

Title	Artin-Schreier coverings of algebraic surface
Author(s)	武田, 好史
Citation	大阪大学, 1990, 博士論文
Version Type	VoR
URL	https://hdl.handle.net/11094/3328
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

1255
13
9050

Artin-Schreier coverings of algebraic surfaces

By Yoshifumi TAKEDA

(Received March 17, 1988)

Introduction.

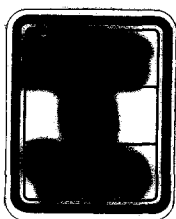
Let k be an algebraically closed field of characteristic $p > 0$ and let X be a nonsingular projective surface defined over k . An *Artin-Schreier covering* of X is a finite morphism $\pi: Y \rightarrow X$ from a normal surface Y onto X such that the field extension $k(Y)/k(X)$ is an Artin-Schreier extension. It is well-known that $k(Y)$ is expressed as $k(X)(\xi)$ with $\xi^p - \xi = f$ and $f \in k(X)$. Since $k(Y)/k(X)$ is a Galois extension with the Galois group $G \cong \mathbb{Z}/p\mathbb{Z}$, G acts on Y so that $X \cong Y/G$. In order to study Artin-Schreier coverings, we have to consider whether or not there exists an affine open covering $\mathfrak{U} = \{U_\lambda\}$ such that $\pi^{-1}(U_\lambda) = \text{Spec } \mathcal{O}_X(U_\lambda)[\xi_\lambda]/(\xi_\lambda^p - s_\lambda \xi_\lambda - t_\lambda)$ with $s_\lambda, t_\lambda \in \mathcal{O}_X(U_\lambda)$. In general, this assertion does not hold (cf. Example 1.5). Under the above circumstance, we shall define an Artin-Schreier covering of simple type (see §1 for the definition), for which the assertion holds. From the definition, every Artin-Schreier covering in characteristic 2 is of simple type.

This article consists of three parts. In Section 1, we consider Artin-Schreier coverings of simple type and give some formulas to compute invariants in the case of nonsingular coverings. In Section 2, we assume that the characteristic is 2 and consider a resolution of singularities for Artin-Schreier coverings with nonsingular branch locus. We give some formulas to compute invariants of nonsingular models of coverings. In Section 3, we consider smooth Artin-Schreier coverings of simple type with ample branch loci which satisfy extra conditions. Especially, we shall determine such coverings with $\kappa = -\infty, 0$, and 1.

The author would like to express his sincere gratitude to Professor M. Miyanishi who initiated him into this problem and gave several key ideas, and to Dr. S. Tsunoda who occasionally gave him useful comments.

§1. Artin-Schreier coverings of simple type.

Let X be a nonsingular projective surface and let $\pi: Y \rightarrow X$ be an Artin-Schreier covering. Since Y is a Cohen-Macaulay scheme and X is regular, π is a flat morphism. Hence $\pi_* \mathcal{O}_Y$ is a locally free \mathcal{O}_X -algebra. Moreover,



PROPOSITION 1.1. *There is a canonical filtration of \mathcal{O}_X -modules of $\pi_*\mathcal{O}_Y$,*

$$\mathcal{O}_X = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_{p-1} = \pi_*\mathcal{O}_Y$$

such that

- (1) \mathcal{F}_i is a locally free sheaf of rank $i+1$,
- (2) $\mathcal{F}_i/\mathcal{F}_0$ is an invertible sheaf and $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a torsion-free \mathcal{O}_X -module of rank 1 for $1 \leq i \leq p-1$.

PROOF. Let $U = \text{Spec } R$ be an affine open subset of X and let $\pi^{-1}(U) = \text{Spec } A$. Then $\pi^{-1}(U)$ is a G -stable set and $U = \pi^{-1}(U)/G$. On the other hand, as a group scheme, G is written as $G = \text{Spec } k[z]/(z^p - z)$ with the comultiplication $\Delta(z) = z \otimes 1 + 1 \otimes z$ and the counit $\varepsilon(z) = 0$. So, the coaction of G on $\text{Spec } A$ is given by an R -algebra homomorphism $\sigma: A \rightarrow A[z]$ with $z^p = z$ such that $(\sigma \otimes 1)\sigma = (1 \otimes \Delta)\sigma$ and $(1 \otimes \varepsilon)\sigma = \text{id}_A$. Write $\sigma(a) = \sigma_0(a) + \sigma_1(a)z + \dots + \sigma_{p-1}(a)z^{p-1}$ for $a \in A$. Then $(1 \otimes \varepsilon)\sigma = \text{id}_A$ implies $\sigma_0 = \text{id}_A$. We have

$$(\sigma \otimes 1)\sigma(a) = \sum_{i=0}^{2p-2} \sum_{j=0}^i \sigma_j \sigma_{i-j}(a) z^j \otimes z^{i-j}$$

and

$$(1 \otimes \Delta)\sigma(a) = \sum_{i=0}^{p-1} \sum_{j=0}^i C_j \sigma_i(a) z^j \otimes z^{i-j}.$$

Thence the relation $(\sigma \otimes 1)\sigma = (1 \otimes \Delta)\sigma$ implies $\sigma_j \sigma_{i-j} = C_j \sigma_i$ for $0 \leq i \leq p-1$ and $\sigma_i = 0$ for $i \geq p$. Set $\sigma_1 = \delta$. Then these relations are equivalent to $\sigma_0 = \text{id}_A$, $\sigma_i = 1/(i!) \delta^i$ ($1 \leq i \leq p$) and $\delta^p = 0$. So, we can write

$$\sigma(a) = a + \delta(a)z + \frac{1}{2!} \delta^2(a)z^2 + \dots + \frac{1}{(p-1)!} \delta^{p-1}(a)z^{p-1}.$$

Set $F_i = \{a \in A \mid \delta^{i+1}(a) = 0\}$ for $0 \leq i \leq p-1$. Then $F_0 = R$ and F_i is an R -module. Since the G -action on A is nontrivial, there exists $a \in A$ such that $\sigma(a) \neq a$. Suppose $\delta^r(a) \neq 0$ and $\delta^{r+1}(a) = 0$ for $0 < r < p$. Then $\delta^r(a) \in F_0$. So, $\sigma(\delta^{r-1}(a)) = \delta^{r-1}(a) + \delta^r(a)z$. Therefore, $F_1 \neq F_0$. Furthermore, for $1 \leq i \leq p-1$,

$$\begin{aligned} \sigma((\delta^{r-1}(a))^i) &= \sigma(\delta^{r-1}(a))^i = (\delta^{r-1}(a) + \delta^r(a)z)^i \\ &= \delta^{r-1}(a)^i + i(\delta^{r-1}(a))^{i-1} \delta^r(a)z + \dots + (\delta^r(a))^i z^i. \end{aligned}$$

This implies that $F_i \neq F_{i-1}$ for $1 \leq i \leq p-1$. On the other hand, F_i is the inverse image by σ of R -module $A + Az + \dots + Az^i$ of $A[z]$. Hence F_i/F_{i-1} is viewed as an R -submodule of $(A + Az + \dots + Az^i)/(A + Az + \dots + Az^{i-1}) \cong A$. Therefore, F_i/F_{i-1} is a torsion-free R -module of rank 1.

Now we sheafify the above observations. Since the operator δ is defined globally, we can define a coherent sheaf \mathcal{F}_i so that, on an affine open subset W , $\mathcal{F}_i|_W = \{a \in \pi_*\mathcal{O}_Y(W) \mid \delta^{i+1}(a) = 0\} \sim$. Then $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a torsion-free \mathcal{O}_X -module of rank 1.

To show that \mathcal{F}_i is a locally free sheaf, we take the double dual \mathcal{F}_i^{**} of

\mathcal{F}_i . Then we have

$$\begin{array}{ccccccc} \mathcal{O}_X & = & \mathcal{F}_0 & \subset & \mathcal{F}_1 & \subset & \dots \subset \mathcal{F}_{p-1} & = & \pi_*\mathcal{O}_Y \\ \downarrow & & & & \downarrow & & & & \downarrow \\ \mathcal{O}_X^{**} & = & \mathcal{F}_0^{**} & \subset & \mathcal{F}_1^{**} & \subset & \dots \subset \mathcal{F}_{p-1}^{**} & = & \pi_*\mathcal{O}_Y^{**}. \end{array}$$

So, we may regard \mathcal{F}_i^{**} as \mathcal{O}_X -submodule of $\pi_*\mathcal{O}_Y$. Hence δ^{i+1} operates on \mathcal{F}_i^{**} and $\delta^{i+1} \in \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_i^{**}, \pi_*\mathcal{O}_Y)$. We know that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_i^{**}, \pi_*\mathcal{O}_Y)$ is a locally free sheaf and that $\delta^{i+1}|_V = 0$, where $V = X - \text{Supp } \mathcal{F}_i^{**}/\mathcal{F}_i$. On the other hand, since \mathcal{F}_i is torsion-free and X is regular, $\mathcal{F}_i^{**}/\mathcal{F}_i$ has support of codimension ≥ 2 . Therefore, $\delta^{i+1}(\mathcal{F}_i^{**}) = 0$. So, we have $\mathcal{F}_i^{**} = \mathcal{F}_i$. Hence \mathcal{F}_i is a locally free sheaf.

Finally we show that $\mathcal{F}_1/\mathcal{F}_0$ is an invertible sheaf. We consider an exact sequence

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_1/\mathcal{F}_0 \longrightarrow 0.$$

We know that $\mathcal{F}_0 \otimes k(P) \rightarrow \mathcal{F}_1 \otimes k(P)$ is injective for an arbitrary point $P \in X$ because $\mathcal{F}_0 \otimes k(P)$ contains the unity of $(\pi_*\mathcal{O}_Y) \otimes k(P)$. Hence we have $\text{rank}(\mathcal{F}_1/\mathcal{F}_0) \otimes k(P) = 1$. This implies that $\mathcal{F}_1/\mathcal{F}_0$ is an invertible sheaf on X . Q. E. D.

