The limit sets of Coxeter groups

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Introduction

1. Asymptotic behavior of roots

The theory of Coxeter groups has been developed from not only combinatorial but also geometrical aspects. Indeed Coxeter groups are realized as reflection groups acting on vector spaces. One of the most fundamental and important object associated with Coxeter groups is the root system. Finite Coxeter groups are nothing but finite reflection groups. In this case the root system corresponds to the finite system of normal vectors to hyperplanes defining Euclidean reflections. In addition for affine reflection groups (which are of infinite order) the root system also corresponds to the (infinite) system of normal vectors to hyperplanes. However most of infinite Coxeter groups can not be realized as affine reflection groups. There are few studies on the root systems for the case of general infinite Coxeter groups. To analyze the root system a new dynamical approach has been introduced by Hohlweg, Labbé and Ripoll in [**18**]. In the same paper they proposed a conjecture [**18**, Conjecture 3.9] which asks whether the distribution of accumulation points of roots of infinite Coxeter groups can be described as some appropriate set of points. We give a proof of this conjecture in the case where associating bi-linear form of a Coxeter group has the signature $(n-1,1)$. We should notice that Dyer, Hohlweg and Ripoll also proved in the positive the conjecture by a different approach at the same time (cf. Remark 2.4). More precisely we will show the following in Chapter 2 (see Theorem 2.2).

Theorem A (Distribution of accumulation points of roots). *Fix a Coxeter group W of rank n equipped with the signature $(n-1,1)$ bi-linear form B. Let $E = E(W)$ be the set of accumulation points of normalized roots. Let $\widehat{Q}$ be an ellipsoid in an affine subspace of the phase space defined by B. Let $\widehat{\Delta}$ be the set of normalized simple roots and* $\mathrm{conv}(\widehat{\Delta})$ *denotes the convex hull of $\widehat{\Delta}$. Then we have the following:*

(a) *When $\widehat{Q} \subset \mathrm{conv}(\widehat{\Delta})$, we have $E = \widehat{Q}$.*
(b) *When $\widehat{Q} \not\subset \mathrm{conv}(\Delta)$, we have $E = \widehat{Q} \setminus \left(\bigcup_{i=1}^{m} W \cdot R_i\right)$, where $R_1, \ldots, R_m$ are connected components of $\widehat{Q}$ out of $\mathrm{conv}(\widehat{\Delta})$ with $1 \leq m \leq n$.*

2. Dynamical approach to Coxeter groups

The approach in [**18**] implicates a study of infinite Coxeter groups from a dynamical viewpoint. We analyze the asymptotic behavior of the orbit of points under the action of a infinite Coxeter group. As is known in the theory of the Kleinian groups (which are discrete groups of Möbius transformation on the Poincaré ball), to study accumulation points is nothing but to study the interaction between ergodic theory and discrete groups. In order to establish that theory, the hyperbolicity of its phase space plays a crucial role. For the case where the associated bilinear forms have the signature $(n-1,1)$, Coxeter groups also act on the hyperbolic space. Following the notion of the theory of the Kleinian groups, we call the set of accumulation points of a point in the phase subspace the limit set of W denoted by $\Lambda(W)$.

Theorem B ($E(W) = \Lambda(W)$). *Let W be a Coxeter groups of rank n whose associating bilinear form B has the signature $(n-1, 1)$. Then the limit set $\Lambda(W)$ of W coincides with the set of accumulation points of roots $E = E(W)$ of W.*

We notice that Hohlweg, Préaux and Ripoll also showed Theorem B independently [**19**]. Our second purpose is to develop the theory of semi-conjugacies (i.e. an equivariant continuous surjection) from the Gromov boundaries of Coxeter groups to the limit sets lying on the boundary (with respect to the Euclidean topology) of the phase spaces.

Theorem C (Cannon-Thurston maps exist). *Let (W, S) be a Coxeter system of rank n whose associating bilinear form B has the signature $(n-1, 1)$. Let $\partial_G(W, S)$ be the Gromov boundary of W and let $\Lambda(W)$ be the limit set of W. There exists a W-equivariant, continuous surjection $F : \partial_G(W, S) \longrightarrow \Lambda(W)$.*

3. Background and Motivations

The Coxeter groups are purely algebraic object. Indeed these are defined by matrices which represent relations of generators. From this reason there are enormous studies of Coxeter groups from the algebraic and combinatoric aspects. Likewise, as noted before, Coxeter groups are studied from the geometric aspect. Actually they canonically have an action as reflections on a vector space.

We study the algebraic structure of Coxeter groups from a dynamical viewpoint. For example the growth series of a Coxeter group is one of the most important invariants from the algebraic viewpoint. On the one hand in dynamical viewpoint it turns into the Poincaré series in the case where the associated bi-linear form has the signature $(n-1, 1)$. This will provide another research direction to study the Coxeter groups using asymptotic quantities such as the Hausdorff dimension.

In this thesis we try to build a step to connect the dynamical aspect with the algebraic and the geometric aspects focussing on the case where the associated bilinear form has the signature $(n-1, 1)$. At first we need Theorem A to see how the roots (which are combinatoric objects) are distributed at infinity of the phase space. This is an observation of the asymptotic behavior of the action on the outside of an invariant ellipsoid defined by the associated bilinear form. The next step is to investigate the asymptotic behavior of the action on the inside. In this case we can define the limit sets of Coxeter groups since reflections on the phase space define the isometric discrete action on the hyperbolic space. It is natural to ask whether the set of accumulation points equals to the limit set. Theorem B answers in the positive this claim. For the next, our interest moves to the existence of the Cannon-Thurston maps.

In general, a continuous equivariant map between boundaries of a discrete group and their limit set is called a Cannon-Thurston map. The importance can be seen by going back to its origin. Cannon and Thurston first gave a geometrically infinite example of a semi-conjugacy map. In [**10**], they invented the existence of such a map between the boundary of the fundamental group of a hyperbolic surface S and the boundary of the fundamental group of a hyperbolic manifold M which is an S-bundle over a circle. In addition they also considered a singly degenerate group with an asymptotically periodic end. In this case its limit set is given by continuous image of the circle. As a consequence Cannon and Thurston concluded that the limit set is locally connected. Motivated this fact, the existence of such map is one of the most interesting question in the group theory from a geometrical viewpoint. For the Kleinian groups several authors contributed to this topic. In particular recently Mj showed that for Kleinan surface groups (in fact for all finitely generated Kleinian groups) there exist the Cannon-Thurston maps and local connectivity of the connected limit sets [**31**].

From more general viewpoint Mitra considered the Cannon-Thurston map for the Gromov hyperbolic groups. Let H be a hyperbolic subgroup of a hyperbolic group G in the sense of Gromov. He asked whether the inclusion map always extends continuously to the equivariant map between the Gromov compactifications $\widehat{H}$ and $\widehat{G}$. Here word metrics on H and G defining their compactifications may differ. For this question he positively answered in the case when H is an infinite normal subgroup of a hyperbolic group G [**29**]. He also proved that the existence of the Cannon-Thurston map when G is a hyperbolic group acting cocompactly on a simplicial tree T such that all vertex and edge stabilizers are hyperbolic, and H is the stabilizer of a vertex or edge of T provided every inclusion of an edge stabilizer in a vertex stabilizer is a quasi isometric embedding [**30**]. On the other hand, Baker and Riley constructed a negative example for Mitra's question. In fact they proved that there exists a free subgroup of rank 3 in a hyperbolic group such that the Cannon-Thurston map is not well-defined [**1**]. Moreover Matsuda and Oguni showed that a similar phenomenon occurs for every non-elementary relatively hyperbolic group [**23**].

Inspired by the above results we shall consider the problem which asks whether the Cannon-Thurston map for the Coxeter groups exists. We give a motivation behind this problem as follows. Since the Gromov boundary of a group is defined by a transcendental way (indeed it is the set of equivalence classes of sequences satisfying an asymptotic property) it is difficult to see its topological properties or geometric structures. On the other hand the limit set of a Coxeter group lies on the boundary of the ellipsoid. Moreover since each limit point is realized by the end point of Euclidean straight line emanating from a fixed base point, it is relatively easy to observe their geometric property. Indeed the limit set of a Coxeter subgroup appears on the intersection of the boundary of the ellipsoid and the simplex defined by the bi-linear form associated to the whole Coxeter group. Conversely the existence of the Cannon-Thurston map indicate that the geometric or the asymptotic information of the action inherits to the algebraic information of the group. As a conclusion we hope that we can derive another research direction to obtain algebraic informations of the Coxeter groups via the dynamical approach.

4. Future prospects

The author studies the action of groups on metric spaces from the asymptotical view point. In this thesis, we deal with actions of Coxeter groups on the hyperbolic space to obtain information on the roots of the given Coxeter group via the limit set. In particular the author would like to derive a kind of rigidity property of Coxeter groups.

We already observed a rigidity at infinity in a different situation. Indeed in a joint work with Miyachi [**27**] the author has shown Royden's theorem which states that any biholomorphic automorphism on Teichmüller space is induced by an orientation preserving homeomorphism on the base surface. Our proof is accomplished by studying the behavior of biholomorphic automorphisms at infinity of Teichmüller space to get the rigidity.

For the future, we can consider the following:

- The relation between the Hausdorff dimension and the growth rate of the given Coxetr group. Actually, the growth function is nothing but the Poincaré series. From this point of view, for a Coxeter group of rank 3 we can see how the deformation of the action affects the Hausdorff dimension of the limit set ([**28**]).

- The asymptotical behavior of Coxeter groups in the case where the signature of associating bilinear form is $(n - m, m)$ for $m \geq 2$. In fact there exist examples which their bilinear forms have the signature $(n - 2, 2)$ ([**8**]). For these groups, we can not apply our argument in this thesis.

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CHAPTER 1

Normalized reflections and Coxeter groups

In this chapter we collect some notations and definitions of notions used throughout this thesis. Especially a crucial assumption (Assumption 1.1) is introduced.

1. Coxeter groups and B-reflections

Recall that W is a *Coxeter group* of rank n if W is generated by the set $S = \{s_1, \ldots, s_n\}$ with the relations $(s_i s_j)^{m_{ij}} = 1$, where $m_{ij} \in \mathbb{Z}_{>1} \cup \{\infty\}$ for $1 \le i < j \le n$ and $m_{ii} = 1$ for $1 \le i \le n$. More precisely, we say that the pair (W, S) is a *Coxeter system*. We refer the reader to [**20**] for the introduction to Coxeter groups.

For a Coxeter system (W, S) of rank n, let V be a real vector space with its orthonormal basis $\Delta = \{\alpha_s | s \in S\}$ with respect to the Euclidean inner product. Note that by identifying V with $\mathbb{R}^n$, we treat V as a Euclidean space. We define a symmetric bilinear form on V by setting

$$B(\alpha_i, \alpha_j) \begin{cases} = -\cos\left(\frac{\pi}{m_{ij}}\right) & \text{if } m_{ij} < \infty, \\ \le -1 & \text{if } m_{ij} = \infty \end{cases}$$

for $1 \le i \le j \le n$, where $\alpha_{s_i} = \alpha_i$, and call the associated matrix B the *Coxeter matrix*. Classically, $B(\alpha_i, \alpha_j) = -1$ if $m_{ij} = \infty$, but throughout this thesis, we allow its value to be any real number less than or equal to -1. This definition derives from [**18**]. Given $\alpha \in V$ such that $B(\alpha, \alpha) \neq 0$, s_α denotes the map $s_\alpha : V \to V$ by

$$s_\alpha(v) = v - 2\frac{B(\alpha, v)}{B(\alpha, \alpha)}\alpha \quad \text{for any } v \in V,$$

which is said to be a *B-reflection*. Then Δ is linearly independent and satisfies that

(i) for all $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$, one has

$$B(\alpha, \beta) \in (-\infty, -1] \cup \left\{ -\cos\left(\frac{\pi}{k}\right) \;\middle|\; k \in \mathbb{Z}_{>1} \right\};$$

(ii) for all $\alpha \in \Delta$, one has $B(\alpha, \alpha) = 1$.

Such a set Δ is called a *simple system* and its elements are *simple roots* of W. The Coxeter group W acts on V associated with its generating set S as compositions of B-reflections $\{s_\alpha \mid \alpha \in \Delta\}$ generated by simple roots. The *root system* Φ of W is defined to be the orbit of Δ under the action of W and its elements are called its *roots*. The pair (Φ, Δ) is said to be a *based root system* in (V, B). We mention that Δ in [**18**, Definition 1.2] is assumed to be positively independent, while we assume the linearly independence throughout this thesis. Let

$$V^+ := \left\{ v \in V \;\middle|\; v = \sum_{i=1}^n v_i\alpha_i, v_i > 0 \right\} \text{ and } V^- := \left\{ v \in V \;\middle|\; v = \sum_{i=1}^n v_i\alpha_i, v_i < 0 \right\}.$$

Assumption 1.1. In this thesis, we always assume the following.

- The bilinear form B has the signature $(n-1,1)$. We call such a group a Coxeter group of type $(n-1,1)$.
- The Coxeter matrix B is not block-diagonal up to permutation of the basis. In that case, the matrix B is said to be *irreducible*.

Recall that a matrix A is *non-negative* if each entry of A is non-negative.

LEMMA 1.2. *Let o be an eigenvector for the negative eigenvalue of B. Then all coordinates of o have the same sign.*

PROOF. This follows from Perron-Frobenius theorem for irreducible non-negative matrices. Let I be the identity matrix of rank n. Then $-B+I$ is irreducible and non-negative. Note that since $-B+I$ and B are symmetric, all eigenvalues are real. By Perron-Frobenius theorem, we have a positive eigenvalue λ' of $-B+I$ such that λ' is the maximum of eigenvalues of $-B+I$ and each entry of corresponding eigenvector u is positive. On the other hand, for each eigenvalue a of B there exists an eigenvalue b of $-B+I$ such that $a = 1-b$. Let λ be the negative eigenvalue of B. Then an easy calculation gives $\lambda = 1-\lambda'$. Therefore $\mathbb{R}u = \mathbb{R}o$. ∎

We fix $o \in V$ be the eigenvector corresponding to the negative eigenvalue of B whose euclidean norm equals to 1 and all coordinates are positive. Hence if we write o in a linear combination $o = \sum_{i=1}^n o_i\alpha_i$ of Δ then $o_i > 0$. Given $v \in V$, we define $|v|_1$ by $\sum_{i=1}^n o_i v_i$ if $v = \sum_{i=1}^n v_i\alpha_i$. Note that a function $|\cdot|_1 : V \to \mathbb{R}$ is actually a norm in the set of vectors having nonnegative coefficients. It is obvious that $|v|_1 > 0$ for $v \in V^+$ and $|v|_1 < 0$ for $v \in V^-$. Let $V_i = \{v \in V \mid |v|_1 = i\}$, where $i = 0, 1$. For $v \in V \setminus V_0$, we write $\widehat{v}$ for the "normalized" vector $\frac{v}{|v|_1} \in V_1$. We also call o the normalized eigenvector (corresponding to the negative eigenvalue of B). Also for a set $A \subset V \setminus V_0$, we write $\widehat{A}$ for the set of all $\widehat{a}$ with $a \in A$. We notice that $B(x,\alpha) = |\alpha|_1 B(x,\widehat{\alpha})$ hence the sign of $B(x,\alpha)$ equals to the sign of $B(x,\alpha)$ for any $x \in V$ and $\alpha \in \Delta$.

REMARK 1.3. It is known that the based root system allows us to define positive roots $\Phi^+ := \Phi \cap V^+$, and then $\Phi = \Phi^+ \sqcup (-\Phi^+)$ (see, for instance, [**3, 22**]). In other words, all roots are contained in $V^+ \cup V^-$ and hence $\Phi \cap V_0 = \emptyset$.

Then by Remark1.3, the set $\widehat{\Phi}$ is well-defined. Let E be the set of accumulation points of $\widehat{\Phi}$ with respect to the Euclidean topology.

It turns out that we only need to work on the case where B is irreducible. If the matrix B is reducible, then we can divide Δ into l subsets $\Delta = \sqcup_{i=1}^l \Delta_i$ so that each corresponding matrix $B_i = \{B(\alpha,\beta)\}_{\alpha,\beta\in\Delta_i}$ is irreducible and B is block diagonal $B = (B_1, \ldots, B_l)$. Then for any distinct i, j, if $\alpha \in \Delta_i$ and $\beta \in \Delta_j$, s_α and s_β commute. In this case we see that W is direct product

$$W = W_1 \times W_2 \times \cdots \times W_l,$$

where W_i is the Coxeter group corresponding to Δ_i. From this, the action of W can be regarded as a direct product of the actions of each W_i. Then for the set E of accumulation points of roots of W we see that $E = \sqcup_{i=1}^l E_i$, where E_i is the set of accumulation points of roots $W_i \cdot \Delta_i$ (see Proposition 2.14 in [**18**]). Moreover if B has the signature $(n-1,1)$, there exists a unique B_k which has the signature $(n_k-1,1)$ and others are positive definite. Since if the Coxeter matrix is positive definite then the corresponding Coxeter group W' is finite, and hence the limit set $\Lambda(W') = \emptyset$ (for the definition of the limit set, see Section 3.3). This ensures that $\Lambda(W)$ is distributed on $\mathrm{conv}(\widehat{\Delta_k})$, where $\mathrm{conv}(\widehat{\Delta_k})$ is the convex hull of $\widehat{\Delta_k}$. Thus $\Lambda(W) = \Lambda(W_k)$. Accordingly, if there exists the Cannon-Thurston map for W_k

then we also have the Cannon-Thurston map for the whole group W. This follows from the fact that the direct product $G_1 \times G_2$ of a finite generated infinite group G_1 and a finite group G_2 has the same Gromov boundary as the Gromov boundary of G_1.

We denote $q(v) = B(v, v)$ for $v \in V$. Let $Q = \{v \in V \mid q(v) = 0\}$, $Q_- = \{v \in V \mid q(v) < 0\}$ then we have

$$\widehat{Q} = V_1 \cap Q, \quad \widehat{Q_-} = V_1 \cap Q_-.$$

Since B is of type $(n-1, 1)$, $\widehat{Q}$ is an ellipsoid. The cone Q_- has two components the "positive side" $Q_-^+ = Q_- \cap V_+$ and the "negative side" $Q_-^- = Q_- \cap V_-$. Similarly we divide Q into two components Q^+ and Q^- so that $Q^+ = \partial Q_-^+$ and $Q^- = \partial Q_-^-$.

Remark 1.4. We have

$$W(V_0) \cap Q = \{\mathbf{0}\},$$

where $\mathbf{0}$ is the origin of $\mathbb{R}^n$. To see this we only need to verify that $V_0 \cap Q = \{\mathbf{0}\}$ since Q is invariant under B-reflections. We notice that $V_0 = \{v \in V \mid B(v, o) = 0\}$. For $i = 1, \ldots, n-1$, let p_i be an eigenvector of B corresponding to a positive eigenvalue λ_i. For any $v \in V_0$, we can express v in a linear combination $v = \sum_i^{n-1} v_i p_i$ since $B(v, o) = 0$. Then we have $B(v, v) = \sum_i^{n-1} \lambda_i v_i^2 \|p_i\|^2 \geq 0$ where $\| * \|$ denotes the euclidean norm. Since $\lambda_i > 0$ for $i = 1, \ldots, n-1$, we have $B(v, v) = 0$ if and only if $v = \mathbf{0}$.

Remark 1.5. It is easy to calculate that each Coxeter matrix arising from a Coxeter group of rank 3 is either positive type or has the signature $(2, 1)$ (cf. [**20**, Section 6.7]). However, for a general Coxeter group of rank n, there exists a bilinear form whose signature is neither positive type nor $(n-1, 1)$. See Example 2.21.

2. The normalized action

2.1. The normalized action of W. Our purpose is to investigate the set of accumulation points of roots $\widehat{\Phi}$ when W is infinite. Since $\widehat{\Phi}$ is defined by the normalization of the orbit of Δ, we need another action of W to analyze the asymptotic aspects. This leads us to consider the normalized action of W on a suitable subset of V including $\widehat{\Phi}$. For $v \in V \setminus V_0$, we defined $\widehat{v} = \frac{v}{|v|_1}$. Then the *normalized action* of $w \in W$ is given by

$$w \cdot v := \widehat{w(v)}, \quad v \in V \setminus W(V_0).$$

where $w(v)$ denotes the action of w as B-reflections. As we mentioned before $\Phi \cap V_0 = \emptyset$. Therefore W acts on Φ by the normalized action. Hohlweg, Labbé and Ripoll showed the following ([**18**, Theorem 2.7]):

Theorem 1.6. *Consider an injective sequence of positive roots $\{\rho_n\}_{n \in \mathbb{N}} \subset \Phi_+$, and suppose that $(\widehat{\rho_n})$ converges to a limit ℓ. Then:*

(i) *the norm $\|\rho_n\|$ tends to infinity (for any norm on V);*
(ii) *the limit ℓ lies in $\widehat{Q} = Q \cap V_1$.*

The claim (ii) in the theorem above says $E \subset \widehat{Q}$. As a consequence of this, the authors also have the following ([**18**, Corollary 2.9]):

Corollary 1.7. *The set of normalized roots of a Coxeter group has no accumulation points in itself, equivalently, $E \cap \widehat{\Phi} = \emptyset$.*

We remark that these two theorems hold for any infinite Coxeter group, i.e., there is no need to assume that the signature of B equals to $(n-1, 1)$.

After Chapeter 2 we consider another set as the target of the normalized action. Define an open set D in V_1 (with respect to the subspace topology) as

$$D = V_1 \cap Q_-.$$

Then by Remark 1.4, the normalized action is a continuous action on D and $\partial D = \widehat{Q}$. In Chapter 3 we will see that this turns out to be an isometric action on D equipped with the Hilbert metric. The region D can protrude from the convex hull $\mathrm{conv}(\widehat{\Delta})$ of $\widehat{\Delta}$. In the case where D is not completely included in the interior of $\mathrm{conv}(\widehat{\Delta})$, let $R := D \setminus \mathrm{conv}(\widehat{\Delta})$. Then we can restrict the normalized action on

$$D' := D \setminus \bigcup_{w \in W} w \cdot R,$$

since $\bigcup_{w \in W} w \cdot R$ is stable under the normalized action of W. We call this the *restricted* normalized action. We will see that D' is the convex hull of the set of accumulation points of roots (see Theorem 2.2), equivalently, the limit set (see Theorem 3.1).

2.2. The word metric. This thesis is devoted to make clear the connection between the geometry of the Coxeter groups themselves and their phase spaces. To see this, it needs to regard finitely generated groups as metric spaces.

Let G be a finitely generated group. Fixing a finite generating set S of G, all elements in G can be represented by a product of elements in S. We say such a representation to be a *word* and let $\langle S \rangle$ be the set of all words generated by S. For a word $w \in \langle S \rangle$ we define the *word length* $\ell_S(w)$ as the number of generators $s \in S$ in w. We denote the minimal word length of $g \in G$ by $|g|_S$. An expression of g realizing $|g|_S$ is called *the reduced expression* or *the geodesic word*. Using the word length, we can define so-called *the word metric* with respect to S on G, i.e. for $g, h \in G$, their distance is $|g^{-1}h|_S$. In this thesis for a Coxeter system (W, S) we always work on the generating set S. For this reason we omit the subscript and denote the word length (resp. the minimal word length) for S simply by ℓ (resp. $| * |$).

3. Fixed points of the normalized action

We first investigate the fixed points of the normalized action. In this section we use Lemma 2.12 and postpone the proof until the next section. We remark that it is proved independently.

Recall that any invertible linear map $f : V \longrightarrow V$ is Lipschitz continuous, i.e., there exists a constant C such that for any $x \in V$ we have

$$C^{-1}\|x\| \leq \|f(x)\| \leq C\|x\|.$$

Lemma 1.8. *For $w \in W$ of infinite order and $x \in \widehat{Q}$, let $\{w^{n_i} \cdot x\}_{n_i}$ be a converging subsequence of $\{w^n \cdot x\}_n$ to $y \in \widehat{Q}$. If $\|w^{n_i}(x)|_1| \longrightarrow \infty$, then for any $k \in \mathbb{Z}$ the sequence $\{w^{n_i+k} \cdot x\}_{n_i}$ also converges to y.*

Proof. Fix $k \in \mathbb{Z}$ arbitrarily. By the remark above, we have a constant $C_k \geq 1$, which depends only on k, such that for each $n \in \mathbb{N}$,

$$C_k{}^{-1}\|w^n(x)\| \leq \|w^{n+k}(x)\| \leq C_k\|w^n(x)\|.$$

Since $\| \cdot |_1|$ is comparable to $\| \cdot \|$ on Q, we have a constant $C \geq 1$ such that for any $z \in Q$,

$$C^{-1}\|z\| \leq \|z|_1| \leq C\|z\|.$$

Then we see that

$$|B(w^{n_i} \cdot x, w^{n_i+k} \cdot x)| = \frac{|B(w^{n_i}(x), w^{n_i+k}(x))|}{\|w^{n_i}(x)|_1| \cdot \|w^{n_i+k}(x)|_1|}$$
$$= \frac{|B(x, w^k(x))|}{\|w^{n_i}(x)|_1| \cdot \|w^{n_i+k}(x)|_1|} \leq C_k C^2 \frac{|B(x, w^k(x))|}{\|w^{n_i}(x)|_1|^2} \longrightarrow 0,$$

as $n_i \longrightarrow \infty$. By Proposition 2.12(a), we have the conclusion. ∎

Lemma 1.9. *For $w \in W$ of infinite order and $x \in \widehat{Q}$, if there exists a converging subsequence of $\{w^n \cdot x\}_n$ such that $\|w^{n_i}(x)|_1| \longrightarrow 0$, then $w \cdot x = x$.*

Proof. Suppose that $w \cdot x \neq x$, hence $B(x, w \cdot x) \neq 0$. From the argument in the first half of the proof of the previous lemma, we have constants C_1 and C such that for any $n \in \mathbb{N}$ and any $z \in Q$

$$C_1{}^{-1}\|w^n(x)\| \leq \|w^{n+1}(x)\| \leq C_1\|w^n(x)\|, \quad \text{and} \quad C^{-1}\|z\| \leq \|z|_1| \leq C\|z\|,$$

respectively. Since $\widehat{Q}$ is compact, there exist a converging subsequence $\{w^{n_{ij}+1} \cdot x\}_{n_{ij}}$ of $\{w^{n_i+1} \cdot x\}_{n_i}$. Then we have

$$|B(w^{n_{ij}} \cdot x, w^{n_{ij}+1} \cdot x)| = \frac{|B(w^{n_{ij}}(x), w^{n_{ij}+1}(x))|}{\|w^{n_{ij}}(x)|_1| \cdot \|w^{n_{ij}+1}(x)|_1|}$$
$$= \frac{|B(x, w(x))|}{\|w^{n_{ij}}(x)|_1| \cdot \|w^{n_{ij}+1}(x)|_1|} \geq \frac{|B(x, w(x))|}{C_1 C^2 \|w^{n_{ij}}(x)|_1|^2} \longrightarrow \infty,$$

as $n_{ij} \longrightarrow \infty$ by the assumption. However since $\widehat{Q} \times \widehat{Q}$ is compact again, there exists $\max_{z,z' \in \widehat{Q}} |B(z, z')| < \infty$. Consequently, all values of $|B(w^{n_{ij}} \cdot x, w^{n_{ij}+1} \cdot x)|$ should be bounded uniformly. This is a contradiction. ∎

For any $w \in W$ if there is a fixed point on $\widehat{Q}$ of the normalized action of w, then such a point is an eigenvector of w corresponding to a real eigenvalue.

