* [7] show that the IPV model with finite signals and communication has the "product structure," which guarantees the existence of a near efficient equilibrium for sufficiently low discounting.[7] In contrast, the present paper uses the continuous signal formulation, which is most common in the auction literature. Since conclusions based on finite signals do not necessarily extend to continuous signals in many mechanism design problems, we think it important to gain insight into the problem for this standard framework.[8]

The severest restriction of the present model is the assumption that the auctioneer uses the same auction format every period. While this may be an adequate description of some actual practices, it would be extremely important to analyze the alternative scenario where the auctioneer has the ability to react to bidder collusion. In the one-shot framework, Mailath and Zemsky [12] and McAfee and McMillan [14] both discuss the choice of the reserve price as the auctioneer's response to collusion. In repeated auctions, the corresponding treatment is to include the auctioneer as a player of the repeated game who chooses the reserve price as a function of history. The analysis of such a model would be very much involved and is left as a topic of future research.

Although this paper will focus on a two-bidder model for simplicity, its qualitative conclusions would go through with three or more bidders as long as attention is restricted to collusion by the grand coalition. When there are more than two bidders, however, a new set of questions will arise concerning the feasibility and profitability of collusion by a subcoalition of bidders.

We assume that the bidders' private signals are independent across periods. While there are many interesting situations that involve serially correlated signals, the following are some difficulties associated with the analysis of such a model. First, with independent

[7]Based on this result, Athey and Bagwell [1] present an explicit characterization of the first-best scheme in the IPV model with binary signals.

[8]The finiteness of actions is critical for the type of argument used in Fudenberg *et al.* [7]. With communication, a mapping from the set of signals to the set of reports is part of a player's action. This suggests that using a finite message space in communication is not sufficient.

signals, bid rotation improves allocative efficiency by making the highest valuation bidder win at the reserve price (at least in the symmetric phase). When signals are correlated over time, on the other hand, the relationship between bid rotation and allocative efficiency is more subtle. In the extreme case of perfect correlation where the signals stay the same throughout the game, for example, rotating bids would only lower allocative efficiency.[9] Another simplification possible with independent signals is that rules of transition between distinct phases of the collusion scheme can be taken as a function of current reports only and independent of past reports. With serially correlated signals, this would no longer be the case and the determination of transition probabilities would be much more difficult. Finally, the introduction of serial correlation also induces some fundamental asymmetries between the bidders during the course of play.[10]

The organization of the paper is as follows: The next section formulates a model of repeated auctions with the center as a mediation device. The dynamic bid rotation scheme is described in Section 3. The main theorem in this section gives a sufficient condition for this scheme to be an equilibrium and describes the equation that characterizes its payoff. In Section 4, its performance is compared with that of static schemes. Section 5 analyzes a collusion scheme based on implicit communication through winning bids.

2. Model

There are two symmetric, risk-neutral bidders 1 and 2 and a *center* which coordinates their bidding in infinitely repeated auctions. A single indivisible object is sold every period through a fixed auction format. In each period, bidder i receives a private signal $s_i \in [0,1]$ about the value of the object. The probability distribution of the signal profile $s = (s_1, s_2)$ is the same in every period and represented by the density function f whose support is the unit square $[0,1]^2$. The signals are *independent* across periods. The conditional density of s_i given s_j is denoted $f_i(\cdot \mid s_j)$, and the corresponding distribution function is denoted $F_i(\cdot \mid s_j)$ $(i \neq j)$. With slight abuse of notation, we also use f_i (resp. F_i) to denote

[9] Bid rotation could still improve the bidders' expected payoffs from the one-shot Nash equilibrium if the higher valuation bidder makes his opponent win from time to time in return for less aggressive bidding in other periods.

[10] Asymmetry is problematic since it implies that incentive compatibility of a collusion scheme is now characterized by a system of (linear) differential equations rather than one (*cf.* (a7) in the Appendix). Such a system typically admits only a numerical solution and does not yield any characterization of the equilibrium payoff.

the marginal density (resp. distribution) of s_i ($i = 1, 2$). Throughout, we assume that the signals $s = (s_1, s_2)$ are *affiliated.* Affiliation includes independent signals as a special case, and is equivalent to the monotone likelihood ratio property in the current framework with only two signals s_1 and s_2. The present analysis will use the following properties of affiliation (*e.g.,* Milgrom and Weber [15]):

$$(1) \qquad \frac{f_i(s_i \mid s_j)}{F_i(s_i \mid s_j)} \text{ is (weakly) increasing in } s_j \text{ for any } s_i \ (i \neq j),$$

and if $H : [0,1]^2 \to \mathsf{R}$ is increasing, then[11]

$$(2) \qquad E[H(\tilde{s}) \mid s_1 \leq \tilde{s}_1 \leq s_1',\ s_2 \leq \tilde{s}_2 \leq s_2'] \text{ is increasing in } s_i \text{ and } s_i' \ (i = 1, 2)$$

Given the signal profile $s = (s_1, s_2)$, the expected value of the object to bidder i is denoted $v_i(s)$. We adopt the convention that the first argument of v_i is s_i (own signal) and the second is s_j (the other bidder's signal). The valuation function v_i is the same for every period and $v_i(0,0)$ is normalized to zero. Symmetry implies $f(s) = f(s')$ and $v_i(s) = v_j(s')$ for every s, $s' \in [0,1]^2$ such that $(s_i', s_j') = (s_j, s_i)$.

We say that the values are *private* if $v_i(s) = s_i$ for every s and $i = 1, 2$, and *interdependent* if for $i = 1, 2$ and $j \neq i$, (i) $v_i : [0,1]^2 \to \mathsf{R}_+$ is continuously differentiable, (ii) $\frac{\partial v_i}{\partial s_i}(s) > 0$ and $\frac{\partial v_i}{\partial s_j}(s) > 0$ for every s, and (iii) $v_i(s) \geq v_j(s)$ if $s_i \geq s_j$. Note that under the symmetry assumption, (iii) is equivalent to the standard single-crossing condition: $\frac{\partial v_i}{\partial s_i}(s) \geq \frac{\partial v_j}{\partial s_i}(s)$ for every $s \in [0,1]^2$ such that $v_i(s) = v_j(s)$.

A *stage auction* is any transaction mechanism which determines the allocation of the good and monetary transfer according to a pair of sealed bids submitted by the bidders. The restriction to a seal-bid mechanism is for simplicity. Participation in the stage auction is voluntary so that the set of each bidder's *generalized bids* is expressed as $B = \{N\} \cup \mathsf{R}_+$, where N represents "no participation." The rule of the auction is summarized by measurable mappings ψ_i and p_i ($i = 1, 2$) on the set B^2 of bid profiles $b = (b_1, b_2)$: $\psi_i(b)$ is the probability that bidder i is awarded the good, and $p_i(b)$ is his expected payment to the auctioneer. The functions ψ_i and p_i ($i = 1, 2$) are symmetric ($\psi_i(b) = \psi_j(b')$ and $p_i(b) = p_j(b')$ if $(b_i', b_j') = (b_j, b_i)$), and satisfy the following conditions.

[11] Throughout, the expectation E is with respect to random variables written with tilde.

Assumption 1: (i) A bidder makes no payment when he chooses not to participate: $p_i(b) = 0$ if $b_i = N$.

(ii) There exists a non-random reserve price $R \in \big[0, v_i(1,1)\big)$ such that a bidder may win the object only if he submits a bid at or above R: $\psi_i(b) = 0$ if $b_i \in \{N\} \cup [0, R)$.

(iii) If only one bidder participates and submits bid R, then he wins the object at price R: $\psi_i(b) = 1$ and $p_i(b) = R$ if $b_i = R$ and $b_j = N$.

(iv) There exists a symmetric Nash equilibrium in the (Bayesian) game in which each bidder's strategy is a mapping $\zeta_i : [0,1] \to B$ and payoff function is

$$E\Big[\psi_i\big(\zeta_i(\tilde{s}_i), \zeta_j(\tilde{s}_j)\big)\, v_i(\tilde{s}) - p_i\big(\zeta_i(\tilde{s}_i), \zeta_j(\tilde{s}_j)\big)\Big].$$

Assumption 1 holds for most standard auctions including the first- and second-price auctions. Let g^0 be the (*ex ante*) symmetric Nash equilibrium payoff to each bidder in the stage auction as described in Assumption 1(iv). Also, let g^* be the expected payoff to each bidder under truthful information sharing and efficient allocation with bidder i winning the object at price R if and only if $s_i > s_j$ and $v_i(s) > R$:

$$g^* = E\big[1_{\{\tilde{s}_i > \tilde{s}_j,\, v_i(\tilde{s}) > R\}}\, \{v_i(\tilde{s}) - R\}\big].$$

Clearly, $2g^*$ gives the first-best joint collusive payoff, and the bidders see a potential gain from collusion if $g^0 < g^*$. This is the case to be studied in what follows.

Collusion in the repeated auction takes the following form: At the beginning of each period, the two bidders report their private signals s_i to the center. Upon receiving the report profile $\hat{s} = (\hat{s}_1, \hat{s}_2) \in [0,1]^2$, the center chooses instruction to each bidder i on what (generalized) bid to submit in the stage auction.

In general, the bidders may report a false signal, and/or disobey the instruction. Bidder i's *reporting rule* $\lambda_i : [0,1] \to [0,1]$ chooses report $\hat{s}_i$ as a function of signal s_i, and his *bidding rule* $\mu_i : [0,1]^2 \times B \to B$ chooses bid b_i in the stage action as a function of his signal, report and instruction. The reporting rule is *honest* if it always reports the true signal, and the bidding rule is *obedient* if it always obeys the instruction. Denote by λ_i^* and μ_i^* bidder i's honest reporting rule and obedient action rule, respectively.

For simplicity, we assume that the (generalized) bids in the stage auction are observ-

able to every party including the center.[12] The observability of bids implies that a bidder's deviation can be classified into two types: A bidder commits an *observable deviation* when he chooses a bid different from the instruction given to him, and commits an *unobservable deviation* when he reports a false signal.

The *center* is formally a communication device as formulated by Forges [6] and Myerson [16]. Its choice of instructions to the bidders given their reports is captured by an *instruction rule* $q = (q_1, q_2) : [0,1]^2 \to B^2$: $q_i(\hat{s})$ is the instruction to bidder i when the report profile is $\hat{s}$. Let $g_i(\lambda, \mu, q)$ denote bidder i's stage payoff resulting from any profile (λ, μ, q) of reporting and bidding rules $(\lambda, \mu) = (\lambda_1, \mu_1, \lambda_2, \mu_2)$ and instruction rule q. The instruction rule q is *one-shot incentive compatible* (one-shot IC) if neither bidder has an incentive to misreport his signal: $g_i(\lambda^*, \mu^*, q) \geq g_i(\lambda_i, \lambda_j^*, \mu^*, q)$ for any reporting rule λ_i, $i = 1$, 2, and $j \neq i$. Note that one-shot incentive compatibility refers only to the incentive in reporting and presumes bidders' obedience to the given instructions. In particular, the instruction rule that instructs bidders to play the one-shot Nash equilibrium of the stage auction is one-shot IC. Lemma 1 in the next section identifies some other instruction rules with this property.

