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Artin-Schreier coverings of algebraic surfaces

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Introduction.

Let k be an algebraically closed field of characteristic $p>0$ and let X be a nonsingular projective surface defined over k. An *Artin-Schreier covering* of X is a finite morphism $\pi: Y \rightarrow X$ from a normal surface Y onto X such that the field extension $k(Y)/k(X)$ is an Artin-Schreier extension. It is well-known that $k(Y)$ is expressed as $k(Y)=k(X)(\xi)$ with $\xi^p-\xi=f$ and $f \in k(X)$. Since $k(Y)/k(X)$ is a Galois extension with the Galois group $G \cong \boldsymbol{Z}/p\boldsymbol{Z}$, G acts on Y so that $X \cong Y/G$. In order to study Artin-Schreier coverings, we have to consider whether or not there exists an affine open covering $\mathfrak{U}=\{U_\lambda\}$ such that $\pi^{-1}(U_\lambda)=\operatorname{Spec} \mathcal{O}_X(U_\lambda)[\xi_\lambda]/(\xi_\lambda{}^p-s_\lambda\xi_\lambda-t_\lambda)$ with $s_\lambda, t_\lambda \in \mathcal{O}_X(U_\lambda)$. In general, this assertion does not hold (cf. Example 1.5). Under the above circumstance, we shall define an Artin-Schreier covering of simple type (see § 1 for the definition), for which the assertion holds. From the definition, every Artin-Schreier covering in characteristic 2 is of simple type.

This article consists of three parts. In Section 1, we consider Artin-Schreier coverings of simple type and give some formulas to compute invariants in the case of nonsingular coverings. In Section 2, we assume that the characteristic is 2 and consider a resolution of singularities for Artin-Schreier coverings with nonsingular branch locus. We give some formulas to compute invariants of nonsingular models of coverings. In Section 3, we consider smooth Artin-Schreier coverings of simple type with ample branch loci which satisfy extra conditions. Especially, we shall determine such coverings with $\kappa=-\infty$, 0, and 1.

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§ 1. Artin-Schreier coverings of simple type.

Let X be a nonsingular projective surface and let $\pi: Y \rightarrow X$ be an Artin-Schreier covering. Since Y is a Cohen-Macaulay scheme and X is regular, π is a flat morphism. Hence $\pi_* \mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-algebra. Moreover, PROPOSITION 1.1. *There is a canonical filtration of $\mathcal{O}_X$-modules of $\pi_*\mathcal{O}_Y$,*

$$\mathcal{O}_X = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{p-1} = \pi_*\mathcal{O}_Y$$

such that

(1) *$\mathcal{F}_i$ is a locally free sheaf of rank $i+1$,*

(2) *$\mathcal{F}_1/\mathcal{F}_0$ is an invertible sheaf and $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a torsion-free $\mathcal{O}_X$-module of rank 1 for $1 \leqq i \leqq p-1$.*

PROOF. Let $U = \operatorname{Spec} R$ be an affine open subset of X and let $\pi^{-1}(U) = \operatorname{Spec} A$. Then $\pi^{-1}(U)$ is a G-stable set and $U = \pi^{-1}(U)/G$. On the other hand, as a group scheme, G is written as $G = \operatorname{Spec} k[z]/(z^p - z)$ with the comultiplication $\Delta(z) = z \otimes 1 + 1 \otimes z$ and the counit $\varepsilon(z) = 0$. So, the coaction of G on $\operatorname{Spec} A$ is given by an R-algebra homomorphism $\sigma: A \to A[z]$ with $z^p = z$ such that $(\sigma \otimes 1)\sigma = (1 \otimes \Delta)\sigma$ and $(1 \otimes \varepsilon)\sigma = \mathrm{id}_A$. Write $\sigma(a) = \sigma_0(a) + \sigma_1(a)z + \cdots + \sigma_{p-1}(a)z^{p-1}$ for $a \in A$. Then $(1 \otimes \varepsilon)\sigma = \mathrm{id}_A$ implies $\sigma_0 = \mathrm{id}_A$. We have

$$(\sigma \otimes 1)\sigma(a) = \sum_{i=0}^{2p-2} \sum_{j=0}^{i} \sigma_j \sigma_{i-j}(a) z^j \otimes z^{i-j}$$

and

$$(1 \otimes \Delta)\sigma(a) = \sum_{i=0}^{p-1} \sum_{j=0}^{i} {}_iC_j \sigma_i(a) z^j \otimes z^{i-j}.$$

Thence the relation $(\sigma \otimes 1)\sigma = (1 \otimes \Delta)\sigma$ implies $\sigma_j \sigma_{i-j} = {}_iC_j \sigma_i$ for $0 \leqq i \leqq p-1$ and $\sigma_i = 0$ for $i \geqq p$. Set $\sigma_1 = \delta$. Then these relations are equivalent to $\sigma_0 = \mathrm{id}_A$, $\sigma_i = 1/(i!)\delta^i$ $(1 \leqq i \leqq p)$ and $\delta^p = 0$. So, we can write

$$\sigma(a) = a + \delta(a)z + \frac{1}{2!}\delta^2(a)z^2 + \cdots + \frac{1}{(p-1)!}\delta^{p-1}(a)z^{p-1}.$$

Set $F_i = \{a \in A \mid \delta^{i+1}(a) = 0\}$ for $0 \leqq i \leqq p-1$. Then $F_0 = R$ and F_i is an R-module. Since the G-action on A is nontrivial, there exists $a \in A$ such that $\sigma(a) \neq a$. Suppose $\delta^r(a) \neq 0$ and $\delta^{r+1}(a) = 0$ for $0 < r < p$. Then $\delta^r(a) \in F_0$. So, $\sigma(\delta^{r-1}(a)) = \delta^{r-1}(a) + \delta^r(a)z$. Therefore, $F_1 \neq F_0$. Furthermore, for $1 \leqq i \leqq p-1$,

$$\begin{aligned}\sigma((\delta^{r-1}(a))^i) &= \sigma(\delta^{r-1}(a))^i = (\delta^{r-1}(a) + \delta^r(a)z)^i \\ &= \delta^{r-1}(a)^i + i(\delta^{r-1}(a))^{i-1}\delta^r(a)z + \cdots + (\delta^r(a))^i z^i.\end{aligned}$$

This implies that $F_i \neq F_{i-1}$ for $1 \leqq i \leqq p-1$. On the other hand, F_i is the inverse image by σ of R-module $A + Az + \cdots + Az^i$ of $A[z]$. Hence F_i/F_{i-1} is viewed as an R-submodule of $(A + Az + \cdots + Az^i)/(A + Az + \cdots + Az^{i-1}) \cong A$. Therefore, F_i/F_{i-1} is a torsion-free R-module of rank 1.

Now we sheafify the above observations. Since the operator δ is defined globally, we can define a coherent sheaf $\mathcal{F}_i$ so that, on an affine open subset W, $\mathcal{F}_i|_W = \{a \in \pi_*\mathcal{O}_Y(W) \mid \delta^{i+1}(a) = 0\}^\sim$. Then $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a torsion-free $\mathcal{O}_X$-module of rank 1.

To show that $\mathcal{F}_i$ is a locally free sheaf, we take the double dual $\mathcal{F}_i^{**}$ of

$\mathcal{F}_i$. Then we have

$$\begin{array}{ccccccc} \mathcal{O}_X & = \mathcal{F}_0 & \subset \mathcal{F}_1 & \subset \cdots \subset & \mathcal{F}_{p-1} & = \pi_*\mathcal{O}_Y \\ \downarrow & & \downarrow & & & \downarrow \\ \mathcal{O}_X^{**} & = \mathcal{F}_0^{**} & \subset \mathcal{F}_1^{**} & \subset \cdots \subset & \mathcal{F}_{p-1}^{**} & = \pi_*\mathcal{O}_Y^{**} . \end{array}$$

So, we may regard $\mathcal{F}_i^{**}$ as $\mathcal{O}_X$-submodule of $\pi_*\mathcal{O}_Y$. Hence δ^{i+1} operates on $\mathcal{F}_i^{**}$ and $\delta^{i+1} \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_i^{**}, \pi_*\mathcal{O}_Y)$. We know that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_i^{**}, \pi_*\mathcal{O}_Y)$ is a locally free sheaf and that $\delta^{i+1}|_V=0$, where $V=X-\mathrm{Supp}\,\mathcal{F}_i^{**}/\mathcal{F}_i$. On the other hand, since $\mathcal{F}_i$ is torsion-free and X is regular, $\mathcal{F}_i^{**}/\mathcal{F}_i$ has support of codimension$\geqq 2$. Therefore, $\delta^{i+1}(\mathcal{F}_i^{**})=0$. So, we have $\mathcal{F}_i^{**}=\mathcal{F}_i$. Hence $\mathcal{F}_i$ is a locally free sheaf.

Finally we show that $\mathcal{F}_1/\mathcal{F}_0$ is an invertible sheaf. We consider an exact sequence

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_1/\mathcal{F}_0 \longrightarrow 0 .$$

We know that $\mathcal{F}_0 \otimes k(P) \to \mathcal{F}_1 \otimes k(P)$ is injective for an arbitrary point $P \in X$ because $\mathcal{F}_0 \otimes k(P)$ contains the unity of $(\pi_*\mathcal{O}_Y) \otimes k(P)$. Hence we have $\mathrm{rank}(\mathcal{F}_1/\mathcal{F}_0) \otimes k(P)=1$. This implies that $\mathcal{F}_1/\mathcal{F}_0$ is an invertible sheaf on X. Q. E. D.

We shall define a good class of Artin-Schreier coverings. Let π, X, Y and $\mathcal{F}_i$ be as above. We call $\pi: Y \to X$ an Artin-Schreier covering *of simple type* if $\mathcal{F}_i/\mathcal{F}_{i-1} \cong (\mathcal{F}_1/\mathcal{F}_0)^{\otimes i}$ for $1 \leqq i \leqq p-1$. From now on, we consider Artin-Schreier coverings of simple type. We have the following fundamental lemmas.