We shall define a good class of Artin-Schreier coverings. Let π, X, Y and \mathcal{F}_i be as above. We call $\pi: Y \rightarrow X$ an Artin-Schreier covering of simple type if $\mathcal{F}_i/\mathcal{F}_{i-1} \cong (\mathcal{F}_1/\mathcal{F}_0)^{\otimes i}$ for $1 \leq i \leq p-1$. From now on, we consider Artin-Schreier coverings of simple type. We have the following fundamental lemmas.

LEMMA 1.2. Suppose that $\pi: Y \rightarrow X$ is an Artin-Schreier covering of simple type. Then there exist an affine open covering $\mathfrak{U} = \{U_\lambda\}$ of X and $s_\lambda, t_\lambda \in H^0(U_\lambda, \mathcal{O}_X)$ such that

$$\pi^{-1}(U_\lambda) = \text{Spec } \mathcal{O}_X(U_\lambda)[[\xi_\lambda]]/(\xi_\lambda^p - s_\lambda \xi_\lambda - t_\lambda).$$

PROOF. Write $\mathcal{L}^{-1} = \mathcal{F}_1/\mathcal{F}_0$. Let $\mathfrak{U} = \{U_\lambda\}$ be an affine open covering of X such that $\mathcal{L}^{-1}|_{U_\lambda} \cong \mathcal{O}_{U_\lambda}$. Take $\xi_\lambda \in \mathcal{F}_1(U_\lambda)$ such that $\mathcal{L}^{-1}|_{U_\lambda} = \mathcal{O}_{U_\lambda} \overline{\xi}_\lambda$, where $\overline{\xi}_\lambda$ is the image of ξ_λ . Then $\sigma(\xi_\lambda) = \xi_\lambda + \alpha_\lambda z$ with $\alpha_\lambda \in H^0(U_\lambda, \mathcal{O}_X)$, where σ and z are the same as in the proof of Proposition 1.1. Since $\sigma(\xi_\lambda^p) = \xi_\lambda^p + \alpha_\lambda^p z$, we have $\xi_\lambda^p \in \mathcal{F}_1(U_\lambda)$. Thus, $\xi_\lambda^p = s_\lambda \xi_\lambda + t_\lambda$ with $s_\lambda, t_\lambda \in H^0(U_\lambda, \mathcal{O}_X)$. On the other hand, $\pi_*\mathcal{O}_Y|_{U_\lambda} = \mathcal{O}_{U_\lambda} + \mathcal{O}_{U_\lambda} \xi_\lambda + \dots + \mathcal{O}_{U_\lambda} \xi_\lambda^{p-1}$ by the hypothesis. The assertion follows from these observations. Q. E. D.

LEMMA 1.3. Under the same assumptions and notations as in the previous lemma, we have $\{s_\lambda\} \in H^0(X, \mathcal{L}^{p-1})$, $s_\lambda = \alpha_\lambda^{p-1}$, $\{\alpha_\lambda\} \in H^0(X, \mathcal{L})$ and $t_\mu - a_{\lambda\mu}^p t_\lambda = b_{\lambda\mu}^p - s_\mu b_{\lambda\mu}$, where $\{a_{\lambda\mu}\}$ is transition functions of \mathcal{L} and $\{b_{\lambda\mu}\} \in H^1(X, \mathcal{L})$.

PROOF. Since $\xi_\lambda^p = s_\lambda \xi_\lambda + t_\lambda$, we obtain $\sigma(\xi_\lambda^p) = \xi_\lambda^p + \alpha_\lambda^p z = (s_\lambda \xi_\lambda + t_\lambda) + s_\lambda \alpha_\lambda z$.

Thus $s_\lambda = \alpha_\lambda^{p-1}$. On $U_\lambda \cap U_\mu$, set $\xi_\mu = a_{\lambda\mu}\xi_\lambda + b_{\lambda\mu}$. Then $\{b_{\lambda\mu}\} \in H^1(X, \mathcal{L})$ and we have

$$\begin{aligned} \xi_\mu^p &= s_\mu \xi_\mu + t_\mu = s_\mu(a_{\lambda\mu}\xi_\lambda + b_{\lambda\mu}) + t_\mu \\ &= a_{\lambda\mu}^p \xi_\lambda^p + b_{\lambda\mu}^p = a_{\lambda\mu}^p(s_\lambda \xi_\lambda + t_\lambda) + b_{\lambda\mu}^p. \end{aligned}$$

Hence, $s_\mu = a_{\lambda\mu}^{p-1} s_\lambda$ and $t_\mu - a_{\lambda\mu}^p t_\lambda = b_{\lambda\mu}^p - s_\mu b_{\lambda\mu}$. So, we have $\{s_\lambda\} \in H^0(X, \mathcal{L}^{p-1})$. Moreover,

$$\sigma(\xi_\mu) = \sigma(a_{\lambda\mu}\xi_\lambda + b_{\lambda\mu}) = a_{\lambda\mu}\xi_\lambda + b_{\lambda\mu} + a_{\lambda\mu}\alpha_\lambda z = \xi_\mu + \alpha_\mu z.$$

Thence $\alpha_\mu = a_{\lambda\mu}\alpha_\lambda$, i.e. $\{\alpha_\lambda\} \in H^0(X, \mathcal{L})$.

Q. E. D.

REMARK 1.4. If $\{b_{\lambda\mu}\} = 0$ in $H^1(X, \mathcal{L})$, we can choose t_λ so that $\{t_\lambda\} \in H^0(X, \mathcal{L}^p)$. In particular, we may assume $\{t_\lambda\} \in H^0(X, \mathcal{L}^p)$ provided $H^1(X, \mathcal{L}) = (0)$.

Let B be the effective divisor corresponding to the section $\{\alpha_\lambda\} \in H^0(X, \mathcal{L})$. Clearly, B is independent of choice of generators $\{\xi_\lambda\}$. Moreover, π is unramified over $X - \text{Supp } B$ and totally ramified over $\text{Supp } B$. We call B the *branch locus* of π .

We shall give an example of an Artin-Schreier covering which is not of simple type.

EXAMPLE 1.5. Assume that $\text{char } k = p > 2$. Consider \mathbf{P}_k^3 with a homogeneous coordinate system (x_0, x_1, x_2, x_3) . Let $X = \{x_3 = 0\} \cong \mathbf{P}^2$, $Y = \{x_3^p - x_1^{p-1}x_2 - x_1^{p-1}x_2 = 0\} \subset \mathbf{P}^3$ and let $\rho: Y \rightarrow X$ be the projection from $(0, 0, 0, 1)$. Then ρ is surjective and finite. Take the normalization $\nu: \tilde{Y} \rightarrow Y$ and denote $\pi = \rho \circ \nu$. Let $U_i = \{x_i \neq 0\} \subset X$ for $i=0, 1, 2$. By the Jacobian criterion, $\rho(\text{Sing } Y) = \{x_1 = 0\}$. Hence $\pi^{-1}(U_1) = \rho^{-1}(U_1)$. On the other hand, $\rho^{-1}(U_0) = \text{Spec } k[x, y, \xi]$, where $\xi^p - x^{p-1}\xi - x^{p-1}y$, and $\rho^{-1}(U_2) = \text{Spec } k[u, v, \zeta]$, where $\zeta^p - u^{p-1}\zeta - u^{p-1}$. It is easy to verify that $\pi^{-1}(U_2) = \text{Spec } k[v, \zeta, \tau]$, where $\tau = u/\zeta$ and $\zeta - \tau^{p-1}\zeta - \tau^{p-1} = 0$. On U_0 , it is a little more difficult. Let $T^p = x$ and $S = \xi/T^{p-1}$. Then T and S are algebraically independent over k and we have $k[T^p, T^{p-1}S, S^p] = k[x, y, \xi]$. Let $\mathcal{O} = k[T^p, T^{p-1}S, T^{p-2}S^2, \dots, TS^{p-1}, S^p]$. Clearly, $T^{p-2}S^2, \dots, TS^{p-1}$ are integral over $k[T^p, T^{p-1}S, S^p]$. Meanwhile, \mathcal{O} is none other than the coordinate ring of the cone of the p -uple embedding of \mathbf{P}^1 in \mathbf{P}^p . So, \mathcal{O} is normal. Hence \mathcal{O} is the integral closure of $k[x, y, \xi]$. Furthermore, $\text{Spec } \mathcal{O}$ has only one singular point whose minimal resolution consists of a curve C such that $C \cong \mathbf{P}^1$ and $(C^2) = -p$.

Since $k(\tilde{Y})/k(X)$ is an Artin-Schreier extension, $\pi: \tilde{Y} \rightarrow X$ is an Artin-Schreier covering. However, π is not of simple type. Suppose π is of simple type. Then \tilde{Y} is locally a hypersurface by Lemma 1.2. Hence every rational singularity is a rational double point. This contradicts the above observation.

By Remark 1.4, if $H^1(X, \mathcal{L})=0$, every Artin-Schreier covering of simple type is defined locally by $\xi_\lambda^p - \alpha_\lambda^{p-1}\xi_\lambda = t_\lambda$, where $\alpha = \{\alpha_\lambda\} \in H^0(X, \mathcal{L})$ and $t = \{t_\lambda\} \in H^0(X, \mathcal{L}^p)$. The set $\{dt_\lambda\}$ of 1-forms defines a section $dt \in H^0(X, \Omega_X \otimes \mathcal{L}^p)$. Applying the Jacobian criterion to the above local defining equations, we obtain

PROPOSITION 1.6. *With the above notations and assumptions, in the characteristic $p > 2$, Y is singular at a point $Q \in Y$ if and only if $\pi(Q) \in \text{Supp } B$ and $dt=0$ at $\pi(Q)$.*

Artin-Schreier coverings of simple type are obtained as follows (cf. [5]). Let L be a line bundle on X associated with invertible sheaf \mathcal{L} and consider L and L^p as smooth X -group schemes. Take a global section s of L^{p-1} and consider a surjective homomorphism of X -group schemes $F-s: L \rightarrow L^p$ defined by $(F-s)(x) = x^p - sx$ for $x \in L$. Let α_s be its kernel. Then we have an exact sequence of X -group schemes in flat topology

$$0 \longrightarrow \alpha_s \longrightarrow L \longrightarrow L^p \longrightarrow 0.$$

Taking the flat cohomologies, we have an exact sequence

$$0 \rightarrow H_{f1}^0(X, \alpha_s) \rightarrow H^0(X, \mathcal{L}) \rightarrow H^0(X, \mathcal{L}^p) \xrightarrow{\partial} H_{f1}^1(X, \alpha_s) \rightarrow H^1(X, \mathcal{L}),$$

where we can interpret $H_{f1}^1(X, \alpha_s)$ as the set of isomorphism classes of α_s -torsors. Suppose that $s \in H^0(X, \mathcal{L}^{p-1})$ is given locally by $\{s_\lambda\}$ as in Lemmas 1.2 and 1.3. Let $t = \{t_\lambda\}$ be local sections of \mathcal{L}^p as in Lemma 1.3 and let $\rho: Z \rightarrow X$ be the α_s -torsor obtained by applying the connecting map ∂ to $t = \{t_\lambda\}$. In other words, Z is locally the fibre product of $F-s: L \rightarrow L^p$ and $t_\lambda: U_\lambda \rightarrow L^p$. Then it is clear that $\rho: Z \rightarrow X$ is isomorphic to $\pi: Y \rightarrow X$. If $H^1(X, \mathcal{L})=0$, all α_s -torsors, hence all Artin-Schreier coverings of simple type, are obtained from global sections of \mathcal{L}^p (cf. Remark 1.4).

In the sequel of this section, we consider an Artin-Schreier covering $\pi: Y \rightarrow X$ of simple type. We fix the notations \mathcal{F}_i ($0 \leq i < p$), \mathcal{L} , and B as in Proposition 1.1 and Lemmas 1.2 and 1.3. By the local description, we know that Y is locally a hypersurface. Therefore Y is a Gorenstein scheme. More precisely, we have

PROPOSITION 1.7. *Y has the dualizing sheaf*

$$\omega_Y = \pi^*(\omega_X \otimes \mathcal{L}^{p-1}).$$

PROOF. Apply the adjunction formula.

We shall compute invariants of Artin-Schreier coverings of simple type. There are the following formulas.

LEMMA 1.8. (1) $(\omega_Y^2) = p\{(K_X^2) + 2(p-1)(B, K_X) + (p-1)^2(B^2)\}$.

$$(2) \chi(\mathcal{O}_Y) = p\left\{\chi(\mathcal{O}_X) + \frac{(p-1)}{4}(B, K_X) + \frac{(p-1)(2p-1)}{12}(B^2)\right\}.$$

(3) If Y is smooth,

$$e(Y) = p\{e(X) + (p-1)(B, K_X) + (p-1)p(B^2)\},$$

where $e(Y)$ is the Euler number of Y .

$$(4) \kappa(Y) = \kappa(X, K_X + (p-1)B).$$

PROOF. (1) Immediate from Proposition 1.7.

(2) By the assumptions, $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{O}(-iB)$. Hence we have $\chi(\mathcal{F}_i) = \chi(\mathcal{F}_{i-1}) + \chi(\mathcal{O}(-iB))$ for $1 \leq i \leq p-1$. Therefore $\chi(\mathcal{O}_Y) = \chi(\mathcal{F}_{p-1}) = \sum_{i=0}^{p-1} \chi(\mathcal{O}(-iB))$, where, by the Riemann-Roch theorem,

$$\begin{aligned} \chi(\mathcal{O}(-iB)) &= (1/2)(-iB, -iB - K_X) + \chi(\mathcal{O}_X) \\ &= (1/2)(i^2(B^2) + i(B, K_X)) + \chi(\mathcal{O}_X). \end{aligned}$$

Thence we obtain the stated formula.

(3) Use Noether's formula: $12\chi(\mathcal{O}_Y) = (K_Y^2) + e(Y)$.

(4) It follows from a fundamental property of the D -dimension. Q.E.D.

In order to construct examples, we need the following:

LEMMA 1.9. Let X, Y and \mathcal{L} be as above. Suppose that \mathcal{L} is ample. If $H^1(X, \mathcal{L}^{-1}) = (0)$, we have $H^1(X, \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y)$.

PROOF. By the exact sequence

$$0 \longrightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_B \longrightarrow 0,$$

we have $H^0(X, \mathcal{O}_X) = H^0(B, \mathcal{O}_B) = k$. The exact sequence

$$0 \longrightarrow \mathcal{L}^{-2} \longrightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{L}^{-1} \otimes \mathcal{O}_B \longrightarrow 0$$

implies $H^1(X, \mathcal{L}^{-2}) = (0)$ because $\mathcal{L} \otimes \mathcal{O}_B$ is ample and $H^1(X, \mathcal{L}^{-1}) = (0)$. Similarly, by the exact sequences

$$0 \longrightarrow \mathcal{L}^{-i} \longrightarrow \mathcal{L}^{-(i-1)} \longrightarrow \mathcal{L}^{-(i-1)}|_B \longrightarrow 0 \quad (i > 1),$$

we obtain inductively $H^1(X, \mathcal{L}^{-i}) = (0)$. Now look at the exact sequences

$$0 \longrightarrow \mathcal{F}_{i-1} \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{L}^{-i} \longrightarrow 0 \quad (0 < i < p).$$

We know $H^1(X, \mathcal{F}_{i-1}) = H^1(X, \mathcal{F}_i)$. Hence $H^1(Y, \mathcal{O}_Y) = H^1(X, \mathcal{O}_X)$. Q.E.D.

EXAMPLE 1.10. Assume that $\text{char } k = p = 3$. Let $X = \mathbf{P}^1 \times \mathbf{P}^1$ and $\mathcal{L} = p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(1)$. Take an affine covering $\{U_i \times V_j\}_{i,j=1,2}$ such that $U_1 = \text{Spec } k[x]$,

$U_2 = \text{Spec } k[u]$, $V_1 = \text{Spec } k[y]$ and $V_2 = \text{Spec } k[v]$, where $u = x^{-1}$, $v = y^{-1}$. Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering such that

$$\begin{aligned} \pi^{-1}(U_1 \times V_1) &= \text{Spec } \mathcal{O}_{U_1 \times V_1}[\xi_{11}] / (\xi_{11}^3 - x^2 y^2 \xi_{11} - (x^2 + y^2 + x + y)), \\ \pi^{-1}(U_1 \times V_2) &= \text{Spec } \mathcal{O}_{U_1 \times V_2}[\xi_{12}] / (\xi_{12}^3 - x^2 \xi_{12} - (x^2 v^3 + v + xv^3 + v^2)), \\ \pi^{-1}(U_2 \times V_1) &= \text{Spec } \mathcal{O}_{U_2 \times V_1}[\xi_{21}] / (\xi_{21}^3 - y^2 \xi_{21} - (u + y^2 u^3 + u^2 + yu^3)), \\ \pi^{-1}(U_2 \times V_2) &= \text{Spec } \mathcal{O}_{U_2 \times V_2}[\xi_{22}] / (\xi_{22}^3 - \xi_{22} - (uv^3 + u^3 v + u^2 v^3 + u^3 v^2)). \end{aligned}$$

Then Y is nonsingular and its dualizing sheaf is

$$\omega_Y = \pi^*(p_1^* \mathcal{O}(-2) \otimes p_2^* \mathcal{O}(-2) \otimes \mathcal{L}^2) \cong \mathcal{O}_Y.$$

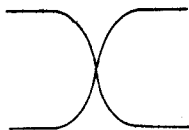
By the previous lemma, we see that $H^1(Y, \mathcal{O}_Y) = H^1(X, \mathcal{O}_X) = (0)$. Hence Y is a $K3$ -surface.

EXAMPLE 1.11. Assume $\text{char } k = p = 3$. Let $X = \mathbf{P}^2$ and $\mathcal{L} = \mathcal{O}(1)$. Then $H^1(\mathbf{P}^2, \mathcal{L}) = (0)$. Let (x, y, z) be a system of homogeneous coordinate of \mathbf{P}^2 . Choose $s = x^2 \in H^0(\mathbf{P}^2, \mathcal{L}^2)$ and $t = xy^2 + x^2 y + y^2 z + yz^2 + z^2 x + zx^2 \in H^0(\mathbf{P}^2, \mathcal{L}^3)$. Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering of simple type obtained from s and t . Then Y is smooth. Moreover, $\omega_Y = \pi^* \mathcal{O}(-1)$ and $(K_Y)^3 = 3$. So, Y is a del Pezzo surface of degree 3, i.e. a smooth cubic hypersurface in \mathbf{P}^3 .

EXAMPLE 1.12. Assume $\text{char } k = p = 2$. Let $X = \mathbf{P}^1 \times \mathbf{P}^1$ and let $\mathcal{L} = p_1^* \mathcal{O}(2) \otimes p_2^* \mathcal{O}(3)$. Take an affine open covering $\{U_i \times V_j\}_{i,j=1,2}$ which is the same as in Example 1.10. Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering such that

$$\begin{aligned} \pi^{-1}(U_1 \times V_1) &= \text{Spec } \mathcal{O}_{U_1 \times V_1}[\xi_{11}] / (\xi_{11}^2 + x^2(y+1)^3 \xi_{11} + (x+x^3)y^3 + y^5 + y^3 + y), \\ \pi^{-1}(U_1 \times V_2) &= \text{Spec } \mathcal{O}_{U_1 \times V_2}[\xi_{12}] / (\xi_{12}^2 + x^2(1+v)^3 \xi_{12} + (x+x^3)v^3 + v^5 + v^3 + v), \\ \pi^{-1}(U_2 \times V_1) &= \text{Spec } \mathcal{O}_{U_2 \times V_1}[\xi_{21}] / (\xi_{21}^2 + (y+1)^3 \xi_{21} + (u+u^3)y^3 + u^4(y^5 + y^3 + y)), \\ \pi^{-1}(U_2 \times V_2) &= \text{Spec } \mathcal{O}_{U_2 \times V_2}[\xi_{22}] / (\xi_{22}^2 + (1+v)^3 \xi_{22} + (u+u^3)v^3 + u^4(v^5 + v^3 + v)). \end{aligned}$$

Then Y is nonsingular. By Proposition 1.7 and Lemma 1.8, we have $\omega_Y = \pi^* p_2^* \mathcal{O}(1)$ and $\kappa(Y) = 1$. Moreover, $f = p_2 \circ \pi: Y \rightarrow \mathbf{P}^1$ is an elliptic fibration and three fibers $f^{-1}(P_0)$, $f^{-1}(P_1)$ and $f^{-1}(P_\infty)$ exhaust singular fibres of f , where P_0 , P_1 and P_∞ are points of \mathbf{P}^1 defined by $y = 0, 1$ and ∞ , respectively. The fibres $f^{-1}(P_0)$ and $f^{-1}(P_\infty)$ are of type



The fibre $f^{-1}(P_1)$ is a cuspidal rational curve.

EXAMPLE 1.13. Let p, X, \mathcal{L} and $\{U_i \times V_j\}$ be as in the previous example. Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering such that

$$\begin{aligned}\pi^{-1}(U_1 \times V_1) &= \text{Spec } \mathcal{O}_{U_1 \times V_1}[\xi_{11}]/(\xi_{11}^2 + x^2(y+1)^3\xi_{11} + x^3y^3 + y^3), \\ \pi^{-1}(U_1 \times V_2) &= \text{Spec } \mathcal{O}_{U_1 \times V_2}[\xi_{12}]/(\xi_{12}^2 + x^2(1+v)^3\xi_{12} + x^3v^3 + v^3), \\ \pi^{-1}(U_2 \times V_1) &= \text{Spec } \mathcal{O}_{U_2 \times V_1}[\xi_{21}]/(\xi_{21}^2 + (y+1)^3\xi_{21} + uy^3 + u^4y^3), \\ \pi^{-1}(U_2 \times V_2) &= \text{Spec } \mathcal{O}_{U_2 \times V_2}[\xi_{22}]/(\xi_{22}^2 + (1+v)^3\xi_{22} + uv^3 + u^4v^3).\end{aligned}$$

Then the branch locus of π is the same as in the previous example. Y has two singular points, which lie over the points $(x=0, y=0)$ and $(x=0, y=\infty)$ of X . It is easy to verify that both points are rational double points of type E_6 . Let $\sigma: \tilde{Y} \rightarrow Y$ be the minimal resolution of singularities of Y . Then we have $\omega_{\tilde{Y}} = \sigma^* \cdot \pi^* \cdot p_2^* \mathcal{O}(1)$ and $\kappa(Y) = 1$. Moreover, the composite $f = p_2 \circ \pi \circ \sigma$ defines a quasi-elliptic fibration $f: \tilde{Y} \rightarrow \mathbf{P}^1$.

§ 2. Canonical resolution of singularities in the case of nonsingular branch locus and in characteristic 2.

In this section, we assume $\text{char } k = p = 2$. Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering, which is necessarily of simple type. Suppose that the branch locus B in the sense of § 1 is a nonsingular curve on X . Since Y is normal, Y has at most isolated singularities. We shall consider a resolution of singularities of Y which we call the canonical resolution of singularities of Y . To begin with, we consider a local ring $\mathfrak{D} = k[[x, y]][\xi]/(\xi^2 + x\xi + t)$ with $t \in k[[x, y]]$, which has at most an isolated singularity. Then \mathfrak{D} is normal. Write $t = c_0 + c_1x + c_2y + c_3xy$ with $c_i \in k[[x^2, y^2]]$. Replacing ξ by $\xi + c_1 + c_3y$, we may assume $t = c_0 + c_2y$. So, we can write $t = d_0(y) + x^2d_1(x^2, y)$, where $d_0(y) \neq 0$ by the hypothesis that \mathfrak{D} is normal. Write $d_0 = a_\nu y^\nu + (\text{terms of higher degree})$, where $\nu \geq 0$, $a_\nu \in k$ and $a_\nu \neq 0$. Clearly, \mathfrak{D} is regular if and only if $\nu = 0$ or $\nu = 1$. Furthermore, it is easy to see that ν is invariant under change of variables $(\xi, x, y) \rightarrow (\xi + f, x, y)$ with $f \in k[[x, y]]$ as long as we keep the condition $t = c_0 + c_2y$. Suppose $\nu \geq 2$. Let $x_1 = x/y$. Then

$$\xi^2 + x\xi + d_0(y) + x^2d_1(x^2, y) = \xi^2 + x_1y\xi + d_0(y) + x_1^2y^2d_1(x_1^2y^2, y).$$

Normalizing this equation, we have

$$\xi_1^2 + x_1\xi_1 + d_0^{(1)}(y) + x_1^2d_1(x_1^2y^2, y) = 0, \quad \text{where } \xi_1 = \xi/y.$$

Inductively, one obtains the following series of local rings

$$\begin{aligned} \mathfrak{D} &= \mathfrak{D}_0 = k[[x, y]][\xi]/(\xi^2 + x\xi + d_0(y) + x^2d_1), \\ \mathfrak{D}_1 &= k[[x_1, y]][\xi_1]/(\xi_1^2 + x_1\xi_1 + d_0^{(1)}(y) + x_1^2d_1), \\ &\vdots \\ \mathfrak{D}_n &= k[[x_n, y]][\xi_n]/(\xi_n^2 + x_n\xi_n + d_0^{(n)}(y) + x_n^2d_1), \end{aligned}$$

and

$$\begin{array}{ccccccc} \mathfrak{D} = \mathfrak{D}_0 & \text{---} & \mathfrak{D}_1 & \text{---} & \dots & \text{---} & \mathfrak{D}_n \\ | & & | & & & & | \\ k[[x, y]] & \text{---} & k[[x_1, y]] & \text{---} & \dots & \text{---} & k[[x_n, y]], \end{array}$$

where $n = \lfloor \nu/2 \rfloor$. Then \mathfrak{D}_n is regular. Globally speaking, we consider a series of blowing-ups $X = X_0 \leftarrow X_1 \leftarrow \dots \leftarrow X_n$ with centres $(x=0, y=0), (x_1=0, y=0), \dots, (x_{n-1}=0, y=0)$ and consider the normalization Y_i of X_i in the function field $k(Y)$. Thus one obtains a commutative diagram

$$\begin{array}{ccccccc} Y = Y_0 & \leftarrow & Y_1 & \leftarrow & \dots & \leftarrow & Y_n \\ \pi \downarrow & & \downarrow \pi_1 & & & & \downarrow \pi_n \\ X = X_0 & \leftarrow & X_1 & \leftarrow & \dots & \leftarrow & X_n. \end{array}$$

We call this process of blowing-ups the *canonical resolution* of the singularity of $\text{Spec } \mathfrak{D}$.

Suppose that ν is even. Then

$$\mathfrak{D}_n \cong k[[x, y]][\eta]/(\eta^2 + x\eta + x + t'(x, y)),$$

where $t'(x, y)$ consists of terms of degree ≥ 2 and $\eta^2 + x\eta + x + t'(x, 0)$ is irreducible. Let E_n be the exceptional curve of the blowing-up $X_n \rightarrow X_{n-1}$ and let $\tilde{E}_n = \pi_n^{-1}(E_n)$. Since $\pi_n^{-1}(E_n) = \{y=0\}$ locally, \tilde{E}_n is an irreducible curve and $(\tilde{E}_n^2) = -2$. Therefore we have the following configuration of exceptional curves which arise from the canonical resolution of the singularity of $\text{Spec } \mathfrak{D}$

$$\circ \text{---} \circ \text{---} \dots \text{---} \underset{\tilde{E}_n}{\circ} \text{---} \dots \text{---} \circ \text{---} \circ \quad \text{type } A_{\nu-1},$$

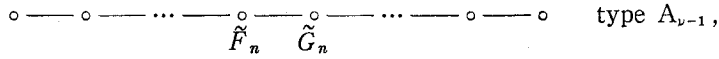
where “ \circ ” stands for a nonsingular rational curve whose self-intersection number is -2 , i.e. a (-2) -curve. In particular, we know the $\text{Spec } \mathfrak{D}$ has a rational double point.

Now suppose that ν is odd. Then

$$\mathfrak{D}_n \cong k[[x, y]][\eta]/(\eta^2 + x\eta + y).$$

Let E_n be as above. Since $\pi^{-1}(E_n) = \{y=0\}$ locally, $\pi^{-1}(E_n)$ splits to two curves $F_n = \{\xi=0\}$ and $G_n = \{\xi+x=0\}$. F_n and G_n intersect transversally at the point $(\xi, x, y) = (0, 0, 0)$. So, $(F_n^2) = (G_n^2) = -2$. Therefore, we have the following

configuration of exceptional curves which arise from the canonical resolution of the singularity of $\text{Spec } \mathfrak{D}$



where “ \circ ” stands for a (-2) -curve as above. In particular, we know that $\text{Spec } \mathfrak{D}$ has a rational double point.

By virtue of the above observations, we conclude

THEOREM 2.1. *Let \mathfrak{D}, ν and t be as above. Then*

- (1) *$\text{Spec } \mathfrak{D}$ has a singularity if and only if $\nu \geq 2$.*
- (2) *If $\text{Spec } \mathfrak{D}$ has a singular point, then it is a rational double point of type*

$A_{\nu-1}$.

We know that ν is an important invariant of a local ring \mathfrak{D} . There is the following explicit formula.

LEMMA 2.2. *With the same notations and assumptions,*

$$\nu = \text{length } k[[x, y]]/(x, t + (\partial t/\partial x)^2).$$

PROOF. Write $t = c_0 + c_1x + c_2y + c_3xy$ with $c_i \in k[[x^2, y^2]]$. Set $\xi' = \xi + c_1 + c_3y$. Then $\xi^2 + x\xi + t = \xi'^2 + x\xi' + c_0 + c_1^2 + c_3^2y^2 + c_2y$. So, $d_0(y) + x^2d_1(x^2, y) = c_0 + c_1^2 + c_3^2y^2 + c_2y$. On the other hand, $t + (\partial t/\partial x)^2 = c_0 + c_1^2 + c_3^2y^2 + c_2y + x(c_1 + c_3y)$. Therefore, we have $(x, d_0(y)) = (x, d_0(y) + x^2d_1) = (x, t + (\partial t/\partial x)^2)$ as ideals in $k[[x, y]]$. Since $\nu = \text{length } k[[x, y]]/(x, d_0(y))$, we obtain the required formula.

Q. E. D.

Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering obtained as an α_s -torsor from a line bundle L on X , a global section s of L and local sections $\{t_\lambda\}$ of L^2 (cf. §1). Suppose that $B = (s)_0$ is a nonsingular curve on X and that $\{t_\lambda\}$ give rise to a global section of L^2 . Take an affine covering $\{U_\lambda\}$ such that $s = x_\lambda e_\lambda$ on U_λ and (x_λ, y_λ) is a local coordinate system on U_λ for $U_\lambda \cap B \neq \emptyset$, where $\mathcal{L}|_{U_\lambda} = \mathcal{O}_{U_\lambda} e_\lambda$. Then $\pi^{-1}(U_\lambda) = \text{Spec } \mathcal{O}_X(U_\lambda)[\xi]/(\xi^2 + x_\lambda\xi + t_\lambda)$. For each closed point $P \in X$, we define $\nu(P)$ after Lemma 2.2 as follows:

$$\nu(P) = \begin{cases} \text{length } (\mathcal{O}_{P, X})^\wedge / (x_\lambda, t_\lambda + (\partial t_\lambda/\partial x_\lambda)^2) & \text{if } P \in B \cap U_\lambda \\ 0 & \text{if } P \notin B. \end{cases}$$

We shall estimate $\sum_{P \in Y, \nu(P) > 0} (\nu(P) - 1)$ as follows.

LEMMA 2.3. $\{(\partial t_\lambda/\partial y_\lambda)|_B\} \in H^0(B, \omega_X \otimes \mathcal{L}^3|_B)$.

PROOF. Since $dt_\mu = (\partial t_\mu/\partial x_\mu)dx_\mu + (\partial t_\mu/\partial y_\mu)dy_\mu$, we have $dx_\mu \wedge dt_\mu = (\partial t_\mu/\partial y_\mu)dx_\mu \wedge dy_\mu = (\partial t_\mu/\partial y_\mu)J_{\mu\lambda}dx_\lambda \wedge dy_\lambda$, where $\{J_{\mu\lambda}\}$ are the transition functions of the canonical bundle of X . Let $\{a_{\lambda\mu}\}$ be transition functions of \mathcal{L} such that

$e_\lambda = a_{\lambda\mu} e_\mu$. Then $x_\mu = a_{\lambda\mu} x_\lambda$ and $t_\mu = a_{\lambda\mu}^2 t_\lambda$. So, $dx_\mu = a_{\lambda\mu} dx_\lambda + x_\lambda da_{\lambda\mu}$ and $dt_\mu = a_{\lambda\mu}^2 dt_\lambda$. Therefore,

$$\begin{aligned} dx_\mu \wedge dt_\mu &= a_{\lambda\mu}^3 dx_\lambda \wedge dt_\lambda + x_\lambda a_{\lambda\mu}^2 da_{\lambda\mu} \wedge dt_\lambda \\ &= a_{\lambda\mu}^3 \frac{\partial t_\lambda}{\partial y_\lambda} dx_\lambda \wedge dy_\lambda + x_\lambda a_{\lambda\mu}^2 da_{\lambda\mu} \wedge dt_\lambda. \end{aligned}$$

Hence we have $(\partial t_\mu / \partial y_\mu)|_B \cdot J_{\mu\lambda}|_B = a_{\lambda\mu}^3 (\partial t_\lambda / \partial y_\lambda)|_B$ on B . This asserts that $\{(\partial t_\lambda / \partial y_\lambda)|_B\} \in H^0(B, \omega_X \otimes \mathcal{L}^3|_B)$. Q. E. D.

PROPOSITION 2.4. $\sum(\nu(P)-1) \leq \max\{2(B^2), 2(B^2)+2p_a(B)-2\}$, where $P \in X$ and $\nu(P) > 0$.

PROOF. Set $\partial_y t = \{(\partial t_\lambda / \partial y_\lambda)|_B\} \in H^0(B, \omega_X \otimes \mathcal{L}^3|_B)$. Suppose $\partial_y t \neq 0$. Let $P \in B$ and $P \in U_\lambda$. We consider t_λ, x_λ and y_λ in $(\mathcal{O}_{P,X})^\wedge$. With the same notations as in Lemma 2.2, $t_\lambda = c_0 + c_1 x_\lambda + c_2 y_\lambda + c_3 x_\lambda y_\lambda$ and $\partial t_\lambda / \partial y_\lambda = c_2 + c_3 x_\lambda$. Since $d_0(y_\lambda) + x_\lambda^2 d_1(x_\lambda^2, y_\lambda) = c_0 + c_1^2 + c_3^2 y_\lambda^2 + c_2 y_\lambda$, we have $\nu(P) \leq$ (multiplicity of $(\partial_\lambda t)_0$ at P) + 1, where $(\partial_y t)_0$ is the effective divisor corresponding to $\partial_y t$. Hence $\sum(\nu(P)-1) \leq (B, 3B + K_X) = 2(B^2) + 2p_a(B) - 2$.

Now, suppose $\partial_y t = 0$, i. e. $\partial t_\lambda / \partial y_\lambda = 0$ on B for all λ . Then $dt_\mu = (\partial t_\mu / \partial x_\mu) dx_\mu$ on B . Since $dx_\mu = (a_{\lambda\mu} + x_\lambda \cdot \partial a_{\lambda\mu} / \partial x_\lambda) dx_\lambda + (\partial x_\mu / \partial y_\lambda) dy_\lambda$,

$$\begin{aligned} dt_\mu &= \frac{\partial t_\mu}{\partial x_\mu} \left[(a_{\lambda\mu} + x_\lambda \cdot \frac{\partial a_{\lambda\mu}}{\partial x_\lambda}) dx_\lambda + \frac{\partial x_\mu}{\partial y_\lambda} dy_\lambda \right] \\ &= \frac{\partial t_\mu}{\partial x_\mu} (a_{\lambda\mu} + x_\lambda \cdot \frac{\partial a_{\lambda\mu}}{\partial x_\lambda}) dx_\lambda + \frac{\partial t_\mu}{\partial x_\mu} \cdot \frac{\partial x_\mu}{\partial y_\lambda} dy_\lambda \quad \text{on } B. \end{aligned}$$

On the other hand, $dt_\mu = a_{\lambda\mu}^2 (\partial t_\lambda / \partial x_\lambda) dx_\lambda + a_{\lambda\mu}^2 (\partial t_\lambda / \partial y_\lambda) dy_\lambda$. Therefore $a_{\lambda\mu}^2 (\partial t_\lambda / \partial x_\lambda) = (\partial t_\mu / \partial x_\mu) \{a_{\lambda\mu} + x_\lambda (\partial a_{\lambda\mu} / \partial x_\lambda)\}$ on B . Namely, we have $(\partial t_\mu / \partial x_\mu)|_B \cdot (a_{\lambda\mu}|_B) = (a_{\lambda\mu}^2|_B) (\partial t_\lambda / \partial x_\lambda)|_B$. Hence $\{\partial t_\lambda / \partial x_\lambda|_B\}$ is a global section of $\mathcal{L}|_B$. Set $\tau = \{[t_\lambda + (\partial t_\lambda / \partial x_\lambda)^2]|_B\}$. Then τ is a global section of $\mathcal{L}^2|_B$. For $P \in B$, we know that $\nu(P) =$ (multiplicity of $(\tau)_0$ at P). So, $\sum_{P \in X} \nu(P) = 2(B^2)$. The assertion follows from these observations. Q. E. D.

Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering with nonsingular branch locus B and let

$$\begin{array}{ccc} Y & \xleftarrow{\rho} & \tilde{Y} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ X & \xleftarrow{\sigma} & \tilde{X} \end{array}$$

be the canonical resolution of singularities of Y .

PROPOSITION 2.5. \tilde{Y} has the dualizing sheaf

$$\omega_{\tilde{Y}} = \rho^* \circ \pi^*(\omega_X \otimes \mathcal{O}(B)).$$

PROOF. We already know that $\omega_Y = \pi^*(\omega_X \otimes \mathcal{O}(B))$. Since every singularity of Y is a rational double point by Theorem 2.1, $\omega_{\tilde{Y}} = \rho^* \omega_Y$. Therefore, $\omega_{\tilde{Y}} = \rho^* \circ \pi^*(\omega_X \otimes \mathcal{O}(B))$. Q. E. D.

COROLLARY 2.6. (1) $(K_{\tilde{Y}^2}) = 2(K_X + B)^2$.

(2) $\kappa(\tilde{Y}) = \kappa(X, K_X + B)$.

PROOF. Straightforward.

In general, \tilde{Y} may not be a minimal surface. However we have

PROPOSITION 2.7. *If K_X is numerically effective, nef in short, \tilde{Y} is a minimal surface.*

PROOF. Let E be a (-1) -curve on \tilde{Y} and write $\phi = \sigma \circ \tilde{\pi}$. Suppose that $\phi_* E = 2C$, where C is the set-theoretic image of E . Since $(E, K_{\tilde{Y}}) = -1$, we have $-1 = 2(C, K_X + B)$ by the projection formula. This is a contradiction. Now, suppose $\phi_* E = C$. Similarly we have $-1 = (C, K_X + B)$. So, $(C, B) = -1 - (C, K_X) \leq -1$ by the assumption. Hence C must be an irreducible component of B . Since B is a disjoint union of nonsingular curves, $(C, B) = (C^2)$. Therefore we have $(C, K_X + C) = -1$ and this is impossible. This completes the proof.

Q. E. D.

We shall compute other invariants of \tilde{Y} .

PROPOSITION 2.8. *Under the same notations, we have*

$$\chi(\mathcal{O}_{\tilde{Y}}) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}(-B)).$$

PROOF. Since Y has only rational double points, one obtains $\chi(\mathcal{O}_{\tilde{Y}}) = \chi(\mathcal{O}_Y)$. Now the assertion follows from Lemma 1.8.(2). Q. E. D.

COROLLARY 2.9. $e(\tilde{Y}) = 2[e(X) + (B, K_X + 2B)]$.

We have already computed the irregularity in the case where Y is nonsingular with an assumption (cf. Lemma 1.9). Here we have another formula

PROPOSITION 2.10. *Under the same notations and assumptions as above,*

$$h^1(X, \mathcal{O}_X) \leq h^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) \leq h^1(X, \mathcal{O}_X) + h^1(X, \mathcal{O}(-B)).$$

PROOF. Since Y has only rational singularities, $H^i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = H^i(Y, \mathcal{O}_Y)$. On the other hand, because $H^0(X, \mathcal{O}(-B)) = (0)$, we have

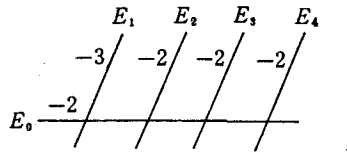
$$0 \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \pi_* \mathcal{O}_{\tilde{Y}}) \rightarrow H^1(X, \mathcal{O}(-B)).$$

Hence $h^1(X, \pi_* \mathcal{O}_{\tilde{Y}}) \leq h^1(X, \mathcal{O}_X) + h^1(X, \mathcal{O}(-B))$. The assertion follows from this.

Q. E. D.

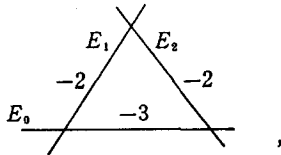
The following two examples of local rings have irrational singularities and appear as the local rings of Artin-Schreier coverings with non-reduced or singular branch loci.

EXAMPLE 2.11. Let $\mathfrak{D} = k[[x, y]][[\xi]]/(\xi^5 + x^2\xi + x^3 + x^2y^3 + xy^6)$, let $Y = \text{Spec } \mathfrak{D}$ and let $\sigma: \tilde{Y} \rightarrow Y$ be the minimal resolution of the singularity of Y . Then the exceptional locus of σ has the following configuration:



where E_0, \dots, E_4 are nonsingular rational curves. The fundamental cycle Z of this singularity is $2E_0 + E_1 + E_2 + E_3 + E_4$. Hence $p_a(Z) = 1$. So, this singularity is not rational.

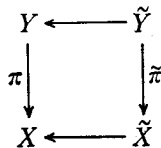
EXAMPLE 2.12. Let $\mathfrak{D} = k[[x, y]][[\xi]]/(\xi^2 + xy\xi + x^3 + y^9)$, let $Y = \text{Spec } \mathfrak{D}$ and let $\sigma: \tilde{Y} \rightarrow Y$ be the minimal resolution of the singularity of Y . Then the exceptional locus of σ has the following configuration:



where E_0, E_1 and E_2 are nonsingular rational curves. The fundamental cycle Z is $E_0 + E_1 + E_2$ and $p_a(Z) = 1$. Hence this singularity is not rational.

To close this section, we shall give an example of the canonical resolution.

EXAMPLE 2.13. Let $X = \mathbf{P}^2$ and let $\mathcal{L} = \mathcal{O}(1)$. Take $s \in H^0(X, \mathcal{L})$. Then $(s)_0$ is a line. Consider an Artin-Schreier covering Y whose branch locus B is $(s)_0$. Since $H^1(X, \mathcal{L}) = 0$, we know that such a covering is obtained as an α_s -torsor from s and a global section of \mathcal{L}^2 . If Y is smooth, then Y is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ (see §3 Theorem 3.2). Suppose Y is singular. Since $2(B^2) = 2$ and $2(B^2) + 2p_a(B) - 2 = 0$, we have $\sum_{P \in X} \nu(P) = 2$ by the proof of Proposition 2.4. Hence Y has only one singular point of type A_1 . Let



be the canonical resolution. Then \tilde{X} is the Hirzebruch surface of degree 1 and

the branch locus of $\tilde{\pi}$ is a fibre of the canonical \mathbf{P}^1 -fibration $\theta: \tilde{X} \rightarrow \mathbf{P}^1$. By the Stein factorization of $\theta \circ \tilde{\pi}$, we obtain a \mathbf{P}^1 -fibration on \tilde{Y} . More precisely, \tilde{Y} is the Hirzebruch surface of degree 2.

§3. Artin-Schreier coverings of simple type with ample branch loci.

In this section, the characteristic p of k is not necessarily 2. Let $\pi: Y \rightarrow X$ be an Artin-Schreier covering of simple type, where X and Y are nonsingular projective surfaces. The branch locus B of π is assumed to be a reduced ample curve satisfying $H^1(X, \mathcal{O}(-B))=0$. We denote by Σ_n the Hirzebruch surface of degree n . We shall fix these notations and assumptions throughout the section. The following lemma is immediately derived from the classification of the divisor $K_X + (\text{ample divisor})$. For the reader's convenience, we shall give the proof.

LEMMA 3.1. *Suppose that the canonical divisor K_Y of Y is not numerically effective, not nef in short. Then the following assertions hold:*

- (1) $p < 5$.
- (2) If $p=2$, then X is either a relatively minimal ruled surface or the projective plane.
- (3) If $p=3$, then X is the projective plane.

PROOF. Since K_X is not nef, there exists a curve C on Y such that $(K_Y, C) < 0$. Set $D = \pi(C)$. Then $(K_X + (p-1)B, D) < 0$ by the canonical divisor formula in Proposition 1.7. Let $\overline{NE}(X)$ be the closed convex cone spanned by all effective divisors on X modulo numerical equivalence. Let $P = \{E \in \overline{NE}(X) \mid (K_X + (p-1)B, E) < 0\}$ and $Q = \{E \in \overline{NE}(X) \mid (K_X, E) < 0\}$. Then $P \subset Q$ and $P \neq \emptyset$. By the Mori theory, Q is polyhedral and so is P . Hence there exists an extremal rational curve l such that $(K_X + (p-1)B, l) < 0$. Moreover, one of the following three cases takes place:

- (1) l is a line on \mathbf{P}^2 ;
- (2) l is a fibre on a relatively minimal ruled surface;
- (3) l is a (-1) -curve.

We consider these three cases separately.

Case (1). Since $X = \mathbf{P}^2$, one obtains $B \sim al$ for some positive integer a and $(B, l) = a$. On the other hand, $(K_X, l) = -3$. So, $(K_X + (p-1)B, l) < 0$ implies $(p-1)a < 3$. Hence $(p-1)a = 1$ or 2. Only three cases can occur: (i) $p=2$ and $a=1$; (ii) $p=2$ and $a=2$; (iii) $p=3$ and $a=1$.

Case (2). We know that $(K_X, l) = -2$. Let $B \equiv aM + bl$ ($a, b \in \mathbf{Z}$), where " \equiv " means the numerical equivalence and M is a cross-section of the \mathbf{P}^1 -fibration given on X . Since B is ample, $a > 0$ and $b + a(M^2) > 0$. The inequality

$(K_X + (p-1)B, l) < 0$ implies $(p-1)a < 2$. Hence $p=2$ and $a=1$.

Case (3). Since l is a (-1) -curve, we have $(K_X, l) = -1$. By the same inequality as above, one obtains $(p-1)(B, l) < 1$. This is impossible. Q. E. D.

We shall specify each case.

THEOREM 3.2. *Suppose $X = \mathbf{P}^2$. Then we have:*

(1) *B is either a line or a conic. If B is a line (resp. conic), then $p=2$ or 3 (resp. $p=2$).*

(2) *If $p=2$ and B is a line, then Y is isomorphic to Σ_0 .*

(3) *If $p=3$ and B is a line, then Y is a del Pezzo surface of degree 3, i.e. Y is a cubic hypersurface in \mathbf{P}^3 .*

(4) *If B is a conic (hence $p=2$), then Y is a del Pezzo surface of degree 2 and the generator of the Galois group of $\pi: Y \rightarrow X$ is the Geiser involution.*

PROOF. (1) The assertion was already verified in the proof of the previous lemma.

(2) Since $K_X + B \sim -2B$, we have $(K_Y^2) = 8$. On the other hand, the irregularity $q(Y)$ of Y equals to that $q(X)$ of X by Lemma 1.9. So, $q(Y) = 0$ and Y is a Hirzebruch surface Σ_n . Consider the canonical divisor K_Y . We can write $K_Y = -2M_0 - (n+2)L$, where M_0 is the minimal section and L is a fibre. Meanwhile, $K_Y = -2\pi^*B$. Hence n is an even number. Write $\pi^*B \sim M_0 + aL$ ($a \in \mathbf{Z}$). Then $2a = n + 2$. Since B is ample, so is π^*B . Thus $a > n$. So, $n = 0$.

(3) and (4) Straightforward. Q. E. D.

THEOREM 3.3. *Suppose that X is a relatively minimal ruled surface with irregularity q . Then the following assertions hold:*

(1) *$B = S + l_1 + \dots + l_r$, where S is a cross-section and l_i 's are fibres.*

(2) *Y is a ruled surface with the \mathbf{P}^1 -fibration $f = \theta \circ \pi: Y \rightarrow A$, where $\theta: X \rightarrow A$ is either induced by the Albanese mapping or the canonical \mathbf{P}^1 -fibration. A general fibre F of f is regarded as an Artin-Schreier covering of $l = \pi(F)$.*

(3) *Any singular fibre of f consists of two (-1) -curves crossing each other transversally.*

(4) *Let N be the number of singular fibres of f . Then $N = 2(S^2) + 4r > 0$.*

(5) *$\tilde{S} = \pi^*(S)$ is an irreducible curve with $p_a(\tilde{S}) = (S^2) + 2q + r - 1$. Moreover, \tilde{S} is a singular curve unless $X \cong \Sigma_n$ ($n \geq 0$) and $B = M + l_1 + \dots + l_{n+1}$, where M is the minimal section and l_i 's are fibres.*

PROOF. (1) The assertion was already shown in the proof of Lemma 3.1.

(2) Straightforward.

(3) Let π^*l_0 be a singular fibre of f , where l_0 is a fibre of θ . Since π is a double covering, π^*l_0 has a form $E_1 + E_2$, where E_i 's are nonsingular rational curves. Moreover, one of the components is a (-1) -curve. Hence so is the

other. From $((\pi^*l_0)^2)=0$, it follows that $(E_1, E_2)=1$.

(4) Note that $p_a(B)=p_a(S)+\sum_{i=1}^r p_a(l_i)+\sum_{i=1}^r (S, l_i)+\sum_{i<j} (l_i, l_j)+1-(1+r)=q$. Since $K_Y=\pi^*(K_X+B)$, we have $(K_Y^2)=2[2p_a(B)-2+8(1-q)+(K_X, B)]=8(1-q)-2(S^2)-4r$. On the other hand, $N=8(1-q)-(K_Y^2)$. Hence $N=2(S^2)+4r$. Furthermore, $(B^2)=(S^2)+2r>0$. Thence follows the assertion.

(5) Suppose $\pi^*(S)=2\tilde{S}$. By (4), there exists a singular fibre $\pi^*(l_0)=E_1+E_2$ on Y . By the projection formula, $2(\tilde{S}, E_1)=(S, l_0)=1$. This is impossible. Hence $\tilde{S}=\pi^*(S)$ is an irreducible curve. Since $(S, K_X+B)=2q-2+r$, we have $p_a(\tilde{S})=(1/2)(\tilde{S}, \tilde{S}+K_Y)+1=(S^2)+2q+r-1$. On the other hand, the restriction $\pi|_{\tilde{S}}: \tilde{S}\rightarrow S$ is a purely inseparable morphism. Hence the geometric genus of \tilde{S} is equal to q . Suppose that \tilde{S} is nonsingular. Then one obtains $q=0$ and $(S^2)+r=1$ since $q\geq 0$ and $(S^2)+r>0$. In particular, X is a rational ruled surface Σ_n . Set $S\sim M+al$, where $a\geq 0$. Then $(S^2)=-n+2a$. Thence $1-r=-n+2a$. Since B is ample, $a+r>n$ and hence $1-a=a+r-n>0$. We therefore have $a=0$ and $r=n+1$. Q. E. D.

By the above results, we have:

PROPOSITION 3.4. *If K_Y is not nef, then $p<5$ and $\kappa(Y)=-\infty$. Hence, if $\kappa(Y)\geq 0$, then K_Y is nef, i. e., Y is relatively minimal.*

We shall now consider the case with $\kappa(Y)=0$ and 1.

THEOREM 3.5. *Suppose $\kappa(Y)=0$. Then we have:*

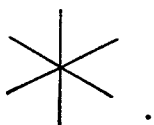
- (1) $p=2$ or 3. Moreover, X is a del Pezzo surface and Y is a K3-surface. Furthermore,
- (2) if $p=3$, then $X=\Sigma_0$ and $B\in|M+l|$, where M is the minimal section and l is a fibre.

PROOF. (1) Since Y is relatively minimal, $K_Y\equiv 0$. This implies $K_X+(p-1)B\equiv 0$. So, $-K_X$ is ample. Hence X is a del Pezzo surface. This implies $(p-1)^2(B^2)=(K_X^2)\leq 9$, whence $p-1\leq 3$, i. e., $p=2$ or $p=3$. Since X is a rational surface, $K_X+(p-1)B\sim 0$. Thence $K_Y\sim 0$. On the other hand, $q(Y)=q(X)=0$ by Lemma 1.9. Hence Y is a K3-surface.

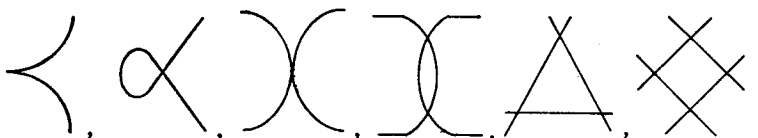
- (2) Suppose $p=3$. Then $K_X=-2B$. Hence X is Σ_0 and $B\sim M+l$. Q.E.D.

THEOREM 3.6. *Suppose $\kappa(Y)=1$. Then we have:*

- (1) Either $p=2$ or 3, and X is a ruled surface. Furthermore, if $\theta: X\rightarrow A$ is a natural P^1 -fibration on X (cf. Theorem 3.3), then $f=\theta\circ\pi: Y\rightarrow A$ is an elliptic or quasi-elliptic fibration.
- (2) If $p=3$, then X is relatively minimal, $B=S+l_1+\dots+l_r$ with a cross-section S , and every fibre of f is reduced. Moreover, any singular fibre is either a cuspidal curve or



(3) If $p=2$, then the horizontal part of components of B consists of either two cross-sections S_1 and S_2 or a single 2-section T . Any singular fibre of θ has the form E_1+E_2 with $(E_1^2)=(E_2^2)=-1$ and $(E_1, E_2)=1$. Any singular fibre of f has one of the following forms:



PROOF. (1) Since $\kappa(Y)=1$, Y has an elliptic or quasi-elliptic fibration f . Let F be a general fibre of f . Since $aK_Y \approx bF$ for some positive integers a and b , we have $b(gF) \approx bF$ for any element g of the Galois group G , where “ \approx ” means the algebraic equivalence. So, gF is also a fibre of f . Let $C = (\pi_*F)_{\text{red}}$. By the projection formula, $(K_Y, F) = (K_X + (p-1)B, \pi_*F) = 0$. Hence $(K_X + (p-1)B, C) = 0$. Since $(B, C) > 0$, we have $(K_X, C) < 0$. On the other hand, we have $(C^2) = 0$ because $\pi^*(C) \approx cF$ for some positive integer c . So, $C \cong \mathbf{P}^1$ and $(K_X, C) = -2$. Thus X is a ruled surface. Let $\theta: X \rightarrow A$ be the canonical \mathbf{P}^1 -fibration if $q(X) > 0$ and the \mathbf{P}^1 -fibration defined by the linear system $|C|$ if $q(X) = 0$. Then f must be the composite $\theta \circ \pi$. Furthermore, if l is a general fibre of θ , then $f^*(l)$ is a general fibre F of f and $\pi|_F: F \rightarrow l$ is an Artin-Schreier covering. We may assume that l is the above C . So, $-2 = -(p-1)(B, l)$. Hence, $p=2$ or 3 .

(2) Suppose F_0 is a reducible singular fibre of f and G is an irreducible component of F_0 . Then $G \cong \mathbf{P}^1$, $(G^2) = -2$ and $(G, K_Y) = 0$. Set $E = (\pi_*G)_{\text{red}}$. Then $(K_X + (p-1)B, E) = 0$. Since $(B, E) > 0$, we have $(K_X, E) < 0$. Hence, if $(E^2) < 0$, then E is a (-1) -curve and $p=2$. From these observations, it follows that the \mathbf{P}^1 -fibration θ has no singular fibres provided $p=3$. Indeed, if H is a singular fibre of θ , then π^*H is a reducible singular fibre of f , whose existence implies $p=2$. Thus X is a relatively minimal ruled surface if $p=3$. Let l be a general fibre of θ . Then $(K_X + 2B, l) = 0$. So, $(B, l) = 1$. Hence we can write $B = S + l_1 + \dots + l_r$, where S is a cross-section and l_i 's are fibres. Now the remaining assertions can be easily verified.

(3) From the same arguments as in (2), it follows that every singular fibre of θ has the form $E_1 + E_2$, where E_1 and E_2 are nonsingular rational curves crossing each other transversally. Moreover, if l is a general fibre of θ , then $(B, l) = 2$. Then the assertions follow from these observations. Q. E. D.

By virtue of the above results, we conclude the following:

COROLLARY 3.7. *If $p > 3$, then Y is a relatively minimal surface of general type. In particular, K_Y is nef.*

In Theorems 3.5 and 3.6, we considered the case where Y has an elliptic or quasi-elliptic fibration. When $p = 3$, we have a more precise result.

THEOREM 3.8. *Assume $p = 3$. Suppose that X is a relatively minimal ruled surface with irregularity q and that $f = \theta \circ \pi: Y \rightarrow A$ is an elliptic or quasi-elliptic fibration, where $\kappa(Y) \geq 0$ and $\theta: X \rightarrow A$ is the natural \mathbf{P}^1 -fibration. Let B be the branch locus of π and write $B = S + l_1 + \dots + l_r$, where S is a cross-section and l_i 's are fibres of θ . Then we have the following:*

- (1) π^*S is reduced.
- (2) $\tilde{S} = \pi^*(S)$ is a singular curve.
- (3) f is an elliptic fibration.
- (4) $\kappa(Y) > 0$ if and only if $2(q-1) + (S^2) + 2r > 0$.

PROOF. (1) Suppose $\pi^*S = 3\tilde{S}$. Then $\pi|_{\tilde{S}}: \tilde{S} \rightarrow S$ is an isomorphism. So, $2q - 2 = (\tilde{S}^2) + (\tilde{S}, K_Y)$. Meanwhile, $(\tilde{S}^2) = (1/3)(S^2)$ and $(\tilde{S}, K_Y) = 2q - 2 + (S^2) + 2r$. Hence $(S^2) + r = -(1/2)r < 0$. However, $(B, S) = (S^2) + r > 0$, a contradiction.

(2) Since $\pi|_{\tilde{S}}: \tilde{S} \rightarrow S$ is a purely inseparable covering, \tilde{S} has the geometric genus q . On the other hand, $p_a(\tilde{S})$ is computed as

$$\begin{aligned} p_a(\tilde{S}) &= (1/2)(\pi^*S, \pi^*S + K_Y) + 1 \\ &= (3/2)[(2q - 2) + 2(S^2) + 2r] + 1 \\ &= 3q + 3((S^2) + r) - 2. \end{aligned}$$

Hence $p_a(\tilde{S}) - q = 2(q - 1) + 3((S^2) + r) > 0$.

(3) Suppose f is a quasi-elliptic fibration. Then \tilde{S} must be the locus of moving singularities on Y . Hence \tilde{S} is nonsingular (cf. Bombieri-Mumford [4]). This contradicts (2).

(4) Compute $(K_X + 2B, S) = 2(q - 1) + (S^2) + 2r$.

Q. E. D.

In characteristic $p = 2$, we have the following partial result:

THEOREM 3.9. *Assume $p = 2$. Suppose that X is a ruled surface and $f = \theta \circ \pi: Y \rightarrow A$ is an elliptic or quasi-elliptic fibration, where $\theta: X \rightarrow A$ is the natural \mathbf{P}^1 -fibration. Suppose that $\kappa(Y) \geq 0$ and that the horizontal part of components of B consists of S_1 and S_2 which are cross-sections (resp. a 2-section T). Then we have:*

- (1) $\pi^*(S_i)$ (resp. $\pi^*(T)$) is reduced.

Suppose, furthermore, that one of the following conditions holds:

- (i) $q(X) \neq 0$;

(ii) $(S_i^2) \geq 0$ for $i=1, 2$ (resp. $(T^2) \geq 0$).

Then

(2) $\tilde{S}_i = \pi^*S_i$ is a singular curve for $i=1, 2$ (resp. $\tilde{T} = \pi^*T$ is a singular curve).

(3) f is an elliptic fibration.

PROOF. At first we consider the case where B contains two cross-sections.

(1) Suppose $\pi^*S_i = 2\tilde{S}_i$. Then $\tilde{S}_i \cong S_i$ via π and $(\tilde{S}_i^2) = (1/2)(S_i^2)$. Meanwhile, $2p_a(\tilde{S}_i) - 2 = (\tilde{S}_i, K_Y + \tilde{S}_i) = (\tilde{S}_i^2) + (S_i, K_X + B) = (\tilde{S}_i^2) + 2p_a(S_i) - 2 + (B - S_i, S_i)$. Hence $(B, S_i) = (1/2)(S_i^2)$. So, $(B, S_i) = -(B - S_i, S_i)$. Since $(B, S_i) > 0$ and $(B - S_i, S_i) > 0$, this is a contradiction.

(2) Suppose \tilde{S}_i is nonsingular. Since the restriction of $\pi: \tilde{S}_i \rightarrow S_i$ is purely inseparable, we have $p_a(\tilde{S}_i) = p_a(S_i)$. On the other hand, $2p_a(\tilde{S}_i) - 2 = 4p_a(S_i) - 4 + 2(B, S_i)$. Hence $(B, S_i) = 1 - p_a(S_i)$, whence $p_a(S_i) = 0$ and $(B, S_i) = 1$. Meanwhile, we have $(K_X + B, S_i) \geq 0$. So, $(K_X, S_i) \geq -1$. This implies $(S_i^2) \leq -1$. This is, however, inconsistent with the hypothesis. Hence \tilde{S}_i is singular.

(3) It is similar to the proof of (3) of the previous lemma.

The case where B contains a 2-section is handled in the same way as above. Q. E. D.

The condition (i) or (ii) in the previous theorem is not necessary to show that f is an elliptic fibration. In fact, we have the following:

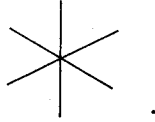
PROPOSITION 3.10. *Let X, Y, A, π, f , and θ be as in the previous theorem. Then f is an elliptic fibration.*

PROOF. Suppose f is a quasi-elliptic fibration. Let Γ be the locus of moving singularities on Y . In view of the construction of the fibration f , we know that $\pi(\Gamma)$ is contained in the horizontal part of B . Take an general fibre l of θ and choose a local parameter y of A so that l is defined by $y=0$. Let $\{P\} = \pi(\Gamma) \cap l$ and $Q = \pi^{-1}(P)$. Assume that B is locally given by $x=0$, where (x, y) is a local coordinate system at P . We consider the completion $(\mathcal{O}_{P, X})^\wedge = k[[x, y]]$. Let \mathfrak{D} be $\mathcal{O}_{Q, Y} \otimes_{\mathcal{O}_{P, X}} k[[x, y]]$. Suppose $\mathfrak{D} = k[[x, y]][\xi]/(\xi^2 + x\xi + t)$ with $t = c_0(y) + xc_1(y) + x^2c_2(x, y) \in k[[x, y]]$. Write $\Phi = \xi^2 + x\xi + t$. Since f is a quasi-elliptic fibration, we must have $\partial\Phi/\partial\xi = \partial\Phi/\partial x = 0$ wherever $x=0$. This implies that $\xi + \partial t/\partial x = \xi + c_1(y) = 0$ wherever $x=0$. Meanwhile, $\xi^2 = c_0(y)$ wherever $x=0$. Therefore, $c_0(y) = c_1(y)^2$. So, we have $\partial\Phi/\partial y = 0$ wherever $x=0$. Hence \mathfrak{D} is not normal, a contradiction. Q. E. D.

In the rest of this section, we shall construct examples of singular fibres of elliptic fibrations.

EXAMPLE 3.11. Assume $\text{char } k = p = 3$. Let $\pi: Y \rightarrow X$ be as in Example 1.10.

Then $f = p_1 \circ \pi : Y \rightarrow \mathbf{P}^1$ is an elliptic fibration and two fibres $f^{-1}(P_0)$ and $f^{-1}(P_\infty)$ exhaust singular fibres of f , where we consider the \mathbf{P}^1 -fibration on $\mathbf{P}^1 \times \mathbf{P}^1$ defined by the first projection p_1 and where P_0 and P_∞ are points of \mathbf{P}^1 defined respectively by $x=0$ and $x=\infty$. Moreover, $f^{-1}(P_0)$ is a cuspidal rational curve and $f^{-1}(P_\infty)$ is of type



EXAMPLE 3.12. Assume $\text{char } k = p = 2$. Let $X = \mathbf{P}^1 \times \mathbf{P}^1$ and let $\{U_i \times V_j\}_{i,j=1,2}$ be the same as in Example 1.12. Take an Artin-Schreier covering $\pi : Y \rightarrow X$ which is defined by

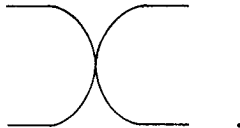
$$\xi^2 + xy(x+y)\xi + (x+y) + ax^3 + by^3 + (x^3y^4 + x^4y^3) = 0$$

$(a, b \in k \text{ and } ab(a+b) \neq 0)$

over $U_1 \times V_1$ and whose branch locus is $L + M + \Delta$, where $L = \{x=0\}$, $M = \{y=0\}$ and $\Delta = \{x+y=0\}$, i.e., the diagonal. Then Y is a smooth K3-surface with an elliptic fibration $p_1 \circ \pi$. Let F_α be the fibre of f defined by $x = \alpha$. We have the following singular fibres:

- F_0 : a rational curve with one cusp.
- F_α : a rational curve with one node, where α satisfies one of the following equations: $1 + b\alpha^4 + (a+b)\alpha^7 + \alpha^{12} = 0$, $1 + \alpha^5 + a\alpha^7 = 0$ or $\alpha = \infty$.

EXAMPLE 3.13. Keep the same assumptions and notations as in the previous example. Let $\sigma : X' \rightarrow X$ be the blowing-up with centre $(x=1, y=0)$ and let $\pi' : Y' \rightarrow X'$ be the normalization of X' in $k(Y)$. Then the branch locus B' of π' is $E + L' + M' + \Delta'$, where L', M' and Δ' are the proper transforms of L, M and Δ , respectively and where E is the exceptional curve of σ . Moreover, the fibre F_0 is replaced by



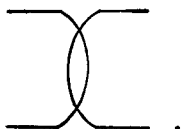
EXAMPLE 3.14. Let X and $\{U_i \times V_j\}$ be as above. Let $\pi : Y \rightarrow X$ be an Artin-Schreier covering which is defined by

$$\xi^2 + xy(x+1)(y+1)\xi + ax + by + x^3 + y^3 = 0$$

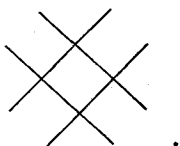
$(a, b \in k, a \neq 0, b \neq 0, a+1+b(b+1) \neq 0, b+1+a(a+1) \neq 0)$

over $U_1 \times V_1$ and whose branch locus is $L_0 + L_1 + M_0 + M_1$, where L_0, L_1, M_0 and

M_1 are defined by $x=0$, $x=1$, $y=0$ and $y=1$, respectively. Then Y is a smooth $K3$ -surface with an elliptic fibration $f=p_1 \circ \pi: Y \rightarrow \mathbf{P}^1$. Let F_∞ be the fibre defined by $x=\infty$. Then F_∞ is of type



Blow up the point $(x=\infty, y=0)$ to obtain $\sigma: X' \rightarrow X$. Let Y' be the normalization of X' in $k(Y)$. Then F_∞ is replaced by



References

- [1] M. Artin, Some numerical criteria for contractability of curves on algebraic surfaces, *Amer. J. Math.*, **84** (1962), 485-496.
- [2] M. Artin, Wildly ramified $\mathbf{Z}/2\mathbf{Z}$ actions in dimension two, *Proc. Amer. Math. Soc.*, **52** (1975), 60-64.
- [3] E. Bombieri and D. Mumford, Enriques' classification of surfaces in char. p , II, in *Complex Analysis and Algebraic Geometry*, (W.L. Baily and T. Shioda, eds), Iwanami Shoten and Cambridge University Press, 1977, pp. 22-42.
- [4] E. Bombieri and D. Mumford, Enriques' classification of surfaces in char. p , III, *Invent. Math.*, **35** (1976), 197-232.
- [5] T. Ekedahl, Canonical models of surfaces of general type in positive characteristic, preprint.
- [6] M. Miyanishi, p -cyclic coverings of the affine space, *J. Algebra*, **63** (1980), 279-284.
- [7] S. Mori, Threefolds whose canonical bundles are not numerically effective, *Ann. of Math.*, **116** (1982), 133-176.
- [8] D. Mumford, Enriques' classification of surfaces in char. p , I, in *Global Analysis*, (D. C. Spencer and S. Iyanaga, eds), Princeton University Press, 1969, pp. 325-339.

Yoshifumi TAKEDA

Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan