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Author(s)	武田, 好史
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Artin-Schreier coverings of algebraic surfaces

By Yoshifumi TAKEDA

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Introduction.

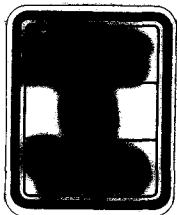
Let k be an algebraically closed field of characteristic $p > 0$ and let X be a nonsingular projective surface defined over k . An *Artin-Schreier covering* of X is a finite morphism $\pi: Y \rightarrow X$ from a normal surface Y onto X such that the field extension $k(Y)/k(X)$ is an Artin-Schreier extension. It is well-known that $k(Y)$ is expressed as $k(Y) = k(X)(\xi)$ with $\xi^p - \xi = f$ and $f \in k(X)$. Since $k(Y)/k(X)$ is a Galois extension with the Galois group $G \cong \mathbb{Z}/p\mathbb{Z}$, G acts on Y so that $X \cong Y/G$. In order to study Artin-Schreier coverings, we have to consider whether or not there exists an affine open covering $\mathfrak{U} = \{U_\lambda\}$ such that $\pi^{-1}(U_\lambda) = \text{Spec } \mathcal{O}_X(U_\lambda)[\xi_\lambda]/(\xi_\lambda^p - s_\lambda \xi_\lambda - t_\lambda)$ with $s_\lambda, t_\lambda \in \mathcal{O}_X(U_\lambda)$. In general, this assertion does not hold (cf. Example 1.5). Under the above circumstance, we shall define an Artin-Schreier covering of simple type (see §1 for the definition), for which the assertion holds. From the definition, every Artin-Schreier covering in characteristic 2 is of simple type.

This article consists of three parts. In Section 1, we consider Artin-Schreier coverings of simple type and give some formulas to compute invariants in the case of nonsingular coverings. In Section 2, we assume that the characteristic is 2 and consider a resolution of singularities for Artin-Schreier coverings with nonsingular branch locus. We give some formulas to compute invariants of nonsingular models of coverings. In Section 3, we consider smooth Artin-Schreier coverings of simple type with ample branch loci which satisfy extra conditions. Especially, we shall determine such coverings with $\kappa = -\infty, 0$, and 1.

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§1. Artin-Schreier coverings of simple type.

Let X be a nonsingular projective surface and let $\pi: Y \rightarrow X$ be an Artin-Schreier covering. Since Y is a Cohen-Macaulay scheme and X is regular, π is a flat morphism. Hence $\pi_* \mathcal{O}_Y$ is a locally free $\mathcal{O}_X$ -algebra. Moreover,



PROPOSITION 1.1. *There is a canonical filtration of $\mathcal{O}_X$ -modules of $\pi_*\mathcal{O}_Y$,*

$$\mathcal{O}_X = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{p-1} = \pi_*\mathcal{O}_Y$$

such that

- (1) $\mathcal{F}_i$ is a locally free sheaf of rank $i+1$,
- (2) $\mathcal{F}_i/\mathcal{F}_0$ is an invertible sheaf and $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a torsion-free $\mathcal{O}_X$ -module of rank 1 for $1 \leq i \leq p-1$.

PROOF. Let $U = \text{Spec } R$ be an affine open subset of X and let $\pi^{-1}(U) = \text{Spec } A$. Then $\pi^{-1}(U)$ is a G -stable set and $U = \pi^{-1}(U)/G$. On the other hand, as a group scheme, G is written as $G = \text{Spec } k[z]/(z^p - z)$ with the comultiplication $\Delta(z) = z \otimes 1 + 1 \otimes z$ and the counit $\varepsilon(z) = 0$. So, the coaction of G on $\text{Spec } A$ is given by an R -algebra homomorphism $\sigma: A \rightarrow A[z]$ with $z^p = z$ such that $(\sigma \otimes 1)\sigma = (1 \otimes \Delta)\sigma$ and $(1 \otimes \varepsilon)\sigma = \text{id}_A$. Write $\sigma(a) = \sigma_0(a) + \sigma_1(a)z + \cdots + \sigma_{p-1}(a)z^{p-1}$ for $a \in A$. Then $(1 \otimes \varepsilon)\sigma = \text{id}_A$ implies $\sigma_0 = \text{id}_A$. We have

$$(\sigma \otimes 1)\sigma(a) = \sum_{i=0}^{2p-2} \sum_{j=0}^i \sigma_j \sigma_{i-j}(a) z^j \otimes z^{i-j}$$

and

$$(1 \otimes \Delta)\sigma(a) = \sum_{i=0}^{p-1} \sum_{j=0}^i C_j \sigma_i(a) z^j \otimes z^{i-j}.$$

Thence the relation $(\sigma \otimes 1)\sigma = (1 \otimes \Delta)\sigma$ implies $\sigma_j \sigma_{i-j} = C_j \sigma_i$ for $0 \leq i \leq p-1$ and $\sigma_i = 0$ for $i \geq p$. Set $\sigma_1 = \delta$. Then these relations are equivalent to $\sigma_0 = \text{id}_A$, $\sigma_i = 1/(i!) \delta^i$ ($1 \leq i \leq p$) and $\delta^p = 0$. So, we can write

$$\sigma(a) = a + \delta(a)z + \frac{1}{2!} \delta^2(a)z^2 + \cdots + \frac{1}{(p-1)!} \delta^{p-1}(a)z^{p-1}.$$

Set $F_i = \{a \in A \mid \delta^{i+1}(a) = 0\}$ for $0 \leq i \leq p-1$. Then $F_0 = R$ and F_i is an R -module. Since the G -action on A is nontrivial, there exists $a \in A$ such that $\sigma(a) \neq a$. Suppose $\delta^r(a) \neq 0$ and $\delta^{r+1}(a) = 0$ for $0 < r < p$. Then $\delta^r(a) \in F_0$. So, $\sigma(\delta^{r-1}(a)) = \delta^{r-1}(a) + \delta^r(a)z$. Therefore, $F_1 \neq F_0$. Furthermore, for $1 \leq i \leq p-1$,

$$\begin{aligned} \sigma((\delta^{r-1}(a))^i) &= \sigma(\delta^{r-1}(a))^i = (\delta^{r-1}(a) + \delta^r(a)z)^i \\ &= \delta^{r-1}(a)^i + i(\delta^{r-1}(a))^{i-1} \delta^r(a)z + \cdots + (\delta^r(a))^i z^i. \end{aligned}$$

This implies that $F_i \neq F_{i-1}$ for $1 \leq i \leq p-1$. On the other hand, F_i is the inverse image by σ of R -module $A + Az + \cdots + Az^i$ of $A[z]$. Hence F_i/F_{i-1} is viewed as an R -submodule of $(A + Az + \cdots + Az^i)/(A + Az + \cdots + Az^{i-1}) \cong A$. Therefore, F_i/F_{i-1} is a torsion-free R -module of rank 1.

Now we sheafify the above observations. Since the operator δ is defined globally, we can define a coherent sheaf $\mathcal{F}_i$ so that, on an affine open subset W , $\mathcal{F}_i|_W = \{a \in \pi_*\mathcal{O}_Y(W) \mid \delta^{i+1}(a) = 0\}^\sim$. Then $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a torsion-free $\mathcal{O}_X$ -module of rank 1.

To show that $\mathcal{F}_i$ is a locally free sheaf, we take the double dual $\mathcal{F}_i^{**}$ of

$\mathcal{F}_i$. Then we have

$$\begin{array}{ccccccc} \mathcal{O}_X & = \mathcal{F}_0 & \subset \mathcal{F}_1 & \subset \cdots \subset \mathcal{F}_{p-1} & = \pi_* \mathcal{O}_Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_X^{**} & = \mathcal{F}_0^{**} & \subset \mathcal{F}_1^{**} & \subset \cdots \subset \mathcal{F}_{p-1}^{**} & = \pi_* \mathcal{O}_Y^{**}. \end{array}$$

So, we may regard $\mathcal{F}_i^{**}$ as $\mathcal{O}_X$ -submodule of $\pi_* \mathcal{O}_Y$. Hence δ^{i+1} operates on $\mathcal{F}_i^{**}$ and $\delta^{i+1} \in \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_i^{**}, \pi_* \mathcal{O}_Y)$. We know that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_i^{**}, \pi_* \mathcal{O}_Y)$ is a locally free sheaf and that $\delta^{i+1}|_V = 0$, where $V = X - \text{Supp } \mathcal{F}_i^{**}/\mathcal{F}_i$. On the other hand, since $\mathcal{F}_i$ is torsion-free and X is regular, $\mathcal{F}_i^{**}/\mathcal{F}_i$ has support of codimension ≥ 2 . Therefore, $\delta^{i+1}(\mathcal{F}_i^{**}) = 0$. So, we have $\mathcal{F}_i^{**} = \mathcal{F}_i$. Hence $\mathcal{F}_i$ is a locally free sheaf.

Finally we show that $\mathcal{F}_1/\mathcal{F}_0$ is an invertible sheaf. We consider an exact sequence

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_1/\mathcal{F}_0 \longrightarrow 0.$$

We know that $\mathcal{F}_0 \otimes k(P) \rightarrow \mathcal{F}_1 \otimes k(P)$ is injective for an arbitrary point $P \in X$ because $\mathcal{F}_0 \otimes k(P)$ contains the unity of $(\pi_* \mathcal{O}_Y) \otimes k(P)$. Hence we have $\text{rank}(\mathcal{F}_1/\mathcal{F}_0) \otimes k(P) = 1$. This implies that $\mathcal{F}_1/\mathcal{F}_0$ is an invertible sheaf on X . Q. E. D.

We shall define a good class of Artin-Schreier coverings. Let π, X, Y and $\mathcal{F}_i$ be as above. We call $\pi: Y \rightarrow X$ an Artin-Schreier covering of simple type if $\mathcal{F}_i/\mathcal{F}_{i-1} \cong (\mathcal{F}_1/\mathcal{F}_0)^{\otimes i}$ for $1 \leq i \leq p-1$. From now on, we consider Artin-Schreier coverings of simple type. We have the following fundamental lemmas.

LEMMA 1.2. Suppose that $\pi: Y \rightarrow X$ is an Artin-Schreier covering of simple type. Then there exist an affine open covering $\mathcal{U} = \{U_\lambda\}$ of X and $s_\lambda, t_\lambda \in H^0(U_\lambda, \mathcal{O}_X)$ such that

$$\pi^{-1}(U_\lambda) = \text{Spec } \mathcal{O}_X(U_\lambda)[\xi_\lambda]/(\xi_\lambda^p - s_\lambda \xi_\lambda - t_\lambda).$$

PROOF. Write $\mathcal{L}^{-1} = \mathcal{F}_1/\mathcal{F}_0$. Let $\mathcal{U} = \{U_\lambda\}$ be an affine open covering of X such that $\mathcal{L}^{-1}|_{U_\lambda} \cong \mathcal{O}_{U_\lambda}$. Take $\xi_\lambda \in \mathcal{F}_1(U_\lambda)$ such that $\mathcal{L}^{-1}|_{U_\lambda} = \mathcal{O}_{U_\lambda} \overline{\xi}_\lambda$, where $\overline{\xi}_\lambda$ is the image of ξ_λ . Then $\sigma(\xi_\lambda) = \xi_\lambda + \alpha_\lambda z$ with $\alpha_\lambda \in H^0(U_\lambda, \mathcal{O}_X)$, where σ and z are the same as in the proof of Proposition 1.1. Since $\sigma(\xi_\lambda^p) = \xi_\lambda^p + \alpha_\lambda^p z$, we have $\xi_\lambda^p \in \mathcal{F}_1(U_\lambda)$. Thus, $\xi_\lambda^p = s_\lambda \xi_\lambda + t_\lambda$ with $s_\lambda, t_\lambda \in H^0(U_\lambda, \mathcal{O}_X)$. On the other hand, $\pi_* \mathcal{O}_Y|_{U_\lambda} = \mathcal{O}_{U_\lambda} + \mathcal{O}_{U_\lambda} \xi_\lambda + \cdots + \mathcal{O}_{U_\lambda} \xi_\lambda^{p-1}$ by the hypothesis. The assertion follows from these observations. Q. E. D.

LEMMA 1.3. Under the same assumptions and notations as in the previous lemma, we have $\{s_\lambda\} \in H^0(X, \mathcal{L}^{p-1})$, $s_\lambda = \alpha_\lambda^{p-1}$, $\{\alpha_\lambda\} \in H^0(X, \mathcal{L})$ and $t_\mu - a_{\lambda\mu}^p t_\lambda = b_{\lambda\mu}^p - s_\mu b_{\lambda\mu}$, where $\{a_{\lambda\mu}\}$ is transition functions of $\mathcal{L}$ and $\{b_{\lambda\mu}\} \in H^1(X, \mathcal{L})$.

PROOF. Since $\xi_\lambda^p = s_\lambda \xi_\lambda + t_\lambda$, we obtain $\sigma(\xi_\lambda^p) = \xi_\lambda^p + \alpha_\lambda^p z = (s_\lambda \xi_\lambda + t_\lambda) + s_\lambda \alpha_\lambda z$.

Thus $s_\lambda = \alpha_\lambda^{p-1}$. On $U_\lambda \cap U_\mu$, set $\xi_\mu = a_{\lambda\mu}\xi_\lambda + b_{\lambda\mu}$. Then $\{b_{\lambda\mu}\} \in H^1(X, \mathcal{L})$ and we have

$$\begin{aligned}\xi_\mu^p &= s_\mu \xi_\mu + t_\mu = s_\mu(a_{\lambda\mu}\xi_\lambda + b_{\lambda\mu}) + t_\mu \\ &= a_{\lambda\mu}^p \xi_\lambda^p + b_{\lambda\mu}^p = a_{\lambda\mu}^p(s_\lambda \xi_\lambda + t_\lambda) + b_{\lambda\mu}^p.\end{aligned}$$

Hence, $s_\mu = a_{\lambda\mu}^{p-1} s_\lambda$ and $t_\mu - a_{\lambda\mu}^p t_\lambda = b_{\lambda\mu}^p - s_\mu b_{\lambda\mu}$. So, we have $\{s_\lambda\} \in H^0(X, \mathcal{L}^{p-1})$. Moreover,

$$\sigma(\xi_\mu) = \sigma(a_{\lambda\mu}\xi_\lambda + b_{\lambda\mu}) = a_{\lambda\mu}\xi_\lambda + b_{\lambda\mu} + a_{\lambda\mu}\alpha_\lambda z = \xi_\mu + \alpha_\mu z.$$

Thence $\alpha_\mu = a_{\lambda\mu}\alpha_\lambda$, i.e. $\{\alpha_\lambda\} \in H^0(X, \mathcal{L})$.

Q. E. D.

REMARK 1.4. If $\{b_{\lambda\mu}\} = 0$ in $H^1(X, \mathcal{L})$, we can choose t_λ so that $\{t_\lambda\} \in H^0(X, \mathcal{L}^p)$. In particular, we may assume $\{t_\lambda\} \in H^0(X, \mathcal{L}^p)$ provided $H^1(X, \mathcal{L}) = (0)$.

Let B be the effective divisor corresponding to the section $\{\alpha_\lambda\} \in H^0(X, \mathcal{L})$. Clearly, B is independent of choice of generators $\{\xi_\lambda\}$. Moreover, π is unramified over $X - \text{Supp } B$ and totally ramified over $\text{Supp } B$. We call B the *branch locus* of π .

We shall give an example of an Artin-Schreier covering which is not of simple type.

EXAMPLE 1.5. Assume that $\text{char } k = p > 2$. Consider $\mathbf{P}_k^3$ with a homogeneous coordinate system (x_0, x_1, x_2, x_3) . Let $X = \{x_3 = 0\} \cong \mathbf{P}^2$, $Y = \{x_3^p - x_1^{p-1}x_3 - x_1^{p-1}x_2 = 0\} \subset \mathbf{P}^3$ and let $\rho: Y \rightarrow X$ be the projection from $(0, 0, 0, 1)$. Then ρ is surjective and finite. Take the normalization $\nu: \tilde{Y} \rightarrow Y$ and denote $\pi = \rho \circ \nu$. Let $U_i = \{x_i \neq 0\} \subset X$ for $i=0, 1, 2$. By the Jacobian criterion, $\rho(\text{Sing } Y) = \{x_1 = 0\}$. Hence $\pi^{-1}(U_1) = \rho^{-1}(U_1)$. On the other hand, $\rho^{-1}(U_0) = \text{Spec } k[x, y, \xi]$, where $\xi^p - x^{p-1}\xi - x^{p-1}y$, and $\rho^{-1}(U_2) = \text{Spec } k[u, v, \zeta]$, where $\zeta^p - u^{p-1}\zeta - u^{p-1}$. It is easy to verify that $\pi^{-1}(U_2) = \text{Spec } k[v, \zeta, \tau]$, where $\tau = u/\zeta$ and $\zeta - \tau^{p-1}\zeta - \tau^{p-1} = 0$. On U_0 , it is a little more difficult. Let $T^p = x$ and $S = \xi/T^{p-1}$. Then T and S are algebraically independent over k and we have $k[T^p, T^{p-1}S, S^p] = k[x, y, \xi]$. Let $\mathcal{O} = k[T^p, T^{p-1}S, T^{p-2}S^2, \dots, TS^{p-1}, S^p]$. Clearly, $T^{p-2}S^2, \dots, TS^{p-1}$ are integral over $k[T^p, T^{p-1}S, S^p]$. Meanwhile, $\mathcal{O}$ is none other than the coordinate ring of the cone of the p -uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^p$. So, $\mathcal{O}$ is normal. Hence $\mathcal{O}$ is the integral closure of $k[x, y, \xi]$. Furthermore, $\text{Spec } \mathcal{O}$ has only one singular point whose minimal resolution consists of a curve C such that $C \cong \mathbf{P}^1$ and $(C^2) = -p$.

Since $k(\tilde{Y})/k(X)$ is an Artin-Schreier extension, $\pi: \tilde{Y} \rightarrow X$ is an Artin-Schreier covering. However, π is not of simple type. Suppose π is of simple type. Then $\tilde{Y}$ is locally a hypersurface by Lemma 1.2. Hence every rational singularity is a rational double point. This contradicts the above observation.