Lemma 1.10. *Let w be an element in W. Suppose that w has distinct eigenvectors p, p' lying on $\widehat{Q}$, and let $\lambda, \lambda' \in \mathbb{R}$ be corresponding eigenvalues respectively. Then $\lambda\lambda' = 1$.*

Proof. We see this by calculating directly;

$$B(p, p') = B(w(p), w(p')) = \lambda\lambda' B(p, p').$$

Since p and p' are distinct and sitting on $\widehat{Q}$, we have $B(p, p') \neq 0$. Hence $\lambda\lambda' = 1$, as required. ∎

This lemma gives us the following observations about eigenvalues of $w \in W$ which are not ± 1 and corresponding eigenvectors are in $\widehat{Q}$:

- There are at most two such eigenvalues, hence w fixes at most two points in $\widehat{Q}$.
- The intersection of Q (not $\widehat{Q}$) and each eigenspace of such eigenvalue is one dimensional.
- If such eigenvalue exists, there are no eigenvectors in $\widehat{Q}$ corresponding to eigenvalues of ± 1.

Moreover we have the following corollary.

Corollary 1.11. *Let w be an element in W. Then w has an eigenvector $v \in \widehat{Q}$ corresponding to the eigenvalue $\lambda \neq \pm 1$ if and only if the normalized action of w has exactly two fixed points on $\widehat{Q}$.*

Proof. The "if" part is Lemma 1.10. We show the converse. By the first observation above, we see that the normalized action of w fixes one or two points. Assume that the normalized action of w fixes only one point v. Now as a linear transformation $w(v) = \lambda v$. We take $u \in \widehat{Q} \setminus \{v\}$ arbitrarily. Since w preserves B, we have $B(u, v) = B(w(u), w(v)) = \lambda B(w(u), v)$ and $B(u, v) = B(w^{-1}(u), w^{-1}(v)) = \lambda^{-1} B(w^{-1}(u), v)$. Hence we obtain

$$B\Big((\lambda - \lambda^{-1})v, w(u) - w^{-1}(u)\Big) = 0.$$

This means that there exists $\lambda'(\neq 0)$ such that

$$\lambda' v = w(u) - w^{-1}(u), \tag{1}$$

by Lemma 2.12(b). Since $w(u)$ and $w^{-1}(u)$ are in Q we have

$$q(w(u) - w^{-1}(u)) = -2B(w(u), w^{-1}(u)) = -2B(w^2(u), u).$$

On the other hand by (1)

$$q(w(u) - w^{-1}(u)) = \lambda'^2 q(v) = 0$$

since $v \in \widehat{Q}$. Consequently there exists $\lambda'' \neq 0$ such that $w^2(u) = \lambda'' u$. Thus the normalized action of w^2 fixes $\widehat{Q}$ pointwise because we took u arbitrarily. This contradicts to Lemma 1.10 (in particular the first observation above) since w^2 is of infinite order and $w^2(v) = \lambda^2 v$. ∎

Proposition 1.12. *Let $w \in W$ be of infinite order and take $x \in \widehat{Q}$ arbitrarily. If w has an eigenvector p in $\widehat{Q}$ corresponding to the eigenvalue $\lambda \neq \pm 1$, then $\{w^n \cdot x\}_n$ converges to a fixed point. In particular, p lies in E.*

Proof. Since $\widehat{Q}$ is compact, there exists a converging subsequence $\{w^{n_i} \cdot x\}_{n_i}$ of $\{w^n \cdot x\}_n$. Let y be the limit point of the sequence above. The value $\lambda \in \mathbb{R}$ always denotes an eigenvalue in the claim of this proposition, hence $\lambda \neq \pm 1$. In addition let $p \in \widehat{Q}$ be the normalized eigenvector corresponding to λ.

Notice that w has at most two eigenvectors in $\widehat{Q}$. In such a case, we denote the other eigenvector by p' and the corresponding eigenvalue by λ'. By Lemma 1.10, $\lambda\lambda' = 1$.

First, we consider the case of $|\lambda| < 1$. Suppose $y \neq p$. Then $0 < |B(y, p)| < \infty$ and we have

$$|B(w^{n_i} \cdot x, p)| = \frac{|B(w^{n_i}(x), p)|}{||w^{n_i}(x)|_1|} = \frac{|B(x, w^{-n_i}(p))|}{||w^{n_i}(x)|_1|} = \frac{|\lambda^{-n_i}||B(x, p)|}{||w^{n_i}(x)|_1|} \longrightarrow |B(y, p)|,$$

as $n_i \longrightarrow \infty$. Here the third equality comes from $w^{-1}(p) = w^{-1}(\lambda^{-1} w(p)) = \lambda^{-1} p$. Since $|\lambda^{-n_i}| \longrightarrow \infty$, we have $||w^{n_i}(x)|_1| \longrightarrow \infty$. Now there exists a constant $C \geq 1$ so that

$$C^{-1}||x|| \leq ||w(x)|| \leq C||x||,$$

which is independent of x. Therefore,

$$\begin{aligned} |B(w^{n_i} \cdot x, w \cdot y)| &= \frac{|B(w^{n_i}(x), w(y))|}{||w^{n_i}(x)|_1| \cdot ||w(y)|_1|} = \frac{|B(w^{n_i - 1}(x), y)|}{||w^{n_i}(x)|_1| \cdot ||w(y)|_1|} \\ &= \frac{C|B(w^{n_i - 1}(x), y)|}{||w^{n_i - 1}(x)|_1| \cdot ||w(y)|_1|} \longrightarrow 0, \end{aligned}$$

as $n_i \longrightarrow \infty$ by Lemma 1.8. This implies that $y = w \cdot y$. Applying this argument for all converging subsequence of $\{w^n \cdot x\}_n$, we deduce that $\{w^n \cdot x\}_n$ converges to y. By Lemma 1.10, we have following two possibilities:

1) If w has only one fixed point in $\widehat{Q}$ then it contradicts to Corollary 1.11, hence $y = p$;

2) if w has two fixed points p, p' in $\widehat{Q}$ then $y = p'$.

Thus we have the conclusion in this case.

Next, we consider the case of $|\lambda| > 1$. Suppose $y \neq p$. Then $0 < |B(y, p)| < \infty$ and we have

$$|B(w^{n_i} \cdot x, p)| = \frac{|B(w^{n_i}(x), p)|}{\|w^{n_i}(x)|_1|} = \frac{|B(x, w^{-n_i}(p))|}{\|w^{n_i}(x)|_1|} = \frac{|\lambda^{-n_i}||B(x, p)|}{\|w^{n_i}(x)|_1|} \longrightarrow |B(y, p)|,$$

as $n_i \longrightarrow \infty$. Since $|\lambda^{-n_i}| \longrightarrow 0$, we have $\|w^{n_i}(x)|_1| \longrightarrow 0$ as $n_i \longrightarrow \infty$. Applying Lemma 1.9, we see that x is a fixed point itself. In particular $w^n \cdot x = x$ for all $n \in \mathbb{N}$. Similar to the first case, by Lemma 1.10 we have two possibilities:

3) If w has only one fixed point in $\widehat{Q}$ then we have a contradiction, hence $y = p$:
4) If w has two fixed points p, p' in $\widehat{Q}$ then $y = p'$.

We also have the conclusion for this case. Here notice that if the case 4) happens then for any $x' \in \widehat{Q} \setminus \{x\}$, $\{w^n \cdot x'\}_n$ converges to p.

The last assertion in this proposition can be seen by taking x from E. ∎

Let $w \in W$ and $x \in \widehat{Q}$ be elements as in Proposition 1.12. Then we have a converging sequence $\{w^{n_i} \cdot x\}_{n_i}$ to $y \in \widehat{Q}$ so that $|w^{n_i}(x)|_1 \longrightarrow \infty$. In the case of $\lambda = \pm 1$, if $\{w^{n_i} \cdot x\}_{n_i}$ does not converge to the eigenvector p of w, then $B(w^{n_i} \cdot x, p) \longrightarrow B(y, p)$. This shows that if $\|w^{n_i}(x)|_1| \longrightarrow \infty$, then the sequence $\{w^{n_i} \cdot x\}_{n_i}$ converges to p since p is an eigenvector of w with eigenvalue ± 1.

Remark 1.13. In the case where W is rank 3, for $\delta_1, \delta_2 \in \Delta$,

- if $B(\delta_1, \delta_2) < -1$, then there exist two real eigenvalues of $s_{\delta_1} s_{\delta_2}$ which are different from ± 1;
- if $B(\delta_1, \delta_2) = -1$, then an easy calculation gives that $\|(s_{\delta_1} s_{\delta_2})^n(v)|_1| \longrightarrow \infty$ for any $v \in \widehat{Q} \setminus \{\frac{1}{2}\widehat{\delta_1} + \frac{1}{2}\widehat{\delta_2}\}$.

CHAPTER 2

Asymptotic behavior of roots

In this chapter we focus on the asymptotic behavior of roots of infinite Coxeter groups. More precisely our main interest is the distribution of accumulation points of roots. In the case of a finite Coxeter group, its root system Φ is finite. When a Coxeter group is of infinite, Φ is also infinite and the bilinear form is not positive definite. So the classical tools developed in the Euclidean geometry are no longer usable.

On the other hand, in a recent paper [**18**], some tools to deal with roots of infinite Coxeter groups were established as the first step of their study. Our motivation to organize this chapter is to contribute further studies of the paper [**18**].

Recall that E is the set of accumulation points of normalized roots $\widehat{\rho}$ for $\rho \in \Phi$, i.e., the set consisting of all possible limits of injective sequences of normalized roots. We denote the normalized action on $V_1 \setminus W(V_0)$ by $w \cdot x$ for $w \in W$ and $x \in V_1 \setminus W(V_0)$. Hohlweg, Labbé and Ripoll proved that $E \subset \widehat{Q}$ [**18**, Theorem 2.7] (see Theorem 1.6) and also proposed the following.

Conjecture 2.1 ([**18**, Section 3.2]). *Given an infinite Coxeter group W of rank n, the following assertions hold:*

(i) *When $\widehat{Q} \subset \mathrm{conv}(\widehat{\Delta})$, we have $E = \widehat{Q}$.*
(ii) *When $\widehat{Q} \not\subset \mathrm{conv}(\widehat{\Delta})$, we have $E = \widehat{Q} \setminus \left(\bigcup_{i=1}^{m} W \cdot R_i\right)$, where $R_1, \dots, R_m$ are connected components of $\widehat{Q}$ out of $\mathrm{conv}(\widehat{\Delta})$ with $m \leq n$.*

Remark that [**18**, Conjecture 3.9] is a stronger version of this conjecture.

In this chapter, we prove this conjecture for the case where the signature of the Coxeter matrix is $(n-1, 1)$. Here the number n is the rank of the corresponding Coxeter group.

Theorem 2.2. *For an infinite Coxeter group of rank n equipped with the signature $(n-1, 1)$ bilinear form, we have the following:*

(a) *When $\widehat{Q} \subset \mathrm{conv}(\widehat{\Delta})$, we have $E = \widehat{Q}$.*
(b) *When $\widehat{Q} \not\subset \mathrm{conv}(\widehat{\Delta})$, we have $E = \widehat{Q} \setminus \left(\bigcup_{i=1}^{m} W \cdot R_i\right)$, where $R_1, \dots, R_m$ are connected components of $\widehat{Q}$ out of $\mathrm{conv}(\widehat{\Delta})$ with $1 \leq m \leq n$.*

Moreover, we also prove the following theorem.

Theorem 2.3. *Fix $x \in E$. Then*

$$\overline{W \cdot x} = E.$$

Remark 2.4. In [**12**], Dyer, Hohlweg and Ripoll also proved Theorem 2.2 and Theorem 2.3 by a different approach ([**12**, Theorem 4.10 (a) and Theorem 3.1 (b)]). In fact, their approach was accomplished by using a method of so-called *imaginary cones* and they do not assume the linearly independence. On the other hand, in this thesis, some other aspects of infinite Coxeter groups (e.g. a metric on $\widehat{Q}$) are investigated.

Notation 2.5. From now on, we work in V_1 with the subspace topology unless otherwise indicated. For a subset $A \subset V_1$, $\mathrm{int}(A)$ means the interior of A and $\overline{A}$ denotes the closure of A with respect to the subspace topology of V_1. After this chapter, we always use these notations.

1. Metrics on $\widehat{Q}$

We have a natural metric on $\widehat{Q}$ defined by the bi-linear form B. The normalized action is actually "semi-contractions" on $\widehat{Q}$ for this metric.

1.1. A metric on $\widehat{Q}$. We define a metric on $\widehat{Q}$ by using the bi-linear form B.

Remark 2.6. Since B is positive-definite on V_0, $q(x-y)^{\frac{1}{2}}$ defines a metric on V_0 and hence V_1. Moreover, since $|q(x-y)| = 2|B(x,y)|$ for $x, y \in \widehat{Q}$, $|B(*,*)|^{1/2} : \widehat{Q} \times \widehat{Q} \to \mathbb{R}_{\geq 0}$ defines a metric on $\widehat{Q}$.

Let c be a piecewise C^∞ curve in $\widehat{Q}$ connecting x and y for $x, y \in \widehat{Q}$. The *length* $\ell_B(c)$ of c is defined by

$$\ell_B(c) = \sup_C \sum_{i=1}^{n} |B(x_{i-1}, x_i)|^{\frac{1}{2}},$$

where the infimum is taken over all chains $C = \{x = x_0, x_1, \ldots, x_n = y\}$ on c with unbounded n. Given $x, y \in \widehat{Q}$ with $x \neq y$, we define

$$d_B(x,y) = \inf\{\ell_B(c) \mid c \text{ is a piecewise } C^\infty \text{ curve joining } x \text{ and } y\}.$$

It is easy to verify that $d_B : \widehat{Q} \times \widehat{Q} \to \mathbb{R}_{\geq 0}$ is a pseudometric on $\widehat{Q}$. The following lemma guarantees that d_B is actually a metric.

Lemma 2.7. *For any $x, y \in \widehat{Q}$, if $d_B(x,y) = 0$, then $x = y$.*

Proof. When $d_B(x,y) = 0$, for any $\epsilon > 0$, there exists a curve c such that $\ell_B(c) < \epsilon$. This means that $\sum_{i=1}^{m} |B(x_{i-1}, x_i)|^{\frac{1}{2}} < \epsilon$ for any chain on c by the definition of ℓ_B. Thus one has $|B(x,y)|^{\frac{1}{2}} < \epsilon$. This implies $B(x,y) = 0$. Therefore, $x = y$ by Proposition 2.12. ∎

It is easy to see that $(\widehat{Q}, d_B)$ is homeomorphic to the metric space $\widehat{Q}$ equipped with the metric mentioned in Remark 2.6.

Let d_E be the metric on $\widehat{Q}$ defined by the same way as d_B using the Euclidean metric instead of B. In what follows, we prove that $(\widehat{Q}, d_E)$ is homeomorphic to $(\widehat{Q}, d_B)$. Now one see that two metrics d_B and d_E are comparable. In fact, $K = \sup_{x,y \in \widehat{Q}} \frac{|B(x,y)|^{\frac{1}{2}}}{\|x-y\|}$ is bounded, where $\|\cdot\|$ denotes the Euclidean norm, because of the following:

$$\begin{aligned}\sup_{x,y\in\widehat{Q}} \frac{\sqrt{2}|B(x,y)|^{\frac{1}{2}}}{\|x-y\|} &= \sup_{x,y\in\widehat{Q}} \frac{\sqrt{2}\left|\frac{1}{2}B(x-y, x-y)\right|^{\frac{1}{2}}}{\|x-y\|} \\ &\leq \sup_{v \in V_0} \frac{|B(v,v)|^{\frac{1}{2}}}{\|v\|} = \sup_{\substack{v\in V_0,\\ \|v\|=1}} |B(v,v)|^{\frac{1}{2}}.\end{aligned}$$

Since the region $\{v \in V_0 \mid \|v\| = 1\}$ is compact and the bilinear map $B(,)$ is continuous, there is $u \in V$ such that $|B(u,u)| = \sup_{v \in V_0, \|v\|=1} |B(v,v)| < \infty$. Then for an arbitrary curve c in $\widehat{Q}$ joining x and y and $\epsilon > 0$, there exists a chain $C = \{x = x_0, x_1, \ldots, x_m = y\}$ such that

$$\ell_B(c) - \epsilon \leq \sum_{i=1}^{m} |B(x_{i-1}, x_i)|^{\frac{1}{2}} \leq K \sum_{i=1}^{m} \|x_{i-1} - x_i\| \leq K\ell_E(c),$$

where ℓ_E denotes the length of c with respect to the Euclidean metric. Thus, for any $x, y \in \widehat{Q}$ and $\epsilon > 0$, we have $d_B(x,y) - \epsilon \leq K d_E(x,y)$. Hence $d_B(x,y) \leq K d_E(x,y)$. This implies the that comparability of d_B and the Euclidean metric.

Since $\widehat{Q}$ is an ellipsoid in $V \cong \mathbb{R}^n$, $\widehat{Q}$ is a C^∞ manifold and its topology induced from d_E coincides with the relative topology of V. Clearly, $\widehat{Q}$ is compact on the relative topology of V, hence $(\widehat{Q}, d_B)$ is also compact. Then by Hopf–Rinow Theorem (cf. [**15**, p. 9]), $(\widehat{Q}, d_B)$ is a geodesic space. Moreover, since each normalized B-reflection is a homeomorphism with respect to $(\widehat{Q}, d_E)$, W acts on $(\widehat{Q}, d_B) \cong (\widehat{Q}, d_E)$ continuously.

Remark 2.8. For $x \in \widehat{Q}$, its tangent space $T_x\widehat{Q}$ is equal to $\{y \in V_1 \mid B(x,y) = 0\}$. Moreover, it is immediate to see that B is positive definite on $T_x\widehat{Q}$. Thus, we conclude that $\widehat{Q}$ is a Riemannian manifold with a Riemannian metric B. This metric coincides with the metric d_B.

1.2. Visibility on $\widehat{Q}$. Let $[x,y]$ denote a geodesic in $(\widehat{Q}, d_B)$ between x and y, that is a curve joining x and y which attains $d_B(x,y)$. The compactness of $(\widehat{Q}, d_B)$ ensures that such a geodesic always exists. Let $L(x,y)$ (resp. $L[x,y]$) denote the Euclidean line in V through x and y (resp. the segment joining x and y). Using this, we define a notion given in [**18**]. We say that $x \in \mathrm{conv}(\widehat{\Delta}) \cap \widehat{Q}$ is *visible* from $\alpha \in V_1$ if $L[x,\alpha] \cap \widehat{Q} = \{x\}$. The set of all visible points of $\widehat{Q}$ from a normalized simple root $\widehat{\alpha}$ is said to be a *visible area* from $\widehat{\alpha}$, denoted by $V_{\widehat{\alpha}}$. Given $\alpha \in V_1$, we call a curve included in $V_{\widehat{\alpha}} \cap \widehat{Q}$ a *visible curve from* α. If there is no confusion, then we simply call it a visible curve.

We recall the following proposition.

Lemma 2.9 ([**18**, Proposition 3.7]). *Let $x \in \widehat{Q}$ and $\alpha \in \Delta$.*

(i) *$x \in V_{\widehat{\alpha}}$ if and only if $B(\alpha, x) \geq 0$.*
(ii) *x and $s_\alpha \cdot x$ lie on the same line $L(x, \widehat{\alpha})$.*
(iii) *$x \in \partial V_{\widehat{\alpha}}$ if and only if $B(\alpha, x) = 0$.*

Proposition 2.10. *For any $z \in \widehat{Q} \cap \mathrm{conv}(\widehat{\Delta})$ there exists a normalized root $\widehat{\alpha} \in \widehat{\Delta}$ such that z is visible from $\widehat{\alpha}$. In other words, $\widehat{Q} \cap \mathrm{conv}(\widehat{\Delta})$ is covered by $\{V_{\widehat{\alpha}} \mid \alpha \in \Delta\}$.*

Proof. Suppose, on the contrary, that there exists an element $z \in \widehat{Q}$ such that z is not visible from any normalized simple root.

Let $\widehat{\alpha}$ and $\widehat{\beta}$ be two distinct normalized simple roots. Then z is not visible from both $\widehat{\alpha}$ and $\widehat{\beta}$. Because of the convexity of $\{v \in V_1 \mid q(v) \leq 0\}$, z is not visible from any element lying in the segment $L[\alpha, \beta]$, either. Similarly, one can see that z is not visible from any element of $\partial(\mathrm{conv}(\widehat{\Delta}))$, where ∂A denotes the boundary of a set $A \subset V$. In particular, $z \notin \partial(\mathrm{conv}(\widehat{\Delta}))$.

Take a point $z' \in \{v \in \mathrm{conv}(\widehat{\Delta}) \setminus \partial(\mathrm{conv}(\widehat{\Delta})) \mid q(v) > 0\}$ such that z' is visible from z. Clearly, such a point should exist. Let us consider a ray $L[z,z') \subset L(z,z')$ starting at z through z'. Then the convexity of $\{v \in V_1 \mid q(v) \leq 0\}$ implies that $L[z,z') \cap \widehat{Q} = \{z\}$. This says that z is visible from some point of $\partial(\mathrm{conv}(\widehat{\Delta}))$. This is a contradiction. ∎

In the following, we prove some lemmas for the proof of Theorem 2.2.

Lemma 2.11. *Let α be a simple root. Then for any $x \in \widehat{Q}$, one has $B(\alpha, x) < \frac{1}{2|\alpha|_1}$.*

Proof. Suppose that $B(\alpha, x) \geq \frac{1}{2|\alpha|_1}$ for some $x \in \widehat{Q}$ and $\alpha \in \Delta$. Then we have $|s_\alpha(x)|_1 = 1 - 2B(\alpha, x)|\alpha|_1 \leq 0$. On the other hand, for $y \in \partial V_{\widehat{\alpha}}$, we have $B(\alpha, y) = 0$ hence $|s_\alpha(y)|_1 = 1 - 2B(y, \alpha)|\alpha|_1 = 1 > 0$. By the continuity of a linear map $B(\alpha, *)$, there should be $z \in \widehat{Q}$ such that $|s_\alpha(z)|_1 = 0$. However, as mentioned in Remark 1.4, $W(V_0) \cap \widehat{Q} = \emptyset$, which is a contradiction. ∎

Proposition 2.12. *For $x, y \in Q \setminus \{0\}$,*

(a) *one has $B(x, y) = 0$ if and only if $\widehat{x} = \widehat{y}$;*
(b) *if $\widehat{x} \neq \widehat{y}$, then one has $B(\widehat{x}, \widehat{y}) < 0$.*

Proof. (a) If $\widehat{x} = \widehat{y}$, then there exists $c \in \mathbb{R} \setminus \{0\}$ such that $x = cy$. By the definition of Q, it is obvious that $B(x, y) = 0$. For the inverse, we suppose that there exist x and y in $Q \setminus \{0\}$ with $B(x, y) = 0$. Then for any $a, b \in \mathbb{R}$, we have

$$q(ax + by) = a^2 q(x) + 2abB(x, y) + b^2 q(y) = 0.$$

This implies that Q includes a Euclidean segment joining x and y, namely, $v(t) = (1-t)x+ty$ lies in Q for any $t \in [0, 1]$. However since $\widehat{Q}$ is strictly convex, for any two vectors $x, y \in Q$, a segment joining them intersects Q only at x and y. This is a contradiction.

(b) Given $x \in Q \setminus \{0\}$, we may assume that $x \in Q^+$ without loss of generality. It suffice to show $B(\widehat{x}, \widehat{y}) < 0$ in the case $y \in Q^+$ with $\widehat{x} \neq \widehat{y}$. Remark that Q^+_- is strictly convex. Let $A_+ = \{v \in V \mid B(x, v) > 0\}$ and $A_- = \{v \in V \mid B(x, v) < 0\}$. Then $V \setminus \{v \in V \mid B(x, v) = 0\} = A_+ \sqcup A_-$. Moreover, the connectedness of Q and the claim (a) of this proposition imply that either $Q^+ \setminus \mathbb{R}_{>0}x \subset A_+$ or $Q^+ \setminus \mathbb{R}_{>0}x \subset A_-$ is satisfied. In particular the hyperplane $\partial A_+ = \partial A_- = \{v \in V \mid B(x, v) = 0\}$ is tangent to Q_+. We show that the latter case only happen. Now we have two cases as follows.

We first assume that there exists a simple root $\alpha \in \Delta$ such that $\widehat{x}$ is visible from $\widehat{\alpha}$ and $s_\alpha(x) \neq x$. The latter condition gives $\alpha \notin \{v \in V \mid B(x, v) = 0\}$. Consider the Euclidean segment $L[\widehat{\alpha}, \widehat{x}]$ connecting x and $\widehat{\alpha}$. The visibility of x from $\widehat{\alpha}$ implies that $L[\widehat{\alpha}, \widehat{x}] \cap Q = \{x\}$ hence the convexity of Q^+_- guarantees that $\widehat{\alpha}$ is contained in the different component from $\widehat{Q} \setminus \widehat{x}$. Since $\widehat{\alpha} \in A_+$ by Lemma 2.9, we conclude that $y \in A_-$ for any $y \in Q^+$ if $\widehat{x} \neq \widehat{y}$. Since $|x|_1 > 0$ and $|y|_1 > 0$, we have $B(\widehat{x}, \widehat{y}) < 0$.