Bidder i's *communication history* in period t in the repeated auction game is the sequence of his reports and instructions in periods $1, \ldots, t-1$. On the other hand, bidder i's *private history* in period t is the sequence of his private signals s_i in periods $1, \ldots, t-1$. Furthermore, the *public history* in period t is a sequence of instruction rules used by the center in periods $1, \ldots, t$ and (generalized) bid profiles in the stage auctions in periods $1, \ldots, t-1$.

Bidder i's (pure) *strategy* σ_i in the repeated auction chooses the pair (λ_i, μ_i) of reporting and bidding rules in each period t as a function of his communication and private histories in t, and the public history in t. Let σ_i^* be bidder i's *honest and obedient* strategy which plays the pair (λ_i^*, μ_i^*) of the honest reporting rule and obedient bidding rule for all histories.

The *collusion scheme* τ describes the center's choice of an instruction rule in every

[12]The conclusions in Sections 3-5 hold if only the identity of the winner is publicly announced by the auctioneer. When we take the interpretation that the center is a simple communication device which does not have any monitoring function, we can replicate the same results by letting the bidders report the outcome of each stage auction to the center.

period as a function of communication and public histories. At the beginning of each period, it publicly informs the bidders which instruction rule is used in that period.

Our analysis will focus on the following class of "grim-trigger" collusion schemes with two phases: The game starts in the *collusion phase,* and reverts to the *punishment phase* forever if and only if there is an observable deviation by either bidder in the sense described above. In the punishment phase, the bidders are instructed to play the one-shot Nash equilibrium of the stage auction specified in Assumption 1(iv).

A collusion scheme τ in this class is *static* if it chooses the same instruction rule in every period during the collusion phase independent of the history, and is *dynamic* otherwise. Furthermore, the collusion scheme τ employs *bid rotation* if no more than one bidder is instructed to participate in each stage auction during the collusion phase.

Let $\delta < 1$ be the bidders' common discount factor, and $\Pi_i(\sigma, \tau, \delta)$ be bidder i's average discounted payoff (normalized by $(1-\delta)$) in the repeated game under the strategy profile (σ, τ). The collusion scheme τ is an *equilibrium* if the pair $\sigma^* = (\sigma_1^*, \sigma_2^*)$ of honest and obedient strategies constitutes a *perfect public equilibrium* of the repeated game: σ_i^* is optimal against (σ_j^*, τ) after any public history of the game.[13] It follows from the definition that if τ is an equilibrium static collusion scheme, then its instruction rule in the collusion phase is one-shot IC.

3. A Dynamic Bid Rotation Scheme

Let q^* be the efficient instruction rule that instructs bidder i to (i) bid R if his report $\hat{s}_i$ is higher than j's report $\hat{s}_j$, *and* if his valuation $v_i(\hat{s})$ (computed from the report profile $\hat{s}$) exceeds R, and (ii) stay out otherwise:

$$q_i^*(\hat{s}) = \begin{cases} R & \text{if } \hat{s}_i > \hat{s}_j \text{ and } v_i(\hat{s}) > R, \\ N & \text{otherwise.} \end{cases}$$

Clearly, the (*ex ante*) payoff $g_i(\lambda^*, \mu^*, q^*)$ associated with q^* equals the first-best level g^* although q^* is not one-shot IC.

Consider next the asymmetric instruction rule q^i that "favors" bidder j over bidder i as follows: Bidder j is instructed to (i) bid R if his valuation $v_j(\hat{s})$ exceeds R, and (ii) stay out otherwise. On the other hand, bidder i is instructed to (i) bid R if his valuation $v_i(\hat{s})$

[13] See Fudenberg *et al.* [7].

exceeds R *and* if bidder j's valuation v_j would not exceed R even when i's signal were 1, and (ii) stay out otherwise:

$$q_i^i(\hat{s}) = \begin{cases} R & \text{if } v_i(\hat{s}) > R \geq v_j(\hat{s}_j, 1), \\ N & \text{otherwise,} \end{cases} \qquad q_j^i(\hat{s}) = \begin{cases} R & \text{if } v_j(\hat{s}) > R, \\ N & \text{otherwise.} \end{cases}$$

We refer to bidder j as the primary bidder under q^i and bidder i as the secondary bidder. Let $\bar{g} = g_j(\lambda^*, \mu^*, q^i)$ and $\underline{g} = g_i(\lambda^*, \mu^*, q^i)$ be the expected stage payoffs to the primary and secondary bidders, respectively, under q^i. By definition, $\bar{g} > \underline{g}$.

Lemma 1. *q^i is one-shot IC.*

Proof: See the Appendix.

Since $2g^*$ is the first-best joint collusive payoff, it can be readily verified that

$$2g^* > \bar{g} + \underline{g}, \tag{3}$$

where the strict inequality is the consequence of the full support of the density function f and $R < v_i(1,1)$. Let τ^d be a dynamic bid rotation scheme such that:

a) The collusion phase consists of three subphases S, A_1 and A_2: S is the original *symmetric* phase where the efficient instruction rule q^* is used, while A_i is the *asymmetric adjustment* phase where the instruction rule q^i is used.

b) Play begins in the symmetric phase S. After each period in phase S, transition to the asymmetric phase A_i ($i = 1, 2$) takes place with probability $\omega_i(\hat{s})$, which is a function of the reported signals in the current period alone and given by

$$\omega_i(\hat{s}) = \begin{cases} x(\hat{s}_i) & \text{if } \hat{s}_i > \hat{s}_j, \\ 0 & \text{otherwise,} \end{cases}$$

for some increasing function $x : [0,1] \to [0,1]$. Play stays in phase S with probability $1 - \omega_1(\hat{s}) - \omega_2(\hat{s})$.

c) Each asymmetric phase A_i lasts exactly for m periods and then play returns to S.

It should be noted that the transition probability $x(\hat{s}_i)$ depends only on the higher of the two reports. There is a clear connection between the above collusion scheme and the one with side transfer in McAfee and McMillan [14]. In McAfee and McMillan [14], each

bidder is discouraged from overstating his signal by the transfer payment that is required from the bidder with the higher report. On the other hand, the deterrent in the above scheme is the possibility of a lower continuation payoff for such a bidder. This is a natural modification of the side transfer scheme in view of the substitutability of continuation payoffs for monetary transfer in repeated games.

Write $v_i(s_i, s_i) = \hat{v}_i(s_i)$ and recall $\hat{v}_i(0) = 0$ by normalization. Let $r \in [0, 1]$ be the (unique) signal such that $\hat{v}_i(r) = R$. Define

$$z_i(\beta) = \frac{f_i(\beta \mid \beta)}{F_i(\beta \mid \beta)}.$$

It can be seen that given any transition probability function x, a bidder's payoff $u^d = \Pi_i(\sigma^*, \tau^d, \delta)$ from the dynamic bid rotation scheme τ^d satisfies the following recursive relationship:

$$\begin{aligned} u = (1-\delta)\, g^* + \delta\, E\Big[& 1_{\{\tilde{s}_i > \tilde{s}_j\}} \left\{ x(\tilde{s}_i)\, \underline{u} + (1 - x(\tilde{s}_i))\, u \right\} \\ & + 1_{\{\tilde{s}_i < \tilde{s}_j\}} \left\{ x(\tilde{s}_j)\, \bar{u} + (1 - x(\tilde{s}_j))\, u \right\} \Big], \end{aligned} \tag{4}$$

where $\bar{u} = (1 - \delta^m)\, \bar{g} + \delta^m\, u$ and $\underline{u} = (1 - \delta^m)\, \underline{g} + \delta^m\, u$. On the other hand, x is obtained as a solution to the incentive compatibility condition for truth-telling in phase S. As seen in the Appendix, the incentive condition reduces to a linear differential equation which has the payoff u^d as a parameter. By substituting its solution into (4) and simplifying, we obtain the following equation of $u = u^d$:

$$\varphi(u) \equiv u - g^* + \frac{2u - \bar{g} - \underline{g}}{u - \underline{g}}\, y(u) = 0, \tag{5}$$

where $y : (\underline{g}, \infty) \to \mathsf{R}_{++}$ is defined by

$$y(u) = \int_r^1 \int_0^{s_i} \int_r^{s_i} \{\hat{v}_i(\beta) - R\}\, z_i(\beta)\, e^{-\frac{\bar{g} - \underline{g}}{u - \underline{g}} \int_\beta^{s_i} z_i(\gamma)\, d\gamma}\, d\beta\, f(s)\, ds_j\, ds_i.$$

Note that the continuity of y implies that of $\varphi : (\underline{g}, \infty) \to \mathsf{R}$. The following is our main theorem.

Theorem 1. *Assume that the values are either private or interdependent. If $\varphi(u) = 0$ has a solution u^d strictly greater than g^0, then for a sufficiently large discount factor δ,*

the dynamic bid rotation scheme τ^d is an equilibrium for some $x(\cdot)$ (transition probability function) and m (duration of phase A_i), and yields payoff u^d.

Proof: See the Appendix.

It should be noted that (5) and its solution u^d are independent of δ although low discounting is required for τ^d to be an equilibrium. By construction, note that the bidders' overall payoff u^d in the dynamic bid rotation scheme is a convex combination of their (*ex ante*) stage payoff g^* in phase S, and the average of their stage payoffs in phases A_1 and A_2. (Note that A_1 and A_2 are equally likely *ex ante*.) While $\varphi(u) = 0$ cannot be solved analytically in general, it is possible to bound its solution from below as follows. Let

$$C = \int_r^1 \{\hat{v}_i(s_i) - R\}\, F_j(s_i \mid s_i)\, f_i(s_i)\, ds_i.$$

Then it is shown in the Appendix that there exists a solution u^d to (5) which satisfies

$$u^d > L \equiv \frac{\bar{g} - \underline{g}}{\bar{g} - \underline{g} + 2C}\, g^* + \frac{2C}{\bar{g} - \underline{g} + 2C}\, \frac{\bar{g} + \underline{g}}{2}. \tag{6}$$

Note that we have $L > (\bar{g} + \underline{g})/2$ since $g^* > (\bar{g} + \underline{g})/2$ by (3). In particular, (6) shows that the weight on the efficient payoff g^* in L is an increasing function of the *gap* between the two asymmetric payoffs $\bar{g}$ and $\underline{g}$. With $\bar{g} + \underline{g}$ fixed, therefore, the lower bound on the efficiency of the dynamic scheme τ^d is increased when we take the payoffs in the two asymmetric phases farther apart.[14] The following corollary is an immediate consequence of the above observation.