LEMMA 1.2. *Suppose that $\pi: Y \to X$ is an Artin-Schreier covering of simple type. Then there exist an affine open covering $\mathfrak{U}=\{U_\lambda\}$ of X and $s_\lambda, t_\lambda \in H^0(U_\lambda, \mathcal{O}_X)$ such that*

$$\pi^{-1}(U_\lambda) = \mathrm{Spec}\,\mathcal{O}_X(U_\lambda)[\xi_\lambda]/(\xi_\lambda^p - s_\lambda \xi_\lambda - t_\lambda) .$$

PROOF. Write $\mathcal{L}^{-1}=\mathcal{F}_1/\mathcal{F}_0$. Let $\mathfrak{U}=\{U_\lambda\}$ be an affine open covering of X such that $\mathcal{L}^{-1}|_{U_\lambda} \cong \mathcal{O}_{U_\lambda}$. Take $\xi_\lambda \in \mathcal{F}_1(U_\lambda)$ such that $\mathcal{L}^{-1}|_{U_\lambda}=\mathcal{O}_{U_\lambda}\overline{\xi_\lambda}$, where $\overline{\xi_\lambda}$ is the image of ξ_λ. Then $\sigma(\xi_\lambda)=\xi_\lambda+\alpha_\lambda z$ with $\alpha_\lambda \in H^0(U_\lambda, \mathcal{O}_X)$, where σ and z are the same as in the proof of Proposition 1.1. Since $\sigma(\xi_\lambda^p)=\xi_\lambda^p+\alpha_\lambda^p z$, we have $\xi_\lambda^p \in \mathcal{F}_1(U_\lambda)$. Thus, $\xi_\lambda^p=s_\lambda \xi_\lambda+t_\lambda$ with $s_\lambda, t_\lambda \in H^0(U_\lambda, \mathcal{O}_X)$. On the other hand, $\pi_*\mathcal{O}_Y|_{U_\lambda}=\mathcal{O}_{U_\lambda}+\mathcal{O}_{U_\lambda}\xi_\lambda+\cdots+\mathcal{O}_{U_\lambda}\xi_\lambda^{p-1}$ by the hypothesis. The assertion follows from these observations. Q. E. D.

LEMMA 1.3. *Under the same assumptions and notations as in the previous lemma, we have $\{s_\lambda\} \in H^0(X, \mathcal{L}^{p-1})$, $s_\lambda=\alpha_\lambda^{p-1}$, $\{\alpha_\lambda\} \in H^0(X, \mathcal{L})$ and $t_\mu - a_{\lambda\mu}^p t_\lambda = b_{\lambda\mu}^p - s_\mu b_{\lambda\mu}$, where $\{a_{\lambda\mu}\}$ is transition functions of $\mathcal{L}$ and $\{b_{\lambda\mu}\} \in H^1(X, \mathcal{L})$.*

PROOF. Since $\xi_\lambda^p=s_\lambda\xi_\lambda+t_\lambda$, we obtain $\sigma(\xi_\lambda^p)=\xi_\lambda^p+\alpha_\lambda^p z=(s_\lambda\xi_\lambda+t_\lambda)+s_\lambda\alpha_\lambda z$.

Thus $s_\lambda=\alpha_\lambda^{p-1}$. On $U_\lambda\cap U_\mu$, set $\xi_\mu=a_{\lambda\mu}\xi_\lambda+b_{\lambda\mu}$. Then $\{b_{\lambda\mu}\}\in H^1(X, \mathcal{L})$ and we have

$$\begin{aligned}\xi_\mu{}^p &= s_\mu\xi_\mu+t_\mu = s_\mu(a_{\lambda\mu}\xi_\lambda+b_{\lambda\mu})+t_\mu\\ &= a_{\lambda\mu}{}^p\xi_\lambda{}^p+b_{\lambda\mu}{}^p = a_{\lambda\mu}{}^p(s_\lambda\xi_\lambda+t_\lambda)+b_{\lambda\mu}{}^p\,.\end{aligned}$$

Hence, $s_\mu=a_{\lambda\mu}{}^{p-1}s_\lambda$ and $t_\mu-a_{\lambda\mu}{}^p t_\lambda=b_{\lambda\mu}{}^p-s_\mu b_{\lambda\mu}$. So, we have $\{s_\lambda\}\in H^0(X, \mathcal{L}^{p-1})$. Moreover,

$$\sigma(\xi_\mu)=\sigma(a_{\lambda\mu}\xi_\lambda+b_{\lambda\mu})=a_{\lambda\mu}\xi_\lambda+b_{\lambda\mu}+a_{\lambda\mu}\alpha_\lambda z=\xi_\mu+\alpha_\mu z\,.$$

Thence $\alpha_\mu=a_{\lambda\mu}\alpha_\lambda$, i.e. $\{\alpha_\lambda\}\in H^0(X, \mathcal{L})$. Q.E.D.

REMARK 1.4. If $\{b_{\lambda\mu}\}=0$ in $H^1(X, \mathcal{L})$, we can choose t_λ so that $\{t_\lambda\}\in H^0(X, \mathcal{L}^p)$. In particular, we may assume $\{t_\lambda\}\in H^0(X, \mathcal{L}^p)$ provided $H^1(X, \mathcal{L})=(0)$.

Let B be the effective divisor corresponding to the section $\{\alpha_\lambda\}\in H^0(X, \mathcal{L})$. Clearly, B is independent of choice of generators $\{\xi_\lambda\}$. Moreover, π is unramified over $X-\operatorname{Supp} B$ and totally ramified over $\operatorname{Supp} B$. We call B the *branch locus* of π.

We shall give an example of an Artin-Schreier covering which is not of simple type.

EXAMPLE 1.5. Assume that char $k=p>2$. Consider $\boldsymbol{P}^3_k$ with a homogeneous coordinate system (x_0, x_1, x_2, x_3). Let $X=\{x_3=0\}\cong\boldsymbol{P}^2$, $Y=\{x_3{}^p-x_1{}^{p-1}x_3-x_1{}^{p-1}x_2=0\}\subset\boldsymbol{P}^3$ and let $\rho: Y\to X$ be the projection from $(0, 0, 0, 1)$. Then ρ is surjective and finite. Take the normalization $\nu: \tilde{Y}\to Y$ and denote $\pi=\rho\circ\nu$. Let $U_i=\{x_i\neq 0\}\subset X$ for $i=0, 1, 2$. By the Jacobian criterion, $\rho(\operatorname{Sing} Y)=\{x_1=0\}$. Hence $\pi^{-1}(U_1)=\rho^{-1}(U_1)$. On the other hand, $\rho^{-1}(U_0)=\operatorname{Spec} k[x, y, \xi]$, where $\xi^p-x^{p-1}\xi-x^{p-1}y$, and $\rho^{-1}(U_2)=\operatorname{Spec} k[u, v, \zeta]$, where $\zeta^p-u^{p-1}\zeta-u^{p-1}$. It is easy to verify that $\pi^{-1}(U_2)=\operatorname{Spec} k[v, \zeta, \tau]$, where $\tau=u/\zeta$ and $\zeta-\tau^{p-1}\zeta-\tau^{p-1}=0$. On U_0, it is a little more difficult. Let $T^p=x$ and $S=\xi/T^{p-1}$. Then T and S are algebraically independent over k and we have $k[T^p, T^{p-1}S, S^p]=k[x, y, \xi]$. Let $\mathcal{O}=k[T^p, T^{p-1}S, T^{p-2}S^2, \cdots, TS^{p-1}, S^p]$. Clearly, $T^{p-2}S^2, \cdots, TS^{p-1}$ are integral over $k[T^p, T^{p-1}S, S^p]$. Meanwhile, $\mathcal{O}$ is none other than the coordinate ring of the cone of the p-uple embedding of $\boldsymbol{P}^1$ in $\boldsymbol{P}^p$. So, $\mathcal{O}$ is normal. Hence $\mathcal{O}$ is the integral closure of $k[x, y, \xi]$. Furthermore, $\operatorname{Spec}\mathcal{O}$ has only one singular point whose minimal resolution consists of a curve C such that $C\cong\boldsymbol{P}^1$ and $(C^2)=-p$.

Since $k(\tilde{Y})/k(X)$ is an Artin-Schreier extension, $\pi: \tilde{Y}\to X$ is an Artin-Schreier covering. However, π is not of simple type. Suppose π is of simple type. Then $\tilde{Y}$ is locally a hypersurface by Lemma 1.2. Hence every rational singularity is a rational double point. This contradicts the above observation.

By Remark 1.4, if $H^1(X, \mathcal{L})=(0)$, every Artin-Schreier covering of simple type is defined locally by $\xi_\lambda{}^p-\alpha_\lambda{}^{p-1}\xi_\lambda=t_\lambda$, where $\alpha=\{\alpha_\lambda\}\in H^0(X, \mathcal{L})$ and $t=\{t_\lambda\}\in H^0(X, \mathcal{L}^p)$. The set $\{dt_\lambda\}$ of 1-forms defines a section $dt\in H^0(X, \Omega_X\otimes\mathcal{L}^p)$. Applying the Jacobian criterion to the above local defining equations, we obtain

PROPOSITION 1.6. *With the above notations and assumptions, in the characteristic $p>2$, Y is singular at a point $Q\in Y$ if and only if $\pi(Q)\in\operatorname{Supp} B$ and $dt=0$ at $\pi(Q)$.*

Artin-Schreier coverings of simple type are obtained as follows (cf. [**5**]). Let L be a line bundle on X associated with invertible sheaf $\mathcal{L}$ and consider L and L^p as smooth X-group schemes. Take a global section s of L^{p-1} and consider a surjective homomorphism of X-group schemes $F-s: L\to L^p$ defined by $(F-s)(x)=x^p-sx$ for $x\in L$. Let α_s be its kernel. Then we have an exact sequence of X-group schemes in flat topology

$$0\longrightarrow \alpha_s\longrightarrow L\longrightarrow L^p\longrightarrow 0\,.$$

Taking the flat cohomologies, we have an exact sequence

$$0\to H^0_{fl}(X, \alpha_s)\to H^0(X, \mathcal{L})\to H^0(X, \mathcal{L}^p)\xrightarrow{\ \partial\ } H^1_{fl}(X, \alpha_s)\to H^1(X, \mathcal{L})\,,$$

where we can interpret $H^1_{fl}(X, \alpha_s)$ as the set of isomorphism classes of α_s-torsors. Suppose that $s\in H^0(X, \mathcal{L}^{p-1})$ is given locally by $\{s_\lambda\}$ as in Lemmas 1.2 and 1.3. Let $t=\{t_\lambda\}$ be local sections of $\mathcal{L}^p$ as in Lemma 1.3 and let $\rho: Z\to X$ be the α_s-torsor obtained by applying the connecting map ∂ to $t=\{t_\lambda\}$. In other words, Z is locally the fibre product of $F-s: L\to L^p$ and $t_\lambda: U_\lambda\to L^p$. Then it is clear that $\rho: Z\to X$ is isomorphic to $\pi: Y\to X$. If $H^1(X, \mathcal{L})=0$, all α_s-torsors, hence all Artin-Schreier coverings of simple type, are obtained from global sections of $\mathcal{L}^p$ (cf. Remark 1.4).