Consider the other case, namely, there is no simple root $\alpha \in \Delta$ such that $\widehat{x}$ is visible from $\widehat{\alpha}$ and $s_\alpha(x) \neq x$. In this case there must exist at least one simple root $\beta \in \Delta$ such that $B(\beta, x) < 0$. Actually the assumption implies that $B(x, \delta) \leq 0$ for any $\delta \in \Delta$. If $B(x, \delta) = 0$ for all $\delta \in \Delta$ then x should be 0 since each eigenvalue of B is not zero which is a contradiction. Now we get $B(s_\beta(x), x) < 0$ by the condition $B(\beta, x) < 0$. Thus we have $Q^+ \setminus \mathbb{R}_{>0}x \subset A_-$. ∎

We notice that for any $x \in \widehat{Q}$ and $y \in D$ we have $B(x, y) < 0$.

Proposition 2.13. *Let $\alpha \in \Delta$ and $x, y \in V_{\widehat{\alpha}}$.*

(i) *Each geodesic between x and y is contained in $V_{\widehat{\alpha}}$.*
(ii) *For any visible curve from $\widehat{\alpha}$, we have*

$$\ell_B(c) \leq \ell_B(s_\alpha \cdot c). \tag{2}$$

Moreover, $d_B(x, y) \leq d_B(s_\alpha \cdot x, s_\alpha \cdot y)$.

(iii) *The equality* (2) *holds if and only if $c \subset \partial V_\alpha$.*

Proof. For the proofs of (i) and (ii), we show that for any curve c in V_α and $x, y \in c$, one has $|B(x, y)| \leq |B(s_\alpha \cdot x, s_\alpha \cdot y)|$.

From Remark 1.4 and Lemma 2.11, one has $0 \leq B(x, \alpha), B(y, \alpha) < \frac{1}{2|\alpha|_1}$. Thus the inequality $0 < |s_\alpha(x)|_1 = |x - 2B(x, \alpha)|_1 \leq 1$ holds. Similarly, $0 < |s_\alpha(y)|_1 \leq 1$. Hence

$$|B(s_\alpha \cdot x, s_\alpha \cdot y)| = \frac{|B(s_\alpha(x), s_\alpha(y))|}{|s_\alpha(x)|_1 |s_\alpha(y)|_1} = \frac{|B(x, y)|}{|s_\alpha(x)|_1 |s_\alpha(y)|_1} \geq |B(x, y)|.$$

Let c' be a geodesic joining $s_\alpha \cdot x$ and $s_\alpha \cdot y$ and $V_{\widehat{\alpha}}$ the visible area from $\widehat{\alpha}$. Let us decompose c' into

$$c' = \bigcup_{i \in I} c_i \cup \bigcup_{j \in J} c_j,$$

where c_i ($i \in I$) is a visible curve from $\widehat{\alpha}$ and c_j ($j \in J$) the others. Set $c'' = \bigcup_{i \in I} c_i \cup \bigcup_{j \in J} s_\alpha \cdot c_j$. Then c'' is a curve joining x and y because each point of the boundary of the visible area from $\widehat{\alpha}$ is fixed by s_α. By the above arguments, we obtain

$$d_B(s_\alpha \cdot x, s_\alpha \cdot y) = \ell_B(c') \geq \ell_B(c'') \geq d_B(x, y).$$

This says that each geodesic between x and y is contained in $V_{\widehat{\alpha}}$.

Next, we prove (iii). If $c \subset \partial V_{\widehat{\alpha}}$, since $B(\alpha, x) = 0$ for any $x \in c$, the equality of (2) directly follows. Assume that $\ell_B(c) = \ell_B(s_\alpha \cdot c)$. Then for an arbitrary curve $c' \subset c$, we also have $\ell_B(c') = \ell_B(s_\alpha \cdot c')$. Decompose c into k curves for an arbitrary fixed $k \in \mathbb{Z}_{>0}$. Let c_1 be one component of such curves. For $\epsilon > 0$, by the definition of ℓ_B, there exists a chain $\{x_1, \ldots, x_m\}$, where x_1 and x_m are the endpoints of c_1, such that

$$\sum_{i=1}^{m} |B(x_{i-1}, x_i)|^{\frac{1}{2}} \geq (1-\epsilon)^{\frac{1}{2}} \ell_B(c_1).$$

Since $\ell_B(c_1) = \ell_B(s_\alpha \cdot c_1)$, one has

$$(1-\epsilon)^{\frac{1}{2}} \ell_B(c_1) = (1-\epsilon)^{\frac{1}{2}} \ell_B(s_\alpha \cdot c_1) = (1-\epsilon)^{\frac{1}{2}} \sum_{i=1}^{m} |B(s_\alpha \cdot x_{i-1}, s_\alpha \cdot x_i)|^{\frac{1}{2}}.$$

Hence,

$$\frac{\sum_{i=1}^{m} |B(x_{i-1}, x_i)|^{\frac{1}{2}}}{\sum_{i=1}^{m} |B(s_\alpha \cdot x_{i-1}, s_\alpha \cdot x_i)|^{\frac{1}{2}}} \geq (1-\epsilon)^{\frac{1}{2}}.$$

Now, in general, for positive real numbers $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$, we see that

$$\max_{i \in \{1, \ldots, m\}} \frac{a_i}{b_i} \geq \frac{\sum_{i=1}^{m} a_i}{\sum_{i=1}^{m} b_i}.$$

Thus, there exists some i such that

$$\begin{aligned} \frac{|B(x_{i-1}, x_i)|^{\frac{1}{2}}}{|B(s_\alpha \cdot x_{i-1}, s_\alpha \cdot x_i)|^{\frac{1}{2}}} \geq (1-\epsilon)^{\frac{1}{2}} &\iff \frac{|B(x_{i-1}, x_i)|}{|B(s_\alpha \cdot x_{i-1}, s_\alpha \cdot x_i)|} \geq 1 - \epsilon \\ &\iff (1 - 2B(x_{i-1}, \alpha)|\alpha|_1)(1 - 2B(x_i, \alpha)|\alpha|_1) \geq 1 - \epsilon. \end{aligned}$$

On the other hand, since $x_{i-1}, x_i \in V_{\widehat{\alpha}}$, one has $1 - 2B(x_{i-1}, \alpha)|\alpha|_1 \leq 1$ and $1 - 2B(x_i, \alpha)|\alpha|_1 \leq 1$. Hence $1 - \epsilon \leq 1 - 2B(x_{i-1}, \alpha)|\alpha|_1 \leq 1$ and $1 - \epsilon \leq 1 - 2B(x_i, \alpha)|\alpha|_1 \leq 1$. Since ϵ is arbitrary, by taking ϵ as $\epsilon \to 0$, we see that x_{i-1} and x_i belong to $\partial V_{\widehat{\alpha}}$. Moreover, since k is also arbitrary, by taking k as $k \to \infty$, we conclude that $c \subset \partial V_{\widehat{\alpha}}$, as desired. ∎

2. The orbit of points in E

In this section, we prove Theorem 2.3. First, we note that $x \in \widehat{Q}$ is fixed by the normalized action of s_α for $\alpha \in \Delta$ if and only if $B(x, \alpha) = 0$.

Lemma 2.14. *Let $K \subset \widehat{Q}$ be a nonempty W-invariant subset of $\widehat{Q}$. Then for each $\alpha \in \Delta$, there is $x_\alpha \in K$ such that $x_\alpha \neq s_\alpha \cdot x_\alpha$.*

Proof. Since, in general, a point $x \in \widehat{Q}$ which is fixed by every normalized action of s_α corresponds to an eigenvector of B whose eigenvalue is 0, there is no such a point when B is definite, in particular, B has the signature $(n-1, 1)$. Thus there is no element in $\widehat{Q}$ which is fixed by every s_α with $\alpha \in \Delta$. Hence K contains x with $x \neq s_{\alpha_0} \cdot x$ for some $\alpha_0 \in \Delta$.

Fix $\alpha \in \Delta$ arbitrarily. Since we assume that W is irreducible, the Coxeter graph associated with W is connected (cf. [**20**, Section 2.2]). Hence there is a path from α_0 to α in the Coxeter graph, that is to say, there is a sequence of simple roots $(\alpha_0, \alpha_1, \ldots, \alpha_k)$ such that $\alpha_k = \alpha$ and $B(\alpha_{i-1}, \alpha_i) \neq 0$ for $i = 1, \ldots, k$, where k is some positive integer.

For $i = 0$, by the above discussions, there is a point $x_0 \in K$ such that x_0 is not fixed by s_{α_0}, i.e., $x_0 \neq s_{\alpha_0} \cdot x_0$. For $i = 1$, we have $B(\alpha_0, \alpha_1) \neq 0$. On the other hand, since

$$\begin{aligned} B(s_{\alpha_0} \cdot x_0, \alpha_1) &= \frac{1}{1 - 2B(x_0, \alpha_0)|\alpha_0|_1} B(x_0 - 2B(x_0, \alpha_0), \alpha_1) \\ &= \frac{1}{1 - 2B(x_0, \alpha_0)|\alpha_0|_1} (B(x_0, \alpha_1) - 2B(x_0, \alpha_0)B(\alpha_0, \alpha_1)), \end{aligned}$$

if $B(x_0, \alpha_1) = 0$, then $B(s_{\alpha_0} \cdot x_0, \alpha_1) \neq 0$ because $B(x_0, \alpha_0) \neq 0$ and $B(\alpha_0, \alpha_1) \neq 0$. Hence either $B(x_0, \alpha_1)$ or $B(s_{\alpha_0} \cdot x_0, \alpha_1)$ is nonzero. This means that either x_0 or $s_{\alpha_0} \cdot x_0$ is not fixed by s_{α_1}. Let x_1 be such a point. Since $x_0 \in K$ and K is W-invariant, we know that $s_{\alpha_0} \cdot x_0 \in K$. Similarly, we obtain that either x_1 or $s_{\alpha_1} \cdot x_1$ is not fixed by s_{α_2}. Let x_2 be such a point. By repeating this procedure, we eventually obtain $x_k \in K$ such that x_k is not fixed by s_α, as required. ∎

The following proposition plays a crucial role in the proof of Theorem 2.3.

Proposition 2.15. *The set E of accumulation points of normalized roots is a minimal W-invariant closed set in $\widehat{Q}$, namely, any W-invariant closed set in $\widehat{Q}$ includes E.*

Proof. Let $K \subset \widehat{Q}$ be a W-invariant closed subset. We may show that $E \subset K$. Recall that $L(x, y)$ for $x, y \in V$ denotes the Euclidean line through x and y. Let

$$\widetilde{K} = \bigcup_{x,y \in K} L(x, y) \setminus W(V_0).$$

We see that $\widetilde{K}$ is also a W-invariant.

For each $\alpha \in \Delta$, when we take $x \in K$ with $x \neq s_\alpha \cdot x$, $L(x, y)$ intersects with $\widehat{\alpha}$. Since x and $s_\alpha \cdot x$ belong to $\widetilde{K}$, $\widehat{\alpha}$ also belongs to $\widetilde{K}$. By Lemma 2.14, we can take such an element of K for every $\alpha \in \Delta$. Hence $\widehat{\Delta} \subset \widetilde{K}$. Since $\widetilde{K}$ is W-invariant, we also have $W \cdot \widehat{\Delta} \subset \widetilde{K}$. Now consider the set $K' = \widetilde{K} \cap \operatorname{conv}(\widehat{\Delta})$. Then K' includes $\widehat{\Phi}$ since $\widehat{\Phi} \subset \operatorname{conv}(\widehat{\Delta})$. Moreover K' is closed in $\operatorname{conv}(\widehat{\Delta}) \setminus W(V_0)$ since K is closed. Thus the accumulation points of $W \cdot \widehat{\Delta}$ (which is nothing but E) should be contained in $K' \cap \widehat{Q}$ since $E \subset \widehat{Q}$. Therefore we have $E \subset K$ since $K' \cap \widehat{Q} \subset K$. ∎

As a consequence of Proposition 2.15, we can prove Theorem 2.3. In fact, for any $x \in E$, it is obvious that $\overline{W \cdot x} \subset E$. Moreover, since $\overline{W \cdot x}$ is W-invariant closed set, from Proposition 2.15, we also have $E \subset \overline{W \cdot x}$, as desired. ∎

3. A proof of Theorem A : the case of rank 3

Before the general case, we first prove Theorem 2.2 for the case of rank 3. Let (W, S) be a Coxeter system of rank 3 with $S = \{s_\alpha, s_\beta, s_\gamma\}$ and $\Delta = \{\alpha, \beta, \gamma\}$ its simple system. For the proof of the case of rank 3, we consider the following two cases:

(a) $-1 \leq B(\alpha, \beta) \leq 0$, $-1 \leq B(\beta, \gamma) \leq 0$ and $-1 \leq B(\alpha, \gamma) \leq 0$;

(b) other cases, i.e., there are distinct δ and δ' in Δ such that $B(\delta, \delta') < -1$.

3.1. The case (a). First, we concentrate on the case where $(s_\alpha s_\beta)^m = 1, (s_\beta s_\gamma)^n = 1, (s_\alpha s_\gamma)^k = 1$ with $m, n, k \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ and $B(\delta, \delta') = -1$ if the order of $s_\delta s_{\delta'}$ is infinity, where $\delta, \delta' \in \Delta$.

Let ρ be an arbitrary accumulation point of normalized roots and

$$\mathcal{R}_\rho = \overline{\{w \cdot \rho \mid w \in W\}} \subset E$$

the closure of a set of limit points. Proving $\mathcal{R}_\rho = \widehat{Q}$ leads us to the desired conclusion $E = \widehat{Q}$ since $\mathcal{R}_\rho \subset E \subset \widehat{Q}$.

First, we prove that there exist at least three points $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}$ in $\mathcal{R}_\rho$ satisfying the following three conditions:

$$\begin{cases} \widetilde{\alpha} \text{ is visible from both } \widehat{\beta} \text{ and } \widehat{\gamma}; \\ \widetilde{\beta} \text{ is visible from both } \widehat{\gamma} \text{ and } \widehat{\alpha}; \\ \widetilde{\gamma} \text{ is visible from both } \widehat{\alpha} \text{ and } \widehat{\beta}. \end{cases}$$

Let $\{\delta_1, \delta_2, \delta_3\} = \Delta$.

- When the order of $s_{\delta_1} s_{\delta_2}$ is infinite, we may set $\widetilde{\delta_3} = \lim_{n\to\infty}(s_{\delta_1} s_{\delta_2})^n \cdot \rho$. Then $\widetilde{\delta_3} \in \mathcal{R}_\rho$ and $\widetilde{\delta_3} = \frac{1}{2}\widehat{\delta_1} + \frac{1}{2}\widehat{\delta_2}$ by Remark 1.13. Moreover, it is easy to check that $\widetilde{\delta_3}$ is visible from both $\widehat{\delta_1}$ and $\widehat{\delta_2}$.
- When the order of $s_{\delta_1} s_{\delta_2}$ is finite, say, $(s_\alpha s_\beta)^m = 1$ with $m < \infty$, the order of the parabolic subgroup W' generated by s_α and s_β is $2m$. Let $T = W' \cdot \rho$ and let $T_\alpha \subset T$ (resp. $T_\beta \subset T$) be the set of the points in T which are visible from $\widehat{\alpha}$ (resp. $\widehat{\beta}$). Suppose that there does not exist $\widetilde{\gamma}$, i.e., $T_\alpha \cap T_\beta = \emptyset$. Then T_α and T_β have the same cardinality m. In fact, since $s_\beta : T_\alpha \to T_\beta$ is well-defined by $T_\alpha \cap T_\beta = \emptyset$ and this is injective, one has $|T_\alpha| \leq |T_\beta|$. Similarly, $|T_\beta| \leq |T_\alpha|$. Hence $|T_\alpha| = |T_\beta|$, denoted by m'. Moreover, since $s_\alpha : T \setminus T_\alpha \to T_\alpha$ is injective, one has $|T_\alpha| \geq |T \setminus T_\alpha| = |T| - |T_\alpha|$. Thus, $|T| \leq 2|T_\alpha| = |T_\alpha| + |T_\beta| \leq |T|$. Hence $T \setminus (T_\alpha \cup T_\beta) = \emptyset$. We write $T_\alpha = \{\rho_1, \ldots, \rho_{m'}\}$ and $T_\beta = \{\rho'_1, \ldots, \rho'_{m'}\}$. Observe that s_α and s_β act on $\{1, \ldots, m'\}$ as permutations $\sigma_\alpha, \sigma_\beta : \{1, \ldots, m'\} \longrightarrow \{1, \ldots, m'\}$ so that $s_\beta \cdot T_\alpha = \{\rho'_{\sigma_\alpha(1)}, \ldots, \rho'_{\sigma_\alpha(m')}\}$ and $s_\alpha \cdot T_\beta = \{\rho_{\sigma_\beta(1)}, \ldots, \rho_{\sigma_\beta(m')}\}$. In particular, we recognize that each image of the points in T_β by s_α must be visible from α and vice versa. Moreover, the permutation $\sigma_{\alpha\beta} := \sigma_\alpha \sigma_\beta$ has order m'. By Proposition 2.13 (i), each B-reflection extends the length of visible curves. So we see that
$$\begin{aligned} d_B(\rho_i, \rho_j) &\leq d_B(\rho_{\sigma_{\alpha\beta}(i)}, \rho_{\sigma_{\alpha\beta}(j)}) \\ &\leq d_B(\rho_{\sigma^2_{\alpha\beta}(i)}, \rho_{\sigma^2_{\alpha\beta}(j)}) \\ &\leq \cdots \\ &\leq d_B(\rho_{\sigma^{m'}_{\alpha\beta}(i)}, \rho_{\sigma^{m'}_{\alpha\beta}(j)}) = d_B(\rho_i, \rho_j), \end{aligned}$$
for any $i, j \in \{1, 2, \ldots, m'\}$. By virtue of Proposition 2.13 (iii), one has $\rho_i = \rho_j$. Thus each of T_α and T_β consists of one element. Let $T_\alpha = \{\rho_1\}$ and $T_\beta = \{\rho'_1\}$. Then four points α, ρ_1, ρ'_1 and β should lie in the same line $L(\alpha, \beta)$. On the other hand, since W' is finite, $L(\alpha, \beta)$ does not intersect with $\widehat{Q}$, a contradiction. Hence $T_\alpha \cap T_\beta$ is not empty. This means that $\widetilde{\gamma}$ exists in $W' \cdot \rho$.

Next, we prove the desired assertion. Suppose, on the contrary, that $\mathcal{R}_\rho \subsetneq \widehat{Q}$. Equivalentry, there exists a geodesic $[a_1, b_1]$ in $\widehat{Q}$ such that no element of $\mathcal{R}_\rho$ is contained in the open interval (a_1, b_1) but $a_1, b_1 \in \mathcal{R}_\rho$. By Proposition 2.10, $\widehat{Q}$ is covered with three visible areas. One of the following holds: $[a_1, b_1] \subset [\widetilde{\alpha}, \widetilde{\beta}]$ or $[a_1, b_1] \subset [\widetilde{\beta}, \widetilde{\gamma}]$ or $[a_1, b_1] \subset [\widetilde{\gamma}, \widetilde{\alpha}]$.

Let, say, $[a_1, b_1] \subset [\widetilde{\alpha}, \widetilde{\beta}]$. Let $a_2 = s_\gamma \cdot a_1$ and $b_2 = s_\gamma \cdot b_1$. Then either $a_2 \in [\widetilde{\alpha}, \widetilde{\gamma}]$ or $a_2 \in [\widetilde{\beta}, \widetilde{\gamma}]$, and so is b_2. Moreover, if $\widetilde{\gamma} \in [a_2, b_2]$, then $s_\gamma \cdot \widetilde{\gamma} \in [a_1, b_1]$, a contradiction. Thus, $\widetilde{\gamma} \notin [a_2, b_2]$. In particular, $[a_2, b_2] \subset [\widetilde{\alpha}, \widetilde{\gamma}]$ or $[a_2, b_2] \subset [\widetilde{\beta}, \widetilde{\gamma}]$ occurs, say, $[a_2, b_2] \subset [\widetilde{\alpha}, \widetilde{\gamma}]$. From Proposition 2.13 (i), we notice that $d_B(a_1, b_1) \leq d_B(a_2, b_2)$.

Similarly, for each $n \geq 1$, if $[a_n, b_n] \subset [\widetilde{\delta_1}, \widetilde{\delta_2}]$, then we set $a_{n+1} = s_{\delta_3} \cdot a_n$ and $b_{n+1} = s_{\delta_3} \cdot b_n$. Moreover, we also have $d_B(a_n, b_n) \leq d_B(a_{n+1}, b_{n+1})$. In addition, $[a_n, b_n] \cap \mathcal{R}_\rho = \{a_n, b_n\}$ and $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ for all i and j with $i \neq j$. Now since $(\widehat{Q}, d_B)$ is compact, $\widehat{Q}$ has a bounded ℓ_B length. Hence there exists a sufficiently large N such that

$$d_B(a_N, b_N) = d_B(a_{N+1}, b_{N+1}) = \cdots .$$

By Proposition 2.13 (iii), one has $a_N = b_N$. Since $d_B(a_N, b_N) = 0$ if $a_N = b_N$, this never happens, we have a contradiction.

Therefore, we conclude that $\mathcal{R}_\rho = \widehat{Q}$, as required. ∎

3.2. The case (b). Next, we consider the case where there exist δ and δ' in Δ such that $B(\delta, \delta') < -1$. In this case, we may assume that one of the following three situations happens:

- Assume that $B(\alpha, \beta) < -1$, $B(\alpha, \gamma) < -1$ and $B(\beta, \gamma) < -1$. Then $\widehat{Q} \cap \text{conv}(\widehat{\Delta})$ consists of three visible arcs and each pair of these arcs has no common point. Let $R = \widehat{Q} \setminus \text{conv}(\widehat{\Delta})$. Moreover, the six endpoints of such three visible arcs belong to E. If we suppose that there exists an arc on $\widehat{Q}$ which is in $\widehat{Q} \setminus (E \cup W \cdot R)$, then we have a contradiction in the same manner as the case (a) by using the six endpoints instead of $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}$.
- Assume that $B(\alpha, \beta) < -1$, $B(\alpha, \gamma) < -1$ and $B(\beta, \gamma) \geq -1$. Then $\widehat{Q} \cap \text{conv}(\widehat{\Delta})$ consists of two arcs c and c' and these arcs have no common point. One arc c is a visible arc from α and the other c' is covered by $V_{\widehat{\beta}}$ and $V_{\widehat{\gamma}}$. Moreover, the four endpoints of such two arcs belong to E. In addition, the endpoints of c are not visible from $\widehat{\beta}$ and $\widehat{\gamma}$. Hence, by the arguments appearing in the proof of the case (a), we have a point of E which is visible from both $\widehat{\beta}$ and $\widehat{\gamma}$ in c'. Similarly, if we suppose that there exists an arc on $\widehat{Q}$ which is in $\widehat{Q} \setminus (E \cup W \cdot R)$, then we have a contradiction by using such five points.
- Assume that $B(\alpha, \beta) < -1$, $B(\alpha, \gamma) \geq -1$ and $B(\beta, \gamma) \geq -1$. Then $\widehat{Q} \cap \text{conv}(\widehat{\Delta})$ consists of an arc which is covered by $V_{\widehat{\alpha}}$, $V_{\widehat{\beta}}$ and $V_{\widehat{\gamma}}$. Moreover, the two endpoints of such arc belong to E. In addition, we have two points of E, one of which is visible from both $\widehat{\alpha}$ and $\widehat{\gamma}$ and the other is visible from both $\widehat{\beta}$ and $\widehat{\gamma}$. Similarly, if we suppose that there exists an arc on $\widehat{Q}$ which is in $\widehat{Q} \setminus (E \cup W \cdot R)$, then we have a contradiction by using such four points.

Finally, by Remark 1.13, each of the endpoints of visible arcs can be realized as an accumulation point of $W' \cdot \rho$, where W' is a certain parabolic subgroup of W and ρ is an arbitrary accumulation point of normalized roots. ∎

4. A proof of Theorem A : the case of an arbitrary rank

Finally, in this section, we prove Theorem 2.2 for the case of an arbitrary rank.

We devide the following two cases:

(a) $\widehat{Q} \subset \text{int}(\text{conv}(\widehat{\Delta}))$;

(b) $\widehat{Q} \not\subset \mathrm{int}(\mathrm{conv}(\widehat{\Delta}))$.

Here $\mathrm{int}(\cdot)$ denotes the interior. Since $|q(x-y)|^{\frac{1}{2}}$ is a metric on $\widehat{Q}$ (Remark 2.6), for the proof of Theorem 2.2, we estimate $|B(x,y)|$ for $x, y \in \widehat{Q}$ in this section.

4.1. The case (a).

LEMMA 2.16. *There exists a constant $C' > 0$ such that for any $x \in \widehat{Q}$, one has $B(x,\alpha) \geq C'$ for some $\alpha \in \Delta$.*

PROOF. By Proposition 2.10, $\widehat{Q}$ is covered by $\{V_{\widehat{\alpha}} \mid \alpha \in \Delta\}$. Note that $V_{\widehat{\alpha}}$ is a closed set. Since $\widehat{Q} \subset \mathrm{int}(\mathrm{conv}(\widehat{\Delta}))$, for any $y = \sum_{i=1}^n y_i\widehat{\alpha_i} \in \widehat{Q}$, one has $y_i > 0$.

