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# Cooper-pair and Bose-Einstein condensations in two dimensions: A critical analysis based on the Nozières and Schmitt-Rink formalism

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The crossover between the Cooper-pair condensation and the Bose-Einstein condensation of "di-electronic" molecules in two-dimensional superconductors is investigated in detail on the basis of the Nozières and Schmitt-Rink formalism. It is shown that temperature dependence of the chemical potential $\mu$ so calculated is classified into two classes as decreasing temperatures; i.e., class (a) where $\mu$ approaches the point of Bose-Einstein condensation of two-dimensional ideal Bose gas of "di-electronic" molecules, and class (b) where $\mu$ diverges positively along the line of BCS-type mean-field pair condensation. This feature is rather universal irrespective of strength $V$ of the attractive interaction of the $s$-wave type. While the former class (a) has been found by Schmitt-Rink, Varma, and Ruckenstein, existence of the latter class (b) is recognized here. In the case where $V$ is fixed, class (a) is realized for electron number density $N$ smaller than $N_{cr}$, which is an increasing function of $V$, and class (b) is realized for $N$ larger than $N_{cr}$. If $N \gg N_{cr}$ in particular, there exists a regime, where the Fermi-liquid-like description is valid, between the BCS-type mean-field transition temperature and the Fermi temperature. In the situation where $V$ is changed with $N$ being fixed, low-temperature states for the strong-coupling case belong to class (a) while those for the weak-coupling case belong to class (b). Therefore, with decreasing $V$, the chemical potential $\mu(T)$, at temperatures far below the Fermi temperature, shows a discontinuous jump at $V = V_{cr}(N)$ corresponding to the transition from class (a) to (b). However, this is in contradiction to a physical picture that the chemical potential should smoothly cross over between the above two limits unless the liquid-gas transition occurs. This shows in turn a necessity of improving the Nozières and Schmitt-Rink formalism itself especially in two dimensions. A preliminary approach beyond their formalism is briefly discussed.

## I. INTRODUCTION

The BCS theory describes the Bose condensation of Cooper pairs and the superconductivity of electrons. It is regarded as the weak limit of attractive force among electrons. In this case, the formation of Cooper pairs and their Bose condensation occurs at the same temperature. If the strength of the attractive force $V$ is increased, electrons will first form "di-electronic" molecules around the temperature corresponding to its binding energy. The system will show the Bose-Einstein condensation of di-electronic molecules at some lower transition temperature. Since it is the gauge symmetry that is broken both in the weak and strong attractive limits, the states in the two limits must cross over smoothly in regard to the strength of the attractive force unless the liquid-gas phase separation occurs.

This crossover problem of the Bose condensation is one of the most fundamental problems of superconductivity,[1] and has been discussed extensively in a variety of physical contexts, e.g., nuclear matter,[2] superfluid $^3$He,[3,4] excitons in semiconductors,[5,6] and so on. In recent years, it has been revived in connection with the problem of oxide superconductors, which have very short coherence lengths comparable to a few times of the lattice constant.[7–11] Since the coherence length represents the size of a "mole-

cule," oxide superconductors can be regarded as being located around the crossover region.

Leggett proposed a general formalism for treating the crossover problem at zero temperature.[3] The key idea is that not only the gap $\Delta$ at $T = 0$ but also the chemical potential $\mu$ of fermions must be determined self-consistently, which is similar in its spirit to that proposed by Eagles.[12] This idea was extended to determine the transition temperature $T_c$ in the three-dimensional (3D) case by Nozières and Schmitt-Rink,[13] who reported that the transition temperature crosses over smoothly between the weak and strong attraction limits. Recently Schmitt-Rink, Varma, and Ruckenstein[7] have investigated two-dimensional (2D) pairing on the basis of the Nozières and Schmitt-Rink formalism and concluded that in a 2D system, $T_c$ is equal to zero and $\mu$ is given by half the energy of the di-electronic bound state even in the weak attraction limit.

It is somewhat surprising that even in the weak attraction limit the fixed point is the Bose condensed state of di-electronic molecules. In such a case, the molecules overlap each other so that the exchange effect among constituent electrons is expected to work to push up the chemical potential towards the Fermi energy of the system. While importance of such an effect has been recognized also by Schmitt-Rink *et al.*, it is still open whether

such an effect is beyond scope of the Nozières and Schmitt-Rink formalism.[13] Schmitt-Rink *et al.* have investigated only the case of dilute Fermi gas and the limited, though rather wide, range of parameters of the problem.[7] A purpose of this paper is to investigate the problem more extensively and to assess a range of applicability of the Nozières and Schmitt-Rink formalism.

Main results are summarized as follows: As temperature decreases, the chemical potential $\mu$ of the system shows two distinct behaviors, depending on the electron number density $N$ or the strength $V$ of the attraction. With fixed $V$, $\mu$ approaches half the binding energy of a di-electronic molecule as $T \rightarrow 0$ when $N$ is less than some critical value $N_{cr}$. The system is said to belong to class (a) in this case. This class has been found by Schmitt-Rink, Varma, and Ruckenstein. When $N > N_{cr}$, $\mu$ becomes large as the temperature approaches the transition temperature $T_c^{MF}$ given by the BCS-type mean-field theory, and tends to positive infinite as $T \rightarrow 0$ along the line of pairing instability. The system is said to belong to class (b) in this case. In particular, when $N \gg N_{cr}$, the system behaves as the Fermi liquid in the regime $T_c^{MF} \lesssim T \ll T_F$. In the case, where $N$ is fixed, the system belongs to class (a) when $V > V_{cr}$, and to class (b) when $V < V_{cr}$. Here $V_{cr}$ is determined by the condition that the critical electron number density $N_{cr}$, which is an increasing function of $V$, is equal to the electron number density $N$. As $N$ increases or $V$ decreases, the system changes its class from (a) to (b) at $N = N_{cr}$ or $V = V_{cr}$. These results indicate the Nozières and Schmitt-Rink's formalism is not sufficient for a low-dimensional system. However, judging by the results for 2D, the formalism seems to be applicable to a three-dimensional system. For a 2D system, it is necessary to take into account the effect of fluctuations.

The organization of this paper is as follows. In Sec. II, the Nozières and Schmitt-Rink formalism is reviewed. In Sec. III, a 2D version of the Norières and Schmitt-Rink formalism is investigated in detail. In Sec. III A, the so-called Thouless condition is investigated. In Sec. III B, the thermodynamic relation among $\mu$, $T$, and $N$, is investigated for two cases, $T \ll -\mu$ and $\mu > 0$, which reveal the existence of two classes in flow patterns of $\mu$-$T$ relations as decreasing temperatures. In Sec. III C, the relation between $N_{cr}$ and $V$ is discussed, and the behavior of the system as changing $N$ or $V$ is described. In Sec. IV, the relationship between the results obtained here and those by Schmitt-Rink, Varma, and Ruckenstein is discussed. The range of applicability of the Nozières and Schmitt-Rink formalism is discussed and how to include the effect beyond their formalism is briefly described.

## II. OUTLINE OF THE NOZIÈRES
## AND SCHMITT-RINK FORMALISM

Let us first recapitulate the Nozières and Schmitt-Rink formalism[13] which tries to describe how the crossover of the superconducting transition temperature occurs between the Cooper pairing and the Bose-Einstein condensation of di-electronic molecules as the strength of the $s$-wave attractive interactions is increased. Their theory is based on the assumption that the transition temperature is given by the so-called Thouless condition, which corresponds to divergence of the pair susceptibility calculated by the ladder-diagram approximation, with the chemical potential $\mu$ being determined self-consistently by taking the process of free propagation of the pair of two electrons into account, which corresponds to the ring-diagram approximation for the thermodynamic potential. This appears to be in some sense a natural extension of Leggett's theory, which treats the problem of the crossover of the ground-state properties, so as to include the degrees of freedom describing the center-of-mass motions of pairs of electrons.

The Thouless condition is expressed as

$$1 = \sum_k V_{kk} \frac{\tanh[(k^2/m - 2\mu)/4T]}{k^2/m - 2\mu} . \tag{1}$$

The pairing interaction $V_{kk'}$ is assumed to be separable and of $s$-wave type:

$$V_{kk'} = \frac{V}{[(1+k^2/k_0^2)(1+k'^2/k_0^2)]^{1/2}} . \tag{2}$$

The Thouless condition (1) is given by the condition for divergence of the pair susceptibility

$$\chi_{pair}(T) \equiv \sum_p \sum_{p'} \int_0^{1/T} d\tau \langle e^{\tau H} c_{p\uparrow} c_{-p\downarrow} e^{-\tau H} c_{-p'\downarrow}^\dagger c_{p'\uparrow}^\dagger \rangle \tag{3}$$

calculated in the ladder-diagram approximation as shown in Fig. 1.

The chemical potential $\mu$ in (1) is determined by the relation $N = -(\partial\Omega/\partial\mu)$ where the thermodynamic potential $\Omega$ is calculated in the ring-diagram approximation as shown in Fig. 2. Analytic expression for $\Omega$ is given as (see also Appendix A)

$$\Omega = \Omega_f(\mu, T) + T \sum_{\omega_m} \sum_q \ln[1 - V\chi_0(q, i\omega_m)] e^{i0^+ \omega_m} , \tag{4}$$

where $\Omega_f(\mu, T)$ is the thermodynamic potential for the free fermions with the chemical potential $\mu$ at the temperature $T$, and $\chi_0(q, i\omega_m)$ is defined as

$$\chi_0(q, i\omega_m) \equiv \sum_k \frac{1 - f(\epsilon_{q/2+k} - \mu) - f(\epsilon_{q/2-k} - \mu)}{(1+k^2/k_0^2)(\epsilon_{q/2+k} + \epsilon_{q/2-k} - i\omega_m - 2\mu)} \tag{5}$$

where $\epsilon_k = k^2/2m$ is the kinetic energy of an electron and $f(\epsilon) = (e^{\epsilon/T} + 1)^{-1}$ is the Fermi factor. Thus, the relation among $N$, $\mu$, and $T$ is obtained from (4) as follows:

$$N = N_f(\mu, T) - \frac{\partial}{\partial\mu}\left[T \sum_{\omega_m} \sum_q \ln[1 - V\chi_0(q, i\omega_m)]\right] , \tag{6}$$

where $N_f(\mu, T) \equiv -(\partial\Omega_f/\partial\mu)$ is the free-fermion part of the electron number. Relation (6) is reduced to a more compact form with use of the phase shift $\delta(q, \omega)$ for the particle-particle scattering channel as follows:

$$N = N_f(\mu, T) + \sum_q P \int_{-\infty}^{+\infty} \frac{d\omega}{\pi} g(\omega) \frac{\partial}{\partial\mu} \delta(q, \omega) , \tag{7}$$

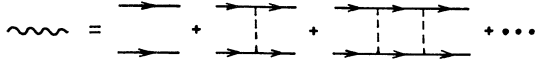FIG. 1. The ladder-diagram approximation for the pair susceptibility $\chi_{\text{pair}}$, (3), which is represented by the wavy line. The solid and dashed lines represent the one-body Green function of noninteracting electrons and the $s$-wave-type attractive interaction of Eq. (2), respectively.

where $g(\omega)=(e^{\omega/T}-1)^{-1}$ is the Bose function, and the phase shift $\delta(q,\omega)$ is defined as

$$\delta(q,\omega)=\tan^{-1}\frac{V\,\text{Im}\chi_0(q,\omega+i0^+)}{1-V\,\text{Re}\chi_0(q,\omega+i0^+)} \ . \tag{8}$$

It should be remarked that the principal part is taken for the $\omega$ integration in (7).

The transition temperature $T_c$ is given by the solution of simultaneous equations (1) and (7) as a function of $N$ and $V$. By analytic calculations Nozières and Schmitt-Rink have shown that, in the three-dimensional case, the chemical potential $\mu(T_c)$ is given in the two extreme limits as

$$\mu(T_c)\simeq\begin{cases}\epsilon_F & \text{for } VN/\epsilon_F\ll 1 \ , \\ -|E_0|/2 & \text{for } VN/\epsilon_F\gg 1 \ , \end{cases} \tag{9}$$

where $\epsilon_F(\equiv\pi N/m)$ is the Fermi energy for noninteracting electrons and $E_0$ is the binding energy of the di-electronic molecule, and that the transition temperature $T_c$ corresponds to the onset of the BCS-type mean-field pair condensation in the weak-coupling limit ($VN/\epsilon_F\ll 1$) and the onset of the Bose-Einstein condensation of di-electronic molecules in the strong-coupling limit ($VN/\epsilon_F\gg 1$), respectively. Then they interpolated numerically between both limits by a Padé-type treatment.[13] Schmitt-Rink, Varma, and Ruckenstein have investigated the two-dimensional case and emphasized that the chemical potential $\mu$ at $T=0$ is given by $-|E_0|/2$, i.e., one for the Bose-Einstein condensation of the two-dimensional ideal gas of di-electronic molecules, for *arbitrary* strength of attractive interaction $V$, and that the transition temperature always remains zero.[7]
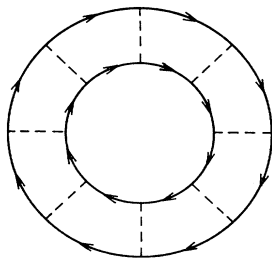


FIG. 2. The ring-diagram approximation for the thermodynamic potential $\Omega$. This represents the contribution from the free propagation of pair fluctuations given by Fig. 1.

## III. PAIRING IN TWO DIMENSIONS

In the two-dimensional case, the existence of a two-particle bound state is a necessary condition for many-body $s$-wave pairing.[4,8] Indeed the two-particle bound state of $s$-wave type always exists for the model pairing interaction (2) so long as $V>0$. This makes an analysis simpler than the three-dimensional case where the strength of attraction must exceed a threshold value in order to form the two-particle bound state.

### A. Thouless condition

Given a pairing potential $V_{kk'}$ as (2), condition (1) is written as

$$1-V\,\text{Re}\chi_0(0,i0^+)=0 \ . \tag{10}$$

Calculation of $\text{Re}\chi_0(q,\omega+i0^+)$ is performed in Appendix B. In the low-temperature regions $T\ll|\mu|$ important in the following discussions, $\text{Re}\chi_0(0,i0^+)$ is given by (B3) for $T\ll-\mu$ or by (B5) for $T\ll\mu$:

$$\text{Re}\chi_0(0,i0^+)\simeq\frac{m}{4\pi}\frac{1-\lambda e^{-|\mu|/T}}{-1+|\mu|/\epsilon_0}\ln\frac{|\mu|}{\epsilon_0}\quad\text{for } T\ll-\mu \tag{11}$$

or

$$\text{Re}\chi_0(0,i0^+)\simeq\frac{m}{4\pi}\frac{1}{1+\mu/\epsilon_0}\ln\frac{\mu(\mu+\epsilon_0)^2e^2}{4T^2\epsilon_0}\quad\text{for } T\ll\mu \ , \tag{12}$$

where $\epsilon_0\equiv k_0^2/2m$. Substituting these expressions into (10), one obtains the relation between $\mu$ and $T$ for varieties of values of the coupling constant $mV/4\pi$. Both in the weak- ($mV/4\pi\ll 1$) and strong- ($mV/4\pi\gg 1$) coupling limits, it is written as

$$T\simeq\frac{e(\mu+\epsilon_0)}{2}\sqrt{\mu/\epsilon_0}\exp\left[-\frac{2\pi(\mu+\epsilon_0)}{mV\epsilon_0}\right]\quad\text{for } T\ll\mu \tag{13}$$

and

$$\mu\simeq\frac{E_0}{2}\left[1-\frac{4\pi\lambda}{mV}\left[1-\frac{|E_0|}{2\epsilon_0}\right]e^{-|E_0|/2T}\right]\quad\text{for } T\ll|E_0| \ , \tag{14}$$

where $E_0$ is the binding energy of the di-electronic molecule. It should be remarked that in the two-dimensional case the bound-state solutions ($\mu<0$ and $T=0$) of (10) is always possible for the *arbitrary* strength $V$ of the attractive interaction (2). The binding energy $E_0$ is given as a bound-state solution of the Schrödinger equation

$$(2\epsilon_k-E_0)\varphi_k=\sum_{k'}V_{kk'}\varphi_{k'} \ . \tag{15}$$

With use of (C3) in Appendix C, $E_0$ is expressed in the weak-coupling limit ($mV/4\pi\ll 1$) as

$$E_0 \simeq -2\epsilon_0 e^{-4\pi/mV}\left[1 + \frac{4\pi}{mV}e^{-4\pi/mV}\right] , \tag{16}$$

and in the strong-coupling limit ($mV/4\pi \gg 1$) as

$$E_0 \simeq -2\epsilon_0 \frac{mV}{4\pi}\zeta(V) , \tag{17}$$

where $\zeta(V)$ satisfies the relation

$$\zeta = \ln\frac{emV\zeta}{4\pi} . \tag{18}$$

Qualitative behavior of the $\mu$ vs $T$ curve for the Thouless condition (10) is drawn in Fig. 3. For $\mu=0$, the condition (10) is written with use of (B1) as

$$1 - \frac{mV}{4\pi}\int_0^\infty d\epsilon\frac{\tanh\epsilon/2T}{(1+\epsilon/\epsilon_0)\epsilon} = 0 , \tag{19}$$

which gives the temperature $T_0$ crossing the $T$ axis ($\mu=0$) in the $\mu$-$T$ plane. In the weak-coupling limit, $T_0$ is given by

$$T_0 \simeq e\frac{\epsilon_0}{2}e^{-4\pi/mV} \tag{20}$$

and in the strong-coupling limit,

$$T_0 \simeq \frac{\epsilon_0}{2}\frac{mV}{4\pi}\zeta(V) . \tag{21}$$

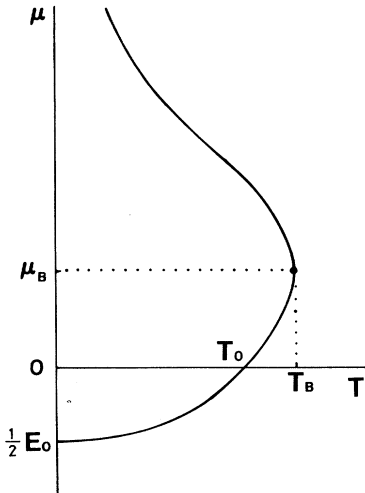The derivative $d\mu/dT$ on the curve of the Thouless condition, (10), is given as



FIG. 3. Behavior of the relation between the chemical potential $\mu$ and the temperature $T$, given by the Thouless condition (10). Its qualitative behavior in (a) the weak-coupling case ($mV/4\pi \ll 1$) and (b) the strong-coupling case ($mV/4\pi \gg 1$) is the same. The behavior at the low-temperature region is given by (13) and (14) with (17) in case (a) and with (18) in case (b). The $\mu$-$T$ curve crosses the $T$ axis at $T=T_0$ given by (19) with (20) in case (a) and with (21) in case (b), and takes right extreme at $\mu=\mu_B$ and $T=T_B$, which is a solution of Eqs. (24) and (25).

$$\frac{d\mu}{dT}\frac{\partial}{\partial\mu}\left[\frac{mV}{4\pi}\int_0^\infty d\epsilon\frac{\tanh(\epsilon-\mu)/2T}{(1+\epsilon/\epsilon_0)(\epsilon-\mu)}\right]$$
$$= \frac{mV}{4\pi}\frac{1}{T^2}\int_0^\infty d\epsilon\frac{\cosh^{-2}(\epsilon-\mu)/2T}{(1+\epsilon/\epsilon_0)} > 0 . \tag{22}$$

Here the coefficient of $d\mu/dT$ is $\partial[V\,\mathrm{Re}\chi_0(0,i0^+)]/\partial\mu$. At $T=T_0$ on the $T$ axis ($\mu=0$), this is reduced to

$$\frac{\partial}{\partial\mu}[V\,\mathrm{Re}\chi_0(0,i0^+)]$$
$$= \frac{mV}{4\pi\epsilon_0}\left[-\int_0^\infty d\epsilon\frac{\tanh(\epsilon/2T_0)}{(1+\epsilon/\epsilon_0)^2\epsilon} + \frac{\epsilon_0}{2T_0}\right] , \tag{23}$$

which is positive for arbitrary value of $\epsilon_0/T_0 > 0$. Then, the derivative $d\mu/dT$ at $T=T_0$ and $\mu=0$ is always positive. The right extreme point of the Thouless condition curve, shown as a solid circle in Fig. 3, corresponds to the condition $\partial\,\mathrm{Re}\chi_0(0,i0^+)/\partial\mu=0$, where we denote $\mu=\mu_B$ and $T=T_B$. The explicit forms of the two conditions, $1-V\,\mathrm{Re}\chi_0(0,0;\mu,T)=0$ and $\partial\,\mathrm{Re}\chi_0(0,0;\mu,T)/\partial\mu=0$, which are satisfied by $\mu_B$ and $T_B$ are

$$1 - \frac{mV}{4\pi}\int_0^\infty d\epsilon\frac{\tanh[(\epsilon-\mu)/2T]}{(1+\epsilon/\epsilon_0)(\epsilon-\mu)} = 0 \tag{24}$$

and

$$\frac{\tanh(\mu/2T)}{\mu} - \frac{1}{\epsilon_0}\int_0^\infty d\epsilon\frac{\tanh[(\epsilon-\mu)/2T]}{(1+\epsilon/\epsilon_0)^2(\epsilon-\mu)} = 0 , \tag{25}$$

respectively. It is remarked that always $\mu_B > 0$ because $d\mu/dT$ is positive at $\mu=0$ and $T=T_0$ and because the Thouless condition curve finally approaches the $\mu$ axis as $\mu$ increases following (13).

## B. Chemical potential $\mu$ as a function of $T$ and $N$

Another ingredient of the Nozières and Schmitt-Rink formalism is the relation between $\mu$ and $T$ obtained from (7), which has a quite different structure for two cases: (1) $T \ll -\mu$, and (2) $\mu > 0$. In any case, the structure of the phase shift $\delta(q,\omega)$, (8), determines its relation.

Case (1): $T \ll -\mu$. The real and imaginary parts of $\chi_0(q,\omega+i0^+)$ are given by (B3) and (B4), respectively. Their characteristics are the logarithmic divergence in $\mathrm{Re}\chi_0$ around $\omega \sim 2|\mu|+q^2/4m$ and vanishing $\mathrm{Im}\chi_0$ for $\omega < 2|\mu|+q^2/4m$. Schematic behaviors of $1-V\,\mathrm{Re}\chi_0$ and $\mathrm{Im}\chi_0$ are shown in Fig. 4(a), from which one can obtain the behavior of the phase shift $\delta(q,\omega)$, (8), as shown in Fig. 4(b). It is noted that the discontinuity (by $\pi$) in $\delta(q,\omega)$ at $\omega=\eta_q$ implies the existence of a di-electronic molecule with momentum $q$, whose energy $\eta_q$ is determined by the relation $1-V\,\mathrm{Re}\chi_0(q,\eta_q+i0^+)=0$.