By Remark 1.4, if $H^1(X, \mathcal{L}) = (0)$, every Artin-Schreier covering of simple type is defined locally by $\xi_\lambda^p - \alpha_\lambda^{p-1} \xi_\lambda = t_\lambda$, where $\alpha = \{\alpha_\lambda\} \in H^0(X, \mathcal{L})$ and $t = \{t_\lambda\} \in H^0(X, \mathcal{L}^p)$. The set $\{dt_\lambda\}$ of 1-forms defines a section $dt \in H^0(X, \Omega_X \otimes \mathcal{L}^p)$. Applying the Jacobian criterion to the above local defining equations, we obtain

PROPOSITION 1.6. *With the above notations and assumptions, in the characteristic $p > 2$, Y is singular at a point $Q \in Y$ if and only if $\pi(Q) \in \text{Supp } B$ and $dt = 0$ at $\pi(Q)$.*

Artin-Schreier coverings of simple type are obtained as follows (cf. [5]). Let L be a line bundle on X associated with invertible sheaf $\mathcal{L}$ and consider L and L^p as smooth X -group schemes. Take a global section s of L^{p-1} and consider a surjective homomorphism of X -group schemes $F - s: L \rightarrow L^p$ defined by $(F - s)(x) = x^p - sx$ for $x \in L$. Let α_s be its kernel. Then we have an exact sequence of X -group schemes in flat topology

$$0 \longrightarrow \alpha_s \longrightarrow L \longrightarrow L^p \longrightarrow 0.$$

Taking the flat cohomologies, we have an exact sequence

$$0 \rightarrow H_{f1}^0(X, \alpha_s) \rightarrow H^0(X, \mathcal{L}) \rightarrow H^0(X, \mathcal{L}^p) \xrightarrow{\partial} H_{f1}^1(X, \alpha_s) \rightarrow H^1(X, \mathcal{L}),$$

where we can interpret $H_{f1}^1(X, \alpha_s)$ as the set of isomorphism classes of α_s -torsors. Suppose that $s \in H^0(X, \mathcal{L}^{p-1})$ is given locally by $\{s_\lambda\}$ as in Lemmas 1.2 and 1.3. Let $t = \{t_\lambda\}$ be local sections of $\mathcal{L}^p$ as in Lemma 1.3 and let $\rho: Z \rightarrow X$ be the α_s -torsor obtained by applying the connecting map ∂ to $t = \{t_\lambda\}$. In other words, Z is locally the fibre product of $F - s: L \rightarrow L^p$ and $t_\lambda: U_\lambda \rightarrow L^p$. Then it is clear that $\rho: Z \rightarrow X$ is isomorphic to $\pi: Y \rightarrow X$. If $H^1(X, \mathcal{L}) = 0$, all α_s -torsors, hence all Artin-Schreier coverings of simple type, are obtained from global sections of $\mathcal{L}^p$ (cf. Remark 1.4).

In the sequel of this section, we consider an Artin-Schreier covering $\pi: Y \rightarrow X$ of simple type. We fix the notations $\mathcal{F}_i$ ($0 \leq i < p$), $\mathcal{L}$, and B as in Proposition 1.1 and Lemmas 1.2 and 1.3. By the local description, we know that Y is locally a hypersurface. Therefore Y is a Gorenstein scheme. More precisely, we have

PROPOSITION 1.7. *Y has the dualizing sheaf*

$$\omega_Y = \pi^*(\omega_X \otimes \mathcal{L}^{p-1}).$$

PROOF. Apply the adjunction formula.

We shall compute invariants of Artin-Schreier coverings of simple type. There are the following formulas.

LEMMA 1.8. (1) $(\omega_Y^2) = p\{(K_X^2) + 2(p-1)(B, K_X) + (p-1)^2(B^2)\}.$

$$(2) \quad \chi(\mathcal{O}_Y) = p\left\{\chi(\mathcal{O}_X) + \frac{(p-1)}{4}(B, K_X) + \frac{(p-1)(2p-1)}{12}(B^2)\right\}.$$

(3) If Y is smooth,

$$e(Y) = p\{e(X) + (p-1)(B, K_X) + (p-1)p(B^2)\},$$

where $e(Y)$ is the Euler number of Y .

$$(4) \quad \kappa(Y) = \kappa(X, K_X + (p-1)B).$$

PROOF. (1) Immediate from Proposition 1.7.

(2) By the assumptions, $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{O}(-iB)$. Hence we have $\chi(\mathcal{F}_i) = \chi(\mathcal{F}_{i-1}) + \chi(\mathcal{O}(-iB))$ for $1 \leq i \leq p-1$. Therefore $\chi(\mathcal{O}_Y) = \chi(\mathcal{F}_{p-1}) = \sum_{i=0}^{p-1} \chi(\mathcal{O}(-iB))$, where, by the Riemann-Roch theorem,

$$\begin{aligned} \chi(\mathcal{O}(-iB)) &= (1/2)(-iB, -iB - K_X) + \chi(\mathcal{O}_X) \\ &= (1/2)(i^2(B^2) + i(B, K_X)) + \chi(\mathcal{O}_X). \end{aligned}$$

Thence we obtain the stated formula.

(3) Use Noether's formula: $12\chi(\mathcal{O}_Y) = (K_Y^2) + e(Y)$.

(4) It follows from a fundamental property of the D -dimension. Q.E.D.

In order to construct examples, we need the following:

LEMMA 1.9. Let X, Y and $\mathcal{L}$ be as above. Suppose that $\mathcal{L}$ is ample. If $H^1(X, \mathcal{L}^{-1}) = (0)$, we have $H^1(X, \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y)$.

PROOF. By the exact sequence

$$0 \longrightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_B \longrightarrow 0,$$

we have $H^0(X, \mathcal{O}_X) = H^0(B, \mathcal{O}_B) = k$. The exact sequence

$$0 \longrightarrow \mathcal{L}^{-2} \longrightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{L}^{-1} \otimes \mathcal{O}_B \longrightarrow 0$$

implies $H^1(X, \mathcal{L}^{-2}) = (0)$ because $\mathcal{L} \otimes \mathcal{O}_B$ is ample and $H^1(X, \mathcal{L}^{-1}) = (0)$. Similarly, by the exact sequences

$$0 \longrightarrow \mathcal{L}^{-i} \longrightarrow \mathcal{L}^{-(i-1)} \longrightarrow \mathcal{L}^{-(i-1)}|_B \longrightarrow 0 \quad (i > 1),$$

we obtain inductively $H^1(X, \mathcal{L}^{-i}) = (0)$. Now look at the exact sequences

$$0 \longrightarrow \mathcal{F}_{i-1} \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{L}^{-i} \longrightarrow 0 \quad (0 < i < p).$$

We know $H^1(X, \mathcal{F}_{i-1}) = H^1(X, \mathcal{F}_i)$. Hence $H^1(Y, \mathcal{O}_Y) = H^1(X, \mathcal{O}_X)$. Q.E.D.

EXAMPLE 1.10. Assume that $\text{char } k = p = 3$. Let $X = \mathbf{P}^1 \times \mathbf{P}^1$ and $\mathcal{L} = p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(1)$. Take an affine covering $\{U_i \times V_j\}_{i,j=1,2}$ such that $U_1 = \text{Spec } k[x]$,

$U_2 = \text{Spec } k[u]$, $V_1 = \text{Spec } k[y]$ and $V_2 = \text{Spec } k[v]$, where $u = x^{-1}$, $v = y^{-1}$. Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering such that

$$\begin{aligned}\pi^{-1}(U_1 \times V_1) &= \text{Spec } \mathcal{O}_{U_1 \times V_1}[\xi_{11}]/(\xi_{11}^3 - x^2 y^2 \xi_{11} - (x^2 + y^2 + x + y)), \\ \pi^{-1}(U_1 \times V_2) &= \text{Spec } \mathcal{O}_{U_1 \times V_2}[\xi_{12}]/(\xi_{12}^3 - x^2 \xi_{12} - (x^2 v^3 + v + x v^3 + v^2)), \\ \pi^{-1}(U_2 \times V_1) &= \text{Spec } \mathcal{O}_{U_2 \times V_1}[\xi_{21}]/(\xi_{21}^3 - y^2 \xi_{21} - (u + y^2 u^3 + u^2 + y u^3)), \\ \pi^{-1}(U_2 \times V_2) &= \text{Spec } \mathcal{O}_{U_2 \times V_2}[\xi_{22}]/(\xi_{22}^3 - \xi_{22} - (u v^3 + u^3 v + u^2 v^3 + u^3 v^2)).\end{aligned}$$

Then Y is nonsingular and its dualizing sheaf is

$$\omega_Y = \pi^*(p_1^* \mathcal{O}(-2) \otimes p_2^* \mathcal{O}(-2) \otimes \mathcal{L}^2) \cong \mathcal{O}_Y.$$

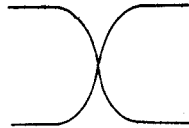
By the previous lemma, we see that $H^1(Y, \mathcal{O}_Y) = H^1(X, \mathcal{O}_X) = (0)$. Hence Y is a $K3$ -surface.

EXAMPLE 1.11. Assume $\text{char } k = p = 3$. Let $X = \mathbf{P}^2$ and $\mathcal{L} = \mathcal{O}(1)$. Then $H^1(\mathbf{P}^2, \mathcal{L}) = (0)$. Let (x, y, z) be a system of homogeneous coordinate of $\mathbf{P}^2$. Choose $s = x^2 \in H^0(\mathbf{P}^2, \mathcal{L}^2)$ and $t = xy^2 + x^2 y + y^2 z + yz^2 + z^2 x + zx^2 \in H^0(\mathbf{P}^2, \mathcal{L}^3)$. Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering of simple type obtained from s and t . Then Y is smooth. Moreover, $\omega_Y = \pi^* \mathcal{O}(-1)$ and $(K_Y)^2 = 3$. So, Y is a del Pezzo surface of degree 3, i.e. a smooth cubic hypersurface in $\mathbf{P}^3$.

EXAMPLE 1.12. Assume $\text{char } k = p = 2$. Let $X = \mathbf{P}^1 \times \mathbf{P}^1$ and let $\mathcal{L} = p_1^* \mathcal{O}(2) \otimes p_2^* \mathcal{O}(3)$. Take an affine open covering $\{U_i \times V_j\}_{i,j=1,2}$ which is the same as in Example 1.10. Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering such that

$$\begin{aligned}\pi^{-1}(U_1 \times V_1) &= \text{Spec } \mathcal{O}_{U_1 \times V_1}[\xi_{11}]/(\xi_{11}^2 + x^2(y+1)^3 \xi_{11} + (x+x^3)y^3 + y^5 + y^3 + y), \\ \pi^{-1}(U_1 \times V_2) &= \text{Spec } \mathcal{O}_{U_1 \times V_2}[\xi_{12}]/(\xi_{12}^2 + x^2(1+v)^3 \xi_{12} + (x+x^3)v^3 + v^5 + v^3 + v), \\ \pi^{-1}(U_2 \times V_1) &= \text{Spec } \mathcal{O}_{U_2 \times V_1}[\xi_{21}]/(\xi_{21}^2 + (y+1)^3 \xi_{21} + (u+u^3)y^3 + u^4(y^5 + y^3 + y)), \\ \pi^{-1}(U_2 \times V_2) &= \text{Spec } \mathcal{O}_{U_2 \times V_2}[\xi_{22}]/(\xi_{22}^2 + (1+v)^3 \xi_{22} + (u+u^3)v^3 + u^4(v^5 + v^3 + v)).\end{aligned}$$

Then Y is nonsingular. By Proposition 1.7 and Lemma 1.8, we have $\omega_Y = \pi^* p_2^* \mathcal{O}(1)$ and $\kappa(Y) = 1$. Moreover, $f = p_2 \circ \pi: Y \rightarrow \mathbf{P}^1$ is an elliptic fibration and three fibers $f^{-1}(P_0)$, $f^{-1}(P_1)$ and $f^{-1}(P_\infty)$ exhaust singular fibres of f , where P_0 , P_1 and P_∞ are points of $\mathbf{P}^1$ defined by $y=0$, 1 and ∞ , respectively. The fibres $f^{-1}(P_0)$ and $f^{-1}(P_\infty)$ are of type



The fibre $f^{-1}(P_1)$ is a cuspidal rational curve.