Suppose that there is $x \in \widehat{Q}$ such that $x \notin \bigcup_{\alpha\in\Delta} \mathrm{int}(V_{\widehat{\alpha}})$. This means from Proposition 2.10 that x should belong to $\bigcap_{i=1}^k \partial V_{\widehat{\alpha_{q_i}}}$ for some $\alpha_{q_1}, \ldots, \alpha_{q_k} \in \Delta$, where $k < n$ by Lemma 2.14. Hence $B(x, \alpha_{q_i}) = 0$ for $i = 1, \ldots, k$. Moreover, $B(x, \alpha') < 0$, where $\alpha' \in \Delta \setminus \{\alpha_{q_1}, \ldots, \alpha_{q_k}\}$. On the other hand, when x can be written as $x = \sum_{i=1}^n x_i\widehat{\alpha_i}$, one has $B(x,x) = 0$ from $x \in Q$, while by $x \notin \bigcup_{\alpha\in\Delta} \mathrm{int}(V_{\widehat{\alpha}})$, one has $B(\widehat{\alpha}, x) = \sum_{i=1}^n x_i B(\widehat{\alpha}, \widehat{\alpha_i}) < 0$ for each $\alpha \in \Delta$, thus we have

$$B(x,x) = \sum_{1\leq i,j\leq n} x_i x_j B(\widehat{\alpha_i}, \widehat{\alpha_j}) = \sum_{i\in I} x_i \sum_{j=1}^n x_j B(\widehat{\alpha_i}, \widehat{\alpha_j}) < 0,$$

where $I = \{1, \ldots, n\} \setminus \{q_1, \ldots, q_k\}$, which is a contradiction.

Hence $x \in \mathrm{int}(V_{\widehat{\alpha}})$ for some $\alpha \in \Delta$. Since $B(x,\alpha) > 0$ for each $x \in \mathrm{int}(V_{\widehat{\alpha}})$, we obtain

$$\min_{x\in\widehat{Q}} \max_{\alpha\in\Delta}\{B(x,\alpha)\} > 0.$$

If we set $C' = \min_{x\in\widehat{Q}} \max_{\alpha\in\Delta}\{B(x,\alpha)\}$, then the assertion holds. ∎

Remark that the constant C' appearing above depends only on B.

In the sequel, we fix $C = C' - \epsilon$ for a sufficiently small $\epsilon > 0$. For each $\alpha \in \Delta$, let $U_\alpha = \{v \in \widehat{Q} \mid B(\alpha, v) > C\}$. In particular, by Lemma 2.9, one has $U_\alpha \subset \mathrm{int}(V_\alpha)$. Thus one can rephrase Lemma 2.16 as follows.

COROLLARY 2.17. *The family of the regions $\{U_\alpha \mid \alpha \in \Delta\}$ covers $\widehat{Q}$.*

Let $T = \frac{1}{1-2C}$. Then $T > 1$.

PROPOSITION 2.18. *For an arbitrary $x \in U_\alpha$ and $y \in V_{\widehat{\alpha}}$, we have*

(i) $|B(s_\alpha \cdot x, s_\alpha \cdot y)| \geq T|B(x,y)|$;
(ii) $|B(s_\alpha \cdot x, y)| \geq T|B(x,y)|$.

PROOF. (i) Since $|s_\alpha(x)|_1 = 1 - 2B(x,\alpha)|\alpha|_1$, one has $C < B(x,\alpha) < \frac{1}{2|\alpha|_1}$. Moreover, we obtain

$$|B(s_\alpha \cdot x, s_\alpha \cdot y)| = \frac{|B(s_\alpha(x), s_\alpha(y))|}{||s_\alpha(x)|_1 |s_\alpha(y)|_1|} > T|B(s_\alpha(x), s_\alpha(y))| = T|B(x,y)|.$$

(ii) We have $B(x,y) \leq 0$ since $x, y \in \widehat{Q}$. When $B(x,y) = 0$, the assertion is obvious. Assume that $B(x,y) < 0$. Since $B(x,\alpha) > 0$ and $B(y,\alpha) > 0$, one has

$$1 - 2\frac{B(y,\alpha)}{B(x,y)} B(x,\alpha)|\alpha|_1^2 > 1.$$

Hence

$$|B(s_\alpha \cdot x, y)| = \left| \frac{1 - 2\frac{B(y,\alpha)}{B(x,y)} B(x,\alpha)|\alpha|_1^2}{1 - 2B(x,\alpha)|\alpha|_1} \right| |B(x,y)| \geq T|B(x,y)|.$$

∎

We now come to the position to prove Theorem 2.2 in the case of $\widehat{Q} \subset \mathrm{int}(\mathrm{conv}(\widehat{\Delta}))$. By [**18**, Theorem 2.7], we know $E \subset \widehat{Q}$. What we must show is another inclusion $\widehat{Q} \subset E$.

Fix $x \in \widehat{Q}$. For x, we choose an element $w_{x,m} = s_{\alpha_m} \cdots s_{\alpha_1} \in W$ of length m as follows:

- For $m = 1$, write $w_{x,1} = s_\alpha$ for some $\alpha \in \Delta$ such that $x \in U_\alpha$. There is at least one such α by Corollary 2.17.
- When we consider $w_{x,m-1} \cdot x$, there exists $\beta \in \Delta$ such that $w_{x,m-1} \cdot x \in U_\beta$. We set $w_{x,m} = s_\beta w_{x,m-1}$.

Remark that $w_{x,m}$ is not uniquely determined. Moreover, for each i, we have $w_{x,i} \neq 1$. By taking $y_i \in U_\alpha$ such that $|B(x, y_i)|$ is sufficiently small for each i, we have $|B(w_{x,i} \cdot x, w_{x,i} \cdot y)| > |B(x, y_i)|$.

Then $w_{x,m} \cdot x \in \widehat{Q} \setminus V_{\widehat{\alpha_m}}$. On the other hand, there exists $y_m \in E \cap (\widehat{Q} \setminus V_{\widehat{\alpha_m}})$. In fact, since there is at least one $y \in E$, $s_\alpha \cdot y \in E \cap (\widehat{Q} \setminus V_{\widehat{\alpha_m}})$ if $y \in V_{\widehat{\alpha_m}}$. By the definition of $w_{x,m}$, we see that $w_{x,m-1} \cdot x \in U_{\alpha_m}$ Moreover, $s_{\alpha_m} \cdot y_m \in V_{\widehat{\alpha_m}}$. If we set $y_{m-1} = s_{\alpha_m} \cdot y_m$, then $y_{m-1} \in E$. By Proposition 2.18 (i), we see that

$$|B(w_{x,m} \cdot x, y_m)| = |B(s_{\alpha_m} \cdot (w_{x,m-1} \cdot x), s_{\alpha_m} \cdot (s_\alpha \cdot y_m))| \geq T|B(w_{x,m-1} \cdot x, y_{m-1})|.$$

Let

$$y_{m-2} = \begin{cases} y_{m-1}, & \text{if } y_{m-1} \in V_{\widehat{\alpha_{m-1}}}, \\ s_{\alpha_{m-1}} \cdot y_{m-1}, & \text{if } y_{m-1} \notin V_{\widehat{\alpha_{m-1}}}. \end{cases}$$

By Proposition 2.18 (ii) if $y_{m-2} = y_{m-1}$ and Proposition 2.18 (i) if $y_{m-2} = s_{\alpha_{m-1}} \cdot y_{m-1}$, we obtain that

$$|B(w_{x,m-1} \cdot x, y_{m-1})| \geq T|B(w_{x,m-2} \cdot x, y_{m-2})|.$$

By repeating this estimate, we conclude that

$$|B(w_{x,m} \cdot x, y_m)| \geq T^m |B(x, y_0)|.$$

Let $M = \max_{u,v \in \widehat{Q}} |B(u, v)|$. Then $M > 0$ and

$$0 \leq |B(x, y_0)| \leq \frac{M}{T^m}.$$

By taking sufficiently large m, one can find $y_0 \in E$ such that $|B(x, y_0)|$ is arbitrarily small. Therefore, we obtain that $x \in E$, as desired. ∎

4.2. The case (b). For $\alpha \in \Delta$, let $\Delta_\alpha = \Delta \setminus \{\alpha\}$, $S_\alpha = S \setminus \{s_\alpha\}$ and let W_α denote the parabolic subgroup of W generated by S_α. When $\alpha = \alpha_j$, we denote Δ_j, S_j and W_j instead of Δ_{α_j}, S_{α_j} and W_{α_j}, respectively. By our assumption $\widehat{Q} \not\subset \mathrm{int}(\mathrm{conv}(\widehat{\Delta}))$, one has $\widehat{Q} \cap \mathrm{int}(\mathrm{conv}(\Delta_j)) \neq \emptyset$ for some j's. Let $\overline{R}_i$ $(i = 1, \ldots, m)$ denote the closure of a connected component R_i of $\widehat{Q} \setminus \mathrm{conv}(\widehat{\Delta})$. We first assume that $R_i \neq \emptyset$ for some i.

Proposition 2.19. *When $\widehat{Q} \not\subset$ int$(\mathrm{conv}(\widehat{\Delta}))$, we have $E = \bigcup_{i=1}^m \partial(W \cdot \overline{R}_i)$.*

Proof. By the result in the previous section (which is Theorem A in the case of rank 3), we can use the induction for the rank of Coxeter groups. Fix some component of $\widehat{Q} \setminus \mathrm{int}(\mathrm{conv}(\widehat{\Delta}))$, denoted by $\overline{R}$.

We see that $\partial(W \cdot \overline{R})$ is W-invariant, i.e., $W \cdot \partial(W \cdot \overline{R}) = \partial(W \cdot \overline{R})$. Let $y \in W \cdot \partial(W \cdot \overline{R})$. Then $y = w \cdot z$ for some $w \in W$ and $z \in \partial(W \cdot \overline{R})$. For any neighborhood O of z in $\widehat{Q}$, one has $O \cap W \cdot \overline{R} \neq \emptyset$. Since each element in W acts on $\widehat{Q}$ as a homeomorphism, any neighbor of y can be expressed as an image by w of some neighbor of z. Thus y should belong to

$\partial(W\cdot\overline{R})$. Hence $W\cdot\partial(W\cdot\overline{R})\subset\partial(W\cdot\overline{R})$. On the other hand, the reverse inclusion is obvious. Thus $\partial(W\cdot\overline{R})$ is W-invariant.

Clearly, $\partial(W\cdot\overline{R})$ is a closed set. Hence, by Proposition 2.15, $\partial(W\cdot\overline{R})$ contains E. Therefore, once we show that $\partial(W\cdot\overline{R})\subset E$, we obtain that $\partial(W\cdot\overline{R})=\bigcup_{i=1}^{m}\partial(W\cdot\overline{R_i})=E$.

For $j=1,\ldots,n$, let $A_j=\operatorname{conv}(\Delta_j)$ and assume that $\overline{R}\cap A_j\neq\emptyset$ for $j=1,\ldots,k$. Then one has $\partial\overline{R}=\widehat{Q}\cap\left(\bigcup_{j=1}^{k}A_j\right)$. Let V^j be the subspace of V spanned by Δ_j and let $H_j=V^j\cap V_1$. Now H_j be the hyperplane in V^j which contains A_j. Let $Q_j=\{v\mid v\in V^j, B_j(v,v)=0\}$, where B_j is the Coxeter matrix associated with W_j. Then $\widehat{Q}\cap H_j=\widehat{Q_j}$ and $\overline{R}\cap H_j=\widehat{Q_j}\setminus A_j$ for each $1\le j\le k$. By the inductive hypothesis, one has $E_j=\widehat{Q_j}\setminus(W_j\cdot(\widehat{Q_j}\setminus A_j))$, where E_j is the accumulation set of normalized roots of W_j.

Let $x\in\partial(W\cdot\overline{R})$. Suppose that $x\notin E$. Since E is a closed set, there exists a neighborhood U of x in $\widehat{Q}$ such that $U\cap E=\emptyset$. Moreover, since $\partial(W\cdot\overline{R})\subset W\cdot\partial\overline{R}$, one has $x\in w\cdot\partial\overline{R}$ for some $w\in W$. Hence $w^{-1}\cdot x\in\partial\overline{R}=\widehat{Q}\cap\left(\bigcup_{j=1}^{k}A_j\right)$. Thus $w^{-1}\cdot x\in\widehat{Q}\cap A_j$ for some $j\in\{1,\ldots,k\}$. Then we have $w^{-1}\cdot U\cap E_j=\emptyset$ from $w^{-1}\cdot U\cap E=\emptyset$. Hence, by the inductive hypothesis, there exists $w'\in W_j$ such that $w^{-1}\cdot U\cap A_j\subset w'\cdot(\widehat{Q}\setminus A_j)$. This means that $w'^{-1}w^{-1}\cdot U\cap A_j\subset\widehat{Q}\setminus A_j=\overline{R}\cap H_j\subset\overline{R}$. On the other hand, x belongs to the boundary of $W\cdot\overline{R}$ and U is a neighborhood of x, a contradiction. Therefore, $x\in E$, as required. ∎

Proposition 2.20. *When $\widehat{Q}\not\subset int(\operatorname{conv}(\widehat{\Delta}))$, we have $\widehat{Q}=\overline{\bigcup_{i=1}^{m}W\cdot\overline{R_i}}$.*

Proof. Since the inclusion $\overline{\bigcup_{i=1}^{m}W\cdot\overline{R_i}}\subset\widehat{Q}$ is obvious, it suffices to show the reverse inclusion. Let $G=\widehat{Q}\setminus\bigcup_{i=1}^{m}W\cdot R_i$ then G is closed. We may show that $\operatorname{int}(G)$ is empty. Note that G is W-invariant.

Suppose that $\operatorname{int}(G)\neq\emptyset$. Since ∂G is W-invariant and closed, we see that $E\subset\partial G$ by Proposition 2.15. Notice that $\max_{v\in G}\min_{u\in E}|B(u,v)|>0$, otherwise $G\subset E$ by Proposition 2.12. Suppose that $x\in G$ and $y\in E$ attain such value, i.e., $|B(x,y)|=\max_{v\in G}\min_{u\in E}|B(u,v)|$. Now x should belong to $\operatorname{int}(G)$ by Proposition2.19. By Proposition 2.10, there is $\alpha\in\Delta$ such that $x\in V_{\widehat{\alpha}}$ hence $B(x,\alpha)\ge 0$. If we suppose that $y\notin V_{\widehat{\alpha}}$, then $s_\alpha\cdot y\in E\cap\operatorname{int}(V_{\widehat{\alpha}})$. Let $z=s_\alpha\cdot y$. Since x is not in E, we have $x\neq z$ hence $B(x,z)<0$ by Lemma 2.12 (b). Adding to this one has $0<B(z,\alpha)<\frac{1}{2|\alpha|_1}$ by Lemma 2.9 and Lemma 2.11. Thus $0<1-2B(z,\alpha)|\alpha|_1<1$. Accordingly we see that

$$\begin{aligned}|B(x,y)|=|B(x,s_\alpha\cdot z)|&=\left|\frac{B(x,z)-2B(x,\alpha)B(z,\alpha)}{1-2B(z,\alpha)|\alpha|_1}\right|\\&=\left|\frac{1-2\frac{B(x,\alpha)}{B(x,z)}B(z,\alpha)}{1-2B(z,\alpha)|\alpha|_1}\right||B(x,z)|\\&>|B(x,s_\alpha\cdot y)|.\end{aligned}$$

However, this is a contradiction to $|B(x,y)|=\min_{u\in E}|B(x,u)|$. Hence, y should belong to $V_{\widehat{\alpha}}$. Moreover, suppose that $x\notin\partial V_{\widehat{\alpha}}$. Since $|B(x,y)|<|B(s_\alpha\cdot x,s_\alpha\cdot y)|$ by the proof of Proposition 2.13, from the maximality of $|B(x,y)|$, there is $z\in E\cap(\widehat{Q}\setminus V_{\widehat{\alpha}})\setminus\{s_\alpha\cdot y\}$ (hence $s_\alpha\cdot z\in V_{\widehat{\alpha}}$) such that $|B(s_\alpha\cdot x,z)|=\min_{u\in E}|B(s_\alpha\cdot x,u)|\le|B(x,y)|$. Similar to the above computation, we obtain that $|B(x,s_\alpha\cdot z)|<|B(s_\alpha\cdot x,z)|\le|B(x,y)|$. This contradicts to $|B(x,y)|=\min_{u\in E}|B(x,u)|$. Hence, x should belong to $\partial V_{\widehat{\alpha}}$.

Therefore, for each $\alpha\in\Delta$, if $x\in V_{\widehat{\alpha}}$, then $x\in\partial V_{\widehat{\alpha}}$. This implies that $B(x,\alpha)=0$ if x is visible from $\widehat{\alpha}$. Moreover, since $x\in G$, x belongs to $\operatorname{int}(\operatorname{conv}(\widehat{\Delta}))$, that is, x can be written as $x = \sum_{\delta\in\Delta} x_\delta\widehat{\delta}$, where $x_\delta > 0$ for each $\delta \in \Delta$. On the other hand, for each $\delta \in \Delta$ such that $x \notin V_{\widehat{\delta}}$, we have $B(x, \delta) < 0$ by Lemma 2.9. Hence

$$B(x, x) = \sum_{\alpha,\beta\in\Delta} x_\alpha x_\beta B(\widehat{\alpha}, \widehat{\beta}) = \sum_{\beta\in\{\delta\in\Delta \mid x\notin V_{\widehat{\delta}}\}} x_\beta \sum_{\alpha\in\Delta} x_\alpha B(\widehat{\alpha}, \widehat{\beta})$$
$$= \sum_{\beta\in\{\delta\in\Delta \mid x\notin V_{\widehat{\delta}}\}} x_\beta B(x, \widehat{\beta}) < 0.$$

It then follows that if $x = \sum_{\delta\in\Delta} x_\delta\widehat{\delta}$ and $x \notin V_{\widehat{\alpha}}$ for some $\alpha \in \Delta$, then $x_\alpha = 0$. This means that x should belong to $x \in \bigcap_{\alpha\in\Delta} \partial V_{\widehat{\alpha}}$ since $x_\alpha > 0$ for every $\alpha \in \Delta$. This contradicts to Lemma 2.14. ∎

We should remark the following. In the case where $\widehat{Q} \not\subset \mathrm{int}(\mathrm{conv}(\widehat{\Delta}))$ and $\widehat{Q} \subset \mathrm{conv}(\widehat{\Delta})$, there exist finite points $p_1, \ldots, p_k$ in $\widehat{Q} \cap \partial\mathrm{conv}(\widehat{\Delta})$ although there are no R_i's. This follows from the fact that $\widehat{Q}$ does not contain any Euclidean line. Notice that $p_1, \ldots, p_k \in E$. In fact for each point p_i there is $\alpha_i \in \Delta$ such that

$$\{p_i\} = \bigcap_{\delta\in\Delta_\alpha} H_\delta$$

where $H_\delta = \{v \in V \mid B(v, \delta) = 0\}$. This means that the set of accumulation points of roots of the subgroup W_α equals to the singleton $\{p_i\}$. In this case by regarding each point $\{p_i\}$ as $\overline{R_i}$, the proof of Proposition 2.20 works and together with $E \subset \widehat{Q}$, we have $\widehat{Q} = \overline{\bigcup_{i=1}^k W \cdot p_i} = E$.

By Proposition 2.19 and Proposition 2.20, we conclude that

$$E = \widehat{Q} \setminus \bigcup_{i=1}^{m} W \cdot \overline{R}_i,$$

as required. ∎

Example 2.21. Let W be a Coxeter system of rank 4 with $S = \{s_1, s_2, s_3, s_4\}$ and $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Let

$$B(\alpha_1, \alpha_2) = -a,\ B(\alpha_2, \alpha_3) = -b,\ B(\alpha_3, \alpha_4) = -c,$$
$$B(\alpha_1, \alpha_3) = B(\alpha_1, \alpha_4) = B(\alpha_2, \alpha_4) = 0,$$

where $a, b, c \in \left\{\cos\left(\frac{\pi}{k}\right) \mid k \in \mathbb{Z}_{>2}\right\} \cup [1, \infty)$. It then follows from an easy computation that the signature of B is $(2, 2)$ if and only if B is not positive type and three positive real numbers a, b, c satisfy $a^2 + b^2 + c^2 - a^2c^2 < 1$. (Consult, e.g., [**20**] for the classification of positive type.) For example, when $(a, b, c) = (2, \frac{1}{2}, 2)$, this condition is satisfied.

Thus, in the case of rank 4, there exists an infinite Coxeter group whose associated bilinear form has its signature $(2, 2)$, while each Coxeter group of rank 3 is either positive type or of type $(2, 1)$.

CHAPTER 3

Normalized actions and the Hilbert metric

We will see that the normalized action defines a discrete action of Coxeter groups of type $(n-1,1)$ on a Gromov hyperbolic space. In general an isometric group action $G \curvearrowright X$ on a metric space X is *discrete* or *properly discontinuous* if for any compact set K the set

$$\{g \in G \mid g(K) \cap K \neq \emptyset\} \subset G$$

is finite. We denote the action $G \curvearrowright X$ by $g.x$ for $g \in G$ and $x \in X$. If X is locally compact and there exists a fundamental region R (see Definition 3.8) then the action is discrete. In fact for any compact set K there exists a finite set $\{g_1, \ldots, g_k\} \subset G$ such that $\bigcup_i g_i.R \supset K$. Since the set

$$\left\{ g \in G \;\middle|\; g.\left(\bigcup_i g_i.R\right) \cap \bigcup_i g_i.R \neq \emptyset \right\}$$

is finite, we have the claim.

As we have already seen, the set of accumulation points of roots is distributed in $\widehat{Q}$. On the other hand, we can define the limit set of W with respect to B and it turns out that the limit set is distributed in $\widehat{Q}$ by the discreteness of the normalized action. It is natural to ask how they are connected to each other. We answer this as follows.

Theorem 3.1. *Let W be a Coxeter groups of rank n whose associating bi-linear form B has signature $(n-1,1)$. Then the limit set $\Lambda(W)$ of W coincides with the set of accumulation points of roots $E = E(W)$ of W.*

1. The Hilbert metric

At the beginning we define a metric on D and show that the normalized action is actually an isometric action.

1.1. The Cross ratio. For four vectors $a, b, c, d \in V$ with $c-d, b-a \notin Q$, we define the *cross ratio* $[a,b,c,d]$ with respect to B by

$$[a,b,c,d] := \frac{q(c-a)\cdot q(b-d)}{q(c-d)\cdot q(b-a)}.$$

We observe that the cross ratio is preserved by the normalization.

Proposition 3.2. *Let a_1, a_2, a_3, a_4 be points in V which are co-linear (namely a_2, a_3 are on the segment connecting a_1 and a_4), and $a_1 - a_4 \notin Q$. Let $b_1, b_2, b_3, b_4 \in V$ satisfying*

- *for each i, b_i lies on a ray R_i connecting a_i and some point $p \in V$,*
- *four vectors b_1, b_2, b_3, b_4 are co-linear and $b_1 - b_4 \notin Q$.*

Then we have

$$[a_1, a_2, a_3, a_4] = [b_1, b_2, b_3, b_4].$$

Proof. From the assumption, all eight points are located on the two dimensional subspace P which is spanned by $a_1 - p$ and $a_4 - p$ in V.

Let ℓ_0 be a line in P through a_1 and a_4. Consider two lines ℓ_2 and ℓ_3 in P parallel to ℓ_0 with $b_2 \in \ell_2$ and $b_3 \in \ell_3$. Let $B_i \in R_i \cap \ell_2$ and $B'_i \in R_i \cap \ell_3$ for $i = 1, 2, 3, 4$. Then we have $b_2 = B_2$ and $b_3 = B'_3$, and there is a positive constant k such that $B'_i - p = k(B_i - p)$ for $i = 1, 2, 3, 4$. Since two triangles with vertices $\{b_4, b_2, B_4\}$ and $\{b_4, b_3, B'_4\}$ are similar,

$$\frac{q(b_2 - b_4)}{q(b_3 - b_4)} = \frac{q(B_2 - B_4)}{q(B'_3 - B'_4)}.$$

By the similar reason, we also have

$$\frac{q(b_2 - b_1)}{q(b_3 - b_1)} = \frac{q(B_2 - B_1)}{q(B'_3 - B'_1)}.$$

In addition since ℓ_0 and ℓ_2 are parallel, there exists a constant m so that

$$B_i - B_j = m(a_i - a_j),$$

for all $i, j \in \{1, 2, 3, 4\}$. Therefore, we obtain

$$\begin{aligned}
[b_1, b_2, b_3, b_4] &= \frac{q(b_3 - b_1)q(b_2 - b_4)}{q(b_3 - b_4)q(b_2 - b_1)} = \frac{q(B'_3 - B'_1)q(B_2 - B_4)}{q(B'_3 - B'_4)q(B_2 - B_1)} \\
&= \frac{q(k(B_3 - B_1))q(B_2 - B_4)}{q(k(B_3 - B_4))q(B_2 - B_1)} = \frac{q(B_3 - B_1)q(B_2 - B_4)}{q(B_3 - B_4)q(B_2 - B_1)} \\
&= \frac{q(a_3 - a_1)q(a_2 - a_4)}{q(a_3 - a_4)q(a_2 - a_1)} = [a_1, a_2, a_3, a_4],
\end{aligned}$$

which implies what we wanted. ∎

1.2. The Hilbert metric. We define a distance d on D as follows. For any $x, y \in D$, take $a, b \in \partial D$ so that the points a, x, y, b lie on the segment connecting a, b in this order. Then $y - b, x - a \notin Q$. We define

$$d(x, y) := \frac{1}{2} \log[a, x, y, b],$$

and call this the *Hilbert metric for B*. The definition of the Hilbert metric for B depends heavily on B. However the following observation tells us that our definition coincides with the ordinary Hilbert metric d_H on D. Recall that the Hilbert metric d_H on D is defined for taking a, x, y, b as above,

$$d_H(x, y) = \log\left(\frac{\|y - a\|\,\|x - b\|}{\|y - b\|\,\|x - a\|}\right)$$

where $\| * \|$ denotes the Euclidean norm.

Observation 3.3. Take arbitrary $x, y \in Q_-$ and pick two points $a, b \in \partial D$ up so that $d(x, y) = \frac{1}{2}\log[a, x, y, b]$. Then we have $\|y - b\| \le \|x - b\|$, $\|x - a\| \le \|y - a\|$ with respect to the Euclidean norm $\| * \|$. From the co-linearity, each pair $\{y - b, x - b\}$ and $\{x - a, y - a\}$ have the same direction respectively. So there exist constants $k, l \ge 1$ such that $x - b = k(y - b)$ and $y - a = l(x - a)$. By the bi-linearity of the function q, we have

$$\begin{aligned}
[a, x, y, b] &= \frac{q(y - a)\, q(x - b)}{q(y - b)\, q(x - a)} = \frac{l^2 q(x - a)\, k^2 q(y - b)}{q(y - b)\, q(x - a)} \\
&= l^2 \cdot k^2 = \left(\frac{l\|x - a\|\, k\|y - b\|}{\|y - b\|\,\|x - a\|}\right)^2 = \left(\frac{\|y - a\|\,\|x - b\|}{\|y - b\|\,\|x - a\|}\right)^2.
\end{aligned}$$

Hence $d(x, y) = d_H(x, y)$ for all $x, y \in D$.