In a way quite similar to obtaining (14), $\eta_q$ is shown to satisfy the following relation:

$$\mu + \frac{\eta_q}{2} - \frac{q^2}{8m} \simeq \frac{E_0}{2}\left[1 - \frac{4\pi\lambda}{mV}\left[1 - \frac{|E_0|}{2\epsilon_0}\right]e^{-|E_0|/2T}\right] , \tag{26}$$
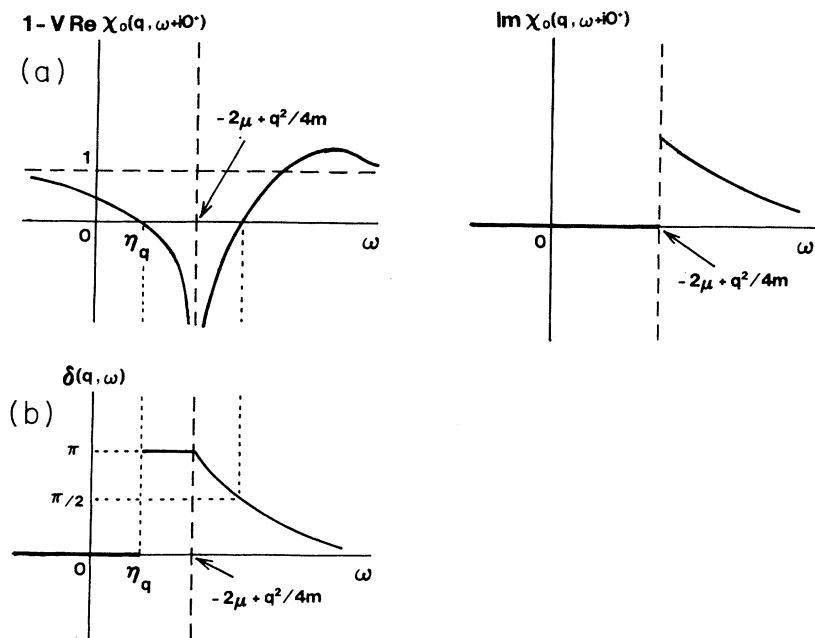
FIG. 4. (a) Schematic behaviors of $1 - V \operatorname{Re}\chi_0(q, \omega + i0^+)$ and $\operatorname{Im}\chi_0(q, \omega + i0^+)$ as a function of $\omega$ for the case $T \ll -\mu$. (b) Schematic behavior of the phase shift $\delta(q, \omega)$, (8), for the case $T \ll -\mu$. A discontinuity of $\delta(q, \omega)$ at $\omega = \eta_q$ implies existence of the di-electronic molecule with the momentum $q$ and the energy $\eta_q$ which is given by the relation $1 - V \operatorname{Re}\chi_0(q, \eta_q + i0^+) = 0$. The chemical potential $\mu$ is determined *a posteriori* so as to satisfy the condition $\eta_q > 0$.

where $E_0$ is given by (16) or (17). Therefore, neglecting the $q$ dependence of $O(e^{-|\mu|/T})$, $\eta_q$ is written as

$$\eta_q = \eta_0 + \frac{q^2}{4m} \, , \tag{27}$$

where $\mu$ and $\eta_0$ satisfy the relation

$$\mu + \frac{\eta_0}{2} \simeq \frac{E_0}{2}\left[1 - \frac{4\pi\lambda}{mV}\left[1 - \frac{|E_0|}{2\epsilon_0}\right]e^{-|E_0|/2T}\right] \, . \tag{28}$$

As discussed by Nozières and Schmitt-Rink, of the contributions to (7), the part arising from the jump in $\delta(q, \omega)$ dominates in the low-temperature region $T \ll -\mu$. Indeed, $N_f(\mu, T)$ and the contributions from the scattering state of $\omega > 2|\mu| + q^2/4m$ are, at most, of $O(e^{-|\mu|/T})$. Around $\omega \sim \eta_q$, the phase shift $\delta(q, \omega)$ is approximated as

$$\delta(q, \omega) \simeq \pi\theta(\omega - \eta_q) \, . \tag{29}$$

Then $\partial\delta(q, \omega)/\partial\mu$ is given as follows:

$$\frac{\partial\delta(q, \omega)}{\partial\mu} = 2\pi\delta(\omega - \eta_q) \, , \tag{30}$$

where the derivative $\partial\eta_q/\partial\mu$ has been substituted by $-2$ with use of (27) and (28), and due to the fact that $E_0$ is in-

dependent of $\mu$.

Substituting (30), relation (7) is expressed in the form of the state equation of the ideal Bose gas:

$$N \simeq 2\sum_q g\left[\eta_0 + \frac{q^2}{4m}\right] + O(e^{-|\mu|/T}) \, . \tag{31}$$

This determines $\eta_0$ as a function of $T$ and $N$ as usual and, in turn, $\mu$ via relation (28):

$$\mu \simeq -\frac{T}{2}\exp\left[-\frac{\pi N}{2mT}\right] \\ + \frac{E_0}{2}\left[1 - \frac{4\pi\lambda}{mV}\left[1 - \frac{|E_0|}{2\epsilon_0}\right]e^{-|E_0|/2T}\right] \, . \tag{32}$$

It is noted that $\eta_0$ should remain positive and approaches zero as $T \to 0$ corresponding to the onset of the Bose-Einstein condensation of di-electronic molecules at $T = 0$.

Case (2): $\mu > 0$. Let us denote the solution of the Thouless condition, (10), as $T = T_c^{\mathrm{MF}}(\mu)$, whose explicit form is given by (13) or (14). On the high-temperature side of the Thouless condition, relation (7) is valid and the $\mu$ derivative of the phase shift is written as follows:

$$\frac{\partial}{\partial\mu}\delta(q, \omega) = \frac{[1 - V\operatorname{Re}\chi_0(q, \omega)](\partial/\partial\mu)[V\operatorname{Im}\chi_0(q, \omega)] + V\operatorname{Im}\chi_0(q, \omega)(\partial/\partial\mu)[V\operatorname{Re}\chi_0(q, \omega)]}{[1 - V\operatorname{Re}\chi_0(q, \omega)]^2 + [V\operatorname{Im}\chi_0(q, \omega)]^2} \, , \tag{33}$$

where (8) has been used. A problem is investigating properties of (33) which determine the $\mu$-$T$ relation through (7). An analysis is simpler in the weak-coupling case $(mV/4\pi \ll 1)$ than the strong-coupling one $(mV/4\pi \gg 1)$. So let us investigate the former case first and argue the latter case later on.

In the weak-coupling case, if there were no singularity in (33), the last term of (7) would be small of the order of $N_f(\mu, T) \times O(mV/4\pi)$ so that the chemical potential would be given essentially by that of noninteracting electrons. However, when $T \to T_c^{\mathrm{MF}}(\mu)$, there must arise the logarithmic divergence in the last term of (7). Such singularity is ex-

pected to appear as an infrared divergence associated with virtual formation of Cooper pairs. This is indeed the case as shown shortly. With use of (B6) for the imaginary part of $\chi_0(0,\omega+i0^+)$, (33) for $q=0$, and $\omega\sim0$ is estimated as

$$\frac{\partial}{\partial\mu}\delta(0,\omega) \simeq \frac{mV}{4\epsilon_0}\tanh\frac{\omega}{4T}\frac{-[1-V\operatorname{Re}\chi_0(0,\omega)]+[1+(\mu+\omega/2)/\epsilon_0]V(\partial/\partial\mu)\operatorname{Re}\chi_0(0,\omega)}{[1+(\mu+\omega/2)/\epsilon_0]^2[1-V\operatorname{Re}\chi_0(0,\omega)]^2+[(mV/4)\tanh(\omega/4T)]^2} , \tag{34}$$

where $\theta(\omega+2\mu)$ in (B6) is replaced by 1. Since $1-V\operatorname{Re}\chi_0(0,0)\to0$ as $T\to T_c^{\mathrm{MF}}(\mu)+0^+$,

$$\lim_{\omega\to0}\lim_{T\to T_c^{\mathrm{MF}}}g(\omega)\frac{\partial}{\partial\mu}\delta(0,\omega)$$

$$\simeq\frac{mV}{16\epsilon_0}\frac{[1+(\mu+\omega/2)/\epsilon_0]V(\partial/\partial\mu)\operatorname{Re}\chi_0(0,0)}{[(mV/4\pi)(\omega/2\mu)]^2+[(mV/4)(\omega/4T_c^{\mathrm{MF}})]^2} , \tag{35}$$

where we have used (B5) for $\operatorname{Re}\chi_0(0,\omega+i0^+)$. Therefore, owing to the $1/\omega^2$ dependence of $g(\omega)\partial\delta(0,\omega)/\partial\mu$ around $\omega\sim0$, the $\omega$ integration with $q=0$ (7) diverges as $T\to T_c^{\mathrm{MF}}(\mu)+0^+$. For finite but small $q$, the $\omega$ integration results in

$$\lim_{T\to T_c^{\mathrm{MF}}}\int\frac{d\omega}{\pi}g(\omega)\frac{\partial}{\partial\mu}\delta(q,\omega)$$

$$\simeq C\frac{(T_c^{\mathrm{MF}})^2}{\epsilon_0}\frac{1}{q^2}\frac{\partial}{\partial\mu}\operatorname{Re}\chi_0(0,0) , \tag{36}$$

where $C$ is a positive constant of $O(1)$. This is seen from the inspection of the structure of (B1) and (B2) where $\omega$ appears pairwise with $q^2/4m$ in its denominator and an argument of the $\delta$ function. Then, for $\omega\sim0$, $q\sim0$, and $T-T_c^{\mathrm{MF}}\gtrsim0$, $1-V\operatorname{Re}\chi_0(q,\omega+i0^+)$ in the denominator of (33) is proportional to $\max\{q^2/4m,|\omega|,(T-T_c^{\mathrm{MF}})\}$ so that the last term of (7) shows logarithmic divergence like $\partial\operatorname{Re}\chi_0(0,0)/\partial\mu\ln[\mu/(T-T_c^{\mathrm{MF}})]$. It should be remarked that the sign of its divergence changes at the right extreme point on the Thouless condition curve, at the solid circle in Fig. 3, where $\partial\operatorname{Re}\chi_0(0,0)/\partial\mu=0$.