Corollary 1. *Assume that the values are either private or interdependent. If $g^0 \leq L$, then for a sufficiently large discount factor δ, the dynamic bid rotation scheme τ^d is an equilibrium for some $x(\cdot)$ and m.*

Example 1: Suppose that the stage auction is second-price sealed-bid with the reserve price R equal to zero. In this case, it is well known that the bidding function in the one-shot Nash equilibrium is given by $\zeta^0(s_i) = \hat{v}_i(s_i)$. Suppose further that the valuation function v_i has the linear form $v_i(s) = cs_i + (1-c)s_j$, where $c \in [1/2, 1]$. As seen, the

[14] Conversely, this also suggests the following: While we may specify any other (static or dynamic) path of play in phase A_i, the performance of τ^d cannot be increased by raising just the efficiency (*e.g.*, $\bar{g} + \underline{g}$) of that phase. It is important to maintain the gap $\bar{g} - \underline{g}$.

values are private if $c = 1$ and interdependent otherwise. The one-shot Nash equilibrium payoff equals

$$g^0 = \int_{\{s:\, s_i > s_j\}} \{v_i(s) - \zeta^0(s_j)\}\, f(s)\, ds = c \int_{\{s:\, s_i > s_j\}} (s_i - s_j) f(s)\, ds,$$

whereas $\underline{g} = 0$ and

$$\bar{g} = \int_{[0,1]^2} \{c s_i + (1-c) s_j\}\, f(s)\, ds.$$

From these, we can verify that

$$\frac{1}{2}\,(\underline{g} + \bar{g}) - g^0 = \frac{1}{2} \int_{[0,1]^2} \{s_j - c|s_j - s_i|\}\, f(s)\, ds.$$

Hence, the sufficient condition in Corollary 1 holds if $E[\tilde{s}_j] \geq cE[|\tilde{s}_j - \tilde{s}_i|]$. With f fixed, therefore, if this inequality holds under private values ($c = 1$), then it also holds under interdependent values ($c < 1$). Furthermore, since $E[\tilde{s}_j] > \frac{1}{2}\, E[|\tilde{s}_i - \tilde{s}_j|]$ for any f as can be readily verified, there exists $\bar{c} > 1/2$ such that if $c \leq \bar{c}$, then there exists an equilibrium dynamic bid rotation scheme.

The last observation in the above example has the following simple generalization. Suppose that the two bidders have almost common values so that their valuation of the object is similar for every signal profile. It then follows that any allocation of the object between them leads to almost the same level of efficiency. In particular, the allocation in phase A_i described above is almost (first-best) efficient, and so is the dynamic bid rotation scheme τ^d. Formally, given any $\epsilon > 0$, we say that the values are *ϵ-common* if $\sup_{s \in [0,1]^2} |v_1(s) - v_2(s)| < \epsilon$.

Corollary 2. *Let $\epsilon > 0$ be given, and assume that the values are interdependent and ϵ-common. If the dynamic bid rotation scheme τ^d is an equilibrium, then it is ϵ-efficient in the sense that its payoff u^d satisfies $u^d \geq g^* - \epsilon$.*

Proof: See the Appendix. //

In particular, when the stage auction is either first- or second-price sealed-bid, the one-shot Nash equilibrium payoff g^0 of an ϵ-common value model is bounded away from g^* in the limit as $\epsilon \to 0$. For $\epsilon \geq 0$ small, therefore, $u^d > g^0$ holds by Corollary 2 so that

by Corollary 1, the dynamic bid rotation scheme τ^d is in fact an (ϵ-efficient) equilibrium for δ close to one.[15]

4. Dynamic vs. Static Collusion Schemes

This section compares the performance of the dynamic bid rotation scheme described in Section 3 with that of static collusion schemes. The characterization of the optimal static collusion scheme is available only in the independent private values (IPV) environment, and our primary focus is on this case. A brief discussion of a more general environment is given at the end of this section. With IPV, we show that the dynamic scheme yields a strictly higher payoff to a bidder than the optimal static scheme as identified by McAfee and McMillan [14].

As a first step, Theorem 2 below states that the dynamic bid rotation scheme is in fact an equilibrium for δ close to one when the stage auction is either first-price or second-price. Recall that the reserve price R is allowed to be any number in the interval $\big[0, v_i(1,1)\big)$.

Theorem 2. *Assume independent private values (IPV). Suppose that the stage auction is either first-price or second-price sealed-bid. Then for a sufficiently large discount factor δ, the dynamic bid rotation scheme τ^d is an equilibrium for some $x(\cdot)$ and m, and its payoff u^d is strictly higher than both $(\bar{g} + \underline{g})/2$ and g^0.*

Proof: See the Appendix. //

McAfee and McMillan [14] identify the highest payoff achieved by a collusion scheme with no side transfer for one-shot first-price sealed-bid auctions in the IPV environment. Their theorem applies directly to the current repeated game framework and yields the characterization of the most efficient static collusion scheme. Let

$$h_i(s_i) = \frac{1 - F_i(s_i)}{f_i(s_i)}$$

be the inverse hazard rate of s_i. The following theorem is stated without a proof as it is a straightforward application of Theorem 1 of McAfee and McMillan [14].

[15] With ϵ-common values, the static bid rotation scheme that randomly allocates the object with probability 1/2 to each bidder during the collusion phase tends to be an equilibrium as well. As seen in Corollary 4, however, such a scheme is dominated by a dynamic scheme.

Theorem 3. *Assume independent private values (IPV). Suppose that the stage auction is first-price sealed-bid. Let τ^s be the most efficient equilibrium static collusion scheme without side transfer for a sufficiently large discount factor δ. Then τ^s is a grim-trigger scheme, and its payoff u^s is as follows:*
(i) If $h_i(\cdot)$ is (weakly) increasing, then u^s equals the one-shot equilibrium payoff g^0.
(ii) If $h_i(\cdot)$ is (weakly) decreasing, then u^s equals $(\bar{g} + \underline{g})/2$.

Note that the instruction rule used in a static collusion scheme must be one-shot IC since no incentive for truthful reporting can be provided through the adjustment in continuation payoffs. McAfee and McMillan [14] show that the efficient static scheme τ^s when $h_i(\cdot)$ is decreasing (case (ii) above) is described as follows: The (mixed) instruction rule $\tilde{q} : [0,1]^2 \to \Delta B^2$ used in the collusion phase is such that

$$(\tilde{q}_i(\hat{s}), \tilde{q}_j(\hat{s})) = \begin{cases} (R, N) & \text{if } \hat{s}_i > R \geq \hat{s}_j, \\ \frac{1}{2}(N, R) + \frac{1}{2}(R, N) & \text{if } \hat{s}_i,\ \hat{s}_j > R, \\ (N, N) & \text{if } \hat{s}_i,\ \hat{s}_j \leq R. \end{cases}$$

In other words, the representative bidder is chosen at random with probability one-half when both valuations exceed R. Clearly, such an instruction rule is one-shot IC. Following any deviation from the instruction, play reverts to the punishment phase where the bidders are instructed to play the one-shot Nash equilibrium ζ^0.

Comparison of Theorems 2 and 3 immediately reveals that when $h_i(\cdot)$ is monotone, the dynamic bid rotation scheme τ^d outperforms any static bid rotation scheme. The following corollary summarizes this observation.

Corollary 3. *Assume independent private values (IPV). Suppose that the stage auction is first-price sealed-bid, and that $h_i(\cdot)$ is monotone. Then for a sufficiently large discount factor δ, the dynamic bid rotation scheme τ^d is an equilibrium for some $x(\cdot)$ and m, and yields a strictly higher payoff than any equilibrium static bid rotation scheme.*

The intuition behind Corollary 3 for a decreasing h_i (case (ii)) is as follows: The instruction rule $\tilde{q}$ described above equals a convex combination of q^i and q^j defined in the previous section: $\tilde{q} = \frac{1}{2} q^i + \frac{1}{2} q^j$. It follows that the bidder's payoff in the optimal static scheme exactly equals the average of his payoffs in phases A_1 and A_2 of the dynamic scheme τ^d. Since the allocation in phase S is efficient, the bidder's overall payoff in τ^d is strictly higher.

Example 2: Suppose that the stage-auction is first-price, and that s_i has the uniform distribution over $[0,1]$. Assume that δ is sufficiently large.[16] Since $h_i'(s_i) < 0$ for every s_i, the best static scheme yields $u^s = (\underline{g} + \bar{g})/2$ by Theorem 3. Table 1 below presents the values of u^d, u^s and the one-shot equilibrium payoff g^0 as fractions of the first-best efficiency level g^* for various values of R. It can be seen that τ^d extracts at least close to 90% of the surplus.

R	g^*	$\frac{u^d}{g^*}$	$\frac{u^s}{g^*}$	$\frac{g^0}{g^*}$
.0	.3333	.886	.750	.500
.1	.2835	.899	.786	.571
.2	.2347	.912	.818	.636
.3	.1878	.926	.848	.696
.4	.1440	.938	.875	.750
.5	.1042	.949	.900	.799
.6	.0693	.961	.923	.847
.7	.0405	.981	.944	.889

Table 1

In a general environment without IPV, little is known about the optimal one-shot scheme, and hence no clear-cut comparison of dynamic and static collusion schemes is possible. Even without IPV, however, any static scheme that uses a mixed instruction rule as in case (ii) above is dominated by a dynamic scheme that belongs to the class described in Section 3. Formally, let $\hat{q}^1$ and $\hat{q}^2$ be any pair of (possibly mixed) asymmetric one-shot IC instruction rules such that $\hat{q}^1_1(s) = \hat{q}^2_2(s')$ and $\hat{q}^1_2(s) = \hat{q}^2_1(s')$ if $(s'_1, s'_2) = (s_2, s_1)$, and

$$g_i(\hat{q}^i, \lambda^*, \mu^*) < g_j(\hat{q}^i, \lambda^*, \mu^*) \quad i = 1, 2,\ j \neq i.$$

In other words, bidder 2 is favored over bidder 1 under q^1 and their roles are exactly reversed under q^2. Suppose that $\hat{\tau}^s$ is a symmetric static bid rotation scheme which uses the mixed instruction rule $\hat{q} = \frac{1}{2}\,\hat{q}^1 + \frac{1}{2}\,\hat{q}^2$ in the collusion phase.

Corollary 4. *Assume that the values are either private or interdependent. Suppose that $\hat{\tau}^s$ is a symmetric static bid rotation scheme as described above which yields payoff*

[16] When $R = 0$, for example, τ^d is an equilibrium if $\delta \geq .95$.

$\hat{u}^s < g^*$ to each bidder. If $\hat{\tau}^s$ is an equilibrium, then there exists a dynamic bid rotation scheme $\hat{\tau}^d$ of the class described in Section 3 which is an equilibrium for some δ and yields a strictly higher payoff than $\hat{u}^s$.