In the sequel of this section, we consider an Artin-Schreier covering $\pi: Y\to X$ of simple type. We fix the notations $\mathcal{F}_i$ $(0\leq i<p)$, $\mathcal{L}$, and B as in Proposition 1.1 and Lemmas 1.2 and 1.3. By the local description, we know that Y is locally a hypersurface. Therefore Y is a Gorenstein scheme. More precisely, we have

PROPOSITION 1.7. *Y has the dualizing sheaf*

$$\omega_Y=\pi^*(\omega_X\otimes\mathcal{L}^{p-1})\,.$$

PROOF. Apply the adjunction formula.

We shall compute invariants of Artin-Schreier coverings of simple type. There are the following formulas.

LEMMA 1.8. (1) $(\omega_Y^2)=p\{(K_X^2)+2(p-1)(B, K_X)+(p-1)^2(B^2)\}$.

(2) $\chi(\mathcal{O}_Y)=p\left\{\chi(\mathcal{O}_X)+\frac{(p-1)}{4}(B, K_X)+\frac{(p-1)(2p-1)}{12}(B^2)\right\}$.

(3) *If Y is smooth,*

$$e(Y)=p\{e(X)+(p-1)(B, K_X)+(p-1)p(B^2)\},$$

where $e(Y)$ *is the Euler number of* Y.

(4) $\kappa(Y)=\kappa(X, K_X+(p-1)B)$.

PROOF. (1) Immediate from Proposition 1.7.

(2) By the assumptions, $\mathcal{F}_i/\mathcal{F}_{i-1}\cong\mathcal{O}(-iB)$. Hence we have $\chi(\mathcal{F}_i)=\chi(\mathcal{F}_{i-1})+\chi(\mathcal{O}(-iB))$ for $1\leqq i\leqq p-1$. Therefore $\chi(\mathcal{O}_Y)=\chi(\mathcal{F}_{p-1})=\sum_{i=0}^{p-1}\chi(\mathcal{O}(-iB))$, where, by the Riemann-Roch theorem,

$$\begin{aligned}\chi(\mathcal{O}(-iB))&=(1/2)(-iB, -iB-K_X)+\chi(\mathcal{O}_X)\\&=(1/2)(i^2(B^2)+i(B, K_X))+\chi(\mathcal{O}_X).\end{aligned}$$

Thence we obtain the stated formula.

(3) Use Noether's formula: $12\chi(\mathcal{O}_Y)=(K_Y^2)+e(Y)$.

(4) It follows from a fundamental property of the D-dimension. Q. E. D.

In order to construct examples, we need the following:

LEMMA 1.9. *Let X, Y and $\mathcal{L}$ be as above. Suppose that $\mathcal{L}$ is ample. If $H^1(X, \mathcal{L}^{-1})=(0)$, we have $H^1(X, \mathcal{O}_X)=H^1(Y, \mathcal{O}_Y)$.*

PROOF. By the exact sequence

$$0\longrightarrow\mathcal{L}^{-1}\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{O}_B\longrightarrow 0,$$

we have $H^0(X, \mathcal{O}_X)=H^0(B, \mathcal{O}_B)=k$. The exact sequence

$$0\longrightarrow\mathcal{L}^{-2}\longrightarrow\mathcal{L}^{-1}\longrightarrow\mathcal{L}^{-1}\otimes\mathcal{O}_B\longrightarrow 0$$

implies $H^1(X, \mathcal{L}^{-2})=(0)$ because $\mathcal{L}\otimes\mathcal{O}_B$ is ample and $H^1(X, \mathcal{L}^{-1})=(0)$. Similarly, by the exact sequences

$$0\longrightarrow\mathcal{L}^{-i}\longrightarrow\mathcal{L}^{-(i-1)}\longrightarrow\mathcal{L}^{-(i-1)}|_B\longrightarrow 0 \qquad (i>1),$$

we obtain inductively $H^1(X, \mathcal{L}^{-i})=(0)$. Now look at the exact sequences

$$0\longrightarrow\mathcal{F}_{i-1}\longrightarrow\mathcal{F}_i\longrightarrow\mathcal{L}^{-i}\longrightarrow 0 \qquad (0<i<p).$$

We know $H^1(X, \mathcal{F}_{i-1})=H^1(X, \mathcal{F}_i)$. Hence $H^1(Y, \mathcal{O}_Y)=H^1(X, \mathcal{O}_X)$. Q. E. D.

EXAMPLE 1.10. Assume that $\operatorname{char} k=p=3$. Let $X=\boldsymbol{P}^1\times\boldsymbol{P}^1$ and $\mathcal{L}=p_1{}^*\mathcal{O}(1)\otimes p_2{}^*\mathcal{O}(1)$. Take an affine covering $\{U_i\times V_j\}_{i,j=1,2}$ such that $U_1=\operatorname{Spec} k[x]$,

$U_2=\operatorname{Spec} k[u]$, $V_1=\operatorname{Spec} k[y]$ and $V_2=\operatorname{Spec} k[v]$, where $u=x^{-1}$, $v=y^{-1}$. Let $\pi: Y\to X$ be an Artin-Schreier covering such that

$$\pi^{-1}(U_1\times V_1)=\operatorname{Spec}\mathcal{O}_{U_1\times V_1}[\xi_{11}]/(\xi_{11}{}^3-x^2y^2\xi_{11}-(x^2+y^2+x+y)),$$
$$\pi^{-1}(U_1\times V_2)=\operatorname{Spec}\mathcal{O}_{U_1\times V_2}[\xi_{12}]/(\xi_{12}{}^3-x^2\xi_{12}-(x^2v^3+v+xv^3+v^2)),$$
$$\pi^{-1}(U_2\times V_1)=\operatorname{Spec}\mathcal{O}_{U_2\times V_1}[\xi_{21}]/(\xi_{21}{}^3-y^2\xi_{21}-(u+y^2u^3+u^2+yu^3)),$$
$$\pi^{-1}(U_2\times V_2)=\operatorname{Spec}\mathcal{O}_{U_2\times V_2}[\xi_{22}]/(\xi_{22}{}^3-\xi_{22}-(uv^3+u^3v+u^2v^3+u^3v^2)).$$

Then Y is nonsingular and its dualizing sheaf is

$$\omega_Y=\pi^*(p_1{}^*\mathcal{O}(-2)\otimes p_2{}^*\mathcal{O}(-2)\otimes\mathcal{L}^2)\cong\mathcal{O}_Y.$$

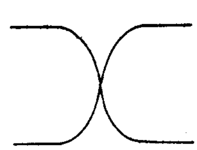
By the previous lemma, we see that $H^1(Y, \mathcal{O}_Y)=H^1(X, \mathcal{O}_X)=(0)$. Hence Y is a $K3$-surface.

EXAMPLE 1.11. Assume $\operatorname{char} k=p=3$. Let $X=\boldsymbol{P}^2$ and $\mathcal{L}=\mathcal{O}(1)$. Then $H^1(\boldsymbol{P}^2, \mathcal{L})=(0)$. Let (x, y, z) be a system of homogeneous coordinate of $\boldsymbol{P}^2$. Choose $s=x^2\in H^0(\boldsymbol{P}^2, \mathcal{L}^2)$ and $t=xy^2+x^2y+y^2z+yz^2+z^2x+zx^2\in H^0(\boldsymbol{P}^2, \mathcal{L}^3)$. Let $\pi: Y\to X$ be an Artin-Schreier covering of simple type obtained from s and t. Then Y is smooth. Moreover, $\omega_Y=\pi^*\mathcal{O}(-1)$ and $(K_Y{}^2)=3$. So, Y is a del Pezzo surface of degree 3, i.e. a smooth cubic hypersurface in $\boldsymbol{P}^3$.

EXAMPLE 1.12. Assume $\operatorname{char} k=p=2$. Let $X=\boldsymbol{P}^1\times\boldsymbol{P}^1$ and let $\mathcal{L}=p_1{}^*\mathcal{O}(2)\otimes p_2{}^*\mathcal{O}(3)$. Take an affine open covering $\{U_i\times V_j\}_{i,j=1,2}$ which is the same as in Example 1.10. Let $\pi: Y\to X$ be an Artin-Schreier covering such that

$$\pi^{-1}(U_1\times V_1)=\operatorname{Spec}\mathcal{O}_{U_1\times V_1}[\xi_{11}]/(\xi_{11}{}^2+x^2(y+1)^3\xi_{11}+(x+x^3)y^3+y^5+y^3+y),$$
$$\pi^{-1}(U_1\times V_2)=\operatorname{Spec}\mathcal{O}_{U_1\times V_2}[\xi_{12}]/(\xi_{12}{}^2+x^2(1+v)^3\xi_{12}+(x+x^3)v^3+v^5+v^3+v),$$
$$\pi^{-1}(U_2\times V_1)=\operatorname{Spec}\mathcal{O}_{U_2\times V_1}[\xi_{21}]/(\xi_{21}{}^2+(y+1)^3\xi_{21}+(u+u^3)y^3+u^4(y^5+y^3+y)),$$
$$\pi^{-1}(U_2\times V_2)=\operatorname{Spec}\mathcal{O}_{U_2\times V_2}[\xi_{22}]/(\xi_{22}{}^2+(1+v)^3\xi_{22}+(u+u^3)v^3+u^4(v^5+v^3+v)).$$

Then Y is nonsingular. By Proposition 1.7 and Lemma 1.8, we have $\omega_Y=\pi^*p_2{}^*\mathcal{O}(1)$ and $\kappa(Y)=1$. Moreover, $f=p_2\circ\pi: Y\to\boldsymbol{P}^1$ is an elliptic fibration and three fibers $f^{-1}(P_0)$, $f^{-1}(P_1)$ and $f^{-1}(P_\infty)$ exhaust singular fibres of f, where P_0, P_1 and P_∞ are points of $\boldsymbol{P}^1$ defined by $y=0$, 1 and ∞, respectively. The fibres $f^{-1}(P_0)$ and $f^{-1}(P_\infty)$ are of type


.

The fibre $f^{-1}(P_1)$ is a cuspidal rational curve.

EXAMPLE 1.13. Let p, X, $\mathcal{L}$ and $\{U_i \times V_j\}$ be as in the previous example. Let $\pi: Y \to X$ be an Artin-Schreier covering such that

$$\pi^{-1}(U_1 \times V_1) = \operatorname{Spec} \mathcal{O}_{U_1 \times V_1}[\xi_{11}]/(\xi_{11}^2 + x^2(y+1)^3\xi_{11} + x^3y^3 + y^3),$$
$$\pi^{-1}(U_1 \times V_2) = \operatorname{Spec} \mathcal{O}_{U_1 \times V_2}[\xi_{12}]/(\xi_{12}^2 + x^2(1+v)^3\xi_{12} + x^3v^3 + v^3),$$
$$\pi^{-1}(U_2 \times V_1) = \operatorname{Spec} \mathcal{O}_{U_2 \times V_1}[\xi_{21}]/(\xi_{21}^2 + (y+1)^3\xi_{21} + uy^3 + u^4y^3),$$
$$\pi^{-1}(U_2 \times V_2) = \operatorname{Spec} \mathcal{O}_{U_2 \times V_2}[\xi_{22}]/(\xi_{22}^2 + (1+v)^3\xi_{22} + uv^3 + u^4v^3).$$

Then the branch locus of π is the same as in the previous example. Y has two singular points, which lie over the points $(x=0,\ y=0)$ and $(x=0,\ y=\infty)$ of X. It is easy to verify that both points are rational double points of type E_6. Let $\sigma: \tilde{Y} \to Y$ be the minimal resolution of singularities of Y. Then we have $\omega_{\tilde{Y}} = \sigma^* \circ \pi^* \circ p_2^* \mathcal{O}(1)$ and $\kappa(Y)=1$. Moreover, the composite $f = p_2 \circ \pi \circ \sigma$ defines a quasi-elliptic fibration $f: \tilde{Y} \to \boldsymbol{P}^1$.

§ 2. Canonical resolution of singularities in the case of nonsingular branch locus and in characteristic 2.

In this section, we assume $\operatorname{char} k = p = 2$. Let $\pi: Y \to X$ be an Artin-Schreier covering, which is necessarily of simple type. Suppose that the branch locus B in the sense of § 1 is a nonsingular curve on X. Since Y is normal, Y has at most isolated singularities. We shall consider a resolution of singularities of Y which we call the canonical resolution of singularities of Y. To begin with, we consider a local ring $\mathfrak{O} = k[[x, y]][\xi]/(\xi^2 + x\xi + t)$ with $t \in k[[x, y]]$, which has at most an isolated singularity. Then $\mathfrak{O}$ is normal. Write $t = c_0 + c_1x + c_2y + c_3xy$ with $c_i \in k[[x^2, y^2]]$. Replacing ξ by $\xi + c_1 + c_3y$, we may assume $t = c_0 + c_2y$. So, we can write $t = d_0(y) + x^2d_1(x^2, y)$, where $d_0(y) \neq 0$ by the hypothesis that $\mathfrak{O}$ is normal. Write $d_0 = a_\nu y^\nu +$(terms of higher degree), where $\nu \geqq 0$, $a_\nu \in k$ and $a_\nu \neq 0$. Clearly, $\mathfrak{O}$ is regular if and only if $\nu = 0$ or $\nu = 1$. Furthermore, it is easy to see that ν is invariant under change of variables $(\xi, x, y) \to (\xi + f, x, y)$ with $f \in k[[x, y]]$ as long as we keep the condition $t = c_0 + c_2y$. Suppose $\nu \geqq 2$. Let $x_1 = x/y$. Then

$$\xi^2 + x\xi + d_0(y) + x^2d_1(x^2, y) = \xi^2 + x_1y\xi + d_0(y) + x_1^2y^2d_1(x_1^2y^2, y).$$

Normalizing this equation, we have

$$\xi_1^2 + x_1\xi_1 + d_0^{(1)}(y) + x_1^2d_1(x_1^2y^2, y) = 0, \qquad \text{where } \xi_1 = \xi/y.$$

Inductively, one obtains the following series of local rings

$$\begin{aligned}\mathfrak{O} = \mathfrak{O}_0 &= k[[x, y]][\xi]/(\xi^2+x\xi+d_0(y)+x^2 d_1),\\ \mathfrak{O}_1 &= k[[x_1, y]][\xi_1]/(\xi_1^2+x_1\xi_1+d_0^{(1)}(y)+x_1^2 d_1),\\ &\vdots \\ \mathfrak{O}_n &= k[[x_n, y]][\xi_n]/(\xi_n^2+x_n\xi_n+d_0^{(n)}(y)+x_n^2 d_1),\end{aligned}$$

and

$$\begin{tikzcd} \mathfrak{O}=\mathfrak{O}_0 \arrow[r, dash] \arrow[d, dash] & \mathfrak{O}_1 \arrow[r, dash] \arrow[d, dash] & \cdots\cdots \arrow[r, dash] & \mathfrak{O}_n \arrow[d, dash] \\ k[[x, y]] \arrow[r, dash] & k[[x_1, y]] \arrow[r, dash] & \cdots\cdots \arrow[r, dash] & k[[x_n, y]], \end{tikzcd}$$

where $n=[\nu/2]$. Then $\mathfrak{O}_n$ is regular. Globally speaking, we consider a series of blowing-ups $X=X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n$ with centres $(x=0,\ y=0)$, $(x_1=0,\ y=0)$, $\cdots$, $(x_{n-1}=0,\ y=0)$ and consider the normalization Y_i of X_i in the function field $k(Y)$. Thus one obtains a commutative diagram

$$\begin{tikzcd} Y=Y_0 \arrow[d, "\pi"'] & Y_1 \arrow[l] \arrow[d, "\pi_1"] & \cdots \arrow[l] & Y_n \arrow[l] \arrow[d, "\pi_n"] \\ X=X_0 & X_1 \arrow[l] & \cdots \arrow[l] & X_n . \arrow[l] \end{tikzcd}$$

We call this process of blowing-ups the *canonical resolution* of the singularity of Spec $\mathfrak{O}$.

Suppose that ν is even. Then

$$\mathfrak{O}_n \cong k[[x, y]][\eta]/(\eta^2+x\eta+x+t'(x, y)),$$

where $t'(x, y)$ consists of terms of degree $\geqq 2$ and $\eta^2+x\eta+x+t'(x, 0)$ is irreducible. Let E_n be the exceptional curve of the blowing-up $X_n \to X_{n-1}$ and let $\tilde{E}_n=\pi_n^{-1}(E_n)$. Since $\pi_n^{-1}(E_n)=\{y=0\}$ locally, $\tilde{E}_n$ is an irreducible curve and $(\tilde{E}_n^2)=-2$. Therefore we have the following configuration of exceptional curves which arise from the canonical resolution of the singularity of Spec $\mathfrak{O}$

$$\circ\!-\!\!-\!\circ\!-\!\!-\cdots-\!\!-\!\underset{\tilde{E}_n}{\circ}\!-\!\!-\cdots-\!\!-\!\circ\!-\!\!-\!\circ \qquad \text{type } \mathrm{A}_{\nu-1},$$

where "$\circ$" stands for a nonsingular rational curve whose self-intersection number is -2, i.e. a (-2)-curve. In particular, we know the Spec $\mathfrak{O}$ has a rational double point.

Now suppose that ν is odd. Then

$$\mathfrak{O}_n \cong k[[x, y]][\eta]/(\eta^2+x\eta+y).$$

Let E_n be as above. Since $\pi^{-1}(E_n)=\{y=0\}$ locally, $\pi^{-1}(E_n)$ splits to two curves $F_n=\{\xi=0\}$ and $G_n=\{\xi+x=0\}$. F_n and G_n intersect transversally at the point $(\xi, x, y)=(0, 0, 0)$. So, $(F_n^2)=(G_n^2)=-2$. Therefore, we have the following configuration of exceptional curves which arise from the canonical resolution of the singularity of Spec$\mathfrak{O}$

$$\circ - \circ - \cdots - \underset{\tilde{F}_n}{\circ} - \underset{\tilde{G}_n}{\circ} - \cdots - \circ - \circ \qquad \text{type } A_{\nu-1},$$

where "$\circ$" stands for a (-2)-curve as above. In particular, we know that Spec$\mathfrak{O}$ has a rational double point.

By virtue of the above observations, we conclude

THEOREM 2.1. *Let* $\mathfrak{O}$, ν *and* t *be as above. Then*

(1) Spec$\mathfrak{O}$ *has a singularity if and only if* $\nu \geqq 2$.

(2) *If* Spec$\mathfrak{O}$ *has a singular point, then it is a rational double point of type* $A_{\nu-1}$.

We know that ν is an important invariant of a local ring $\mathfrak{O}$. There is the following explicit formula.

LEMMA 2.2. *With the same notations and assumptions,*

$$\nu = \text{length}\, k[[x, y]]/(x, t+(\partial t/\partial x)^2).$$

PROOF. Write $t=c_0+c_1x+c_2y+c_3xy$ with $c_i \in k[[x^2, y^2]]$. Set $\xi'=\xi+c_1+c_3y$. Then $\xi^2+x\xi+t=\xi'^2+x\xi'+c_0+c_1^2+c_3^2y^2+c_2y$. So, $d_0(y)+x^2d_1(x^2, y)=c_0+c_1^2+c_3^2y^2+c_2y$. On the other hand, $t+(\partial t/\partial x)^2=c_0+c_1^2+c_3^2y^2+c_2y+x(c_1+c_3y)$. Therefore, we have $(x, d_0(y))=(x, d_0(y)+x^2d_1)=(x, t+(\partial t/\partial x)^2)$ as ideals in $k[[x, y]]$. Since $\nu=\text{length}\, k[[x, y]]/(x, d_0(y))$, we obtain the required formula.

Q.E.D.

Let $\pi: Y \to X$ be an Artin-Schreier covering obtained as an α_s-torsor from a line boundle L on X, a global section s of L and local sections $\{t_\lambda\}$ of L^2 (cf. §1). Suppose that $B=(s)_0$ is a nonsingular curve on X and that $\{t_\lambda\}$ give rise to a global section of L^2. Take an affine covering $\{U_\lambda\}$ such that $s=x_\lambda e_\lambda$ on U_λ and (x_λ, y_λ) is a local coordinate system on U_λ for $U_\lambda \cap B \neq \emptyset$, where $\mathcal{L}|_{U_\lambda}=\mathcal{O}_{U_\lambda}e_\lambda$. Then $\pi^{-1}(U_\lambda)=\text{Spec}\,\mathcal{O}_X(U_\lambda)[\xi]/(\xi^2+x_\lambda\xi+t_\lambda)$. For each closed point $P \in X$, we define $\nu(P)$ after Lemma 2.2 as follows:

$$\nu(P)=\begin{cases} \text{length}\,(\mathcal{O}_{P,X})\hat{}/(x_\lambda, t_\lambda+(\partial t_\lambda/\partial x_\lambda)^2) & \text{if } P \in B \cap U_\lambda \\ 0 & \text{if } P \notin B. \end{cases}$$

We shall estimate $\sum_{P \in Y, \nu(P)>0}(\nu(P)-1)$ as follows.

LEMMA 2.3. $\{(\partial t_\lambda/\partial y_\lambda)|_B\} \in H^0(B, \omega_X \otimes \mathcal{L}^3|_B)$.

PROOF. Since $dt_\mu=(\partial t_\mu/\partial x_\mu)dx_\mu+(\partial t_\mu/\partial y_\mu)dy_\mu$, we have $dx_\mu \wedge dt_\mu = (\partial t_\mu/\partial y_\mu)dx_\mu \wedge dy_\mu=(\partial t_\mu/\partial y_\mu)J_{\mu\lambda}dx_\lambda \wedge dy_\lambda$, where $\{J_{\mu\lambda}\}$ are the transition functions of the canonical bundle of X. Let $\{a_{\lambda\mu}\}$ be transition functions of $\mathcal{L}$ such that

$e_\lambda = a_{\lambda\mu} e_\mu$. Then $x_\mu = a_{\lambda\mu} x_\lambda$ and $t_\mu = a_{\lambda\mu}{}^2 t_\lambda$. So, $dx_\mu = a_{\lambda\mu} dx_\lambda + x_\lambda da_{\lambda\mu}$ and $dt_\mu = a_{\lambda\mu}{}^2 dt_\lambda$. Therefore,

$$\begin{aligned} dx_\mu \wedge dt_\mu &= a_{\lambda\mu}{}^3 dx_\lambda \wedge dt_\lambda + x_\lambda a_{\lambda\mu}{}^2 da_{\lambda\mu} \wedge dt_\lambda \\ &= a_{\lambda\mu}{}^3 \frac{\partial t_\lambda}{\partial y_\lambda} dx_\lambda \wedge dy_\lambda + x_\lambda a_{\lambda\mu}{}^2 da_{\lambda\mu} \wedge dt_\lambda \,. \end{aligned}$$

Hence we have $(\partial t_\mu/\partial y_\mu)|_B \cdot J_{\mu\lambda}|_B = a_{\lambda\mu}{}^3|_B \cdot (\partial t_\lambda/\partial y_\lambda)|_B$ on B. This asserts that $\{(\partial t_\lambda/\partial y_\lambda)|_B\} \in H^0(B, \omega_X \otimes \mathcal{L}^3|_B)$. Q. E. D.

PROPOSITION 2.4. $\sum(\nu(P)-1) \leqq \max\{2(B^2), 2(B^2)+2p_a(B)-2\}$, *where* $P \in X$ *and* $\nu(P)>0$.

PROOF. Set $\partial_y t = \{(\partial t_\lambda/\partial y_\lambda)|_B\} \in H^0(B, \omega_X \otimes \mathcal{L}^3|_B)$. Suppose $\partial_y t \neq 0$. Let $P \in B$ and $P \in U_\lambda$. We consider t_λ, x_λ and y_λ in $(\mathcal{O}_{P,X})^\wedge$. With the same notations as in Lemma 2.2, $t_\lambda = c_0 + c_1 x_\lambda + c_2 y_\lambda + c_3 x_\lambda y_\lambda$ and $\partial t_\lambda/\partial y_\lambda = c_2 + c_3 x_\lambda$. Since $d_0(y_\lambda) + x_\lambda{}^2 d_1(x_\lambda{}^2, y_\lambda) = c_0 + c_1{}^2 + c_3{}^2 y_\lambda{}^2 + c_2 y_\lambda$, we have $\nu(P) \leqq$(multiplicity of $(\partial_\lambda t)_0$ at P)$+1$, where $(\partial_y t)_0$ is the effective divisor corresponding to $\partial_y t$. Hence $\sum(\nu(P)-1) \leqq (B, 3B+K_X) = 2(B^2)+2p_a(B)-2$.

Now, suppose $\partial_y t = 0$, i.e. $\partial t_\lambda/\partial y_\lambda = 0$ on B for all λ. Then $dt_\mu = (\partial t_\mu/\partial x_\mu) dx_\mu$ on B. Since $dx_\mu = (a_{\lambda\mu} + x_\lambda \cdot \partial a_{\lambda\mu}/\partial x_\lambda) dx_\lambda + (\partial x_\mu/\partial y_\lambda) dy_\lambda$,

$$\begin{aligned} dt_\mu &= \frac{\partial t_\mu}{\partial x_\mu}\left[\left(a_{\lambda\mu} + x_\lambda \cdot \frac{\partial a_{\lambda\mu}}{\partial x_\lambda}\right) dx_\lambda + \frac{\partial x_\mu}{\partial y_\lambda} dy_\lambda\right] \\ &= \frac{\partial t_\mu}{\partial x_\mu}\left(a_{\lambda\mu} + x_\lambda \cdot \frac{\partial a_{\lambda\mu}}{\partial x_\lambda}\right) dx_\lambda + \frac{\partial t_\mu}{\partial x_\mu} \cdot \frac{\partial x_\mu}{\partial y_\lambda} dy_\lambda \qquad \text{on } B \,. \end{aligned}$$

On the other hand, $dt_\mu = a_{\lambda\mu}{}^2 (\partial t_\lambda/\partial x_\lambda) dx_\lambda + a_{\lambda\mu}{}^2 (\partial t_\lambda/\partial y_\lambda) dy_\lambda$. Therefore $a_{\lambda\mu}{}^2 (\partial t_\lambda/\partial x_\lambda) = (\partial t_\mu/\partial x_\mu)\{a_{\lambda\mu} + x_\lambda (\partial a_{\lambda\mu}/\partial x_\lambda)\}$ on B. Namely, we have $(\partial t_\mu/\partial x_\mu|_B) \cdot (a_{\lambda\mu}|_B) = (a_{\lambda\mu}{}^2|_B)(\partial t_\lambda/\partial x_\lambda|_B)$. Hence $\{\partial t_\lambda/\partial x_\lambda|_B\}$ is a global section of $\mathcal{L}|_B$. Set $\tau = \{[t_\lambda + (\partial t_\lambda/\partial x_\lambda)^2]|_B\}$. Then τ is a global section of $\mathcal{L}^2|_B$. For $P \in B$, we know that $\nu(P) =$(multiplicity of $(\tau)_0$ at P). So, $\sum_{P \in X} \nu(P) = 2(B^2)$. The assertion follows from these observations. Q. E. D.

Let $\pi\colon Y \to X$ be an Artin-Schreier covering with nonsingular branch locus B and let

$$\begin{tikzcd} Y \arrow[d, "\pi"'] & \tilde{Y} \arrow[l, "\rho"'] \arrow[d, "\tilde{\pi}"] \\ X & \tilde{X} \arrow[l, "\sigma"] \end{tikzcd}$$

be the canonical resolution of singularities of Y.

PROPOSITION 2.5. $\tilde{Y}$ *has the dualizing sheaf*

$$\omega_{\tilde{Y}} = \rho^* \circ \pi^*(\omega_X \otimes \mathcal{O}(B)).$$

PROOF. We already know that $\omega_Y = \pi^*(\omega_X \otimes \mathcal{O}(B))$. Since every singularity of Y is a rational double point by Theorem 2.1, $\omega_{\tilde{Y}} = \rho^* \omega_Y$. Therefore, $\omega_{\tilde{Y}} = \rho^* \circ \pi^*(\omega_X \otimes \mathcal{O}(B))$. Q. E. D.

COROLLARY 2.6. (1) $(K_{\tilde{Y}}^2) = 2(K_X + B)^2$.
(2) $\kappa(\tilde{Y}) = \kappa(X, K_X + B)$.

PROOF. Straightforward.