EXAMPLE 1.13. Let $p, X, \mathcal{L}$ and $\{U_i \times V_j\}$ be as in the previous example. Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering such that

$$\pi^{-1}(U_1 \times V_1) = \text{Spec } \mathcal{O}_{U_1 \times V_1}[\xi_{11}]/(\xi_{11}^2 + x^2(y+1)^3\xi_{11} + x^3y^3 + y^3),$$

$$\pi^{-1}(U_1 \times V_2) = \text{Spec } \mathcal{O}_{U_1 \times V_2}[\xi_{12}]/(\xi_{12}^2 + x^2(1+v)^3\xi_{12} + x^3v^3 + v^3),$$

$$\pi^{-1}(U_2 \times V_1) = \text{Spec } \mathcal{O}_{U_2 \times V_1}[\xi_{21}]/(\xi_{21}^2 + (y+1)^3\xi_{21} + uy^3 + u^4y^3),$$

$$\pi^{-1}(U_2 \times V_2) = \text{Spec } \mathcal{O}_{U_2 \times V_2}[\xi_{22}]/(\xi_{22}^2 + (1+v)^3\xi_{22} + uv^3 + u^4v^3).$$

Then the branch locus of π is the same as in the previous example. Y has two singular points, which lie over the points $(x=0, y=0)$ and $(x=0, y=\infty)$ of X . It is easy to verify that both points are rational double points of type E_6 . Let $\sigma: \tilde{Y} \rightarrow Y$ be the minimal resolution of singularities of Y . Then we have $\omega_{\tilde{Y}} = \sigma^* \circ \pi^* \circ p_2^* \mathcal{O}(1)$ and $\kappa(Y)=1$. Moreover, the composite $f = p_2 \circ \pi \circ \sigma$ defines a quasi-elliptic fibration $f: \tilde{Y} \rightarrow \mathbf{P}^1$.

§ 2. Canonical resolution of singularities in the case of nonsingular branch locus and in characteristic 2.

In this section, we assume $\text{char } k = p = 2$. Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering, which is necessarily of simple type. Suppose that the branch locus B in the sense of § 1 is a nonsingular curve on X . Since Y is normal, Y has at most isolated singularities. We shall consider a resolution of singularities of Y which we call the canonical resolution of singularities of Y . To begin with, we consider a local ring $\mathfrak{D} = k[[x, y]][\xi]/(\xi^2 + x\xi + t)$ with $t \in k[[x, y]]$, which has at most an isolated singularity. Then $\mathfrak{D}$ is normal. Write $t = c_0 + c_1x + c_2y + c_3xy$ with $c_i \in k[[x^2, y^2]]$. Replacing ξ by $\xi + c_1 + c_3y$, we may assume $t = c_0 + c_2y$. So, we can write $t = d_0(y) + x^2d_1(x^2, y)$, where $d_0(y) \neq 0$ by the hypothesis that $\mathfrak{D}$ is normal. Write $d_0 = a_\nu y^\nu + (\text{terms of higher degree})$, where $\nu \geq 0$, $a_\nu \in k$ and $a_\nu \neq 0$. Clearly, $\mathfrak{D}$ is regular if and only if $\nu = 0$ or $\nu = 1$. Furthermore, it is easy to see that ν is invariant under change of variables $(\xi, x, y) \rightarrow (\xi + f, x, y)$ with $f \in k[[x, y]]$ as long as we keep the condition $t = c_0 + c_2y$. Suppose $\nu \geq 2$. Let $x_1 = x/y$. Then

$$\xi^2 + x\xi + d_0(y) + x^2d_1(x^2, y) = \xi^2 + x_1y\xi + d_0(y) + x_1^2y^2d_1(x_1^2y^2, y).$$

Normalizing this equation, we have

$$\xi_1^2 + x_1\xi_1 + d_0^{(1)}(y) + x_1^2d_1(x_1^2y^2, y) = 0, \quad \text{where } \xi_1 = \xi/y.$$

Inductively, one obtains the following series of local rings

$$\begin{aligned}\mathfrak{D} &= \mathfrak{D}_0 = k[[x, y]][\xi]/(\xi^2 + x\xi + d_0(y) + x^2d_1), \\ \mathfrak{D}_1 &= k[[x_1, y]][\xi_1]/(\xi_1^2 + x_1\xi_1 + d_0^{(1)}(y) + x_1^2d_1), \\ &\vdots \\ \mathfrak{D}_n &= k[[x_n, y]][\xi_n]/(\xi_n^2 + x_n\xi_n + d_0^{(n)}(y) + x_n^2d_1),\end{aligned}$$

and

$$\begin{array}{ccccccc}\mathfrak{D} = \mathfrak{D}_0 & \text{---} & \mathfrak{D}_1 & \text{---} & \cdots & \text{---} & \mathfrak{D}_n \\ | & & | & & & & | \\ k[[x, y]] & \text{---} & k[[x_1, y]] & \text{---} & \cdots & \text{---} & k[[x_n, y]],\end{array}$$

where $n = [\nu/2]$. Then $\mathfrak{D}_n$ is regular. Globally speaking, we consider a series of blowing-ups $X = X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n$ with centres $(x=0, y=0)$, $(x_1=0, y=0)$, $\dots$, $(x_{n-1}=0, y=0)$ and consider the normalization Y_i of X_i in the function field $k(Y)$. Thus one obtains a commutative diagram

$$\begin{array}{ccccccc}Y = Y_0 & \leftarrow & Y_1 & \leftarrow & \cdots & \leftarrow & Y_n \\ \pi \downarrow & & \downarrow \pi_1 & & & & \downarrow \pi_n \\ X = X_0 & \leftarrow & X_1 & \leftarrow & \cdots & \leftarrow & X_n.\end{array}$$

We call this process of blowing-ups the *canonical resolution* of the singularity of $\text{Spec } \mathfrak{D}$.

Suppose that ν is even. Then

$$\mathfrak{D}_n \cong k[[x, y]][\eta]/(\eta^2 + x\eta + x + t'(x, y)),$$

where $t'(x, y)$ consists of terms of degree ≥ 2 and $\eta^2 + x\eta + x + t'(x, 0)$ is irreducible. Let E_n be the exceptional curve of the blowing-up $X_n \rightarrow X_{n-1}$ and let $\tilde{E}_n = \pi_n^{-1}(E_n)$. Since $\pi_n^{-1}(E_n) = \{y=0\}$ locally, $\tilde{E}_n$ is an irreducible curve and $(\tilde{E}_n^2) = -2$. Therefore we have the following configuration of exceptional curves which arise from the canonical resolution of the singularity of $\text{Spec } \mathfrak{D}$

$$\circ \text{---} \circ \text{---} \cdots \text{---} \underset{\tilde{E}_n}{\circ} \text{---} \cdots \text{---} \circ \text{---} \circ \quad \text{type } A_{\nu-1},$$

where “ $\circ$ ” stands for a nonsingular rational curve whose self-intersection number is -2 , i.e. a (-2) -curve. In particular, we know the $\text{Spec } \mathfrak{D}$ has a rational double point.

Now suppose that ν is odd. Then

$$\mathfrak{D}_n \cong k[[x, y]][\eta]/(\eta^2 + x\eta + y).$$

Let E_n be as above. Since $\pi^{-1}(E_n) = \{y=0\}$ locally, $\pi^{-1}(E_n)$ splits to two curves $F_n = \{\xi=0\}$ and $G_n = \{\xi+x=0\}$. F_n and G_n intersect transversally at the point $(\xi, x, y) = (0, 0, 0)$. So, $(F_n^2) = (G_n^2) = -2$. Therefore, we have the following

configuration of exceptional curves which arise from the canonical resolution of the singularity of $\text{Spec } \mathfrak{O}$

$$\circ \text{---} \circ \text{---} \cdots \text{---} \underset{\tilde{F}_n}{\circ} \text{---} \underset{\tilde{G}_n}{\circ} \text{---} \cdots \text{---} \circ \text{---} \circ \quad \text{type } A_{\nu-1},$$

where “ $\circ$ ” stands for a (-2) -curve as above. In particular, we know that $\text{Spec } \mathfrak{O}$ has a rational double point.

By virtue of the above observations, we conclude

THEOREM 2.1. *Let $\mathfrak{O}$, ν and t be as above. Then*

(1) *$\text{Spec } \mathfrak{O}$ has a singularity if and only if $\nu \geq 2$.*

(2) *If $\text{Spec } \mathfrak{O}$ has a singular point, then it is a rational double point of type $A_{\nu-1}$.*

We know that ν is an important invariant of a local ring $\mathfrak{O}$. There is the following explicit formula.

LEMMA 2.2. *With the same notations and assumptions,*

$$\nu = \text{length } k[[x, y]]/(x, t + (\partial t / \partial x)^2).$$

PROOF. Write $t = c_0 + c_1x + c_2y + c_3xy$ with $c_i \in k[[x^2, y^2]]$. Set $\xi' = \xi + c_1 + c_3y$. Then $\xi^2 + x\xi + t = \xi'^2 + x\xi' + c_0 + c_1^2 + c_3^2y^2 + c_2y$. So, $d_0(y) + x^2d_1(x^2, y) = c_0 + c_1^2 + c_3^2y^2 + c_2y$. On the other hand, $t + (\partial t / \partial x)^2 = c_0 + c_1^2 + c_3^2y^2 + c_2y + x(c_1 + c_3y)$. Therefore, we have $(x, d_0(y)) = (x, d_0(y) + x^2d_1) = (x, t + (\partial t / \partial x)^2)$ as ideals in $k[[x, y]]$. Since $\nu = \text{length } k[[x, y]]/(x, d_0(y))$, we obtain the required formula.

Q. E. D.

Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering obtained as an α_s -torsor from a line bundle L on X , a global section s of L and local sections $\{t_\lambda\}$ of L^2 (cf. §1). Suppose that $B = (s)_0$ is a nonsingular curve on X and that $\{t_\lambda\}$ give rise to a global section of L^2 . Take an affine covering $\{U_\lambda\}$ such that $s = x_\lambda e_\lambda$ on U_λ and (x_λ, y_λ) is a local coordinate system on U_λ for $U_\lambda \cap B \neq \emptyset$, where $\mathcal{L}|_{U_\lambda} = \mathcal{O}_{U_\lambda} e_\lambda$. Then $\pi^{-1}(U_\lambda) = \text{Spec } \mathcal{O}_X(U_\lambda)[\xi]/(\xi^2 + x_\lambda \xi + t_\lambda)$. For each closed point $P \in X$, we define $\nu(P)$ after Lemma 2.2 as follows:

$$\nu(P) = \begin{cases} \text{length}(\mathcal{O}_{P, X})^\wedge / (x_\lambda, t_\lambda + (\partial t_\lambda / \partial x_\lambda)^2) & \text{if } P \in B \cap U_\lambda \\ 0 & \text{if } P \notin B. \end{cases}$$

We shall estimate $\sum_{P \in Y, \nu(P) > 0} (\nu(P) - 1)$ as follows.

LEMMA 2.3. $\{(\partial t_\lambda / \partial y_\lambda)|_B\} \in H^0(B, \omega_X \otimes \mathcal{L}^3|_B)$.

PROOF. Since $dt_\mu = (\partial t_\mu / \partial x_\mu) dx_\mu + (\partial t_\mu / \partial y_\mu) dy_\mu$, we have $dx_\mu \wedge dt_\mu = (\partial t_\mu / \partial y_\mu) dx_\mu \wedge dy_\mu = (\partial t_\mu / \partial y_\mu) J_{\mu\lambda} dx_\lambda \wedge dy_\lambda$, where $\{J_{\mu\lambda}\}$ are the transition functions of the canonical bundle of X . Let $\{a_{\lambda\mu}\}$ be transition functions of $\mathcal{L}$ such that

$e_\lambda = a_{\lambda\mu} e_\mu$. Then $x_\mu = a_{\lambda\mu} x_\lambda$ and $t_\mu = a_{\lambda\mu}^2 t_\lambda$. So, $dx_\mu = a_{\lambda\mu} dx_\lambda + x_\lambda da_{\lambda\mu}$ and $dt_\mu = a_{\lambda\mu}^2 dt_\lambda$. Therefore,

$$\begin{aligned} dx_\mu \wedge dt_\mu &= a_{\lambda\mu}^3 dx_\lambda \wedge dt_\lambda + x_\lambda a_{\lambda\mu}^2 da_{\lambda\mu} \wedge dt_\lambda \\ &= a_{\lambda\mu}^3 \frac{\partial t_\lambda}{\partial y_\lambda} dx_\lambda \wedge dy_\lambda + x_\lambda a_{\lambda\mu}^2 da_{\lambda\mu} \wedge dt_\lambda. \end{aligned}$$

Hence we have $(\partial t_\mu / \partial y_\mu)|_B \cdot J_{\mu\lambda}|_B = a_{\lambda\mu}^3 |_B \cdot (\partial t_\lambda / \partial y_\lambda)|_B$ on B . This asserts that $\{(\partial t_\lambda / \partial y_\lambda)|_B\} \in H^0(B, \omega_X \otimes \mathcal{L}^3|_B)$. Q. E. D.

PROPOSITION 2.4. $\sum(\nu(P)-1) \leq \max\{2(B^2), 2(B^2)+2p_a(B)-2\}$, where $P \in X$ and $\nu(P) > 0$.

PROOF. Set $\partial_y t = \{(\partial t_\lambda / \partial y_\lambda)|_B\} \in H^0(B, \omega_X \otimes \mathcal{L}^3|_B)$. Suppose $\partial_y t \neq 0$. Let $P \in B$ and $P \in U_\lambda$. We consider t_λ, x_λ and y_λ in $(\mathcal{O}_{P,X})^\wedge$. With the same notations as in Lemma 2.2, $t_\lambda = c_0 + c_1 x_\lambda + c_2 y_\lambda + c_3 x_\lambda y_\lambda$ and $\partial t_\lambda / \partial y_\lambda = c_2 + c_3 x_\lambda$. Since $d_0(y_\lambda) + x_\lambda^2 d_1(x_\lambda^2, y_\lambda) = c_0 + c_1^2 + c_3^2 y_\lambda^2 + c_2 y_\lambda$, we have $\nu(P) \leq (\text{multiplicity of } (\partial_\lambda t)_0 \text{ at } P) + 1$, where $(\partial_y t)_0$ is the effective divisor corresponding to $\partial_y t$. Hence $\sum(\nu(P)-1) \leq (B, 3B + K_X) = 2(B^2) + 2p_a(B) - 2$.

Now, suppose $\partial_y t = 0$, i. e. $\partial t_\lambda / \partial y_\lambda = 0$ on B for all λ . Then $dt_\mu = (\partial t_\mu / \partial x_\mu) dx_\mu$ on B . Since $dx_\mu = (a_{\lambda\mu} + x_\lambda \cdot \partial a_{\lambda\mu} / \partial x_\lambda) dx_\lambda + (\partial x_\mu / \partial y_\lambda) dy_\lambda$,

$$\begin{aligned} dt_\mu &= \frac{\partial t_\mu}{\partial x_\mu} \left[(a_{\lambda\mu} + x_\lambda \cdot \frac{\partial a_{\lambda\mu}}{\partial x_\lambda}) dx_\lambda + \frac{\partial x_\mu}{\partial y_\lambda} dy_\lambda \right] \\ &= \frac{\partial t_\mu}{\partial x_\mu} (a_{\lambda\mu} + x_\lambda \cdot \frac{\partial a_{\lambda\mu}}{\partial x_\lambda}) dx_\lambda + \frac{\partial t_\mu}{\partial x_\mu} \cdot \frac{\partial x_\mu}{\partial y_\lambda} dy_\lambda \quad \text{on } B. \end{aligned}$$

On the other hand, $dt_\mu = a_{\lambda\mu}^2 (\partial t_\lambda / \partial x_\lambda) dx_\lambda + a_{\lambda\mu}^2 (\partial t_\lambda / \partial y_\lambda) dy_\lambda$. Therefore $a_{\lambda\mu}^2 (\partial t_\lambda / \partial x_\lambda) = (\partial t_\mu / \partial x_\mu) \{a_{\lambda\mu} + x_\lambda (\partial a_{\lambda\mu} / \partial x_\lambda)\}$ on B . Namely, we have $(\partial t_\mu / \partial x_\mu)|_B \cdot (a_{\lambda\mu}|_B) = (a_{\lambda\mu}^2|_B) (\partial t_\lambda / \partial x_\lambda)|_B$. Hence $\{\partial t_\lambda / \partial x_\lambda|_B\}$ is a global section of $\mathcal{L}|_B$. Set $\tau = \{[t_\lambda + (\partial t_\lambda / \partial x_\lambda)^2]|_B\}$. Then τ is a global section of $\mathcal{L}^2|_B$. For $P \in B$, we know that $\nu(P) = (\text{multiplicity of } (\tau)_0 \text{ at } P)$. So, $\sum_{P \in X} \nu(P) = 2(B^2)$. The assertion follows from these observations. Q. E. D.

Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering with nonsingular branch locus B and let

$$\begin{array}{ccc} Y & \xleftarrow{\rho} & \tilde{Y} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ X & \xleftarrow{\sigma} & \tilde{X} \end{array}$$

be the canonical resolution of singularities of Y .

PROPOSITION 2.5. $\tilde{Y}$ has the dualizing sheaf

$$\omega_{\tilde{Y}} = \rho^* \circ \pi^*(\omega_X \otimes \mathcal{O}(B)).$$

PROOF. We already know that $\omega_Y = \pi^*(\omega_X \otimes \mathcal{O}(B))$. Since every singularity of Y is a rational double point by Theorem 2.1, $\omega_{\tilde{Y}} = \rho^* \omega_Y$. Therefore, $\omega_{\tilde{Y}} = \rho^* \circ \pi^*(\omega_X \otimes \mathcal{O}(B))$. Q. E. D.

COROLLARY 2.6. (1) $(K_{\tilde{Y}})^2 = 2(K_X + B)^2$.

(2) $\kappa(\tilde{Y}) = \kappa(X, K_X + B)$.

PROOF. Straightforward.