Thus, *if* there is no other singular contribution to the $q$ and $\omega$ integrations in the last term of (7), relation (7) is reduced to

$$N=N_f(\mu,T)\left[1+C_0\frac{mV}{4\pi}+C_1\frac{(T_c^{\mathrm{MF}})^2}{m\epsilon_0}\frac{\partial}{\partial\mu}\operatorname{Re}\chi_0(0,0)\right.$$

$$\left.\times\ln\frac{\mu}{T-T_c^{\mathrm{MF}}}\right] , \tag{37}$$

where $C_0$ and $C_1$ are positive constants of $O(1)$. Then, in the weak-coupling case, the $\mu$-$T$ curves for different electron number behave as Fig. 5. The right extreme point on the Thouless condition curve, which we denote $B$, is a branch point which divides the $\mu$-$T$ curves for a variety of electron numbers into two classes (a) and (b). It is remarked that on point $B$ there is no logarithmic divergence because $\partial\operatorname{Re}\chi_0(0,0)/\partial\mu=0$ there. The critical electron number $N_{\mathrm{cr}}$, for which the flow line approaches the branch point $B$ as temperatures decrease, is given by

the condition

$$N_{\mathrm{cr}}=N_f(\mu_B,T_B)$$

$$+\sum_q P\int_{-\infty}^{+\infty}\frac{d\omega}{\pi}g(\omega)\frac{\partial}{\partial\mu}\delta(q,\omega;\mu_B,T_B) , \tag{38}$$

where $\mu_B$ and $T_B$ are the solution of simultaneous equations (24) and (25). For class (a), where $N<N_{\mathrm{cr}}$ and $\partial\operatorname{Re}\chi_0(0,0)/\partial\mu>0$, the flow line approaches the point $\mu=-\frac{1}{2}|E_0|$ and $T=0$ along the Thouless condition curve. For class (b), where $N>N_{\mathrm{cr}}$ and $\partial\operatorname{Re}\chi_0(0,0)/\partial\mu<0$, it approaches $\mu=\infty$ and $T=0$ again along the Thouless condition line. In the high-density limit, $N\gg N_{\mathrm{cr}}$, the flow line first approaches from the Boltzmann gas regime to a region where the Fermi-liquid description is valid. In this region, the second term of Eq. (7) can be neglected compared to the first term except
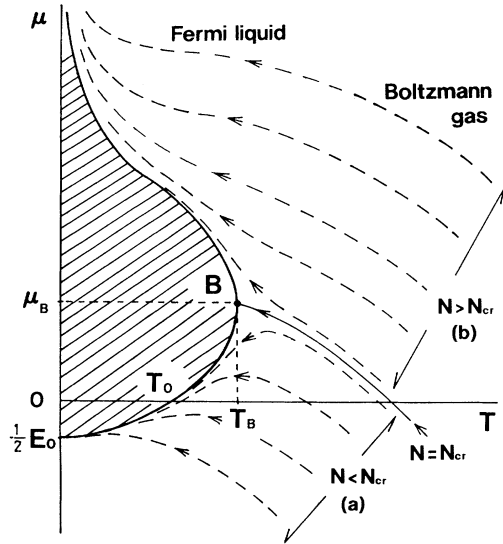


FIG. 5. Chemical potential $\mu$ as a function of temperature $T$ for different electron number $N$. They are schematic curves of solutions of Eqs. (32) and (37) for the weak-coupling case. The shaded part surrounded by the line satisfying the Thouless condition (24) cannot be reached as decreasing temperature. Patterns of the $\mu$-$T$ curve (dashed line) are divided into two classes (a) and (b) by the critical line approaching the branch point $B$ which is given by a solution of Eqs. (24) and (25). Its critical line corresponds to a critical electron number $N_{\mathrm{cr}}$ determined by (38). In class (a), where $N<N_{\mathrm{cr}}$, "flow lines" approach a point of Bose condensation of di-electronic molecules. In class (b), where $N>N_{\mathrm{cr}}$, they approach $\mu=\infty$ and $T=0$ along the Thouless condition line. In case $N\gg N_{\mathrm{cr}}$, there exists a Fermi-liquid region.

for case $T$ is extremely (exponentially) close to $T_c^{MF}$ because it depends logarithmically on $T - T_c^{MF}$. Such a behavior of the flow line is physically expected because the binding energy of the two-electron bound state (which always exists for a 2D attractively interacting system) measured from the bottom of the continuum of the scattering is much smaller than the mean energy of the scattering state which is comparable to $\epsilon_F$. As a result, the scattering state is expected to dominate the bound state here. The flow line deviates from the Fermi liquid around the temperature $T \gtrsim T_c^{MF}$, and finally goes up along the Thouless condition curve. While the former branch ($N < N_{cr}$) has been found by Schmitt-Rink, Varma, and Ruckenstein[7] on the basis of numerical study of the relation (7), the latter one ($N \geq N_{cr}$) has not been recognized yet. It is shown in Appendix D that there is no divergent contributions to the $q$ and $\omega$ integrations of (7) other than the last term in the square bracket of (37).

Also in the strong-coupling case, analysis about the divergent contribution to $q$ and $\omega$ integrations of (7) can be performed in a way quite similar to that leading to Eqs. (34)–(36). Other contributions to $q$ and $\omega$ integrations of (7) are estimated in Appendix D for the region $\mu \gg T$. Then, with use of (D16), relation (7) is reduced to

$$N = N_f(\mu, T) - \frac{m\epsilon_0}{2\pi} \frac{mV}{4\pi} \zeta(V)$$
$$+ C_2 \frac{(T_c^{MF})^2}{m\epsilon_0} \frac{\partial}{\partial\mu} \mathrm{Re}\chi_0(0,0) \ln \frac{\mu}{T - T_c^{MF}} N_f(\mu, T) ,$$
$$\tag{39}$$

where $\zeta(V)$ is a solution of (18) and $C_2$ is a positive constant of $O(1)$. Analysis of the region around $\mu \sim T$ is difficult to perform analytically. However, qualitative behavior of the $\mu$-$T$ curve with fixed $N$ is expected to be the same as Fig. 5 because existence of the branch point $B$ is only due to that of the divergent contribution like $\partial \mathrm{Re}\chi_0(0,0)/\partial\mu \ln[\mu/T - T_c^{MF})]$ both in (37) and (39).

### C. Equations of state on the $\mu$-$T$ plane

As shown in Sec. III B, the behavior of the system at low temperature depends on its electron number density $N$: the system belongs to class (a) when $N < N_{cr}$, and it belongs to class (b) when $N > N_{cr}$. It is crucial to see the $V$ dependence of $N_{cr}$. Since $N_{cr}$ is defined by (38) with $\mu_B$ and $T_B$, we first examine the $V$ dependence of $\mu_B$ and $T_B$ which is given by (24) and (25). They are rewritten as

$$\frac{4\pi}{mV} = z \int_0^\infty dx \frac{\tanh[y(zx - 1)]}{(1 + x)(zx - 1)} \tag{40}$$

and

$$\tanh y = \int_0^\infty dx \frac{\tanh[y(zx - 1)]}{(1 + x)^2(zx - 1)} , \tag{41}$$

respectively. Here we put $y \equiv \mu_B/2T_B$ and $z \equiv \epsilon_0/\mu_B$. Both $y$ and $z$ are positive since $\mu_B > 0$ as discussed in Sec. III A. The main contribution to the integration in (40) comes from the regions $x \sim 1/z$ and $x \lesssim 1$. The factor $\tanh[y(zx - 1)]/(zx - 1)$ has a large peak around $x \sim 1/z$

when $y \gg 1$. So the right-hand side of (40) is large when $0 < 1/z \ll 1$ and $y \gg 1$, and is small when $1 \ll 1/z$ and $y \ll 1$.

First we discuss the weak-coupling case ($mV/4\pi \ll 1$). Since the left-hand side of (40) is large in the weak-coupling limit, $y \gg 1$ and $z \gg 1$ in order to keep relation (40). To obtain the relation between $V$, $y$, and $z$, $\tanh[y(zx - 1)]$ is approximated as $-1$ for $y(zx - 1) < -1$, as $y(zx - 1)$ for $|y(zx - 1)| \leq 1$, and as 1 for $y(zx - 1) > 1$. It is noted that such an approximation gives upper bounds of the integrations in (40) and (41). With this approximation, (40) and (41) are evaluated as

$$\frac{4\pi}{mV} = \frac{z}{1 + z} \ln|[1 + y(1 + z)][1 - y(1 + z)]|$$
$$+ yz \ln \left| \frac{1 + y(1 + z)}{1 - y(1 + z)} \right|$$
$$\simeq \ln(y^2 z) \tag{42}$$

and

$$\tanh y = \frac{1}{1 + z} + \frac{z}{(1 + z)^2} \ln|[1 + y(1 + z)][1 - y(1 + z)]/z|$$
$$\simeq \frac{1}{z} \ln(y^2 z)$$
$$\simeq \frac{4\pi}{mV} \frac{1}{z} , \tag{43}$$

respectively. Here we have used the fact $y \gg 1$ and $z \gg 1$ to derive the second equations in (42) and (43), and (42) to derive the last equation in (43). Equation (42) is rewritten as

$$\frac{\mu_B}{2T_B} \simeq \frac{2T_B}{\epsilon_0} e^{4\pi/mV} . \tag{44}$$

Since $y \gg 1$ ($\mu_B \gg 2T_B$), $\mu_B = \epsilon_0/z$ is evaluated as $\mu_B \simeq (mV/4\pi)\epsilon_0$ from (43). In the definition of $N_{cr}$ (38), the integral part is of the order of $N_f(\mu_B, T_B) \times O(mV/4\pi)$ in the weak-coupling limit, as in (37). Therefore, in the weak-coupling limit,

$$N_{cr} \simeq N_f(\mu_B, T_B) \simeq \frac{m}{\pi} \mu_B \simeq \frac{m^2 \epsilon_0}{4\pi^2} V . \tag{45}$$

Next we discuss the strong-coupling limit ($mV/4\pi \gg 1$). In this case, $y$ and $z$ must be much smaller than unity in order to keep relation (40). With the same approximation for $\tanh[y(zx - 1)]$ as used in the weak-coupling case, (40) and (41) are evaluated as

$$\frac{4\pi}{mV} = yz \ln\{[1 + y(1 + z)]/yz\} + \frac{z}{1 + z} \ln[1 + y(1 + z)]$$
$$\simeq yz[1 - \ln(yz)] \tag{46}$$

and

$$\tanh y = \frac{y}{1 + y} + \frac{z}{(1 + z)^2} \ln[1 + y(1 + z)]$$
$$\simeq y - \frac{1}{2}y^2 z + \frac{1}{3}y^3 z(1 + z) \cdots , \tag{47}$$

respectively. Second relations in (46) and (47) have been derived on the fact $y \ll 1$ and $z \ll 1$. As mentioned

above, the approximation we used gives upper bounds of the integrations. The term $-y^2z/2$ in (47) indicates that the right-hand side of (41) is smaller than $\tanh y$ if $0 < y \ll 1$. For $y \gg 1$, the right-hand side of (41) diverges as $\ln y$. So there exists a solution $y$ for (41). From numerical calculation of (41), $y$ is approximately given as $y \simeq 2.4z + 0.25$ in the range $0.02 < z < 0.1$, which is somewhat larger than the solution of (47), $y \sim 3z/2 \ll 1$, and crosses over to $y \sim 3z/2$ around $z \sim 0.02$. Though the right-hand side of (46) is the upper bound of the integration in (40), it is confirmed with numerical calculation that (46) gives an approximate value of the integration in (40) with enough accuracy for $0.1 \lesssim y \lesssim 1.0$ and $0.001 \lesssim z \lesssim 0.1$. Since $\epsilon_0 \ll \mu_B \lesssim 2T_B$ (or $z \ll 1$ and $y \lesssim 1$), $T_B$ is expressed with use of (46) as $T_B = m\epsilon_0 V\zeta(V)/8\pi$, where $\zeta(V)$ is determined by (18). In the 2D system, the first term $N_f(\mu_B, T_B)$ in (38) is evaluated as

$$N_f = \frac{m}{\pi}[\mu_B + T_B\ln(1 + e^{-\mu_B/T_B})] .$$

To see the order of magnitude of the second term in (38), we examine (33), (B1), and (B2). Both (B1) and (B2) contain a factor $1 - f(\epsilon_{q/2+k} - \mu) - f(\epsilon_{q/2-k} - \mu)$, which is equal to $(\epsilon_{q/2+k} + \epsilon_{q/2-k} - 2\mu)/4T$ at high temperatures. Therefore, $\mathrm{Re}\chi_0(q,\omega)$ and $\mathrm{Im}\chi_0(q,\omega)$ are of the order of $(m\epsilon_0/8\pi T_B)\ln(T_B/\epsilon_0)$ and $\omega m\epsilon_0/2\pi\mu_B T_B$, respectively. Since $T_B$ is of the order of $V$ within logarithmic accuracy as shown above, both $V\mathrm{Re}\chi_0(q,\omega)$ and $V\mathrm{Im}\chi_0(q,\omega)$ are $O(mV/4\pi)^0$. Substituting these relations into (33), $\partial\delta/\partial\mu$ is $O(mV/4\pi)^0$, and so is the integral part. Hence the first term in (38) is dominant compared with the second one:

$$N_{\mathrm{cr}} \simeq \frac{mT_B}{\pi}\left[\frac{\mu_B}{T_B} + \ln(1 + e^{-\mu_B/T_B})\right]$$

$$\sim \frac{m^2\epsilon_0}{8\pi^2}V\zeta(V) . \tag{48}$$

$N_{\mathrm{cr}}$ increases in proportion to $V$ within logarithmic accuracy both in the strong- and weak-coupling limits as shown in (45) and (48). Therefore, we can expect that $N_{\mathrm{cr}}$ is a smoothly increasing function of $V$ over the whole range of $V$ connecting the two limits. Schematic behavior of $N_{\mathrm{cr}}$ as a function of $V$ can be drawn as in Fig. 6.
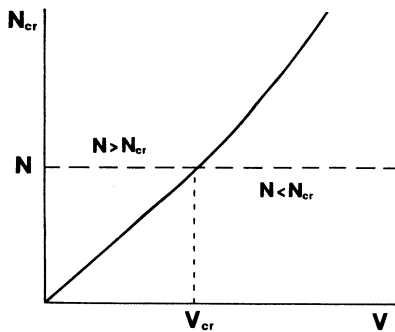


FIG. 6. Critical electron number $N_{\mathrm{cr}}$ as a function of strength $V$ of attractive interaction.
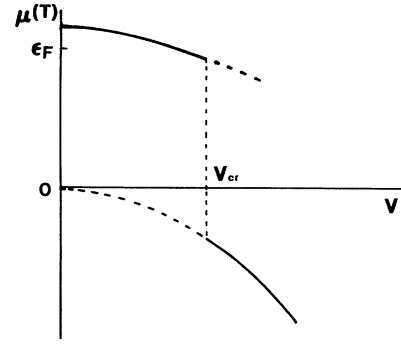


FIG. 7. Chemical potential $\mu(T)$ at temperature $T$, low enough compared to the Fermi temperature $T_F$, as a function of strength $V$ of attractive interaction.

From these observations, we can describe the behavior of $\mu$ as a function of the strength of the attraction $V$. In the case where the electron number density $N$ is fixed, the critical value $V_{\mathrm{cr}}$ of the attraction is determined by the condition that $N = N_{\mathrm{cr}}(V)$. When $V < V_{\mathrm{cr}}$, the system belongs to class (b) because $N_{\mathrm{cr}}(V) < N$; $\mu$ tends to positive infinite as $T \to 0$. Beyond $V_{\mathrm{cr}}$, the system belongs to class (a); $\mu$ approaches half the binding energy of a di-electronic molecule as $T \to 0$. As $V$ decreases, the system changes its class from (a) to (b) at $V = V_{\mathrm{cr}}$, where $\mu$ at $T < T_B(V_{\mathrm{cr}})$ changes its value discontinuously. The chemical potential $\mu$ as a function of $V$ at finite temperature but much lower than $T_F$ is shown in Fig. 7.

The transition temperature $T_c$ and the chemical potential at $T = T_c$ are given by the coincident point of the curves of conditions (1) and (7). When the system belongs to class (a), they coincide at the point $\mu = E_0/2 < 0$ and $T = 0$, so that the transition temperature is zero in this case. When the system belongs to class (b), the $\mu$-$T$ curve of condition (7) goes up along the Thouless condition curve as the temperature decreases. Therefore, the two conditions cannot be satisfied simultaneously even at $T = 0$, indicating that there is no phase transition even at $T = 0$ in this case.

## IV. DISCUSSIONS

As seen in Sec. III, there exists a branch point $B$ on the Thouless conditions curve (Fig. 5), so that the curves representing (37) or (39) are divided into two classes (a) and (b). To which class the system belongs depends on the electron number density $N$ or the strength $V$ of the attraction. When the system belongs to class (a), i.e., $N < N_{\mathrm{cr}}$ or $V > V_{\mathrm{cr}}$, the Bose-Einstein condensation of bound pairs is realized at $T = 0$. When the system belongs to class (b), i.e., $N > N_{\mathrm{cr}}$ or $V < V_{\mathrm{cr}}$, $\mu$ becomes positive infinite as $T \to 0$. When $N \gg N_{\mathrm{cr}}$ or $V \ll V_{\mathrm{cr}}$, the system behaves as the Fermi liquid in the region $T_c^{\mathrm{MF}} \lesssim T \ll T_F$. As increasing $N$ or decreasing $V$, the system changes its class from (a) to (b) at $N = N_{\mathrm{cr}}$ or $V = V_{\mathrm{cr}}$.

Schmitt-Rink, Varma, and Ruckenstein[7] have used the Nozières and Schmitt-Rink formalism on the assumption that the range of the interaction, $l_0(\sim 1/\sqrt{2m\epsilon_0})$, is

much shorter than both the size of a molecule $\xi(\sim 1/\sqrt{2m|E_0|})$ and the interparticle distance $r_0(=1/\sqrt{N})$. Such an assumption makes it possible to eliminate the interaction $V$ from the formalism in favor of corresponding binding energy. They have performed numerical calculations with electron number density up to $N=2m|E_0|$, where $r_0 \sim \xi$, i.e., up to the crossover region.[7] In the weak-coupling limit $(mV/4\pi \ll 1)$, $|E_0| \simeq 2\epsilon_0 \exp(-4\pi/mV)$, (16), and $N_{cr} \sim m^2 \epsilon_0 V/4\pi^2$, (45). When $N=2m|E_0|$,

$$N/N_{cr} \sim 8\pi^2 |E_0|/m\epsilon_0 V \sim 4\pi(4\pi/mV)\exp(-4\pi/mV) ,$$

which is much smaller than unity. When $N/N_{cr} \sim 1$, $l_0^2/r_0^2 \sim mV/8\pi^2 \ll 1$ and

$$N \sim (1/2\pi)(mV/4\pi)\exp(4\pi/mV)m|E_0| \gg 2m|E_0| ,$$

so that $\xi \gg r_0 \gg l_0$. Hence a change of the class from (a) to (b) should have been observed even on the assumption $l_0 \ll \xi, r_0$, if higher density cases were investigated. In the intermediate case $(mV/4\pi \sim 1)$, $|E_0| \simeq 2\epsilon_0$ from (C3). Hence $l_0 \sim \xi$ and the assumption $l_0 \ll \xi, r_0$ is not applicable. In the strong-coupling limit $(mV/4\pi \gg 1)$, $|E_0| \simeq 2\epsilon_0(mV/4\pi)\zeta(V)$. Hence $l_0^2 \gg \xi^2(mV/4\pi)\zeta(V) \gg \xi^2$, i.e., the assumption is not applicable in this case either. Therefore, the region Schmitt-Rink et al. have investigated is the low-density limit for the weak-coupling case. So, their results cannot be applied to the intermediate or more strong-coupling case.

Within the framework of the Nozières and Schmitt-Rink formalism,[13] when applied to the 2D problem, the system changes its class from one to the other as $N$ or $V$ changes. However, $\mu$ must be a finite and continuous function of the number density. The reason $\mu$ tends to infinite or half the binding energy is existence of the logarithmic divergence in (37) or (39) near the Thouless condition. This divergence reflects the two-dimensionality of the system. To avoid the unphysical divergence it seems necessary to include the effect of fluctuations around the mean-field treatment. Preliminary results along such direction have already been reported.[9,10] The effect of fluctuation is to press the Thouless condition curve against the $\mu$ axis in the case $\mu \gg T$. In the weak-coupling case, the chemical potential $\mu$ is expected to reach the Fermi energy corresponding to its number density at $T=0$. This is indeed the case at least in the high-density limit. Detailed calculation will be reported elsewhere. At present, however, it is still open whether the system crosses over between two classes continuously within the method we employed. A possibility of phase separation of the gas-liquid should also be explored.

Nozières and Schmitt-Rink's work was for a 3D system.[13] In such a case there is no divergence: the line of conditions (7) reaches that of the Thouless condition safely, so that there is no branch point separating the solutions into two classes. Therefore, the Nozières and Schmitt-Rink formalism is considered to be applicable to 3D systems, but it cannot be applied in its original form to 2D systems where fluctuation effect is dominant. Difficulty of the results by Schmitt-Rink et al., and of the Nozières and Schmitt-Rink formalism, has been pointed

out by Serene[14] in a somewhat different point of view from ours.

The crossover problem in Bose condensation has intimate connection with that in magnetism, i.e., the crossover between the itinerant-electron magnetism and localized spin magnetism.[15] As is well known, the negative-$U$ Hubbard model can be mapped onto the positive-$U$ Hubbard model with magnetic field by means of the canonical transformation.[16–18] The superconducting order in the former corresponds to the transverse magnetic order in the latter. The weak and strong attraction limits in the former correspond to the itinerant and the localized spin limits in the latter, respectively. A gas-liquid transition in the former, which must be always paid attention to when the strength of the attraction increases, can be observed as a metamagnetic transition in the latter. These relations show that making clear how the system crosses over in the case of superconductivity is equivalent to making clear the crossover problem in magnetism, and vice versa.

In conclusion we have discussed the crossover between the Cooper-pair and the Bose-Einstein condensation in a 2D system, on the basis of the Nozières and Schmitt-Rink formalism. The systems are separated into two classes, (a) and (b), depending on their electron number density $N$ or the strength of the attraction $V$. When the system belongs to class (a) where $N < N_{cr}$, Bose-Einstein condensation of di-electronic molecules is set in at zero temperature. When the system belongs to class (b) where $N > N_{cr}$, the chemical potential $\mu$ tends to positive infinite as temperature decreases along the line of pairing instability. When $N \gg N_{cr}$ in particular, $\mu$ is of $O(\epsilon_F)$ in the temperature region $T_c^{MF} \lesssim T \ll T_F$. As $N$ or $V$ is changed, the system changes its class abruptly. Then a discontinuous change of the chemical potential $\mu$ occurs below $T_B$. Such an abrupt change of the system indicates that the Nozières and Schmitt-Rink formalism cannot be applied to a system with strong fluctuations. It seems necessary to include the effect of fluctuations in order to construct a smooth crossover formalism for low-dimensional systems.

## APPENDIX A

Instead of (4), some use a different form for the thermodynamic potential:[14,19]

$$\Omega = \Omega_f(\mu,T) + T \sum_{\omega_m} \sum_{q} \{\ln[1-V\chi_0(q,i\omega_m)]$$

$$+ V\chi_0(q,i\omega_m)\} . \tag{A1}$$

This form does not contain the first-order perturbation term. Two reasons can be imagined for the use of this form: to avoid the discontinuity of the thermodynamic

potential originating from the discontinuity of Green's function at $t = 0$; disappearance of the electron-phonon interaction with carrying no momentum. The discontinuity does not appear if the logarithmic term in (4) is calculated *without* expanding it in powers of $V\chi_0(\mathbf{q}, i\omega_m)$, as Nozières and Schmitt-Rink showed, and we also showed it in the text. It can be avoided, as shown below, even when the logarithmic term is expanded if the convention for the Green's function at $t = 0$ is used. The second reason does not fit for the case considered in this paper since we do not restrict the interaction to the electron-phonon case. With use of the postulated Hamiltonian (1), the matrix element of the interaction $V_{\mathbf{k},\mathbf{k}'}$ does not vanish even at $\mathbf{k} = \mathbf{k}'$.

Here we show that the first-order term of the expanded logarithmic term in (4) is equivalent to the Hartree term. Adding positive infinitesimal $\delta$ according to the convention of the Green's function at $t = 0$, the Hartree term is transformed as follows:

$$\lim_{\delta \to 0} \sum_q \sum_k V_{k,k} T \sum_{\epsilon_n} G(-k+q/2, i\epsilon_n) e^{i\epsilon_n \delta} T \sum_{\epsilon_n'} G(k+q/2, i\epsilon_n') e^{i\epsilon_n' \delta}$$

$$= \lim_{\delta \to 0} T^2 \sum_{\omega_m} \sum_{\epsilon_n} \sum_q \sum_k V_{k,k} G(-k+q/2, -i\epsilon_n) G(k+q/2, i\omega_m + i\epsilon_n) e^{i\omega_m \delta}$$

$$= \lim_{\delta \to 0} \sum_q \sum_k V_{k,k} \int \frac{d\omega}{2\pi} \frac{1-f(\epsilon_{q/2+k}-\mu)-f(\epsilon_{q/2-k}-\mu)}{\epsilon_{q/2+k}+\epsilon_{q/2-k}-\omega-2\mu} \frac{e^{\delta\omega}}{e^{\beta\omega}-1}$$

$$= T \sum_{\omega_m} \sum_q \sum_k V_{k,k} \frac{1-f(\epsilon_{q/2+k}-\mu)-f(\epsilon_{q/2-k}-\mu)}{\epsilon_{q/2+k}+\epsilon_{q/2-k}-i\omega_m-2\mu} e^{i0^+\omega_m} . \tag{A2}$$

It is clear from (5) that this is equivalent to $T \sum_{\omega_m} \sum_q V\chi_0(\mathbf{q}, i\omega_m) e^{i0^+\omega_m}$.

## APPENDIX B

In this appendix the properties of $\chi_0(q, \omega + i0^+)$ are investigated. From (5) the real and imaginary parts of $\chi_0(q, \omega + i0^+)$ are written as follows:

$$\text{Re}\chi_0(q, \omega + i0^+) = P \int_0^\infty \frac{dk\, k}{2\pi} \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{[1-f(\epsilon_{\mathbf{q}/2+\mathbf{k}}-\mu)-f(\epsilon_{\mathbf{q}/2-\mathbf{k}}-\mu)]}{(1+k^2/k_0^2)(k^2/m+q^2/4m-\omega-2\mu)} , \tag{B1}$$

and

$$\text{Im}\chi_0(q, \omega + i0^+) = \int_0^\infty \frac{dk\, k}{2\pi} \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{[1-f(\epsilon_{\mathbf{q}/2+\mathbf{k}}-\mu)-f(\epsilon_{\mathbf{q}/2-\mathbf{k}}-\mu)]}{(1+k^2/k_0^2)} \pi\delta(k^2/m+q^2/4m-\omega-2\mu) , \tag{B2}$$

where $\varphi$ is the angle between $\mathbf{k}$ and $\mathbf{q}$. Integrations in (B1) and (B2) are complicated in general. However, it is relatively easy to perform the integrations in the low-temperature regions $T \ll |\mu|$ important for the discussions in the text.

Furthermore, the calculations in the case $\mu < 0$ can be simply performed. In the region $T \ll -\mu$, (B1) and (B2) are calculated up to the terms of $O(e^{-|\mu|/T})$ as

$$\text{Re}\chi_0(q, \omega + i0^+) \simeq \frac{m}{4\pi} \frac{[1-\lambda(q, \omega, \mu, T)e^{-|\mu|/T}]}{-1+(|\mu|+q^2/8m-\omega/2)/\epsilon_0} \ln|(|\mu|+q^2/8m-\omega/2)/\epsilon_0| \tag{B3}$$

and

$$\text{Im}\chi_0(q, \omega + i0^+) \simeq \frac{m}{4} \frac{[1-\nu(q, \omega, \mu, T)e^{-|\mu|/T}]}{1-(|\mu|+q^2/8m-\omega/2)/\epsilon_0} \theta(\omega-2|\mu|-q^2/4m) , \tag{B4}$$

where $\epsilon_0 \equiv k_0^2/2m$, and $\lambda$ and $\nu$ are of $O(1)$ and complicated functions of $q$, $\omega$, $\mu$, and $T$. It is noted that $\text{Re}\chi_0(q, \omega + i0^+)$ shows logarithmic divergence around $\omega \sim 2|\mu| + q^2/4m$ and $\text{Im}\chi_0(q, \omega + i0^+)$ is vanishing for $\omega < 2|\mu| + q^2/4m$.

In the case $\mu > 0$, the integrations in (B1) and (B2) are cumbersome even in the low-temperature region $T \ll \mu$. So we present the results for special values of parameters $q$ and $T$. Then we interpolate among them by physical arguments. $\chi_0(0, \omega + i0^+)$ in the region $T \ll \mu$ is calculated analytically up to the terms of $O(T/\mu, T/\epsilon_0)$:

$$\mathrm{Re}\chi_0(0,\omega+i0^+) \simeq \frac{m}{4\pi}\frac{1}{1+(\mu+\omega/2)/\epsilon_0}\{[1-(\mu+\epsilon_0)/2T]\ln[1+(\mu-2T)/\epsilon_0]$$

$$+[1+(\mu+\epsilon_0)/2T]\ln[1+(\mu+2T)/\epsilon_0]+\ln|(\mu+\omega/2)/\epsilon_0|$$

$$-(1+\omega/4T)\ln|(2T+\omega/2)/\epsilon_0|-(1-\omega/4T)\ln|(2T-\omega/2)/\epsilon_0|\} \qquad (B5)$$

and

$$\mathrm{Im}\chi_0(0,\omega+i0^+)=\frac{m}{4}\frac{\tanh\omega/4T}{1+(\mu+\omega/2)/\epsilon_0}\theta(\omega+2\mu) . \qquad (B6)$$

It is remarked that (B6) is always valid as long as $\mu > 0$.

At $T=0$, $\chi_0(q,\omega+0^+)$ can be calculated analytically.  The expressions are different for two cases:  (i) $q^2/8m < \mu$, and (ii) $q^2/8m > \mu$.

Case (i),  $q^2/8m < \mu$:

$$\mathrm{Re}\chi_0(q,\omega+i0^+)=\frac{m}{4\pi}\frac{1}{1+(\mu+\omega/2-q^2/8m)/\epsilon_0}$$

$$\times\left[\ln\left|\left|\mu+\frac{\omega}{2}-\frac{q^2}{8m}\right|\right/\epsilon_0\right|$$

$$+\ln\left\{\left|\left|\mu-\frac{q^2}{8m}\right|^2+2\mu\epsilon_0+\epsilon_0^2\right|+\left|\mu-\frac{q^2}{8m}+\epsilon_0\right|\left[\left|\mu-\frac{q^2}{8m}+\epsilon_0\right|^2+\frac{q^2\epsilon_0}{2m}\right]^{1/2}\right\}$$

$$-\ln\left\{\left|\left[\frac{\omega}{2}\right]^2-\frac{q^2}{4m}\left|\mu+\frac{\omega}{2}-\frac{q^2}{8m}\right|\right|+\left|\frac{\omega}{2}\right|\left[\left|\left[\frac{\omega}{2}-\frac{q^2}{4m}\right]^2-\frac{q^2\mu}{2m}\right]^{1/2}\right\}\right] \qquad (B7)$$

and

$$\mathrm{Im}\chi_0(q,\omega+i0^+)=\frac{m}{4}\frac{1}{1+(\mu+\omega/2-q^2/8m)/\epsilon_0}\begin{cases}0, & \mu+\dfrac{\omega}{2}-\dfrac{q^2}{8m}<0 , \\[2mm] -1, & 0<\mu+\dfrac{\omega}{2}-\dfrac{q^2}{8m}<\dfrac{(\sqrt{2m\mu}-q/2)^2}{2m} , \\[2mm] -A, & \dfrac{(\sqrt{2m\mu}-q/2)^2}{2m}<\mu+\dfrac{\omega}{2}-\dfrac{q^2}{8m}<\mu-\dfrac{q^2}{8m} , \\[2mm] A, & \mu-\dfrac{q^2}{8m}<\mu+\dfrac{\omega}{2}-\dfrac{q^2}{8m}<\dfrac{(\sqrt{2m\mu}+q/2)^2}{2m} , \\[2mm] 1, & \dfrac{(\sqrt{2m\mu}+q/2)^2}{2m}<\mu+\dfrac{\omega}{2}-\dfrac{q^2}{8m} ,\end{cases} \qquad (B8)$$

where $A$ is defined as

$$A\equiv\frac{2}{\pi}\sin^{-1}\left\{\frac{|\omega|}{2}\left[\frac{q^2}{2m}\left|\mu+\frac{\omega}{2}-\frac{q^2}{8m}\right|\right]^{-1/2}\right\} . \qquad (B9)$$

Case (ii), $q^2/8m > \mu$:

$$\mathrm{Re}\chi_0(q,\omega+i0^+)=\frac{m}{4\pi}\frac{1}{1+(\mu+\omega/2-q^2/8m)/\epsilon_0}\left\{-\ln\left|\left|\mu+\frac{\omega}{2}-\frac{q^2}{8m}\right|\right/\epsilon_0\right|$$

$$+\int_{\theta_0}^{\pi/2}\frac{d\theta}{\pi/2}\ln\left|\frac{[\epsilon_+(\theta)+\epsilon_0][\epsilon_-(\theta)-(\mu+\omega/2-q^2/8m)]}{[\epsilon_-(\theta)+\epsilon_0][\epsilon_+(\theta)-(\mu+\omega/2-q^2/8m)]}\right|\right\}$$

$$(B10)$$

and

$$\mathrm{Im}\chi_0(q,\omega+i0^+)=\frac{m}{4}\frac{1}{1+(\mu+\omega/2-q^2/8m)/\epsilon_0}\begin{cases}0, & \mu+\dfrac{\omega}{2}-\dfrac{q^2}{8m}<0\ ,\\[2mm] 1, & 0<\mu+\dfrac{\omega}{2}-\dfrac{q^2}{8m}<\dfrac{(q/2-\sqrt{2m\mu})^2}{2m}\ ,\\[2mm] A, & \dfrac{(\sqrt{2m\mu}-q/2)^2}{2m}<\mu+\dfrac{\omega}{2}-\dfrac{q^2}{8m}<\dfrac{(q/2+\sqrt{2m\mu})^2}{2m}\ ,\\[2mm] 1, & \dfrac{(\sqrt{2m\mu}+q/2)^2}{2m}<\mu+\dfrac{\omega}{2}-\dfrac{q^2}{8m}\ ,\end{cases}\tag{B11}$$

where $\theta_0\equiv\cos^{-1}\sqrt{8m\mu/q^2}$ and $\epsilon_\pm(\theta)$ is defined as

$$\epsilon_\pm(\theta)\equiv-\frac{q^2}{8m}\cos2\theta+\mu$$
$$\pm\frac{q}{\sqrt{2m}}\sin\theta\left[\mu-\frac{q^2}{8m}\cos^2\theta\right]^{1/2}\ .\tag{B12}$$

It is noted that the term $\ln|(\mu+\omega/2-q^2/8m)/\epsilon_0|$, singular around $\omega\sim-2\mu+q^2/4m$, appears both in (B7) and (B10), and the last term of (B7) shows singularity like $-\ln|\omega|$ around $\omega\sim0$, while the remaining terms in (B7) and (B10) are regular. The singularity of the term $\ln|(\mu+\omega/2-q^2/8m)/\epsilon_0|$ arises from the integration of (B1) near $k\sim0$. So this singular term can be calculated also at $T>0$ by approximating the numerator of (B1) as

$$1-2f(\epsilon_{q/2}-\mu)=\tanh(\epsilon_{q/2}-\mu)/T\ .$$

Then, in general, the singular behavior of $\mathrm{Re}\chi_0(q,\omega+i0^+)$ around $\omega\sim-2\mu+q^2/4m$ is written as

$$\mathrm{Re}\chi_0(q,\omega+i0^+)\simeq\frac{m}{4\pi}\frac{1}{1+(\mu+\omega/2-q^2/8m)/\epsilon_0}$$
$$\times\tanh\frac{(\mu-q^2/8m)}{2T}$$
$$\times\ln\left|\left[\mu+\frac{\omega}{2}-\frac{q^2}{8m}\right]\Big/\epsilon_0\right|\ .\tag{B13}$$

### APPENDIX C

The $s$-wave bound-state solution $(E_0<0)$ of the Schrödinger equation (15) is discussed. For the separable $s$-wave interaction (2), Eq. (15) is transformed to

$$1-V\sum_{\mathbf{k}}\frac{1}{(1+k^2/k_0^2)(2\epsilon_k+|E_0|)}=0\ .\tag{C1}$$

This has the same structure as (10) with $T=0$ and $\mu$ being substituted by $-|E_0|/2$; so that (C1) is reduced to

$$1-\frac{mV}{4\pi}\frac{1}{-1+|E_0|/2\epsilon_0}\ln\frac{|E_0|}{2\epsilon_0}=0\ .\tag{C2}$$

Thus, $E_0$ is given by a solution of the following algebraic equation:

$$E_0=-2\epsilon_0\exp\left[-\frac{4\pi}{mV}\left[1-\frac{|E_0|}{2\epsilon_0}\right]\right]\ .\tag{C3}$$

### APPENDIX D

In this appendix, we show that there is no divergence in the last term of (7) other than the one associated with virtual formation of the Cooper pairs as discussed in Sec. III B.[14] Structures of the real and imaginary part of $\chi_0(q,\omega+i0^+)$ are qualitatively different for the two cases, (i) $q^2/8m<\mu$ and (ii) $q^2/8m>\mu$, as discussed in Appendix B. So, we have to discuss these two cases separately.

For case (i), $q^2/8m<\mu$, schematic behaviors of $1-V\,\mathrm{Re}\chi_0$ and $\mathrm{Im}\chi_0$ are shown in Fig. 8(a), from which one can see the behavior of the phase shift $\delta(q,\omega)$, (8), as shown in Fig. 8(b). Although $\mathrm{Re}\chi_0$, (B7), contains the logarithmic divergence around $\omega\sim-2\mu+q^2/4m$, it causes no singularity in $\delta(q,\omega)$. This is in contrast to class (a) where the logarithmic divergence in $\mathrm{Re}\chi_0(q,\omega)$ gives the discontinuity in $\delta(q,\omega)$ corresponding to the existence of the di-electronic bound state. The difference arises from that of the sign of the singularity $\ln|(\mu+\omega/2-q^2/8m)/\epsilon_0|$ in $\mathrm{Re}\chi_0$.

The analytic form of the integrand $\partial\delta(q,\omega)/\partial\mu$ in (7) is still complicated. So here we show the calculation of $\partial\delta(0,\omega)/\partial\mu$ and argue the case for finite $q$ $(<\sqrt{8m\mu})$. The real part of $\lim_{T\to0}\chi_0(0,\omega+i0^+)$ is obtained from (B5) or (B7) as

$$\lim_{T\to0}\mathrm{Re}\chi_0(0,\omega+i0^+)$$
$$\simeq\frac{m}{4\pi}\frac{\epsilon_0}{\epsilon_0+\mu+\omega/2}\ln\left|\frac{(\epsilon_0+\mu)^2(\mu+\omega/2)}{(\omega/2)^2\epsilon_0}\right|\ .\tag{D1}$$

The imaginary part of $\lim_{T\to0}\chi_0(0,\omega+i0^+)$ is given by (B6) or (B8) as

$$\lim_{T\to0}\mathrm{Im}\chi_0(0,\omega+i0^+)=\frac{m}{4}\frac{\epsilon_0\mathrm{sgn}\{\omega\}}{\epsilon_0+\mu+\omega/2}\theta(\omega+2\mu)\ .\tag{D2}$$

Substituting (D1) and (D2) into (8), the phase shift $\lim_{T\to0}\delta(0,\omega)$ is expressed as

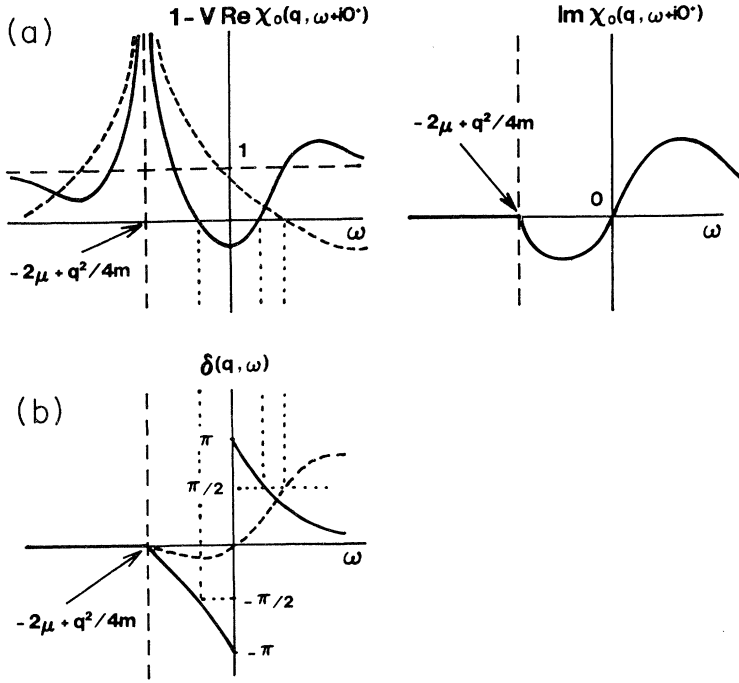AKIO TOKUMITU, KAZUMASA MIYAKE, AND KAZUO YAMADA



FIG. 8. (a) Schematic behaviors of $1-V\,\mathrm{Re}\chi_0(q,\omega+i0^+)$ and $\mathrm{Im}\chi_0(q,\omega+i0^+)$ as a function of $\omega$ for the case $\mu\gg T\rightarrow 0$ and $q^2/8m<\mu$. (b) Schematic behavior of the phase shift $\delta(q,\omega)$, (8), for the case $\mu\gg T\rightarrow 0$ and $q^2/8m<\mu$. Solid and dashed lines represent two different typical cases, respectively.

$$\lim_{T\rightarrow 0}\delta(0,\omega)\simeq\tan^{-1}\left\{\left[\frac{mV}{4}\frac{\epsilon_0\mathrm{sgn}\{\omega\}}{\epsilon_0+\mu+\omega/2}\theta(\omega+2\mu)\right]\bigg/\left[1-\frac{mV}{4\pi}\frac{\epsilon_0}{\epsilon_0+\mu+\omega/2}\ln\left|\frac{(\epsilon_0+\mu)^2(\mu+\omega/2)}{(\omega/2)^2\epsilon_0}\right|\right]\right\}. \tag{D3}$$

Differentiating this by $\mu$, $\lim_{T\rightarrow 0}\partial\delta(0,\omega)/\partial\mu$ is written as follows:

$$\lim_{T\rightarrow 0}\frac{\partial\delta(0,\omega)}{\partial\mu}\simeq-\frac{\omega}{|\omega|}\left[1-\frac{mV}{4\pi}\left[\frac{\epsilon_0}{\mu+\omega/2}+\frac{2\epsilon_0}{\mu+\epsilon_0}\right]\right]\frac{mV}{4\epsilon_0}\theta(\omega+2\mu)$$

$$\times\left\{\left[1+\frac{\mu+\omega/2}{\epsilon_0}-\frac{mV}{4\pi}\ln\left|\frac{(\epsilon_0+\mu)^2(\mu+\omega/2)}{(\omega/2)^2\epsilon_0}\right|\right]^2+\left[\frac{mV}{4}\theta(\omega+2\mu)\right]^2\right\}^{-1}. \tag{D4}$$

The apparent divergence in (D4) around $\omega\sim-2\mu$ is smeared out by the $\omega$ integration in (7) to give the term

$$P\int_{-2\mu}\frac{d\omega}{\pi}g(\omega)\frac{\partial}{\partial\mu}\delta(0,\omega)\simeq P\int_{-2\mu}^0 d\omega\left[\frac{mV}{4\pi}\right]^2\frac{1}{\mu+\omega/2}$$

$$\times\frac{1}{\left[(mV/4\pi)\ln\left|[(\epsilon_0+\mu)^2(\mu+\omega/2)/\mu^2\epsilon_0]\right|-1\right]^2+\left[mV/4\right]^2}$$

$$\simeq\frac{2}{\pi}\left\{\pi+\tan^{-1}\frac{\ln[(\epsilon_0+\mu)^2/\mu\epsilon_0]-4\pi/mV}{\pi}\right\},$$

$$\xrightarrow[mV/4\pi\ll 1]{}\frac{1}{2}\frac{mV}{4\pi}, \tag{D5}$$

which is of $O(mV/4\pi)$ in the weak-coupling limit. It is noted that the singularity around $\omega\sim 0$ due to the Bose factor gives no divergence to the $\omega$ integration in (7) because the integration $\int d\omega\,1/|\omega|(\ln\omega)^2$ converges around $\omega\sim 0$. For a general value $V$ of the strength of the attractive interaction, the $\omega$ integration in (7) with (D4) is difficult to perform analytically. However, in the weak-coupling limit ($mV/4\pi\ll 1$), (D4) is reduced to

$$\lim_{T\rightarrow 0}\frac{\partial\delta(0,\omega)}{\partial\mu}\simeq-\frac{\pi}{\epsilon_0}\frac{\omega}{|\omega|}\delta\left[1+\frac{\mu+\omega/2}{\epsilon_0}-\frac{mV}{4\pi}\ln\left|\frac{(\epsilon_0+\mu)^2(\mu+\omega/2)}{(\omega/2)^2\epsilon_0}\right|\right], \tag{D6}$$

except for the term leading to (D5). In deriving (D6), the formula $\pi\delta(x)=\lim_{a\rightarrow 0}a/(x^2+a^2)$ has been used. The $\delta$ function in (D6) is further transformed to

$$\lim_{T \to 0} \frac{\partial \delta(0,\omega)}{\partial \mu} \simeq -\frac{2\pi}{\epsilon_0}\frac{\omega}{|\omega|}\left| \frac{1}{\epsilon_0} - \frac{mV}{2\pi}\frac{1}{2\mu+\omega} + \frac{mV}{\pi}\frac{1}{\omega} \right|^{-1} \delta(\omega-\omega_0) \ , \tag{D7}$$

where $\omega_0$ is given by the condition that the argument in the $\delta$ function in (D6) is zero:

$$1 + \frac{\mu+\omega_0/2}{\epsilon_0} = \frac{mV}{4\pi}\ln\left| \frac{(\epsilon_0+\mu)^2(\mu+\omega_0/2)}{(\omega_0/2)^2\epsilon_0} \right| . \tag{D8}$$

Since $\mu$ is of $O(\epsilon_F)$ or larger *a posteriori*, $\omega_0$ is obtained as

$$\omega_0 \simeq \pm 2(\mu+\epsilon_0)\sqrt{\mu/\epsilon_0}\exp\left[ -\frac{2\pi(1+\mu/\epsilon_0)}{mV} \right] . \tag{D9}$$

Thus, with use of (D7) and (D9), the $q=0$ term in (7) is given as

$$-\frac{4\pi}{mV}\frac{(\mu+\epsilon_0)}{\epsilon_0}\sqrt{\mu/\epsilon_0}\exp\left[ -\frac{2\pi(1+\mu/\epsilon_0)}{mV} \right] , \tag{D10}$$

which is exponentially small in the weak-coupling limit ($mV/4\pi \ll 1$). Therefore, in the weak-coupling limit, the main contribution (except for the divergent one discussed in Sec. III B) to the $\omega$ integration for the $q=0$ term in (7) is given by (D5).

For finite $q$ ($< \sqrt{8m\mu}$), the expression for $\partial\delta(q,\omega)/\partial\mu$ becomes far more complicated. However, it is seen that the main contribution to the $\omega$ integration in (7) comes from the singular term (around $\omega \sim -2\mu+q^2/4m$) stemmed from the $\mu$ derivative of $\ln|(\mu+\omega/2-q^2/8m)/\epsilon_0|$ in (B7) which gave the term of $O(mV/4\pi)$ just as (D5).

For case (ii), $q^2/8m > \mu$, the schematic behaviors of $1-V\,\mathrm{Re}\chi_0$ and $\mathrm{Im}\chi_0$ are shown in Fig. 9 (a), from which one can see the behavior of the phase shift $\delta(q,\omega)$, (8), as shown in Fig. 9(b). It is noted that the phase shift has the discontinuity (by $\pi$) at $\omega=\eta_q < q^2/4m - 2\mu$ corresponding to the existence of the bound state, where $\eta_q$ satisfies the relation $1-V\,\mathrm{Re}\chi_0(q,\eta_q+i0^+)=0$. This was recognized in the work of Schmitt-Rink, Varma, and Ruckenstein.[7] The energy $q$ of the bound state can be either positive or negative depending on the value of $q$. Indeed in the weak-coupling limit ($mV/4\pi \ll 1$), with use of (B10) $\eta_q$ is given as

$$\eta_q \simeq -2\mu + \frac{q^2}{4m} - B\epsilon_0 e^{-4\pi/mV} , \tag{D11}$$

where $B$ is a positive constant of $O(1)$.

Here one might suspect that such a bound state gives a divergent contribution to (7) from the Bose condensation of the bound state as in the case $T \ll -\mu$, i.e., $\mu < 0$. However, as discussed shortly, this is not the case. The crucial point is that the principal part is taken in the $\omega$ integration in (7). So, the bound-state contribution to the second part of (7) is calculated using a relation the same as (29) and (30):
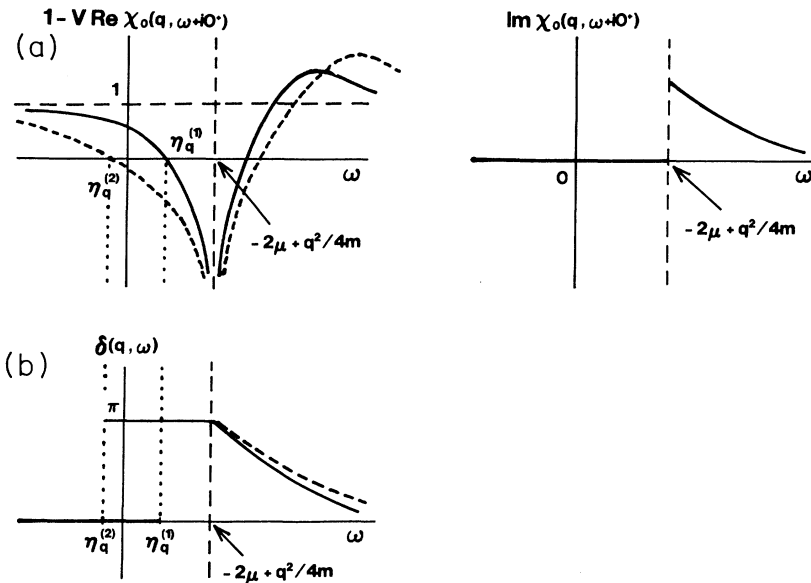


FIG. 9. (a) Schematic behaviors of $1-V\,\mathrm{Re}\chi_0(q,\omega+i0^+)$ and $\mathrm{Im}\chi_0(q,\omega+i0^+)$ as a function of $\omega$ for the case $\mu \gg T \to 0$ and $q^2/8m > \mu$. (b) Schematic behavior of the phase shift $\delta(q,\omega)$, (8), for the case $\mu \gg T \to 0$ and $q^2/8m > \mu$. A discontinuity of $\delta(q,\omega)$ at $\omega=\eta_q$ represents the existence of the di-electronic molecule with the momentum $q(>\sqrt{8m\mu})$ and the energy $\eta_q$ which is given by the relation $1-V\,\mathrm{Re}\chi_0(q,\eta_q+i0^+)=0$. The value of $\eta_q$, (D11), can be either positive or negative, which is in marked contrast with Fig. 4. Solid and dashed lines represent two different typical cases, respectively.

$$
\sum_{q>\sqrt{8m\mu}} P \int_{-\infty}^{\infty} \frac{d\omega}{\pi} g(\omega) \frac{\partial \delta_b(q,\omega)}{\partial \mu} \simeq P \sum_{q>\sqrt{8m\mu}} g(\eta_q)
$$

$$
= \frac{m}{4\pi} P \int_{\eta(\sqrt{8m\mu})}^{\infty} d\eta \frac{1}{e^{\eta/T}-1}
$$

$$
\xrightarrow[T\to 0]{} -\frac{m}{4\pi} \epsilon_0 B e^{-4\pi/mV} , \qquad (D12)
$$

where the lower bound of the $\eta$ integration $\eta(\sqrt{8m\mu})= -\epsilon_0 B \exp(-4\pi/mV)$ from (D11). The scattering-state contribution to the same term of (7) should be calculated carefully as follows. Just as the singularity around $\omega \sim -2\mu$ in (D4), $\partial\delta(q,\omega)/\partial\mu$ has an apparently diverging term around $\omega \sim -2\mu + q^2/4m$ like

$$
-\pi \left[\frac{mV}{4\pi}\right]^2 \frac{\partial}{\partial\mu} \left[\tanh\frac{\mu-q^2/8m}{2T} \ln|(\mu-q^2/8m+\omega/2)/\epsilon_0|\right] \theta(\omega+2\mu-q^2/4m)
$$

$$
\times \left\{\left[\frac{mV}{4\pi}\ln|(\mu-q^2/8m+\omega/2)/\epsilon_0|-1\right]^2 + \left[\frac{mV}{4}\right]^2\right\}^{-1} , \qquad (D13)
$$

which can be derived from (8) with use of (B10), (B11), and (B13) as in the case deriving (D4). Therefore,

$$
\sum_{q>\sqrt{8m\mu}} P \int_{-2\mu+q^2/4m} \frac{d\omega}{\pi} g(\omega) \frac{\partial\delta_s(q,\omega)}{\partial\mu}
$$

$$
\simeq \sum_{q>\sqrt{8m\mu}} \left[\frac{mV}{4\pi}\right]^2 P \int_{-2\mu+q^2/4m} d\omega \frac{-1}{e^{\omega/T}-1} \frac{\partial}{\partial\mu} \left[\tanh\frac{\mu-q^2/8m}{2T} \ln|(\mu-q^2/8m+\omega/2)/\epsilon_0|\right]
$$

$$
\times \left\{\left[\frac{mV}{4\pi}\ln|(\mu-q^2/8m+\omega/2)/\epsilon_0|-1\right]^2 + \left[\frac{mV}{4}\right]^2\right\}^{-1}
$$

$$
\simeq \sum_{q>\sqrt{8m\mu}} \left[\frac{mV}{4\pi}\right]^2 \tanh\frac{\mu-q^2/8m}{2T} \int_0 d\widetilde{\omega} \frac{-1}{\exp[(q^2/4m-2\mu+\widetilde{\omega})/T]-1}
$$

$$
\times \frac{2}{\widetilde{\omega}} \left\{\left[\frac{mV}{4\pi}\ln|\widetilde{\omega}/2\epsilon_0|-1\right]^2 + \left[\frac{mV}{4}\right]^2\right\}^{-1} . \qquad (D14)
$$

This integral is convergent and gives a positive constant of $O(mV/4\pi)$ in the weak-coupling limit as in (D5) because the singular behavior around $q \gtrsim \sqrt{8m\mu}$ cancels out after the $\widetilde{\omega}$ integration.[14] Except for the above two contributions, (D12) and (D14), to the second term of (7), only an exponentially small one like (D10) is expected to remain in the weak-coupling limit $(mV/4\pi \ll 1)$. Thus, in the weak-coupling limit, relation (37) is valid.

Next let us discuss how the above discussions are modified as the strength of attractive interaction $V$ increases.

For case (i), $q^2/8m < \mu$, one can see by inspection of (B7) and (B8) that $\lim_{T\to 0}\partial\delta(q,\omega)/\partial\mu$ is regular except for a term like (D13). Then the contributions to the second term of (7) are given in a way quite similar to (D14) but with the $q$ summation being taken in the region $q < \sqrt{8m\mu}$, which tends to a finite and positive constant as the coupling constant $mV/4\pi$ grows up as can be seen in the calculation of (D5).

For case (ii), $q^2/8m > \mu$, the scattering-state contribution to the second term of (7) is given by (D14) which is also positive and of $O(1)$ as in case (i). On the other hand, the bound-state contribution to (7) is given by the

same expression as (D12). However, the lower bound of $\eta$ integration, $\eta(\sqrt{8m\mu})$, is not determined by (D11), the expression for the weak-coupling limit. In the strong-coupling limit, the bound-state energy $\eta_q$ is given, with use of (B10), (C2), and (17), as

$$
\eta_q \simeq -2\mu + \frac{q^2}{4m} - 2\epsilon_0 \frac{mV}{4\pi} \zeta(V) . \qquad (D15)
$$

So, the bound-state contribution is calculated just as in (D12):

$$
\sum_{q>\sqrt{8m\mu}} P \int_{-\infty}^{\infty} \frac{d\omega}{\pi} g(\omega) \frac{\partial\delta_b(q,\omega)}{\partial\mu} \xrightarrow[T\to 0]{} -\frac{m\epsilon_0}{2\pi} \frac{mV}{4\pi} \zeta(V) ,
$$

$$
\qquad (D16)
$$

where $\zeta(V)$ satisfies relation (18).

Therefore, in the strong-coupling limit $(mV/4\pi \gg 1)$, contribution (D16) is a dominant one to the second term of (7), except for the logarithmically divergent term associated with virtual formation of the Cooper pairs discussed in Sec. III B. Thus, in the strong-coupling limit, relation (39) is valid as long as $\mu \gg T$.

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