In general, $\tilde{Y}$ may not be a minimal surface. However we have

PROPOSITION 2.7. *If K_X is numerically effective, nef in short, $\tilde{Y}$ is a minimal surface.*

PROOF. Let E be a (-1)-curve on $\tilde{Y}$ and write $\phi = \sigma \circ \tilde{\pi}$. Suppose that $\phi_* E = 2C$, where C is the set-theoretic image of E. Since $(E, K_Y) = -1$, we have $-1 = 2(C, K_X + B)$ by the projection formula. This is a contradiction. Now, suppose $\phi_* E = C$. Similarly we have $-1 = (C, K_X + B)$. So, $(C, B) = -1 - (C, K_X) \leqq -1$ by the assumption. Hence C must be an irreducible component of B. Since B is a disjoint union of nonsingular curves, $(C, B) = (C^2)$. Therefore we have $(C, K_X + C) = -1$ and this is impossible. This completes the proof. Q. E. D.

We shall compute other invariants of $\tilde{Y}$.

PROPOSITION 2.8. *Under the same notations, we have*

$$\chi(\mathcal{O}_{\tilde{Y}}) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}(-B)).$$

PROOF. Since Y has only rational double points, one obtains $\chi(\mathcal{O}_{\tilde{Y}}) = \chi(\mathcal{O}_Y)$. Now the assertion follows from Lemma 1.8.(2). Q. E. D.

COROLLARY 2.9. $\mathrm{e}(\tilde{Y}) = 2[\mathrm{e}(X) + (B, K_X + 2B)]$.

We have already computed the irregularity in the case where Y is nonsingular with an assumption (cf. Lemma 1.9). Here we have another formula

PROPOSITION 2.10. *Under the same notations and assumptions as above,*

$$h^1(X, \mathcal{O}_X) \leqq h^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) \leqq h^1(X, \mathcal{O}_X) + h^1(X, \mathcal{O}(-B)).$$

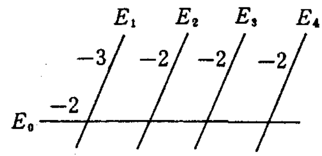
PROOF. Since Y has only rational singularities, $H^i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = H^i(Y, \mathcal{O}_Y)$. On the other hand, because $H^0(X, \mathcal{O}(-B)) = (0)$, we have

$$0 \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \pi_* \mathcal{O}_Y) \longrightarrow H^1(X, \mathcal{O}(-B)).$$

Hence $h^1(X, \pi_* \mathcal{O}_Y) \leqq h^1(X, \mathcal{O}_X) + h^1(X, \mathcal{O}(-B))$. The assertion follows from this. Q. E. D.

The following two examples of local rings have irrational singularities and appear as the local rings of Artin-Schreier coverings with non-reduced or singular branch loci.

EXAMPLE 2.11. Let $\mathfrak{O}=k[[x, y]][\xi]/(\xi^2+x^2\xi+x^3+x^2y^3+xy^6)$, let $Y=\operatorname{Spec}\mathfrak{O}$ and let $\sigma:\tilde{Y}\to Y$ be the minimal resolution of the singularity of Y. Then the exceptional locus of σ has the following configuration:



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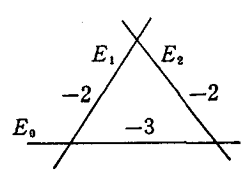
where $E_0, \cdots, E_4$ are nonsingular rational curves. The fundamental cycle Z of this singularity is $2E_0+E_1+E_2+E_3+E_4$. Hence $p_a(Z)=1$. So, this singularity is not rational.

EXAMPLE 2.12. Let $\mathfrak{O}=k[[x, y]][\xi]/(\xi^2+xy\xi+x^3+y^9)$, let $Y=\operatorname{Spec}\mathfrak{O}$ and let $\sigma:\tilde{Y}\to Y$ be the minimal resolution of the singularity of Y. Then the exceptional locus of σ has the following configuration:



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where E_0, E_1 and E_2 are nonsingular rational curves. The fundamental cycle Z is $E_0+E_1+E_2$ and $p_a(Z)=1$. Hence this singularity is not rational.

To close this section, we shall give an example of the canonical resolution.

EXAMPLE 2.13. Let $X=\boldsymbol{P}^2$ and let $\mathcal{L}=\mathcal{O}(1)$. Take $s\in H^0(X, \mathcal{L})$. Then $(s)_0$ is a line. Consider an Artin-Schreier covering Y whose branch locus B is $(s)_0$. Since $H^1(X, \mathcal{L})=0$, we know that such a covering is obtained as an α_s-torsor from s and a global section of $\mathcal{L}^2$. If Y is smooth, then Y is isomorphic to $\boldsymbol{P}^1\times\boldsymbol{P}^1$ (see §3 Theorem 3.2). Suppose Y is singular. Since $2(B^2)=2$ and $2(B^2)+2p_a(B)-2=0$, we have $\sum_{P\in X}\nu(P)=2$ by the proof of Proposition 2.4. Hence Y has only one singular point of type A_1. Let

$$\begin{tikzcd} Y \arrow[d, "\pi"'] & \tilde{Y} \arrow[l] \arrow[d, "\tilde{\pi}"] \\ X & \tilde{X} \arrow[l] \end{tikzcd}$$

be the canonical resolution. Then $\tilde{X}$ is the Hirzebruch surface of degree 1 and the branch locus of $\tilde{\pi}$ is a fibre of the canonical $\boldsymbol{P}^1$-fibration $\theta: \tilde{X} \to \boldsymbol{P}^1$. By the Stein factorization of $\theta \circ \tilde{\pi}$, we obtain a $\boldsymbol{P}^1$-fibration on $\tilde{Y}$. More precisely, $\tilde{Y}$ is the Hirzebruch surface of degree 2.

§ 3. Artin-Schreier coverings of simple type with ample branch loci.

In this section, the characteristic p of k is not necessarily 2. Let $\pi: Y \to X$ be an Artin-Schreier covering of simple type, where X and Y are nonsingular projective surfaces. The branch locus B of π is assumed to be a reduced ample curve satisfying $H^1(X, \mathcal{O}(-B))=0$. We denote by Σ_n the Hirzebruch surface of degree n. We shall fix these notations and assumptions throughout the section. The following lemma is immediately derived from the classification of the divisor K_X+(ample divisor). For the reader's convenience, we shall give the proof.

LEMMA 3.1. *Suppose that the canonical divisor K_Y of Y is not numerically effective, not nef in short. Then the following assertions hold:*

(1) $p < 5$.

(2) *If $p=2$, then X is either a relatively minimal ruled surface or the projective plane.*

(3) *If $p=3$, then X is the projective plane.*

PROOF. Since K_X is not nef, there exists a curve C on Y such that $(K_Y, C) < 0$. Set $D=\pi(C)$. Then $(K_X+(p-1)B, D)<0$ by the canonical divisor formula in Proposition 1.7. Let $\overline{NE}(X)$ be the closed convex cone spanned by all effective divisors on X modulo numerical equivalence. Let $P=\{E \in \overline{NE}(X) \mid (K_X+(p-1)B, E)<0\}$ and $Q=\{E \in \overline{NE}(X) \mid (K_X, E)<0\}$. Then $P \subset Q$ and $P \neq \emptyset$. By the Mori theory, Q is polyhedral and so is P. Hence there exists an extremal rational curve l such that $(K_X+(p-1)B, l)<0$. Moreover, one of the following three cases takes place:

(1) l is a line on $\boldsymbol{P}^2$;

(2) l is a fibre on a relatively minimal ruled surface;

(3) l is a (-1)-curve.

We consider these three cases separately.

Case (1). Since $X=\boldsymbol{P}^2$, one obtains $B \sim al$ for some positive integer a and $(B, l)=a$. On the other hand, $(K_X, l)=-3$. So, $(K_X+(p-1)B, l)<0$ implies $(p-1)a<3$. Hence $(p-1)a=1$ or 2. Only three cases can occur: (i) $p=2$ and $a=1$; (ii) $p=2$ and $a=2$; (iii) $p=3$ and $a=1$.

Case (2). We know that $(K_X, l)=-2$. Let $B \equiv aM+bl$ $(a, b \in \boldsymbol{Z})$, where "$\equiv$" means the numerical equivalence and M is a cross-section of the $\boldsymbol{P}^1$-fibration given on X. Since B is ample, $a>0$ and $b+a(M^2)>0$. The inequality

$(K_X+(p-1)B, l)<0$ implies $(p-1)a<2$. Hence $p=2$ and $a=1$.

Case (3). Since l is a (-1)-curve, we have $(K_X, l)=-1$. By the same inequality as above, one obtains $(p-1)(B, l)<1$. This is impossible. Q.E.D.

We shall specify each case.

THEOREM 3.2. *Suppose $X=\boldsymbol{P}^2$. Then we have:*

(1) *B is either a line or a conic. If B is a line (resp. conic), then $p=2$ or 3 (resp. $p=2$).*

(2) *If $p=2$ and B is a line, then Y is isomorphic to Σ_0.*

(3) *If $p=3$ and B is a line, then Y is a del Pezzo surface of degree 3, i.e. Y is a cubic hypersurface in $\boldsymbol{P}^3$.*

(4) *If B is a conic (hence $p=2$), then Y is a del Pezzo surface of degree 2 and the generator of the Galois group of $\pi: Y\rightarrow X$ is the Geiser involution.*

PROOF. (1) The assertion was already verified in the proof of the previous lemma.

(2) Since $K_X+B\sim-2B$, we have $(K_Y{}^2)=8$. On the other hand, the irregularity $q(Y)$ of Y equals to that $q(X)$ of X by Lemma 1.9. So, $q(Y)=0$ and Y is a Hirzebruch surface Σ_n. Consider the canonical divisor K_Y. We can write $K_Y=-2M_0-(n+2)L$, where M_0 is the minimal section and L is a fibre. Meanwhile, $K_Y=-2\pi^*B$. Hence n is an even number. Write $\pi^*B\sim M_0+aL$ ($a\in\boldsymbol{Z}$). Then $2a=n+2$. Since B is ample, so is π^*B. Thus $a>n$. So, $n=0$.

(3) and (4) Straightforward. Q.E.D.

THEOREM 3.3. *Suppose that X is a relatively minimal ruled surface with irregularity q. Then the following assertions hold:*

(1) *$B=S+l_1+\cdots+l_r$, where S is a cross-section and l_i's are fibres.*

(2) *Y is a ruled surface with the $\boldsymbol{P}^1$-fibration $f=\theta\circ\pi: Y\rightarrow A$, where $\theta: X\rightarrow A$ is either induced by the Albanese mapping or the canonical $\boldsymbol{P}^1$-fibration. A general fibre F of f is regarded as an Artin-Schreier covering of $l=\pi(F)$.*

(3) *Any singular fibre of f consists of two (-1)-curves crossing each other transversally.*

(4) *Let N be the number of singular fibres of f. Then $N=2(S^2)+4r>0$.*

(5) *$\tilde{S}=\pi^*(S)$ is an irreducible curve with $p_a(\tilde{S})=(S^2)+2q+r-1$. Moreover, $\tilde{S}$ is a singular curve unless $X\cong\Sigma_n$ ($n\geqq 0$) and $B=M+l_1+\cdots+l_{n+1}$, where M is the minimal section and l_i's are fibres.*

PROOF. (1) The assertion was already shown in the proof of Lemma 3.1.

(2) Straightforward.

(3) Let π^*l_0 be a singular fibre of f, where l_0 is a fibre of θ. Since π is a double covering, π^*l_0 has a form E_1+E_2, where E_i's are nonsingular rational curves. Moreover, one of the components is a (-1)-curve. Hence so is the

other. From $((\pi^*l_0)^2)=0$, it follows that $(E_1, E_2)=1$.

(4) Note that $p_a(B)=p_a(S)+\sum_{i=1}^r p_a(l_i)+\sum_{i=1}^r(S, l_i)+\sum_{i<j}(l_i, l_j)+1-(1+r)=q$. Since $K_Y=\pi^*(K_X+B)$, we have $(K_Y^2)=2[2p_a(B)-2+8(1-q)+(K_X, B)]=8(1-q)-2(S^2)-4r$. On the other hand, $N=8(1-q)-(K_Y^2)$. Hence $N=2(S^2)+4r$. Furthermore, $(B^2)=(S^2)+2r>0$. Thence follows the assertion.

(5) Suppose $\pi^*(S)=2\tilde{S}$. By (4), there exists a singular fibre $\pi^*(l_0)=E_1+E_2$ on Y. By the projection formula, $2(\tilde{S}, E_1)=(S, l_0)=1$. This is impossible. Hence $\tilde{S}=\pi^*(S)$ is an irreducible curve. Since $(S, K_X+B)=2q-2+r$, we have $p_a(\tilde{S})=(1/2)(\tilde{S}, \tilde{S}+K_Y)+1=(S^2)+2q+r-1$. On the other hand, the restriction $\pi|_{\tilde{S}}: \tilde{S}\to S$ is a purely inseparable morphism. Hence the geometric genus of $\tilde{S}$ is equal to q. Suppose that $\tilde{S}$ is nonsingular. Then one obtains $q=0$ and $(S^2)+r=1$ since $q\geqq 0$ and $(S^2)+r>0$. In particular, X is a rational ruled surface Σ_n. Set $S\sim M+al$, where $a\geqq 0$. Then $(S^2)=-n+2a$. Thence $1-r=-n+2a$. Since B is ample, $a+r>n$ and hence $1-a=a+r-n>0$. We therefore have $a=0$ and $r=n+1$. Q. E. D.

By the above results, we have:

PROPOSITION 3.4. *If K_Y is not nef, then $p<5$ and $\kappa(Y)=-\infty$. Hence, if $\kappa(Y)\geqq 0$, then K_Y is nef, i.e., Y is relatively minimal.*

We shall now consider the case with $\kappa(Y)=0$ and 1.

THEOREM 3.5. *Suppose $\kappa(Y)=0$. Then we have:*

(1) *$p=2$ or 3. Moreover, X is a del Pezzo surface and Y is a $K3$-surface. Furthermore,*

(2) *if $p=3$, then $X=\Sigma_0$ and $B\in|M+l|$, where M is the minimal section and l is a fibre.*

PROOF. (1) Since Y is relatively minimal, $K_Y\equiv 0$. This implies $K_X+(p-1)B\equiv 0$. So, $-K_X$ is ample. Hence X is a del Pezzo surface. This implies $(p-1)^2(B^2)=(K_X^2)\leqq 9$, whence $p-1\leqq 3$, i.e., $p=2$ or $p=3$. Since X is a rational surface, $K_X+(p-1)B\sim 0$. Thence $K_Y\sim 0$. On the other hand, $q(Y)=q(X)=0$ by Lemma 1.9. Hence Y is a $K3$-surface.

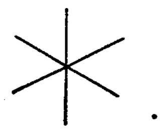
(2) Suppose $p=3$. Then $K_X=-2B$. Hence X is Σ_0 and $B\sim M+l$. Q.E.D.

THEOREM 3.6. *Suppose $\kappa(Y)=1$. Then we have:*

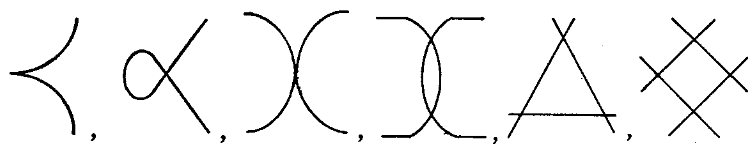
(1) *Either $p=2$ or 3, and X is a ruled surface.*

Furthermore, if $\theta: X\to A$ is a natural $\boldsymbol{P}^1$-fibration on X (cf. Theorem 3.3), *then $f=\theta\circ\pi: Y\to A$ is an elliptic or quasi-elliptic fibration.*

(2) *If $p=3$, then X is relatively minimal, $B=S+l_1+\cdots+l_r$ with a cross-section S, and every fibre of f is reduced. Moreover, any singular fibre is either a cuspidal curve or*



(3) *If $p=2$, then the horizontal part of components of B consists of either two cross-sections S_1 and S_2 or a single 2-section T. Any singular fibre of θ has the form E_1+E_2 with $(E_1^2)=(E_2^2)=-1$ and $(E_1, E_2)=1$. Any singular fibre of f has one of the following forms:*



PROOF. (1) Since $\kappa(Y)=1$, Y has an elliptic or quasi-elliptic fibration f. Let F be a general fibre of f. Since $aK_Y \approx bF$ for some positive integers a and b, we have $b(gF)\approx bF$ for any element g of the Galois group G, where "$\approx$" means the algebraically equivalence. So, gF is also a fibre of f. Let $C=(\pi_*F)_{\mathrm{red}}$. By the projection formula, $(K_Y, F)=(K_X+(p-1)B, \pi_*F)=0$. Hence $(K_X+(p-1)B, C)=0$. Since $(B, C)>0$, we have $(K_X, C)<0$. On the other hand, we have $(C^2)=0$ because $\pi^*(C)\approx cF$ for some positive integer c. So, $C\cong \boldsymbol{P}^1$ and $(K_X, C)=-2$. Thus X is a ruled surface. Let $\theta: X\to A$ be the canonical $\boldsymbol{P}^1$-fibration if $q(X)>0$ and the $\boldsymbol{P}^1$-fibration defined by the linear system $|C|$ if $q(X)=0$. Then f must be the composite $\theta\circ\pi$. Furthermore, if l is a general fibre of θ, then $f^*(l)$ is a general fibre F of f and $\pi|_F: F\to l$ is an Artin-Schreier covering. We may assume that l is the above C. So, $-2=-(p-1)(B, l)$. Hence, $p=2$ or 3.

(2) Suppose F_0 is a reducible singular fibre of f and G is an irreducible component of F_0. Then $G\cong \boldsymbol{P}^1$, $(G^2)=-2$ and $(G, K_Y)=0$. Set $E=(\pi_*G)_{\mathrm{red}}$. Then $(K_X+(p-1)B, E)=0$. Since $(B, E)>0$, we have $(K_X, E)<0$. Hence, if $(E^2)<0$, then E is a (-1)-curve and $p=2$. From these observations, it follows that the $\boldsymbol{P}^1$-fibration θ has no singular fibres provided $p=3$. Indeed, if H is a singular fibre of θ, then π^*H is a reducible singular fibre of f, whose existence implies $p=2$. Thus X is a relatively minimal ruled surface if $p=3$. Let l be a general fibre of θ. Then $(K_X+2B, l)=0$. So, $(B, l)=1$. Hence we can write $B=S+l_1+\cdots+l_r$, where S is a cross-section and l_i's are fibres. Now the remaining assertions can be easily verified.

(3) From the same arguments as in (2), it follows that every singular fibre of θ has the form E_1+E_2, where E_1 and E_2 are nonsingular rational curves crossing each other transversally. Moreover, if l is a general fibre of θ, then $(B, l)=2$. Then the assertions follow from these observations. Q.E.D.

By virtue of the above results, we conclude the following:

COROLLARY 3.7. *If $p>3$, then Y is a relatively minimal surface of general type. In particular, K_Y is nef.*

In Theorems 3.5 and 3.6, we considered the case where Y has an elliptic or quasi-elliptic fibration. When $p=3$, we have a more precise result.

THEOREM 3.8. *Assume $p=3$. Suppose that X is a relatively minimal ruled surface with irregularity q and that $f=\theta\circ\pi: Y\to A$ is an elliptic or quasi-elliptic fibration, where $\kappa(Y)\geqq 0$ and $\theta: X\to A$ is the natural $\boldsymbol{P}^1$-fibration. Let B be the branch locus of π and write $B=S+l_1+\cdots+l_r$, where S is a cross-section and l_i's are fibres of θ. Then we have the following:*

(1) *π^*S is reduced.*

(2) *$\hat{S}=\pi^*(S)$ is a singular curve.*

(3) *f is an elliptic fibration.*

(4) *$\kappa(Y)>0$ if and only if $2(q-1)+(S^2)+2r>0$.*

PROOF. (1) Suppose $\pi^*S=3\tilde{S}$. Then $\pi|_{\tilde{S}}: \tilde{S}\to S$ is an isomorphism. So, $2q-2=(\tilde{S}^2)+(\tilde{S}, K_Y)$. Meanwhile, $(\tilde{S}^2)=(1/3)(S^2)$ and $(\tilde{S}, K_Y)=2q-2+(S^2)+2r$. Hence $(S^2)+r=-(1/2)r<0$. However, $(B, S)=(S^2)+r>0$, a contradiction.

(2) Since $\pi|_{\tilde{S}}: \tilde{S}\to S$ is a purely inseparable covering, $\tilde{S}$ has the geometric genus q. On the other hand, $p_a(\tilde{S})$ is computed as

$$\begin{aligned} p_a(\tilde{S}) &= (1/2)(\pi^*S, \pi^*S+K_Y)+1 \\ &= (3/2)[(2q-2)+2(S^2)+2r]+1 \\ &= 3q+3((S^2)+r)-2\,. \end{aligned}$$

Hence $p_a(\tilde{S})-q=2(q-1)+3((S^2)+r)>0$.

(3) Suppose f is a quasi-elliptic fibration. Then $\tilde{S}$ must be the locus of moving singularities on Y. Hence $\tilde{S}$ is nonsingular (cf. Bombieri-Mumford [**4**]). This contradicts (2).

(4) Compute $(K_X+2B, S)=2(q-1)+(S^2)+2r$. Q.E.D.

In characteristic $p=2$, we have the following partial result:

THEOREM 3.9. *Assume $p=2$. Suppose that X is a ruled surface and $f=\theta\circ\pi: Y\to A$ is an elliptic or quasi-elliptic fibration, where $\theta: X\to A$ is the natural $\boldsymbol{P}^1$-fibration. Suppose that $\kappa(Y)\geqq 0$ and that the horizontal part of components of B consists of S_1 and S_2 which are cross-sections (resp. a 2-section T). Then we have:*

(1) *$\pi^*(S_i)$ (resp. $\pi^*(T)$) is reduced.*

Suppose, furthermore, that one of the following conditions holds:

(i) *$q(X)\neq 0$;*

(ii) $(S_i^2)\geqq 0$ *for* $i=1, 2$ (*resp.* $(T^2)\geqq 0$).

Then

(2) $\tilde{S}_i=\pi^*S_i$ *is a singular curve for* $i=1, 2$ (*resp.* $\tilde{T}=\pi^*T$ *is a singular curve*).

(3) f *is an elliptic fibration.*

PROOF. At first we consider the case where B contains two cross-sections.

(1) Suppose $\pi^*S_i=2\tilde{S}_i$. Then $\tilde{S}_i\cong S_i$ via π and $(\tilde{S}_i^2)=(1/2)(S_i^2)$. Meanwhile, $2p_a(\tilde{S}_i)-2=(\tilde{S}_i, K_Y+\tilde{S}_i)=(\tilde{S}_i^2)+(S_i, K_X+B)=(\tilde{S}_i^2)+2p_a(S_i)-2+(B-S_i, S_i)$. Hence $(B, S_i)=(1/2)(S_i^2)$. So, $(B, S_i)=-(B-S_i, S_i)$. Since $(B, S_i)>0$ and $(B-S_i, S_i)>0$, this is a contradiction.

(2) Suppose $\tilde{S}_i$ is nonsingular. Since the restriction of $\pi:\tilde{S}_i\to S_i$ is purely inseparable, we have $p_a(\tilde{S}_i)=p_a(S_i)$. On the other hand, $2p_a(\tilde{S}_i)-2=4p_a(S_i)-4+2(B, S_i)$. Hence $(B, S_i)=1-p_a(S_i)$, whence $p_a(S_i)=0$ and $(B, S)=1$. Meanwhile, we have $(K_X+B, S_i)\geqq 0$. So, $(K_X, S_i)\geqq -1$. This implies $(S_i^2)\leqq -1$. This is, however, inconsistent with the hypothesis. Hence $\tilde{S}_i$ is singular.

(3) It is similar to the proof of (3) of the previous lemma.

The case where B contains a 2-section is handled in the same way as above. Q.E.D.

The condition (i) or (ii) in the previous theorem is not neccessary to show that f is an elliptic fibration. In fact, we have the following:

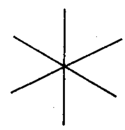
PROPOSITION 3.10. *Let* X, Y, A, π, f, *and* θ *be as in the previous theorem. Then* f *is an elliptic fibration.*

PROOF. Suppose f is a quasi-elliptic fibration. Let Γ be the locus of moving singularities on Y. In view of the construction of the fibration f, we know that $\pi(\Gamma)$ is contained in the horizontal part of B. Take an general fibre l of θ and choose a local parameter y of A so that l is defined by $y=0$. Let $\{P\}=\pi(\Gamma)\cap l$ and $Q=\pi^{-1}(P)$. Assume that B is locally given by $x=0$, where (x, y) is a local coordinate system at P. We consider the completion $(\mathcal{O}_{P,X})^{\wedge}=k[[x, y]]$. Let $\mathfrak{O}$ be $\mathcal{O}_{Q,Y}\otimes_{\mathcal{O}_{P,X}}k[[x, y]]$. Suppose $\mathfrak{O}=k[[x, y]][\xi]/(\xi^2+x\xi+t)$ with $t=c_0(y)+xc_1(y)+x^2c_2(x, y)\in k[[x, y]]$. Write $\Phi=\xi^2+x\xi+t$. Since f is a quasi-elliptic fibration, we must have $\partial\Phi/\partial\xi=\partial\Phi/\partial x=0$ wherever $x=0$. This implies that $\xi+\partial t/\partial x=\xi+c_1(y)=0$ wherever $x=0$. Meanwhile, $\xi^2=c_0(y)$ wherever $x=0$. Therefore, $c_0(y)=c_1(y)^2$. So, we have $\partial\Phi/\partial y=0$ wherever $x=0$. Hence $\mathfrak{O}$ is not normal, a contradiction. Q.E.D.

In the rest of this section, we shall construct examples of singular fibres of elliptic fibrations.

EXAMPLE 3.11. Assume char $k=p=3$. Let $\pi: Y\to X$ be as in Example 1.10.

Then $f=p_1\circ\pi: Y\to\boldsymbol{P}^1$ is an elliptic fibration and two fibres $f^{-1}(P_0)$ and $f^{-1}(P_\infty)$ exhaust singular fibres of f, where we consider the $\boldsymbol{P}^1$-fibration on $\boldsymbol{P}^1\times\boldsymbol{P}^1$ defined by the first projection p_1 and where P_0 and P_∞ are points of $\boldsymbol{P}^1$ defined respectively by $x=0$ and $x=\infty$. Moreover, $f^{-1}(P_0)$ is a cuspidal rational curve and $f^{-1}(P_\infty)$ is of type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EXAMPLE 3.12. Assume char $k=p=2$. Let $X=\boldsymbol{P}^1\times\boldsymbol{P}^1$ and let $\{U_i\times V_j\}_{i,j=1,2}$ be the same as in Example 1.12. Take an Artin-Schreier covering $\pi: Y\to X$ which is defined by

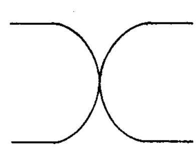
$$\xi^2+xy(x+y)\xi+(x+y)+ax^3+by^3+(x^3y^4+x^4y^3)=0$$
$$(a,\ b\in k \text{ and } ab(a+b)\neq 0)$$

over $U_1\times V_1$ and whose branch locus is $L+M+\varDelta$, where $L=\{x=0\}$, $M=\{y=0\}$ and $\varDelta=\{x+y=0\}$, i.e., the diagonal. Then Y is a smooth $K3$-surface with an elliptic fibration $p_1\circ\pi$. Let F_α be the fibre of f defined by $x=\alpha$. We have the following singular fibres:

F_0 : a rational curve with one cusp.

F_α : a rational curve with one node, where α satisfies one of the following equations: $1+b\alpha^4+(a+b)\alpha^7+\alpha^{12}=0$, $1+\alpha^5+a\alpha^7=0$ or $\alpha=\infty$.

EXAMPLE 3.13. Keep the same assumptions and notations as in the previous example. Let $\sigma: X'\to X$ be the blowing-up with centre $(x=1,\ y=0)$ and let $\pi': Y'\to X'$ be the normalization of X' in $k(Y)$. Then the branch locus B' of π' is $E+L'+M'+\varDelta'$, where L', M' and $\varDelta'$ are the proper transforms of L, M and $\varDelta$, respectively and where E is the exceptional curve of σ. Moreover, the fibre F_0 is replaced by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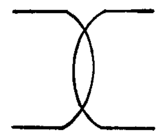
EXAMPLE 3.14. Let X and $\{U_i\times V_j\}$ be as above. Let $\pi: Y\to X$ be an Artin-Schreier covering which is defined by

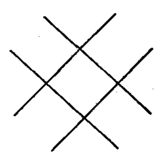
$$\xi^2+xy(x+1)(y+1)\xi+ax+by+x^3+y^3=0$$
$$(a,\ b\in k,\ a\neq 0,\ b\neq 0,\ a+1+b(b+1)\neq 0,\ b+1+a(a+1)\neq 0)$$

over $U_1\times V_1$ and whose branch locus is $L_0+L_1+M_0+M_1$, where L_0, L_1, M_0 and

M_1 are defined by $x=0$, $x=1$, $y=0$ and $y=1$, respectively. Then Y is a smooth $K3$-surface with an elliptic fibration $f=p_1\circ\pi: Y\to \boldsymbol{P}^1$. Let F_∞ be the fibre defined by $x=\infty$. Then F_∞ is of type



Blow up the point ($x=\infty$, $y=0$) to obtain $\sigma: X'\to X$. Let Y' be the normalization of X' in $k(Y)$. Then F_∞ is replaced by



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