In general, $\tilde{Y}$ may not be a minimal surface. However we have

PROPOSITION 2.7. *If K_X is numerically effective, nef in short, $\tilde{Y}$ is a minimal surface.*

PROOF. Let E be a (-1) -curve on $\tilde{Y}$ and write $\phi = \sigma \circ \tilde{\pi}$. Suppose that $\phi_* E = 2C$, where C is the set-theoretic image of E . Since $(E, K_Y) = -1$, we have $-1 = 2(C, K_X + B)$ by the projection formula. This is a contradiction. Now, suppose $\phi_* E = C$. Similarly we have $-1 = (C, K_X + B)$. So, $(C, B) = -1 - (C, K_X) \leq -1$ by the assumption. Hence C must be an irreducible component of B . Since B is a disjoint union of nonsingular curves, $(C, B) = (C^2)$. Therefore we have $(C, K_X + C) = -1$ and this is impossible. This completes the proof. Q. E. D.

We shall compute other invariants of $\tilde{Y}$.

PROPOSITION 2.8. *Under the same notations, we have*

$$\chi(\mathcal{O}_{\tilde{Y}}) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}(-B)).$$

PROOF. Since Y has only rational double points, one obtains $\chi(\mathcal{O}_{\tilde{Y}}) = \chi(\mathcal{O}_Y)$. Now the assertion follows from Lemma 1.8.(2). Q. E. D.

COROLLARY 2.9. $e(\tilde{Y}) = 2[e(X) + (B, K_X + 2B)]$.

We have already computed the irregularity in the case where Y is non-singular with an assumption (cf. Lemma 1.9). Here we have another formula

PROPOSITION 2.10. *Under the same notations and assumptions as above,*

$$h^1(X, \mathcal{O}_X) \leq h^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) \leq h^1(X, \mathcal{O}_X) + h^1(X, \mathcal{O}(-B)).$$

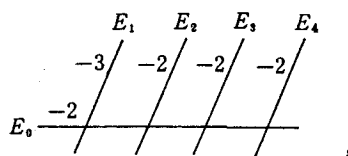
PROOF. Since Y has only rational singularities, $H^i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = H^i(Y, \mathcal{O}_Y)$. On the other hand, because $H^0(X, \mathcal{O}(-B)) = (0)$, we have

$$0 \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \pi_* \mathcal{O}_{\tilde{Y}}) \longrightarrow H^1(X, \mathcal{O}(-B)).$$

Hence $h^1(X, \pi_* \mathcal{O}_{\tilde{Y}}) \leq h^1(X, \mathcal{O}_X) + h^1(X, \mathcal{O}(-B))$. The assertion follows from this. Q. E. D.

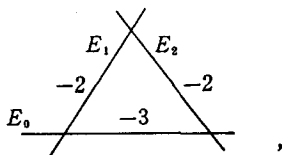
The following two examples of local rings have irrational singularities and appear as the local rings of Artin-Schreier coverings with non-reduced or singular branch loci.

EXAMPLE 2.11. Let $\mathfrak{D} = k[[x, y]][\xi]/(\xi^5 + x^2\xi + x^3 + x^2y^3 + xy^6)$, let $Y = \text{Spec } \mathfrak{D}$ and let $\sigma: \tilde{Y} \rightarrow Y$ be the minimal resolution of the singularity of Y . Then the exceptional locus of σ has the following configuration:



where $E_0, \dots, E_4$ are nonsingular rational curves. The fundamental cycle Z of this singularity is $2E_0 + E_1 + E_2 + E_3 + E_4$. Hence $p_a(Z)=1$. So, this singularity is not rational.

EXAMPLE 2.12. Let $\mathfrak{D} = k[[x, y]][\xi]/(\xi^2 + xy\xi + x^3 + y^9)$, let $Y = \text{Spec } \mathfrak{D}$ and let $\sigma: \tilde{Y} \rightarrow Y$ be the minimal resolution of the singularity of Y . Then the exceptional locus of σ has the following configuration:



where E_0, E_1 and E_2 are nonsingular rational curves. The fundamental cycle Z is $E_0 + E_1 + E_2$ and $p_a(Z)=1$. Hence this singularity is not rational.

To close this section, we shall give an example of the canonical resolution.

EXAMPLE 2.13. Let $X = \mathbf{P}^2$ and let $\mathcal{L} = \mathcal{O}(1)$. Take $s \in H^0(X, \mathcal{L})$. Then $(s)_0$ is a line. Consider an Artin-Schreier covering Y whose branch locus B is $(s)_0$. Since $H^1(X, \mathcal{L})=0$, we know that such a covering is obtained as an α_s -torsor from s and a global section of $\mathcal{L}^2$. If Y is smooth, then Y is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ (see §3 Theorem 3.2). Suppose Y is singular. Since $2(B^2)=2$ and $2(B^2)+2p_a(B)-2=0$, we have $\sum_{P \in X} \nu(P)=2$ by the proof of Proposition 2.4. Hence Y has only one singular point of type A_1 . Let

$$\begin{array}{ccc} Y & \xleftarrow{\quad} & \tilde{Y} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ X & \xleftarrow{\quad} & \tilde{X} \end{array}$$

be the canonical resolution. Then $\tilde{X}$ is the Hirzebruch surface of degree 1 and

the branch locus of $\tilde{\pi}$ is a fibre of the canonical P^1 -fibration $\theta: \tilde{X} \rightarrow P^1$. By the Stein factorization of $\theta \circ \tilde{\pi}$, we obtain a P^1 -fibration on $\tilde{Y}$. More precisely, $\tilde{Y}$ is the Hirzebruch surface of degree 2.

§3. Artin-Schreier coverings of simple type with ample branch loci.

In this section, the characteristic p of k is not necessarily 2. Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering of simple type, where X and Y are nonsingular projective surfaces. The branch locus B of π is assumed to be a reduced ample curve satisfying $H^1(X, \mathcal{O}(-B)) = 0$. We denote by Σ_n the Hirzebruch surface of degree n . We shall fix these notations and assumptions throughout the section. The following lemma is immediately derived from the classification of the divisor $K_X + (\text{ample divisor})$. For the reader's convenience, we shall give the proof.

LEMMA 3.1. *Suppose that the canonical divisor K_Y of Y is not numerically effective, not nef in short. Then the following assertions hold:*

- (1) $p < 5$.
- (2) If $p=2$, then X is either a relatively minimal ruled surface or the projective plane.
- (3) If $p=3$, then X is the projective plane.

PROOF. Since K_X is not nef, there exists a curve C on Y such that $(K_Y, C) < 0$. Set $D = \pi(C)$. Then $(K_X + (p-1)B, D) < 0$ by the canonical divisor formula in Proposition 1.7. Let $\overline{NE}(X)$ be the closed convex cone spanned by all effective divisors on X modulo numerical equivalence. Let $P = \{E \in \overline{NE}(X) \mid (K_X + (p-1)B, E) < 0\}$ and $Q = \{E \in \overline{NE}(X) \mid (K_X, E) < 0\}$. Then $P \subset Q$ and $P \neq \emptyset$. By the Mori theory, Q is polyhedral and so is P . Hence there exists an extremal rational curve l such that $(K_X + (p-1)B, l) < 0$. Moreover, one of the following three cases takes place:

- (1) l is a line on P^2 ;
- (2) l is a fibre on a relatively minimal ruled surface;
- (3) l is a (-1) -curve.

We consider these three cases separately.

Case (1). Since $X = P^2$, one obtains $B \sim al$ for some positive integer a and $(B, l) = a$. On the other hand, $(K_X, l) = -3$. So, $(K_X + (p-1)B, l) < 0$ implies $(p-1)a < 3$. Hence $(p-1)a = 1$ or 2. Only three cases can occur: (i) $p=2$ and $a=1$; (ii) $p=2$ and $a=2$; (iii) $p=3$ and $a=1$.

Case (2). We know that $(K_X, l) = -2$. Let $B \equiv aM + bl$ ($a, b \in \mathbb{Z}$), where " $\equiv$ " means the numerical equivalence and M is a cross-section of the P^1 -fibration given on X . Since B is ample, $a > 0$ and $b + a(M^2) > 0$. The inequality

$(K_X + (p-1)B, l) < 0$ implies $(p-1)a < 2$. Hence $p=2$ and $a=1$.

Case (3). Since l is a (-1) -curve, we have $(K_X, l) = -1$. By the same inequality as above, one obtains $(p-1)(B, l) < 1$. This is impossible. Q.E.D.

We shall specify each case.

THEOREM 3.2. Suppose $X = \mathbf{P}^2$. Then we have:

(1) B is either a line or a conic. If B is a line (resp. conic), then $p=2$ or 3 (resp. $p=2$).

(2) If $p=2$ and B is a line, then Y is isomorphic to Σ_0 .

(3) If $p=3$ and B is a line, then Y is a del Pezzo surface of degree 3, i.e. Y is a cubic hypersurface in $\mathbf{P}^3$.

(4) If B is a conic (hence $p=2$), then Y is a del Pezzo surface of degree 2 and the generator of the Galois group of $\pi: Y \rightarrow X$ is the Geiser involution.

PROOF. (1) The assertion was already verified in the proof of the previous lemma.

(2) Since $K_X + B \sim -2B$, we have $(K_Y^2) = 8$. On the other hand, the irregularity $q(Y)$ of Y equals to that $q(X)$ of X by Lemma 1.9. So, $q(Y) = 0$ and Y is a Hirzebruch surface Σ_n . Consider the canonical divisor K_Y . We can write $K_Y = -2M_0 - (n+2)L$, where M_0 is the minimal section and L is a fibre. Meanwhile, $K_Y = -2\pi^*B$. Hence n is an even number. Write $\pi^*B \sim M_0 + aL$ ($a \in \mathbf{Z}$). Then $2a = n+2$. Since B is ample, so is π^*B . Thus $a > n$. So, $n=0$.

(3) and (4) Straightforward.

Q.E.D.

THEOREM 3.3. Suppose that X is a relatively minimal ruled surface with irregularity q . Then the following assertions hold:

(1) $B = S + l_1 + \dots + l_r$, where S is a cross-section and l_i 's are fibres.