From this observation, d is actually a metric and we simply call it the Hilbert metric. An advantage of our definition of the Hilbert metric for B will appear in the proof of Proposition 3.7.

1.3. Some properties of the Hilbert metric. In this section we collect known geometric properties of a space with the Hilbert metric.

Let (X, d) be a metric space. We define the *length* of an arc $\gamma : [0, t] \longrightarrow (X, d)$ by

$$len(\gamma) = \sup_C \sum_{i=1}^{k} d(\gamma(t_{i-1}), \gamma(t_i)),$$

where the infimum is taken over all chains $C = \{0 = t_0, t_1, \ldots, t_n = t\}$ on $[0, t] \subset \mathbb{R}$ with unbounded k. A metric space is a *geodesic space* if for any two points there exists at least one arc connecting them whose length equals to their distance. Such an arc is called a *geodesic*. More generally an arc γ is *quasi geodesic* connecting $x, y \in X$ if there exist constants $a \geq 1, b > 0$ so that

$$a^{-1} d(x, y) - b \leq len(\gamma) \leq a d(x, y) + b.$$

Proposition 3.4. *(D, d) is*

(i) *a proper (i.e. any closed ball is compact) complete metric space and,*

(ii) *a uniquely geodesic space.*

Proof. (i) We denote d_E be the Euclidean metric on D. Then the identity map id : $(D, d_E) \longrightarrow (D, d)$ is continuous. In fact, fix a point x in D and consider a sequence $\{y_i\}_i$ in D converging to x. For each $i \in \mathbb{N}$, take $a_i, b_i \in \partial D$ so that four points a_i, x, y_i, b_i are collinear. Then since $y_i \longrightarrow x$ $(i \longrightarrow \infty)$, we have

$$\frac{\|y_i - a_i\| \, \|x - b_i\|}{\|y_i - b_i\| \, \|x - a_i\|} \longrightarrow 1.$$

This shows that $d(x, y_i) \longrightarrow 0$, hence id is continuous. Furthermore any closed ball in (D, d) is an image of a compact set in (D, d_E). In fact it is bounded closed set in (D, d_E) since D is bounded with respect to the Euclidean metric and the identity map is continuous. Therefore any closed ball in (D, d) is compact.

By the properness of (D, d), any Cauchy sequence $\{x_m\}_m$ in (D, d) has at least one converging subsequence in D since the Cauchy sequences are bounded. This implies that $\{x_m\}_m$ converges in D.

(ii) We can see that the Hilbert metric is a geodesic space by the following so-called *straightness property*. For any $x, y \in V$, $[x, y]$ denotes the segment connecting x and y.

If three points $x, y, z \in D$ are on one segment $[a, b]$ $(a, b \in \partial D)$ in this order, then $d(x, z) = d(x, y) + d(y, z)$.

In fact, we have

$$\begin{aligned} d(x, y) + d(y, z) &= \frac{1}{2} \left(\log[a, x, y, b] + \log[a, y, z, b] \right) \\ &= \frac{1}{2} \log \left(\frac{q(y - a)\, q(x - b)}{q(y - b)\, q(x - a)} \cdot \frac{q(z - a)\, q(y - b)}{q(z - b)\, q(y - a)} \right) \\ &= \frac{1}{2} \log \left(\frac{q(z - a)\, q(x - b)}{q(z - b)\, q(x - a)} \right) \\ &= d(x, z). \end{aligned}$$

Thus the length of the segment $[x, z]$ realizes the metric $d(x, z)$. Furthermore since D is strictly convex, geodesics are unique. ∎

Let (X, d) be a geodesic space. For $x, y, p \in X$, we define the *Gromov product* $(x|y)_p$ of x and y with respect to p by the equality

$$(x|y)_p = \frac{1}{2}\left(d(x, p) + d(y, p) - d(x, y)\right).$$

Using this, the hyperbolicity in the sense of Gromov is defined as follows. Let $\delta \geq 0$. The space X is *δ-hyperbolic* if

$$(x|z)_p \geq \min\{(x|y)_p, (y|z)_p\} - \delta$$

for all $x, y, z, p \in X$. We say the space is simply *Gromov hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$.

A *geodesic triangle* $T \subset X$ with vertices $x, y, z \in X$ is a union of three geodesic curves with end points x, y, z. We call these curves the *sides* of T. A *triangle map* is a map $f : T \longrightarrow \mathbb{R}^2$ from geodesic triangle onto a Euclidean triangle whose sides have the same length as corresponding sides of T, such that the restriction of f to any one side is an isometry. We always have triangle maps and they are unique up to the composition of isometry of $\mathbb{R}^2$ for a geodesic triangle. A geodesic space is called a *CAT(0) space* if for any geodesic triangle T, $d(x, y) \leq \|f(x) - f(y)\|$ for all $x, y \in T$ whenever $f : T \longrightarrow \mathbb{R}^2$ is a triangle map.

A metric space (D, d) is a CAT(0) and Gromov hyperbolic space since the region D is an ellipsoid. The former derived from a result given in [**13**] by Egloff.

Theorem 3.5 (Egloff). *Let $H \subset \mathbb{R}^n$ be a convex open set with the Hilbert metric d_H. Then (H, d_H) is a CAT(0) space if and only if H is an ellipsoid.*

The latter owe to a result of Karlsson-Noskov [**21**].

Theorem 3.6 (Karlsson-Noskov). *Let $H \subset \mathbb{R}^n$ be a convex open set with the Hilbert metric d_H. If H is an ellipsoid, then (H, d_H) is hyperbolic in the sense of Gromov.*

The point of our definition of the Hilbert metric can be seen in the proof of the following proposition.

Proposition 3.7. *Let W be a Coxeter group of type $(n-1, 1)$. W acts on (D, d) isometrically by the normalized action.*

Proof. It suffices to show that the cross ratio defining the Hilbert metric d is invariant under any normalized B-reflection s_α $(\alpha \in \Delta)$. We take $x, y \in D$ arbitrary and let $a, b \in \partial D$ be the points satisfying $d(x, y) = (1/2)\log[a, x, y, b]$.

We check that B-reflection s_α preserves q. Actually, for any $v \in V$, we have

$$q(s_\alpha(v)) = q(v) - 4B(v, \alpha)^2 + 4B(v, \alpha)^2 q(\alpha) = q(v),$$

because $q(\alpha) = 1$ for all $\alpha \in \Delta$. This means that $[a, x, y, b] = [s_\alpha(a), s_\alpha(x), s_\alpha(y), s_\alpha(b)]$.

Our remaining task is to show that $[s_\alpha(a), s_\alpha(x), s_\alpha(y), s_\alpha(b)]$ does not vary under the normalization for $|*|_1$ in Q_-^+. This follows from Proposition 3.2. In fact, since s_α is linear, a segment is mapped to a segment. So the image $s_\alpha([a, b])$ coincides with $[s_\alpha(a), s_\alpha(b)]$. In particular four points $\{s_\alpha(a), s_\alpha(x), s_\alpha(y), s_\alpha(b)\}$ are co-linear. Furthermore $s_\alpha(x)$ and $s_\alpha(y)$ are in Q_-^+ because the image of a segment in Q_-^+ by s_α does not include $\mathbf{0}$. This means that $s_\alpha(a) - s_\alpha(b) \notin Q$. At last, recall that for any $v \in Q_-^+$, $\widehat{v}$ lies on the ray through $\mathbf{0}$ and v. Therefore for each $z \in \{s_\alpha(a), s_\alpha(x), s_\alpha(y), s_\alpha(b)\}$, we have a ray through $\mathbf{0}$ and $\widehat{z}$. ∎

2. Discreteness of the normalized action

Recall our definition of two sets

$$D := V_1 \cap Q_- \qquad \text{and} \qquad D' := D \setminus \bigcup_{w \in W} w \cdot R$$

where $R := D \setminus \mathrm{conv}(\widehat{\Delta})$. Note that by Theorem 2.2 $D' \cup \partial D'$ is the convex hull of the set E of accumulation points of roots in the sense of the Euclidean topology. D' is endowed with the subspace topology of D and hence D' is also complete proper CAT(0) Gromov hyperbolic space.

We define two open sets (with respect to the subspace topology of V_1)

$$K := \{v \in D \mid \forall \alpha \in \Delta, B(\alpha, v) < 0\} \qquad \text{and} \qquad K' := K \cap D'.$$

For $\alpha \in \Delta$ we set $P_\alpha = \{v \in V_1 \mid \alpha\text{-th coordinate of } v \text{ is } 0\}$ and $H_\alpha = \{v \in V_1 \mid B(v, \alpha) = 0\}$. We define

$$\mathcal{P} = \{v \in V_1 \mid \forall \alpha \in \Delta, B(\alpha, v) < 0\} \qquad \text{and} \qquad \mathcal{P}' = \mathcal{P} \cap \mathrm{int}(\mathrm{conv}(\widehat{\Delta})).$$

Then clearly $K = \mathcal{P} \cap D$. Moreover, we will see that $K' = \mathcal{P}' \cap D$ (Lemma 3.11). Since $\mathcal{P}$ (resp. $\mathcal{P}'$) is bounded by finitely many $n-1$ dimensional subspaces $\{H_\alpha \mid \alpha \in \Delta\}$ (resp. $\{H_\alpha \mid \alpha \in \Delta\}$ and $\{P_\alpha \mid \alpha \in \Delta\}$), actually $\overline{\mathcal{P}}$ (resp. $\overline{\mathcal{P}'}$) is a polyhedron. In general, $\mathcal{P}$ is not a simplex. The following example of W such that $\mathcal{P}$ is not a simplex is given by Yohei Komori.

$$W = \langle s_1, \ldots, s_5 \mid s_i^2, (s_{i-1}s_i)^4 \rangle,$$

where $i = 1, \ldots, 5$ and $s_0 = s_5$. In fact the Coxeter graph of this does not appear in the list given by Schlettwein [**32**].

Definition 3.8. We assume that a group G acts on a metric space X isometrically. We denote the action $G \curvearrowright X$ by $g.x$ for $g \in G$ and $x \in X$. Then an open set $A \subset X$ is

- a *fundamental region* if $\overline{G.A} = X$ and $g.A \cap A = \emptyset$ for any $g \in G$ where $\overline{G.A}$ is the topological closure of $G.A$;
- the *Dirichlet region* at $o \in A$ if A equals to the set

$$\{x \in D \mid d(o, x) < d(o, w \cdot x) \text{ for } w \in W \setminus \{\mathrm{id}\}\}.$$

We will show that K (resp. K') is the Dirichlet region at any $x \in K$ hence a fundamental region for the (resp. restricted) normalized action of W on D.

Remark 3.9. By [**2**, Proposition 4.2.5], for $w \in W$ and $s_\alpha \in S$ if $|sw| > |w|$ then all coordinates of $w^{-1}(\alpha)$ are non-negative.

Proposition 3.10. *For any $z \in K$, we have the followings.*

(i) *For any $w \in W \setminus \{id\}$, there exists $\alpha \in \Delta$ so that $B(w \cdot z, \alpha) > 0$:*
(ii) *For any $w \in W$, $|w(z)|_1 > 0$. Moreover, if $z \in \mathrm{int}(\mathrm{conv}(\widehat{\Delta}))$ then all coordinates of $w(z)$ are positive.*

Proof. We prove (i) and (ii) at the same time by the induction for the word length.

In the case $|w| = 1$, there exists $\alpha \in \Delta$ so that $w = s_\alpha$. Then we have

$$|s_\alpha(z)|_1 = |z|_1 - 2B(z, \alpha)|\alpha|_1 > 0.$$

Therefore

$$B(s_\alpha \cdot z, \alpha) = \frac{B(z, -\alpha)}{|s_\alpha(z)|_1} > 0,$$

since $s_\alpha(\alpha) = -\alpha$.

If $|w| > 1$, there exist $\alpha \in \Delta$ and $w' \in W$ satisfying $w = s_\alpha w'$. In particular $|w'| = |w|-1$. We have $|w'(z)|_1 > 0$ by the assumption of the induction. From Remark 3.9 we have $v_\beta \geq 0$ if $w'^{-1}(\alpha) = \sum_{\beta\in\Delta} v_\beta \beta$. Then we see that

$$\begin{aligned} |w(z)|_1 &= |s_\alpha(w'(z))|_1 = |w'(z)|_1 - 2B(z, w'^{-1}(\alpha))|\alpha|_1 \\ &= |w'(o)|_1 - 2|\alpha|_1 \sum_{\beta\in\Delta} v_\beta B(z,\beta) > 0, \end{aligned} \tag{3}$$

because $z \in K$. This shows (ii). In addition, we have

$$B(w \cdot z, \alpha) = \frac{-B(w'(z), \alpha)}{|s_\alpha(w'(z))|_1} = \frac{-\sum_{\beta\in\Delta} v_\beta B(z,\beta)}{|s_\alpha(w'(o))|_1} > 0.$$

Hence we have (i). ∎

This lemma ensures that K and K' are not empty.

Lemma 3.11. *We have the following:*

(i) $K' = K \cap \mathrm{int}(\mathrm{conv}(\widehat{\Delta})) = \mathcal{P}' \cap D$.
(ii) *K' is not empty.*

Proof. (i) Recall that $R = D \setminus \mathrm{conv}(\widehat{\Delta})$. We set $K_{int} = K \cap \mathrm{int}(\mathrm{conv}(\widehat{\Delta}))$. Then clearly $K' \subset K_{int}$. To see the inverse inclusion, it suffices to show that $w \cdot R \cap K_{int} = \emptyset$ for any $w \in W$. Take $x \in K_{int}$ arbitrarily. Then $x_\alpha > 0$ for any $\alpha \in \Delta$ if we write $x = \sum_{\alpha\in\Delta} x_\alpha \alpha$. Now we assume that $w \cdot x \in R$ then there exists $\alpha \in \Delta$ such that α-th coordinate of $w \cdot x$ is non-positive. This contradicts to the latter claim of Proposition 3.10 (ii).

(ii) Let o be the normalized eigenvector for the negative eigenvalue $-\lambda$ of B. Then all coordinates of o are positive by the definition and Lemma 1.2. For any $\alpha \in \Delta$, we have

$$B(o, \alpha) = -\lambda(o, \alpha) < 0.$$

Thus $o \in K$. Furthermore by the same argument as the proof of (i), we also have $o \in K'$ since all coordinates of o are positive. ∎

As a consequence of Proposition 3.10, we have the following.

Lemma 3.12. *For any $w \in W \setminus \{\mathrm{id}\}$, we have $w \cdot K \cap K = \emptyset$.*

Lemma 3.13. *For any $x \in K$ and $\xi \in \partial D$ (or in $\partial D' \setminus D$) the Euclidean segment $[x, \xi]$ joining x and ξ is not contained in any hyperplane $w \cdot H_\alpha$ ($w \in W$, $\alpha \in \Delta$).*

Lemma 3.14. *For any $x \in K$, K is the Dirichlet region at x.*

Proof. We assume that there exists a point y in the Dirichlet region at x such that $y \notin K$. Then by the definition of K we have $\alpha \in \Delta$ satisfying $B(\alpha, y) \geq 0$. If $B(\alpha, y) = 0$ then $s_\alpha(y) = y$ and hence $d(x, y) = d(x, s_\alpha \cdot y)$ which is a contradiction. For the other case $B(\alpha, y) > 0$, then the Euclidean segment $[o, x]$ joining x and y intersects with H_α. Let z be the intersection point. Since z fixed by s_α, $d(s_\alpha \cdot y, z) = d(s_\alpha \cdot y, s_\alpha \cdot z) = d(o, z)$. Then we have $d(x, y) = d(x, z) + d(z, y) = d(x, z) + d(z, s_\alpha \cdot y)$, hence $d(x, y) \geq d(x, s_\alpha \cdot y)$ by the triangle inequality. This contradicts to the hypothesis that y is in the Dirichlet region at x.

For the inverse, assume that $y \in K$ is not in the Dirichlet region at x. By Lemma 3.12 there exists an element $w \in W \setminus \{\mathrm{id}\}$ that attains $\min_{w\in W\setminus\{\mathrm{id}\}} d(y, w \cdot x)$ and satisfies $w \cdot x \notin K$. Consequently there exists $\alpha \in \Delta$ such that the Euclidean segment $[w \cdot x, y]$ joining $w \cdot x$ and y intersects with H_α. The intersection point z is fixed by s_α hence $d(s_\alpha \cdot x, z) = d(x, z)$. The uniqueness of the geodesic between y and $(s_\alpha w) \cdot x$ gives $d(y, (s_\alpha w) \cdot x) < d(y, z) + d(z, (s_\alpha w) \cdot x) = d(y, w \cdot x)$. This contradicts to the minimality of $d(y, w \cdot x)$. ∎

Lemma 3.14 shows also that K is connected. In fact, assume that K has more than two components. Then we can decompose K into $K_1 \sqcup K_2$ and assume that $o \in K_1$. Take $v \in K_2$ and consider the geodesic γ from o to v. Then γ should pass through at least one hyperplane H_α. Let u be an intersection point. Since $u \in H_\alpha$, we have $w \cdot u = u$. Now we see that $d(o, s_\alpha \cdot v) < d(o,u) + d(s_\alpha \cdot u, s_\alpha \cdot v) = d(o,u) + d(u,v) = d(o,v)$. This contradicts to Lemma 3.14.

Proposition 3.15. *K is a fundamental region for the normalized action.*

Proof. Take $y \in D$ arbitrary. Let $w \cdot o$ be the nearest orbit of o from y. Then we see that $w^{-1} \cdot y \in \overline{K}$ by Lemma 3.14. The second assertion of the definition of the fundamental region is Lemma 3.12. ∎

The following corollary is originally proved by Floyd [**14**, Lemma in p.213] for geometrically finite Kleinnian groups without parabolic elements. Here we assume the Coxeter groups W acts cocompactly, i.e., the quotient space of D by the normalized action is compact. In that case, polytope $\overline{K}$ is contained in D.

Corollary 3.16. *Let o be the normalized eigenvector for the negative eigenvalue of B. If the fundamental region K is bounded, then there are constants $k, k' > 0$ so that $k|w| \leq d(w \cdot o, o) \leq k'|w|$ for all $w \in W$.*

Proof. Take $w \in W$ arbitrary. Taking $k' = \max\{d(o, s_\alpha \cdot o) \mid \alpha \in \Delta\}$, we see that the inequality $d(w \cdot o, o) \leq k'|w|$ holds by the triangle inequality.

From Proposition 3.14, the set K is a fundamental region including x. By the assumption, the diameter of K is finite. Let $d = \operatorname{diam}(K)$ and let $c = \max\{|u| \in W \mid d(u \cdot o, o) \leq 3d\}$. We divide the geodesic between o and $w \cdot o$ into intervals of length d and one shorter interval. Connect each end point of each intervals to the closest point the orbit of o. Then we have the estimate $|w| \leq Cd(w \cdot o, o)/d$. ∎

Definition 3.17. Let (W, S) be a Coxeter system.

- We call a sequence $\{w_k\}_k$ in W a *short sequence* if for each $n \in \mathbb{N}$ there exists $s \in S$ such that $w_{k+1} = sw_k$ and $|w_k| = k$.
- For a sequence $\{w_k\}_k$ in W, a path in V_1 is a *sequence path* for $\{w_k\}_k$ if the path is given by connecting Euclidean segments $[w_k \cdot o, w_{k+1} \cdot o]$ for all $k \in \mathbb{N}$.

Remark 3.18. A *reflection* in W is an element of the form wsw^{-1} for $s \in S$ and $w \in W$. We see that $w \cdot H_\alpha = H_{w \cdot \widehat{\alpha}}$. We remark that each reflection $ws_\alpha w^{-1}$ corresponds to the normalized B-reflection with respect to $H_{w \cdot \widehat{\alpha}}$. We say that the normalized action of $ws_\alpha w^{-1}$ to be the reflection for $w \cdot H_\alpha$.

Proposition 3.19. *Suppose that W acts on D cocompactly. For any $\xi \in \Lambda(W)$ there exists a short sequence $\{w_k\}_k$ so that $w_k \cdot o$ converges to ξ. Furthermore the sequence path for $\{w_k\}_k$ lies in c-neighborhood of a segment $[o, \xi]$ connecting o and ξ for some $c > 0$ with respect to the Hilbert metric.*

Proof. A Euclidean segment $\gamma = [o, \xi]$ is a geodesic ray with respect to the Hilbert metric. The segment γ intersects with infinitely many hyperplanes $\{w_k \cdot H_{\alpha_k}\}$ ($\alpha_k \in \Delta$, $w_k \in W$ for $k \in \mathbb{N}$) transversely since it is not contained any hyperplane $w_k \cdot H_{\alpha_k}$ by Lemma 3.13. We notice that γ pass through each $\{w_k \cdot H_{\alpha_k}\}$ only once because the Euclidean straight line cannot pass through any hyperplane twice. If γ intersects with some hyperplanes at the same point x, then by perturbing subpath of γ in the ϵ ball $B(x, \epsilon)$ centered at x we have a quasi geodesic ray γ' toward ξ which is in ϵ neighborhood of γ for sufficiently small

$\epsilon > 0$. Then γ' intersects with the hyperplanes passing through x only once. In particular, γ' intersects with distinct hyperplanes at distinct points.

We renumber the hyperplanes $\{w_k \cdot H_{\alpha_k}\}$ with which γ' intersects so that if γ' intersects with some hyperplanes $w_k \cdot H_{\alpha_k}$, $w_{k'} \cdot H_{\alpha_{k'}}$ at $\gamma'(t)$, $\gamma'(t')$ respectively for $t < t'$, then we have $k < k'$. Thus we have a sequence $\{w_k\}_k$. Considering the B-reflection s_{α_i} with respect to H_{α_i} for each $i \in \mathbb{N}$, we see that $w_k = s_{\alpha_k} s_{\alpha_{k-1}} \cdots s_{\alpha_1}$ for each $k \in \mathbb{N}$. Let $w_0 = id$ and $r_i = w_{i-1} s_{\alpha_i} w_{i-1}^{-1}$ for $i \in \mathbb{N}$. Then $w_k = r_k \cdots r_1$. Now each r_i is the reflection for $w_i \cdot H_{\alpha_i}$ for all i. Since γ' meets each hyperplane $w_i \cdot H_{\alpha_i}$ ($i \in \mathbb{N}$) only once, in the sequence $\{r_k, \ldots, r_1\}$ no reflection occurs more than once for all $k \in \mathbb{N}$. This shows that the word $s_{\alpha_k} s_{\alpha_{k-1}} \cdots s_{\alpha_1}$ is a geodesic for w_k ([**11**, Corollary 3.2.7]). Therefore, the sequence $\{w_k\}_k$ is a short sequence.

Furthermore by the construction, we see that the sequence path for $\{w_k\}_k$ is included in c-neighborhood of γ, where c equals to the diameter of K. ∎

3. The limit set and the set of accumulation points of roots

In this section we give a proof of Theorem 3.1.

Definition 3.20. For a Coxeter system (W, S) of type $(n-1, 1)$, let o be the normalized eigenvector corresponding to the negative eigenvalue of the corresponding Coxeter matrix. The *limit set* $\Lambda_B(W)$ of W with respect to B is the set of accumulation points of the orbit of o by the normalized action of W on D in the Euclidean topology. The limit set depends on the Coxeter matrix B. If B is understood, then we simply denote the limit set by $\Lambda(W)$.

Now, we claim $\Lambda(W) = E$. Before proving this, we need to confirm the definition of the limit set is independent of the choice of the base point o.

Lemma 3.21. *Let $\{x_k\}_k$ and $\{y_k\}_k$ be two sequences in $D \cap \mathrm{int}(\mathrm{conv}(\widehat{\Delta}))$ converging to the points x and y in ∂D with respect to the Euclidean metric. If there exist a constant C so that $d(x_k, y_k) \leq C$ for all $k \in \mathbb{N}$ then $x = y$.*

Proof. Recall that $q(x-y) = 0$ if and only if $x = y$.

Let $\{a_k\}_k$ and $\{b_k\}_k$ be two sequences in ∂D associating with $\{x_k\}_k$ and $\{y_k\}_k$ so that

$$d(x_k, y_k) = \frac{1}{2} \log[a_k, x_k, y_k, b_k] = \frac{1}{2} \log\left(\frac{q(y_k - a_k) \cdot q(x_k - b_k)}{q(y_k - b_k) \cdot q(x_k - a_k)}\right)$$

for all $k \in \mathbb{N}$. Now there exists a constant $C' > 0$ so that for any $z, z' \in D \cup \partial D$, $q(z'-z) \leq C'$ since $D \cup \partial D$ is compact. Then we have

$$q(y_k - a_k) \cdot q(x_k - b_k) \leq e^{2C} q(y_k - b_k) \cdot q(x_k - a_k) \leq e^{2C} C' q(x_k - a_k).$$

We have $a_k \longrightarrow x$ since $x_k \longrightarrow x \in \partial D \subset Q$. Hence the right hand side of the inequality above tends to 0. Hence $q(y_k - a_k)$ or $q(x_k - b_k)$ converges to 0. If $q(y_k - a_k)$ tends 0 then $y_k \longrightarrow x$ since $\{x_k\}_k$ and $\{a_k\}_k$ converge to the same point x. For the other case, we also have $y_k \longrightarrow x$ since y_k is on the segment joining a_k and b_k for all $k \in \mathbb{N}$. ∎

Theorem 3.1 immediately follows form the next proposition. To show this we remember that $B(x, y) = 0$ if and only if $x = y$ for $x, y \in \partial D$ (Proposition 2.12).

Proposition 3.22. *Let $(w_n)_{n \in \mathbb{N}}$ be a sequence of elements in W. For any $\delta \in \Delta$ and $y \in D$, $w_n \cdot \widehat{\delta} \longrightarrow z \in \partial D$ if and only if $w_n \cdot y \longrightarrow z \in \partial D$.*

Proof. It suffices to show that in the case $y = o$ where o is the normalized negative eigenvector of B from Lemma 3.21. By [**18**, Theorem 2.7], we have that for any injective sequence $\{w_n\}_n$ in W and $\delta \in \Delta$, $\|w_n(\delta)|_1| \longrightarrow \infty$. This implies $\|w_n(\widehat{\delta})|_1| \longrightarrow \infty$. On the other hand by Proposition 3.14, the normalized action is discrete. This implies $w_n \cdot o$ tends to ∂D, hence $q(w_n \cdot o) \longrightarrow 0$, equivalently $|w_n(o)|_1 \longrightarrow \infty$.

Since $B(w(p), w(p')) = B(p, p')$ for any $p, p' \in V$ and $w \in W$, it holds that

$$\begin{aligned}
B(w_n \cdot \widehat{\delta}, w_n \cdot o) &= B\left(\frac{w_n(\widehat{\delta})}{|w_n(\widehat{\delta})|_1}, \frac{w_n(o)}{|w_n(o)|_1}\right) \\
&= \frac{1}{|w_n(\widehat{\delta})|_1 |w_n(o)|_1} B(w_n(\widehat{\delta}), w_n(o)) \\
&= \frac{1}{|w_n(\widehat{\delta})|_1 |w_n(o)|_1} B(\delta, o) \to 0 \quad (n \to \infty).
\end{aligned}$$

We have the conclusion. ∎

CHAPTER 4

The Cannon-Thurston maps

In this chapter we give a proof of Theorem C in the introduction.

THEOREM 4.1. *Let W be a rank n Coxeter groups whose associating bi-linear form B has the signature $(n-1, 1)$. Let $\partial_G(W, S)$ be the Gromov boundary of W and let $\Lambda(W)$ be the limit set of W. There exists a W-equivariant, continuous surjection $F : \partial_G(W, S) \longrightarrow \Lambda(W)$.*

We remark that the Gromov boundary is ordinary defined on a hyperbolic metric space. In this thesis we extend the definition to an arbitrary metric space by taking the transitive closure.

1. Three cases

At the beginning of this chapter we divide what happens by the normalized action into three cases: cocompact, convex cocompact, with cusps. We recall that $\mathrm{conv}(\widehat{\Delta})$ is a simplex. It can happen three distinct situations due to the bilinear form B;

(i) the region $D \cup \partial D$ is included in $\mathrm{int}(\mathrm{conv}(\widehat{\Delta}))$;
(ii) there exist some n' $(< n)$ dimensional faces of $\mathrm{conv}(\widehat{\Delta})$ which are tangent to the boundary ∂D;
(iii) $D \cup \partial D \not\subset \mathrm{int}(\mathrm{conv}(\widehat{\Delta}))$ and no faces of $\mathrm{conv}(\widehat{\Delta})$ tangent to ∂D.

We argue the cases (i) and (iii) simultaneously. For the case (ii), we can not apply the same argument as (i) and (iii). The most general case will be discussed in Section 4.2.

REMARK 4.2. By [**16**, Corollary 2.2], we see that a Coxeter subsystem (W', S') satisfying $S' \subset S$ is either of type $(|S'|-1, 1)$ or $(|S'|-1, 0)$ or positive definite. Let B' be the bilinear form corresponding to (W', S'). If B' has the signature $(|S'|-1, 1)$ (resp. $(|S'|-1, 0)$), then by the same argument as Lemma 1.2, we have an eigenvector $o' \in \mathrm{span}(\Delta')$ of the negative eigenvector (resp. 0 eigenvalue) such that all coordinates of o' for Δ' are positive where $\mathrm{span}(\Delta')$ denotes the subspace spanned by Δ'. This shows that $Q' = \{v \in \mathrm{span}(\Delta') \mid B'(v, v) = 0\}$ should intersect with $\mathrm{conv}(\widehat{\Delta'})$. Since the Coxeter matrix of B' is a principal submatrix of the Coxeter matrix of B, we see that $\partial D \cap \mathrm{conv}(\widehat{\Delta'}) = Q' \cap \mathrm{conv}(\widehat{\Delta'})$. Thus we have the followings:

(1) B' has the signature $(|S'|-1, 1)$ if and only if $D \cap \mathrm{conv}(\Delta') \neq \emptyset$;
(2) B' has the signature $(|S'|-1, 0)$ if and only if $\partial D \cap \mathrm{conv}(\Delta') = Q' \cap \mathrm{conv}(\widehat{\Delta'})$, which is a singleton;
(3) B' is positive definite if and only if $(D \cup \partial D) \cap \mathrm{conv}(\widehat{\Delta'}) = \emptyset$.

If B' has the signature $(|S'|-1, 1)$ then H_α for $\alpha \in \Delta'$ intersects with $D \cap \mathrm{conv}(\Delta')$. In fact if not, then $D \cap \mathrm{conv}(\widehat{\Delta'})$ is not preserved by s_α for $\alpha \in \Delta'$. Moreover, by the compactness of Q, $Q' \cap V_0 = \mathbf{0}$ for any Coxeter subsystem (W', S').

We say a rank n Coxeter system is *affine* if its associating bi-linear form B has the signature $(n-1,0)$. Fixing a generating set S we simply say Coxeter group W is affine if the Coxeter system (W,S) is affine. An affine Coxeter group is of infinite order and its limit set is a singleton ([**18**, Corollary 2.15]). We notice that for any affine Coxeter group if its rank is more than 2 then there are no simple roots $\alpha,\beta \in \Delta$ with $B(\alpha,\beta) \leq -1$ if B is irreducible. In fact if $B(\alpha,\beta) \leq -1$ then the subgroup generated by s_α, s_β is of infinite order hence $E \cap \mathrm{conv}(\{\alpha,\beta\}) \neq \emptyset$. This implies that $E \subset \mathrm{conv}(\{\alpha,\beta\})$ since E is a singleton. Hence $B(\alpha,\beta) < -1$ can not happen because that if so then the limit set of the subgroup generated by s_α, s_β consists of two points. For the case where $B(\alpha,\beta) = -1$, let x be the limit point, i.e, $E = \{x\}$. Since $x \in \mathrm{conv}(\{\alpha,\beta\})$, for any $\gamma \in \Delta \setminus \{\alpha,\beta\}$, the γ-th coordinate of x equals to 0. For the α-th coordinate and the β-th coordinate of x are not 0. Since B is irreducible, $B(x,\gamma) \neq 0$ for $\gamma \in \Delta \setminus \{\alpha,\beta\}$. This shows that $s_\gamma \cdot x \neq x$. However since $s_\gamma \cdot x$ is in E, we have a contradiction.

Remark 4.3. We remark that for $x \in \partial D$ and $\alpha \in \Delta$ we have $\{x, s_\alpha \cdot x\} = L(\widehat{\alpha}, x) \cap \partial D$ where $L(\widehat{\alpha}, x)$ is the Euclidean line passing through $\widehat{\alpha}$ and x. This is because that $s_\alpha \cdot x$ is a linear combination of $\widehat{\alpha}$ and x, and ∂D is preserved by s_α (Lemma 2.9 (ii)).

Proposition 4.4. *Assume that (W,S) is Coxeter system of type $(n-1,1)$.*

(a) *The case (i) happens if and only if every Coxeter subgroup of W of rank $n-1$ generated by a subset of S is finite.*
(b) *The case (ii) happens if and only if there exists a rank n' $(< n)$ affine Coxeter subgroup of W generated by a subset of S.*
(c) *The case (iii) happens if and only if every Coxeter subgroup of W of rank n' $(< n)$ generated by a subset of S is of type $(n'-1,1)$ or $(n',0)$.*

Proof. Note that for $\Delta' \subset \Delta$ we can restrict the bi-linear form B to Δ'. We denote such a bi-linear form as B', namely, the Coxeter matrix with respect to B' is a principal submatrix of the Coxeter matrix with respect to B.

(a) Let W' be a Coxeter subgroup of rank $n-1$ and let B' be the bilinear form for W'. Recall a classical result that W' is finite if and only if B' is positive definite (see [**20**, Theorem 6.4]). This is equivalent to that $\widehat{Q}$ does not intersect with $\mathrm{conv}(\widehat{\Delta'})$.

(b) Assume that there exists an affine rank n' $(< n)$ Coxeter subgroup W' of W generated by a subset S' of S. Let Δ' and B' be the subset of Δ and the bilinear form corresponding to W' respectively. As we have mentioned before, the normalized limit set $\widehat{\Lambda(W')}$ is a singleton $\{\xi\}$ and it equals to $\widehat{Q'} \subset \widehat{Q}$. This shows that an n' dimensional face $\mathrm{conv}(\widehat{\Delta'})$ is tangent to $\widehat{Q}$. Let $p \in \partial D \cap \mathrm{conv}(\widehat{\Delta})$ be such a point. Then by Remark 4.3 we see that W' fixes p since $B'(p,\alpha) = 0$ for any $\alpha \in \Delta$.

For the converse we assume that an $n'(< n)$ dimensional face $\mathrm{conv}(\widehat{\Delta'})$ is tangent to $\widehat{Q}$ for some $\Delta' \subset \Delta$. We also assume that face is minimal. Let $S' \subset S$ and W' be the set of simple B'-reflection corresponding to Δ' and the Coxeter subgroup of W generated by S' respectively. Then for the corresponding bilinear form B' the set $\widehat{Q'}$ consists of one point v. Then $s_\alpha \cdot v = v$ for any $\alpha \in \Delta'$ by Remark 4.3. Therefore $B'(v,\alpha) = 0$ for any $\alpha \in \Delta'$ and hence v is an eigenvector of 0 eigenvalue of B'. This means that B' has the signature $(n'-1,0)$ and hence W' is affine.

(c) If every infinite rank n' $(< n)$ Coxeter subgroup of W generated by a proper subset S' of S is of type $(n'-1,1)$ or positive definite then $\widehat{Q'}$ is either an ellipsoid or empty. Obviously $\widehat{Q'} \subset \partial D$ we have the case (iii). If the case (iii) happens, then ∂D should intersect with a face of $\mathrm{conv}(\widehat{\Delta})$. Let $\mathrm{conv}(\widehat{\Delta'})$ be such a face and let B' be the bilinear form corresponding to Δ'. Then there exists $v \in D \cap \mathrm{conv}(\widehat{\Delta'})$. Since $B(v,v) = B'(v,v) < 0$, we see that B' has the signature $(|\Delta'| - 1, 1)$. For $\Delta'' \subset \Delta$ if $\mathrm{conv}(\widehat{\Delta''})$ does not intersect with $D \cup \partial D$ then there are no elements $v \in \mathrm{conv}(\widehat{\Delta''})$ such that $B''(v,v) = 0$ where B'' is the bilinear form corresponding to Δ''. This is because that B'' is a principal submatrix of B. Thus B'' is positive definite. ∎

Before moving to the next proposition, we remark the following observation:

Remark 4.5. For $v = \sum_{\alpha\in\Delta} v_\alpha \alpha$ we have

$$q(v) = \sum_{\alpha\in\Delta} v_\alpha B(v,\alpha).$$

From this equality, if there exists $v \in \mathcal{P}$ such that $q(v) \geq 0$ then $v_\alpha < 0$ for some $\alpha \in \Delta$.

Proposition 4.6. *For each case, we have followings:*

(a) *The case* $(i) \iff \overline{\mathcal{P}} = \overline{\mathcal{P}'} \subset D$*;*
(b) *the case* $(ii) \iff \overline{\mathcal{P}'}$ *has some vertices in* ∂D*;*
(c) *the case* $(iii) \iff \overline{\mathcal{P}} \neq \overline{\mathcal{P}'}$ *and* $\overline{\mathcal{P}'} \subset D$*.*

Proof. (a) Remark 4.5 shows that if (i) then $\mathcal{P} \subset D$. Moreover, if there exists a vertex v of $\overline{\mathcal{P}}$ such that $q(v) = 0$ then $v_\alpha = 0$ for some $\alpha \in \Delta$ by the definition. This implies that $v \in \mathrm{conv}(\widehat{\Delta \setminus \{\alpha\}})$ and hence ∂D is tangent to the face $\mathrm{conv}(\widehat{\Delta \setminus \{\alpha\}})$. This is a contradiction. Thus $\overline{\mathcal{P}} \subset D \subset \mathrm{conv}(\widehat{\Delta})$ and $\mathcal{P} = \mathcal{P}'$.

Conversely, we assume that $\overline{\mathcal{P}} \subset D$. Consider $\Delta' \subset \Delta$ and let S' be the corresponding subset of S. Let B' be the bilinear form corresponding to S'. If $\mathrm{conv}(\widehat{\Delta'}) \cap D \neq \emptyset$ or $\mathrm{conv}(\widehat{\Delta'}) \cap \partial D \neq \emptyset$ then the Coxeter subgroup W' generated by S' is infinite. In particular, any Coxeter element in W' has infinite order. Moreover for $\alpha \in \Delta'$, H_α should intersect with $D \cap \mathrm{conv}(\widehat{\Delta'})$. Let v be a point of $\overline{\mathcal{P}}$ such that $v \in \bigcap_{\alpha\in\Delta'} H_\alpha$. Then $v \in D$ by our assumption. However since v is fixed by any element in W', we have an accumulation point in D. This contradicts to the discreteness of the normalized action of W on D.

(b) We have a face of $\mathrm{conv}(\widehat{\Delta})$ which is tangent to $\partial D = \widehat{Q}$. Let Δ' be a subset of Δ such that $\mathrm{conv}(\widehat{\Delta'})$ is tangent to ∂D and let v be the point of tangency. We set $S' = \{s_\alpha \mid \alpha \in \Delta'\}$. Then for any $\alpha \in \Delta'$, we have $B(v,\alpha) = 0$ since $v = s_\alpha \cdot v$ by Remark 4.3. Thus $v \in \bigcap_{\alpha\in\Delta'} H_\alpha$. Furthermore since $v \in \mathrm{conv}(\widehat{\Delta'})$ we have $v_\alpha = 0$ for $\alpha \in \Delta \setminus \Delta'$ if we write $v = \sum_{\alpha\in\Delta} v_\alpha \widehat{\alpha}$. Consequently we obtain

$$\{v\} = \bigcap_{\alpha\in\Delta'} H_\alpha \cap \bigcap_{\beta\in\Delta\setminus\Delta'} P_\beta. \tag{4}$$

This shows that v is a vertex of $\overline{\mathcal{P}'}$.

Conversely, we assume that there exists a vertex v of $\overline{\mathcal{P}'}$ on ∂Q written in (4) for some $\Delta' \subset \Delta$. Then $v \in \mathrm{span}(\Delta')$ where $\mathrm{span}(\Delta')$ is the subspace of V spanned by Δ'. Adding to this, since $v \in \bigcap_{\alpha\in\Delta'} H_\alpha$, we see that v is an eigenvector of the Coxeter matrix B' for Δ' corresponding to the 0 eigenvalue. Then we have that $\mathrm{conv}(\widehat{\Delta'})$ is tangent to ∂D.

(c) We assume that $\mathrm{conv}(\widehat{\Delta'}) \cap D \neq \emptyset$ for some $\Delta' \subset \Delta$. Obviously $\mathcal{P} \neq \mathcal{P}'$. Moreover, by Remark 4.5 we see that every vertex of $\overline{\mathcal{P}'}$ belongs to ∂D or D. However if there exists a vertex lying on ∂D then the case (ii) happens by the proof of (b). Thus all vertices of $\overline{\mathcal{P}'}$ are in D. The converse is clear by (a) and (b). ∎

From Proposition 4.6 we deduce that the fundamental region K (resp.K') is bounded if the case (i) (resp. the case (ii)) happens. If $\overline{K'}$ is not compact, then ∂D must be tangent to some faces of $\mathrm{conv}(\widehat{\Delta})$. In this case K' has some cusps at points of tangency of ∂D. This happens if and only if (ii). Because of this we call each cases as follows: The normalized action of W on D is

- *cocompact* if the case (i) happens;
- *with cusps* if the case (ii) happens;
- *convex cocompact* if the case (iii) happens.

For the case (ii) the *rank* of cusp v is the minimal rank of the affine Coxeter subgroup generated by a subset of S which fixes v.

Note that we can find easily that there exist Coxeter groups corresponding to each cases (i), (ii) and (iii).

Example 4.7. We see that classical hyperbolic Coxeter groups are in the case (i). For the case (iii) one of the simplest example is a triangle group $W = \langle s_1, s_2, s_3 \mid s_i^2\ (i = 1, 2, 3)\rangle$ with bi-linear form satisfying $B(\alpha_i, \alpha_j) < -1$ for $i \neq j$. At last it is in the case (ii) that $W = \langle s_1, s_2, s_3, s_4 \mid s_i^2, (s_1 s_2)^6, (s_1 s_3)^3, (s_j s_k)^2 (j \neq k \in \{2, 3, 4\})\rangle$ with the matrix $\{B(\alpha_i, \alpha_j)\}_{i,j}$ equals to

$$\begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & T \\ -\frac{\sqrt{3}}{2} & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ T & 0 & 0 & 1 \end{bmatrix}$$

where $T < -1$. In fact W is of type $(3, 1)$ although a subgroup generated by $\{s_1, s_2, s_3\}$ is of type $(2, 0)$.

2. The Gromov boundary and the CAT(0) boundary

2.1. The Gromov boundaries. The Gromov boundary of a hyperbolic space is one of the most studied boundary at infinity. In this section we define it for an arbitrary metric space due to [**7**].

Let (X, d, o) be a metric space with a base point o. We denote simply $(*|*)$ as the Gromov product with respect to the base point o. A sequence $x = \{x_i\}_i$ in X is a *Gromov sequece* if $(x_i|x_j)_z \longrightarrow \infty$ as $i, j \longrightarrow \infty$ for any base point $z \in X$. Note that if $(x_i|x_j)_z \longrightarrow \infty$ $(i, j \longrightarrow \infty)$ for some $z \in X$ then for any $z' \in X$ we have $(x_i|x_j)_{z'} \longrightarrow \infty$ $(i, j \longrightarrow \infty)$.

We define a binary relation $\sim_G$ on the set of Gromov sequences as follows. For two Gromov sequences $x = \{x_i\}_i, y = \{y_i\}_i$, $x \sim_G y$ if $\liminf_{i,j\longrightarrow\infty}(x_i|y_j) = \infty$. Then we say that two Gromov sequences x and y are equivalent $x \sim y$ if there exist a finite sequence $\{x = x_0, \ldots, x_k = y\}$ such that

$$x_{i-1} \sim_G x_i \ \text{ for } \ i = 1, \ldots, k.$$

It is easy to see that the relation $\sim$ is an equivalence relation on the set of Gromov sequences. The *Gromov boundary* $\partial_G X$ is the set of all equivalence classes $[x]$ of Gromov sequences x. If the space X is a finitely generated group G then the Gromov boundary of G depends on the choice of the generating set in general. In this thesis we always define the Gromov boundary of a Coxeter group W using the generating set of the Coxeter system (W, S). We shall use without comment the fact that every Gromov sequence is equivalent to each of its subsequences. To simplify the statement of the following definition, we denote a point $x \in X$ by the singleton equivalence class $[x] = [\{x_i\}_i]$ where $x_i = x$ for all i. We extend the Gromov product with base point o to $(X \cup \partial_G X) \times (X \cup \partial_G X)$ via the equations

$$(a|b) = \begin{cases} \inf\left\{\liminf_{i,j\longrightarrow\infty}(x_i|y_j) \;\middle|\; [x] = a,\ [y] = b\right\}, & \text{if } a \neq b, \\ \infty, & \text{if } a = b. \end{cases}$$

We set

$$U(x,r) := \{y \in \partial_G X \mid (x|y) > r\}$$

for $x \in \partial_G X$ and $r > 0$ and define $\mathcal{U} = \{U(x,r) \mid x \in \partial_G X, r > 0\}$. The Gromov boundary $\partial_G X$ can be regarded as a topological space with a subbasis $\mathcal{U}$.

If the space X is δ-hyperbolic in the sense of Gromov, then this topology is equivalent to a topology defined by the following metric. For $\epsilon > 0$ satisfying $\epsilon\delta \leq 1/5$, we define d_ϵ as follows:

$$d_\epsilon(a,b) = e^{-(a|b)} \quad (a, b \in \partial_G X).$$

Then it follows from 5.13 and 5.16 in [**33**] that d_ϵ is actually a metric. In this thesis, we always take ϵ so that $\epsilon\delta \leq 1/5$ for all δ hyperbolic spaces X and assume that $\partial_G X$ is equipped with d_ϵ-topology.

If an isometric group action $G \curvearrowright X$ on a Gromov hyperbolic space X is properly discontinuous and cocompact then the group G is also hyperbolic in the sense of Gromov and it is called a hyperbolic group (see [**33**]).

2.2. CAT(0) boundaries. The map we want is given via the *CAT(0) boundary* $\partial_I D$ (or $\partial_I D'$) of D (or D'). That is a space of geodesic rays emanating from a base point.

Assume that (X, d) is a complete geodesic space. Fix a point o in X. We denote $GR(X, o)$ to be the set of geodesic rays emanating from o:

$$GR(X,o) := \{\gamma \in C([0,\infty), X) \mid \gamma(0) = o, d(o,\gamma(t)) = len(\gamma|_{[0,t]})\ \forall t \in [0,\infty)\},$$

where $C([0,\infty), X)$ denotes the class of continuous maps from $[0,\infty)$ to X. Then we set $GR(X) := \bigcup_{o\in X} GR(X,o)$. Two rays $\gamma, \eta \in GR(X)$ are equivalent $\gamma \sim \eta$ if the supremum $\sup_{t\geq 0} d(\gamma(t), \eta(t))$ is finite. Let $\partial_I X$ be the coset $GR(X)/\sim$ and call this the ideal boundary of X. If X is $CAT(0)$ in addition, then for any point ξ in $\partial_I X$ there exists a unique geodesic γ emanating from o so that the equivalence class of γ equals to ξ (consult with [**5**]). Hence we can identify $GR(X, o)$ and $\partial_I X$ for some fixed $o \in X$ whenever X is CAT(0). In this case we call $\partial_I X$ *the CAT(0) boundary* of X. Since all geodesic rays in $GR(X, o)$ are unbounded, $\partial_I X$ appears at infinitely far from any point in X.

We assume that (X, d) is complete CAT(0) space. We attach the *cone topology* τ_C to the union $X \cup \partial_I X$ then it coincides with original topology in X. This topology is Hausdorff and compact whenever X is proper. We briefly define τ_C here. For the detail of the cone topology, see [**5**]. This is defined by using a base point $o \in X$ but is independent of the choice of o. First, notice that for any $x \in X \cup \partial_I X$ there exists a unique geodesic γ_x from o to x. In the case where $x \in \partial_I X$, we merely mean that x equals to the equivalence class of γ_x. For $r \in (0,\infty)$ set $X_r = \partial_I X \cup (X \setminus \overline{Ball(o,r)})$ where $\overline{Ball(o,r)}$ is the closure of an open ball $Ball(o,r)$ centered at o whose radius is r. Let $S(o,r)$ be the boundary of $Ball(o,r)$ and let $p_r : X_r \longrightarrow S(o,r)$ be the projection defined by $p_r(x) = \gamma_x(r)$ and let the set $U(a,r,s)$, $r, s > 0$, consist of all $x \in X_r$ such that $d(p_r(x), p_r(a)) < s$. We notice that $U(x,r,s)$ consists of geodesics passing through the intersection of $S(o,r)$ and $Ball(p_r(x), s)$. Then τ_C has as a local base at $a \in \partial_I X$ the sets $U(a,r,s)$, $r, s > 0$.

We return to our situation. Since the region D' and D are both complete CAT(0) space, CAT(0) boundaries for each space are well defined. We use the eigenvector o for the negative eigenvalue as the base point in the definition of CAT(0) boundary and the cone topology. Furthermore since D' is a subspace of D, its CAT(0) boundary $\partial_I(D')$ is a subspace of $\partial_I D$.

Proposition 4.8. *$\partial_I D$ (resp. $\partial_I D'$) is homeomorphic to ∂D (resp. $\partial D' \setminus D$).*

Proof. It suffices to see this for the case where the entire space D. Fix a base point $o \in D$. For any $\xi \in \partial_I D$, ξ is a geodesic ray from $o \in D$ and is also a geodesic segment with respect to the Euclidean metric in D. Hence ξ defines a unique endpoint x in ∂D. Conversely for any $y \in \partial D$ take a segment $[o, y]$ from o to y. Then $[o, y]$ is a geodesic with respect to the Hilbert metric which tends to infinity. Therefore we have a bijection $h : \partial_I D \longrightarrow \partial D$.

For any $\gamma \in \partial_I D = \partial D$ we identify the geodesic γ emanating from o in the topology of d and a (half-open) segment $[o, \gamma] \setminus \{\gamma\}$ in the Euclidean topology parametrized by $[0, \infty)$ so that $h(\gamma(t)) = \gamma(t)$.

Let U be an open ball with respect to the Euclidean subspace topology centered at some point in ∂D. We set $\widetilde{U} = \bigcup_{\gamma \in U,\ t \in (0,\infty)} \gamma(t)$. Obviously $\widetilde{U}$ is open in the Euclidean topology. Then for any γ in U and any $t \in [0, \infty)$ there exists a Euclidean open ball $Ball_E(\gamma(t))$ centered at $\gamma(t)$ included in $\widetilde{U}$. Since the identity map $(D, d) \longrightarrow (D, d_E)$ is a homeomorphism, we have an open ball $Ball(\gamma(t), s)$ centered at $\gamma(t)$ in $Ball_E(\gamma(t))$ with respect to the topology of d. Considering the intersection T of sphere $S(o, t)$ and $Ball(\gamma(t), s)$, we see that geodesics from o through T is included in U. This shows that h is a continuous bijection from a compact set to a Hausdorff space and hence it is a homeomorphism. ∎

Remark 4.9. If the case space X is a complete proper hyperbolic CAT(0) space then $\partial_G X \simeq \partial_I X$ ([**7**, Theorem 2.2 (d)]). Because of this, if the case (i) (resp. the case (iii)) happens then $\partial_I D \simeq \partial_G D$ (resp. $\partial_I D' \simeq \partial_G D'$).

Remark 4.10. If the case (iii) happens, then $\Lambda(W)$ is homeomorphic to $\partial D' \setminus D$ by Theorem 2.2 and Theorem 3.1. Together with this and Proposition 4.8, we see that $\Lambda(W) = \partial D' \setminus D \simeq \partial_I D' \simeq \partial_G D'$.

3. The Cannon-Thurston maps

In this section, we give a proof of Theorem 4.1 for case (i),(iii) and with rank 2 cusps. Throughout this section, a vector o denotes the normalized eigenvector corresponding to the negative eigenvalue of B.

3.1. The case of W acting without cusps. We consider when W acts cocompactly or convex cocompactly. In this case W is hyperbolic in the sense of Gromov. Moreover for the case (*iii*), K' is bounded. Together with the convexity of D', we see that Proposition 3.19 also holds in this case.

For simplicity, we mean $\widetilde{D}$ for D or D'. Our purpose in this section is actually to construct a homeomorphism from $\partial_G(W, S)$ to $\partial\widetilde{D}$ via Remark 4.9, 4.10.

We define the map $f : W \longrightarrow \widetilde{D}$ by $w \mapsto w \cdot o$ where o is the eigenvector of the negative eigenvalue. This map is a quasi-isometry by Lemma 3.16.

It is well known that f extends to a homeomorphism between $\partial_G(W, S) \cup W$ and $\partial_G\widetilde{D} \cup \widetilde{D}$ (conf. [**33**]). Let $\overline{f}$ be the restriction of the homeomorphism above to $\partial_G W$. Now we recall following two maps. By the result of Buckley and Kokkendorff [**7**], we know that there exists a homeomorphism $g : \partial_G\widetilde{D} \longrightarrow \partial_I\widetilde{D}$. Moreover, for a Gromov sequence $\xi \in \partial_G\widetilde{D}$ any unbounded sequence given as a subset of a geodesic ray $g(\xi)$ is equivalent to ξ. On the other hand by Proposition 4.8 we have a homeomorphism $h : \partial_I\widetilde{D} \longrightarrow \partial\widetilde{D}$.

We compose these homeomorphisms. Let $F = h \circ g \circ \overline{f}$. Then we have a homeomorphism from $\partial_G(W, S)$ to $\partial\widetilde{D}$. We verify that F sends $\omega \in \partial_G(W, S)$ to the limit point defined by $\{w_k \cdot o\}_k$ for $\{w_k\}_k \in \omega$. If this is true, then we see that F is W-equivariant by the construction. To see this, we inspect the details of the maps g and h. For our situation, the proof in [**7**] says that for a Gromov sequence $\{w_k \cdot o\}_k \in F([\{w_k\}_k])$ in W, there exists a ξ such that a sequence $\{u_i \cdot o\}_i$ constructed by the same way as in the proof of Proposition 3.19 is a short sequence included in a bounded neighborhood of ξ. The image of ξ by h is equivalent to $\{u_i \cdot o\}_i$ in the sense of Gromov. Adding to this, Buckley and Kokkendorff showed that $\{u_i \cdot o\}_i$ equivalent to the original sequence $\{w_k \cdot o\}_k$ and hence they converge to the same point in $\partial_G\widetilde{D} \setminus D$. By Remark 4.10 F is the map we want.

3.2. The case of *W* acting with cusps. We know that there exist some Coxeter groups acting on D with cusps. By Proposition 4.6, this happens when ∂D is tangent to some faces of conv(Δ). We divide this case into following three cases;

(i) there exists at least one pair of simple roots $\alpha, \beta \in \Delta$ so that $B(\alpha, \beta) = -1$,
(ii) there exists at least one subset $\Delta' \subset \Delta$ whose cardinality is more than 3 so that the corresponding matrix B' is positive semidefinite (not positive definite) where B' is the matrix obtained by restricting B to Δ',
(iii) or (i) and (ii) happen simultaneously.

We only deal with the case (i). In this case, the dihedral subgroup of W generated by s_α and s_β is infinite and its limit set is one point. This means that D is tangent to the segment connecting α and β. Hence the fundamental region of W is unbounded.

For the cases (ii) and (iii), we have to see other geometric aspects of the Coxeter groups. We will discuss the existence of the Cannon-Thurston maps for the excepted cases in the next section.

Recall that the number n is the rank of W and hence equals to the dimension of V. Let $\{A_m\}_m$ be a sequence of $n \times n$ matrices which are defined as follows. For each $m \in \mathbb{N}$, we define A_m so that

$$A_m(\alpha, \beta) = \begin{cases} 1/m, & \text{if } B(\alpha, \beta) = -1, \\ 0, & \text{if otherwise,} \end{cases}$$

for each $\alpha, \beta \in \Delta$. Then let $B_m = B - A_m$.

If B has the signature $(n-1, 1)$, then B_m also has the signature $(n-1, 1)$ for sufficiently large $m \in \mathbb{N}$. Therefore for sufficiently large m, our definitions of Q,D, D', L, K can be extended to the bilinear form defined by B_m. We define Q_m, D_m, D'_m, L_m, K_m each of them by using B_m instead of B in their definitions.

Clearly B_m converges to B as m tends to ∞. It is not trivial that their normalized eigenvectors converge to the normalized eigenvectors of B. This is an well-known fact. However we briefly review the proof as a remark. In this remark, there is no need to assume that B has the the signature $(n-1, 1)$.

Remark 4.11. In this remark we assume that all vectors are normalized with respect to some norm. Since A_m converges to 0 matrix, by Rouche's theorem, the characteristic equation of B_m has the same number of roots as the characteristic equation of B around each eigenvalues of B for sufficiently large m in $\mathbb{C}$. Hence the eigenvalues of B_m converge to the eigenvalues of B. Let C_i $(i = 1, \ldots, n)$ be boundaries of balls $Ball(\lambda_i, \epsilon)$ in $\mathbb{C}$ centered at each eigenvalues λ_i of B for sufficiently small $\epsilon > 0$. For sufficiently large m, we can assume that each eigenvalue of B_m is included in $Ball(\lambda_i, \epsilon)$ for some i. Set linear operators

$P_{m,i}$ and P_i on V by

$$P_{m,i}(v) = \frac{1}{2\pi\sqrt{-1}}\int_{C_i}(\lambda - B_m)^{-1}v d\lambda, \quad P_i(v) = \frac{1}{2\pi\sqrt{-1}}\int_{C_i}(\lambda - B)^{-1}v d\lambda.$$

Then it turns out that each operator $P_{m,i}$ is the projection to the eigenspace of some eigenvalues of B_m near by λ_i. Since $(\lambda - B_m)^{-1}v$ is bounded on C_i for each $v \in V$ and i, $P_{m,i}(v)$ converges to $P_i(v)$ by Lebesgue's theorem. This shows that the eigenvectors of B_m converges to the eigenvectors of B. In particular $\{o_m\}_m$ converges to o since $|*|_1$ is a norm in the cone spanned by Δ. Even if there exist some eigenvalues of B whose multiplicity is more than 1, these convergence preserves linearly independence of eigenvectors (by taking a subsequence if necessary).

Let $v_1, \ldots, v_n$ be eigenvectors of B normalized with respect to the Euclidean norm so that the matrix $(v_1, \ldots, v_n)$ diagonalize B. Then since each $P_{m,i}(v_i)$ converges to v_i, the matrix diagonalizing B_m also converges to $(v_1, \ldots, v_n)$. This fact shows that the sequence $\{D_m\}_m$ converges to D.

We can consider the B_m-reflection of W on V with respect to B_m. We denote this action by ρ_m. For example, the simple B_m-reflection of $\alpha \in \Delta$ can be calculated as

$$\rho_m(s_\alpha)(x) = x - 2B_m(x, \alpha)\alpha, \quad (x \in V).$$

The normalized action with respect to B_m is defined in the same way as B. We denote this also by ρ_m. Furthermore if B_m has the signature $(n-1, 1)$, then all our Lemmas and Propositions can be proved by using the normalized eigenvector o_m corresponding to the negative eigenvalue of B_m instead of o. Therefore if the normalized action ρ_m is (convex) cocompact, then there exists a map F_m from the Gromov boundary $\partial_G(W, S)$ of W to the limit set $\Lambda_{B_m}(W)$ which is homeomorphic. In fact we have a W-equivariant homeomorphism $F_m : \partial_G(W, S) \longrightarrow \Lambda_{B_m}(W)$ for each m since the case (iii) happens. Note that for sufficiently large m, we have $V_0 \cap Q_m = \{\mathbf{0}\}$. Hence we can define the Hilbert metric on $V_1 \cap Q_{m-}$ where $Q_{m-} = \{v \in V \mid B_m(v, v) < 0\}$. Consider the correspondence between $x \in D_m$ and $y = \mathbb{R}x \cap V_1 \cap Q_{m-}$. Then we see that this is an isometry between D_m and $V_1 \cap Q_{m-}$ and W equivaliant. Thus we can regard the normalized action ρ_m as an action of W on $V_1 \cap Q_{m-}$.

We remark that for any $\alpha \in \Delta$ and $m \in \mathbb{N}$, we have $B_m(o, \alpha) = B(o, \alpha) - A_m(o, \alpha) < 0$ since $B(o, \alpha) < 0$ and all coordinates of o are positive. Hence o is in K_m for any $m \in \mathbb{N}$.

Lemma 4.12. *Let o be the normalized eigenvector corresponding to the negative eigenvalue of B. There exists a constant $C_1 > 0$ such that $|w(o)|_1 \geq C_1|w|$ for any $w \in W$.*

Proof. Let $\lambda > 0$ be the absolute value of the negative eigenvector of B hence $Bo = -\lambda o$. Note that all coordinates of o are positive by Lemma 1.2. If $|w| = 1$ then there exists $\alpha \in \Delta$ such that $w = s_\alpha$. Then we have

$$|s_\alpha(o)|_1 = |o|_1 - 2B(o, \alpha)|\alpha|_1 = 1 + 2\lambda(o, \alpha)|\alpha|_1 > 1 = |s_\alpha|.$$

Before moving to the inductive step we remark the following. By [**18**, Lemma 2.10 (ii)] there exists a constant C' such that for $w \in W$ and $\alpha \in \Delta$ with $w(\alpha) \in \mathrm{cone}(\Delta)$, $|w(\alpha)|_1 \geq C'|w|^{\frac{1}{2}}$ where $\mathrm{cone}(\Delta)$ is the cone spanned by Δ. Since $o(= \sum_{\delta\in\Delta} o_\delta \delta)$ is in the convex hull of Δ each coordinate o_δ of o satisfies $0 \leq o_\delta \leq 1$. Letting $\lambda' = \min_{\delta\in\Delta} o_\delta$, for $w \in W$ and $\alpha \in \Delta$ with $w(\alpha) \in \mathrm{cone}(\Delta)$ we have

$$-B(o, w(\alpha)) = \lambda \sum_\beta o_\beta w(\alpha)_\beta \geq \lambda\lambda'|w(\alpha)|_1 \geq \lambda\lambda' C'|w|^{\frac{1}{2}}$$

where $w(\alpha)_\beta$ denotes the β-th coordinate of $w(\alpha)$.

For the inductive step we take an arbitrary $w \in W$ with $|w| = k+1$ ($k \in \mathbb{N}$) and assume that for any $w' \in W$ with $|w'| \leq k$ we have $|w'(o)|_1 \geq C|w'|$ for some universal constant $C \geq 1$. We take $w' \in W$ so that $w = s_\alpha w'$ with $|w'| = k$ for some $\alpha \in \Delta$. Then by Remark 3.9 all coordinates of $w'^{-1}(\alpha)$ are non-negative. From the argument above, we have

$$|w(o)|_1 = |w'(o)|_1 - 2B(o, w'^{-1}(\alpha)) \geq Ck + 2\lambda\lambda' C' k^{\frac{1}{2}} \geq C(k+1).$$

if $C \leq 2\lambda\lambda' C'$. Thus taking C_1 so that $C_1 \leq \min\{1, 2\lambda\lambda' C'\}$, we have the conclusion. ∎

Let $c_0 > 1$ be the maximum operator norm of S. More precisely, we set $c_0 = \max_{s\in S} \max_{x \in S^{n-1}} \|s(x)\|$ where S^{n-1} is the sphere in V centered at 0. Then for any $w \in W$ with $|w| = k$, we have $c^k \geq \|w(o)\|$. Since the Euclidean norm $\| * \|$ is comparable to $| * |_1$ in the cone Q_-^+, there exists a constant $C_{2,0}$ such that $C_{2,0} c_0^k \geq |w(o)|_1$. We can take these constants $C_{2,m}$ and c_m for each $\rho_m(W)$ ($m \in \mathbb{N}$). Since the sequence $\{B_m\}_m$ converges to B, sequences $\{C_{2,m}\}_m$ and (c_m) must converge to $C_{2,0}$ and c_0. Thus there must exist the maximum

$$C_2 = \max_{m \in \mathbb{N}\cup\{0\}} C_{2,m}, \quad \text{and} \quad c = \max_{m\in\mathbb{N}\cup\{0\}} c_m.$$

Proposition 4.13. *Assume that the normalized action of W includes rank 2 cusps. There exists a continuous W-equivariant surjection $\iota : \Lambda(\rho_1(W)) \longrightarrow \Lambda(W)$.*

Proof. Since Q and V_0 meet only at 0, B is positive definite on V_0. Hence B defines an inner product on V_0 and it gives a metric on V_0 by $q(x-y)^{\frac{1}{2}}$. It is easy to see that this metric induces to V_1 and it is comparable to the Euclidean metric.

Let o be the normalized eigenvector for the negative eigenvalue $-\lambda$ of B. Notice that $o \in K_m$ for any $m \in \mathbb{N}$ since $B_m(o,\alpha) = B(o,o) - A_m(o,\alpha) < 0$ and $D \subset D_m$. We claim that for any short sequence $\{w_k\}_k$ in W, if $\rho_m(w_k)\cdot o \longrightarrow \xi \in \partial D$ as $k, m \longrightarrow \infty$ then $w_k \cdot o \longrightarrow \xi$ as $k \longrightarrow \infty$. This ensures that the correspondence $\iota(\xi_1) = \xi$ for each $\xi_1 \in \Lambda(\rho_1(W))$ is actually a map where $\xi \in \Lambda(W)$ is the equivalence class of the sequence $\{w_k \cdot o\}_k$ for $\{w_k\}_k$ defining ξ_1. If ι is well-defined then it is obviously W-equivariant and surjective. To see that ι is continuous, it suffices to show that $q(w\cdot o - \rho_m(w)\cdot o) \longrightarrow 0$ as $k, m \longrightarrow \infty$ uniformly.

Fix $m \in \mathbb{N}$ arbitrarily. We write $A_m(v, v') := {}^t v A_m v'$ for $v, v' \in V$ where ${}^t v$ denotes transpose of a vector v. For any $x \in \mathrm{cone}(\Delta)$ and any $\alpha \in \Delta$ we have

$$|\rho_m(s_\alpha)(x)|_1 = |s_\alpha(x) + 2A_m(x,\alpha)\alpha|_1 \geq |s_\alpha(x)|_1 \qquad \text{and} \qquad A_m(x,\alpha) \leq \frac{|x|_1}{m}.$$

The first inequality shows that for any $x \in D$ whose orbit $W(x)$ is included in $\mathrm{cone}(\Delta)$, we have $|\rho_m(w)(x)|_1 \geq |w(x)|_1$ for any $w \in W$. The second inequality implies the following;

$$\begin{aligned} -B(s_\alpha(x), \rho_m(s_\alpha)(x)) &= -B(s_\alpha(x), s_\alpha(x) + 2A_m(x,\alpha)\alpha) \\ &\leq -q(x) - 2A_m(x,\alpha)B(x,\alpha) \\ &\leq -q(x) - 2C\frac{|x|_1}{m}B(x,\alpha), \end{aligned}$$

for any $x \in \mathrm{cone}(\Delta)$ and any $\alpha \in \Delta$ where C is a constant depending on B.

We show that $-B(w(o), \rho_m(w)(o)) \geq -q(o)$ for any $w \in W$ by the induction. For any $\alpha \in \Delta$ the argument above gives an inequality

$$-B(s_\alpha(o), \rho_m(s_\alpha)(o)) \leq -q(o) - \frac{2\lambda o_\alpha}{m} \leq -q(o),$$

where o_α denotes the α-th coordinate of o. Take $w \in W$ so that $|w| > 1$ arbitrarily. We assume that $-B(w(o), \rho_m(w(o)) \leq -q(o)$. For $\alpha \in \Delta$ if $|s_\alpha w| = |w| + 1$ then we have

$$\begin{aligned}
-B(s_\alpha w(o), \rho_m(s_\alpha)\rho_m(w)(o)) &= -B(s_\alpha w(o), s_\alpha(\rho_m(w)(o)) + 2A_m(\rho_m(w)(o), \alpha)\alpha) \\
&= -B(s_\alpha w(o), s_\alpha(\rho_m(w)(o))) - 2A_m(\rho_m(w)(o), \alpha)B(s_\alpha w(o), \alpha) \\
&= -B(s_\alpha w(o), s_\alpha(\rho_m(w)(o))) + 2A_m(\rho_m(w)(o), \alpha)B(w(o), \alpha) \\
&= -B(s_\alpha w(o), s_\alpha(\rho_m(w)(o))) - 2\lambda A_m(\rho_m(w)(o), \alpha)(o, w^{-1}(\alpha)) \\
&\leq -q(o),
\end{aligned}$$

where $(,)$ denotes the Euclidean inner product. Furthermore we have

$$\begin{aligned}
q(\rho_m(w)(o)) = B(\rho_m(w)(o), \rho_m(w)(o)) &= B_m(\rho_m(w)(o), \rho_m(w)(o)) + A_m(\rho_m(w)(o), \rho_m(w)(o)) \\
&= B_m(o, o) + A_m(\rho_m(w)(o), \rho_m(w)(o)) \\
&= q(o) - A_m(o, o) + A_m(\rho_m(w)(o), \rho_m(w)(o)) \\
&\leq q(o) + C^2 \frac{|\rho_m(w)(o)|_1^2}{m}
\end{aligned}$$

for any $w \in W$. Together with these inequalities we deduce that for $w \in W$ with $|w| = k$,

$$\begin{aligned}
q(w \cdot o - \rho_m(w) \cdot o) &= \frac{q(o)}{|w(o)|_1^2} - 2\frac{B(w(o), \rho_m(w)(o))}{|w(o)|_1 |\rho_m(w)(o)|_1} + \frac{q(\rho_m(w)(o))}{|\rho_m(w)(o)|_1^2} \\
&\leq \frac{q(o)}{|w(o)|_1^2} - 2\frac{q(o)}{|w(o)|_1 |\rho_m(w)(o)|_1} + \frac{q(o)}{|\rho_m(w)(o)|_1^2} + \frac{1}{m} \\
&\leq \frac{q(o)}{C_1 k^2} - 2\frac{q(o)}{C_2 k^2} + \frac{q(o)}{C_1 k^2} + \frac{1}{m}.
\end{aligned}$$

This shows that the convergence of $q(w \cdot o - \rho_m(w) \cdot o) \longrightarrow 0$ as $k, m \longrightarrow \infty$ does not depend on the short sequence (w_k). Thus ι is well-defined and continuous. ∎

Considering the composition $F' = \iota \circ F_1$, we have the map which is surjective, continuous and W-equivariant.

Remark 4.14. If $B(\alpha, \beta) = -1$ for some $\alpha, \beta \in \Delta$ then the Coxeter subgroup W' generated by $\{s_\alpha, s_\beta\}$ is affine. Since an affine Coxeter group has only one limit point, $\{(s_\alpha s_\beta)^k \cdot o\}_k$ and $\{(s_\beta s_\alpha)^k \cdot o\}_k$ converges to the same limit point. However in the Gromov boundary of (W, S), $\{(s_\alpha s_\beta)^k\}_k$ and $\{(s_\beta s_\alpha)^k\}_k$ lie in distinct equivalence classes. In fact, considering another action of (W, S) defined by another bi-linear form B' such that $B'(\alpha, \beta) < -1$, then the limit set $\Lambda_{B'}(W') \subset \Lambda_{B'}(W)$ consists of two points. In this case the limit points of $\{(s_\alpha s_\beta)^k \cdot o\}_k$ and $\{(s_\beta s_\alpha)^k \cdot o\}_k$ are distinct. On the other hand the map $\partial_G(W, S) \longrightarrow \Lambda_{B'}(W)$ is well defined hence F' cannot be an injection.

4. Coxeter groups with higher rank cusps

In this section we discuss the case where the Coxeter groups have higher rank cusps.

4.1. The Cannon-Thurston maps for W with higher rank cusps. Let (W, S) be a Coxeter system of type $(n-1, 1)$. We recall that $\mathrm{conv}(\widehat{\Delta})$ is a polytope. It can happen three distinct situations due to the bilinear form B;

(i) the region $D \cup \partial D$ is included in $\mathrm{int}(conv(\widehat{\Delta}))$;
(ii) there exist some faces of $\mathrm{conv}(\widehat{\Delta})$ which are tangent to the boundary ∂D;
(iii) No face of $\mathrm{conv}(\widehat{\Delta})$ is tangent to ∂D.

We focus on the case (ii) in this section. Hence we assume that W includes at least one affine Coxeter subgroup W' with generating set $S' \subset S$.

Before discussing this case we see that affine subgroups contain so-called parabolic isometries. For a metric space, its isometries are classified into three types by the translation length. The *translation length* of an isometry γ of a metric space (X, d) is the value $trans(\gamma) := \inf\{d(x, \gamma(x)) \mid x \in X\}$.

Definition 4.15. For a metric space X an isometry γ of X is called

(1) *elliptic* if $trans(\gamma) = 0$ and attains in X,
(2) *hyperbolic* if $trans(\gamma)$ attains a strictly positive minimum,
(3) *parabolic* if $trans(\gamma)$ does not attain its minimum.

If the space X is CAT(0) then one can rephrase this classification by using fixed points on $X \cup \partial_I X$, where $\partial_I X$ is the CAT(0) boundary of X.

Remark 4.16. Let X be a CAT(0) space and γ be an isometry on X. It is clear that γ is elliptic if and only if there exists at least one fixed point of γ in X. In the case where X is proper and the group $\langle\gamma\rangle$ acts on X discretely, if γ is elliptic then $\langle\gamma\rangle$ is finite. For the hyperbolic isometry, we have a fixed geodesic line in X ([**5**, Theorem 6.8(1)]). This means that if γ is hyperbolic then there are at least two fixed points by γ on $\partial_I X$. Accordingly if γ is of infinite order and has only one fixed point in $\partial_I X$ then it is parabolic.

Lemma 4.17. *Consider the action of W on V as B-reflections. Assume that $w \in W$ has the infinite order and an eigenvector ξ on $\widehat{Q}$ corresponding to the eigenvalue* 1. *Then any eigenvector of w corresponding to the eigenvalue* 1 *on $\widehat{Q}$ lies on the set $\mathbb{R}\xi$.*

Proof. We assume that w has another eigenvector $\eta \neq \xi$ in $\widehat{Q}$ for the eigenvalue λ. Then by Lemma 1.10, $\lambda = 1$.

Consider the normalized action of W on D. Notice that we have $w \cdot \widehat{\xi} = \widehat{\xi}$ and $w \cdot \widehat{\eta} = \widehat{\eta}$. In this case a point $t\widehat{\xi} + (1 - t)\widehat{\eta}$ for $t \in [0, 1]$ is fixed by w and lies in D. This contradicts to the discreteness of the normalized action of W. ∎

Proposition 4.18. *Any infinite order element w of an affine Coxeter subgroup W' of W with generating set $S' \subset S$ is a parabolic isometry of (D, d).*

Proof. Let Δ' be a subset of Δ corresponding to S' and let B' be the submatrix corresponding to Δ'. We remark that B-reflections on $\mathrm{conv}(\widehat{\Delta'})$ coincide with B'-reflections. The discreteness of the normalized action ensures that w is either hyperbolic or parabolic.

Now by [**18**, Corollary 2.15] and Theorem 3.1 the limit set of W' is a singleton $\{\xi\}$. Moreover it equals to $\widehat{Q'} = \{v \in V_1 \mid B'(v, v) = 0\}$. Then ξ should be a fixed point in ∂D by w since B'-reflections preserve the bilinear form B'. Considering the action of w as a composition of B-reflections, we see that ξ is an eigenvector of w for the eigenvalue λ. On the other hand, by the definition of $\widehat{Q'}$, ξ is also an eigenvector of B' corresponding to the eigenvalue 0. This shows that $\lambda = 1$. Lemma 4.17 says that such an element in W fixes only one point in ∂D. By Remark 4.16, we see that w is parabolic. ∎

By Proposition 4.6 (b) a point of tangency $p \in \mathrm{conv}(\widehat{\Delta'}) \cap \partial D$ for some $\Delta' \subset \Delta$ can be characterized as a vertex of $\overline{\mathcal{P}'}$, that is, the intersection of hyperplanes $\{H_\alpha \mid \alpha \in \Delta'\}$ and $\{P_\beta \mid \beta \in \Delta \setminus \Delta'\}$. We define PF to be the set of all of such points:

$$PF := \left\{ p \in \partial D \;\middle|\; \exists \Delta' \subset \Delta \text{ s.t. } \{p\} = \left(\bigcap_{\alpha \in \Delta'} H_\alpha \right) \cap \left(\bigcap_{\beta \in \Delta \setminus \Delta'} P_\beta \right) \right\}.$$

By the definition, PF is finite.

REMARK 4.19. For $p \in PF$ so that $\{p\} = (\bigcap_{\alpha\in\Delta'} H_\alpha) \cap \left(\bigcap_{\delta\in\Delta\setminus\Delta'} P_\delta\right)$, the affine subgroup W' generated by $\{s_\alpha \mid \alpha \in \Delta'\}$ fixes p by Proposition 4.4 (b). Conversely the minimal Coxeter subgroup fixing p is W'. In fact for $w \in W \setminus W'$ there exists at least one $\beta \in \Delta \setminus \Delta'$ such that the β-th coordinate of $w \cdot p$ is not 0. This can be seen by the induction on the word length. For $\beta \in \Delta \setminus \Delta'$, then the β-th coordinate of $s_\beta(p)$ is $-2B(p,\beta) > 0$ since $p \in \overline{K}$ hence $B(p,\beta) \leq 0$. We assume that the claim holds for all elements in W whose word length are less than or equal to k. For $w \in W$ satisfying $|w| = k+1$ and $w = s_\beta w'$ where w' is of $|w'| = k$, we have $w(p) = w'(p) - 2B(w'(p),\beta)\beta$. Then by the inductive assumption there exists at least one non-zero coordinate of $w'(p)$ for some $\gamma \in \Delta \setminus \Delta'$. If $\gamma \neq \beta$ then we obtain the claim. For the case where $\gamma = \beta$ and the β-th coordinate of $w'(p)$ is not 0, we have $B(w'(p),\beta) = B(p, w'^{-1}(\beta)) \leq 0$ since all coordinates of $w'^{-1}(\beta)$ are non-zero (Remark 3.9) and $p \in \overline{K}$.

DEFINITION 4.20. Let (X,d) be a CAT(0) space. Fix a point $o \in X$ and take $k \in \mathbb{R}$. For $\xi \in \partial_I X$, we take a geodesic c from o to ξ. A *horoball* at ξ with k (based at o) is a set

$$O_{\xi,k} = \left\{ x \in X \;\middle|\; \lim_{t\longrightarrow\infty} d(c(t),x) - t < k \right\}.$$

The boundary of a horoball $\partial O_{\xi,k}$ is called a *horosphere*, that is,

$$\partial O_{\xi,k} = \left\{ x \in X \;\middle|\; \lim_{t\longrightarrow\infty} d(c(t),x) - t = k \right\}.$$

The function $b_c(x) := \lim_{t\longrightarrow\infty} d(c(t),x) - t$ defining the horoball is said to be a *Busemann function* associated with c. It is known that Busemann function is well defined, convex and 1-Lipschitz. We remark that $O_{\xi,k} \subset O_{\xi,k'}$ for $k < k'$ and $O_{p,k}$ tends to p for $k \longrightarrow -\infty$. In this thesis, we always take the normalized eigenvector for the negative eigenvalue of B as the base point o.

LEMMA 4.21. *There exists $k \in \mathbb{R}$ such that for any $p, p' \in PF$ and $w \in W$, if $O_{p,k} \neq w \cdot O_{p',k}$ then*

$$O_{p,k} \cap w \cdot O_{p',k} = \emptyset.$$

PROOF. For any k and $x \notin O_{p,k}$ we have $k \leq \lim_{t\longrightarrow\infty} d(c(t),x) - t$ where c is the geodesic from o to p. If $x \in K \setminus O_{p,k}$ then since K equals to the Dirichlet region at x we have

$$k \leq d(c(t),x) - t \leq d(c(t), w\cdot x) - t$$

for all $w \in W$ and $t \in \mathbb{R}_{\geq 0}$. Hence we have $W \cdot x \notin O_{p,k}$.

Take a closed ball $B(o,r)$ centered at o and the radius r satisfying the condition $r > \max\{\{d(o,[\alpha,\beta]) \mid \alpha,\beta \in \Delta\}, \{d(o,P_\alpha) \mid \alpha \in \Delta\}\}$ where $[\alpha,\beta]$ is the segment connecting α and β. This maximum always exists since Δ is finite. By the definition of r, each component of $\overline{K'} \setminus B(o,r)$ includes just one vertex of $\overline{K'}$. Since the Busemann function is continuous and $B(o,r)$ is compact, we have $k < 0$ such that $O_{p,k)} \cap B(o,r) = \emptyset$ for all $p \in PF$.

If there exists $w \in W$ such that $O_{p,k} \neq w \cdot O_{p',k}$ and $O_{p,k} \cap w \cdot O_{p',k} \neq \emptyset$ then there must be $x \in K \cap O_{p,k} \cap w'w \cdot O_{p',k}$ for some $w' \in W$ by the above argument. Let ξ be the Euclidean segment from p to x and let η be the Euclidean segment from $w'w \cdot p$ to x. By the definition of $k > 0$, ξ does not intersect with any face of $\partial K'$ which does not contain p. η also does not intersect with any face of $w'w \cdot \partial K'$ which does not contain $w'w \cdot p'$. Thus we have $x \in K' \cap w'w \cdot \overline{K'}$. This contradicts to the discreteness of the normalized action. ∎

REMARK 4.22. For $p \in PF$, let W' be the minimal affine subgroup of W which fixes p. We notice that $O_{p,k}$ is covered by the $W' \cdot \overline{K}$. In fact for any $y \in O_{p,k}$ the Euclidean segment $[p, y]$ is included in $O_{p,k}$. If $y \notin H_\alpha$ then $[p, y]$ does not intersect with all H_α such that $p \in H_\alpha$. If $y \in H_\alpha$ then $[p, y] \subset H_\alpha$. Thus y is included in an orbit $w \cdot K$ for some $w \in W$ such that $w \cdot K$ contains p as a vertex. Since if $w \in W$ fixes p then $w \in W'$ by Remark 4.19, the above argument shows that $w \in W'$.

Fix a constant k which smaller than the constant in the claim of Lemma 4.21. Set $O := \{O_{p,k}\}_{p \in PF}$. We remove the orbits of O from D and denote it by D'':

$$D'' = D' \setminus W \cdot O.$$

Then we have the following.

LEMMA 4.23. *The set D'' is invariant under the normalized action of W.*

We define $K'' = K \cap D''$ then K'' is not empty because $o \in K''$ by the proof of Lemma 4.21. Recall that O contains all horoballs at the vertices of $\overline{K}$ which lie on ∂D. This indicates that $\overline{K''}$ is bounded closed set hence compact since D is proper. Since K is a fundamental region of the normalized action, Lemma 4.23 says that K'' is a fundamental region of the normalized action restricted on D''. Define a metric d' on D'' by letting $d'(x, y)$ be the minimum length of a path in D'' connecting x and y.

LEMMA 4.24. *Under the notations above, there exist constants l and l' so that*

$$l|w| \leq d'(o, w \cdot o) \leq l'|w|$$

for all $w \in W$.

We see this by the same way as Corollary 3.16.

PROPOSITION 4.25. *W acts on (D'', d') isometrically.*

PROOF. Fix $w \in W$ and $a, b \in D''$ arbitrary. Let σ be a path in D'' connecting a and b. Then for any $\epsilon > 0$ there exists a partition $\{c_0 = a, c_1, \ldots, c_n = b\}$ of σ such that

$$\ell(\sigma) \leq \sum_{i=1}^{n} d(c_{i-1}, c_i) + \epsilon.$$

Since W acts on (D, d) isometrically we have

$$\sum_{i=1}^{n} d(c_{i-1}, c_i) = \sum_{i=1}^{n} d(w \cdot c_{i-1}, w \cdot c_i) \leq \ell(w \cdot \sigma).$$

Hence $\ell(\sigma) \leq \ell(w \cdot \sigma)$. This implies that $d'(a, b) \leq d'(w \cdot a, w \cdot b)$ and the reverse is showed in the same way. Thus we have $d'(a, b) = d'(w \cdot a, w \cdot b)$. ∎

We need to compute how the metric d' differs from the metric d. To see this we deform the region D. Now B is conjugate to the diagonal matrix $A = (1, \ldots, 1, -1)$ by a linear transformation. Then the region D is deformed into the region $\mathcal{H} = \{v \in V \mid A(v, v) < 0, v_n = 1\}$ where $A(,)$ is the bilinear form defined by A and v_n is the n-th coordinate of v. The region $\mathcal{H}$ can be regard as the projective model of the hyperbolic space which is equipped with the Hilbert metric d_A defined in the same way as d using A instead of B. We see that such a map gives an isometry from (D, d) to $(\mathcal{H}, d_A)$. To prove this we prepare a lemma.

Lemma 4.26. *Let $\overline{D}$ be the closure of a bounded convex domain D. Any linear transformation $L : V \longrightarrow V$ preserves the cross ratio $[a, x, y, b]$ for any $\{a, x, y, b\}$ which are collinear in this order and $a, b \in \partial D$, $x, y \in D$.*

Proof. For any collinear four points $\{a, x, y, b\}$ in V we have $\|y-b\| \leq \|x-b\|$, $\|x-a\| \leq \|y-a\|$ with respect to the Euclidean norm $\|*\|$. From the collinearity, each pair $\{y-b, x-b\}$ and $\{x-a, y-a\}$ have the same direction respectively. Hence there exist constants $k, l \geq 1$ such that $x - b = k(y - b)$ and $y - a = l(x - a)$. Since a linear transformation maps lines to lines, we have

$$\begin{aligned}[L(a), L(x), L(y), L(b)] &= \frac{\|L(y-a)\|\,\|L(x-b)\|}{\|L(y-b)\|\,\|L(x-a)\|} = \frac{l\|L(x-a)\|\,k\|L(y-b)\|}{\|L(y-b)\|\,\|L(x-a)\|} = lk \\ &= \frac{l\|x-a\|\,k\|y-b\|}{\|y-b\|\,\|x-a\|} = \frac{\|y-a\|\,\|x-b\|}{\|y-b\|\,\|x-a\|} = [a, x, y, b].\end{aligned}$$

∎

We can diagonalize the Coxeter matrix $\{B(\alpha_i, \alpha_j)\}_{i,j}$ by an orthogonal transformation L since B is a symmetric bi-linear form. Let $\{\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n\}$ be the eigenvalues of B and let L' be a diagonal matrix $\left(\frac{1}{\sqrt{\lambda_1}}, \ldots, \frac{1}{\sqrt{\lambda_n}}\right)$. We have ${}^t(LL')B(LL') = A$ where ${}^tM$ denotes the transpose of a matrix M. We set a linear transformation $g(v) = L'^{-1}L^{-1}v$ for $v \in V$. Then we notice that $g(D) \subset \{v \in V \mid 0 < v_n, A(v, v) < 0\}$. We define a projection $p : g(D) \longrightarrow \mathcal{H}$ by $v \mapsto v/v_n$ where v_n is the n-th coordinate of v. Obviously $p \circ g$ is a bijection. Take any $x, y \in D$ and $a, b \in \partial D$ such that $d_D(x, y) = (1/2)\log[a, x, y, b]$. Then by Lemma 4.26 we have $[a, x, y, b] = [g(a), g(x), g(y), g(b)]$. Adding to this by Proposition 3.2 we have $[g(a), g(x), g(y), g(b)] = [p \circ g(a), p \circ g(x), p \circ g(y), p \circ g(b)]$. Thus we have the following:

Lemma 4.27. *The metric space (D, d) is isometric to $(\mathbb{H}^{n-1}, d_{\mathbb{H}})$ where $d_{\mathbb{H}}$ denotes the hyperbolic metric.*

As a consequence of this lemma together with [**4**, Definition(GF3),p.289] we obtain the following.

Theorem 4.28. *Let (W, S) be a rank n Coxeter system whose associating bilinear form has the signature $(n-1, 1)$ is embedded into* Isom$(\mathbb{H}^{n-1})$*. Moreover, it is geometrically finite.*

Lemma 4.27 guarantees us that horoballs in D are mapped to horoballs in $\mathcal{H}$. It is well known that $(\mathcal{H}, d_A)$ is isometric to the hyperbolic space $(\mathbb{H}^{n-1}, d_{\mathbb{H}})$ of the upper half plane model. In $(\mathbb{H}^{n-1}, d_{\mathbb{H}})$ we can compare the hyperbolic distance of two points on a horosphere and the length of a path on that horosphere (for the precise computation see [**14**, p.214-p.215]). The estimate is as follows. For x, y on horosphere in $(\mathbb{H}^{n-1}, d_{\mathbb{H}})$ we denote c as an arc on horosphere joining x and y. It holds that

$$\text{the hyperbolic length of } c \leq \exp\left(\frac{d_{\mathbb{H}}(x, y)}{2}\right)$$

and hence

$$2\left(\log d'(x, y)\right) \leq d(x, y). \tag{5}$$

Consequently we obtain a constant $C > 0$ from Lemma 4.24, we have the following.

Lemma 4.29. *For a Coxeter group W of type $(n-1,1)$, there exists a constant $C > 0$ so that*

$$2(\log |w|) - C \leq d(o, w \cdot o)$$

for all $w \in W$.

Remark 4.30. Next we consider the geodesic γ on D between two points x and y on a horosphere. A horoball in D is mapped isometrically to a horoball $H = \{(x_1, \dots, x_{n-1}) \in \mathbb{H}^{n-1} \mid x_{n-1} > c\}$ for some $c > 0$. We denote $o' \in \mathbb{H}^{n-1}$ as the image of the base point $o \in D$. Let $x', y' \in \mathbb{H}^{n-1}$ be the image of the end points x, y of γ and let z' be the image of the nearest point z of γ from o in D. The geodesic γ' joining x', y' in $\mathbb{H}^{n-1} \setminus H$ for the length metric is a straight line on ∂H and it is nothing but the image of the nearest point projection of γ to the plane $\{(x_1, \dots, x_{n-1}) \in \mathbb{H}^{n-1} \mid x_{n-1} = c\}$. Therefore the distance from o' to γ' is bounded above by $2d_{\mathbb{H}}(o', z')$.

More generally we consider a geodesic with respect to d' of $x, y \in D''$. We denote ξ as the geodesic on (D, d) connecting x, y. Let $\{O_i\}$ be the set of horoballs which intersects with ξ. Then by [**24**, Theorem 8.1] we see that the geodesic η with respect to d' lies in r neighborhood of $\xi \cup \bigcup_i O_i$ where r is a universal constant. Now we consider each segment $\xi_i = \xi \cap O_i$ and let x_i, y_i be the end points of ξ_i. Let ξ_i' be the geodesic in D'' connecting x_i, y_i and let ξ' be the path given by replacing each ξ_i with ξ_i'. Recall that the geodesic in $\mathbb{H}^{n-1}$ is unique and the fact that the geodesic between two points on a horoball $U = \{(x_1, \dots, x_{n-1}) \in \mathbb{H}^{n-1} \mid x_{n-1} > c\}$ for some $c > 0$ is a Euclidean straight line on ∂U. Because of this, we see that there exists another universal constant r' such that the geodesic η lie in the r' neighborhood of ξ'.

Let $F : W \longrightarrow D''$ be the quasi isometry defined by $F(w) = w \cdot o$ for every $w \in W$ and if $w = w's$ for some $s \in S$ then F maps the edge joining the vertices $w, w' \in W$ to the geodesic between $w \cdot o$ and $w' \cdot o$.

Lemma 4.31. *There exists a constant $c > 0$ satisfying following. For any $x, y \in W \cdot o$ there exists a geodesic γ in W which $F(\gamma)$ connects x, y such that is in bounded Hausdorff distance by c from a geodesic connecting x, y in D''.*

Proof. Take the geodesic ξ joining x and y in D. Then ξ crosses some orbits of K and some horoballes. We claim that a curve τ given by replacing all segments of ξ which cross horoballs with the geodesics in D'' connecting each end points crosses the same orbits of K'' as ξ.

We remark that ξ gives a geodesic γ in W in the following way. Let $w_1 \cdot K, \dots, w_k \cdot K$ be the orbits which are crossed by ξ. Then for each $i \in \{1, \dots, k-1\}$, there exists a B-reflection $s_i \in S$ such that $w_{i+1} = s_i w_i$. We see that $\gamma = s_{k-1} \cdots s_1$ is a geodesic in W by the same way as the proof of Proposition 3.19. We see that $F(\gamma)$ equals to the sequence path $w_1 \cdot o, \dots, w_k \cdot o$.

Let H be a horoball in D which ξ crosses and let H' be the horoball $\{(x_1, \dots, x_{n-1}) \in \mathbb{H}^{n-1} \mid x_{n-1} > c\}$ in $\mathbb{H}^{n-1}$ corresponding to H (up to isometry of $\mathbb{H}^{n-1}$). We denote x and y as the endpoints of a segment ξ_0 in ξ which crosses H, and let $x', y' \in \mathbb{H}^{n-1}$ be the corresponding points to x, y each other. We notice that x' and y' lie on $\partial H' = \{(x_1, \dots, x_{n-1}) \in \mathbb{H}^{n-1} \mid x_{n-1} = c\}$. Consider the geodesic l in $\mathbb{H}^{n-1} \setminus H'$ connecting x' and y' for the length metric. Then l is the Euclidean straight line connecting x', y' and it crosses the same image of orbits of K as the image of ξ_0. This is because all orbits of K crossing to H are the orbits of an affine Coxeter subgroup which fixes the point of tangency of H, and l is given by the orthogonal projection of a hyperbolic geodesic to $\partial H'$ with respect to the Euclidean inner product.

Thus resulting curve τ crosses the same orbits of K'' as ξ. More of this since τ lies in the orbit of $\overline{K''}$ and the diameter of K'' is bounded, the Hausdorff distance between the sequence path $F(\gamma)$ and τ is bounded by the diameter of K''.

By the remark described before this lemma, we see that the geodesic in D'' between x, y is on bounded distance with universal constant r' from τ.

Putting $c = r' +$ (diameter of K''), we have the conclusion. ∎

In [**21**, Theorem 5.2], Karlsson and Noskov showed following useful theorem.

Theorem 4.32 (Karlsson-Noskov). *Let $\{x_n\}_n$ and $\{z_n\}_n$ be two sequences of points in D. Assume that $x_n \longrightarrow \overline{x} \in \partial D$, $z_n \longrightarrow \overline{z} \in \partial D$ and $[\overline{x}, \overline{z}] \not\subseteq \partial D$, where $[\overline{x}, \overline{z}]$ is a segment connecting $\overline{x}$ and $\overline{z}$. Then there exists a constant $M = M(\overline{x}, \overline{z})$ such that for the Gromov product $(x_n|z_n)_y$ in Hilbert distance relative to some fixed point $y \in D$ we have*

$$\limsup_{n\longrightarrow\infty}(x_n|z_n)_y \leq M.$$

This implies that if two Gromov sequences $\{x_n\}_n$ and $\{z_n\}_n$ in D are in the relation $\{x_n\}_n \sim_G \{z_n\}_n$, then these sequences converge to the same point in ∂D.

We have the Cannon-Thurston map for a Coxeter group with higher rank cusps directly. We remind the following fact. Let (X, d) be a δ-hyperbolic space. For any $x, y, o \in X$, let z be an arbitrary point on a geodesic connecting x, y. In a δ-hyperbolic space, we have $\delta \geq \min\{d(z, [o, x]), d(z, [o, y])\}$. Then we have $d(o, z) \geq (x|y)_o$. If z is the nearest point of a geodesic $[x, y]$ from o, then we have $(x|y)_o \geq d(o, z) - \delta$ ([**33**, 2.33]). Thus

$$d(o, z) \geq (x|y)_o \geq d(o, z) - \delta$$

for such a point.

Proposition 4.33. *Assume that W includes rank $m > 2$ cusps. Let $F : W \longrightarrow D''$ be the quasi isometry defined by $F(w) = w \cdot o$ for every $w \in W$. Then F extends to $\widetilde{F} : \partial_G(W, S) \longrightarrow \Lambda(W)$ continuously. Moreover $\widetilde{F}$ is surjective and W-equivariant.*

Proof. In this proof we denote by C a generic constant whose value may change line to line. We show that the Gromov product of any two orbits $w \cdot o, w' \cdot o$ with respect to o for the metric d is bounded below by the Gromov product of w, w' with respect to the unit $id \in W$ for the word metric. If this is true, by Theorem 4.32 we have the well definedness of $\widetilde{F}$ and the continuity since $\partial_G D$ (resp. $\partial_G D'$) and $\partial D = \Lambda(W)$ (resp. $\partial D' \setminus D$) are homeomorphic. More of this for any limit point ξ which is not in $W \cdot PF$ by taking the geodesic on (D, d) from o to ξ we can construct a sequence path. The corresponding sequence for that sequence path is actually a geodesic in W by Lemma 4.31. If $\xi \in W \cdot PF$ then $w \cdot \xi \in PF$ for some $w \in W$. In this case there exists a Coxeter subsystem (W', S') of (W, B) such that $S' \subset S$ and W' fixes ξ by Proposition 4.4. Since W' is affine, there exists at least one Gromov sequence. Then for any Gromov sequence $\{w'_i\}_i$ consists of elements in W', the sequence $\{w'_i \cdot o\}_i$ converges to ξ. Thus we see that $\widetilde{F}^{-1}(\xi)$ is not empty and hence $\widetilde{F}$ is surjective. The W-equivariantness of $\widetilde{F}$ is trivial by the construction.

Take $x, y \in W \cdot o$ arbitrarily and let τ be the geodesic on (D, d) connecting $x = w_x \cdot o$ and $y = w_y \cdot o$. Let z be the nearest point in D from o to τ. Let γ be a geodesic in W which is constructed in the same way of the proof of Lemma 4.31. We construct a path τ' in D'' by replacing segments of τ which cross horoballs with the geodesics in D'' connecting each end points. We denote by z' the nearest point from o to τ'. Now we have $d(o, z) \geq Cd(o, z')$ by Remark 4.30. Adding to this we put $z'' = w_z \cdot o \in W \cdot o$ as the nearest point from z' to $F(\gamma)$. Then $d(o, z') \geq d(o, z'') - C$ since the diameter of K' is bounded.

Furthermore by the inequality (5) we have

$$\log\left(Cd'(p,q)^2\right) \leq d(p,q)$$

for any $p, q \in D''$. Then we have

$$\begin{aligned}(x|y)_o &\geq d(o,z) - C \geq Cd(o,z') - C \geq Cd(o,z'') - C \\ &\geq \log\left(Cd'(o,z'')^2\right) \geq \log\left(C|w_z|^2\right) \\ &\geq \log\left(C(w_x|w_y)_{\mathrm{id}}^2\right).\end{aligned}$$

This shows that for any two sequences $\{w_k\}_k$ and $\{w'_k\}_k$ in W if $\liminf_{i,j\to\infty}(w_i|w'_j)_{\mathrm{id}} = \infty$ then $\liminf_{i,j\to\infty}(w_i \cdot o|w'_j \cdot o)_o = \infty$. ∎

REMARK 4.34. The author does not know whether the preimage of p is a point or not in the case where the rank of p is not 2.

Bibliography

[1] O. Baker and T. R. Riley, *Cannon-Thurston maps do not always exist*, Forum of Mathematics, Sigma, Volume 1, 2013, e3.

[2] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, volume 231 of Graduate Texts in Mathematics. Springer, New York, 2005.

[3] C. Bonnafé and M. Dyer, *Semidirect product decomposition of Coxeter groups*, Comm. Algebra, 38(4):1549-1574, 2010. http://dx.doi.org/10.1080/00927870902980354.

[4] B. Bowditch, *Geometrical finiteness for hyperbolic groups*, J. Funct. Anal. **113**, 245-317 (1993)

[5] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*. Grundlehren der math. Wiss. 319, Springer-Verlag, Berlin, 1999.

[6] S. M. Buckley and K.Falk, *The boundary at in infinity of a rough CAT(0) space*, preprint arXiv:1209.6557.

[7] S. M. Buckley and S. L. Kokkendorff, *Comparing the Floyd and ideal boundaries of a metric space*, Trans. Amer. Math. Soc. **361** (2009), no. 2, 715–734.

[8] P-E. Caprace, *Buildings with isolated subspaces and relatively hyperbolic Coxeter groups*, Innov. Incidence Geom. 10 (2009), 15–31.

[9] P-E. Caprace, *Erratum to 'Buildings with isolated subspaces and relatively hyperbolic Coxeter groups'*.

[10] J. W. Cannon and W. P. Thurston, *Group invariant Peano curves*, Geom. Topol., 11:1315:-1355, 2007.

[11] M. Davis, *The Geometry and Topology of Coxeter Groups*, London Mathematical Society Monographs Series, Princeton University Press, Princeton, 2008.

[12] M. Dyer, C. Hohlweg, and V. Ripoll, *Imaginary cones and limit roots of infinite Coxeter groups* Preprint arXiv:1303.6710.

[13] D. Egloff, *Uniform Finsler Hadamard manifolds*, Ann. Inst. H. Poincaré Phys. Théor. **66** (1997), no. 3, 323-357.

[14] W. J. Floyd, *Group Completions and limit sets of Kleinian groups*, Invent. Math. **57**(1980), 205-218.

[15] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, English ed. Modern Birkhauser Classics, Birkhauser Boston Inc., Boston, MA, 2007.

[16] W. Haemers, *Interlacing eigenvalues and graphs*, Linear Algebra Appl. 226/228 (1995) 593-616.

[17] A. Higashitani, R. Mineyama, and N. Nakashima, A metric analysis of infinite Coxeter groups : the case of type $(n-1,1)$ Coxeter matrices, preprint arXiv:1212.6617.

[18] C. Hohlweg, J-P. Labbé, and V. Ripoll, *Asymptotical behaviour of roots of infinite Coxeter groups*, Preprint arXiv:1112.5415.

[19] C. Hohlweg, J-P. Préaux, V. Ripoll, *On the Limit Set of Root Systems of Coxeter Groups and Kleinian Groups*, preprint arXiv:1305.0052.

[20] J. E. Humphreys, *Reflection groups and Coxeter groups*, volume 29 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990.

[21] A. Karlsson and G. A. Noskov, *The Hilbert metric and Gromov hyperbolicity*, Enseign. Math. **48** (2002), 73-89.

[22] D. Krammer, *The conjugacy problem for Coxeter groups.* Groups Geom. Dyn., 3(1):71-171, 2009. http://dx.doi.org/10.4171/GGD/52.

[23] Y. Matsuda and S. Oguni, *On Cannon-Thurston maps for relatively hyperbolic groups*, Preprint arXiv:1206.5868,2012

[24] C. T. McMullen, *Local connectivity, Kleinian groups and geodesics on the blowup of the torus.* Invent. Math., **146** (2001), 35-91.

[25] R. Mineyama, *Cannon-Thurston maps for Coxeter groups with signature* $(n-1,1)$, preprint arXiv:1312.3174.

[26] R. Mineyama, *Cannon-Thurston maps for Coxeter groups with affine reflection subgroups*, preprint arXiv:1312.5017 .

[27] R. Mineyama and H. Miyachi, *A Characterization of Biholomorphic Automorphisms of Teichmüller Space*, Mathematical Proceedings of the Cambridge Philosophical Society, 154 (2013), 71–83.

[28] R. Mineyama and K. Shimomura, *Deformation of a triangle group and Hausdorff dimension of the limit set*, preprint.

[29] M. Mitra, *Cannon-Thurston Maps for Hyperbolic Group Extensions*, Topology **37**, 527-538, 1998.

[30] M. Mitra, *Cannon-Thurston Maps for Trees of Hyperbolic Metric Spaces*, J.Differential Geom. 48, pages 135-164, 1998.

[31] M. Mj *Cannon-Thurston maps for surface groups*, Annals of Math. Pages 1-80 from Volume 179 (2014), Issue 1.

[32] L. Schlettwein, *Hyperbolische Simplexe*, Diplomarbeit, Universität Basel, 1995.

[33] J. Väisälä, *Gromov hyperbolic spaces*, Expo. Math. **23** (2005), 187-231.