Proof: Denote $\underline{g}' = g_i(\hat{q}^i, \lambda^*, \mu^*)$ and $\bar{g}' = g_j(\hat{q}^i, \lambda^*, \mu^*)$ $(i = 1,\ 2,\ j \neq i)$. Let $\hat{\tau}^d$ be a dynamic bid rotation scheme as described in Section 3 with the exception that it uses the instruction rule $\hat{q}^i$ in phase A_i $(i = 1,\ 2)$. It then follows from (6) that its payoff $\hat{u}^d > (\underline{g}' + \bar{g}')/2 = \hat{u}^s$. Since $\hat{u}^s \geq g^0$ by assumption, we also have $\hat{u}^d > g^0$ so that $\hat{\tau}^d$ is an equilibrium for some δ by Corollary 1. //

5. Implicit Communication through Winning Bids

In the previous sections, bidder communication is explicit in the sense that reporting of private signals is done separately from bidding in the stage auction. This section shows that even if there is no explicit communication, collusion may be sustained by *implicit* communication through bids submitted to the stage auction when the identity of the winner as well as his bid is publicly announced by the auctioneer. In other words, communication studied in this section is *signaling* of private information with bids. It is *ex post* in the sense that the messages are exchanged only after the stage auction is concluded, and will be used to determine continuation play only.[17] For simplicity, we assume that the stage auction is first-price sealed-bid, and that a public randomization device is available.

Let $w_i(s_i) = E[v_i(\tilde{s}) \mid \tilde{s}_i = s_i,\ \tilde{s}_j < s_i]$ be the expected value of the object to bidder i with signal s_i conditional on the knowledge that j's signal is lower than s_i. The function w_i is strictly increasing over $(0, 1]$ since for $s_i' > s_i > 0$,

$$\begin{aligned} w_i(s_i) &= \int_0^{s_i} v_i(s_i, s_j)\, f(s_j \mid s_i)\, ds_j \\ &< \int_0^{s_i} v_i(s_i', s_j)\, f(s_j \mid s_i)\, ds_j \\ &= E[v_i(s_i', \tilde{s}_j) \mid \tilde{s}_i = s_i,\ \tilde{s}_j < s_i] \\ &\leq E[v_i(s_i', \tilde{s}_j) \mid \tilde{s}_i = s_i',\ \tilde{s}_j < s_i'] = w_i(s_i'), \end{aligned}$$

where the first inequality follows from the strict monotonicity of v_i, and the second from (2). We make the regularity assumption that the derivative w_i' is continuous. Suppose that

[17] This model is a special case of the general formulation in Section 3 where the instruction to bidder i is independent of j's report.

the reserve price R satisfies $R < w_i(1)$, and let $\rho \in [0,1]$ be the signal such that $w_i(\rho) = R$. Consider the bidding function $\zeta_\epsilon : [0,1] \to B$ such that

$$\zeta_\epsilon(s_i) = \begin{cases} R + \epsilon \displaystyle\int_\rho^{s_i} \{w_i(\alpha) - R\}\, z_i(\alpha)\, e^{-\int_\alpha^{s_i} z_i(\beta)\, d\beta}\, d\alpha & \text{if } s_i \geq \rho \\ N & \text{otherwise.} \end{cases} \tag{7}$$

When $s_i \geq \rho$, partial integration of the right-hand side of (7) yields

$$\zeta_\epsilon(s_i) = (1-\epsilon)\, R + \epsilon\, w_i(s_i) - \epsilon \int_\rho^{s_i} w_i'(\alpha)\, e^{-\int_\alpha^{s_i} z_i(\beta)\, d\beta}\, d\alpha.$$

It follows that $\zeta_\epsilon(s_i) < (1-\epsilon)\, R + \epsilon\, w_i(s_i)$ for $s_i > \rho$. When the bidders bid according to ζ_ϵ, hence, the winning bid is always within $\epsilon\,\{w_i(1) - R\}$ of R. Furthermore, ζ_ϵ is differentiable over $[\rho, 1]$ and satisfies

$$\zeta_\epsilon'(s_i) = z_i(s_i)\{\epsilon\, w_i(s_i) + (1-\epsilon)\, R - \zeta_\epsilon(s_i)\}. \tag{8}$$

Since (8) implies $\zeta_\epsilon'(s_i) > 0$ for $s_i > \rho$ from the above observation, the private signal of the winner (if any) is fully revealed. It can also be seen that when both bidders use ζ_ϵ, i's (*interim*) expected payoff from winning the stage auction with signal $s_i > \rho$ and bid $b_i = \zeta_\epsilon(s_i)$ is strictly positive since it equals $w_i(s_i) - \zeta_\epsilon(s_i) > 0$.

Let $\bar{v}_i(s_i) = E[v_i(\tilde{s}) \mid \tilde{s}_i = s_i]$ be the expected valuation conditional only on one's own signal. The collusion scheme with implicit communication through bids is described as follows:

a) The collusion phase consists of the symmetric phase S, and the asymmetric phases A_1 and A_2. In phase S, bidder i with signal s_i bids $b_i = \zeta_\epsilon(s_i)$. In phase A_i, bidder j is favored over bidder i and bids are determined as follows:

$$b_i = \begin{cases} R & \text{if } \bar{v}_i(s_i) \geq R \\ N & \text{otherwise,} \end{cases} \quad \text{and} \quad b_j = \begin{cases} R + \epsilon & \text{if } \bar{v}_j(s_j) \geq R + \epsilon \\ N & \text{otherwise.} \end{cases}$$

b) Play begins in phase S, and transition to phase A_i takes place with probability $x(b_i)$ (based on the public randomization) if bidder i wins with bid $b_i \in [R, \zeta_\epsilon(1)]$.

c) Each asymmetric phase A_i lasts for m periods and then play returns to S.

In this scheme, deviations are observable if either bidder wins with a bid above $\zeta_\epsilon(1)$ in phase S, or if the primary (resp. secondary) bidder wins with a bid different from

$R+\epsilon$ (resp. R) in phase A_1 or A_2. As before, reversion to the punishment phase takes place following any such deviation. On the other hand, deviations are unobservable if, for example, bidder i bids $b_i = \zeta_\epsilon(\hat{s}_i)$ in phase S when his true type is $s_i \neq \hat{s}_i$. Of course, neither type of deviation should be profitable in equilibrium.

It can be seen that the absence of pre-auction information sharing entails the following inefficiencies: First, when the values are interdependent, the winner's valuation conditional on both bidders' signals could be lower than his bid (or even the reserve price). Second, any participating bidders in phase S must bid strictly above the reserve price R (albeit by a small amount) in order to achieve an efficient allocation. Third, the primary bidder in phase A_1 or A_2 must win with a bid strictly above R in order to avoid a tie.

For any $\epsilon \geq 0$, let

$$\begin{aligned} g^*(\epsilon) &= E\big[1_{\{\tilde{s}_i > \max\{\tilde{s}_j, \rho\}\}}\,\{v_i(\tilde{s}) - \zeta_\epsilon(\tilde{s}_i)\}\big], \\ \bar{g}(\epsilon) &= E\big[1_{\{\overline{v}_i(\tilde{s}_i) \geq R+\epsilon\}}\,\{v_i(\tilde{s}) - R - \epsilon\}\big], \\ \underline{g}(\epsilon) &= E\big[1_{\{\overline{v}_i(\tilde{s}_i) \geq R,\, \overline{v}_j(\tilde{s}_j) < R+\epsilon\}}\,\{v_i(\tilde{s}) - R\}\big]. \end{aligned}$$

When $\epsilon > 0$, $g^*(\epsilon)$ represents each bidder's (ex ante) expected stage payoff in phase S, and $\bar{g}(\epsilon)$ and $\underline{g}(\epsilon)$ represent the (ex ante) expected stage payoffs of the primary and secondary bidders, respectively, in the asymmetric phases. We assume that $\bar{g}$ and $\underline{g}$ are both differentiable as functions of ϵ.

Consider the following equation of u parameterized by $\epsilon < 1$:

$$\varphi(u,\epsilon) \equiv u - g^*(\epsilon) + \frac{2u - \bar{g}(\epsilon) - \underline{g}(\epsilon)}{u - \underline{g}(\epsilon)}\, y(u,\epsilon) = 0, \tag{9}$$

where

$$y(u,\epsilon) = \int_\rho^1 \int_0^{s_i} \int_\rho^{s_i} \{\hat{v}_i(\beta) - (1-\epsilon)\,R - \epsilon\, w_i(\beta)\}\, z_i(\beta)\, e^{-\frac{\bar{g}(\epsilon)-\underline{g}(\epsilon)}{u-\underline{g}(\epsilon)} \int_\beta^{s_i} z_i(\gamma)\, d\gamma}\, d\beta\, f(s)\, ds_j\, ds_i.$$

As in Section 3, (9) characterizes the equilibrium payoff of the above dynamic bid-rotation scheme with implicit communication. Suppose now that $\varphi(u,0) = 0$ (with $\epsilon = 0$) has a solution $u(0)$ strictly greater than the one-shot equilibrium payoff g^0. By the same argument as in Section 3, we have $u(0) > \{\bar{g}(0) + \underline{g}(0)\}/2 > \underline{g}(0)$, and hence $\dfrac{\partial \varphi}{\partial u}(u(0),0) > 0$ as can be readily verified. It then follows from the implicit function theorem that if $\epsilon > 0$ is small, $\varphi(u,\epsilon) = 0$ has a solution $u(\epsilon)$ close to $u(0)$. Theorem 4 describes the conditions for $u(\epsilon)$ to be an equilibrium payoff of the above collusion scheme.

Theorem 4. *Assume that the values are either private or interdependent, and that the stage auction is first-price sealed-bid with the reserve price $R < w_i(1)$. If the equation $\varphi(u, 0) = 0$ has a solution $u(0)$ strictly greater than g^0, then the following holds for a sufficiently large discount factor δ: Given any $\kappa > 0$, there exists a dynamic bid rotation scheme with implicit communication as described above that is an equilibrium and yields payoff $u > u(0) - \kappa$.*

Proof: See the Appendix.

It should be noted that the discount factor δ required is independent of the level of approximation κ. When the values are private or when the reserve price equals zero, there is no inefficiency associated with bidding based only on one's own signal. In this case, therefore, the efficiency loss caused by the absence of information sharing can be made negligible when we take a small $\epsilon > 0$.

Corollary 5. *Suppose that the stage auction is first-price sealed-bid, and that one of the following holds: (i) The values are private and the reserve price $R < w_i(1)$; (ii) The values are interdependent and the reserve price $R = 0$. If the dynamic bid rotation scheme τ^d with explicit communication in Section 3 is an equilibrium and yields payoff u^d (for some δ), then the following holds for a sufficiently large discount factor: Given any $\kappa > 0$, there exists a dynamic bid rotation scheme with implicit communication as described above that is an equilibrium and yields payoff $u > u^d - \kappa$.*

Proof: When the values are private or when $R = 0$, we have $g^*(0) = g^*$, $\bar{g}(0) = \bar{g}$, and $\underline{g}(0) = \underline{g}$, where g^*, $\bar{g}$, and $\underline{g}$ are as defined in Section 3. It follows that (9) with $\epsilon = 0$ is equivalent to (5), and hence that $u^d = u(0)$. The conclusion then follows from Theorem 5. //

Appendix

The following notation is used for the discussion of the interdependent values case in the Appendix. For each $s_i \in [0, 1]$, let $k_i(s_i)$ be the opponent's signal s_j such that $v_i(s_i, s_j) = R$ if there exists any such $s_j \in [0, 1]$. Let $k_i(s_i) = 1$ if $v_i(s_i, s_j) < R$ for every $s_j \in [0, 1]$, and $k_i(s_i) = 0$ if $v_i(s_i, s_j) > R$ for every $s_j \in [0, 1]$. It follows from the

continuity and strict monotonicity of v_i that k_i is well-defined. Also, we have by definition

$$v_i(s_i, k_i(s_i)) \begin{cases} \geq R & \text{if } k_i(s_i) = 0, \\ = R & \text{if } k_i(s_i) \in (0,1), \\ \leq R & \text{if } k_i(s_i) = 1. \end{cases}$$

Furthermore, since v_i is continuously differentiable, $k_i(s_i)$ is continuously differentiable itself at any s_i such that $k_i(s_i) \in (0,1)$ by the implicit function theorem. Figure 1 depicts k_i for a generic (interdependent) valuation function v_i.

Proof of Lemma 1: The conclusion is straightforward if the values are private. We will show below that q^i is one-shot IC for bidder i in the interdependent values case. A similar argument proves the same for bidder j. Let $G_i^i(s_i, \hat{s}_i)$ be bidder i's *interim* stage payoff under q^i when he has signal s_i and reports $\hat{s}_i$.

Let k_i be as defined above, and $\xi \in [0,1]$ denote the minimum of s_j such that $v_j(s_j, 1) \geq R$ (Figure 1). Note first that if $\hat{s}_i$ is such that $k_i(\hat{s}_i) \geq \xi$, then $G_i^i(s_i, \hat{s}_i) = 0$ for any s_i since $v_i(\hat{s}_i, s_j) \geq R$ implies $s_j \geq k_i(\hat{s}_i) \geq \xi$ so that $v_j(s_j, 1) \geq R$. Suppose now that $k_i(\hat{s}_i) < \xi$. In this case, we have

$$G_i^i(s_i, \hat{s}_i) = \int_{k_i(\hat{s}_i)}^{\xi} \{v_i(s) - R\} f_j(s_j \mid s_i)\, ds_j.$$

Since $v_i(s_i, s_j) \geq R$ for any $s_j \geq k_i(s_i)$ and $v_i(s_i, s_j) \leq R$ for any $s_j \leq k_i(s_i)$, it follows from the above equation that $G_i^i(s_i, \hat{s}_i)$ is maximized when $\hat{s}_i = s_i$. //

Proof of Theorem 1: Let $u > g^0$ be a solution to (5). Take any $m \in \mathsf{N}$ such that

$$m > \frac{1}{u - \underline{g}} \int_r^1 \{\hat{v}_i(\alpha) - R\}\, z_i(\alpha)\, d\alpha.$$

We claim that there exists $\underline{\delta} < 1$ such that for any $\delta > \underline{\delta}$,

$$\text{(a1)} \qquad (1-\delta)\{\hat{v}_i(1) - R\} + \delta g^0 < (1 - \delta^m)\, \underline{g} + \delta^m u,$$

and

$$\text{(a2)} \qquad \frac{1-\delta}{\delta(1-\delta^m)(u - \underline{g})} \int_r^1 \{\hat{v}_i(\alpha) - R\}\, z_i(\alpha)\, d\alpha < 1.$$

This is clear for (a1) since $u > g^0$ by assumption. For m as specified, (a2) also holds since

$$\lim_{\delta \to 1-} \frac{1-\delta}{\delta(1-\delta^m)} = \lim_{\delta \to 1-} \frac{1}{\delta(1+\delta+\cdots+\delta^{m-1})} = \frac{1}{m}.$$

Recall that $x(\alpha)$ denotes the probability of transition to phase A_i when i's report α is higher than that of bidder j. Take any $\delta > \underline{\delta}$ and let x be given by

$$\text{(a3)} \quad x(\alpha) = \begin{cases} \dfrac{1-\delta}{\delta(1-\delta^m)(u-\underline{g})} \displaystyle\int_r^\alpha \{\hat{v}_i(\beta) - R\}\, z_i(\beta)\, e^{-\frac{\bar{g}-\underline{g}}{u-\underline{g}} \int_\beta^\alpha z_i(\gamma)\, d\gamma}\, d\beta & \text{if } \alpha > r, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $x(\alpha) \in [0,1]$ by our choice of m and δ so that it is indeed a probability.

In what follows, we fix u, m, $\delta > \underline{\delta}$, x as above and prove the theorem in three steps: Step 1 shows that each bidder's payoff is u when they play the honest and obedient strategy σ_i^* under τ^d. Steps 2 and 3 then prove the non-profitability of observable and unobservable deviations, respectively. By the principle of optimality in dynamic programming, the latter two steps can be accomplished by checking the profitability of one-step deviations. For simplicity, we write $\bar{u} = (1-\delta^m)\,\bar{g} + \delta^m\, u$ and $\underline{u} = (1-\delta^m)\,\underline{g} + \delta^m\, u$ for the expected payoffs of the primary and secondary bidders, respectively, at the beginning of the asymmetric phase A_1 or A_2.

Step 1. $\Pi_i(\sigma^*, \tau^d, \delta) = u$.

Using symmetry, we can rewrite (4) as

$$\text{(a4)} \qquad u = g^* - \frac{\delta(1-\delta^m)}{1-\delta}\,(2u - \bar{g} - \underline{g}) \int_r^1 \int_0^{s_i} x(s_i)\, f(s)\, ds_j\, ds_i.$$

Substitution of (a3) into (a4) shows that the above recursive equation is equivalent to (5). Since u solves (5) by assumption, the desired conclusion follows.

Step 2. No observable deviation (in bidding) is profitable.

When bidder i with any signal or report (whether truthful or not) disobeys the instruction, the maximal instantaneous gain from the deviation is bounded above by $\hat{v}_i(1) - R$ and the continuation payoff equals g^0. On the other hand, the lowest payoff along the path of play equals $\underline{u} = (1-\delta^m)\underline{g} + \delta^m u$ (by Step 1) when phase A_i is just beginning. Hence, (a1) implies that no observable deviation is profitable.

Step 3. No unobservable deviation (in reporting) is profitable.

Since a bidder is made strictly worse off by disobeying the instruction, a profitable deviation is possible only when he misreports his signal and then obeys the instruction. Since the instructions rules q^1 and q^2 are one-shot IC by Lemma 1, bidder i has no incentive to misreport a signal in phases A_i and A_j. Likewise, he has no incentive for misreporting in the punishment phase. It remains to check the incentive for misreporting in phase S.

Let $\pi_i(s_i, \hat{s}_i)$ denote bidder i's *interim* (intertemporal) expected payoff in any period in phase S when he has signal s_i and reports $\hat{s}_i$ while bidder j reports his signal truthfully.

The discussion below assumes interdependent values. Derivation in the case of private values is similar and hence omitted. We can express π_i using the continuation payoffs u, $\underline{u}$ and $\bar{u}$ as follows: For $\hat{s}_i > r$,

$$\begin{aligned}\pi_i(s_i, \hat{s}_i) &= (1-\delta) \int_{k_i(\hat{s}_i)}^{\hat{s}_i} \{v_i(s) - R\} f_j(s_j \mid s_i)\, ds_j \\ &\quad + \delta \Bigg[\int_0^{\hat{s}_i} \big[x(\hat{s}_i)\, \underline{u} + \{1 - x(\hat{s}_i)\}\, u\big] f_j(s_j \mid s_i)\, ds_j \\ &\quad + \int_{\hat{s}_i}^1 \big[x(s_j)\, \bar{u} + \{1 - x(s_j)\}\, u\big] f_j(s_j \mid s_i)\, ds_j \Bigg],\end{aligned} \tag{a5}$$

and for $\hat{s}_i \leq r$,

$$\begin{aligned}\pi_i(s_i, \hat{s}_i) &= \delta \Bigg[\int_0^{\hat{s}_i} \big[x(\hat{s}_i)\, \underline{u} + \{1 - x(\hat{s}_i)\}\, u\big] f_j(s_j \mid s_i)\, ds_j \\ &\quad + \int_{\hat{s}_i}^1 \big[x(s_j)\, \bar{u} + \{1 - x(s_j)\}\, u\big] f_j(s_j \mid s_i)\, ds_j \Bigg].\end{aligned} \tag{a6}$$

Upon simplification, we see that (a5) is equivalent to

$$\begin{aligned}&\pi_i(s_i, \hat{s}_i) \\ &= (1-\delta) \int_{k_i(\hat{s}_i)}^{\hat{s}_i} \{v_i(s) - R\} f_j(s_j \mid s_i)\, ds_j \\ &+ \delta \Bigg[u - (u - \underline{u})\, x(\hat{s}_i)\, F_j(\hat{s}_i \mid s_i) + (\bar{g} - \bar{u}) \int_{\hat{s}_i}^1 x(s_j)\, f_j(s_j \mid s_i)\, ds_j \Bigg].\end{aligned}$$

Since k_i is differentiable almost everywhere, we can differentiate the above with respect to

$\hat{s}_i$ to obtain

$$
\begin{aligned}
&\frac{\partial \pi_i}{\partial \hat{s}_i}(s_i, \hat{s}_i) \\
&= (1-\delta)\Big[\{v_i(s_i, \hat{s}_i) - R\}\, f_j(\hat{s}_i \mid s_i) - k_i'(\hat{s}_i)\{v_i(s_i, k_i(\hat{s}_i)) - R\}\, f_j(k_i(\hat{s}_i) \mid s_i)\Big] \\
&- \delta\Big[(u-\underline{u})\big\{x'(\hat{s}_i)F_j(\hat{s}_i \mid s_i) + x(\hat{s}_i)f_j(\hat{s}_i \mid s_i)\big\} + (\bar{u}-u)x(\hat{s}_i)f_j(\hat{s}_i \mid s_i)\Big] \\
&= (1-\delta)\Big[\{v_i(s_i, \hat{s}_i) - R\}\, f_j(\hat{s}_i \mid s_i) - k_i'(\hat{s}_i)\{v_i(s_i, k_i(\hat{s}_i)) - R\}\, f_j(k_i(\hat{s}_i) \mid s_i)\Big] \\
&- \delta\Big[(u-\underline{u})\, x'(\hat{s}_i)F_j(\hat{s}_i \mid s_i) + (\bar{u}-\underline{u})\, x(\hat{s}_i)f_j(\hat{s}_i \mid s_i)\Big].
\end{aligned}
$$

On the other hand, x given in (a3) satisfies the following linear differential equation:

$$
\text{(a7)} \qquad x'(\alpha) + \frac{\bar{g}-\underline{g}}{u-\underline{g}}\, z_j(\alpha)\, x(\alpha) = \begin{cases} \dfrac{1-\delta}{\delta(u-\underline{u})}\, \{v_i(\alpha,\alpha) - R\}\, z_j(\alpha) & \text{if } \alpha > r, \\ 0 & \text{otherwise.} \end{cases}
$$

(Heuristically, (a3) is derived as the (unique) solution to (a7) with the initial condition $x(r) = 0$. (a7) is obtained as follows: The optimality of truth-telling implies the following first-order condition for $s_i > r$:

$$
\begin{aligned}
\frac{\partial \pi_i}{\partial \hat{s}_i}(s_i, s_i) = (1-\delta)&\Big[\{\hat{v}_i(s_i) - R\}\, f_j(s_i \mid s_i) - k_i'(s_i)\{v_i(s_i, k_i(s_i)) - R\}\, f_j(k_i(s_i) \mid s_i)\Big] \\
&- \delta\Big[(u-\underline{u})\, x'(s_i)F_j(s_i \mid s_i) + (\bar{u}-\underline{u})\, x(s_i)f_j(s_i \mid s_i)\Big] = 0.
\end{aligned}
$$

The first line of (a7) follows from the fact that $k'(s_i)\{v_i(s_i, k_i(s_i)) - R\} = 0$ almost everywhere. The similar first-order condition for $s_i \le r$ yields the second line of (a7).)

To verify that truth-telling is (globally) optimal under (a3), we rewrite $\dfrac{\partial \pi_i}{\partial \hat{s}_i}(s_i, \hat{s}_i)$ using (a7) as follows when $\hat{s}_i > r$:

$$
\begin{aligned}
\frac{\partial \pi_i}{\partial \hat{s}_i}(s_i, \hat{s}_i) &= (1-\delta)\Big[\big\{v_i(s_i, \hat{s}_i) - R\big\}\, f_j(\hat{s}_i \mid s_i) - k_i'(\hat{s}_i)\, \big\{v_i(s_i, k_i(\hat{s}_i)) - R\big\}\, f_j(k_i(\hat{s}_i) \mid s_i)\Big] \\
&+ \delta(\bar{u}-\underline{u})F_j(\hat{s}_i \mid s_i)\, x(\hat{s}_i)\left[z_j(\hat{s}_i) - \frac{f_j(\hat{s}_i \mid s_i)}{F_j(\hat{s}_i \mid s_i)}\right] \\
&- (1-\delta)\, \big\{\hat{v}_i(\hat{s}_i) - R\big\}\, F_j(\hat{s}_i \mid s_i)\, z_j(\hat{s}_i) \\
&= (1-\delta)\, f_j(\hat{s}_i \mid s_i)\, \{v_i(s_i, \hat{s}_i) - v_i(\hat{s}_i, \hat{s}_i)\} \\
&+ (1-\delta)\left[\frac{f_j(\hat{s}_i \mid s_i)}{F_j(\hat{s}_i \mid s_i)} - z_j(\hat{s}_i)\right]\left[\hat{v}_i(\hat{s}_i) - R - \frac{\delta}{1-\delta}\, (\bar{u}-\underline{u})x(\hat{s}_i)\right] F_j(\hat{s}_i \mid s_i) \\
&- (1-\delta)k_i'(\hat{s}_i)\, \big\{v_i(s_i, k_i(\hat{s}_i)) - R\big\}\, f_j(k_i(\hat{s}_i) \mid s_i)
\end{aligned}
$$

Note that it follows from (a3) that

$$\hat{v}_i(\hat{s}_i) - R - \frac{\delta}{1-\delta}(\bar{u} - \underline{u})x(\hat{s}_i) > 0$$

for any $\hat{s}_i > r$. If $\hat{s}_i > s_i$, then $k_i'(\hat{s}_i)\left\{v_i(s_i, k_i(\hat{s}_i)) - R\right\} \geq 0$, $v_i(s_i, \hat{s}_i) - v_i(\hat{s}_i, \hat{s}_i) < 0$, and $\dfrac{f_j(\hat{s}_i \mid s_i)}{F_j(\hat{s}_i \mid s_i)} - z_j(\hat{s}_i) \leq 0$. Therefore, $\dfrac{\partial \pi_i}{\partial \hat{s}_i}(s_i, \hat{s}_i) \leq 0$. Similarly, $\dfrac{\partial \pi_i}{\partial \hat{s}_i}(s_i, \hat{s}_i) \geq 0$ when $\hat{s}_i \in (r, s_i)$. When $\hat{s}_i < r$, we can similarly show from (a6) and (a7) that $\dfrac{\partial \pi_i}{\partial \hat{s}_i}(s_i, \hat{s}_i) \leq 0$ (resp. ≥ 0) for $\hat{s}_i > s_i$ (resp. $< s_i$). In either case, $\pi_i(s_i, \cdot)$ is single-peaked at $\hat{s}_i = s_i$. This completes the proof of the theorem. //

Proof of Inequality (6): We first show that (5) has a solution u^d between $(\bar{g} + \underline{g})/2$ and g^*. Since $y(u) > 0$ for any $u > \underline{g}$, it follows from (3) that

$$\varphi(g^*) = \frac{2g^* - \bar{g} - \underline{g}}{g^* - \underline{g}}\, y(g^*) > 0 \quad \text{and} \quad \varphi\Big(\frac{\bar{g} + \underline{g}}{2}\Big) = \frac{\bar{g} + \underline{g}}{2} - g^* < 0.$$

By the intermediate value theorem, hence, there exists $u^d \in \big((\bar{g} + \underline{g})/2, g^*\big)$ that solves $\varphi(u) = 0$. Take any such solution u^d. For $s_i > r$, we then have

$$\begin{aligned}
&\int_r^{s_i} \{\hat{v}_i(\beta) - R\}\, z_i(\beta)\, e^{-\frac{\bar{g}-\underline{g}}{u^d - \underline{g}} \int_\beta^{s_i} z_i(\gamma)\, d\gamma}\, d\beta \\
&= \left[\{\hat{v}_i(\beta) - R\}\, \frac{u^d - \underline{g}}{\bar{g} - \underline{g}}\, e^{-\frac{\bar{g}-\underline{g}}{u^d - \underline{g}} \int_\beta^{s_i} z_i(\gamma)\, d\gamma}\right]_r^{s_i} \\
&- \int_r^{s_i} \hat{v}_i'(\beta)\, \frac{u^d - \underline{g}}{\bar{g} - \underline{g}}\, e^{-\frac{\bar{g}-\underline{g}}{u^d - \underline{g}} \int_\beta^{s_i} z_i(\gamma)\, d\gamma}\, d\beta \\
&\leq \{\hat{v}_i(s_i) - R\}\, \frac{u^d - \underline{g}}{\bar{g} - \underline{g}}.
\end{aligned}$$

It follows that

$$y(u^d) \leq \int_r^1 \int_0^{s_i} \{\hat{v}_i(s_i) - R\}\, \frac{u^d - \underline{g}}{\bar{g} - \underline{g}}\, f(s)\, ds_j\, ds_i = \frac{u^d - \underline{g}}{\bar{g} - \underline{g}}\, C,$$

where C is as defined in the text. Therefore, (5) implies that

$$u^d \geq g^* - \frac{2u^d - \bar{g} - \underline{g}}{\bar{g} - \underline{g}}\, C.$$

Solving this for u^d, we obtain (6). //

Proof of Corollary 2: Note that

$$\begin{aligned}
2g^* - \bar{g} &= E[1_{\{\tilde{s}_i > \tilde{s}_j,\, v_i(\tilde{s}) > R\}} \{v_i(\tilde{s}) - R\}] + E[1_{\{\tilde{s}_i < \tilde{s}_j,\, v_j(\tilde{s}) > R\}} \{v_j(\tilde{s}) - R\}] \\
&\quad - E[1_{\{\tilde{s}_i > \tilde{s}_j,\, v_i(\tilde{s}) > R\}} \{v_i(\tilde{s}) - R\}] - E[1_{\{\tilde{s}_i < \tilde{s}_j,\, v_i(\tilde{s}) > R\}} \{v_i(\tilde{s}) - R\}] \\
&= E[1_{\{\tilde{s}_i < \tilde{s}_j,\, v_j(\tilde{s}) > R\}} \{v_j(\tilde{s}) - R\}] - E[1_{\{\tilde{s}_i < \tilde{s}_j,\, v_i(\tilde{s}) > R\}} \{v_i(\tilde{s}) - R\}] \\
&= E[1_{\{v_j(\tilde{s}) > R \geq v_i(\tilde{s})\}} \{v_j(\tilde{s}) - R\}] + E[1_{\{\tilde{s}_i < \tilde{s}_j,\, v_i(\tilde{s}) > R\}} \{v_j(\tilde{s}) - v_i(\tilde{s})\}] \\
&\leq E[1_{\{v_j(\tilde{s}) < R + \epsilon\}} \{v_j(\tilde{s}) - R\}] + E[1_{\{\tilde{s}_i < \tilde{s}_j,\, v_i(\tilde{s}) > R\}} \{v_j(\tilde{s}) - v_i(\tilde{s})\}] \\
&< \epsilon\, P(v_j(\tilde{s}) < R + \epsilon) + \epsilon\, P(\tilde{s}_i < \tilde{s}_j,\, v_i(\tilde{s}) > R) \\
&\leq 2\epsilon.
\end{aligned}$$

Therefore, $\bar{g} > 2(g^* - \epsilon)$, and hence $(\bar{g} + \underline{g})/2 > g^* - \epsilon$. The desired conclusion then follows from (6). //

Proof of Theorem 2: It is clear from the discussion after Theorem 1 that $u^d > (\bar{g} + \underline{g})/2$. It will be shown below that $u^d > g^0$. It follows from (a7) that for $\alpha > R$,

$$x(\alpha) = \frac{1-\delta}{\delta(1-\delta^m)(\bar{g} - \underline{g})}\,(\alpha - R) - \frac{u - \underline{g}}{(\bar{g} - \underline{g}) z_j(\alpha)} x'(\alpha).$$

Substituting this by noting $g^* = \int_R^1 (\alpha - R) F_i(\alpha) f_i(\alpha)\, d\alpha$, we have

$$\begin{aligned}
&\int_R^1 x(\alpha)\, F_i(\alpha)\, f_i(\alpha)\, d\alpha \\
&= \frac{1-\delta}{\delta(1-\delta^m)(\bar{g} - \underline{g})}\, g^* - \frac{u - \underline{g}}{\bar{g} - \underline{g}} \int_R^1 F_i(\alpha)^2\, x'(\alpha)\, d\alpha \\
&= \frac{1-\delta}{\delta(1-\delta^m)(\bar{g} - \underline{g})}\, g^* - \frac{u - \underline{g}}{\bar{g} - \underline{g}} \left[x(1) - 2 \int_R^1 x(\alpha)\, F_i(\alpha)\, f_i(\alpha)\, d\alpha \right],
\end{aligned}$$

where the second equality follows from integration by parts. It follows that

$$(\bar{g} + \underline{g} - 2u) \int_R^1 x(\alpha)\, F_i(\alpha)\, f_i(\alpha)\, d\alpha = \frac{1-\delta}{\delta(1-\delta^m)}\, g^* - (u - \underline{g})\, x(1).$$

By (a4), $\varphi(u)$ equals

$$\begin{aligned}
\text{(a8)} \qquad \varphi(u) &= u - g^* + (2u - \bar{g} - \underline{g})\, \frac{\delta(1-\delta^m)}{1-\delta} \int_R^1 x(\alpha)\, F_i(\alpha)\, f_i(\alpha)\, d\alpha \\
&= u - 2g^* + \frac{\delta(1-\delta^m)}{1-\delta}\, (u - \underline{g})\, x(1).
\end{aligned}$$

Substituting

$$x(1) = \frac{1-\delta}{\delta(1-\delta^m)(\bar{g}-\underline{g})} \int_R^1 \left\{1 - F_i(\alpha)^{\frac{\bar{g}-\underline{g}}{u-\underline{g}}}\right\} d\alpha$$

from (a3) into (a8), we obtain

$$\varphi(u) = u - \int_R^1 \left\{1 - F_i(\alpha)^2\right\} d\alpha + \frac{u-\underline{g}}{\bar{g}-\underline{g}} \int_R^1 \left\{1 - F_i(\alpha)^{\frac{\bar{g}-\underline{g}}{u-\underline{g}}}\right\} d\alpha.$$

We now show that $\varphi(u) = 0$ has a solution in (g^0, g^*). It suffices to show that $\varphi(g^0) < 0$ since we already know from the remark before Corollary 1 that $\varphi(g^*) > 0$. It can be readily verified that in both first-price or second-price sealed-bid auctions, the (symmetric) one-shot Nash equilibrium payoff is given by

$$g^0 = \int_R^1 \left\{F_i(\alpha) - F_i(\alpha)^2\right\} d\alpha.$$

Substituting this into the first term of $\varphi(g^0)$, we get

$$\varphi(g^0) = -\int_R^1 \left\{1 - F_i(\alpha)\right\} d\alpha + \frac{g^0-\underline{g}}{\bar{g}-\underline{g}} \int_R^1 \left\{1 - F_i(\alpha)^{\frac{\bar{g}-\underline{g}}{g^0-\underline{g}}}\right\} d\alpha.$$

If we define the function $\phi_\alpha(\cdot) : \mathsf{R}_+ \to \mathsf{R}$ by

$$\phi_\alpha(z) = \frac{1}{z}\left\{1 - F_i(\alpha)^z\right\}$$

for each $\alpha \in [0,1]$, then it follows from the above that $\varphi(g^0)$ can be rewritten as

$$\varphi(g^0) = \int_R^1 \left\{\phi_\alpha\left(\frac{\bar{g}-\underline{g}}{g^0-\underline{g}}\right) - \phi_\alpha(1)\right\} d\alpha.$$

Since $\phi'_\alpha(z) < 0$ for each $\alpha \in [0,1)$ and $z \in \mathsf{R}_+$, $\phi_\alpha(z)$ is a strictly decreasing function of z for every $\alpha \in [R,1)$. It follows that $\varphi(g^0) < 0$ since $(\bar{g}-\underline{g})/(g^0-\underline{g}) > 1$. This completes the proof of the theorem. //

Proof of Theorem 4: Since $\underline{g}(\epsilon)$ is continuous in ϵ and $u(0) > \underline{g}(0)$, we can take $\bar{\epsilon} \in (0,1)$ such that $\bar{\epsilon} < \inf_{s_i > \rho} \hat{v}'_i(s_i)/w'_i(s_i)$, and $\underline{g}(\epsilon) < u(0)$ for any $\epsilon \le \bar{\epsilon}$. Choose an integer m such that

$$m > \frac{2}{u(0) - \underline{g}(\bar{\epsilon})} \int_\rho^1 \left\{\hat{v}_i(\alpha) - R\right\} z_i(\alpha)\, d\alpha.$$

As in the proof of Theorem 1, we can take $\underline{\delta} < 1$ such that for any $\delta > \underline{\delta}$,

$$(1-\delta)\{\hat{v}_i(1) - R\} + \delta g^0 < (1-\delta^m)\,\underline{g}(0) + \frac{\delta^m}{2}\,\{u(0) + g^0\},$$

and

$$\frac{1-\delta}{\delta(1-\delta^m)}\,\frac{2}{u(0)-\underline{g}(\bar{\epsilon})}\int_\rho^1 \{\hat{v}_i(\alpha) - R\}\,z_i(\alpha)\,d\alpha < 1.$$

Fix any $\delta > \underline{\delta}$, and let $\kappa > 0$ be given. By the argument preceding the statement of Theorem 5 in the text, we can take $\epsilon < \bar{\epsilon}$ such that (9) with this ϵ has a solution $u(\epsilon)$ greater than each one of $u(0) - \kappa$, $\{u(0) + \underline{g}(\bar{\epsilon})\}/2$ and $\{u(0) + g^0\}/2$. Let $\hat{x} : [\rho, 1] \to [0, 1]$ be defined by

$$\begin{aligned}\hat{x}(s_i) = {} & \frac{1-\delta}{\delta(1-\delta^m)}\,\frac{1}{u(\epsilon)-\underline{g}(\epsilon)} \\ & \times \int_\rho^{s_i} \{\hat{v}_i(\alpha) - (1-\epsilon)\,R - \epsilon\,w_i(\alpha)\}\,z_i(\alpha)\,e^{-\frac{\bar{g}(\epsilon)-\underline{g}(\epsilon)}{u(\epsilon)-\underline{g}(\epsilon)}\int_\alpha^{s_i} z_i(\beta)\,d\beta}\,d\alpha.\end{aligned}$$

Since $\underline{g}(\epsilon) < \underline{g}(\bar{\epsilon})$, we have

$$\frac{1}{u(\epsilon)-\underline{g}(\epsilon)} < \frac{1}{u(\epsilon)-\underline{g}(\bar{\epsilon})} < \frac{2}{u(0)-\underline{g}(\bar{\epsilon})}.$$

Furthermore, $\hat{v}_i(s_i) \geq w_i(s_i)$ by (2) so that $\hat{x}(s_i) \in [0, 1]$ from our choice of m and δ. We now define the transition probability function $x : [R, \zeta_\epsilon(1)] \to [0, 1]$ by

$$x(b_i) = \hat{x}(\zeta_\epsilon^{-1}(b_i)) \text{ for } b_i \in [R, \zeta_\epsilon(1)].$$

In other words, $\hat{x}(s_i)$ is the transition probability when the winner's signal inferred from his bid is s_i. The recursive relationship corresponding to (a4) in the proof of Theorem 1 is now given by

$$u(\epsilon) = g^*(\epsilon) - \frac{\delta(1-\delta^m)}{1-\delta}\,\{2u(\epsilon) - \bar{g}(\epsilon) - \underline{g}(\epsilon)\}\int_\rho^1\int_0^{s_i} \hat{x}(s_i)\,f(s)\,ds_j\,ds_i.$$

Substitution of $\hat{x}$ into the above yields (9). This shows that the solution $u(\epsilon)$ to (9) is the payoff generated by the given scheme with x as defined above.

We next check the bidders' incentive. Note first that no observable deviation is profitable: Bidder i's payoff from any observable deviation at the beginning of phase A_i is less than $(1-\delta)\{\hat{v}_i(1) - R\} + \delta g^0$, and his payoff from following the specified path equals

$$\begin{aligned}(1-\delta^m)\,\underline{g}(\epsilon) + \delta^m\,u(\epsilon) &> (1-\delta^m)\,\underline{g}(0) + \frac{\delta^m}{2}\,\{u(0) + g^0\} \\ &> (1-\delta)\{\hat{v}_i(1) - R\} + \delta g^0.\end{aligned}$$

It is also clear that the bidders have no incentive to commit unobservable deviations in the asymmetric phases A_1 and A_2. It therefore remains to show that no unobservable deviation is profitable in phase S. Let $\pi_i(s_i, \hat{s}_i)$ denote bidder i's (*interim*) payoff in phase S when he has signal s_i but bids $b_i = \zeta_\epsilon(\hat{s}_i)$, while bidder j uses the bidding function ζ_ϵ. It suffices to show that $\pi(s_i, s_i) \geq \pi(s_i, \hat{s}_i)$ for any s_i, $\hat{s}_i \in [0, 1]$. For simplicity, we write $u = u(\epsilon)$, $\bar{u} = (1 - \delta^m)\, \bar{g}(\epsilon) + \delta^m\, u(\epsilon)$, and $\underline{u} = (1 - \delta^m)\, \underline{g}(\epsilon) + \delta^m\, u(\epsilon)$.

i) Suppose first that $\hat{s}_i \geq \rho$ so that $b_i = \zeta_\epsilon(\hat{s}_i) \in [R, \zeta_\epsilon(1)]$. In this case, $\pi_i(s_i, \hat{s}_i)$ can be written as:

$$\begin{aligned}\pi_i(s_i, \hat{s}_i) = (1-\delta)\, E\Big[1_{\{\tilde{s}_j < \hat{s}_i\}} \{v_i(\tilde{s}) - \zeta_\epsilon(\hat{s}_i)\} \,\Big|\, \tilde{s}_i = s_i\Big] \\ + \delta \{(1 - \hat{x}(\hat{s}_i))\, u + \hat{x}(\hat{s}_i)\, \underline{u}\}\, P(\tilde{s}_j < \hat{s}_i \mid \tilde{s}_i = s_i) \\ + \delta\, E\Big[1_{\{\tilde{s}_j > \hat{s}_i\}} \{(1 - \hat{x}(\tilde{s}_j))\, u + \hat{x}(\tilde{s}_j)\, \bar{u}\} \,\Big|\, \tilde{s}_i = s_i\Big].\end{aligned}$$

Differentiation of π_i with respect to $\hat{s}_i$ yields

$$\begin{aligned}\frac{\partial \pi_i}{\partial \hat{s}_i}(s_i, \hat{s}_i) = F_i(\hat{s}_i \mid s_i) \Bigg[&(1-\delta) \{v_i(s_i, \hat{s}_i) - \zeta_\epsilon(\hat{s}_i)\} \frac{f_i(\hat{s}_i \mid s_i)}{F_i(\hat{s}_i \mid s_i)} \\ &- (1-\delta)\, \zeta_\epsilon'(\hat{s}_i) - \delta(u - \underline{u})\, \hat{x}'(\hat{s}_i) \\ &- \delta(\bar{u} - \underline{u})\, \hat{x}(\hat{s}_i) \frac{f_i(\hat{s}_i \mid s_i)}{F_i(\hat{s}_i \mid s_i)} \Bigg].\end{aligned} \tag{a9}$$

Note that $\hat{x}$ satisfies

$$\hat{x}'(s_i) = \frac{1-\delta}{\delta} \frac{1}{u - \underline{u}} \{\hat{v}_i(s_i) - (1-\epsilon)\, R - \epsilon\, w_i(s_i)\}\, z_i(s_i) - \frac{\bar{u} - \underline{u}}{u - \underline{u}}\, z_i(s_i)\, \hat{x}(s_i).$$

Substitution of this and (8) into (a9) yields upon simplification

$$\begin{aligned}\frac{\partial \pi_i}{\partial \hat{s}_i}(s_i, \hat{s}_i) = F_i(\hat{s}_i \mid s_i) \Bigg[&(1-\delta) \Big\{ v_i(s_i, \hat{s}_i) \frac{f_i(\hat{s}_i \mid s_i)}{F_i(\hat{s}_i \mid s_i)} - \hat{v}_i(\hat{s}_i) z_i(\hat{s}_i) \Big\} \\ &- \{(1-\delta)\, \zeta_\epsilon(\hat{s}_i) + \delta(\bar{u} - \underline{u})\, \hat{x}(\hat{s}_i)\} \Big\{ \frac{f_i(\hat{s}_i \mid s_i)}{F_i(\hat{s}_i \mid s_i)} - z_i(\hat{s}_i) \Big\} \Bigg] \\ = F_i(\hat{s}_i \mid s_i) \Bigg[&(1-\delta) \{ v_i(s_i, \hat{s}_i) - \hat{v}_i(\hat{s}_i) \} \frac{f_i(\hat{s}_i \mid s_i)}{F_i(\hat{s}_i \mid s_i)} \\ &+ \Big\{ \frac{f_i(\hat{s}_i \mid s_i)}{F_i(\hat{s}_i \mid s_i)} - z_i(\hat{s}_i) \Big\} \Big[(1-\delta) \{ \hat{v}_i(\hat{s}_i) - \zeta_\epsilon(\hat{s}_i) \} - \delta(\bar{u} - \underline{u})\, \hat{x}(\hat{s}_i) \Big] \Bigg].\end{aligned}$$

From the definitions of ζ_ϵ and $\hat{x}$, we can verify that

$$\begin{aligned}
&\hat{v}_i(\hat{s}_i) - \zeta_\epsilon(\hat{s}_i) - \frac{\delta}{1-\delta}\,(\bar{u} - \underline{u})\,\hat{x}(\hat{s}_i) \\
&= \epsilon \int_\rho^{\hat{s}_i} w_i'(\alpha)\, e^{-\int_\alpha^{\hat{s}_i} z_i(\beta)\, d\beta}\, d\alpha + \{\hat{v}_i(\rho) - R\}\, e^{-\frac{\bar{u}-\underline{u}}{u-\underline{u}} \int_\rho^{\hat{s}_i} z_i(\beta)\, d\beta} \\
&+ \int_\rho^{\hat{s}_i} \{\hat{v}_i'(\alpha) - \epsilon\, w_i'(\alpha)\}\, e^{-\frac{\bar{u}-\underline{u}}{u-\underline{u}} \int_\alpha^{\hat{s}_i} z_i(\beta)\, d\beta}\, d\alpha \\
&> 0.
\end{aligned}$$

It then follows from (1) that $\dfrac{\partial \pi_i}{\partial \hat{s}_i}(s_i, \hat{s}_i) > 0$ (resp. < 0) if $\hat{s}_i < s_i$ (resp. $\hat{s}_i > s_i$). Hence, for any $\hat{s}_i \geq \rho$, we have (a) $\pi_i(s_i, \rho) \geq \pi_i(s_i, \hat{s}_i)$ for $s_i < \rho$, and (b) $\pi_i(s_i, s_i) \geq \pi_i(s_i, \hat{s}_i)$ for $s_i \geq \rho$.

ii) Suppose next that $\hat{s}_i < \rho$ so that $b_i = \zeta_\epsilon(\hat{s}_i) = N$. In this case, i's payoff equals

$$\pi_i(s_i, \hat{s}_i) = \delta(\bar{u} - u)\, E\big[1_{\{\tilde{s}_j \geq \rho\}}\, \hat{x}(\tilde{s}_j) \mid \tilde{s}_i = s_i\big] + \delta u.$$

On the other hand, bidding $\zeta_\epsilon(\rho) = R$ would yield bidder i

$$\begin{aligned}
\pi_i(s_i, \rho) &= E\big[1_{\{\tilde{s}_j < \rho\}} \big\{(1-\delta)\,\{v_i(\tilde{s}) - R\} - \delta(u - \underline{u})\,\hat{x}(\rho)\big\} \mid \tilde{s}_i = s_i\big] \\
&\quad + \delta(\bar{u} - u) E\big[1_{\{\tilde{s}_j \geq \rho\}}\, \hat{x}(\tilde{s}_j) \mid \tilde{s}_i = s_i\big] + \delta u.
\end{aligned}$$

Since $\hat{x}(\rho) = 0$, we have

$$\begin{aligned}
\pi_i(s_i, \rho) - \pi_i(s_i, \hat{s}_i) &= (1-\delta)\, E\big[1_{\{\tilde{s}_j < \rho\}}\, \{v_i(\tilde{s}) - R\} \mid \tilde{s}_i = s_i\big] \\
&= (1-\delta)\, E\big[v_i(\tilde{s}) - R \mid \tilde{s}_j < \rho,\, \tilde{s}_i = s_i\big]\, P(\tilde{s}_j < \rho \mid \tilde{s}_i = s_i).
\end{aligned}$$

It then follows from the definition of ρ that $\pi_i(s_i, \rho) \geq \pi_i(s_i, \hat{s}_i)$ if and only if $s_i \geq \rho$. This combined with the conclusion in case (i) suggests that if $s_i \geq \rho$, then $\pi_i(s_i, s_i) \geq \pi_i(s_i, \rho) \geq \pi_i(s_i, \hat{s}_i)$.

The desired conclusion follows from (i) and (ii). //

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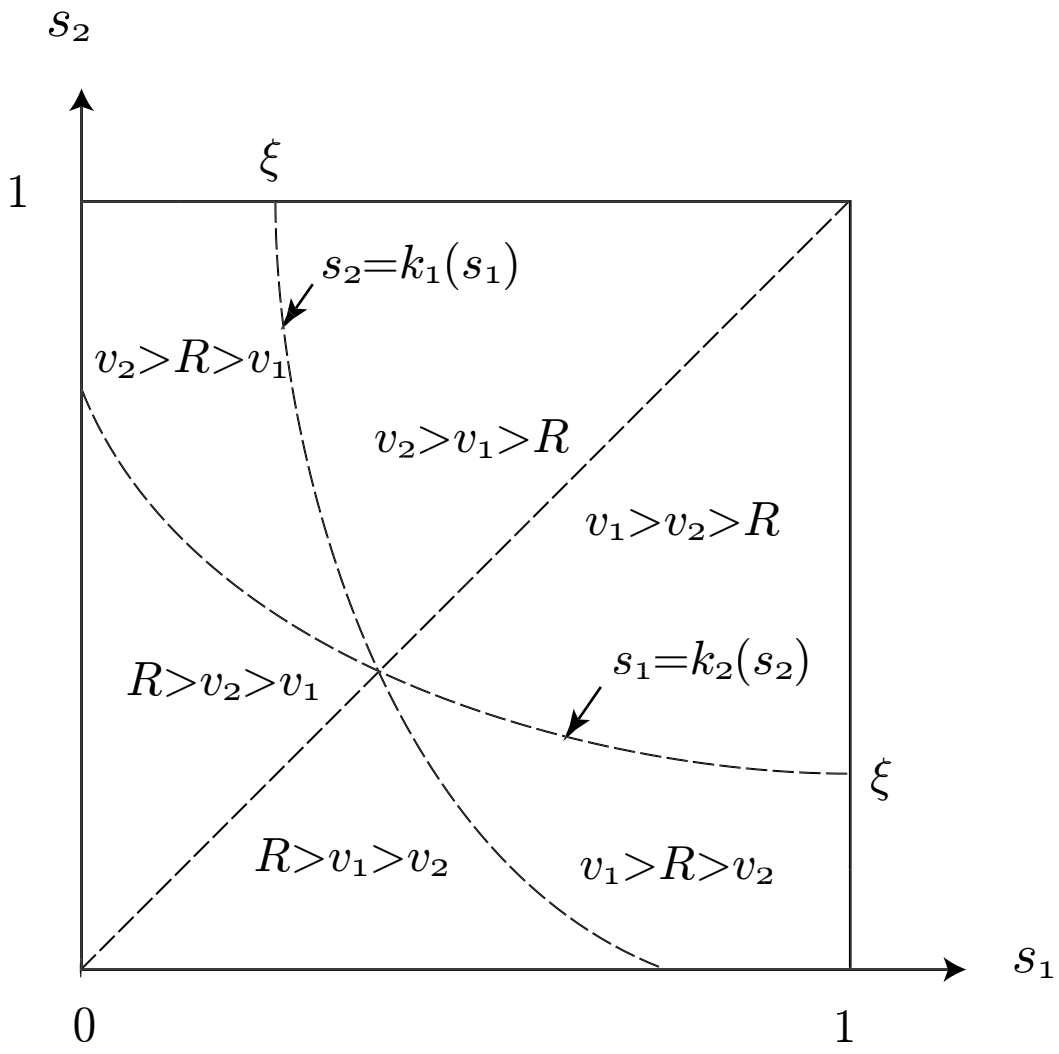
s_2
1
ξ
$s_2{=}k_1(s_1)$
$v_2{>}R{>}v_1$
$v_2{>}v_1{>}R$
$v_1{>}v_2{>}R$
$R{>}v_2{>}v_1$
$s_1{=}k_2(s_2)$
ξ
$R{>}v_1{>}v_2$
$v_1{>}R{>}v_2$
s_1
0
1

Figure 1