(2) Y is a ruled surface with the $\mathbf{P}^1$ -fibration $f = \theta \circ \pi: Y \rightarrow A$, where $\theta: X \rightarrow A$ is either induced by the Albanese mapping or the canonical $\mathbf{P}^1$ -fibration. A general fibre F of f is regarded as an Artin-Schreier covering of $l = \pi(F)$.

(3) Any singular fibre of f consists of two (-1) -curves crossing each other transversally.

(4) Let N be the number of singular fibres of f . Then $N = 2(S^2) + 4r > 0$.

(5) $\tilde{S} = \pi^*(S)$ is an irreducible curve with $p_a(\tilde{S}) = (S^2) + 2q + r - 1$. Moreover, $\tilde{S}$ is a singular curve unless $X \cong \Sigma_n$ ($n \geq 0$) and $B = M + l_1 + \dots + l_{n+1}$, where M is the minimal section and l_i 's are fibres.

PROOF. (1) The assertion was already shown in the proof of Lemma 3.1.

(2) Straightforward.

(3) Let π^*l_0 be a singular fibre of f , where l_0 is a fibre of θ . Since π is a double covering, π^*l_0 has a form $E_1 + E_2$, where E_i 's are nonsingular rational curves. Moreover, one of the components is a (-1) -curve. Hence so is the

other. From $((\pi^*l_0)^2)=0$, it follows that $(E_1, E_2)=1$.

(4) Note that $p_a(B)=p_a(S)+\sum_{i=1}^r p_a(l_i)+\sum_{i=1}^r (S, l_i)+\sum_{i < j} (l_i, l_j)+1-(1+r)=q$. Since $K_Y=\pi^*(K_X+B)$, we have $(K_Y^2)=2[2p_a(B)-2+8(1-q)+(K_X, B)]=8(1-q)-2(S^2)-4r$. On the other hand, $N=8(1-q)-(K_Y^2)$. Hence $N=2(S^2)+4r$. Furthermore, $(B^2)=(S^2)+2r>0$. Thence follows the assertion.

(5) Suppose $\pi^*(S)=2\tilde{S}$. By (4), there exists a singular fibre $\pi^*(l_0)=E_1+E_2$ on Y . By the projection formula, $2(\tilde{S}, E_1)=(S, l_0)=1$. This is impossible. Hence $\tilde{S}=\pi^*(S)$ is an irreducible curve. Since $(S, K_X+B)=2q-2+r$, we have $p_a(\tilde{S})=(1/2)(\tilde{S}, \tilde{S}+K_Y)+1=(S^2)+2q+r-1$. On the other hand, the restriction $\pi|_{\tilde{S}}: \tilde{S} \rightarrow S$ is a purely inseparable morphism. Hence the geometric genus of $\tilde{S}$ is equal to q . Suppose that $\tilde{S}$ is nonsingular. Then one obtains $q=0$ and $(S^2)+r=1$ since $q \geq 0$ and $(S^2)+r>0$. In particular, X is a rational ruled surface Σ_n . Set $S \sim M+al$, where $a \geq 0$. Then $(S^2)=-n+2a$. Thence $1-r=-n+2a$. Since B is ample, $a+r>n$ and hence $1-a=a+r-n>0$. We therefore have $a=0$ and $r=n+1$. Q. E. D.

By the above results, we have:

PROPOSITION 3.4. *If K_Y is not nef, then $p < 5$ and $\kappa(Y) = -\infty$. Hence, if $\kappa(Y) \geq 0$, then K_Y is nef, i. e., Y is relatively minimal.*

We shall now consider the case with $\kappa(Y)=0$ and 1.

THEOREM 3.5. *Suppose $\kappa(Y)=0$. Then we have:*

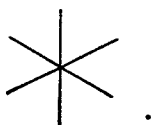
- (1) $p=2$ or 3. Moreover, X is a del Pezzo surface and Y is a K3-surface. Furthermore,
- (2) if $p=3$, then $X=\Sigma_0$ and $B \in |M+l|$, where M is the minimal section and l is a fibre.

PROOF. (1) Since Y is relatively minimal, $K_Y \equiv 0$. This implies $K_X+(p-1)B \equiv 0$. So, $-K_X$ is ample. Hence X is a del Pezzo surface. This implies $(p-1)^2(B^2)=(K_X^2) \leq 9$, whence $p-1 \leq 3$, i. e., $p=2$ or $p=3$. Since X is a rational surface, $K_X+(p-1)B \sim 0$. Thence $K_Y \sim 0$. On the other hand, $q(Y)=q(X)=0$ by Lemma 1.9. Hence Y is a K3-surface.

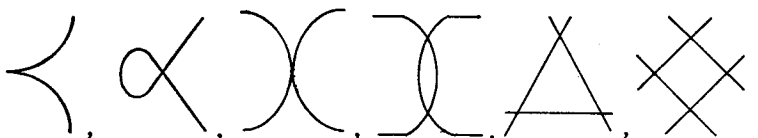
- (2) Suppose $p=3$. Then $K_X=-2B$. Hence X is Σ_0 and $B \sim M+l$. Q.E.D.

THEOREM 3.6. *Suppose $\kappa(Y)=1$. Then we have:*

- (1) Either $p=2$ or 3, and X is a ruled surface. Furthermore, if $\theta: X \rightarrow A$ is a natural $\mathbf{P}^1$ -fibration on X (cf. Theorem 3.3), then $f=\theta \circ \pi: Y \rightarrow A$ is an elliptic or quasi-elliptic fibration.
- (2) If $p=3$, then X is relatively minimal, $B=S+l_1+\dots+l_r$ with a cross-section S , and every fibre of f is reduced. Moreover, any singular fibre is either a cuspidal curve or



(3) If $p=2$, then the horizontal part of components of B consists of either two cross-sections S_1 and S_2 or a single 2-section T . Any singular fibre of θ has the form E_1+E_2 with $(E_1^2)=(E_2^2)=-1$ and $(E_1, E_2)=1$. Any singular fibre of f has one of the following forms:



PROOF. (1) Since $\kappa(Y)=1$, Y has an elliptic or quasi-elliptic fibration f . Let F be a general fibre of f . Since $aK_Y \approx bF$ for some positive integers a and b , we have $b(gF) \approx bF$ for any element g of the Galois group G , where " $\approx$ " means the algebraic equivalence. So, gF is also a fibre of f . Let $C = (\pi_* F)_{\text{red}}$. By the projection formula, $(K_Y, F) = (K_X + (p-1)B, \pi_* F) = 0$. Hence $(K_X + (p-1)B, C) = 0$. Since $(B, C) > 0$, we have $(K_X, C) < 0$. On the other hand, we have $(C^2) = 0$ because $\pi^*(C) \approx cF$ for some positive integer c . So, $C \cong P^1$ and $(K_X, C) = -2$. Thus X is a ruled surface. Let $\theta: X \rightarrow A$ be the canonical P^1 -fibration if $q(X) > 0$ and the P^1 -fibration defined by the linear system $|C|$ if $q(X) = 0$. Then f must be the composite $\theta \circ \pi$. Furthermore, if l is a general fibre of θ , then $f^*(l)$ is a general fibre F of f and $\pi|_F: F \rightarrow l$ is an Artin-Schreier covering. We may assume that l is the above C . So, $-2 = -(p-1)(B, l)$. Hence, $p=2$ or 3 .

(2) Suppose F_0 is a reducible singular fibre of f and G is an irreducible component of F_0 . Then $G \cong P^1$, $(G^2) = -2$ and $(G, K_Y) = 0$. Set $E = (\pi_* G)_{\text{red}}$. Then $(K_X + (p-1)B, E) = 0$. Since $(B, E) > 0$, we have $(K_X, E) < 0$. Hence, if $(E^2) < 0$, then E is a (-1) -curve and $p=2$. From these observations, it follows that the P^1 -fibration θ has no singular fibres provided $p=3$. Indeed, if H is a singular fibre of θ , then π^*H is a reducible singular fibre of f , whose existence implies $p=2$. Thus X is a relatively minimal ruled surface if $p=3$. Let l be a general fibre of θ . Then $(K_X + 2B, l) = 0$. So, $(B, l) = 1$. Hence we can write $B = S + l_1 + \dots + l_r$, where S is a cross-section and l_i 's are fibres. Now the remaining assertions can be easily verified.

(3) From the same arguments as in (2), it follows that every singular fibre of θ has the form $E_1 + E_2$, where E_1 and E_2 are nonsingular rational curves crossing each other transversally. Moreover, if l is a general fibre of θ , then $(B, l) = 2$. Then the assertions follow from these observations. Q. E. D.

By virtue of the above results, we conclude the following:

COROLLARY 3.7. *If $p > 3$, then Y is a relatively minimal surface of general type. In particular, K_Y is nef.*

In Theorems 3.5 and 3.6, we considered the case where Y has an elliptic or quasi-elliptic fibration. When $p=3$, we have a more precise result.

THEOREM 3.8. *Assume $p=3$. Suppose that X is a relatively minimal ruled surface with irregularity q and that $f=\theta \circ \pi: Y \rightarrow A$ is an elliptic or quasi-elliptic fibration, where $\kappa(Y) \geq 0$ and $\theta: X \rightarrow A$ is the natural $\mathbf{P}^1$ -fibration. Let B be the branch locus of π and write $B=S+l_1+\cdots+l_r$, where S is a cross-section and l_i 's are fibres of θ . Then we have the following:*

- (1) π^*S is reduced.
- (2) $\tilde{S}=\pi^*(S)$ is a singular curve.
- (3) f is an elliptic fibration.
- (4) $\kappa(Y) > 0$ if and only if $2(q-1)+(S^2)+2r > 0$.

PROOF. (1) Suppose $\pi^*S=3\tilde{S}$. Then $\pi|_{\tilde{S}}: \tilde{S} \rightarrow S$ is an isomorphism. So, $2q-2=(\tilde{S}^2)+(\tilde{S}, K_Y)$. Meanwhile, $(\tilde{S}^2)=(1/3)(S^2)$ and $(\tilde{S}, K_Y)=2q-2+(S^2)+2r$. Hence $(S^2)+r=-(1/2)r < 0$. However, $(B, S)=(S^2)+r > 0$, a contradiction.

(2) Since $\pi|_{\tilde{S}}: \tilde{S} \rightarrow S$ is a purely inseparable covering, $\tilde{S}$ has the geometric genus q . On the other hand, $p_a(\tilde{S})$ is computed as

$$\begin{aligned} p_a(\tilde{S}) &= (1/2)(\pi^*S, \pi^*S + K_Y) + 1 \\ &= (3/2)[(2q-2) + 2(S^2) + 2r] + 1 \\ &= 3q + 3((S^2) + r) - 2. \end{aligned}$$

Hence $p_a(\tilde{S}) - q = 2(q-1) + 3((S^2) + r) > 0$.

(3) Suppose f is a quasi-elliptic fibration. Then $\tilde{S}$ must be the locus of moving singularities on Y . Hence $\tilde{S}$ is nonsingular (cf. Bombieri-Mumford [4]). This contradicts (2).

(4) Compute $(K_X + 2B, S) = 2(q-1) + (S^2) + 2r$.

Q. E. D.

In characteristic $p=2$, we have the following partial result:

THEOREM 3.9. *Assume $p=2$. Suppose that X is a ruled surface and $f=\theta \circ \pi: Y \rightarrow A$ is an elliptic or quasi-elliptic fibration, where $\theta: X \rightarrow A$ is the natural $\mathbf{P}^1$ -fibration. Suppose that $\kappa(Y) \geq 0$ and that the horizontal part of components of B consists of S_1 and S_2 which are cross-sections (resp. a 2-section T). Then we have:*

- (1) $\pi^*(S_i)$ (resp. $\pi^*(T)$) is reduced.

Suppose, furthermore, that one of the following conditions holds:

- (i) $q(X) \neq 0$;

(ii) $(S_i^2) \geq 0$ for $i=1, 2$ (resp. $(T^2) \geq 0$).

Then

(2) $\tilde{S}_i = \pi^* S_i$ is a singular curve for $i=1, 2$ (resp. $\tilde{T} = \pi^* T$ is a singular curve).

(3) f is an elliptic fibration.

PROOF. At first we consider the case where B contains two cross-sections.

(1) Suppose $\pi^* S_i = 2\tilde{S}_i$. Then $\tilde{S}_i \cong S_i$ via π and $(\tilde{S}_i^2) = (1/2)(S_i^2)$. Meanwhile, $2p_a(\tilde{S}_i) - 2 = (\tilde{S}_i, K_Y + \tilde{S}_i) = (\tilde{S}_i^2) + (S_i, K_X + B) = (\tilde{S}_i^2) + 2p_a(S_i) - 2 + (B - S_i, S_i)$. Hence $(B, S_i) = (1/2)(S_i^2)$. So, $(B, S_i) = -(B - S_i, S_i)$. Since $(B, S_i) > 0$ and $(B - S_i, S_i) > 0$, this is a contradiction.

(2) Suppose $\tilde{S}_i$ is nonsingular. Since the restriction of $\pi: \tilde{S}_i \rightarrow S_i$ is purely inseparable, we have $p_a(\tilde{S}_i) = p_a(S_i)$. On the other hand, $2p_a(\tilde{S}_i) - 2 = 4p_a(S_i) - 4 + 2(B, S_i)$. Hence $(B, S_i) = 1 - p_a(S_i)$, whence $p_a(S_i) = 0$ and $(B, S_i) = 1$. Meanwhile, we have $(K_X + B, S_i) \geq 0$. So, $(K_X, S_i) \geq -1$. This implies $(S_i^2) \leq -1$. This is, however, inconsistent with the hypothesis. Hence $\tilde{S}_i$ is singular.

(3) It is similar to the proof of (3) of the previous lemma.

The case where B contains a 2-section is handled in the same way as above. Q. E. D.

The condition (i) or (ii) in the previous theorem is not necessary to show that f is an elliptic fibration. In fact, we have the following:

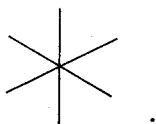
PROPOSITION 3.10. *Let X, Y, A, π, f , and θ be as in the previous theorem. Then f is an elliptic fibration.*

PROOF. Suppose f is a quasi-elliptic fibration. Let Γ be the locus of moving singularities on Y . In view of the construction of the fibration f , we know that $\pi(\Gamma)$ is contained in the horizontal part of B . Take an general fibre l of θ and choose a local parameter y of A so that l is defined by $y=0$. Let $\{P\} = \pi(\Gamma) \cap l$ and $Q = \pi^{-1}(P)$. Assume that B is locally given by $x=0$, where (x, y) is a local coordinate system at P . We consider the completion $(\mathcal{O}_{P, X})^\wedge = k[[x, y]]$. Let $\mathfrak{D}$ be $\mathcal{O}_{Q, Y} \otimes_{\mathcal{O}_{P, X}} k[[x, y]]$. Suppose $\mathfrak{D} = k[[x, y]][\xi]/(\xi^2 + x\xi + t)$ with $t = c_0(y) + xc_1(y) + x^2c_2(y) \in k[[x, y]]$. Write $\Phi = \xi^2 + x\xi + t$. Since f is a quasi-elliptic fibration, we must have $\partial\Phi/\partial\xi = \partial\Phi/\partial x = 0$ wherever $x=0$. This implies that $\xi + \partial t/\partial x = \xi + c_1(y) = 0$ wherever $x=0$. Meanwhile, $\xi^2 = c_0(y)$ wherever $x=0$. Therefore, $c_0(y) = c_1(y)^2$. So, we have $\partial\Phi/\partial y = 0$ wherever $x=0$. Hence $\mathfrak{D}$ is not normal, a contradiction. Q. E. D.

In the rest of this section, we shall construct examples of singular fibres of elliptic fibrations.

EXAMPLE 3.11. Assume $\text{char } k = p = 3$. Let $\pi: Y \rightarrow X$ be as in Example 1.10.

Then $f = p_1 \circ \pi: Y \rightarrow \mathbf{P}^1$ is an elliptic fibration and two fibres $f^{-1}(P_0)$ and $f^{-1}(P_\infty)$ exhaust singular fibres of f , where we consider the $\mathbf{P}^1$ -fibration on $\mathbf{P}^1 \times \mathbf{P}^1$ defined by the first projection p_1 and where P_0 and P_∞ are points of $\mathbf{P}^1$ defined respectively by $x=0$ and $x=\infty$. Moreover, $f^{-1}(P_0)$ is a cuspidal rational curve and $f^{-1}(P_\infty)$ is of type



EXAMPLE 3.12. Assume $\text{char } k = p = 2$. Let $X = \mathbf{P}^1 \times \mathbf{P}^1$ and let $\{U_i \times V_j\}_{i,j=1,2}$ be the same as in Example 1.12. Take an Artin-Schreier covering $\pi: Y \rightarrow X$ which is defined by

$$\xi^2 + xy(x+y)\xi + (x+y) + ax^3 + by^3 + (x^3y^4 + x^4y^3) = 0$$

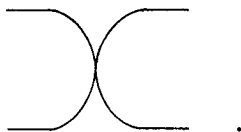
$$(a, b \in k \text{ and } ab(a+b) \neq 0)$$

over $U_1 \times V_1$ and whose branch locus is $L + M + \Delta$, where $L = \{x=0\}$, $M = \{y=0\}$ and $\Delta = \{x+y=0\}$, i.e., the diagonal. Then Y is a smooth K3-surface with an elliptic fibration $p_1 \circ \pi$. Let F_α be the fibre of f defined by $x=\alpha$. We have the following singular fibres:

F_0 : a rational curve with one cusp.

F_α : a rational curve with one node, where α satisfies one of the following equations: $1 + b\alpha^4 + (a+b)\alpha^7 + \alpha^{12} = 0$, $1 + \alpha^5 + a\alpha^7 = 0$ or $\alpha = \infty$.

EXAMPLE 3.13. Keep the same assumptions and notations as in the previous example. Let $\sigma: X' \rightarrow X$ be the blowing-up with centre $(x=1, y=0)$ and let $\pi': Y' \rightarrow X'$ be the normalization of X' in $k(Y)$. Then the branch locus B' of π' is $E + L' + M' + \Delta'$, where L' , M' and Δ' are the proper transforms of L , M and Δ , respectively and where E is the exceptional curve of σ . Moreover, the fibre F_0 is replaced by



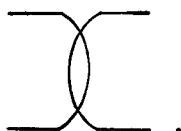
EXAMPLE 3.14. Let X and $\{U_i \times V_j\}$ be as above. Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering which is defined by

$$\xi^2 + xy(x+1)(y+1)\xi + ax + by + x^3 + y^3 = 0$$

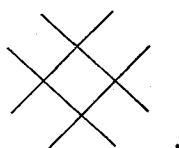
$$(a, b \in k, a \neq 0, b \neq 0, a+1+b(b+1) \neq 0, b+1+a(a+1) \neq 0)$$

over $U_1 \times V_1$ and whose branch locus is $L_0 + L_1 + M_0 + M_1$, where L_0 , L_1 , M_0 and

M_1 are defined by $x=0$, $x=1$, $y=0$ and $y=1$, respectively. Then Y is a smooth $K3$ -surface with an elliptic fibration $f=p_1\circ\pi: Y\rightarrow\mathbf{P}^1$. Let F_∞ be the fibre defined by $x=\infty$. Then F_∞ is of type



Blow up the point $(x=\infty, y=0)$ to obtain $\sigma: X'\rightarrow X$. Let Y' be the normalization of X' in $k(Y)$. Then F_∞ is replaced by



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Yoshifumi TAKEDA

Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan