On the diffusive structure for the damped wave equation with variable coefficients

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Doctoral thesis

2014

Contents

CHAPTER 1

Introduction

1.1. Damped wave equations

The damped wave equation

$$u_{tt} - \Delta u + u_t = 0$$

is a model which describes the propagation of the wave with friction. This equation is also known as the telegraph equation developed by Oliver Heaviside(1850-1925).

It has been investigated by many mathematicians that the solution of the damped wave equation has so-called *diffusion phenomenon*, that is, the solution behaves like that of the corresponding heat equation

$$v_t - \Delta v = 0$$

as $t \to +\infty$.

In this thesis we are concerned with the damped wave equation with variable coefficients

$$u_{tt} - \Delta u + a(t,x)u_t = f(u).$$

Our aim is to study whether this equation still has a diffusive structure. Here $a(t,x)$ denotes a coefficient of the damping depending on time and space variables and $f(u)$ is a nonlinear term.

Roughly speaking, it is expected that if $a(t,x)$ does not decay fast, then the damping is *effective* and the solution behaves like that of the corresponding heat equation

$$-\Delta v + a(t,x)v_t = f(v);$$

if $a(t,x)$ decays sufficiently fast, then the damping becomes *non-effective* and the solution behaves like that of the wave equation without damping

$$w_{tt} - \Delta w = f(w).$$

There are many literature on the damped wave equation with variable coefficients and the above conjecture has been confirmed in several situations. In this thesis, we focus on the damping of the form

$$a(t,x) = (1 + |x|^2)^{-\alpha/2}(1+t)^{-\beta}$$

with $\alpha, \beta \geq 0$. In this case, roughly speaking, it is known that if $\alpha + \beta < 1$, then $a(t,x)$ is effective damping and if $\alpha + \beta > 1$, then $a(t,x)$ is non-effective damping.

Here we give an intuitive observation for understanding the diffusion phenomenon. Consider the linear damped wave equation

$$u_{tt} - \Delta u + u_t = 0 \tag{1.1.1}$$

with initial data $(u, u_t)(0,x) = (0,g)(x)$. Using the Fourier transform, we have

$$\hat{u}_{tt} + |\xi|^2 \hat{u} + \hat{u}_t = 0.$$

Solving this ordinary differential equation, one can obtain

$$\hat{u}(t,\xi) = \frac{1}{\sqrt{1-4|\xi|^2}} \left(e^{-t(1-\sqrt{1-4|\xi|^2})/2} + e^{-t(1+\sqrt{1-4|\xi|^2})/2} \right) \hat{g}(\xi).$$

When $|\xi|$ is sufficiently small, we can consider

$$\frac{1-\sqrt{1-4|\xi|^2}}{2} = \frac{2|\xi|^2}{1+\sqrt{1-4|\xi|^2}} \sim |\xi|^2$$

and hence, if t is sufficiently large, ignoring some terms decaying exponentially, we can see that

$$\hat{u}(t,\xi) \sim e^{-t|\xi|^2} \hat{g}(\xi).$$

The right-hand side is of course the Fourier transform of the solution of the corresponding heat equation with initial data g. Thus, we can naturally expect the diffusion phenomenon.

We also give an another observation by scaling argument. For a solution $u(t,x)$ of (1.1.1), putting

$$u(t,x) = \phi(\lambda t, \lambda^{1/2} x), \quad \lambda t = s, \lambda^{1/2} x = y$$

with a parameter $\lambda > 0$, we have

$$\lambda \phi_{ss}(s,y) - \Delta\phi(s,y) + \phi_s(s,y) = 0.$$

Thus, letting $\lambda \to 0$, we obtain the heat equation

$$-\Delta\phi + \phi_s = 0.$$

We note that $\lambda \to 0$ is corresponding to $t \to +\infty$.

This observation is also applicable to variable coefficient cases. Let $u(t,x)$ be a solution of

$$u_{tt} - \Delta u + |x|^{-\alpha} t^{-\beta} u_t = 0.$$

When $-1 < \beta < 1$, $\alpha < 2$ and $\alpha + \beta < 1$, we put

$$u(t,x) = \phi(\lambda^{1/(1+\beta)} t, \lambda^{1/(2-\alpha)} x), \quad \lambda^{1/(1+\beta)} t = s, \lambda^{1/(2-\alpha)} x = y$$

with a parameter $\lambda > 0$ and have

$$\lambda^{2/(1+\beta)} \phi_{ss}(s,y) - \lambda^{2/(2-\alpha)} \Delta\phi(s,y) + \lambda^{\alpha/(2-\alpha)+1} |y|^{-\alpha} s^{-\beta} \phi_s(s,y) = 0.$$

We can rewrite this equation as

$$\lambda^{2/(1+\beta)-2/(2-\alpha)} \phi_{ss} - \Delta\phi + |y|^{-\alpha} s^{-\beta} \phi_s = 0.$$

Note that

$$\frac{2}{1+\beta} - \frac{2}{2-\alpha} = \frac{2(1-\alpha-\beta)}{(1+\beta)(2-\alpha)} > 0.$$

Therefore, letting $\lambda \to 0$ again, we obtain the corresponding heat equation

$$-\Delta\phi + |y|^{-\alpha} s^{-\beta} \phi_s = 0.$$

On the other hand, when $\alpha + \beta > 1$, we put

$$u(t,x) = \phi(\lambda t, \lambda x), \quad \lambda t = s, \lambda x = y$$

and have

$$\phi_{ss}(s,y) - \Delta\phi(s,y) + \lambda^{\alpha+\beta-1} |y|^{-\alpha} s^{-\beta} \phi_s(s,y) = 0.$$

In this case, letting $\lambda \to 0$, we obtain the wave equation without damping

$$\phi_{ss} - \Delta\phi = 0.$$

This indicates that the asymptotic behavior of solutions essentially depends on the behavior of the coefficient of damping. In this thesis, we investigate in what way the damping influences the behavior of solutions.

This thesis is organized as follows. In the next section, we introduce selected results described in the following chapters. Then, in Section 1.3, we give a review of previous results related to ours. We also explain the method used for the proof of main results in Section 1.4.

In Chapter 2, we introduce some basic results on the study of damped wave equations, including solution representation formula, asymptotic behavior of solutions and some semilinear problems.

Chapter 3 concerns with the diffusion phenomena for the linear wave equation with space-dependent damping. We prove the asymptotic profile of the solution is given by a solution of the corresponding heat equation in the L^2-sense. We also give weighted energy estimates of solutions for higher order derivatives.

Chapter 4 is devoted to the existence of global solutions for the semilinear wave equation with damping depending on time and space variables. In this case we can find an appropriate weight function related to the corresponding heat kernel and we can obtain an a priori estimate of the solution by a weighted energy method. As a corollary of this a priori estimate, we can see that the energy of the solution is concentrated in some parabolic region much smaller than the light cone. This behavior is quite a contrast to that of the wave equation without damping.

In Chapter 5, we consider the critical exponent problem to the semilinear wave equation with scale-invariant damping $\frac{\mu}{1+t}u_t$ with $\mu > 0$. This equation is invariant under the hyperbolic scaling and known as the threshold between *effective* and *non-effective* damping. In this case the asymptotic behavior of the solution is very delicate and the coefficient μ plays an essential role. We prove an $L^2 - L^1$ type decay estimate of solutions and a small data global existence result for sufficiently large μ. We also show some blow-up results for all $\mu > 0$ by using a modified test function method. Moreover, we prove that when $\mu < 1$, the critical exponent is larger than that of the corresponding heat equation. This shows that the equation loses the parabolic structure and recovers its hyperbolic structure as μ gets smaller.

Chapter 6 concerns with the blow-up of solutions to the one-dimensional semilinear wave equation with time and space variables. In this case we cannot apply the test function method directly. However, in one-dimensional case, we can construct an appropriately multiplier function by the method of characteristics. We prove that when the damping is non-effective, the critical exponent agrees with that of the wave equation without damping, that is, the small data blow-up holds for any power of the nonlinearity.

In Chapter 7, we prove upper estimates of the lifespan of solutions to the semilinear wave equation with several types of damping in subcritical case. In particular, our results give almost optimal estimates of the lifespan from both above and below in the constant and time-dependent coefficient cases. This is a joint work with Mr. Masahiro Ikeda.

In Chapter 8, we prove the L^p-L^q estimates of the solution to the linear damped wave equation. These estimates, which show the diffusion phenomenon, has been already proved by several mathematicians. In this chapter we introduce an improvement of these estimates in higher dimensional cases and give a simpler proof by using the solution representation formula. This is a joint work with Mr. Shigehiro Sakata.

Finally, in Appendix, we explain the notation used in this thesis, some useful lemmas, definitions of solutions and the proof of local existence theorem.

1.2. Results

In this section, we collect the selected results described in this thesis. For the sake of simplicity, we introduce only simplified results. We state the results more precisely in the following chapters. We consider the Cauchy problem of the semilinear damped wave equation

$$\begin{cases} u_{tt} - \Delta u + a(t,x)u_t = f(u), & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.2.1}$$

where $u = u(t,x)$ is real-valued unknown, $a(t,x)$ is nonnegative smooth function and $f(u) = 0$ or $|u|^p$. In what follows, we assume that $1 < p$ and $(u_0,u_1) \in C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$.

The first result is about the asymptotic profile of solutions to the linear wave equation with the damping having the form $\langle x\rangle^{-\alpha}$ with $0 \le \alpha < 1$. In this case the damping can be seen as effective and it is conjectured the asymptotic profile is given by a solution of the corresponding heat equation. The following result gives an affirmative answer and is explained in Chapter 3:

Theorem 1.1 ([**112**]). *Let $f(u) = 0$ and $a(t,x) = \langle x\rangle^{-\alpha}$ with $0 \le \alpha < 1$ and let u be the solution of* (1.2.1). *Then we have*

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^2} = o(t^{-\frac{n-2\alpha}{2(2-\alpha)}})$$

as $t \to +\infty$, where $v(t,x)$ is the solution of the corresponding heat equation

$$\langle x\rangle^{-\alpha} v_t - \Delta v = 0$$

with the initial data $v(0,x) = u_0(x) + a(x)^{-1}u_1(x)$.

The second result is the existence of global solutions to the semilinear wave equation with damping depending on time and space variables. We consider the damping of the form $\langle x\rangle^{-\alpha}(1+t)^{-\beta}$ with $\alpha,\beta \ge 0, \alpha+\beta < 1$ and the nonlinearity $|u|^p$. We prove that if $p > 1 + 2/(n-\alpha)$, then the global solution exists for small data having the finite weighted energy. We note that the exponent $1 + 2/(n-\alpha)$ agrees with the critical exponent of the corresponding heat equation.

Theorem 1.2 ([**109**]). *Let $f(u) = |u|^p$ and $a(t,x) = a_0\langle x\rangle^{-\alpha}(1+t)^{-\beta}$ with $a_0 > 0,\ \alpha,\beta \ge 0,\ \alpha+\beta < 1$, and let*

$$\psi(t,x) = A\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}, \quad A = \frac{(1+\beta)a_0}{(2-\alpha)^2(2+\delta)}$$

with $\delta > 0$. If

$$1 + \frac{2}{n-\alpha} < p \le \frac{n}{n-2}\ (n \ge 3), \quad 1 + \frac{2}{n-\alpha} < p < \infty\ (n = 1,2),$$

then there exists a small positive number $\delta_0 > 0$ such that for any $0 < \delta \le \delta_0$ the following holds: If

$$I_0^2 := \int_{\mathbf{R}^n} e^{2\psi(0,x)}(u_0(x)^2 + |\nabla u_0(x)|^2 + u_1(x)^2)dx$$

is sufficiently small, then there exists a unique solution $u \in C([0,\infty); H^1(\mathbf{R}^n)) \cap C^1([0,\infty); L^2(\mathbf{R}^n))$ *to* (1.2.1) *satisfying*

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)} u(t,x)^2 dx \le C_\delta I_0^2 (1+t)^{-(1+\beta)\frac{n-2\alpha}{2-\alpha}+\varepsilon},$$

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)} (u_t(t,x)^2 + |\nabla u(t,x)|^2) dx \le C_\delta I_0^2 (1+t)^{-(1+\beta)(\frac{n-\alpha}{2-\alpha}+1)+\varepsilon},$$

where

$$\varepsilon = \varepsilon(\delta) := \frac{3(1+\beta)(n-\alpha)}{2(2-\alpha)(2+\delta)}\delta$$

and C_δ *is a constant depending on* δ.

The following two theorems are about the critical exponent problem for the semilinear wave equation with the scale-invariant damping $a = \mu/(1+t)$. It is known that if μ is sufficiently large (resp. small), the solution of the equation with $f(u) = 0$ behaves like the corresponding heat equation (resp. the free wave equation). We give a global existence result for $p > 1 + 2/n$, provided that μ is sufficiently large. The exponent $1 + 2/n$ is known as critical for the corresponding heat equation. Thus, in view of the linear problem, the assumption on μ is reasonable. We also give a blow-up result for all $\mu > 0$. We remark that if $\mu < 1$, then the blow-up result holds even when $p > 1 + 2/n$. This phenomena can be interpreted as that if μ is small, then the equation recover the hyperbolic structure and the critical exponent rises.

THEOREM 1.3 ([**110**]). *Let* $f(u) = |u|^p$ *and* $a(t,x) = \frac{\mu}{1+t}$ *with* $\mu > 0$, *and let*

$$\psi(t,x) = A\frac{|x|^2}{(1+t)^2}, \quad A = \frac{\mu}{2(2+\delta)}.$$

If $1 + \frac{2}{n} < p \le n/(n-2)$ $(n \ge 3)$, $1 + \frac{2}{n} < p < \infty$ $(n = 1, 2)$ *and* $0 < \varepsilon < 2n(p - (1 + \frac{2}{n}))/(p-1)$, *then there exist constants* $\delta > 0$ *and* $\mu_0 > 1$ *having the following property: if* $\mu \ge \mu_0$ *and*

$$I_0^2 = \int_{\mathbf{R}^n} e^{2\psi(0,x)} (u_0(x)^2 + |\nabla u_0(x)|^2 + u_1(x)^2) dx$$

is sufficiently small, then there exists a unique solution $u \in C([0,\infty); H^1(\mathbf{R}^n)) \cap C^1([0,\infty); L^2(\mathbf{R}^n))$ *of* (1.2.1) *satisfying*

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)} u(t,x)^2 dx \le C_{\mu,\varepsilon} I_0^2 (1+t)^{-n+\varepsilon},$$

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)} (u_t(t,x)^2 + |\nabla u(t,x)|^2) dx \le C_{\mu,\varepsilon} I_0^2 (1+t)^{-n-2+\varepsilon}$$

for $t \ge 0$, *where* $C_{\mu,\varepsilon}$ *is a positive constant depending on* μ *and* ε.

THEOREM 1.4 ([**110**]). *Let* $f(u) = |u|^p$ *and* $a(t,x) = \frac{\mu}{1+t}$, $\mu > 0$.
(i) $1 < p \le 1 + 2/n$ *and* $\mu > 1$. *Moreover, we assume that*

$$\liminf_{R\to\infty} \int_{|x|<R} (\mu - 1)u_0 + u_1 dx > 0.$$

Then there is no global solution for (1.2.1).

(ii)Let $0 < \mu \leq 1$ *and*

$$1 < p \leq 1 + \frac{2}{n + (\mu - 1)}.$$

We also assume

$$\liminf_{R\to\infty} \int_{|x|<R} u_1(x)dx > 0.$$

Then there is no global solution for (1.2.1).

The next one is a blow-up result for the one-dimensional semilinear wave equation with damping depending on time and space variables. We consider a non-effective damping and prove a blow-up result for any $1 < p < \infty$. This is corresponds to the result of the corresponding semilinear wave equation without damping.

Theorem 1.5 ([**111**]). *Let* $n = 1$ *and* $f(u) = |u|^p$*, and assume that* $a(t, x) \in C^\infty([0, \infty) \times \mathbf{R})$ *satisfies*

$$|\partial_t^\alpha \partial_x^\beta a(t, x)| \leq \frac{\delta}{(1 + t)^{k+\alpha}} \quad (\alpha, \beta = 0, 1)$$

with some $k > 1$ *and sufficiently small* $\delta > 0$*. If* $1 < p < \infty$ *and*

$$u_0 = 0, \quad u_1 \geq 0, \quad \liminf_{R\to\infty} \int_{|x|<R} u_1(x)dx > 0,$$

then there is no global solution (1.2.1).

The following result is about estimates of the lifespan of solutions to the semilinear wave equation with time or space dependent damping. Even for the constant damping case, to obtain the estimates of lifespan was open problem for higher dimensional case $n \geq 4$. Here we give the optimal estimate from above for the constant and time-dependent damping case. We also give an upper estimate for space-dependent damping, which does not seem to be optimal, but the first result for this area.

Theorem 1.6 ([**26**]). *Consider the initial data* $\varepsilon(u_0, u_1)$ *instead of* (u_0, u_1) *in* (1.2.1)*, where* $\varepsilon > 0$ *is small parameter. Let* $f(u) = |u|^p$ *and* $a(t, x) = \langle x\rangle^{-\alpha}(1 + t)^{-\beta}$ *with* $\alpha \in [0, 1), \beta \in (-1, 1), \alpha\beta = 0$*, and let* $1 < p < 1 + 2/(n - \alpha)$*. We assume that the initial data* (u_0, u_1) *satisfy*

$$\liminf_{R\to\infty} \int_{|x|<R} (\langle x\rangle^{-\alpha} B u_0(x) + u_1(x))dx > 0,$$

where

$$B = \left(\int_0^\infty e^{-\int_0^t (1+s)^{-\beta} ds} dt \right)^{-1}.$$

Then there exists $C > 0$ *depending only on* n, p, α, β *and* (u_0, u_1) *such that the lifespan* T_ε *is estimated as*

$$T_\varepsilon \leq C \begin{cases} \varepsilon^{-1/\kappa} & \text{if} \quad 1 + \alpha/(n - \alpha) < p < 1 + 2/(n - \alpha), \\ \varepsilon^{-(p-1)}(\log(\varepsilon^{-1}))^{p-1} & \text{if} \quad \alpha > 0,\ p = 1 + \alpha/(n - \alpha), \\ \varepsilon^{-(p-1)} & \text{if} \quad \alpha > 0,\ 1 < p < 1 + \alpha/(n - \alpha) \end{cases}$$

for any $\varepsilon \in (0,1]$, *where*

$$\kappa = \frac{2(1+\beta)}{2-\alpha}\left(\frac{1}{p-1} - \frac{n-\alpha}{2}\right).$$

The final result is the L^p-L^q estimate of the solution to the linear damped wave equation. This has already shown by Marcati and Nishihara [**58**], Hosono and Ogawa [**23**] and Nishihara [**78**] for the case $n = 1, 2, 3$, respectively. Narazaki [**73**] proved the same type estimate with arbitrary small loss of decay rate ε. In this thesis we prove that the loss ε can be removed and give a simper proof by using the solution representation formula. Let u, v be the solutions to the linear damped wave equation and the corresponding heat equation

$$\begin{cases} u_{tt} - \Delta u + u_t = 0, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u, u_t)(0,x) = (0, g)(x), & x \in \mathbf{R}^n, \end{cases}$$

$$\begin{cases} v_t - \Delta v = 0, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ v(0,x) = g(x), & x \in \mathbf{R}^n, \end{cases}$$

respectively.

THEOREM 1.7 ([**98**]). *For* $1 \le q \le p \le \infty$ *and* $t \ge 1$, *we have*

$$\|u(t) - v(t) - e^{-t/2}\mathbf{W}_n(t)g\|_{L^p} \le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-1}\|g\|_{L^q},$$

where

$$\mathbf{W}_1(t)g(x) = \frac{1}{2}\int_{x-t}^{x+t} g(s)ds$$

if $n = 1$,

$$\mathbf{W}_n(t)g = \frac{1}{(n-2)!!|S^{n-1}|}\sum_{l=0}^{(n-3)/2}\frac{1}{8^l}\frac{1}{l!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2-l}\left(\frac{1}{t}\int_{|x-y|=t} g(y)dS_y\right)$$

if $n \ge 3$ *and an odd number,*

$$\mathbf{W}_n(t)g = \frac{1}{(n-1)!!|S^n|}\sum_{l=0}^{(n-2)/2}\frac{1}{8^l}\frac{1}{l!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2-l} \times \int_{|x-y|\le t}\frac{1}{\frac{1}{2}\sqrt{t^2-|x-y|^2}}g(y)dy$$

if n *is an even number, where* $|S^n|$ *denotes the measure of the* n*-dimensional unit sphere.*

1.3. A review of some previous results

In this section, we give a review of previous study for the damped wave equation with variable coefficients from the view point of the diffusion phenomenon. We clarify the relation between the results stated in the previous section and some earlier literature.

1.3.1. Linear damped wave equations. We consider the linear damped wave equation

$$\begin{cases} u_{tt} - \Delta u + a(t,x)u_t = 0, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.3.1}$$

where $u = u(t,x)$ is a real-valued unknown function and (u_0, u_1) is given initial data. The simplest case is the constant coefficient case, that is

$$\begin{cases} u_{tt} - \Delta u + u_t = 0, \\ (u,u_t)(0,x) = (u_0,u_1)(x). \end{cases} \tag{1.3.2}$$

As we mentioned in the first section, it is well-known that the solution of (1.3.2) has the *diffusion phenomenon*. This means that the solution of (1.3.2) behaves like the solution of the corresponding heat equation

$$\begin{cases} v_t - \Delta v = 0, \\ v(0,x) = u_0(x) + u_1(x) \end{cases} \tag{1.3.3}$$

as $t \to +\infty$. First, we note that by the Duhamel principle, the solution u of (1.3.2) is expressed by

$$u(t,x) = S_n(t)(u_0 + u_1) + \partial_t(S_n(t)u_0),$$

where $S_n(t)$ denotes the solution operator of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + u_t = 0 & (t,x) \in \mathbf{R} \times \mathbf{R}^n, \\ (u,u_t)(0,x) = (0,g)(x) & x \in \mathbf{R}^n, \end{cases}$$

that is, $S_n(t)g$ is the solution to the above problem.

The asymptotic behavior of solutions to the equation (1.3.2) has been initiated by Matsumura [**59**]. He proved the following estimates by using the Fourier transform:

$$\begin{aligned} \left\|\partial_t^i \partial_x^\alpha S_n(t)g\right\|_{L^\infty} &\le C(1+t)^{-n/(2m)-i-|\alpha|/2}(\|g\|_{L^m} + \|g\|_{H^{[n/2]+i+|\alpha|}}), \\ \left\|\partial_t^i \partial_x^\alpha S_n(t)g\right\|_{L^2} &\le C(1+t)^{n/4-n/(2m)-i-|\alpha|/2}(\|g\|_{L^m} + \|g\|_{H^{i+|\alpha|-1}}), \end{aligned} \tag{1.3.4}$$

where $1 \le m \le 2$. The decay rates above are the same as those of the corresponding heat equation (1.3.3).

The study of the precise asymptotic profile of solutions to (1.3.2) was triggered by the observation by Hsiao and Liu [**24**]. They considered a system of hyperbolic conservation laws with damping

$$\begin{cases} V_t - U_x = 0, \\ U_t + p(V)_x = -\alpha U, \\ (V,U)(0,x) = (V_0,U_0)(x) \to (V_\pm, U_\pm), & x \to \pm\infty, \end{cases} \tag{1.3.5}$$

where $(t,x) \in (0,\infty) \times \mathbf{R}$, $\alpha > 0$, $p(V) > 0, p'(V) < 0$ for $V > 0$ and $V_0, V_\pm > 0$. They proved that the asymptotic profile of the solution (V,U) of (1.3.5) is given by a solution of a system given by Darcy's law:

$$\begin{cases} \bar{V}_t = -\frac{1}{\alpha}p(\bar{V})_{xx}, \\ p(\bar{V})_x = -\alpha\bar{U}, \end{cases} \quad \text{i.e.,} \quad \begin{cases} \bar{V}_t - \bar{U}_x = 0, \\ p(\bar{V})_x = -\alpha\bar{U} \end{cases} \tag{1.3.6}$$

with $\bar{V}(0,\pm\infty) = V_\pm$. By putting $W(t,x) = \int_{-\infty}^x (V(t,y) - \bar{V}(t,y+x_0) - \hat{V}(t,y))dy$ with some auxiliary function $\hat{V}$ and suitably chosen x_0, they reduced the system

(1.3.5) to a quasilinear second order hyperbolic equation with damping

$$\begin{cases} W_{tt} + \alpha W_t + (p(W_x + \bar{V} + \hat{V}) - p(W_x))_x = \frac{1}{\alpha} p(\bar{V}), \\ (W, W_t)(0, x) = (W_0, W_1)(x) \to 0 \text{ as } x \to \pm\infty. \end{cases} \tag{1.3.7}$$

Using this formalization, they obtained

$$\|(V - \bar{V})(t)\|_{L^\infty \cap L^2} = O(t^{-1/2}), \quad t \to +\infty,$$

here $\|f\|_{L^\infty \cap L^2} := \max\{\|f\|_{L^\infty}, \|f\|_{L^2}\}$. Under the additional assumption

$$(V_0, U_0)(\pm\infty) = (V_\infty, 0), \quad \int_{-\infty}^{\infty} (V_0(x) - V_\infty) dx = 0$$

with some $V_\infty > 0$, this convergence rate was improved by Nishihara [**76**] to

$$\|(V - \bar{V})(t)\|_{L^\infty} = O(t^{-1}).$$

Moreover, he also consider in [**77**] a generalization of (1.3.7) in one space dimension

$$\begin{cases} u_{tt} + \alpha u_t - (a(u_x))_x = 0, \\ (u, u_t)(0, x) = (u_0, u_1)(x) \end{cases} \tag{1.3.8}$$

and proved that the solution of (1.3.8) behaves like that of the corresponding linear parabolic problem

$$\begin{cases} \alpha v_t - a'(0) v_{xx} = 0, \\ v(0, x) = u_0(x) + \frac{1}{\alpha} u_1(x) \end{cases}$$

as $t \to +\infty$.

Yang and Milani [**121**] further extended the result of Nishihara [**77**] to any space dimension. In particular, for the linear damped wave equation (1.3.2), using Matsumura's estimates (1.3.4), they showed that

$$\|u(t) - v(t)\|_{L^\infty(\mathbf{R}^n)} = O(t^{-n/2-1}) \quad (t \to +\infty),$$

where u is the solution of (1.3.2) with $(u_0, u_1) \in (H^{[n/2]+3}(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)) \times (H^{[n/2]+2}(\mathbf{R}^n) \cap L^1(\mathbf{R}^n))$ and v is the solution of (1.3.3). In general, $\|v(t)\|_{L^\infty}$ does not decay faster than $t^{-n/2}$. Therefore, the estimate above shows that the asymptotic profile of u is actually given by v.

After that, Marcati and Nishihara [**58**] proved the following L^p-L^q decay estimates of the difference of u and v in the one-dimensional case $n = 1$:

$$\|u(t) - v(t) - e^{-t/2}(u_0(\cdot + t) + u_0(\cdot - t))/2\|_{L^p} \le C t^{-\frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right) - 1} \|(u_0, u_1)\|_{L^q} \tag{1.3.9}$$

for $t \ge 2$ and $1 \le q \le p \le \infty$. The estimate above implies that u can be expressed asymptotically by

$$u(t, x) \sim v(t, x) + e^{-t/2} \frac{u_0(x + t) + u_0(x - t)}{2}$$

as $t \to +\infty$. Here the term $(u_0(x + t) + u_0(x - t))/2$ is the solution of the free wave equation

$$w_{tt} - w_{xx} = 0$$

with the initial data $(u_0(x), 0)$. Noting that in the estimate (1.3.4), we need some regularity on (u_0, u_1), we can interpret that the term $e^{-t/2}(u_0(x + t) + u_0(x - t))/2$

has the singularity of u. The estimate (1.3.9) was extended by Hosono and Ogawa [**23**] to $n = 2$ and by Nishihara [**78**] to $n = 3$. More precisely, they obtained

$$\left\| u(t,\cdot) - v(t,\cdot) - e^{-t/2}\left(\left(\frac{1}{2} + \frac{t}{8}\right) W_n(t)u_0 + \partial_t W_n(t)u_0 + W_n(t)u_1 \right) \right\|_{L^p} \le Ct^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}\|(u_0,u_1)\|_{L^q}$$

for $1 \le q \le p \le \infty$, where $W_n(t)g$ denotes the solution of the Cauchy problem of the free wave equation

$$\begin{cases} w_{tt} - \Delta w = 0, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (w, w_t)(0,x) = (0,g)(x), & x \in \mathbf{R}^n \end{cases}$$

In particular, we obtain the following decomposition formula of $S_n(t)$ as $t \to +\infty$:

$$S_n(t)g \sim e^{t\Delta}g + e^{-t/2}W_n(t)g,$$

where $e^{t\Delta}$ denotes the evolution operator of the heat equation. For higher dimensional cases, the corresponding L^p-L^q estimates were given by Narazaki [**73**]. He proved the following estimate for the lower frequency of the difference of solutions to the damped wave equation and the heat equation when $n \ge 2$:

$$\|\mathcal{F}^{-1}\{\chi(\xi)(\hat{u}(t) - \hat{v}(t))\}\|_{L^p} \le C(1+t)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1+\varepsilon}\|(u_0,u_1)\|_{L^q}, \tag{1.3.10}$$

where $1 \le q \le p \le \infty$, ε is an arbitrary small positive number and $\chi(\xi)$ is a compactly supported smooth radial function satisfying $\chi(\xi) = 1$ near $\xi = 0$. Moreover, in the case where $1 < q < p < \infty$, $(p,q) = (2,2)$ or $(p,q) = (\infty,1)$, we may take $\varepsilon = 0$. He also proved the following high frequency estimate

$$\|\mathcal{F}^{-1}\{(1-\chi(\xi))(\hat{u}(t) - e^{-t/2}(\mathbf{M}_0(t,\cdot)\hat{u_0} + \mathbf{M}_1(t,\cdot)\hat{u_1}))\}\|_{L^p} \le Ce^{-\delta t}\|(u_0,u_1)\|_{L^q}$$

where $\delta > 0$, $1 < q \le p < \infty$, $\chi(\xi)$ is as above and

$$\begin{aligned} \mathbf{M}_1(t,\xi) &= \frac{1}{\sqrt{|\xi|^2 - 1/4}} \Bigg(\sin(t|\xi|) \sum_{0 \le k < (n-1)/4} \frac{(-1)^k}{(2k)!} t^{2k}\Theta(\xi)^{2k} \\ &\qquad - \cos(t|\xi|) \sum_{0 \le k < (n-3)/4} \frac{(-1)^k}{(2k+1)!} t^{2k+1}\Theta(\xi)^{2k+1} \Bigg), \\ \mathbf{M}_0(t,\xi) &= \cos(t|\xi|) \sum_{0 \le k < (n+1)/4} \frac{(-1)^k}{(2k)!} t^{2k}\Theta(\xi)^{2k} \\ &\qquad + \sin(t|\xi|) \sum_{0 \le k < (n-1)/4} \frac{(-1)^k}{(2k+1)!} t^{2k+1}\Theta(\xi)^{2k+1} + \frac{1}{2}\mathbf{M}_1(t,\xi) \end{aligned}$$

with $\Theta(\xi) = |\xi| - \sqrt{|\xi|^2 - 1/4}$. We note that his proof is based on an argument using the Fourier transform and there is a loss ε of decay rate in (1.3.10). Theorem 1.7 shows that the loss ε can be removable and in Chapter 8, we shall give a simpler proof based on the representation of the solution.

Matsumura [**60**] also investigated when the energy of solutions to (1.3.1) decays to 0 as $t \to +\infty$. He proved that if $a(t,x)$ satisfies

$$a_0(1 + |x| + t)^{-1} \le a(t,x) \le a_1$$

with some $a_0, a_1 > 0$ and $a_t(t,x) \le 0$, then the energy of the solution

$$E(t) := \frac{1}{2}\int_{\mathbf{R}^n} (u_t(t,x)^2 + |\nabla u(t,x)|^2)dx$$

tends to 0 as $t \to +\infty$. This result was extended by Mochizuki and Nakazawa [**66**]. They put the following condition on $a(t,x)$:

$$a_0\{(e_m+|x|+t)\log(e_m+|x|+t)\cdots\log^{[m]}(e_m+|x|+t)\}^{-1} \le a(t,x) \le a_1 \tag{1.3.11}$$

with some $a_0, a_1 > 0$ and $m \in \mathbf{Z}_{\ge 0}$, where

$$e_0 := 1,\ e_1 := e, \dots, e_m := e^{e_{m-1}},$$
$$\log^{[0]} y := y,\ \log^{[1]} y := \log y, \dots, \log^{[m]} y := \log(\log^{[m-1]} y).$$

They proved that if $a(t,x)$ satisfies (1.3.11) and $a_t(t,x) \le 0$, then the energy of solution decays as

$$\|(\nabla u, u_t)(t)\|_{L^2} \le C\{\log^{[m]}(e_m+t)\}^{-\min\{a_0/2,1\}}.$$

On the other hand, for (1.3.1), Mochizuki [**63**] proved if $n \ne 2$ and $a(t,x)$ satisfies

$$0 \le a(t,x) \le C(1+|x|)^{-1-\delta}$$

with some $\delta > 0$ and $a_t(t,x)$ is bounded and continuous, then the energy of solution does not decay in general. Moreover, he proved that the Møller wave operator exists and not identically zero. The scattering solution $u(t,x)$ is asymptotically equivalent to a solution $w(t,x)$ of the free wave equation in the following sense:

$$\lim_{t\to\infty} \|(\nabla u, u_t)(t) - (\nabla w, w_t)(t)\|_{L^2} = 0.$$

For the case $n = 2$, we refer the reader to [**72**]. When $n \ge 3$, this result was also extended by [**66**] to $a(t,x)$ satisfying

$$\begin{aligned}0 &\le a(t,x)\\ &\le a_2\left\{(e_m+|x|)\log(e_m+|x|)\cdots\log^{[m-1]}(e_m+|x|)\left[\log^{[m]}(e_m+|x|)\right]^{\gamma}\right\}^{-1}\end{aligned}$$

with some $a_2 > 0$, $\gamma > 1$ and $m \in \mathbf{Z}_{\ge 0}$.

The energy decay problem in general exterior domain $\Omega \subset \mathbf{R}^n$ has been investigated for a long time. It is well known that for the wave equation without damping, if Ω is nontrapping, then the local energy

$$E_R(t) := \frac{1}{2}\int_{\Omega\cap B_R} (u_t(t,x)^2 + |\nabla u(t,x)|^2)dx$$

decays exponentially fast if n is odd and polynomially fast if n is even, where $B_R := \{x \in \mathbf{R}^n \mid |x| < R\}$, $R > 0$. This is reasonable because the energy propagates along the wave front and the motion in the bounded region stops after time passes. Shibata [**100**] considered the initial-boundary value problem of the damped wave equation

$$\begin{cases} u_{tt} - \Delta u + u_t = 0, & (t,x) \in (0,\infty)\times\Omega,\\ u = 0, & (t,x) \in (0,\infty)\times\partial\Omega,\\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \Omega, \end{cases} \tag{1.3.12}$$

where $n \geq 3$ and $\partial\Omega$ is smooth and bounded. Let $R > 0$ be a constant such that $\Omega^c \subset B_R$. He proved the following estimate of the local energy:

$$E_R(t) + \|u(t)\|^2_{L^2(\Omega\cap B_R)} \leq C(1+t)^{-n}(\|u_0\|^2_{H^4(\Omega)} + \|u_1\|^2_{H^3(\Omega)}),$$

provided that $\operatorname{supp}(u_0, u_1) \subset \Omega \cap B_R$. We note that his result does not require any geometrical condition of Ω. Because the trapped energy also decreases by virtue of the dissipation term u_t. After that, Dan and Shibata [**9**] extended the result above to $n \geq 2$ and improved the estimate as

$$E_R(t) + \|u(t)\|^2_{L^2(\Omega\cap B_R)} \leq C(1+t)^{-n}(\|u_0\|^2_{H^1} + \|u_1\|^2_{L^2}),$$

provided that $\operatorname{supp}(u_0, u_1) \subset \Omega \cap B_R$. Nakao [**68**] considered (1.3.12) with the localized dissipation $a(x)u_t$ instead of u_t, where $a(x) \equiv 0$ for $|x| > L$ with sufficiently large $L > 0$ and $a(x) \geq c > 0$ with some constant $c > 0$ on a neighborhood of the closure of

$$\Gamma(x_0) = \{x \in \partial\Omega \mid \nu(x)\cdot(x - x_0) > 0\} \tag{1.3.13}$$

with some $x_0 \in \mathbf{R}^n$. Here $\nu(x)$ denotes the unit outward normal vector of Ω at the point $x \in \partial\Omega$. We note that if Ω^c is star-shaped with respect to x_0, then $\Gamma(x_0)$ is empty. He proved that if $\operatorname{supp}(u_0, u_1) \subset B_L$, then the local energy decays as

$$E_{L+\varepsilon t}(t) \leq C_{\varepsilon,\delta}(1+t)^{-1+\delta}$$

with arbitrary $0 < \varepsilon, \delta < 1$. Moreover, for the case of odd dimensions, the local energy decays exponentially. This result says that we need only the dissipation on a part of $\partial\Omega$ for the local energy decay. Matsuyama [**62**] considered a dissipation depending on time and space variables $a(t,x)u_t$ and removed δ in the above rate and relaxed the assumption on $a(t,x)$ as $0 \leq a(t,x) \leq a_1\{(e_m + |x|)\log(e_m + |x|)\cdots[\log^{[m]}(e_m + |x|)]^\gamma\}^{-1}$ with some $a_1 > 0, \gamma > 1$ satisfying $a_1 < \gamma^{-1}(\gamma - 1)$, instead of the condition $a \equiv 0$ for large $|x|$. Moreover, under some suitable additional assumptions on $a(t,x)$ and the initial data, he proved that the total energy of solutions does not decay in general and the solution is asymptotically free as $t \to +\infty$.

For the total energy decay of solutions to

$$\begin{cases} u_{tt} - \Delta u + a(x)u_t = 0, & (t,x) \in (0,\infty)\times\Omega, \\ u = 0, & (t,x) \in (0,\infty)\times\partial\Omega, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \Omega \end{cases}$$

with an exterior domain $\Omega \subset \mathbf{R}^n$ has been also considered by Nakao [**69**] (see also [**71**]). He put the following assumptions on $a(x)$: (i) $a(x) \geq c > 0$ holds on some neighborhood of the closure of $\Gamma(x_0)$ with some $c > 0$ and $x_0 \in \mathbf{R}^n$. Here $\Gamma(x_0)$ is defined by (1.3.13). (ii) $a(x) \geq c > 0$ for all $|x| \geq L$ with some $c, L > 0$. Under these assumptions, he proved

$$\|u(t)\|_{L^2} \leq C(\|u_0\|_{H^1_0} + \|u_1\|_{L^2}),$$
$$E(t) \leq CE(0)(1+t)^{-1},$$

where $E(t)$ is the total energy of u, that is,

$$E(t) := \frac{1}{2}\int_\Omega (u_t(t,x)^2 + |\nabla u(t,x)|^2)dx.$$

When Ω^c includes the origin and is star-shaped with respect to the origin and $a(x) \ge c > 0$ holds for all $|x| \ge L$ with some $c, L > 0$, Ikehata [**28**] improved the decay rates in the above estimates as

$$\|u(t)\|_{L^2}^2 \le C(1+t)^{-1}(\|(u_0,u_1)\|_{H^1\times L^2}^2 + \|d(\cdot)(u_1 + a(\cdot)u_0)\|_{L^2}^2),$$
$$E(t) \le C(1+t)^{-2}(\|(u_0,u_1)\|_{H^1\times L^2}^2 + \|d(\cdot)(u_1 + a(\cdot)u_0)\|_{L^2}^2),$$

where

$$d(x) = \begin{cases} |x| & (n \ge 3), \\ |x|\log(B|x|) & (n = 2) \end{cases}$$

with some constant $B > 0$ satisfying $B\inf_{x\in\Omega}|x| \ge 2$.

Ikehata [**30**] considered the special dissipative term $a_0|x|^{-\alpha}u_t$ with $0 \le \alpha < 1$. He obtained the boundedness of the weighted energy

$$\int_\Omega e^{2a_0(2-\alpha)^{-2}\frac{|x|^{2-\alpha}}{t}}(u_t(t,x)^2 + |\nabla u(t,x)|^2)dx \le C.$$

Todorova and Yordanov [**107**] improved the above estimate by introducing a new weighted energy method. They considered the Cauchy problem for the wave equation with space-dependent damping

$$\left\{ \begin{array}{ll} u_{tt} - \Delta u + a(x)u_t = 0, & (t,x) \in (0,\infty)\times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n. \end{array} \right. \tag{1.3.14}$$

They assumed that

$$a(x) \ge a_0(1+|x|)^{-\alpha}$$

with some $\alpha \in [0,1)$ and there exists a solution of the Poisson equation

$$\Delta A(x) = a(x) \tag{1.3.15}$$

having the following properties:

$$A(x) \ge 0, \tag{a1}$$
$$A(x) = O(|x|^{2-\alpha}) \quad \text{as } |x| \to +\infty, \tag{a2}$$
$$m(a) := \liminf_{|x|\to\infty} \frac{a(x)A(x)}{|\nabla A(x)|^2} > 0. \tag{a3}$$

It is known that such solutions $A(x)$ exist if $a(x)$ is radially symmetric and satisfies $a_0(1+|x|)^{-\alpha} \le a(x) \le a_1(1+|x|)^{-\alpha}$ with some $a_0, a_1 > 0$ and $\alpha \in [0,1)$. Using the function $A(x)$, they constructed a weight function of the form

$$e^{(m(a)-\varepsilon)A(x)/t}$$

and obtained the following weighted energy estimates:

$$\int_{\mathbf{R}^n} e^{(m(a)-\varepsilon)A(x)/t}a(x)u(t,x)^2dx \le C_\varepsilon(\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2)t^{-m(a)+\varepsilon}, \tag{1.3.16}$$

$$\int_{\mathbf{R}^n} e^{(m(a)-\varepsilon)A(x)/t}(u_t(t,x)^2 + |\nabla u(t,x)|^2)dx \tag{1.3.17}$$
$$\le C_\varepsilon(\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2)t^{-m(a)-1+\varepsilon}$$

for large $t > 0$ and any $\varepsilon > 0$, provided that the data (u_0, u_1) have compact support. In particular, if $a(x)$ is radially symmetric and behaves like $a(x) \sim a_0|x|^{-\alpha}$ as $|x| \to +\infty$, then it follows that

$$A(x) \sim \frac{a_0}{(2-\alpha)(n-\alpha)}|x|^{2-\alpha} \quad \text{as } |x| \to +\infty,$$
$$m(a) = \frac{n-\alpha}{2-\alpha}$$

and hence,

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)} a(x) u(t,x)^2 dx \le C_\varepsilon (\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2) t^{-\frac{n-\alpha}{2-\alpha}+\varepsilon},$$
$$\int_{\mathbf{R}^n} e^{2\psi(t,x)} (u_t(t,x)^2 + |\nabla u(t,x)|^2) dx \le C_\varepsilon (\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2) t^{-\frac{n-\alpha}{2-\alpha}-1+\varepsilon}$$

for large $t > 0$ and any $\varepsilon > 0$, where

$$\psi(t,x) = \frac{a_0|x|^{2-\alpha}}{2(2-\alpha+\varepsilon)^2 t}.$$

Their method is also applicable to the corresponding heat equation

$$a(x)v_t - \Delta v = 0$$

and we can obtain the same decay rate. This indicates that in this case the equation (1.3.14) still has the diffusive structure. However, the precise asymptotic profile was remained as an open problem. Recently, Nishiyama [**88**] proved the diffusion phenomenon for the abstract damped wave equation

$$u'' + Au + Bu' = 0.$$

His result includes space-dependent damping which does not decay near infinity. Due to the authors knowledge, Theorem 1.1 stated in the previous section is the first result for the precise asymptotic profile of solutions to the damped wave equation with space-dependent decaying potential.

We also mention that the above result by Todorova and Yordanov was extended to damping depending on time and space variables $a(t,x) = a(x)b(t)$ satisfying

$$a_0(1+|x|)^{-\alpha} \le a(x) \le a_1(1+|x|)^{-\alpha}, \quad b_0(1+t)^{-\beta} \le b(t) \le b_1(1+t)^{-\beta}$$

with $\alpha \in [0,1), \beta \in [0,1), \alpha+\beta \in [0,1)$ by J. S. Kenigson and J. J. Kenigson [**45**].

Recently, Ikehata, Todorova and Yordanov [**37**] considered the critical case

$$a_0\langle x\rangle^{-1} \le a(x) \le a_1\langle x\rangle^{-1}.$$

They obtained several optimal energy estimates of solutions with compactly supported data. More precisely, they proved

$$\|(u_t, \nabla u)(t)\|_{L^2} = O(t^{-\frac{1}{2}\min(a_0,n)+\varepsilon})$$

as $t \to +\infty$, where ε is arbitrary small positive number. When $n \ge 3, 1 < a_0 < n$ or $n = 1, 2, n < a_0$, we can remove ε in the above estimate. Moreover, if $a(x)$ is radially symmetric, $a(x) \sim a_0|x|^{-1}$ as $|x| \to \infty$ and $0 < a_0 < n$, then the decay rate $-a_0/2$ in the above inequality is optimal. We note that when $a_0 \ge n$, the decay rate $-n/2$ agrees with that of the corresponding heat equation. Therefore, we expect that this decay rate is also optimal. However, this optimality is still open and the asymptotic profile of solutions is completely open as far as the author's knowledge.

For the time dependent damping cases, more specific asymptotic behavior of solutions was investigated by Wirth [**114, 115, 116, 117, 119**]. He considered the linear damped wave equation

$$u_{tt} - \Delta u + b(t)u_t = 0. \tag{1.3.18}$$

For simplicity, we assume that $b(t)$ is positive, smooth, monotone and satisfies

$$\left|\frac{d^k}{dt^k}b(t)\right| \le C_k(1+t)^{-k}b(t)$$

for $k \in \mathbf{Z}_{\ge 0}$. A typical example of $b(t)$ is $(1+t)^{-\beta}$ with $\beta \in \mathbf{R}$. We also consider the free wave equation

$$w_{tt} - \Delta w = 0 \tag{1.3.19}$$

and the corresponding heat equation

$$b(t)v_t - \Delta v = 0. \tag{1.3.20}$$

We denote by

$$\lambda(t) = \exp\left(\frac{1}{2}\int_0^t b(s)ds\right)$$

an auxiliary function. He determine the behavior of solutions to (1.3.18) as time tends to infinity in the following five cases:

(i) (scattering) If $b(t) \in L^1((0,\infty))$, then the solution to (1.3.18) is asymptotically free. More precisely, there exists an isomorphism W_+ on $\dot{H}^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ such that for the solution $u(t,x)$ to (1.3.18) with initial data (u_0, u_1) and the solution $w(t,x)$ to (1.3.19) with the initial data $W_+(u_0,u_1)$, the asymptotic equivalence

$$\lim_{t\to\infty} \|(\nabla u, u_t)(t) - (\nabla w, w_t)(t)\|_{L^2} = 0$$

holds.

(ii) (non-effective dissipation) If $\limsup_{t\to\infty} tb(t) < 1$, then the solution u to (1.3.18) satisfies the L^p-L^q estimate

$$\|(\nabla u, u_t)(t)\|_{L^p} \le \frac{C}{\lambda(t)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}(\|u_0\|_{W^{s+1,q}} + \|u_1\|_{W^{s,q}})$$

for $p \in [2,\infty)$, q is the dual of p and $s > n(1/q - 1/p)$. Moreover, $\lambda(t)u$ is asymptotically free in the sense that there exists a solution w of (1.3.19) satisfying

$$\lim_{t\to\infty} \|\lambda(t)(\nabla u, u_t)(t) - (\nabla w, w_t(t))\|_{L^2} = 0.$$

(iii) (scale-invariant weak dissipation) If $b(t) = \mu/(1+t)$ with $\mu > 0$, then the solution u of (1.3.18) satisfies the L^p-L^q estimate

$$\|(\nabla u, u_t)(t)\|_{L^p} \le C(1+t)^{\max\{-\frac{n-1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{\mu}{2}, -n\left(\frac{1}{q}-\frac{1}{p}\right)-1\}}(\|u_0\|_{W^{s+1,q}} + \|u_1\|_{W^{s,q}})$$

for $p \in [2,\infty)$, q is the dual of p and $s > n(1/q - 1/p)$.

(iv) (effective dissipation) If $tb(t) \to +\infty$ as $t \to +\infty$, then the solution u of (1.3.18) satisfies the L^p-L^q estimate

$$\|(\nabla u, u_t)(t)\|_{L^p} \le C\left(1 + \int_0^t b(s)^{-1}ds\right)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}}(\|u_0\|_{W^{s+1,q}} + \|u_1\|_{W^{s,q}})$$

for $p \in [2, \infty)$, q is the dual of p and $s > n(1/q - 1/p)$. Moreover, if $b(t)^{-3} \notin L^1((0,\infty)$, then the lower frequency part of the solution u of (1.3.18) is asymptotically equivalent to that of a solution v of (1.3.20) in the L^2-sense. This is called the *local diffusion phenomenon*. It is also follows that when $b(t)^{-3} \in L^1((0,\infty))$, u is for each frequency asymptotically equivalent to that of v. This is called the *global diffusion phenomenon*.

(v) (overdamping) If $b(t)^{-1} \in L^1((0,\infty))$, then the solution u of (1.3.18) with data from $L^2(\mathbf{R}^n) \times H^{-1}(\mathbf{R}^n)$ converges as $t \to \infty$ to the asymptotic state

$$u(\infty, x) = \lim_{t\to\infty} u(t,x)$$

in $L^2(\mathbf{R}^n)$. Furthermore, this limit is non-zero for non-zero initial data.

He also treated in [**119**] the time periodic dissipation, that is, $b(t+T) = b(t) > 0$ for $t \geq 0$ with some $T > 0$, and proved that Matsumura's estimates are still true in this case.

We remark that for the time-dependent speed and damping case

$$u_{tt} - a(t)\Delta u + b(t)u_t = 0,$$

recently, D'Abbicco and Ebert [**6**] gave an extension of the results (i) and (ii) above.

We also mention the abstract damped equation

$$\begin{cases} u'' + u' + Au = 0, \\ u(0) = u_0, \ u'(0) = u_1 \end{cases} \tag{1.3.21}$$

and the corresponding the heat equation

$$\begin{cases} v' + Av = 0, \\ v(0) = u_0 + u_1 \end{cases} \tag{1.3.22}$$

in a separable Hilbert space H. Here A is a closed, self-adjoint and nonnegative operator on H with a dense domain $D(A)$. The diffusion phenomenon for the abstract equation (1.3.21) is closely related to the problem on exterior domains. Ikehata [**27**] considered the concrete case $H = L^2(\Omega)$, $A = -\Delta$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ with an exterior domain $\Omega \subset \mathbf{R}^n$ having compact smooth boundary, and proved for solutions u to (1.3.21) and v to (1.3.22) that

$$\|u(t) - v(t)\|_{L^2(\Omega)} = O((\sqrt{t}\log t)^{-1})$$

as $t \to +\infty$. After that, Ikehata and Nishihara [**32**] considered abstract case and improved the above estimate as

$$\|u(t) - v(t)\|_H = O(t^{-1}(\log t)^{1/2+\varepsilon})$$

with arbitrary small $\varepsilon > 0$. They conjectured the optimal rate is given by $O(t^{-1})$. Chill and Haraux [**3**] solved this conjecture. They proved that

$$\|u(t) - v(t)\|_{D(A^{1/2})} \leq Ct^{-1}(\|u_0\|_{D(A^{1/2})} + \|u_1\|_H) \tag{1.3.23}$$

for $t \geq 1$, where $\|\cdot\|_{D(A^{1/2})}$ denotes the graph norm of $A^{1/2}$ (note that this operator is well-defined):

$$\|u\|^2_{D(A^{1/2})} := \|A^{1/2}u\|^2_H + \|u\|^2_H.$$

Moreover, they proved the decay rate above is optimal. Radu, Todorova and Yordanov [**97**] obtained the following estimates, which are stronger than (1.3.23):

$$\begin{aligned}\|u(t)-v(t)\|_H &\le Ct^{-1}(\|e^{-tA/2}u_0\|_H+\|e^{-tA/2}u_1\|_H)\\&\quad+Ce^{-t/16}(\|u_0\|_H+\|(A^{1/2}+1)^{-1}u_1\|_H),\\ \|A^k(u(t)-v(t))\|_H &\le Ct^{-1-k}(\|e^{-tA}u_0\|_H+\|e^{-tA}u_1\|_H)\\&\quad+Ce^{-t/16}(\|A^ku_0\|_H+\|(A^{1/2}+1)^{-1}A^ku_1\|_H)\end{aligned}$$

for any $t\ge 1$ and $k\ge 0$. These estimates allow us to transfer the decay from the heat equation to the hyperbolic equation. They applied these estimates to operators A generating a Markov semigroup on $L^1(\Omega,\mu)$ with some σ-finite measure space (Ω,μ). They obtained faster decay of the difference $u(t)-v(t)$ with L^1 data and they also proved an abstract version of Matsumura's estimates (1.3.4). Recently, Ikehata, Todorova and Yordanov [**38**] proved similar estimates for the strongly damped wave equation

$$u''+Au+Au'=0.$$

Yamazaki [**120**] extended the results of Chill and Haraux [**3**] to time-dependent damping cases:

$$\begin{cases} u''+b(t)u'+Au=0,\\ u(0)=u_0,\ u'(0)=u_1\end{cases}\tag{1.3.24}$$

and

$$\begin{cases} b(t)v'+Av=0,\\ v(0)=v_0,\end{cases}\tag{1.3.25}$$

where $b(t)$ is a C^1 function satisfying

$$b_0(1+t)^{-\beta}\le b(t)\le b_1(1+t)^{-\beta},\quad 0<\beta<1,\quad |b'(t)|\le b_2(1+t)^{-\beta-1}$$

and

$$v_0=u_0+\frac{u_1}{b(0)}-u_1\int_0^\infty\frac{b'(s)}{b(s)^2}\exp\left(-\int_0^s b(\sigma)d\sigma\right)ds.$$

Let $k,l\ge 0$ and $(u_0,u_1)\in(D(A^{k+1/2})\cap R(A^l))\times(D(A^k)\cap R(A^l))$, where $R(A^l)$ denotes the range of A^l. Then she proved that

$$\begin{aligned}&\|A^k(u(t)-v(t))\|_{D(A^{1/2})}\\&\quad\le Ct^{\beta-1-(1+\beta)(k+l)}(\|u_0\|_{D(A^{k+1/2})}+\|\tilde{u_0}\|_H+\|u_1\|_{D(A^k)}+\|\tilde{u_1}\|_H),\\&\|A^k(u(t)-v(t))\|_H\\&\quad\le Ct^{\beta-2-(1+\beta)(k+l)}(\|u_0\|_{D(A^{k+1/2})}+\|\tilde{u_0}\|_H+\|u_1\|_{D(A^k)}+\|\tilde{u_1}\|_H)\end{aligned}$$

for $t\ge 1$, where $\|\cdot\|_{D(A^{k+1/2})}$, $\|\cdot\|_{D(A^k)}$ are the graph norm of $A^{k+1/2}$, A^k, respectively, and $\tilde{u_0},\tilde{u_1}$ are elements of H such that $A^l\tilde{u_0}=u_0, A^l\tilde{u_1}=u_1$, respectively. Wirth [**118**] treated the critical case $0\le b(t)\le b_1(1+t)^{-1}$. He proved that if $b(t)$ satisfies $|b'(t)|\le b_2(1+t)^{-2}$, $b\notin L^1((0,\infty))$, $\limsup_{t\to\infty} tb(t)<1$ and $\ker A=\{0\}$, then for the solution $u(t)$ of (1.3.24), it holds that

$$\lim_{t\to\infty}\|\lambda(t)(u(t),u'(t))-(w(t),w'(t))\|_E=0,$$

where $w(t)$ is the a solution of the free wave equation

$$w''+Aw=0,$$

$\lambda(t) = \exp(\int_0^t b(s)ds)$ and $\|(\phi,\psi)\|_E$ is the energy norm $\|(\phi,\psi)\|_E^2 = \|A^{1/2}\phi\|_H + \|\psi\|_H$. Moreover, the operator mapping (u_0, u_1) to $(w(0), w'(0))$ is injective.

1.3.2. Damped wave equations with nonlinear source terms. We consider the Cauchy problem for the semilinear damped wave equation

$$\begin{cases} u_{tt} - \Delta u + a(t,x)u_t = f(u), & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.3.26}$$

where $f(u)$ denotes the nonlinear term. Here we treat source semilinear terms, namely

$$f(u) = f_1(u) = |u|^{p-1}u \quad \text{or} \quad f_2(u) = |u|^p$$

with $p > 1$. First we mention the constant coefficient case $a(t,x) \equiv 1$. In this case, for the corresponding semilinear heat equation

$$\begin{cases} v_t - \Delta v = f(u), & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ v(0,x) = v_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.3.27}$$

there are many literature on the structure of solutions. In particular, it is well known that there is the *critical exponent*

$$p_F = 1 + \frac{2}{n}$$

dividing the behavior of solutions into the following way (see [**13, 15, 52, 10**]):

(i)If $p > p_F$, then for any small data $v_0 \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ there is a unique global solution v of (1.3.27) satisfying

$$v(t,x) \sim \theta_0 G(t,x)$$

as $t \to +\infty$, where

$$\theta_0 = \int_{\mathbf{R}^n} v_0(x)dx + \int_0^\infty \int_{\mathbf{R}^n} f(v)(t,x)dxdt.$$

(ii)If $p \le p_F$, then for the data v_0 satisfying $v_0 \ge 0$ and $v_0 \ne 0$, the locally-in-time solution $v(t,x)$ blows up in finite time, that is,

$$\lim_{t \to T^*} v(t,x) = +\infty$$

for some $x \in \mathbf{R}^n$ and $T^* > 0$. Moreover, for the data εv_0 with small parameter $\varepsilon > 0$ and fixed v_0, the lifespan of the solution

$$T_\varepsilon = \sup\{T \in (0,\infty] \mid v(t,x) < +\infty \text{ for } t \in [0,T]\}$$

is estimated as

$$T_\varepsilon \sim \begin{cases} e^{C\varepsilon^{-(p-1)}} & (p = p_F), \\ C\varepsilon^{-1/\kappa} & (1 < p < p_F) \end{cases} \tag{1.3.28}$$

with

$$\kappa = \frac{1}{p-1} - \frac{n}{2}$$

and some $C > 0$.

From the viewpoint of the diffusion phenomenon, it is expected that for the damped wave equation

$$\begin{cases} u_{tt} - \Delta u + u_t = f(u), & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.3.29}$$

the same results hold. For $f_1(u) = |u|^{p-1}u$, existence of classical solutions has been investigated for a long time (see [**99, 59, 60**]). For weak solutions, it is well known that if $(u_0, u_1) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ and $1 < p \le n/(n-2)$ $(1 < p < \infty$ if $n = 1, 2)$, then there are some $T > 0$ and a unique solution

$$u \in C([0,T); H^1(\mathbf{R}^n)) \cap C^1([0,T); L^2(\mathbf{R}^n)).$$

Moreover, the finite propagation speed property holds, that is, if

$$\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \{x \in \mathbf{R}^n \mid |x| < L\}$$

with some $L > 0$, then it is true that

$$\operatorname{supp} u(t) \subset \{x \in \mathbf{R}^n \mid |x| < t + L\}$$

(see [**101, 39**]). Levine [**51**] proved that if the initial data satisfy

$$\frac{1}{2}(\|u_1\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2) - \frac{1}{p+2}\|u_0\|_{L^{p+2}}^{p+2} < 0,$$

then the local solution blows up in finite time. This result shows that the existence of global solution requires some smallness condition on the data. Nakao and Ono [**67**] proved the existence of global solutions with suitably small and compactly supported data in $H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ when

$$1 + \frac{4}{n} \le p < \frac{n+2}{n-2}.$$

Their proof was based on the so-called (modified) potential well method which was originally introduced by Payne and Sattinger [**94**]. When $n \le 3$, Ikehata, Miyaoka and Nakatake [**31**] proved that if the initial data $(u_0, u_1) \in (H^1 \cap L^1) \times (L^2 \cap L^1)$ are sufficiently small and

$$1 + \frac{2}{n} < p < \infty \ (n = 1, 2), \quad 2 < p \le \frac{n}{n-2} \ (n = 3),$$

then there exists a unique global solution satisfying the decay estimates

$$\|u(t)\|_{L^2} \le C(1+t)^{-n/4},$$
$$\|(u_t, \nabla u)(t)\|_{L^2} \le C(1+t)^{-n/4-1/2}.$$

Their method is also applicable for $f_2(u) = |u|^p$. We note that there is a gap between the exponent $p = 2$ and the critical exponent $p = p_F = 1 + 2/3$ when $n = 3$. Moreover, in view of Matsumura's estimate (1.3.4), we can expect that the decay rate of $\|u_t(t)\|_{L^2}$ is faster than that of $\|\nabla u(t)\|_{L^2}$. On the other hand, Li and Zhou [**53**] obtained small data blow-up results for f_1 and f_2 when $n = 1, 2$ and $1 < p \le 1 + 2/n$. These results show that when $n = 1, 2$, the critical exponent for (1.3.29) with the nonlinearity f_1 and f_2 is actually given by $p_F = 1 + 2/n$. After that, Nishihara [**78, 79**] determined the critical exponent for (1.3.29) with f_1, f_2 when $n = 3$. He proved that if the initial data

$$(u_0, u_1) \in (W^{1,1} \cap W^{1,\infty}) \times (L^1 \cap L^\infty)$$

are sufficiently small and $p > 1 + 2/3$, then the Cauchy problem (1.3.26) admits a unique global solution

$$u \in C([0,\infty); L^1 \cap L^\infty) \cap C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$$

satisfying the decay estimates

$$\|u(t)\|_{L^q} \le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{q}\right)}$$

for $1 \le q \le \infty$ and

$$\|\nabla u(t)\|_{L^2} \le C(1+t)^{-3/4-1/2},$$
$$\|u_t(t)\|_{L^2} \le C(1+t)^{-3/4-1}.$$

Taking into account the linear estimates described in the previous subsection, we can expect the above decay estimates are optimal. When $n = 2$, Hosono and Ogawa [**23**] obtained the corresponding results. They proved that if the initial data

$$(u_0, u_1) \in (B^1_{2,1} \cap W^{1,1}) \times (B^0_{2,1} \cap L^1)$$

are sufficiently small and $p > 2$, then there exists a unique global solution

$$u \in C([0,\infty); H^1 \cap L^\infty) \cap C^1([0,\infty); L^2)$$

of (1.3.26) satisfying

$$\|u(t)\|_{L^q} \le C(1+t)^{-\left(1-\frac{1}{q}\right)}$$

for $1 \le q \le \infty$ and

$$\|\nabla u(t)\|_{L^2} \le C(1+t)^{-1}.$$

Here $B^s_{2,1}$ denotes the Besov space defined by

$$B^s_{2,1}(\mathbf{R}^n) := \left\{ f : \mathbf{R}^n \to \mathbf{R}; \|f\|_{B^s_{2,1}} := \sum_{j \ge 0} 2^{js} \|\phi_j * f\|_{L^2} < +\infty \right\},$$

where $\{\phi_j\}$ is the Littlewood-Paley dyadic decomposition (see [**1**] for detail).

Ono [**92**] relaxed the assumption on the data of [**78**] to $(u_0, u_1) \in (H^1 \cap L^1) \times (L^2 \cap L^1)$ and proved that the problem (1.3.26) admits a unique global solution $u \in C([0,\infty); H^1 \cap L^1) \cap C^1([0,\infty); L^2)$ satisfying

$$\|u(t)\|_{L^q} \le C(1+t)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}, \tag{1.3.30}$$
$$\|\nabla u(t)\|_{L^2} \le C(1+t)^{-\frac{n}{4}-1/2}, \tag{1.3.31}$$
$$\|u_t(t)\|_{L^2} \le C(1+t)^{-\frac{n}{4}-1} \tag{1.3.32}$$

for $1 \le q \le \infty$ when $n \le 3$ and $p > 1 + 2/n$.

For $n = 4, 5$, Narazaki [**73**] obtained the global existence results for f_1, f_2. He proved that if $p_F < p \le n/(n-2)$, $p < 2$ and the initial data satisfy

$$(u_0, u_1) \in (H^2 \cap W^{1,p/(p-1)} \cap W^{1,p} \cap L^1) \times (H^1 \cap L^{p/(p-1)} \cap L^p \cap L^1)$$

and sufficiently small, then there exists a unique global solution

$$u \in C([0,\infty); H^2 \cap L^{p/(p-1)} \cap L^p) \cap C^1([0,\infty); H^1) \cap C^2([0,\infty); L^2)$$

satisfying the estimates (1.3.30)-(1.3.32) for $p \le q \le p/(p-1)$. Ono [**93**] obtained the estimate (1.3.30) for $1 \le q \le 2n/(n-2)$ under the assumption

$$(u_0, u_1) \in (H^1 \cap W^{2,1}) \times (L^2 \cap W^{1,1}).$$

For higher dimensional cases with $f_2(u) = |u|^p$, Todorova and Yordanov [**105, 106**] developed a new weighted energy method and proved the existence of global solutions with small and compactly supported data when $p_F < p \le n/(n-2)$ ($p_F < p < \infty$ when $n = 1, 2$) and local solution blows up in finite time if the initial data satisfy

$$\int_{\mathbf{R}^n} u_i(x)dx > 0$$

for $i = 0, 1$ and $1 < p < p_F$. After that, Qi S. Zhang [**122**] and Kirane and Qafsaoui [**47**] proved that the critical exponent $p = p_F$ belongs to the blow-up case. In particular, Kirane and Qafsaoui [**47**] obtained blow-up results for more general semilinear dissipative wave equation of the form

$$u_{tt} + (-1)^m |x|^\alpha \Delta^m u + u_t = f(t,x)|u|^p + w(t,x),$$

where $m \geq 1$, $p > 1$, $f(t,x) \geq 0$ satisfies $f(R^2 t, R^{1/m} x) \geq C R^\lambda$ for large $R > 0$ with some $\lambda \geq 0$, $0 \leq \alpha < m(\lambda+2)$, and $w(t,x)$ satisfies $|x|^{-\alpha} w(t,x) \in L^1([0,\infty) \times \mathbf{R}^n)$ and $\iint |x|^{-\alpha} w(t,x) dx dt \geq 0$. In this case, they proved that if $1 < p \leq 1 + (m(\lambda + 2) - \alpha)/n$ and $\int_{\mathbf{R}^n} |x|^{-\alpha}(u_0 + u_1) dx > 0$, then there is no global-in-time solution.

The method by Todorova and Yordanov [**106**] for proving the global existence result is a weighted energy estimate by using a weight function of the form $e^{\psi(t,x)}$. They pointed out the following identity:

$$\begin{aligned} e^{2\psi} u_t (u_{tt} - \Delta u + u_t) &= \partial_t \left[\frac{e^{2\psi}}{2} (u_t^2 + |\nabla u|^2) \right] - \nabla \cdot (e^{2\psi} u_t \nabla u) \\ &\quad + \frac{e^{2\psi}}{-\psi_t} |\psi_t \nabla u - u_t \nabla \psi|^2 + e^{2\psi} \left(1 + \frac{|\nabla \psi|^2}{-\psi_t} + (-\psi_t) \right) u_t^2. \end{aligned}$$

They chose the weight function ψ so that

$$\psi_t < 0, \quad \psi_t + |\nabla \psi|^2 - \psi_t^2 = 0.$$

More precisely, they put

$$\psi(t,x) = \frac{1}{2} \left(t + \rho - \sqrt{(t+\rho)^2 - |x|^2} \right)$$

with $\rho > 0$. To make sense the definition of ψ, they need the compactness of the support of the data, that is, $\operatorname{supp}(u_0, u_1) \subset \{x \in \mathbf{R}^n \mid |x| \leq \rho\}$. Ikehata and Tanizawa [**35**] chose the function ψ as

$$\psi(t,x) = \frac{|x|^2}{4(1+t)}$$

and succeeded to remove the assumption on the compactness of the data.

For the estimate of the lifespan in the critical and subcritical cases $1 < p \leq p_F$, Li and Zhou [**53**] and Nishihara[**79**] obtained the same estimate as (1.3.28) when $n = 1, 2$ and $n = 3$, respectively. Theorem 1.6 described in the previous section shows when the subcritical case $1 < p < p_F$, the same estimate as (1.3.28) holds for solutions to (1.3.29) with $f = f_2$. However, to estimate the lifespan in the critical case $p = p_F$ with $n \geq 4$ remains open.

The asymptotic profile of the global solution in supercritical cases $p > p_F$ was investigated by Nishihara [**78**], Hosono and Ogawa [**23**] and Hayashi, Kaikina and Naumkin [**16**]. Nishihara [**78**] and Hosono and Ogawa [**23**] proved that for suitably small data, the corresponding solutions behave as

$$\|u(t) - \theta G(t)\|_{L^q} = o(t^{-\frac{n}{2}\left(1 - \frac{1}{q}\right)})$$

for $1 \leq q \leq \infty$ when $n = 3$ and $n = 2$, respectively. Here

$$G(t,x) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$$

and

$$\theta = \int_{\mathbf{R}^n} (u_0 + u_1)(x) dx + \int_0^\infty \int_{\mathbf{R}^n} f(u) dx dt.$$

Hayashi, Kaikina and Naumkin [**16**] extended these results to any space dimensions. Let $H^{l,m}$ be a weighted Sobolev space defined by

$$H^{l,m} = \{\phi \in L^2 \mid \|\langle x\rangle^m \langle i\partial_x\rangle^l \phi\|_{L^2} < \infty\}.$$

They proved that for f_1 and f_2, if $p > p_F$ and the initial data belong to

$$u_0 \in H^{\alpha,0} \cap H^{0,\delta}, \quad u_1 \in H^{\alpha-1,0} \cap H^{0,\delta} \tag{1.3.33}$$

and sufficiently small, where $\delta > \frac{n}{2}, [\alpha] < p$; $\alpha \geq \frac{n}{2} - \frac{1}{p-1}$ for $n \geq 2$ and $\alpha \in [\frac{1}{2} - \frac{1}{2(p-1)}, 1)$ for $n = 1$, then there exists a unique solution

$$u \in C([0,\infty); H^{\alpha,0} \cap H^{0,\delta})$$

satisfying

$$\|u(t) - \theta G(t,x)\|_{L^q} \leq Ct^{-\frac{n}{2}(1-\frac{1}{q}) - \min(1, \frac{\delta}{2} - \frac{n}{4}, \frac{n}{2}(p-1)-1)} \tag{1.3.34}$$

for $t > 0$ and $q \in [2, 2n/(n-2\alpha)]$ $(\alpha < n/2)$, $q \in [2,\infty)$ $(\alpha = n/2)$, $q \in [2,\infty]$ $(\alpha > n/2)$. We note that their result also does not require the compactness of the support of the data. Kawakami and Ueda [**43**] obtained more precise asymptotic expansion of solutions to (1.3.26) when $n \leq 3$. For $k \geq 0$ and $l \in \mathbf{Z}_{\geq}$, we define

$$\|\phi\|_{L^1_k} = \int_{\mathbf{R}^n} (1+|x|)^k |\phi(x)| dx,$$

$\|\phi\|_{W^{l,1}_k} = \sum_{|\alpha|\leq l} \|\partial_x^\alpha \phi\|_{L^1_k}$ and $L^1_k = \{\phi \in L^1 \mid \|\phi\|_{L^1_k} < +\infty\}$, $W^{l,1}_k = \{\phi \in W^{l,1} \mid \|\phi\|_{W^{l,1}_k}$. They proved that if $(u_0, u_1) \in (W^{1,\infty} \cap W^{1,1}_k) \times (L^\infty \cap L^1_k)$ and $u \in C([0,\infty); L^1) \cap L^\infty(0,\infty; L^\infty)$ is a solution of (1.3.26), then it follows that

$$\begin{aligned} &t^{\frac{n}{2}(1-\frac{1}{q})} \|u(t) - V_0(t)\|_{L^q} \\ &= \begin{cases} O(t^{-k/2}) + O(t^{-\frac{n}{2}(p-1)-1}) + O(t^{-1}) & (\frac{n}{2}(p-1) - 1 \neq \frac{k}{2}), \\ O(t^{-k/2}\log t) + O(t^{-1}) & (\frac{n}{2}(p-1) - 1 = \frac{k}{2}) \end{cases} \end{aligned}$$

as $t \to +\infty$ for $q \in [1,\infty]$. Here

$$V_0(t) := \sum_{|\alpha| \leq [k]} M_\alpha(v(t),t) g_\alpha(t,x),$$

where $g_\alpha(t,x) := \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha G(1+t,x)$,

$$\begin{aligned} M_0(v(t),t) &:= \int_{\mathbf{R}^n} v(t,x) dx, \\ M_\alpha(v(t),t) &:= \int_{\mathbf{R}^n} x^\alpha v(t,x) dx \quad (|\alpha| = 1), \\ M_\alpha(v(t),t) &:= \int_{\mathbf{R}^n} x^\alpha v(t,x) dx \\ &\quad - \sum_{\beta<\alpha} M_\beta(v(t),t) \int_{\mathbf{R}^n} x^\alpha g_\beta(t,x) dx \quad (|\alpha| \geq 2), \end{aligned}$$

and $v(t,x)$ is the solution of the inhomogeneous linear heat equation

$$\begin{cases} v_t - \Delta v = f(u), & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ v(0,x) = u_0(x) + u_1(x), & x \in \mathbf{R}^n. \end{cases}$$

We also mention the critical exponent for (1.3.26) with initial data not belonging to $L^1(\mathbf{R}^n)$. Ikehata and Ohta [**34**] proved that if the initial data belong to

$$(u_0, u_1) \in (H^1 \cap L^m) \times (L^2 \cap L^m) \tag{1.3.35}$$

and

$$1 + \frac{2m}{n} < p < \infty \ (n = 1, 2), \quad 1 + \frac{2m}{n} < p \le \frac{n}{n-2} \ (3 \le n \le 6),$$

where m lies in

$$1 \le m \le 2 \ (n = 1, 2), \qquad \frac{\sqrt{n^2 + 16n} - n}{4} \le m < \min\left(2, \frac{n}{n-2}\right) \ (3 \le n \le 6),$$

then there exists a unique global solution $u \in C([0, \infty); H^1) \cap C^1([0, \infty); L^2)$ of (1.3.26) satisfying

$$\|u(t)\|_{L^2} \le C(1 + t)^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right)},$$
$$\|(u_t, \nabla u)(t)\|_{L^2} \le C(1 + t)^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{2}\right) - 1/2}.$$

Moreover, they also showed the nonexistence of global solution even for small data in the subcritical case $1 < p < 1+2m/n$ for all $n \ge 1$. This indicates that the critical exponent of (1.3.26) for the data belonging to (1.3.35) is given by $p_c = 1 + 2m/n$ at least when $n \le 6$. We expect that this is also true for higher dimensional cases. After that, Narazaki and Nishihara [**75**] considered the initial data satisfying

$$u_0, u_1 = O(\langle x \rangle^{-kn})$$

as $|x| \to +\infty$ with some $k \in (0, 1]$. For the corresponding heat equation (1.3.27), Lee and Ni [**50**] determined the critical exponent as

$$p_c = 1 + \frac{2}{kn}$$

for the data satisfying $v_0 = O(\langle x \rangle^{-kn})$ as $|x| \to +\infty$. Narazaki and Nishihara [**75**] obtained the asymptotic profile of the solution v of (1.3.27) with the initial data satisfying $v_0 \in C(\mathbf{R}^n)$, $\langle x \rangle^{kn} v_0(x) \in L^\infty$ and $\lim_{|x|\to\infty} \langle x \rangle^{kn} v_0(x) = c_1 \ne 0$. They proved that the profile of v is given by $V_0(t, x) = c_1 \int_{\mathbf{R}^n} G(t, x - y) \langle y \rangle^{-kn} dy$, where G denotes the heat kernel $G = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$. They also proved that if the initial data $(u_0, u_1) \in C^{[n/2]}(\mathbf{R}^n) \times C(\mathbf{R}^n)$ satisfies $\langle x \rangle^{kn} |\partial_x^\alpha u_0| \in L^\infty$ $(|\alpha| \le [n/2])$, $\langle x \rangle^{kn} u_1 \in L^\infty$ and $p > 1 + 2/(kn)$, then the problem (1.3.26) admits a unique global solution $u \in C([0, \infty) \times \mathbf{R}^n)$ satisfying $\langle x \rangle^{kn} u(t, x) \in L^\infty(\mathbf{R}^n)$ for all $t \ge 0$. Moreover, if the initial data satisfy $\lim_{|x|\to\infty} \langle x \rangle^{kn} (u_0, u_1)(x) = c_1 \ne 0$, then the asymptotic profile of u is also given by $V_0(t, x)$ defined above.

Next, we consider the variable coefficient cases. Ikehata, Todorova and Yordanov [**36**] considered the semilinear wave equation with space-dependent damping

$$\begin{cases} u_{tt} - \Delta u + a(x) u_t = |u|^p, & (t, x) \in (0, \infty) \times \mathbf{R}^n, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbf{R}^n. \end{cases} \tag{1.3.36}$$

Under the assumption that $a(x)$ is radially symmetric and $a(x) \sim a_0 |x|^{-\alpha}$ as $|x| \to +\infty$, they proved that the critical exponent is given by

$$p_c = 1 + \frac{2}{n - \alpha}$$

for compactly supported initial data (the global existence part is also true for $f_1(u) = |u|^{p-1}u$). This exponent agrees with that of the corresponding heat equation. The proof of global existence part is based on a weighted energy method by using the weight function

$$w(t,x) = t^{-m(a)/2-\varepsilon} e^{-(m(a)/2-\varepsilon/2)A(x)/t},$$

where ε is arbitrary positive number, $A(x)$ is the positive solution of the Poisson equation (1.3.15) satisfying (a1)-(a3) and $m(a)$ is defined by (a3). The proof of the blow-up part is done by the test function method developed by Qi S. Zhang [**122**]. This method requires the positivity of the nonlinearity and is not applicable to $f_1(u) = |u|^{p-1}u$. We mention that Theorem 1.6 gives an estimate of the lifespan from above.

For the time-dependent damping case

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = |u|^p, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n \end{cases} \tag{1.3.37}$$

with $b(t) = b_0(1+t)^{-\beta}$, $-1 < \beta < 1$, Nishihara [**83**] and Lin, Nishihara and Zhai [**56**] determined the critical exponent as

$$p_c = 1 + \frac{2}{n},$$

which also coincides with that of the corresponding heat equation

$$b(t)v_t - \Delta v = |v|^p. \tag{1.3.38}$$

Note that for (1.3.38), by changing the variable as

$$s = B(t) := \int_0^t b(s)^{-1} ds, \quad u(t,x) = \phi(s,x) = \phi(B(t),x),$$

the equation (1.3.38) is reduced to

$$\phi_s - \Delta\phi = |\phi|^p.$$

The global existence part was proved by a weighted energy method by using a weight function

$$w(t,x) = B(t)^{-n/4+\varepsilon/2} e^{|x|^2/(4(2+\delta)B(t))},$$

where ε is arbitrary small positive number and δ is a suitably chosen small positive parameter. To obtain the blow-up result, they introduced a modified test function method. By multiplying the equation (1.3.37) by a nonnegative function $g(t) \in C^1([0,\infty))$, it follows that

$$(g(t)u)_{tt} - \Delta(gu) - (g'(t)u)_t + (-g'(t) + b(t)g(t))u = g(t)|u|^p.$$

They chose $g(t)$ by the solution of the Cauchy problem of the ordinary differential equation

$$\begin{cases} -g'(t) + b(t)g(t) = 1, \\ g(0) = \int_0^\infty \exp\left(-\int_0^t b(s)ds\right) dt. \end{cases}$$

Then we can obtain the equation of the divergence form and so we can apply the test function method by Zhang [**122**]. D'Abbicco and Lucente [**7**] applied this method to more general dissipative wave equations with time-dependent coefficients. This idea is also based on Theorem 1.5 in which we consider the one-dimensional semilinear wave equation with damping depending on time and space variables. D'Abbicco, Lucente and Reissig extended the global existence result in the above one [**56**] to more general effective damping.

The author [**110**] considered the critical exponent problem for the critical case $b(t) = \frac{\mu}{1+t}$ with $\mu > 0$ and proved that if $p > p_F$ and μ is sufficiently large depending on p, then there exists a unique global solution for the small data decaying very fast near the infinity (see Theorem 1.3). After that, D'Abbicco [**5**] improved the global existence result for $\mu \geq n+2$ and $p > p_F$.

For the semilinear damped wave equation with time and space dependent coefficients

$$\begin{cases} u_{tt} - \Delta u + a_0 \langle x \rangle^{-\alpha}(1+t)^{-\beta} u_t = |u|^p, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.3.39}$$

the author [**109**] proved the small data global existence of solutions under the assumption

$$\alpha, \beta \geq 0,\ \alpha + \beta < 1,\ p > 1 + \frac{2}{n-\alpha},$$

provided that the initial data decay sufficiently fast near the infinity (see Theorem 1.2). Recently, a similar result is obtained by Khader [**46**]. The exponent above is expected to be critical. However, this problem is still open. The difficulty is that when the damping depends on time and space variables, we cannot apply the test function method directly. The author [**111**] proved that in the one dimensional case, if the damping term decays sufficiently fast as $t \to +\infty$, then we can transform the equation into divergence form and apply the test function method (see Theorem 1.5).

1.3.3. Damped wave equations with nonlinear absorbing terms. Here we consider the semilinear damped wave equation with absorbing nonlinearity

$$\begin{cases} u_{tt} - \Delta u + u_t = -|u|^{p-1}u, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n. \end{cases} \tag{1.3.40}$$

In this case, there is a unique global solution any large data in $L^1(\mathbf{R}^n)$. For the corresponding heat equation

$$\begin{cases} v_t - \Delta v = -|v|^{p-1}v, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ v(0,x) = v_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.3.41}$$

it is known that the asymptotic behavior of solutions divide in the following way:

$$\begin{aligned} p > p_F &\Rightarrow v(t,x) \sim \theta_0 G(t,x), \\ p = p_F &\Rightarrow v(t,x) \sim \theta_0 (\log t)^{-n/2} G(t,x), \\ p < p_F &\Rightarrow v(t,x) \sim w_b(t,x) = t^{-1/(p-1)} f_b(x/\sqrt{t}), \end{aligned}$$

where $p_F = 1 + 2/n$, $G(t,x) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$,

$$\theta_0 = \int_{\mathbf{R}^n} v_0(x)dx - \int_0^\infty \int_{\mathbf{R}^n} |v|^{p-1} v dx dt$$

and $w_b(t,x)$ $(b \geq 0)$ is the self-similar solution with the profile $f_b(t,x)$ satisfying

$$-f'' - \left(\frac{r}{2} + \frac{n-1}{r}\right) f' + |f|^{p-1} f = \frac{1}{p-1} f, \quad \lim_{r \to \infty} r^{2/(p-1)} f(r) = b.$$

For (1.3.40), it is expected that the same results hold. Kawashima, Nakao and Ono [**44**] proved that if $(u_0, u_1) \in H^1 \times L^2$, then there exists a unique global solution

$$u \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$$

for (1.3.40). Moreover, if $u_0, u_1 \in L^1$ and $p > 1 + 4/n$, then the solution decays as

$$\|u(t)\|_{L^2} \le C(1+t)^{-n/4}$$

for $n \le 4$. Using this result, Karch [**40**] proved

$$\|u(t) - \theta_0 G(t)\|_{L^2} = o(t^{-n/4})$$

as $t \to +\infty$ when $p > 1 + 4/n$. After that, Nishihara and Zhao [**87**], Nishihara [**80**] and Ikehata, Nishihara and Zhao [**33**] improved the results above as follows:

(i)Let $(u_0, u_1) \in H^1 \times L^2$ satisfy

$$u_0, u_1, \nabla u_0, |u_0|^{(p+1)/2} \in H^{0,m},$$

where

$$m = \frac{2}{p-1} - \frac{n-\delta}{2}$$

with some $\delta > 0$. We assume that $2m > n$ if $p > 1 + 2/n$. If $1 < p < n/[n-2]_+$ and $p \le 1 + 4/n$, then the solution decays as

$$\|u(t)\|_{L^q} \le C(1+t)^{-1/(p-1)+n/(2q)} \tag{1.3.42}$$

for $1 \le q \le \infty$, $1 \le q < \infty$, $1 \le q \le 2n/(n-2)$ when $n = 1$, $n = 2$, $n \ge 3$, respectively.

(ii)If $n \le 3$ (resp. $n = 4$), $(u_0, u_1) \in H^2 \times H^1$ $(\Delta u_0, \nabla u_1) \in H^{0,m}$ and $p_F < p \le 1 + 4/n$ (resp. $p_F < p < 1 + 4/n$), then it follows that

$$\|u(t) - \theta_0 G(t,x)\|_{L^q} = o(t^{-\frac{n}{2}(1-1/q)})$$

for $1 \le q \le \infty$, where

$$\theta_0 = \int_{\mathbf{R}^n} (u_0 + u_1)(x)dx - \int_0^\infty \int_{\mathbf{R}^n} |u|^{p-1} u dx dt.$$

The decay rate in (1.3.42) agrees with that of the self-similar solution $w_b(t,x)$. Thus, we expect this rate is optimal in the subcritical case $1 < p < p_F$.

Hayashi, Kaikina and Naumkin [**16, 17, 18, 19, 20, 21**] and Hayashi, Naumkin and Rodriguez-Ceballos [**22**] proved several results on the asymptotic profile of solutions. In [**16**], it is proved that for the initial data satisfying (1.3.33), the solution u satisfies (1.3.34) (the condition on α, δ, p are the same as before). This result shows that our expectation in the supercritical case is true. In the subcritical case with $n = 1$ and $p \in (3-\varepsilon, 3)$ for some small $\varepsilon > 0$, in [**18**] they obtained the following asymptotic behavior:

$$u(t,x) \sim (t\eta)^{-1/(p-1)} V(x/\sqrt{t}) + O(t^{-1/(p-1)-\gamma})$$

for the small data satisfying

$$(u_0, (1 + \partial_x^{-1} u_1)) \in (L^\infty \cap L^{1,a}) \times (L^\infty \cap L^{1,a})$$

with $a \in (0,1)$, where $\gamma = \frac{1}{2}\min(a, 1 - \frac{p-1}{2})$, $V \in L^\infty \cap L^{1,a}$ is the solution of the integral equation

$$V(\xi) = \frac{1}{(4\pi)^{1/2}} e^{-\xi^2/4} - \frac{1}{\eta(4\pi)^{1/2}} \int_0^1 \frac{1}{z(1-z)^{1/2}} \int_{\mathbf{R}} e^{(\xi - y\sqrt{z})^2/(4(1-z))} F(y) dy dz,$$

$$\eta = \frac{p-1}{1-(p-1)/2}\int_{\mathbf{R}} V(y)^p dy$$

and

$$F(y) = V(y)^p - V(y)\int_{\mathbf{R}} V(\xi)^p d\xi.$$

Here $L^{1,a}$ denotes the weighted Lebesgue space

$$L^{1,a} = \{\phi \in L^1 \mid \|\langle\cdot\rangle^a \phi\|_{L^1} < \infty\}.$$

In the critical case, they considered in [**17**] the initial-boundary value problem on a half line

$$(1.3.43)\qquad \begin{cases} u_{tt} - u_{xx} + u_t + |u|u = 0, & (t,x) \in (0,\infty)\times(0,\infty) \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in (0,\infty), \\ u(t,0) = 0, & t \in (0,\infty). \end{cases}$$

They put the following assumption on the initial data

$$u_0 \in H_0^{\alpha,0}\cap H^{0,2}, \quad u_1 \in H^{0,2}$$

and

$$\theta = 2\int_0^\infty x(u_0(x)+u_1(x))dx > 0,$$

where

$$H_0^{\alpha,0} = \{\phi \in L^2 \mid \|\langle i\partial_x\rangle^\alpha \phi\|_{L^2} < \infty,\ \phi(0) = 0\}.$$

Under the above assumption, they proved if the data are sufficiently small, then there exists a unique global solution $u \in C([0,\infty); H_0^{\alpha,0}\cap H^{0,2})$ satisfying

$$\left\| u(t) - \theta(4\pi)^{-1/2}\frac{x}{2t}(\log t)^{-1}e^{-\frac{x^2}{4t}}\right\|_{L^q(0,\infty)} \le C(\log t)^{-2}(1+t)^{-1+1/(2q)}$$

as $t \to +\infty$, where $1 \le q \le 2/(1-2\alpha)$ if $0 \le \alpha < 1/2$, $1 \le q < \infty$ if $\alpha = 1/2$, and $1 \le q \le \infty$ if $1/2 < \alpha \le 1$.

For the Cauchy problem on the whole space in the critical case

$$(1.3.44)\qquad \begin{cases} u_{tt} - \Delta u + u_t + |u|^{2/n}u = 0, & (t,x) \in (0,\infty)\times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n, \end{cases}$$

they proved in [**19, 20, 21**] that if the initial data are in $u_0 \in H^{2,k}(\mathbf{R}^n)\cap C(\mathbf{R}^n)$, $u_1 \in H^{1,k}(\mathbf{R}^n)$ with $k > n/2+1$, then there exists a unique global solution $u \in C([0,\infty); H^{1,k}(\mathbf{R}^n)\cap C^1([0,\infty); H^{0,k}(\mathbf{R}^n))$ having the only one of the following asymptotic behavior as $t \to \infty$:

$$\left\| (\log t)^{n/2+\gamma}\left(u(t) - \left(\frac{n}{2\eta}\right)^{n/2}(\log t)^{-n/2}G(t,x)\right)\right\|_X \le C$$

or

$$\|(\log t)^{n/2+1}u(t)\|_X \le C,$$

where $\eta = (4\pi)^{-1}(1+2/n)^{-n/2}$, $0 < \gamma < 1/n$ and

$$\|\phi\|_X = \sup_{t\ge 0}\Big(\langle t\rangle^{n/4-k/2}\|\phi(t)\|_{H^{0,k}} + \langle t\rangle^{n/4}\|\phi(t)\|_{L^2} + \langle t\rangle^{n/4+1/2-k/2}\|\nabla\phi(t)\|_{H^{0,k}}$$
$$+\langle t\rangle^{n/4+1/2}\|\nabla\phi(t)\|_{L^2}\Big).$$

Hayashi, Naumkin and Rodriguez-Ceballos [**22**] obtained the asymptotic formula for the periodic problem

$$(1.3.45)\qquad \begin{cases} u_{tt} - u_{xx} + u_t + |u|^{p-1}u = 0, & (t,x) \in (0,\infty) \times [-\pi,\pi], \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in [-\pi,\pi], \\ u(t,x) = u(t,2\pi + x), & (t,x) \in (0,\infty) \times [-\pi,\pi]. \end{cases}$$

They proved for the initial data $(u_0, u_1) \in H^1 \times L^2$, where

$$H^1 = \{\phi \in L^2([-\pi,\pi]) \mid \sum_{-\infty}^{\infty} (1+|n|)^2 |\hat{\phi}(n)|^2 < \infty\},$$

there exists a unique global solution $u \in C([0,\infty); H^1)$ satisfying

$$\|u(t)\|_{L^\infty} \le C(1+t)^{-1/(p-1)}, \quad \|\partial u(t)\|_{L^\infty} \le C(1+t)^{-1/(p-1)-1/2}.$$

Moreover, if the initial data are sufficiently small and $\hat{\phi}(0) > 0$, then the solution has the asymptotic formula

$$u(t,x) = At^{-1/(p-1)} + O(t^{-1/(p-1)-1/2})$$

with $A = 2(2/(p-1))^{1/(p-1)}$.

Next, we consider the variable coefficient damping cases. Nishihara and Zhai [**86**] and Nishihara [**84**] treated the time-dependent damping

$$(1.3.46)\qquad \begin{cases} u_{tt} - \Delta u + b(t)u_t + |u|^{p-1}u = 0, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n, \end{cases}$$

where $1 < p < (n+2)/[n-2]_+$, $b(t) = b_0(1+t)^{-\beta}$, $b_0 > 0$, $\beta \in (-1,1)$ and the initial data $(u_0,u_1) \in H^1 \times L^2$ have compact support. They obtained the existence of global solution and the decay estimates

$$\|u(t)\|_{L^1} \le C(1+t)^{-(1/(p-1)-n/2)(1+\beta)},$$

$$\|u(t)\|_{L^2} \le \begin{cases} C(1+t)^{-(1/(p-1)-n/4)(1+\beta)} & (1 < p \le 1+2/n), \\ C(1+t)^{-(1+\beta)n/4+\varepsilon} & (1+2/n < p < (n+2)/[n-2]_+) \end{cases}$$

with arbitrary small $\varepsilon > 0$. Moreover, in one-dimensional case, Nishihara [**84**] proved the asymptotic profile of the solution with small data is given by $\theta G_B(t,x)$, where

$$G_B(t,x) = (4\pi B(t))^{-n/2} e^{-|x|^2/(4B(t))}, \quad B(t) = \int_0^t b(s)^{-1} ds$$

and

$$\theta = \int_{\mathbf{R}} \left(\frac{1}{b_0} u_1(x) + \left(1 - \frac{\beta}{b_0}\right) u_0(x) \right) dx + \int_0^\infty \left[\frac{\beta(1-\beta)}{b_0} (1+\tau)^{-(2-\beta)} \int_{\mathbf{R}} u(\tau,x)dx - \frac{1}{b_0}(1+\tau)^{\beta} \int_{\mathbf{R}} |u|^{p-1}u dx \right] d\tau.$$

Nishihara [**81**] considered the space-dependent damping case

$$u_{tt} - \Delta u + a(x)u_t + |u|^{p-1}u = 0$$

with compactly supported initial data $(u_0,u_1) \in H^1 \times L^2$, where $1 < p < (n+2)/[n-2]_+$ and $a(x)$ satisfies

$$a_0 \langle x \rangle^{-\alpha} \le a(x) \le a_1 \langle x \rangle^{-\alpha}$$

with some $a_0, a_1 > 0$ and $\alpha \in [0,1)$. He proved that there exists a unique global solution

$$u \in C([0,\infty); H^1(\mathbf{R}^n)) \cap C^1([0,\infty); L^2(\mathbf{R}^n))$$

satisfying

$$\|u(t)\|_{L^2} \le \begin{cases} C(1+t)^{-\frac{1}{p-1}+\frac{\alpha}{2(2-\alpha)}} & (1 < p < 1 + \frac{2\alpha}{n-\alpha}), \\ C(1+t)^{-\frac{1}{p-1}+\frac{\alpha}{2(2-\alpha)}} \log(2+t) & (p = 1 + \frac{2\alpha}{n-\alpha}), \\ C(1+t)^{-\frac{2}{2-\alpha}(\frac{1}{p-1}-\frac{n}{4})} & (1 + \frac{2\alpha}{n-\alpha} < p < \frac{n+2}{[n-2]_+}). \end{cases}$$

Lin, Nishihara and Zhai [**54, 55**] further extended the above results to space and time dependent damping

$$u_{tt} - \Delta u + a(x)b(t)u_t + |u|^{p-1}u = 0$$

with $a(x) = a_0\langle x\rangle^{-\alpha}$, $b(t) = (1+t)^{-\beta}$, $\alpha \in [0,1), \beta \in [0,1)$, $\alpha+\beta \in [0,1)$ and $a_0 > 0$. They obtained

$$\|u(t)\|_{L^2} \le \begin{cases} C(1+t)^{-\left(\frac{1}{p-1}-\frac{\alpha}{2(2-\alpha)}\right)(1+\beta)} & (1 < p < 1 + \frac{2\alpha}{n-\alpha}), \\ C(1+t)^{-\left(\frac{1}{p-1}-\frac{\alpha}{2(2-\alpha)}\right)(1+\beta)} \log(2+t) & (p = 1 + \frac{2\alpha}{n-\alpha}), \\ C(1+t)^{-\frac{2}{2-\alpha}(\frac{1}{p-1}-\frac{n}{4})(1+\beta)} & (1 + \frac{2\alpha}{n-\alpha} < p \le 1 + \frac{2}{n-\alpha}), \\ C(1+t)^{-\frac{n-2\alpha}{2(2-\alpha)}(1+\beta)+\varepsilon} & (1 + \frac{2}{n-\alpha} < p < \frac{n+2}{[n-2]_+}). \end{cases}$$

We expect that the above decay rates are almost optimal (if we can remove ε, then the decay rate will be optimal). However, the optimality of the above decay estimates and the precise asymptotic profiles are still open.

1.3.4. Systems of damped wave equations. First, we consider the weakly coupled system of damped wave equations

$$\begin{cases} u_{tt} - \Delta u + u_t = g(v), \\ v_{tt} - \Delta v + v_t = f(u), \\ (u, u_t, v, v_t)(0,x) = (u_0, u_1, v_0, v_1)(x), \end{cases} \tag{1.3.47}$$

where $u = u(t,x), v = v(t,x)$ are real-valued unknown functions of $(t,x) \in (0,\infty) \times \mathbf{R}^n$ and $(f(u), g(v))$ denotes the nonlinear term.

We also consider the corresponding system of heat equations

$$\begin{cases} u_t - \Delta u = g(v), \\ v_t - \Delta v = f(u), \\ (u, v)(0,x) = (u_0, v_0)(x). \end{cases} \tag{1.3.48}$$

Escobedo and Herrero [**11**] proved for (1.3.48) that when the data (u_0, v_0) is non-negative, bounded continuous, $(f(u), g(v)) = (u^p, v^q)$ with $p, q > 0, pq > 1$, the exponents p, q satisfying

$$\alpha := \max\left\{\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\right\} = \frac{n}{2} \tag{1.3.49}$$

are critical. Here the "critical" means that (i) if $\alpha < n/2$, then the local-in-time solution can be extended globally for suitably small data and (ii) if $\alpha \ge n/2$, then every local-in-time solution blows up in finite time. We remark that they also proved the existence of global solutions when $0 < pq \le 1$ without smallness assumption on the data and the blow-up of solutions for large data when $pq > 1, \alpha < n/2$.

Sun and Wang [**103**] obtained the corresponding results to that of [**11**] for (1.3.47) for $(f(u), g(v)) = (|u|^p, |v|^q)$ with $p, q > 1$ in the one and three dimensional cases (for the case $n = 2$, see Narazaki [**74**]). Nishihara [**85**] considered the asymptotic behavior of solutions to (1.3.47) and (1.3.48). For (1.3.48) with small data and the nonlinear term $(f(u), g(u))$ satisfying

$$|f(u) - f(v)| \leq C(|u| + |v|)^{p-1}|u - v|, \quad |g(u) - g(v)| \leq C(|u| + |v|)^{q-1}|u - v|,$$

he assorted the supercritical case to two cases. The first one is the case $p, q > 1 + 2/n$. In this case he proved that the asymptotic profile of solution is given by a constant multiple of the Gaussian. In the second case $\alpha > n/2$ and $q > 1 + 2/n \geq p$ (without loss of generality, we may assume that $p \leq q$), he proved that the asymptotic profile of v is given by a constant multiple of the Gaussian and that of u is given by the solution of the linear heat equation with the data u_0 in some sense. He also obtained the same results for the system of damped wave equations (1.3.47) when $n \leq 3$.

Takeda [**104**] considered a generalization of (1.3.47)

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + \partial_t u_1 = |u_k|^{p_1}, \\ \partial_t^2 u_2 - \Delta u_2 + \partial_t u_2 = |u_1|^{p_2}, \\ \vdots \\ \partial_t^2 u_k - \Delta u_k + \partial_t u_k = |u_{k-1}|^{p_k}, \end{cases} \tag{1.3.50}$$

where $k \geq 2$ and $p_j > 1$ for $j = 1, \dots, k$. We put

$$P = \begin{pmatrix} 0 & 0 & \cdots & 0 & p_1 \\ p_2 & 0 & \cdots & 0 & 0 \\ 0 & p_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p_k & 0 \end{pmatrix}$$

and

$$\alpha = (\alpha_1, \dots, \alpha_k) = (P - Id)^{-1} \cdot {}^t(1, \dots, 1), \quad \alpha_{max} = \max(\alpha_1, \dots, \alpha_k).$$

He proved that when $n \leq 3$, the critical exponent of the system (1.3.50) is given by

$$\alpha_{max} = n/2,$$

that is, if $\alpha_{max} < n/2$, then there exists a unique global solution for small data; if $\alpha_{max} \geq n/2$, then the local-in-time solution blows up in finite time (see [**108**] for the corresponding result for the system of heat equations). Moreover, he obtained blow-up results for any space dimensions. Ogawa and Takeda [**90**] further extended the above result to the strongly coupled system of damped wave equations

$$u_{tt} - \Delta u + u_t = F(u), \tag{1.3.51}$$

where $u = (u_1, \dots, u_k)$, $F(u) = (F_1(u), \dots, F_k(u))$ and

$$F_j(u) = \prod_{l=1}^{k} |u_l|^{p_{j,l}}.$$

As before, we put $P = (p_{j,l})_{1 \leq j,l \leq k}$ and $\alpha = (P - Id)^{-1} \cdot {}^t(1, \dots, 1)$, $\alpha_{max} = \max(\alpha_1, \dots, \alpha_k)$. When $n \leq 3$, they proved that if $p_{j,l} \in [1, \infty) \cup \{0\}$, $\sum_{l=1}^{k} p_{j,l} > 1$, $\det(P - Id) \neq 0$ and $0 < \alpha_j < n/2$, then the system (1.3.51) admits a unique global solution for suitably small data. Moreover, they obtained blow-up results for $\max_{1 \leq j \leq k} \alpha_j \geq n/2$ and all $n \geq 1$. In [**91**] they also obtained the decay estimates of global solutions and that the asymptotic profile is given by a constant multiple of the Gaussian under the condition

$$\min_{1\le j\le k}\sum_{l=1}^{k} p_{j,l} > 1+2/n.$$

We conjecture that the above results can be suitably extended to higher dimensional cases $n \ge 4$. We are also interested in the damping depending on variable coefficients

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + a_1(t,x)\partial_t u_1 = F_1(u), \\ \partial_t^2 u_2 - \Delta u_2 + a_2(t,x)\partial_t u_2 = F_2(u), \\ \vdots \\ \partial_t^2 u_k - \Delta u_k + a_k(t,x)\partial_t u_k = F_k(u). \end{cases}$$

However, as of now, these problems are open as far as the author's knowledge.

1.3.5. Other nonlinearities. Here we mention some results on other nonlinearities. We first consider the wave equation with nonlinear damping

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{p-1}u_t = 0, & (t,x)\in(0,\infty)\times\mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x\in\mathbf{R}^n, \end{cases} \tag{1.3.52}$$

where $p > 1$. It is known that for the initial data (u_0, u_1) belonging to $H^2 \times (H^1 \cap L^{2p})$, there exists a unique global solution $u \in C([0,\infty);H^1)\cap C^1([0,\infty);L^2)$ satisfying the energy identity

$$E(t) + \int_0^t \int_{\mathbf{R}^n} |u_t(s,x)|^{p+1} dxds = E(0),$$

where

$$E(t) = \frac{1}{2}\int_{\mathbf{R}^n} (u_t(t,x)^2 + |\nabla u(t,x)|^2)dx$$

(see Lions and Strauss [**57**]). Mochizuki and Motai [**64, 65**] considered the energy decay problem for (1.3.52) and proved the following.

(i) If $n \ge 1$ and $1 < p \le 1 + 2/n$, then $\lim_{t\to\infty} E(t) = 0$.
(ii) If $n \ge 2$ and $p > 1 + 2/(n-1)$, then the energy does not decay to 0 in general. Moreover, if

$$1 + \frac{2}{n-1} < p < \begin{cases} \infty & (1\le n\le 6), \\ \dfrac{n}{n-6} & (n\ge 7), \end{cases}$$

the solution is asymptotically free as $t \to +\infty$.

Recently, Katayama, Matsumura and Sunagawa [**41**] proved that if $n = 2$ and $p = 3$ (note that in this case $p = 1 + 2/(n-1)$ holds), then the energy of solutions decays to 0 as $t \to +\infty$. From this, we expect that the threshold of the energy decay problem is given by $p = 1 + 2/(n-1)$. However, the problem that whether or not the energy decays to 0 when $1+2/n < p \le 1+2/(n-1)$ in general space dimensions is still open.

Mochizuki and Motai [**64, 65**] also considered more general nonlinear wave equations

$$\begin{cases} u_{tt} - \Delta u + \lambda u + \beta(t,x,u_t(t,x))u_t = 0, & (t,x)\in(0,\infty)\times\mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x\in\mathbf{R}^n, \end{cases} \tag{1.3.53}$$

where $\lambda \geq 0$ and β is given by

$$\beta(t, x, u_t) = b(t, x)|u_t(t, x)|^{p-1} \tag{1.3.54}$$

with $b \geq 0, p > 1$ or

$$\beta(t, x, u_t) = (|x|^{-\gamma} * u_t^2) = \int_{\mathbf{R}^n} |x - y|^{-\gamma} u_t(t, y)^2 dy \tag{1.3.55}$$

with $0 < \gamma < n$. They investigated when the energy of solutions

$$E_\lambda(t) = \frac{1}{2} \int_{\mathbf{R}^n} (u_t(t, x)^2 + |\nabla u(t, x)|^2 + \lambda u(t, x)^2) dx$$

decays to 0. They proved that for the power type nonlinear damping (1.3.54),

(i) if $n \geq 1$, $1 < p \leq 1 + 2(1 - \delta)/n$ and

$$b_1(1 + t + |x|)^{-\delta} \leq b(t, x) \leq b_2$$

with some $b_1, b_2 > 0$ and $\delta \in [0, 1)$, then the energy of solutions decays to 0 as $t \to +\infty$;

(ii) if

$$\begin{cases} p > 1 + \dfrac{2(1-\delta)}{n-1} & (n \geq 2, \lambda = 0), \\ p > 1 + \dfrac{2(1-\delta)}{n} & (n \geq 1, \lambda > 0) \end{cases}$$

and

$$0 \leq b(t, x) \leq b_3(1 + |x|)^{-\delta}$$

with some $\delta \in [0, 1]$, then the energy of solutions does not decay to 0 in general.

This result shows that in the case that $\lambda > 0$, the number

$$p = 1 + \frac{2(1-\delta)}{n}$$

is a critical value which divides the energy decay and nondecay. However, in the case that $\lambda = 0$, the problems remain unsolved for

$$1 + \frac{2(1-\delta)}{n} < p \leq 1 + \frac{2(1-\delta)}{n-1}.$$

For the non-local nonlinear damping (1.3.55), they proved that

(i) if $n \geq 1$ and $0 < \gamma < n, \gamma \leq 1$, then the energy decay occurs;

(ii) if

$$\begin{cases} \frac{n}{n-1} < \gamma < n & (n \geq 3, \lambda = 0), \\ 1 < \gamma < n & (n \geq 2, \lambda > 0), \end{cases}$$

then the energy does not decay to 0 in general.

This result indicates that in the case that $\lambda > 0$, the number $\gamma = 1$ is critical. However, in the case that $1 < \gamma \leq n/(n-1)$, the energy decay problem is still open.

Racke [**95**] considered the following nonlinear damped wave equation in unbounded domain Ω:

$$\begin{cases} u_{tt} - \Delta u + u_t = f(t, x, u, u_t, \nabla u, \nabla u_t, \nabla^2 u), & (t, x) \in (0, \infty) \times \Omega, \\ u = 0, & (t, x) \in (0, \infty) \times \partial\Omega, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \Omega, \end{cases}$$

where $\Omega = \mathbf{R}^3$ or $\Omega \subset \mathbf{R}^3$ is an unbounded domain with bounded star-shaped complement and smooth boundary. He proved that if $f(t,x,\lambda)$ is smooth and

$$f(t,x,\lambda) = O(|\lambda|^3)$$

near $\lambda = 0$, then there exists a unique global solution for sufficiently small and smooth suitable initial data.

Recently, Watanabe [**113**] considered damped wave equations with nonuniform dissipative term and quasilinear term

$$\begin{cases} u_{tt} - \Delta u + B(x)u_t = N(u,u), & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n, \end{cases}$$

where $u = (u^1, \dots, u^n)$, $N(u,u)$ is a quasilinear term whose i-th component has the form

$$(N(u,u))^i = \sum_{j,k,a,b,c=1}^{n} N_{abc}^{ijk} \partial_{x_a}(\partial_{x_b} u^j \partial_{x_c} u^k),$$

and $B(x)$ is a smooth, bounded, nonnegative, symmetric $n \times n$ matrix-valued function satisfying $B(x) \geq c_0$ for $|x| \geq R$ with some $c_0, R > 0$. He proved the existence of global classical solution for small data and some energy decay estimates of solutions.

The damped wave equation with nonlinear memory

$$\begin{cases} u_{tt} - \Delta u + u_t = \int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p ds, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n \end{cases} \tag{1.3.56}$$

has been also investigated. Here $\gamma \in (0,1)$ and $p > 1$. We note that it holds that

$$\lim_{\gamma \to 1} \Gamma(1-\gamma) \int_0^t (t-s)^{-\gamma} |u(s,x)|^p ds = |u(t,x)|^p$$

almost everywhere $x \in \mathbf{R}^n$. Therefore, it is expected that as $\gamma \to 1$, the structure of the equation (1.3.56) gets close to the damped wave equation with the power nonlinearity $|u|^p$. The Cauchy problem (1.3.56) is related to that of the corresponding heat equation

$$\begin{cases} v_t - \Delta v = \int_0^t (t-s)^{-\gamma} v(s,\cdot)^p ds, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ v(0,x) = v_0(x) \geq 0, & x \in \mathbf{R}^n. \end{cases} \tag{1.3.57}$$

Cazenave, Dickstein and Weissler [**2**] proved that the critical exponent for (1.3.57) is given by

$$\bar{p}(n,\gamma) := \max\{p_\gamma(n), \gamma^{-1}\},$$

where

$$p_\gamma(n) := 1 + \frac{2(2-\gamma)}{[n-2(1-\gamma)]_+},$$

that is, if $p > \bar{p}(n,\gamma)$ then the global-in-time solution exists for small data; if $1 < p \leq \bar{p}(n,\gamma)$, then there exists no global-in-time solution in general even for small data. Fino [**12**] proved that when $p \in (1, p_\gamma]$ and $\bar{p}(n,\gamma) = p_\gamma(n)$, there is no global-in-time solution. He also gives a partial result for the global existence of solutions in the case that $n \leq 3$. After that, D'Abbicco [**4**] remarked that the blow-up results still holds when $p \in (1, n/(n-2)]$ and $\bar{p}(n,\gamma) = \gamma^{-1}$ by the same proof of [**12**]. He also improved the global existence result of [**12**] to any $p > \bar{p}(n,\gamma)$ and $n \leq 5$.

1.4. Methods for proof

For the convenience of the reader, we introduce some methods used for the proof of the results. We give only very simple examples but it might well suffice to make out the idea.

1.4.1. Weighted energy method. We first introduce a weighted energy method which was originally developed by Todorova and Yordanov [**105, 106**]. This method will be used to prove the existence of global solutions and to obtain time-decay estimates of the solution. We consider the linear heat equation

$$\begin{cases} v_t - \Delta v = 0 & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ v(0,x) = v_0(x) & x \in \mathbf{R}^n. \end{cases} \tag{1.4.1}$$

We use a weight function $e^{2\psi(t,x)}$ with

$$\psi(t,x) = \frac{|x|^2}{8(1+t)}.$$

For the sake of simplicity, we assume that $v_0 \in C_0^\infty(\mathbf{R}^n)$. We will show how to determine the weight function ψ in the proof of the following proposition.

PROPOSITION 1.8. *Let $v_0 \in C_0^\infty(\mathbf{R}^n)$. Then it is true that*

$$(1+t)^{n/2} \int_{\mathbf{R}^n} \frac{e^{2\psi(t,x)}}{2} v^2(t,x)dx \leq \int_{\mathbf{R}^n} \frac{e^{2\psi(0,x)}}{2} v_0^2(x)dx.$$

PROOF. By multiplying (1.4.1) by $e^{2\psi}v$, one can obtain

$$\partial_t \left[\frac{e^{2\psi}}{2} v^2 \right] - \nabla \cdot \left(e^{2\psi} v \nabla v \right) + e^{2\psi} \{ (-\psi_t) v^2 + |\nabla v|^2 + 2\nabla\psi \cdot v\nabla v \} = 0.$$

We note that

$$\begin{aligned} 2e^{2\psi}\nabla\psi \cdot v\nabla v &= 4e^{2\psi}\nabla\psi \cdot v\nabla v - 2e^{2\psi}\nabla\psi \cdot v\nabla v \\ &= 4e^{2\psi}\nabla\psi \cdot v\nabla v - \nabla \cdot \left(e^{2\psi}\nabla\psi v^2 \right) + 2e^{2\psi}|\nabla\psi|^2 v^2 + e^{2\psi}(\Delta\psi)v^2. \end{aligned}$$

Substituting this, we have

$$\begin{aligned} &\partial_t \left[\frac{e^{2\psi}}{2} v^2 \right] - \nabla \cdot \left(e^{2\psi}(v\nabla v + \nabla\psi v^2) \right) \\ &\qquad + e^{2\psi} \{ |\nabla v|^2 + 4\nabla\psi \cdot (v\nabla v) + ((-\psi_t) + 2|\nabla\psi|^2)v^2 + (\Delta\psi)v^2 \} = 0. \end{aligned}$$

We assume that

$$-\psi_t \geq 2|\nabla\psi|^2. \tag{1.4.2}$$

Then we obtain

$$\begin{aligned} &\partial_t \left[\frac{e^{2\psi}}{2} v^2 \right] - \nabla \cdot \left(e^{2\psi}(v\nabla v + \nabla\psi v^2) \right) \\ &\qquad + e^{2\psi} \{ |\nabla v|^2 + 4\nabla\psi \cdot (v\nabla v) + 4|\nabla\psi|^2 v^2 + (\Delta\psi)v^2 \} = 0. \end{aligned}$$

Noting

$$|\nabla v|^2 + 4\nabla\psi \cdot (v\nabla v) + 4|\nabla\psi|^2 v^2 = |\nabla v + 2(\nabla\psi)v|^2 \geq 0,$$

we have

$$\partial_t \left[\frac{e^{2\psi}}{2} v^2 \right] - \nabla \cdot \left(e^{2\psi}(v\nabla v + \nabla\psi v^2) \right) + e^{2\psi}(\Delta\psi)v^2 \leq 0.$$

Integrating the above inequality, we obtain

$$\frac{d}{dt}\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}v^2dx+\int_{\mathbf{R}^n}e^{2\psi}(\Delta\psi)v^2dx\le 0.$$

Multiplying this by $(1+t)^k$ with $k\ge 0$, one can see that

$$\frac{d}{dt}\left[(1+t)^k\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}v^2dx\right] -\frac{k}{2}(1+t)^{k-1}\int_{\mathbf{R}^n}e^{2\psi}v^2dx+(1+t)^k\int_{\mathbf{R}^n}(\Delta\psi)v^2dx\le 0.$$

We also assume that

$$\Delta\psi(t,x)\ge\frac{k}{2(1+t)}. \tag{1.4.3}$$

Then we obtain

$$\frac{d}{dt}\left[(1+t)^k\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}v^2dx\right]\le 0.$$

and hence,

$$(1+t)^k\int_{\mathbf{R}^n}\frac{e^{2\psi(t,x)}}{2}v(t,x)^2dx\le\int_{\mathbf{R}^n}\frac{e^{2\psi(0,x)}}{2}v_0(x)^2dx.$$

Therefore, it suffices to find ψ and k such that both (1.4.2) and (1.4.3) hold. We put

$$\psi(t,x)=\frac{k|x|^2}{4n(1+t)}.$$

Then (1.4.3) holds with equality. We calculate

$$-\psi_t(t,x)=\frac{k|x|^2}{4n(1+t)^2},\quad \nabla\psi(t,x)=\frac{kx}{2n(1+t)},\quad |\nabla\psi(t,x)|^2=\frac{k^2|x|^2}{4n^2(1+t)^2}.$$

Therefore, (1.4.2) is true if

$$k\le\frac{n}{2}.$$

Thus, taking $k=\frac{n}{2}$ and $\psi(t,x)=\frac{|x|^2}{8(1+t)}$, we completes the proof. □

1.4.2. Test function method. Here we shall explain a test function method developed by Zhang [**122**]. We give an application of this method for the semilinear heat equation

$$\begin{cases} v_t-\Delta v=|v|^p, & (t,x)\in(0,\infty)\times\mathbf{R}^n,\\ v(0,x)=v_0(x), & x\in\mathbf{R}^n.\end{cases} \tag{1.4.4}$$

It is well known that the critical exponent of (1.4.4) is given by so-called Fujita's critical exponent (see [**13**])

$$p_c=1+\frac{2}{n}.$$

Here we give a proof of the subcritical part of this fact. We shall prove the following:

PROPOSITION 1.9. *If* $1<p<1+2/n$ *and* $v_0\in L^1(\mathbf{R}^n)$ *satisfies*

$$\int_{\mathbf{R}^n}v_0(x)dx>0,$$

then the classical solution of (1.4.4) *does not exists globally.*

Proof. Suppose that v is a global classical solution of (1.4.4). We define test functions

$$\phi(x) = \begin{cases} 1 & (|x| \le 1/2) \\ \dfrac{\exp(-1/(1-|x|^2))}{\exp(-1/(|x|^2-1/4)) + \exp(-1/(1-|x|^2))} & (1/2 < |x| < 1), \\ 0 & (|x| \ge 1), \end{cases}$$

$$\eta(t) = \begin{cases} 1 & (0 \le t \le 1/2), \\ \dfrac{\exp(-1/(1-t^2))}{\exp(-1/(t^2-1/4)) + \exp(-1/(1-t^2))} & (1/2 < t < 1), \\ 0 & (t \ge 1). \end{cases}$$

It is obvious that $\phi \in C_0^\infty(\mathbf{R}^n), \eta \in C_0^\infty([0,\infty))$. We also see that

$$|\eta'(t)| \lesssim \eta(t)^{1/p}, \quad |\Delta\phi(x)| \lesssim \phi(x)^{1/p}.$$

In fact, we put q, r so that $1/p+1/q = 1, 1/p+2/r = 1$ and let $\mu(t) = \eta(t)^{1/q}, \nu(x) = \phi(x)^{1/r}$. Then we have

$$|\eta'(t)| = |(\mu^q)'| = |q\mu^{q-1}\mu'| \lesssim \mu^{q-1} = \eta^{1/p},$$
$$|\Delta\phi(x)| = |\Delta(\nu^r)| \lesssim |\Delta\nu|\nu^{r-1} + |\nabla\nu|^2\nu^{r-2} \lesssim \nu^{r-2} = \phi^{1/p}.$$

Let $R > 0$ be a large parameter and let

$$\psi_R(t,x) = \eta(t/R^2)\phi(x/R).$$

We also define

$$I_R = \int_0^\infty \int_{\mathbf{R}^n} |v|^p \psi_R dxdt.$$

We note that by the Lebesgue dominated convergence theorem, it follows that

$$\int_{\mathbf{R}^n} v_0(x)\phi(x/R)dx > 0$$

for sufficiently large $R > 0$ Then by the equation, integration by parts and the Hölder inequality, we can calculate

$$\begin{aligned} I_R &= \int_0^\infty \int_{\mathbf{R}^n} (v_t - \Delta v)\psi_R dxdt \\ &= \int_0^\infty \int_{\mathbf{R}^n} v(-\partial_t - \Delta)\psi_R dxdt - \int_{\mathbf{R}^n} v_0(x)\phi(x/R)dx \\ &\le \int_0^\infty \int_{\mathbf{R}^n} |v|(|\partial_t\psi_R| + |\Delta\psi_R|)dxdt \\ &\lesssim R^{-2}\left(\int_0^\infty \int_{\mathbf{R}^n} |v|^p\psi_R dxdt\right)^{1/p} \left(\int_0^{R^2} \int_{B_R} dxdt\right)^{1/q} \\ &\lesssim R^{-2+(n+2)/q} I_R^{1/p}, \end{aligned}$$

where q denotes the Hölder conjugate of p, that is $1/p + 1/q = 1$ and $B_R = \{x \in \mathbf{R}^n \mid |x| < R\}$. We can rewrite the above as

$$I_R^{1-1/p} \lesssim R^{-2+(n+2)/q}.$$

Noting that $p < 1 + 2/n$ if and only if $-2 + (n+2)/q < 0$, the right-hand side of the above inequality tends to 0 as $R \to +\infty$. In particular, I_R is bounded when $R \to +\infty$ and this implies $v \in L^p((0,\infty) \times \mathbf{R}^n)$ and $\lim_{R\to\infty} I_R = \|v\|^p_{L^p((0,\infty)\times\mathbf{R}^n)}$.

However, by using the above estimate again, it follows that $\lim_{R\to\infty} I_R = 0$. This means $v = 0$, which contradicts $v_0 \neq 0$. □

CHAPTER 2

Basic facts and their proof

2.1. Linear damped wave equations

We consider the Cauchy problem for the damped wave equation

$$\left\{\begin{array}{ll} u_{tt} - \Delta u + u_t = 0, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^n. \end{array}\right. \tag{2.1.1}$$

We denote the fundamental solution of (2.1.1) by $S_n(t)$, that is, $S_n(t)g$ stands for the solution of (2.1.1) with $(u_0, u_1) = (0, g)$. By the Duhamel principle, the solution u of (2.1.1) can be written by

$$u = S_n(t)(u_0 + u_1) + \partial_t(S_n(t)u_0). \tag{2.1.2}$$

2.1.1. Decay estimates. First, we prove the following estimates obtained by Matsumura [**59**].

Proposition 2.1 (Matsumura [**59**]). *For $i \in \mathbf{Z}_{\geq 0}$ and $\alpha \in \mathbf{Z}^n_{\geq 0}$, we have*

$$\begin{aligned} \left\|\partial_t^i \partial_x^\alpha S_n(t) g\right\|_{L^\infty} &\leq C(1+t)^{-n/(2m)-i-|\alpha|/2}(\|g\|_{L^m} + \|g\|_{H^{[n/2]+i+|\alpha|}}), \\ \left\|\partial_t^i \partial_x^\alpha S_n(t) g\right\|_{L^2} &\leq C(1+t)^{n/4-n/(2m)-i-|\alpha|/2}(\|g\|_{L^m} + \|g\|_{H^{i+|\alpha|-1}}) \end{aligned} \tag{2.1.3}$$

for $t \geq 0$, where $1 \leq m \leq 2$.

Proof. We use the representation of $S_n(t)$ by the Fourier transform. By applying the Fourier transform to the equation (2.1.1), we have

$$\left\{\begin{array}{l} \hat{u}_{tt} + \hat{u}_t + |\xi|^2 \hat{u} = 0, \\ (\hat{u}, \hat{u}_t)(0,\xi) = (0, \hat{g})(\xi). \end{array}\right. \tag{2.1.4}$$

Solving this ordinary differential equation, we obtain

$$\begin{aligned} \hat{u}(t,\xi) &= \left\{\begin{array}{ll} \dfrac{2e^{-t/2}}{\sqrt{1-4|\xi|^2}} \sinh\left(\dfrac{\sqrt{1-4|\xi|^2}}{2} t\right) \hat{g}(\xi), & |\xi| \leq 1/2, \\ \dfrac{2e^{-t/2}}{\sqrt{4|\xi|^2 - 1}} \sin\left(\dfrac{\sqrt{4|\xi|^2-1}}{2} t\right) \hat{g}(\xi), & |\xi| > 1/2. \end{array}\right. \\ &\equiv R(t,\xi)\hat{g}(\xi). \end{aligned} \tag{2.1.5}$$

We first prove the L^∞ estimate of (2.1.3). From Lemma 9.6, we have

$$\|f\|_{L^\infty} \leq C\|\hat{f}\|_{L^1}$$

and obtain

$$\left\|\partial_t^i\partial_x^\alpha S_n(t)g\right\|_{L^\infty} \le \left\|(i\xi)^\alpha \frac{d^i}{dt^i}R(t,\xi)\hat{g}\right\|_{L^1} \le C\int_{\mathbf{R}^n}|\xi|^{|\alpha|}\left|\frac{d^i}{dt^i}R(t,\xi)\right||\hat{g}(\xi)|d\xi. \tag{2.1.6}$$

By a simple calculation, we can see that

$$\left|\frac{d^i}{dt^i}R(t,\xi)\right| \le Ce^{-t/2}\left(\frac{1}{\sqrt{1-4|\xi|^2}}\sinh\left(\frac{\sqrt{1-4|\xi|^2}}{2}t\right)+\cosh\left(\frac{\sqrt{1-4|\xi|^2}}{2}t\right)\right.$$
$$\left.+\cdots+(\sqrt{1-4|\xi|^2})^{i-1}H\left(\frac{\sqrt{1-4|\xi|^2}}{2}t\right)\right)$$

for $|\xi|\le 1/2$, where H denotes cosh or sinh. We also have

$$\left|\frac{d^i}{dt^i}R(t,\xi)\right| \le Ce^{-t/2}\left(\frac{1}{\sqrt{4|\xi|^2-1}}\left|\sin\left(\frac{\sqrt{4|\xi|^2-1}}{2}t\right)\right|+\left|\cos\left(\frac{\sqrt{4|\xi|^2-1}}{2}t\right)\right|\right.$$
$$\left.+\cdots+(\sqrt{4|\xi|^2-1})^{i-1}\left|T\left(\frac{\sqrt{4|\xi|^2-1}}{2}t\right)\right|\right)$$

for $|\xi|>1/2$, where T is given by sin or cos. By Lemma 9.7, we have an elementary inequality

$$\int_0^\delta r^k e^{-cr^2t}dr \le C(1+t)^{-(k+1)/2}, \tag{2.1.7}$$

where $\delta>0, k\ge 0, c>0, t\ge 0$. We take $\delta\in(0,1/2)$ arbitrarily and divide the integral of (2.1.6) as

$$\begin{aligned}&\int_{\mathbf{R}^n}|\xi|^{|\alpha|}\left|\frac{d^i}{dt^i}R(t,\xi)\right||\hat{g}(\xi)|d\xi\\ &=\int_{|\xi|\ge 1}+\int_{1/2<|\xi|<1}+\int_{\delta<|\xi|\le 1/2}+\int_{|\xi|\le\delta}\\ &\equiv I_1+I_2+I_3+I_4.\end{aligned}$$

We first estimate I_1. By the Schwarz inequality, it follows that

$$\begin{aligned}I_1 &\le Ce^{-t/2}\int_{|\xi|\ge 1}\frac{|\xi|^{|\alpha|}(1+\sqrt{4|\xi|^2-1})^i}{\sqrt{4|\xi|^2-1}}|\hat{g}(\xi)|d\xi\\ &\le Ce^{-t/2}\sup_{|\xi|>1}\left(\frac{(1+\sqrt{4|\xi|^2-1})^i}{|\xi|^{i-1}\sqrt{4|\xi|^2-1}}\right)\left(\int_{|\xi|\ge 1}|\xi|^{-2[\frac{n}{2}]-2}d\xi\right)^{1/2}\\ &\quad\times\left(\int_{|\xi|\ge 1}|\xi|^{2[\frac{n}{2}]+2|\alpha|+2i}|\hat{g}(\xi)|^2d\xi\right)^{1/2}\\ &\le Ce^{-t/2}\|g\|_{H^{[\frac{n}{2}]+i+|\alpha|}}.\end{aligned}$$

The second term I_2 is estimated by

$$
\begin{aligned}
I_2 &\le Ce^{-t/2}\left\{1+\sup_{1/2<|\xi|<1}\left(\frac{\sin\left(\frac{\sqrt{4|\xi|^2-1}}{2}t\right)}{\sqrt{4|\xi|^2-1}}\right)\right\}\left(\int_{1/2<|\xi|<1}|\hat{g}(\xi)|^2d\xi\right)^{1/2}\\
&\le C(1+t)e^{-t/2}\|g\|_{L^2}.
\end{aligned}
$$

Next, let us treat I_3. We see that

$$
\begin{aligned}
I_3 &\le Ce^{-t/2}\left\{1+\sup_{\delta<|\xi|\le 1/2}\left(\frac{\sinh\left(\frac{\sqrt{1-4|\xi|^2}}{2}t\right)}{\sqrt{1-4|\xi|^2}}\right)\right.\\
&\quad\left.+\sup_{\delta<|\xi|\le 1/2}\cosh\left(\frac{\sqrt{1-4|\xi|^2}}{2}t\right)\right\}\left(\int_{\delta<|\xi|\le 1/2}|\hat{g}(\xi)|^2d\xi\right)^{1/2}\\
&\le C(1+t)e^{-t(1-\sqrt{1-4\delta^2})/2}\|g\|_{L^2}.
\end{aligned}
$$

Finally, we estimate I_4. In this region, it is also easy to see that

$$
\begin{aligned}
\left|\frac{d^i}{dt^i}R(t,\xi)\right| &\le C\frac{(1-\sqrt{1-4|\xi|^2})^i}{\sqrt{1-4|\xi|^2}}e^{-t(1-\sqrt{1-4|\xi|^2})/2}\\
&\quad+C\frac{(1+\sqrt{1-4|\xi|^2})^i}{\sqrt{1-4|\xi|^2}}e^{-t(1+\sqrt{1-4|\xi|^2})/2}.
\end{aligned}
$$

Using this, we have

$$
\begin{aligned}
I_4 &\le C\int_{|\xi|\le\delta}\frac{(1-\sqrt{1-4|\xi|^2})^i|\xi|^{|\alpha|}}{\sqrt{1-4|\xi|^2}}e^{-t(1-\sqrt{1-4|\xi|^2})/2}|\hat{g}(\xi)|d\xi\\
&\quad+C\int_{|\xi|\le\delta}\frac{(1+\sqrt{1-4|\xi|^2})^i|\xi|^{|\alpha|}}{\sqrt{1-4|\xi|^2}}e^{-t(1+\sqrt{1-4|\xi|^2})/2}|\hat{g}(\xi)|d\xi.
\end{aligned}
$$

Since

$$
1-\sqrt{1-4|\xi|^2}=\frac{4|\xi|^2}{1+\sqrt{1-4|\xi|^2}},
$$

it is obvious that

$$
2|\xi|^2\le 1-\sqrt{1-4|\xi|^2}\le 4|\xi|^2
$$

for $|\xi| < 1/2$. Therefore, by the Hölder inequality, Lemma 9.6 and (2.1.6), we have

$$\begin{aligned}
I_4 \le C &\int_{|\xi|\le\delta} |\xi|^{2i+|\alpha|} e^{-t|\xi|^2} |\hat{g}(\xi)| d\xi \\
&+ Ce^{-t/2} \int_{|\xi|\le\delta} |\hat{g}(\xi)| d\xi \\
&\le C \left(\int_{|\xi|\le\delta} |\xi|^{m(2i+|\alpha|)} e^{-mt|\xi|^2} d\xi \right)^{1/m} \left(\int_{|\xi|\le\delta} |\hat{g}(\xi)|^{m'} d\xi \right)^{1/m'} \\
&+ Ce^{-t/2} \left(\int_{|\xi|\le\delta} |\hat{g}(\xi)|^{m'} d\xi \right)^{1/m'} \\
&\le C \left(\int_0^\delta r^{m(2i+|\alpha|)+n-1} e^{-mtr^2} dr \right)^{1/m} \|g\|_{L^m} \\
&\le (1+t)^{-n/(2m)-i-|\alpha|/2} \|g\|_{L^m},
\end{aligned}$$

which shows the desired estimate for $\|\partial_t^i \partial_x^\alpha S_n(t) g\|_{L^\infty}$.

Next, we enter into the estimate of the L^2-norm. The proof is almost same way as before. We use the Plancherel theorem and have

$$\begin{aligned}
\left\|\partial_t^i \partial_x^\alpha S_n(t) g\right\|_{L^2}^2 &= \left\| (i\xi)^\alpha \frac{d^i}{dt^i} R(t,\xi) \hat{g} \right\|_{L^2}^2 \\
&\le \int_{\mathbf{R}^n} |\xi|^{2|\alpha|} \left| \frac{d^i}{dt^i} R(t,\xi) \right|^2 |\hat{g}(\xi)|^2 d\xi \\
&= \int_{|\xi|\ge 1} + \int_{1/2<|\xi|<1} + \int_{\delta<|\xi|\le 1/2} + \int_{|\xi|\le\delta} \\
&\equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

The estimate of the terms I_2 and I_3 is the same as before. We have

$$\begin{aligned}
I_1 &\le Ce^{-t} \int_{|\xi|\ge 1} |\xi|^{2|\alpha|} \frac{(1+\sqrt{4|\xi|^2-1})^{2i}}{4|\xi|^2-1} |\hat{g}(\xi)|^2 d\xi \\
&\le Ce^{-t} \int_{|\xi|\ge 1} (1+|\xi|^2)^{|\alpha|+i-1} |\hat{g}(\xi)|^2 d\xi \\
&\le Ce^{-t} \|g\|_{H^{i+|\alpha|-1}}.
\end{aligned}$$

To treat I_4, we use an another elementary inequality

$$\sup_{0\le r\le\delta} r^k e^{-cr^2 t} \le C(1+t)^{-k/2} \tag{2.1.8}$$

for $\delta > 0, k \geq 0, c > 0, t \geq 0$. We give a proof of this inequality in Appendix (see Lemma 9.7). As before, we can deduce that

$$\begin{aligned}
I_4 \leq & C \int_{|\xi| \leq \delta} |\xi|^{2|\alpha|} \frac{(1-\sqrt{1-4|\xi|^2})^{2i}}{1-4|\xi|^2} e^{-t(1-\sqrt{1-4|\xi|^2})} |\hat{g}(\xi)|^2 d\xi \\
& + C \int_{|\xi| \leq \delta} |\xi|^{2|\alpha|} \frac{(1+\sqrt{1-4|\xi|^2})^{2i}}{1-4|\xi|^2} e^{-t(1+\sqrt{1-4|\xi|^2})} |\hat{g}(\xi)|^2 d\xi \\
\leq & C \int_{|\xi| \leq \delta} |\xi|^{2|\alpha+4i} e^{-2|\xi|^2 t} |\hat{g}(\xi)|^2 d\xi \\
& + Ce^{-t} \int_{|\xi| \leq \delta} |\hat{g}(\xi)|^2 d\xi.
\end{aligned}$$

The second term is estimated by

$$e^{-t} \int_{|\xi| \leq \delta} |\hat{g}(\xi)|^2 d\xi = e^{-t} \int_{|\xi| \leq \delta} (1+|\xi|^2)(1+|\xi|^2)^{-1} |\hat{g}(\xi)|^2 d\xi \leq Ce^{-t} \|g\|_{H^{-1}}^2.$$

When $m = 2$, we apply (2.1.8) and have

$$\begin{aligned}
\int_{|\xi| \leq \delta} |\xi|^{2|\alpha+4i} e^{-2|\xi|^2 t} |\hat{g}(\xi)|^2 d\xi & \leq \sup_{|\xi| \leq \delta} \left(|\xi|^{2|\alpha|+4i} e^{-2|\xi|^2 t} \right) \|g\|_{L^2}^2 \\
& \leq C(1+t)^{-|\alpha|-2i} \|g\|_{L^2}^2.
\end{aligned}$$

When $m \neq 2$, putting $p = m'/2$ and using Lemma 9.6 again, we have

$$\begin{aligned}
\int_{|\xi| \leq \delta} |\xi|^{2|\alpha+4i} e^{-2|\xi|^2 t} |\hat{g}(\xi)|^2 d\xi & \leq \left(\int_{|\xi| \leq \delta} |\xi|^{(2|\alpha|+4i)p'} e^{-p'|\xi|^2 t} d\xi \right)^{1/p'} \|\hat{g}\|_{L^{m'}}^2 \\
& \leq C \left(\int_0^\delta r^{(2|\alpha|+4i)p'+n-1} e^{-p' r^2 t} dr \right)^{1/p'} \|g\|_{L^m}^2 \\
& \leq C(1+t)^{-n/(2p')-|\alpha|-2i} \|g\|_{L^m}^2 \\
& = C(1+t)^{n/2-n/m-2i-|\alpha|} \|g\|_{L^m}^2,
\end{aligned}$$

since

$$p' = \frac{p}{p-1} = \frac{m'}{m'-2} = \frac{m}{2-m}.$$

This completes the proof. □

2.1.2. Solution representation formula. In this section, we give a solution representation formula for the solution operator $S_n(t)$ of (2.1.1). Let $I_\nu(s)$ denote the modified Bessel function of order ν, that is to say,

$$I_\nu(s) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m+\nu+1)} \left(\frac{s}{2} \right)^{2m+\nu}$$

By using I_ν, we can express $S_n(t)$ as follows:

PROPOSITION 2.2. (i) *When $n = 1$, we have*

$$S_n(t)g(x) = \frac{e^{-t/2}}{2} \int_{|x-y| \leq t} I_0 \left(\frac{1}{2} \sqrt{t^2 - |x-y|^2} \right) g(y) dy.$$

(ii) *When $n \geq 3$ and is an odd number, we have*

$$S_n(t)g(x) = \frac{e^{-t/2}}{(n-2)!!|S^{n-1}|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-1)/2}\int_{|x-y|\leq t} I_0\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)g(y)dy.$$

(iii) *When $n \geq 2$ and is an even number, we have*

$$S_n(t)g(x) = \frac{2e^{-t/2}}{(n-1)!!|S^n|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\int_{|x-y|\leq t}\frac{\cosh(\frac{1}{2}\sqrt{t^2-|x-y|^2})}{\sqrt{t^2-|x-y|^2}}g(y)dy.$$

Here we use the notation $(n-1)!! = (n-1)(n-3)\cdots 1$ and $|S^n|$ denotes the measure of the n-dimensional unit sphere S^n, that is, $|S^n| = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$.

Proof. The argument is the same as in the survey article by Nishihara [**82**]. Consider the Cauchy problem of the wave equation

$$\begin{cases} w_{tt} - \Delta w = 0, & (t,x) \in (0,\infty)\times \mathbf{R}^n, \\ (w,w_t)(0,x) = (0,h)(x), & x \in \mathbf{R}^n. \end{cases}$$

It is well known that the solution $w(t,x)$ is given by

$$w(t,x) = \frac{1}{2}\int_{x-t}^{x+t} h(r)dr$$

for $n=1$,

$$w(t,x) = \frac{1}{(n-2)!!|S^{n-1}|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2}\left(\frac{1}{t}\int_{|x-y|=t} h(y)dS_y\right),$$

for $n \geq 3$ and odd,

$$w(t,x) = \frac{2}{(n-1)!!|S^n|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\left(\int_{|x-y|\leq t}\frac{1}{\sqrt{t^2-|x-y|^2}}h(y)dy\right)$$

for enen n. From these formulae, we obtain the representation of $S_n(t)g$ by the method of descent. We denote $\mathbf{x}_{n+1} = (x_1,\ldots,x_n,x_{n+1}) = (\mathbf{x}_n, x_{n+1})$ and consider the Cauchy problem of the $n+1$-dimensional wave equation

$$\begin{cases} w_{tt} - \Delta_{n+1} w = 0, & (t,\mathbf{x}_{n+1}) \in (0,\infty)\times \mathbf{R}^{n+1}, \\ (w,w_t)(0,\mathbf{x}_{n+1}) = (0,h)(\mathbf{x}_{n+1}), & \mathbf{x}_{n+1} \in \mathbf{R}^{n+1}. \end{cases} \tag{2.1.9}$$

Here Δ_{n+1} denotes the $n+1$-dimensional Laplacian. Let

$$w(t,\mathbf{x}_{n+1}) = e^{(x_{n+1}+t)/2}S_n(t)g(\mathbf{x}_n).$$

Then w is the solution to (2.1.9) with the initial data

$$h(\mathbf{x}_{n+1}) = e^{x_{n+1}/2}g(\mathbf{x}_n).$$

Thus, if we write the solution operator of (2.1.9) by $W_{n+1}(t)$, then we have the relation

$$W_{n+1}(t)\left[e^{x_{n+1}/2}g(\mathbf{x}_n)\right] = e^{(x_{n+1}+t)/2}S_n(t)g(\mathbf{x}_n).$$

Therefore, we have

$$S_n(t)g(\mathbf{x}_n) = e^{-(x_{n+1}+t)/2}W_{n+1}(t)\left[e^{x_{n+1}/2}g(\mathbf{x}_n)\right]. \tag{2.1.10}$$

Using this relation, we can calculate the formula for $S_n(t)$. By Lemma 9.2, we have

$$\int_{-a}^{a}\frac{e^{ct}}{\sqrt{a^2-t^2}}dt = \pi I_0(ca). \tag{2.1.11}$$

When $n = 1$, it holds that

$$W_2(t)h(\mathbf{x}_2) = \frac{1}{2\pi}\int_{|\mathbf{x}_2-\mathbf{y}_2|\le t}\frac{h(\mathbf{y}_2)}{\sqrt{t^2-|\mathbf{x}_2-\mathbf{y}_2|^2}}d\mathbf{y}_2.$$

Noting (2.1.10) and (2.1.11), we have

$$\begin{aligned}
S_n(t)g(x_1) &= \frac{e^{-t/2}}{2\pi}\int_{|\mathbf{x}_2-\mathbf{y}_2|\le t}\frac{e^{(y_2-x_2)/2}g(y_1)}{\sqrt{(t^2-|x_1-y_1|^2)-|x_2-y_2|^2}}d\mathbf{y}_2\\
&= \frac{e^{-t/2}}{2\pi}\int_{|x_1-y_1|\le t} g(y_1)\\
&\quad\times\left(\int_{x_2-\sqrt{t^2-|x_1-y_1|^2}}^{x_2+\sqrt{t^2-|x_1-y_1|^2}}\frac{e^{(y_2-x_2)/2}}{\sqrt{(t^2-|x_1-y_1|^2)-|x_2-y_2|^2}}dy_2\right)dy_1\\
&= \frac{e^{-t/2}}{2\pi}\int_{|x_1-y_1|\le t} g(y_1)\\
&\quad\times\left(\int_{-\sqrt{t^2-|x_1-y_1|^2}}^{\sqrt{t^2-|x_1-y_1|^2}}\frac{e^{y_2/2}}{\sqrt{(t^2-|x_1-y_1|^2)-y_2^2}}dy_2\right)dy_1\\
&= \frac{e^{-t/2}}{2}\int_{|x_1-y_1|\le t} I_0\left(\frac{1}{2}\sqrt{t^2-|x_1-y_1|^2}\right)g(y_1)dy_1.
\end{aligned}$$

Changing variables imply

$$S_n(t)g(x) = \frac{e^{-t/2}}{2}\int_{|x-y|\le t} I_0\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)g(y)dy$$

and we obtain the desired formula.

When $n \ge 3$ and is an odd number, we have

$$\begin{aligned}
W_{n+1}(t)h(\mathbf{x}_{n+1}) &= \frac{2}{n!!|S^{n+1}|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-1)/2}\\
&\quad\times\left(\int_{|\mathbf{x}_{n+1}-\mathbf{y}_{n+1}|\le t}\frac{h(\mathbf{y}_{n+1})}{\sqrt{t^2-|\mathbf{x}_{n+1}-\mathbf{y}_{n+1}|^2}}d\mathbf{y}_{n+1}\right).
\end{aligned}$$

By noting $|S^{n+1}| = \frac{2\pi}{n}|S^{n-1}|$ and using (2.1.10), one can see that

$$\begin{aligned}
&S_n(t)g(\mathbf{x}_n)\\
&= \frac{e^{-t/2}}{(n-2)!!|S^{n-1}|\pi}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-1)/2}\\
&\quad \times \int_{|\mathbf{x}_{n+1}-\mathbf{y}_{n+1}|\le t} \frac{e^{(y_{n+1}-x_{n+1})/2} g(\mathbf{y}_n)}{\sqrt{(t^2-|\mathbf{x}_n-\mathbf{y}_n|^2) - |x_{n+1}-y_{n+1}|^2}} d\mathbf{y}_{n+1}\\
&= \frac{e^{-t/2}}{(n-2)!!|S^{n-1}|\pi}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-1)/2} \int_{|\mathbf{x}_n-\mathbf{y}_n|\le t} g(\mathbf{y}_n)\\
&\quad \times \left(\int_{x_{n+1}-\sqrt{t^2-|\mathbf{x}_n-\mathbf{y}_n|^2}}^{x_{n+1}+\sqrt{t^2-|\mathbf{x}_n-\mathbf{y}_n|^2}} \frac{e^{(y_{n+1}-x_{n+1})/2}}{\sqrt{(t^2-|\mathbf{x}_n-\mathbf{y}_n|^2) - |x_{n+1}-y_{n+1}|^2}} dy_{n+1}\right) d\mathbf{y}_n\\
&= \frac{e^{-t/2}}{(n-2)!!|S^{n-1}|\pi}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-1)/2} \int_{|\mathbf{x}_n-\mathbf{y}_n|\le t} g(\mathbf{y}_n)\\
&\quad \times \left(\int_{-\sqrt{t^2-|\mathbf{x}_n-\mathbf{y}_n|^2}}^{\sqrt{t^2-|\mathbf{x}_n-\mathbf{y}_n|^2}} \frac{e^{y_{n+1}/2}}{\sqrt{(t^2-|\mathbf{x}_n-\mathbf{y}_n|^2) - y_{n+1}^2}} dy_{n+1}\right) d\mathbf{y}_n\\
&= \frac{e^{-t/2}}{(n-2)!!|S^{n-1}|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-1)/2}\\
&\quad \times \int_{|\mathbf{x}_n-\mathbf{y}_n|\le t} I_0\left(\frac{1}{2}\sqrt{t^2-|\mathbf{x}_n-\mathbf{y}_n|^2}\right) g(\mathbf{y}_n) d\mathbf{y}_n.
\end{aligned}$$

and hence,

$$S_n(t)g(x) = \frac{e^{-t/2}}{(n-2)!!|S^{n-1}|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-1)/2} \int_{|x-y|\le t} I_0\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right) g(y)dy.$$

Finally, when n is an even number, we have

$$\begin{aligned}
&W_{n+1}(t)h(\mathbf{x}_{n+1})\\
&= \frac{1}{(n-1)!!|S^n|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\left(\frac{1}{t}\int_{|\mathbf{x}_{n+1}-\mathbf{y}_{n+1}|=t} h(\mathbf{y}_{n+1}) dS_{\mathbf{y}_{n+1}}\right).
\end{aligned}$$

We divide the integral into two parts:

$$\begin{aligned}
&\frac{1}{t}\int_{|\mathbf{x}_{n+1}-\mathbf{y}_{n+1}|=t} h(\mathbf{y}_{n+1}) dS_{\mathbf{y}_{n+1}}\\
&\quad = \frac{1}{t}\left(\int_{\substack{|\mathbf{x}_{n+1}-\mathbf{y}_{n+1}|=t,\\ y_{n+1}-x_{n+1}\ge 0}} + \int_{\substack{|\mathbf{x}_{n+1}-\mathbf{y}_{n+1}|=t,\\ y_{n+1}-x_{n+1}<0}}\right) h(\mathbf{y}_{n+1}) dS_{\mathbf{y}_{n+1}}.
\end{aligned}$$

Since the surface of the integral region is given by the relation $\phi(\mathbf{y}_{n+1}) = 0$ with the function

$$\phi(\mathbf{y}_{n+1}) = t^2 - |\mathbf{x}_{n+1}-\mathbf{y}_{n+1}|^2,$$

it follows that

$$dS_{\mathbf{y}_{n+1}} = \frac{|\nabla\phi(\mathbf{y}_{n+1})|}{|\partial_{y_{n+1}}\phi(\mathbf{y}_{n+1})|} d\mathbf{y}_n = \frac{t}{\sqrt{t^2-|\mathbf{x}_n-\mathbf{y}_n|^2}} d\mathbf{y}_n.$$

Moreover, we have $e^{(y_{n+1}-x_{n+1})/2} = e^{\frac{1}{2}\sqrt{t^2-|\mathbf{x}_n-\mathbf{y}_n|^2}}$ for $y_{n+1}-x_{n+1} \geq 0$ and $e^{(y_{n+1}-x_{n+1})/2} = e^{-\frac{1}{2}\sqrt{t^2-|\mathbf{x}_n-\mathbf{y}_n|^2}}$ for $y_{n+1}-x_{n+1} < 0$. Therefore, taking into account (2.1.10), substituting $h(\mathbf{y}_{n+1}) = e^{y_{n+1}/2}g(\mathbf{y}_n)$, we can calculate

$$\begin{aligned}
&e^{-x_{n+1}/2}\frac{1}{t}\int_{|\mathbf{x}_{n+1}-\mathbf{y}_{n+1}|=t} h(\mathbf{y}_{n+1})dS_{\mathbf{y}_{n+1}} \\
&= \int_{|\mathbf{x}_n-\mathbf{y}_n|\leq t}\left(e^{\frac{1}{2}\sqrt{t^2-|\mathbf{x}_n-\mathbf{y}_n|^2}} + e^{-\frac{1}{2}\sqrt{t^2-|\mathbf{x}_n-\mathbf{y}_n|^2}}\right)\frac{g(\mathbf{y}_n)}{\sqrt{t^2-|\mathbf{x}_n-\mathbf{y}_n|^2}}d\mathbf{y}_n \\
&= \int_{|\mathbf{x}_n-\mathbf{y}_n|\leq t} 2\cosh\left(\frac{1}{2}\sqrt{t^2-|\mathbf{x}_n-\mathbf{y}_n|^2}\right)\frac{g(\mathbf{y}_n)}{\sqrt{t^2-|\mathbf{x}_n-\mathbf{y}_n|^2}}d\mathbf{y}_n
\end{aligned}$$

and hence,

$$S_n(t)g(x) = \frac{2e^{-t/2}}{(n-1)!!|S^n|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\int_{|x-y|\leq t}\frac{\cosh\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)}{\sqrt{t^2-|x-y|^2}}g(y)dy.$$

□

2.2. Diffusion phenomenon

2.2.1. L^∞-estimate. In this subsection, we prove the result of Yang and Milani [**121**]. By using Matsumura's estimates (2.1.3), they proved that the solution u of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + u_t = 0, & (t,x) \in (0,\infty)\times\mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n \end{cases} \tag{2.2.1}$$

is asymptotically equivalent with the solution v of the Cauchy problem

$$\begin{cases} v_t - \Delta v = 0, & (t,x) \in (0,\infty)\times\mathbf{R}^n, \\ v(0,x) = u_0(x) + u_1(x), & x \in \mathbf{R}^n \end{cases} \tag{2.2.2}$$

in L^∞-sense, namely,

Theorem 2.3 (Yang and Milani [**121**]). *If $u_0 \in H^{[n/2]+3}(\mathbf{R}^n)\cap L^1(\mathbf{R}^n)$, $u_1 \in H^{[n/2]+2}(\mathbf{R}^n)\cap L^1(\mathbf{R}^n)$ and u, v are the classical solutions of* (2.2.1), (2.2.2), *respectively, then we have*

$$\|u(t)-v(t)\|_{L^\infty} = O(t^{-n/2-1}) \tag{2.2.3}$$

as $t \to +\infty$.

Remark 2.1. *In general, solutions of the heat equation* (2.2.2) *do not decay faster than $O(t^{-n/2})$. Indeed, if we take a initial data $f(x)$ satisfying $f \in C_0^\infty(\mathbf{R}^n)$, $f \geq 0, f \neq 0$, then it follows that for $t \geq 1$*

$$\begin{aligned}
\|v(t)\|_{L^\infty} &= \sup_x \frac{1}{(4\pi t)^{n/2}}\int_{\mathbf{R}^n} e^{-\frac{|x-y|^2}{4t}}f(y)dy \\
&\geq \sup_x \frac{1}{(4\pi t)^{n/2}}\int_{\mathbf{R}^n} e^{-\frac{|x-y|^2}{4}}f(y)dy \\
&\geq \frac{1}{(4\pi t)^{n/2}}\int_{\mathbf{R}^n} e^{-\frac{|y|^2}{4}}f(y)dy \\
&= Ct^{-n/2}.
\end{aligned}$$

Proof of Theorem 2.3. From Matsumura's estimates (2.1.3), we have

$$\|u_t(t)\|_{L^2} = O(t^{-n/4-1}), \tag{2.2.4}$$

$$\|u_{tt}(t)\|_{L^\infty} = O(t^{-n/2-2}), \tag{2.2.5}$$

$$\|\hat{u}(t)\|_{L^\infty} = O(1), \tag{2.2.6}$$

$$\|u(t)\|_{L^2} = O(t^{-n/4}). \tag{2.2.7}$$

The third inequality immediately follows by (2.1.2), (2.1.5) and

$$\|\hat{u}(t)\|_{L^\infty} \leq C(\|\hat{u}_0\|_{L^\infty} + \|\hat{u}_1\|_{L^\infty}) \leq C(\|u_0\|_{L^1} + \|u_1\|_{L^1}).$$

We denote $G(t,x) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$. It is also well known that

$$\|G(t)\|_{L^1} = O(1), \tag{2.2.8}$$

$$\|\partial_t^k G(t)\|_{L^2} = O(t^{-n/4-k}), \tag{2.2.9}$$

$$\|\partial_t^k G(t)\|_{L^\infty} = O(t^{-n/2-k}) \tag{2.2.10}$$

(see [**14**]). By the Duhamel principle, we can write

$$u(t) - v(t) = -G(t) * u_1 - \int_0^t G(t-\tau) * u_{tt}(\tau) d\tau.$$

By the integration by prats, we obtain

$$\begin{aligned}\int_0^t G(t-\tau) * u_{tt}(\tau) d\tau = &\int_{t/2}^t G(t-\tau) * u_{tt}(\tau) d\tau \\ &+ G(t/2) * u_t(t/2) - G(t) * u_1 \\ &+ \int_0^{t/2} \partial_t G(t-\tau) * u_t(\tau) d\tau\end{aligned}$$

and hence,

$$\begin{aligned}u(t) - v(t) = &-\int_{t/2}^t G(t-\tau) * u_{tt}(\tau) d\tau \\ &- G(t/2) * u_t(t/2) \\ &- \int_0^{t/2} \partial_t G(t-\tau) * u_t(\tau) d\tau \\ \equiv\ & I_1 + I_2 + I_3.\end{aligned}$$

It follows that

$$|I_1| \leq \int_{t/2}^t \|G(t-\tau)\|_{L^1} \|u_{tt}(\tau)\|_{L^\infty} d\tau \leq C \int_{t/2}^t (1+\tau)^{-n/2-2} d\tau \leq (1+t)^{-n/2-1}$$

and

$$|I_2| \leq t^{-n/4}(1+t)^{-n/4-1} = O(t^{-n/2-1}).$$

To estimate I_3, we use the integration by parts once more and have

$$\begin{aligned}I_3 &= -\partial_t G(t/2) * u(t/2) + \partial_t G(t) * u_0 - \int_0^{t/2} \partial_t^2 G(t-\tau) * u(\tau) d\tau \\ &\equiv I_{31} + I_{32} + I_{33}.\end{aligned}$$

By (2.2.7) and (2.2.9), we immediately obtain

$$|I_{31}| \le \|\partial_t G(t/2)\|_{L^2}\|u(t/2)\|_{L^2} = O(t^{-n/2-1}).$$

From (2.2.10), it is also easy to see that

$$|I_{32}| \le \|\partial_t G(t)\|_{L^\infty}\|u_0\|_{L^1} = O(t^{-n/2-1}).$$

Therefore, it suffices to prove that

$$|I_{33}| = O(t^{-n/2-1}). \tag{2.2.11}$$

Noting

$$G(t) * G(s) = G(t+s),$$

which is proved by using $\hat{G}(t,\xi) = e^{-t|\xi|^2}$, we have

$$G(t-\tau) = G(t/4) * G(3t/4-\tau).$$

By the Leibniz rule, it follows that

$$\begin{aligned}\partial_t^2 G(t-\tau) = a_0\partial_t^2 G(t/4) * G(3t/4-\tau)\\ + a_1\partial_t G(t/4) * \partial_t G(3t/4-\tau)\\ + a_2 G(t/4) * \partial_t^2 G(3t/4-\tau),\end{aligned}$$

where $a_0 = 1/16, a_1 = 3/8, a_2 = 9/16$. Thus, we can rewrite I_{33} as

$$I_{33} = -\sum_{i=0}^{2}\int_0^{t/2} a_i\partial_t^{2-i}G(t/4) * \partial_t^i G(3t/4-\tau) * u(\tau)d\tau \equiv \sum_{i=0}^{2} J_i.$$

We can see that

$$|J_i| = O(t^{-n/2-1})$$

for $i = 0, 1, 2$. Indeed, we first have

$$\begin{aligned}|J_0| &\le \int_0^{t/2}\|\partial_t^2 G(t/4)\|_{L^2}\|G(3t/4-\tau) * u\|_{L^2}d\tau\\ &\le Ct^{-n/4-2}\int_0^{t/2}\|\hat{G}(3t/4-\tau)\hat{u}(\tau)\|_{L^2}d\tau\\ &\le Ct^{-n/4-2}\int_0^{t/2}\|\hat{G}(3t/4-\tau)\|_{L^2}\|\hat{u}(\tau)\|_{L^\infty}d\tau\\ &\le Ct^{-n/2-1}.\end{aligned}$$

Here we have used (2.2.6) and the fact

$$\|\hat{G}(3t/4-\tau)\|_{L^2}^2 \le C\int_{\mathbf{R}^n} e^{-2(3t/4-\tau)|\xi|^2}d\xi \le C\int_{\mathbf{R}^n} e^{-t|\xi|^2/2}d\xi = O(t^{-n/2}).$$

Next, we estimate J_1 as

$$
\begin{aligned}
|J_1| &\leq C\int_0^{t/2} \|\partial_t G(t/4)\|_{L^2}\|\partial_t G(3t/4-\tau)*u(\tau)\|_{L^2}d\tau \\
&\leq Ct^{-n/4-1}\int_0^{t/2} \|\partial_t \hat{G}(3t/4-\tau)\hat{u}(\tau)\|_{L^2}d\tau \\
&\leq Ct^{-n/4-1}\int_0^{t/2} \|\partial_t \hat{G}(3t/4-\tau)\|_{L^2}\|\hat{u}(\tau)\|_{L^\infty}d\tau \\
&\leq Ct^{-n/4-1}\int_0^{t/2} \||\cdot|^2\hat{G}(3t/4-\tau,\cdot)\|_{L^2}d\tau \\
&\leq Ct^{-n/2-1}.
\end{aligned}
$$

Here we have used (2.2.6) and

$$
\begin{aligned}
\int_{\mathbf{R}^n}|\xi|^4e^{-2|\xi|^2(3t/4-\tau)}d\xi &\leq \int_{\mathbf{R}^n}|\xi|^4e^{-|\xi|^2t/2}d\xi \\
&\leq t^{-n/2-2}\int_{\mathbf{R}^n}|\eta|^4e^{-|\eta|^2/2}d\eta = O(t^{-n/2-2}).
\end{aligned}
$$

In the same way, by using the fact

$$
\int_{\mathbf{R}^n}|\xi|^8e^{-2|\xi|^2(3t/4-\tau)}d\xi = O(t^{-n/2-4}),
$$

we can see that

$$
\begin{aligned}
|J_2| &\leq C\int_0^{t/2} \|G(t/4)\|_{L^2}\|\partial_t^2 \hat{G}(3t/4-\tau)\hat{u}(\tau)\|_{L^2}d\tau \\
&\leq Ct^{-n/4}\int_0^{t/2} \||\cdot|^4\hat{G}(3t/4-\tau)\|_{L^2}\|\hat{u}\|_{L^\infty}d\tau \\
&\leq Ct^{-n/2-1},
\end{aligned}
$$

which completes the proof. □

2.2.2. L^p-L^q estimates. In this subsection, we prove another result which also represents the diffusion phenomenon in the L^p-L^q sense. For the sake of simplicity, we consider the linear damped wave equation in three space dimension

$$
\left\{
\begin{array}{ll}
u_{tt}-\Delta u+u_t=0, & (t,x)\in(0,\infty)\times\mathbf{R}^3, \\
(u,u_t)(0,x)=(u_0,u_1)(x), & x\in\mathbf{R}^3.
\end{array}
\right. \tag{2.2.12}
$$

We also consider the corresponding heat equation with the initial data u_0+u_1:

$$
\left\{
\begin{array}{ll}
v_t-\Delta v=0, & (t,x)\in(0,\infty)\times\mathbf{R}^3, \\
v(0,x)=u_0(x)+u_1(x), & x\in\mathbf{R}^3.
\end{array}
\right. \tag{2.2.13}
$$

We put

$$
\widetilde{\mathbf{W}}_3(t;u_0,u_1)=\left(\frac{1}{2}+\frac{t}{8}\right)W_3(t)u_0+\partial_t(W_3(t)u_0)+W_3(t)u_1,
$$

where $W_3(t)$ denotes the solution operator of the three dimensional free wave equation, that is,

$$
W_3(t)g(x)=\frac{1}{4\pi t}\int_{|x-y|=t}g(y)dS_y.
$$

THEOREM 2.4 (Nishihara [**78**]). *Let $u_0, u_1 \in L^q(\mathbf{R}^3)$ with $q \geq 1$ and let u and v be the solution of* (2.2.12) *and* (2.2.13)*, respectively. Then the following L^p-L^q estimate holds:*

$$\left\| u(t) - v(t) - e^{-t/2}\widetilde{\mathbf{W}}(t; u_0, u_1) \right\|_{L^p} \leq Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}(\|u_0\|_{L^q} + \|u_1\|_{L^q}) \tag{2.2.14}$$

for $t > 0$.

To prove this theorem, we decompose the solution u into two parts. The first one is $e^{-t/2}\widetilde{\mathbf{W}}(t; u_0, u_1)$ and the other one behaves like $v(t, x)$. By Proposition 2.2 and the Duhamel principle, the solution of (2.2.12) is given by

$$u(t, x) = S_3(t)(u_0 + u_1) + \partial_t(S_3(t)u_0) \tag{2.2.15}$$

with

$$S_3(t)g = \frac{e^{-t/2}}{4\pi t}\partial_t \int_{|x-y|\leq t} I_0\left(\frac{1}{2}\sqrt{t^2 - |x-y|^2}\right) g(y)dy.$$

Using $I(0) = 1, I_0'(s) = I_1(s)$, we obtain

$$\begin{aligned} S_3(t)g = {} & \frac{e^{-t/2}}{4\pi t}\int_{|x-y|=t} g(y)dS_y \\ & + \frac{e^{-t/2}}{8\pi}\int_{|x-y|\leq t} \frac{I_1\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)}{\sqrt{t^2-|x-y|^2}} g(y)dy \\ =: {} & e^{-t/2}W_3(t)g + J_3(t)g. \end{aligned} \tag{2.2.16}$$

By noting that $I_1(s)/s\ |_{s=0} = 1/2$, it follows that

$$\begin{aligned} \partial_t(S_3(t)g) = {} & e^{-t/2}\left(\left(-\frac{1}{2} + \frac{t}{8}\right)W_3(t)g + \partial_t(W_3(t)g)\right) \\ & + \frac{1}{8\pi}\int_{|x-y|\leq t} \partial_t\left(e^{-t/2}\frac{I_1\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)}{\sqrt{t^2-|x-y|^2}}\right) g(y)dS_y \\ =: {} & e^{-t/2}\left(\left(-\frac{1}{2} + \frac{t}{8}\right)W_3(t)g + \partial_t(W_3(t)g)\right) + \tilde{J}_3(t)g. \end{aligned} \tag{2.2.17}$$

Substituting (2.2.16) and (2.2.17) into (2.2.15), we can see that

$$u(t, x) = e^{-t/2}\widetilde{\mathbf{W}}_3(t; u_0, u_1) + J_3(t)(u_0 + u_1) + \tilde{J}_3(t)u_0. \tag{2.2.18}$$

Hence

$$u - v - e^{-t/2}\widetilde{\mathbf{W}}_3(t; u_0, u_1) = (J_3(t) - e^{t\Delta})(u_0 + u_1) + \tilde{J}_3(t)u_0,$$

where $e^{t\Delta}$ denotes the solution operator of the heat equation (2.2.13), that is, $v(t, x) = e^{t\Delta}(u_0 + u_1)$. Thus, it suffices to prove that there exists a constant $C > 0$ such that

$$\|(J_3(t) - e^{t\Delta})g\|_{L^p} \leq Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}\|g\|_{L^q}, \tag{2.2.19}$$

$$\|\tilde{J}_3(t)g\|_{L^p} \leq Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}\|g\|_{L^q} \tag{2.2.20}$$

for $t > 0$. More generally, we shall prove the following estimates:

LEMMA 2.5. *For any $1 \le q \le p \le \infty$, there exists some constant $C > 0$ such that*

$$\|J_3(t)g\|_{L^p} \le C(1+t)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|g\|_{L^q} \tag{2.2.21}$$

$$\|\tilde{J}_3(t)g\|_{L^p} \le C(1+t)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}\|g\|_{L^q} \tag{2.2.22}$$

for $t \ge 0$ and

$$\|(J_3(t) - e^{t\Delta})g\|_{L^p} \le Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}\|g\|_{L^q} \tag{2.2.23}$$

for $t > 0$.

PROOF. We first prove (2.2.21) and (2.2.23). The proof is divided into the case $t \ge 1$ and $0 \le t < 1$. We first assume that $t \ge 1$. Let $\varepsilon \in (0, 1/2)$. We note that the assumption $t \ge 1$ yields $t^{(1+\varepsilon)/2} \le t$. We divide the integral into three parts:

$$\begin{aligned}
&(J_3(t) - e^{t\Delta})g \\
&= \int_{|x-y|\le t^{1/2+\varepsilon}} \left(\frac{e^{-t/2} I_1\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)}{8\pi\sqrt{t^2-|x-y|^2}} - \frac{e^{-|x-y|^2/(4t)}}{(4\pi t)^{3/2}}) \right) g(y)dy \\
&\quad + \int_{t^{1/2+\varepsilon}\le|x-y|\le t} \left(\frac{e^{-t/2} I_1\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)}{8\pi\sqrt{t^2-|x-y|^2}} - \frac{e^{-|x-y|^2/(4t)}}{(4\pi t)^{3/2}} \right) g(y)dy \\
&\quad - \int_{t\le|x-y|} \frac{e^{-|x-y|^2/(4t)}}{(4\pi t)^{3/2}} g(y)dy \\
&=: X_1 + X_2 + X_3.
\end{aligned}$$

By the Hausdorff-Young inequality (see Lemma 9.9), it is easy to see that

$$\|X_3\|_{L^p} \le Ct^{-3/2}\|e^{-|x|^2/(4t)}\|_{L^r(\{|x|\ge t\})}\|g\|_{L^q} \le Ce^{-c_1 t}\|g\|_{L^q}$$

with some $c_1 \in (0, 1/4)$, where $1/q - 1/p = 1 - 1/r$. In the same way, we deduce that

$$\begin{aligned}
\|X_2\|_{L^p} \le C &\left(\int_{t^{(1+\varepsilon)/2}\le|y|\le t} \left(\frac{e^{-t/2} I_1\left(\frac{1}{2}\sqrt{t^2-|y|^2}\right)}{\sqrt{t^2-|y|^2}} \right)^r dy \right)^{1/r} \|g\|_{L^q} \\
&+ Ct^{-3/2}\|e^{-|x|^2/(4t)}\|_{L^r(|x|\ge t^{(1+\varepsilon)/2})}\|g\|_{L^q}.
\end{aligned}$$

We further divide the integral region of the first term into $\{y \in \mathbf{R}^3 \mid \sqrt{t^2-1} \le |y| \le t\}$ and $\{y \in \mathbf{R}^3 \mid t^{(1+\varepsilon)/2} \le |y| \le \sqrt{t^2-1}\}$. When $\sqrt{t^2-1} \le |y| \le t$, noting that $I_1(s)/s$ is bounded for $s \le 1/2$, we can obtain

$$\frac{e^{-t/2} I_1\left(\frac{1}{2}\sqrt{t^2-|y|^2}\right)}{\sqrt{t^2-|y|^2}} \le Ce^{-t/2}.$$

Next, we note that if $|y| \le \sqrt{t^2-1}$, then $\sqrt{t^2-|y|^2} \ge 1$ holds. By using the monotonicity of I_ν and the asymptotic expansion

$$I_\nu(s) = \frac{1}{\sqrt{2\pi s}} e^s (1 + O(s^{-1})) \quad (s \to +\infty),$$

we can estimate

$$\frac{e^{-t/2}I_1\left(\frac{1}{2}\sqrt{t^2-|y|^2}\right)}{\sqrt{t^2-|y|^2}} \le Ce^{-t/2+\sqrt{t^2-t^{1+\varepsilon}}}$$

if $t^{(1+\varepsilon)/2} \le |y| \le \sqrt{t^2-1}$. Since

$$-\frac{t}{2}+\frac{1}{2}\sqrt{t^2-|y|^2} = -\frac{|y|^2}{2(t+\sqrt{t^2-|y|^2})} \le -\frac{t^{1+\varepsilon}}{2(t+\sqrt{t^2-t^{1+\varepsilon}})} \le -\frac{t^\varepsilon}{4},$$

we see that

$$\frac{e^{-t/2}I_1\left(\frac{1}{2}\sqrt{t^2-|y|^2}\right)}{\sqrt{t^2-|y|^2}} \le Ce^{-t^\varepsilon/4}$$

for $t^{(1+\varepsilon)/2} \le |y| \le \sqrt{t^2-1}$. Consequently, we have

$$\|X_2\|_{L^p} \le Ce^{-c_2 t^\varepsilon}\|g\|_{L^q}$$

with some $c_2 \in (0,1/4)$. Thus, it suffices to estimate X_1. Let $\rho = |x-y|$ and we rewrite X_1 as

$$X_1 = \frac{1}{(4\pi t)^{3/2}}\int_{\rho\le t^{(1+\varepsilon)/2}} e^{-\rho^2/(4t)}D(t,\rho)g(y)dy,$$

where

$$D(t,\rho) = \sqrt{\pi}e^{\rho^2/(4t)-t/2}t^{3/2}\frac{1}{\sqrt{t^2-\rho^2}}I_1\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)-1.$$

Since

$$I_1\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right) = \frac{1}{\sqrt{\pi}(t^2-\rho^2)^{1/4}}e^{\sqrt{t^2-\rho^2}/2}\left(1+O\left(\frac{1}{\sqrt{t^2-\rho^2}}\right)\right),$$

we obtain

$$D(t,\rho) = e^{\rho^2/(4t)-t/2+\sqrt{t^2-\rho^2}/2}\left(\frac{t}{\sqrt{t^2-\rho^2}}\right)^{3/2}\left(1+O\left(\frac{1}{\sqrt{t^2-\rho^2}}\right)\right)-1.$$

Here we note that

$$\begin{aligned}\frac{t}{\sqrt{t^2-\rho^2}} &= \frac{1}{\sqrt{1-(\rho/t)^2}}\\ &= 1+O\left(\frac{\rho^2}{t^2}\right)\\ &= 1+\frac{1}{t}O\left(\frac{\rho^2}{t}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\rho^2}{4t}-\frac{t}{2}+\frac{1}{2}\sqrt{t^2-\rho^2} &= \frac{\rho^2}{4t}-\frac{\rho^2}{2(t+\sqrt{t^2-\rho^2})}\\ &= -\frac{\rho^2(t-\sqrt{t^2-\rho^2})}{4t(t+\sqrt{t^2-\rho^2})}\\ &= -\frac{\rho^4}{4t(t+\sqrt{t^2-\rho^2})^2}\end{aligned}$$

and hence, by the mean value theorem,

$$e^{\rho^2/(4t)-t/2+\sqrt{t^2-\rho^2}/2} = 1 + \frac{1}{t} O\left(\frac{\rho^4}{t^2}\right).$$

Therefore, we can obtain

$$\begin{aligned} D(t,\rho) &= \left(1 + \frac{1}{t} O\left(\frac{\rho^4}{t^2}\right)\right)\left(1 + \frac{1}{t} O\left(\frac{\rho^2}{t}\right)\right)^{3/2}\left(1 + O\left(\frac{1}{t}\right)\right) - 1 \\ &\leq C\frac{1}{t}\left(1 + \frac{\rho^2}{t} + \cdots + \left(\frac{\rho^2}{t}\right)^N\right) \end{aligned}$$

for some $C > 0$ and $N \in \mathbf{N}$. Using this inequality and Lemma 9.9, we can deduce that

$$\begin{aligned} \|X_1\|_{L^p} &\leq Ct^{-3/2}\left(\int_{|y|\leq t^{(1+\varepsilon)/2}} e^{-r|y|^2/(4t)}\frac{1}{t^r}\left(1 + \frac{|y|^2}{t} + \cdots + \left(\frac{|y|^2}{t}\right)^N\right)^r dy\right)^{1/r} \\ &\quad \times \|g\|_{L^q} \\ &\leq Ct^{-3/2-1+3/(2r)}\left(\int_{\mathbf{R}^n} e^{-r|z|^2/4}\left(1 + |z|^2 + \cdots + |z|^{2N}\right)^r dz\right)^{1/r}\|g\|_{L^q} \\ &\leq Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}\|g\|_{L^q}, \end{aligned}$$

which implies (2.2.23) for $t \geq 1$. Moreover, by using the well-known fact

$$\|e^{t\Delta}g\|_{L^p} \leq Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|g\|_{L^q} \quad (t > 0)$$

(see [**14**]), we can immediately obtain

$$\begin{aligned} \|J_3(t)g\|_{L^p} &\leq \|J_3(t)g - e^{t\Delta}g\|_{L^p} + \|e^{t\Delta}g\|_{L^p} \\ &\leq Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|g\|_{L^q}, \end{aligned}$$

which proves (2.2.21) for $t \geq 1$. Next, we treat the case $0 \leq t < 1$. Noting that $I_1(s)/s$ is bounded for $\leq s \leq 1/2$, we have

$$\frac{e^{-t/2}I_1\left(\frac{1}{2}\sqrt{t^2-|y|^2}\right)}{8\pi\sqrt{t^2-|y|^2}} \leq C$$

for $0 \leq t < 1$ and $|y| \leq t$ and the Hausdorff-Young inequality (see Lemma 9.9), we obtain

$$\begin{aligned} \|J_3(t)g\|_{L^p} &\leq C\left(\int_{|y|<t}\left(\frac{e^{-t/2}I_1\left(\frac{1}{2}\sqrt{t^2-|y|^2}\right)}{8\pi\sqrt{t^2-|y|^2}}\right)^r dy\right)^{1/r}\|g\|_{L^q} \\ &\leq Ct^{3/r}\|g\|_{L^q} \end{aligned}$$

for $0 \leq t < 1$, where r is determined by $1/q - 1/p = 1 - 1/r$. This implies (2.2.21) for $0 \leq t < 1$. Moreover, we can see that

$$\begin{aligned} \|J_3(t)g - e^{t\Delta}g\|_{L^p} &\leq \|J_3(t)g\|_{L^p} + \|e^{t\Delta}g\|_{L^p} \\ &\leq Ct^{3/r}\|g\|_{L^q} + Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|g\|_{L^q} \\ &\leq Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|g\|_{L^q}, \end{aligned}$$

which yields (2.2.23) for $0 < t < 1$.

Next, we prove (2.2.22). As before, we first assume that $t \geq 1$ and put $\rho = |x-y|$. By noting

$$I_1'(s) = I_0(s) - \frac{1}{s} I_1(s),$$

which is proved in Lemma 9.3, we can see that

$$\begin{aligned}
\tilde{J}_3(t)g = & \frac{1}{16\pi t^{3/2}} \int_{t^{(1+\varepsilon)/2} \leq \rho \leq t} e^{-t/2} \left\{ I_0\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right) \frac{t^{5/2}}{t^2-\rho^2} \right. \\
& \left. -I_1\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right) \left(\frac{t^{3/2}}{\sqrt{t^2-\rho^2}} + \frac{4t^{5/2}}{(t^2-\rho^2)^{3/2}} \right) \right\} g(y)dy \\
& + \frac{1}{16\pi t^{3/2}} \int_{\rho \leq t^{(1+\varepsilon)/2}} e^{-t/2} \left\{ I_0\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right) \frac{t^{5/2}}{t^2-\rho^2} \right. \\
& \left. -I_1\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right) \left(\frac{t^{3/2}}{\sqrt{t^2-\rho^2}} + \frac{4t^{5/2}}{(t^2-\rho^2)^{3/2}} \right) \right\} g(y)dy \\
\equiv & X_4 + X_5.
\end{aligned}$$

In the same way as the estimating X_2, one can prove that

$$\|X_4\|_{L^p} \leq Ce^{-c_3 t^\varepsilon} \|g\|_{L^q}$$

with some $c_3 \in (0, 1/4)$. In order to estimate X_5, we rewrite as

$$X_5 = \frac{1}{16\pi t^{3/2}} \int_{\rho \leq t^{(1+\varepsilon)/2}} e^{-\rho^2/(4t)} D_1(t,\rho) g(y)dy$$

with

$$\begin{aligned}
D_1(t,\rho) = & e^{\rho^2/(4t)-t/2} \left\{ I_0\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right) \frac{t^{5/2}}{t^2-\rho^2} \right. \\
& \left. -I_1\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right) \left(\frac{t^{3/2}}{\sqrt{t^2-\rho^2}} + \frac{4t^{5/2}}{(t^2-\rho^2)^{3/2}} \right) \right\}.
\end{aligned}$$

Since,

$$\begin{aligned}
D_1(t,\rho) = & \frac{1}{\sqrt{\pi}} e^{\rho^2/(4t)-t/2+\sqrt{t^2-\rho^2}/2} \left\{ \left(\frac{t}{\sqrt{t^2-\rho^2}} \right)^{5/2} - \left(\frac{t}{\sqrt{t^2-\rho^2}} \right)^{3/2} \right. \\
& \left. - \frac{t^{5/2}}{(\sqrt{t^2-\rho^2})^{7/2}} \right\} \left(1 + O\left(\frac{1}{t}\right) \right) \\
= & \frac{1}{\sqrt{\pi}} \left(1 + \frac{1}{t} O\left(\frac{\rho^2}{t}\right) \right) \left\{ \left(1 + \frac{1}{t} O\left(\frac{\rho^2}{t}\right) \right)^{5/2} - \left(1 + \frac{1}{t} O\left(\frac{\rho^2}{t}\right) \right)^{3/2} \right. \\
& \left. + \frac{1}{t} \left(1 + \frac{1}{t} O\left(\frac{\rho^2}{t}\right) \right)^{5/2} \right\} \left(1 + O\left(\frac{1}{t}\right) \right) \\
\leq & \frac{C}{t} \left(1 + \frac{\rho^2}{t} + \cdots + \left(\frac{\rho^2}{t}\right)^N \right)
\end{aligned}$$

for some $C>0$ and $N\in\mathbf{N}$, we can deduce that

$$\begin{aligned}\|X_5\|_{L^p} &\le Ct^{-3/2-1}\left(\int_{|y|\le t^{(1+\varepsilon)/2}} e^{-r|y|^2/(4t)}\left(1+\frac{|y|^2}{t}+\cdots+\left(\frac{|y|^2}{t}\right)^N\right)^r dy\right)^{1/r}\\ &\quad\times\|g\|_{L^q}\\ &\le Ct^{-3/2-1+3/2r}\left(\int_{\mathbf{R}^n} e^{-r|z|^2/4}\left(1+|z|^2+\cdots+|z|^{2N}\right)^r dz\right)^{1/r}\|g\|_{L^q}\\ &\le Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}\|g\|_{L^q},\end{aligned}$$

which implies (2.2.22) for $t\ge 1$. Finally, we prove (2.2.22) for $0\le t<1$. We put

$$\begin{aligned}\tilde{J}_3(t)g &= \int_{|x-y|\le t} k(t,x-y)g(y)dy\\ &:= \frac{e^{-t/2}}{16\pi}\int_{\rho\le t}\left\{I_0\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)\frac{t}{t^2-\rho^2}\right.\\ &\qquad\left.-I_1\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)\left(\frac{1}{\sqrt{t^2-\rho^2}}+\frac{4t}{(t^2-\rho^2)^{3/2}}\right)\right\}g(y)dy,\end{aligned}$$

where $\rho=|x-y|$. By the definition of the modified Bessel functions

$$I_0(s)=\sum_{m=0}^{\infty}\frac{1}{(m!)^2}\left(\frac{s}{2}\right)^{2m},\quad I_1(s)=\sum_{m=0}^{\infty}\frac{1}{m!(m+1)!}\left(\frac{s}{2}\right)^{2m+1},$$

we can deduce that

$$I_0\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)\frac{t}{t^2-\rho^2}=\frac{t}{t^2-\rho^2}+O(t)$$

and

$$I_1\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)\frac{4t}{(t^2-\rho^2)^{3/2}}=\frac{t}{t^2-\rho^2}+O(t)$$

as $t\to 0$. Noting this, we can see that $k(t,y)$ is bounded for $0\le t<1$ and $|y|\le t$. Therefore, by Lemma 9.9, we have

$$\begin{aligned}\|\tilde{J}_3(t)g\|_{L^p} &\le C\left(\int_{|y|\le t} k(t,y)^r dy\right)^{1/r}\|g\|_{L^q}\\ &\le Ct^{3/r}\|g\|_{L^q},\end{aligned}$$

which implies (2.2.22) for $0\le t<1$. □

2.3. Semilinear damped wave equations with a source nonlinearity

In this section, we shall give the critical exponent for the Cauchy problem of the semilinear wave equation

$$\begin{cases} u_{tt}-\Delta u+u_t=|u|^p, & (t,x)\in(0,\infty)\times\mathbf{R}^n,\\ (u,u_t)(0,x)=(u_0,u_1)(x), & x\in\mathbf{R}^n.\end{cases}\tag{2.3.1}$$

We prove that the critical exponent for (2.3.1) is given by $1+2/n$. Namely, if $p>1+2/n$, then there exists a unique global solution and if $1<p\le 1+2/n$, then the local solution with suitable data blows up in finite time. We give a two types of proof for global existence result. The first one is due to Ikehata and Tanizawa

[**35**]. This proof is based on Matsumura's estimate 2.1.3 and a weighted energy estimate developed by Todorova and Yordanov [**106**]. Here we shall prove a better estimate for $\|u_t(t)\|_{L^2}$ than that of [**35**]. This is remarked by D'Abbicco, Lucente and Reissig [**8**]. Another proof is given by Nishihara [**78**]. This proof is done by the L^p-L^q estimates described in Theorem 2.4. We also give blow-up results for (2.3.1). We introduce the results by Li and Zhou [**53**] and Zhang [**122**]. The proof of [**53**] is based on the method of differential inequality and the proof of [**122**] is due to the test function method, which is explained in Section 1.4.2.

2.3.1. Global existence by Matsumura's estimates. Here we prove a global existence result by following [**35**]. For $\alpha \in (0, 1/4]$, we put

$$I_\alpha^2 = \int_{\mathbf{R}^n} e^{2\alpha|x|^2}(u_0(x)^2 + |\nabla u_0(x)|^2 + u_1(x)^2)dx$$

and

$$\psi(t,x) = \frac{\alpha|x|^2}{(1+t)}.$$

We first note that for the data with $I_\alpha < \infty$, there exists a unique local-in-time solution

$$u \in C([0,T^*); H^1(\mathbf{R}^n)) \cap C^1([0,T^*); L^2(\mathbf{R}^n))$$

for some $T^* > 0$. Moreover,

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)}(u(t,x)^2 + |\nabla u(t,x)|^2 + u_t(t,x)^2)dx < \infty$$

holds for any $0 \le t < T^*$, the above integral is a continuous function of $t \in [0, T^*)$ and if $T^* < +\infty$, then we have

$$\liminf_{t \to T^*-0} \int_{\mathbf{R}^n} e^{2\psi(t,x)}(u(t,x)^2 + |\nabla u(t,x)|^2 + u_t(t,x)^2)dx = \infty$$

(see Proposition 9.21).

Therefore, it suffices to prove a priori estimate for solutions to (2.3.1).

THEOREM 2.6. *Let $n \ge 1$ and $p > 1 + 2/n$. Moreover, let $p \le n/(n-2)$ if $n \ge 3$. Let $\alpha \in (0, 1/4]$. Then there exists an $\varepsilon_0 > 0$ such that if $I_\alpha \le \varepsilon_0$, then there exists a unique global solution*

$$u \in C([0,\infty); H^1(\mathbf{R}^n)) \cap C^1([0,\infty); L^2(\mathbf{R}^n))$$

to (2.3.1). *Moreover, there exists some constant $C > 0$ such that the solution u satisfies the decay estimates*

$$\|u(t)\|_{L^2} \le C(1+t)^{-n/4} I_\alpha,$$
$$\|\nabla u(t)\|_{L^2} \le C(1+t)^{-n/4-1/2} I_\alpha,$$
$$\|u_t(t)\|_{L^2} \le C(1+t)^{-n/4-1} I_\alpha.$$

REMARK 2.2. *This theorem is still true for other nonlinearities, for example, $\pm|u|^{p-1}u, -|u|^p$. More generally, we can obtain the same global existence result for the nonlinearity $f(u)$ satisfying $f(0) = 0$ and the estimate* (9.5.2) (*see* Section 9.5).

Proof. We shall prove a priori estimate for the functional

$$W(t) = \sup_{0\le s\le t}\Big\{\|e^{\psi}(u_t,\nabla u)(t)\|_{L^2} + (1+t)^{n/4+1/2}\|\nabla u(t)\|_{L^2}$$
$$+(1+t)^{n/4+1}\|u_t(t)\|_{L^2} + (1+t)^{n/4}\|u(t)\|_{L^2}\Big\},$$

First, we prove an a estimate for a weighted energy:

Lemma 2.7. *Let u be a local solution of* (2.3.1) *whose existence is guaranteed by Proposition 9.21 and let $\delta > 0$ be a small number such that $\gamma := 2/(p+1)+\delta < 1$. Then there exists some constant $C = C_\delta > 0$ such that we have*

(2.3.2)
$$\|e^{\psi(t)}(u_t,\nabla u)(t)\|_{L^2} \le CI_\alpha + CI_\alpha^{(p+1)/2} + C\sup_{0\le s\le t}\left((1+s)^{\delta}\|e^{\gamma\psi(s)}u(s)\|_{L^{p+1}}^{(p+1)/2}\right).$$

Proof. A simple calculation gives

$$-\psi_t(t,x) = \frac{\alpha|x|^2}{(1+t)^2},\quad \nabla\psi(t,x) = \frac{2\alpha x}{1+t},\quad \Delta\psi(t,x) = \frac{2n\alpha}{1+t}, \tag{2.3.3}$$

$$-\psi_t(t,x) = \frac{1}{4\alpha}|\nabla\psi(t,x)|^2 \ge |\nabla\psi(t,x)|^2, \tag{2.3.4}$$

since $\alpha \in (0,1/4]$. Multiplying (2.3.1) by $e^{2\psi}u_t$, we obtain

$$\begin{aligned}&\partial_t\left[\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)\right] - \nabla\cdot(e^{2\psi}u_t\nabla u)\\ &\quad + e^{2\psi}\left(1+(-\psi_t)-\frac{|\nabla\psi|^2}{-\psi_t}\right)u_t^2 + \frac{e^{2\psi}}{-\psi_t}|\nabla\psi u_t - \psi_t\nabla u|^2\\ &= \partial_t\left[\frac{e^{2\psi}}{p+1}|u|^p u\right] + (-\psi_t)\frac{e^{2\psi}}{p+1}|u|^p u.\end{aligned} \tag{2.3.5}$$

Noting (2.3.4), one can easily see that

$$e^{2\psi}\left(1+(-\psi_t)-\frac{|\nabla\psi|^2}{-\psi_t}\right)u_t^2 + \frac{e^{2\psi}}{-\psi_t}|\nabla\psi u_t - \psi_t\nabla u|^2 \ge 0.$$

Integrating (2.3.5) over $\mathbf{R}^n$ with using the divergence theorem, which can be applied by $e^{2\psi}u_t\nabla u \in L^1(\mathbf{R}^n)$, we can see that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbf{R}^n} e^{2\psi(t,x)}(u_t(t,x)^2+|\nabla u(t,x)|^2)dx\\ &\le \frac{d}{dt}\int_{\mathbf{R}^n} e^{2\psi(t,x)}|u(t,x)|^{p+1}dx + \int_{\mathbf{R}^n} e^{2\psi(t,x)}(-\psi_t)|u(t,x)|^{p+1}dx.\end{aligned}$$

By integrating over $[0,t]$, we have

$$\begin{aligned}&\int_{\mathbf{R}^n} e^{2\psi(t,x)}(u_t(t,x)^2+|\nabla u(t,x)|^2)dx\\ &\quad\le I_\alpha^2 + CI_\alpha^{p+1} + C\int_{\mathbf{R}^n} e^{2\psi(t,x)}|u(t,x)|^{p+1}dx\\ &\qquad + C\int_0^t\int_{\mathbf{R}^n} e^{2\psi(s,x)}(-\psi_t(s,x))|u(s,x)|^{p+1}dxds.\end{aligned} \tag{2.3.6}$$

Here we used the inequality

$$\int_{\mathbf{R}^n} e^{2\alpha|x|^2}|u_0|^{p+1}dx \le CI_\alpha^{p+1},$$

which is easily obtained from the Gagliardo-Nirenberg inequality (Lemma 9.10). Indeed, we have

$$\begin{aligned}\int_{\mathbf{R}^n} e^{2\alpha|x|^2}|u_0|^{p+1}dx &= \|e^{2\alpha|x|^2/(p+1)}u_0\|_{L^{p+1}}^{p+1}\\ &\le C(\|e^{2\alpha|x|^2/(p+1)}u_0\|_{L^2} + \|\nabla(e^{2\alpha|x|^2/(p+1)}u_0)\|_{L^2})^{p+1}\\ &\le CI_\alpha^{p+1},\end{aligned}$$

since $|x|^2e^{4\alpha|x|^2/(p+1)} \le Ce^{2\alpha|x|^2}$. From (2.3.3), we see that

$$(-\psi_t(s,x))e^{(2-\gamma(p+1))\psi(s,x)} \le \frac{\psi(s,x)}{1+s}e^{-\delta(p+1)\psi(s,x)} \le C_\delta(1+s)^{-1}$$

and hence,

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}^n} e^{2\psi(s,x)}(-\psi_t(s,x))|u(s,x)|^{p+1}dxds\\ &\le C_\delta\int_0^t(1+s)^{-1}\int_{\mathbf{R}^n} e^{\gamma(p+1)\psi(s,x)}|u(s,x)|^{p+1}dxds\\ &\le C_\delta \sup_{0\le s\le t}\left((1+s)^\delta\|e^{\gamma\psi(s)}u(s)\|_{L^{p+1}}^{p+1}\right).\end{aligned}$$

Substituting this into (2.3.6), we reach the conclusion. □

The next lemma is a Gagliardo-Nirenberg type inequality, which helps us to control nonlinear terms.

LEMMA 2.8. *Let $T>0$, $\theta(q)=n(1/2-1/q)$, $0\le\theta(q)<1$ and let $\sigma\in(0,1]$. If $v\in H^1(\mathbf{R}^n)$ satisfies $e^{\psi(t)}v, e^{\psi(t)}\nabla v\in L^2(\mathbf{R}^n)$ for all $t\in[0,T)$, then it follows that*

$$\|e^{\sigma\psi(t)}v\|_{L^q} \le C_\sigma(1+t)^{(1-\theta(q))/2}\|\nabla v\|_{L^2}^{1-\sigma}\|e^{\psi(t)}\nabla v\|_{L^2}^\sigma$$

for $t\in[0,T)$ and some $C_\sigma>0$.

PROOF. From the Gagliardo-Nirenberg inequality (Lemma 9.10), we have

$$\|e^{\sigma\psi(t)}v\|_{L^q} \le C\|e^{\sigma\psi(t)}v\|_{L^2}^{1-\theta(q)}\|\nabla(e^{\sigma\psi(t)}v)\|_{L^2}^{\theta(q)}.$$

On the other hand, using

$$e^{\sigma\psi(t)}\nabla v = \nabla(e^{\sigma\psi(t)}v) - \sigma e^{\sigma\psi(t)}v\nabla\psi(t),$$

and (2.3.3), we can deduce that

$$\begin{aligned}&\|e^{\sigma\psi(t)}\nabla v\|_{L^2}^2\\ &= \int_{\mathbf{R}^n}\left(|\nabla(e^{\sigma\psi(t)}v)|^2 + \sigma^2e^{2\sigma\psi(t)}|v|^2|\nabla\psi(t)|^2 - 2\sigma e^{\sigma\psi(t)}v\nabla(e^{\sigma\psi(t)}v)\nabla\psi\right)dx\\ &= \int_{\mathbf{R}^n}\left(|\nabla(e^{\sigma\psi(t)}v)|^2 + \sigma^2e^{2\sigma\psi(t)}|v|^2|\nabla\psi(t)|^2 + \sigma\Delta\psi(t)(e^{\sigma\psi(t)}v)^2\right)dx\\ &\ge \|\nabla(e^{\sigma\psi(t)}v)\|_{L^2}^2 + \frac{2n\sigma\alpha}{1+t}\|e^{\sigma\psi(t)}v\|_{L^2}^2.\end{aligned}$$

Thus, we have

$$\|e^{\sigma\psi(t)}v\|_{L^q} \le C_{\sigma,\alpha}(1+t)^{(1-\theta(q))/2}\|e^{\sigma\psi(t)}\nabla v\|_{L^2} \tag{2.3.7}$$

for some positive constant $C_{\sigma,\alpha}$ depending on σ and α. By using the Hölder inequality with $\sigma/2+(1-\sigma)/2=1/2$, one can obtain

$$\begin{aligned}\|e^{\sigma\psi(t)}\nabla v\|_{L^2} &= \|e^{\sigma\psi(t)}|\nabla v|^{\sigma}|\nabla v|^{1-\sigma}\|_{L^2}\\ &\le \|e^{\psi(t)}\nabla v\|_{L^2}^{\sigma}\|\nabla v\|_{L^2}^{1-\sigma}.\end{aligned}$$

Substituting this into (2.3.7), we have the desired estimate. □

Now we estimate components of $W(t)$:

LEMMA 2.9. *Let u be a local solution whose existence is guaranteed by Proposition 9.21 and let $\varepsilon > 0$. Then we have the following estimate:*

$$\begin{aligned}&(1+t)^{n/4+1/2}\|\nabla u(t)\|_{L^2} + (1+t)^{n/4+1}\|u_t(t)\|_{L^2} + (1+t)^{n/4}\|u(t)\|_{L^2}\\ &\quad \le CI_\alpha + C_\varepsilon \sup_{0\le s\le t}\left((1+s)^{n/4+1+\varepsilon}\|e^{\varepsilon\psi(s)}u(s)\|_{L^{2p}}^{p}\right).\end{aligned} \tag{2.3.8}$$

PROOF. By the Duhamel principle, the solution u of (2.3.1) satisfies the integral equation

$$u(t) = S_n(t)(u_0+u_1) + \partial_t(S_n(t)u_0) + \int_0^t S_n(t-s)|u(s)|^p ds \equiv u_L(t) + u_N(t),$$

where

$$u_L(t) = S_n(t)(u_0+u_1) + \partial_t(S_n(t)u_0), \quad u_N(t) = \int_0^t S_n(t-s)|u(s)|^p ds$$

and $S_n(t)$ denotes the solution operator for the linear damped wave equation $u_{tt} - \Delta u + u_t = 0$, which maps a function $g(x)$ to the solution $u(t,x)$ with the initial data $u(0,x) = 0, u_t(0,x) = g(x)$. By Matsumura's estimate (2.1.3), one can calculate

$$\|u_L(t)\|_{L^2} \le C(\|u_0\|_{L^2} + \|u_1\|_{L^2}) \le CI_\alpha$$

and

$$
\begin{aligned}
&\int_0^{t/2} \|S_n(t-s)|u(s)|^p\|_{L^2} ds \\
&\quad \le C \int_0^{t/2} (1+t-s)^{-n/4} (\||u(s)|^p\|_{L^1} + \||u(s)|^p\|_{L^2}) ds \\
&\quad \le C \sup_{0\le s\le t/2} \left((1+s)^{1+\varepsilon}(\|u(s)\|_{L^p}^p + \|u(s)\|_{L^{2p}}^p)\right) \\
&\qquad \times \int_0^{t/2} (1+t-s)^{-n/4}(1+s)^{-1-\varepsilon} ds \\
&\quad \le C(1+t)^{-n/4} \sup_{0\le s\le t/2} \left((1+s)^{1+\varepsilon}(\|u(s)\|_{L^p}^p + \|u(s)\|_{L^{2p}}^p)\right), \\
&\int_{t/2}^{t} \|S_n(t-s)|u(s)|^p\|_{L^2} ds \\
&\quad \le C \int_{t/2}^{t} \||u(s)|^p\|_{L^2} ds \\
&\quad \le C(1+t)^{-n/4-1} \int_{t/2}^{t} (1+s)^{n/4+1}\|u(s)\|_{L^{2p}}^p ds \\
&\quad \le C(1+t)^{-n/4} \sup_{t/2\le s\le t} \left((1+s)^{n/4+1}\|u(s)\|_{L^{2p}}^p\right).
\end{aligned}
$$

To control the bad term $\|u(s)\|_{L^p}^p$, we use the inequality

$$
\|u(s)\|_{L^p}^p \le C_\varepsilon (1+s)^{n/4} \|e^{\varepsilon\psi(s)}u(s)\|_{L^{2p}}^p,
$$

which follows from

$$
\begin{aligned}
\|u(s)\|_{L^p}^p &\le \left(\int_{\mathbf{R}^n} e^{-2p\varepsilon\psi(s,x)} dx\right)^{1/2} \left(\int_{\mathbf{R}^n} e^{2p\varepsilon\psi(s,x)} |u(s,x)|^{2p} dx\right)^{1/2} \\
&\le C(1+s)^{n/4}\|e^{\varepsilon\psi(s)}u(s)\|_{L^{2p}}^p.
\end{aligned}
$$

Consequently, we have

$$
\|u_N(t)\|_{L^2} \le C(1+t)^{-n/4} \sup_{0\le s\le t} \left((1+s)^{n/4+1+\varepsilon}\|e^{\varepsilon\psi(s)}u(s)\|_{L^{2p}}^p\right).
$$

We can prove the estimates of $\|\nabla u(t)\|_{L^2}$ and $\|u_t(t)\|_{L^2}$ in a similar way by noting

$$
\begin{aligned}
&\int_{t/2}^{t} \|\nabla S(t-s)|u(s)|^p\|_{L^2} ds \\
&\quad \le C \int_{t/2}^{t} (1+t-s)^{-1/2}\|u(s)\|_{L^{2p}}^p ds \\
&\quad \le C \sup_{0\le s\le t} \left((1+s)^{n/4+1}\|e^{\varepsilon\psi(s)}u(s)\|_{L^{2p}}^p\right) \int_{t/2}^{t} (1+t-s)^{-1/2}(1+s)^{-n/4-1} ds \\
&\quad \le C(1+t)^{-n/4-1/2} \sup_{0\le s\le t} \left((1+s)^{n/4+1}\|e^{\varepsilon\psi(s)}u(s)\|_{L^{2p}}^p\right)
\end{aligned}
$$

and

$$\begin{aligned}
&\int_{t/2}^{t} \|\partial_t S(t-s)|u(s)|^p\|_{L^2} ds \\
&\quad \le C\int_{t/2}^{t} (1+t-s)^{-1}\|u(s)\|_{L^{2p}}^p ds \\
&\quad \le C \sup_{0\le s\le t}\left((1+s)^{n/4+1+\varepsilon}\|e^{\varepsilon\psi(s)}u(s)\|_{L^{2p}}^p\right)\int_{t/2}^{t}(1+t-s)^{-1}(1+s)^{-n/4-1-\varepsilon}ds \\
&\quad \le C(1+t)^{-n/4-1-\varepsilon}\log(1+t/2)\sup_{0\le s\le t}\left((1+s)^{n/4+1+\varepsilon}\|e^{\varepsilon\psi(s)}u(s)\|_{L^{2p}}^p\right) \\
&\quad \le C(1+t)^{-n/4-1}\sup_{0\le s\le t}\left((1+s)^{n/4+1+\varepsilon}\|e^{\varepsilon\psi(s)}u(s)\|_{L^{2p}}^p\right).
\end{aligned}$$

□

Now we give an a priori estimate for $W(t)$. We assume that $I_\alpha \le 1$. By Lemmas 2.7 and 2.9, we can estimate

$$\begin{aligned}
W(t) \le CI_\alpha &+ C_\delta \sup_{0\le s\le t}\left((1+s)^{\delta}\|e^{\gamma\psi(s)}u(s)\|_{L^{p+1}}^{(p+1)/2}\right) \\
&+ C_\varepsilon \sup_{0\le s\le t}\left((1+s)^{n/4+1+\varepsilon}\|e^{\varepsilon\psi(s)}u(s)\|_{L^{2p}}^p\right)
\end{aligned}$$

for any small $\delta > 0$, $\varepsilon > 0$, where $\gamma = 2/(p+1)+\delta < 1$. By Lemma 2.8, one can obtain

$$\begin{aligned}
\|e^{\varepsilon\psi(s)}u(s)\|_{L^{2p}} &\le C(1+t)^{(1-\theta(2p))/2}\|\nabla u(s)\|_{L^2}^{1-\varepsilon}\|e^{\psi(s)}u(s)\|_{L^2}^{\varepsilon} \\
&\le C(1+t)^{(1-\theta(2p))/2-(1-\varepsilon)(n/4+1/2)}W(s)
\end{aligned}$$

and, similarly,

$$\|e^{\gamma\psi(s)}u(s)\|_{L^{p+1}} \le C(1+t)^{(1-\theta(p+1))/2-(1-\gamma)(n/4+1/2)}W(s).$$

A straightforward calculation shows that if $p > 1+2/n$, then both of the powers of $(1+t)$ in the above two inequalities are negative, provided that δ and ε are sufficiently small. Therefore, putting

$$M(t) = \sup_{0\le s\le t} W(s),$$

we obtain

$$M(t) \le CI_\alpha + CM(t)^{(p+1)/2} + CM(t)^p \tag{2.3.9}$$

for $0 \le t < T^*$. We note that $W(t)$ is a continuous function of $t \in [0,T)$ and so is $M(t)$. By taking I_α sufficiently small, we can deduce that

$$M(t) \le CI_\alpha \tag{2.3.10}$$

for $0 \le t < T^*$. Indeed, in (2.3.9), we may assume that $C \ge 1$. We take I_α sufficiently small so that

$$2CI_\alpha > CI_\alpha + C(2CI_\alpha)^{(p+1)/2} + C(2CI_\alpha)^p. \tag{2.3.11}$$

Let $M_1, M_2 (M_1 < M_2)$ be the positive roots of the identity

$$M = CI_\alpha + CM^{(p+1)/2} + CM^p.$$

Note that $M_1 < 2CI_\alpha$ holds by (2.3.11). By noting $M(0) = I_\alpha \le CI_\alpha \le M_1$ and the continuity of $M(t)$, the estimate (2.3.9) implies that $M(t) \le M_1 < 2CI_\alpha$, which proves (2.3.10). Finally, by (2.3.10), we have

$$\limsup_{t\to T^*-0} \int_{\mathbf{R}^n} e^{2\psi(t,x)}(u(t,x)^2 + |\nabla u(t,x)|^2 + u_t(t,x)^2)dx < \infty,$$

which means $T^* = +\infty$ (see Proposition 9.21). □

2.3.2. Global existence by L^p-L^q estimates. In this subsection, we introduce another proof of global existence of solutions to the semilinear damped wave equation in three space dimensions

$$\begin{cases} u_{tt} - \Delta u + u_t = f(u), & (t,x) \in (0,\infty) \times \mathbf{R}^3, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^3. \end{cases} \tag{2.3.12}$$

We shall introduce a proof by using L^p-L^q estimate (Theorem 2.4). This proof was given by Nishihara [**78**]. We assume that the semilinear term $f(u)$ satisfies

$$|f(u) - f(v)| \le C(|u| + |v|)^{p-1}|u - v| \tag{2.3.13}$$

with some $p > 1$ and some constant $C > 0$. We define

$$Z_0 := (W^{1,1} \cap W^{1,\infty}) \times (L^1 \cap L^\infty)$$
$$\|(f,g)\|_{Z_0} := \|f\|_{W^{1,1}} + \|f\|_{W^{1,\infty}} + \|g\|_{L^1} + \|g\|_{L^\infty}$$

and

$$X = \{u \in C([0,\infty); L^1 \cap L^\infty) \mid \|u\|_X < +\infty\}.$$
$$\|u\|_X = \sup_{t\in[0,\infty)} \left\{ \|u(t)\|_{L^1} + (1+t)^{3/2}\|u(t)\|_{L^\infty} \right\}.$$

Here we call a function u a mild solution of (2.3.12) if u satisfies the integral equation

$$u(t,x) = S_3(t)(u_0 + u_1) + \partial_t(S_3(t)u_0) + \int_0^t S_3(t-s)f(u)(s,x)ds. \tag{2.3.14}$$

THEOREM 2.10 (Nishihara [**78**]). *If the semilinear term $f(u)$ satisfies* (2.3.13) *with some $p > 1 + 2/3$ and the initial data $(u_0, u_1) \in Z_0$ is sufficiently small, then there exists a unique mild solution $u \in X$ of* (2.3.12).

REMARK 2.3. *In Theorem 2.6, we need the boundedness of the power of the nonlinearity $p \le n/(n-2)$ when $n \ge 3$. However, the proof of Theorem 2.10 does not require any boundedness for p.*

PROOF. Let

$$\begin{aligned} u^{(0)}(t,x) &= S_3(t)(u_0 + u_1) + \partial_t(S_3(t)u_0) \\ &= e^{-t/2}\widetilde{\mathbf{W}}_3(t;u_0,u_1) + J_3(t)(u_0+u_1) + \tilde{J}_3(t)u_0, \end{aligned}$$

where

$$\widetilde{\mathbf{W}}_3(t;u_0,u_1) = \left(\frac{1}{2}+\frac{t}{8}\right)W_3(t)u_0 + \partial_t(W_3(t)u_0) + W_3(t)u_1,$$

$$W_3(t)u_0(x) = \frac{1}{4\pi t}\int_{|x-y|=t} u_0(y)dS_y,$$

$$J_3(t)u_0(x) = \frac{e^{-t/2}}{8\pi}\int_{|x-y|\le t} \frac{I_1\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)}{\sqrt{t^2-|x-y|^2}} u_0(y)dy,$$

$$\tilde{J}_3(t)u_0(x) = \frac{1}{8\pi}\int_{|x-y|\le t} \partial_t\left(e^{-t/2}\frac{I_1\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)}{\sqrt{t^2-|x-y|^2}}\right) u_0(y)dy,$$

and we define an approximation of the solution inductively by

$$u^{(n+1)}(t,x) = u^{(0)}(t,x) + \int_0^t S_3(t-s)f(u^{(n)})(s,x)ds \equiv u^{(0)}(t,x) + u_N^{(n)}(t,x).$$

It suffices to prove the following three claims:

(i) There exists a constant $C_0 > 0$ such that $\|u^{(0)}\|_X \le C_0\|(u_0,u_1)\|_{Z_0}$.

(ii) If $\|(u_0,u_1)\|_{Z_0}$ is sufficiently small, then $\|u^{(n)}\|_X \le 2C_0\|(u_0,u_1)\|_{Z_0}$ implies $\|u^{(n+1)}\|_X \le 2C_0\|(u_0,u_1)\|_{Z_0}$.

(iii) When $\|(u_0,u_1)\|_{Z_0}$ is sufficiently small, it follows that

$$\|u^{(n+1)} - u^{(n)}\|_X \le \frac{1}{2}\|u^{(n)} - u^{(n-1)}\|_X.$$

In order to prove (i), we use the following lemma:

LEMMA 2.11. *For $q=1$ or $q=\infty$, we have*

$$\|W_3(t)g\|_{L^q} \le Ct\|g\|_{L^q},$$
$$\|\partial_t(W_3(t)g)\|_{L^q} \le Ct\|g\|_{W^{1,q}}.$$

PROOF. This lemma can be easily proved by noting

$$\|W_3(t)g\|_{L^1} \le Ct\int_{S^2}\|g\|_{L^1}d\omega \le Ct\|g\|_{L^1}$$

and the same estimate is true if we replace L^1 in L^∞. The second assertion can be also proved by a similar way. □

By the above lemma and the estimates (2.2.19), (2.2.20), we can immediately obtain (i).

Next, we prove the claim (ii). We have

$$\|u^{(n+1)}\|_X \le \|u^{(0)}\|_X + \|u_N^{(n)}\|_X.$$

By (i), it holds that $\|u^{(0)}\|_X \le C_0\|(u_0,u_1)\|_{Z_0}$. Therefore, it suffices to prove that $\|u_N^{(n)}\|_X \le C_0\|(u_0,u_1)\|_{Z_0}$ under the assumption $\|u^{(n)}\|_X \le 2C_0\|(u_0,u_1)\|_{Z_0}$ and $\|(u_0,u_1)\|_{Z_0}$ is sufficiently small. We write

$$u_N^{(n)}(t,x) = \int_0^t e^{-(t-s)/2}W_3(t-s)f(u^{(n)})(s,x)ds + \int_0^t J_3(t-s)f(u^{(n)})(s,x)ds.$$

By Lemma 2.11, we see that

$$\begin{aligned}
&\| \int_0^t e^{-(t-s)/2} W_3(t-s) f(u^{(n)})(s,x) ds \|_{L^1} \\
&\quad \le C \int_0^t e^{-(t-s)/2} (t-s) \| f(u^{(n)})(s) \|_{L^1} ds \\
&\quad \le C \int_0^t \| u^{(n)}(s) \|_{L^\infty}^{p-1} \| u^{(n)}(s) \|_{L^1} ds \\
&\quad \le \| u^{(n)} \|_X^p \int_0^t (1+s)^{-\frac{3}{2}(p-1)} ds \le C \| u^{(n)} \|_X^p,
\end{aligned}$$

since $p > 1 + 3/2$. It also follows from (2.2.19) that

$$\begin{aligned}
&\| \int_0^t J_3(t-s) f(u^{(n)})(s,x) ds \|_{L^1} \\
&\quad \le \int_0^t \| J_3(t-s) f(u^{(n)})(s) \|_{L^1} ds \\
&\quad \le C \int_0^t \| f(u^{(n)})(s) \|_{L^1} ds \\
&\quad \le C \int_0^t \| u^{(n)}(s) \|_{L^\infty}^{p-1} \| u^{(n)}(s) \|_{L^1} ds \\
&\quad \le \| u^{(n)} \|_X^p \int_0^t (1+s)^{-\frac{3}{2}(p-1)} ds \le C \| u^{(n)} \|_X^p.
\end{aligned}$$

Similarly, we deduce that

$$\begin{aligned}
&\| \int_0^t e^{-(t-s)/2} W_3(t-s) f(u^{(n)})(s,x) ds \|_{L^\infty} \\
&\quad \le \int_0^t \| e^{-(t-s)/2} W_3(t-s) f(u^{(n)})(s) \|_{L^\infty} ds \\
&\quad \le C \int_0^t e^{-(t-s)/2} (t-s) \| f(u^{(n)})(s) \|_{L^\infty} ds \\
&\quad \le C \| u^{(n)} \|_X^p \int_0^t (1+t-s)^{-\frac{3}{2}} (1+s)^{-\frac{3}{2}p} ds \\
&\quad \le C (1+t)^{-\frac{3}{2}} \| u^{(n)} \|_X^p
\end{aligned}$$

and, for $p \ge 2$,

$$\begin{aligned}
&\| \int_0^t J_3(t-s) f(u^{(n)})(s,x) ds \|_{L^\infty} \\
&\quad \le \int_0^t \| J_3(t-s) f(u^{(n)})(s) \|_{L^\infty} ds \\
&\quad \le C \int_0^t (1+t-s)^{-\frac{3}{2}} \| f(u^{(n)})(s) \|_{L^1} ds \\
&\quad \le C \| u^{(n)} \|_X^p \int_0^t (1+t-s)^{-\frac{3}{2}} (1+s)^{-\frac{3}{2}(p-1)} ds \\
&\quad \le C (1+t)^{-\frac{3}{2}} \| u^{(n)} \|_X^p,
\end{aligned}$$

and, for $1 + 2/3 < p < 2$,

$$\begin{aligned}
&\| \int_0^t J_3(t-s) f(u^{(n)})(s,x) ds \|_{L^\infty} \\
&\quad \le \int_0^t \| J_3(t-s) f(u^{(n)})(s) \|_{L^\infty} ds \\
&\quad \le C \int_0^{t/2} (1+t-s)^{-\frac{3}{2}} \| f(u^{(n)})(s) \|_{L^1} ds \\
&\qquad + C \int_{t/2}^t (1+t-s)^{-\frac{3}{2}(p-1)} \| f(u^{(n)})(s) \|_{L^{1/(p-1)}} \\
&\quad \le C \| u^{(n)} \|_X^p \int_0^{t/2} (1+t-s)^{-\frac{3}{2}} (1+s)^{-\frac{3}{2}(p-1)} ds \\
&\qquad + C \| u^{(n)} \|_X^p \int_{t/2}^t (1+t-s)^{-\frac{3}{2}(p-1)} (1+s)^{-\frac{3}{2}} ds \\
&\quad \le C (1+t)^{-\frac{3}{2}} \| u^{(n)} \|_X^p .
\end{aligned}$$

Here we have used Lemma 9.8 and the inequality

$$\| f(u^{(n)})(s) \|_{L^{1/(p-1)}} \le C \| |u^{(n)}(s)|^p \|_{L^{1/(p-1)}} \le C \| u^{(n)}(s) \|_{L^\infty} \| u^{(n)}(s) \|_{L^1}^{p-1} .$$

Consequently, we obtain

$$\begin{aligned}
\| u^{(n+1)} \|_X &\le C_0 \| (u_0, u_1) \|_{Z_0} + C \| u^{(n)} \|_X^p \\
&\le C_0 \| (u_0, u_1) \|_{Z_0} + C (2C_0 \| (u_0, u_1) \|_{Z_0})^p \\
&\le 2C_0 \| (u_0, u_1) \|_{Z_0} ,
\end{aligned}$$

provided that $\| (u_0, u_1) \|_{Z_0}$ is sufficiently small. This proves the claim (ii).

Let us turn to (iii). We see that

$$\begin{aligned}
u^{(n+1)} - u^{(n)} = &\int_0^t e^{-\frac{t-s}{2}} W_3(t-\tau) (f(u^{(n)}) - f(u^{(n-1)})) ds \\
&+ \int_0^t J_3(t-s) (f(u^{(n)}) - f(u^{(n-1)})) ds
\end{aligned}$$

and hence,

$$\begin{aligned}
\| u^{(n+1)} - u^{(n)} \|_{L^1} &\le C \int_0^t \left(e^{-\frac{t-s}{2}} (t-s) + 1 \right) \| f(u^{(n)}) - f(u^{(n-1)}) \|_{L^1} ds \\
&\le C \int_0^t (\| u^{(n)} \|_{L^\infty}^{p-1} + \| u^{(n-1)} \|_{L^\infty}^{p-1}) \| u^{(n)} - u^{(n-1)} \|_{L^1} ds \\
&\le C \int_0^t (1+s)^{-\frac{3}{2}(p-1)} (\| u^{(n)} \|_X^{p-1} + \| u^{(n-1)} \|_X^{p-1}) \\
&\qquad \times \| u^{(n)} - u^{(n-1)} \|_{L^1} ds \\
&\le C (2C_0 \| u_0, u_1 \|_{Z_0})^{p-1} \| u^{(n)} - u^{(n-1)} \|_X .
\end{aligned}$$

Similarly, we have for $p \geq 2$,

$$\begin{aligned}
&\|u^{(n+1)} - u^{(n)}\|_{L^\infty} \\
&\leq C\left(\int_0^t e^{-\frac{t-s}{2}}(t-s)(1+s)^{-\frac{3}{2}p}ds + \int_0^t (1+t-s)^{-\frac{3}{2}}(1+s)^{-\frac{3}{2}(p-1)}ds\right) \\
&\qquad \times (\|u^{(n)}\|_X + \|u^{(n-1)}\|_X)^{p-1}\|u^{(n)} - u^{(n-1)}\|_X \\
&\leq C(1+t)^{-\frac{3}{2}}(2C_0\|u_0, u_1\|_{Z_0})^{p-1}\|u^{(n)} - u^{(n-1)}\|_X,
\end{aligned}$$

and, for $1 + 2/3 < p < 2$,

$$\begin{aligned}
&\|u^{(n+1)} - u^{(n)}\|_{L^\infty} \\
&\leq C\int_0^t e^{-\frac{t-s}{2}}(t-s)\|f(u^{(n)}) - f(u^{(n-1)})\|_{L^\infty}ds \\
&\qquad + C\int_0^{t/2}(1+t-s)^{-\frac{3}{2}}\|f(u^{(n)}) - f(u^{(n-1)})\|_{L^1}ds \\
&\qquad + C\int_{t/2}^t (1+t-s)^{-\frac{3}{2}(p-1)}\|f(u^{(n)}) - f(u^{(n-1)})\|_{L^{1/(p-1)}}ds \\
&\leq C\Bigg((1+t)^{-\frac{3}{2}p} + \int_0^{t/2}(1+t-s)^{-\frac{3}{2}}(1+s)^{-\frac{3}{2}(p-1)}ds \\
&\qquad + \int_{t/2}^t (1+t-s)^{-\frac{3}{2}(p-1)}(1+s)^{-\frac{3}{2}}ds\Bigg) \\
&\qquad \times (2C_0\|u_0, u_1\|_{Z_0})^{p-1}\|u^{(n)} - u^{(n-1)}\|_X \\
&\leq C(1+t)^{-\frac{3}{2}}(2C_0\|u_0, u_1\|_{Z_0})^{p-1}\|u^{(n)} - u^{(n-1)}\|_X.
\end{aligned}$$

Here we have used the inequality

$$\|f(u^{(n)}) - f(u^{(n-1)})\|_{L^{1/(p-1)}} \leq C(\|u^{(n)}\|_{L^1} + \|u^{(n-1)}\|_{L^1})^{p-1}\|u^{(n)} - u^{(n-1)}\|_{L^\infty}.$$

Consequently, we obtain

$$\|u^{(n+1)} - u^{(n)}\|_X \leq C(2C_0\|u_0, u_1\|_{Z_0})^{p-1}\|u^{(n)} - u^{(n-1)}\|_X.$$

Therefore, choosing $\|u_0, u_1\|_{Z_0}$ sufficiently small so that

$$\|u^{(n+1)} - u^{(n)}\|_X \leq \frac{1}{2}\|u^{(n)} - u^{(n-1)}\|_X,$$

we have the claim (iii).

By the claims (i), (ii), (iii), we can find a solution of (2.3.14) $u \in X$ for small data $(u_0, u_1) \in Z_0$ by $u = \lim_{n\to\infty} u^{(n)}$. The uniqueness of solution is immediately obtained from the above proof. In fact, if we have two solutions u and v in X. From the above proof, we can see that

$$\|u(t) - v(t)\|_{L^1} \leq C\int_0^t (\|u\|_X + \|v\|_X)^{p-1}\|u(s) - v(s)\|_{L^1}ds.$$

By the Gronwall inequality (Lemma 9.11), we have $\|u(t) - v(t)\|_{L^1} = 0$. □

2.3.3. Blow-up by differential inequalities. We shall give some blow-up results for the damped wave equation with a source semilinear term

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u,u_t)(0,x) = \varepsilon(u_0,u_1)(x), & x \in \mathbf{R}^n, \end{cases} \tag{2.3.15}$$

where $\varepsilon > 0$. We first introduce a useful comparison lemma developed by Li and Zhou [**53**].

LEMMA 2.12 (Li and Zhou [**53**]). *Let $\alpha \geq 0, T > 0$ and let $k, h \in C^2([0,T))$ satisfy the differential inequality*

$$\begin{aligned} a(t)k''(t) + k'(t) &\geq b(t)|k(t)|^\alpha k(t), \\ a(t)h''(t) + h'(t) &\leq b(t)|h(t)|^\alpha h(t) \end{aligned} \tag{2.3.16}$$

with some positive functions $a(t), b(t)$. Moreover, we assume

$$k(0) > h(0), \quad k'(0) \geq h'(0)$$

or

$$k(0) \geq h(0), \quad k'(0) > h'(0).$$

Then

$$k'(t) > h'(t) \tag{2.3.17}$$

holds for all $0 < t < T$.

PROOF. Without loss of generality, we may assume that

$$k(0) > h(0), \quad k'(0) > h'(0).$$

Indeed, if $k(0) = h(0), k'(0) > h'(0)$, then there exists a small time $t_0 > 0$ such that $k(t_0) > h(t_0), k'(t_0) > h'(t_0)$ and we can consider t_0 as the initial time. In the case that $k(0) > h(0), k'(0) = h'(0)$, using the inequality (2.3.16), we obtain $k''(0) > h''(0)$. Thus, taking a small time $t_0 > 0$, we can see that $k(t_0) > h(t_0), k'(t_0) > h'(t_0)$.

We suppose that the conclusion of the lemma fails, that is, there exists a time $t_* \in (0,T)$ such that

$$\begin{cases} k'(t_*) = h'(t_*) \\ k'(t) > h'(t) & (0 \leq t < t_*). \end{cases}$$

This implies

$$k''(t_*) \leq h''(t_*) \quad \text{and} \quad k(t_*) > h(t_*).$$

However, by using the inequality (2.3.16) and we can deduce that

$$a(t_*)\left(k''(t_*) - h''(t_*)\right) \geq b(t_*)\left(|k|^\alpha k(t_*) - |h|^\alpha h(t_*)\right) > 0,$$

which leads to a contradiction. □

Using the above lemma, we can prove the nonexistence of global solutions for some ordinary differential inequalities. Our proof is based on the result of Todorova and Yordanov [**106**] (see also Nishihara [**83**]).

PROPOSITION 2.13. *Let $0 \leq \gamma \leq 1$ and let $F(t)$ be a C^2-function satisfying the differential inequality*

$$F''(t) + F'(t) \geq c_0(1+t)^{-\gamma}|F(t)|^\alpha F(t) \tag{2.3.18}$$

with some $\alpha > 0, c_0 > 0$. Moreover, we assume that

$$F(0) \geq 0, \quad F'(0) \geq 0, \quad F(0) + F'(0) > 0.$$

Then $F(t)$ cannot exist globally. Furthermore, if $F(0) = \varepsilon$ with sufficiently small $\varepsilon > 0$, then the lifespan

$$T_\varepsilon := \sup\{T \in (0, \infty) \mid F(t) < +\infty \ (t \in (0, T))\}$$

of $F(t)$ is estimated as

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-\alpha/(1-\gamma)} & (\gamma < 1), \\ e^{C\varepsilon^{-\alpha}} & (\gamma = 1) \end{cases} \tag{2.3.19}$$

with some constant $C > 0$ depending only on α, c_0, γ.

Proof. Without loss of generality, we may assume that $F(0) > 0$. Because, if $F(0) = 0$, then the assumption implies $F'(0) > 0$. Hence there exists a small time t_0 such that $F(t_0) > 0, F'(t_0) > 0$ and we can consider t_0 as the initial time. In what follows, we assume that $F(0) > 0, F'(0) > 0$ and we put $F(0) = \varepsilon > 0$. For this ε, we consider the ordinary differential equation

$$y'(t) = \delta c_0 \varepsilon^{\alpha/2} (1+t)^{-\gamma} y(t)^{1+\alpha/2}$$

with the initial data $y(0) = \varepsilon$, where δ is a small positive constant satisfying

$$\delta \left(\delta c_0 \varepsilon^\alpha \left(1 + \frac{\alpha}{2}\right) + 1 \right) \leq 1 \tag{2.3.20}$$

and

$$\delta c_0 \varepsilon^{1+\alpha} < F'(0). \tag{2.3.21}$$

Noting

$$-\frac{2}{\alpha} \frac{d}{dt} \left(y(t)^{-\alpha/2} \right) = \delta c_0 \varepsilon^{\alpha/2} (1+t)^{-\gamma},$$

we can immediately solve the equation and obtain

$$y(t) = \begin{cases} \left[\varepsilon^{-\alpha/2} - \frac{\alpha \delta c_0 \varepsilon^{\alpha/2}}{2(1-\gamma)} ((1+t)^{1-\gamma} - 1) \right]^{-2/\alpha} & (\gamma < 1), \\ \left[\varepsilon^{-\alpha/2} - \frac{\alpha \delta c_0 \varepsilon^{\alpha/2}}{2} \log(1+t) \right]^{-2/\alpha} & (\gamma = 1). \end{cases}$$

We put

$$T_0 := \begin{cases} \left[\frac{2(1-\gamma)}{\alpha \delta c_0} \varepsilon^{-\alpha} + 1 \right]^{1/(1-\gamma)} - 1 & (\gamma < 1), \\ e^{\frac{2}{\alpha \delta c_0} \varepsilon^{-\alpha}} - 1 & (\gamma = 1). \end{cases} \tag{2.3.22}$$

Then we have

$$\lim_{t \uparrow T_0} y(t) = +\infty.$$

We calculate

$$\begin{aligned} y''(t) &= \delta c_0 \varepsilon^{\alpha/2} (-\gamma) (1+t)^{-\gamma-1} y(t)^{1+\alpha/2} \\ &\quad + \delta c_0 \varepsilon^{\alpha/2} (1+t)^{-\gamma} \left(1 + \frac{\alpha}{2}\right) y(t)^{\alpha/2} y'(t) \\ &\leq \delta^2 c_0^2 \varepsilon^\alpha \left(1 + \frac{\alpha}{2}\right) (1+t)^{-2\gamma} y(t)^{1+\alpha} \end{aligned}$$

and hence,

$$\begin{aligned}
y''(t) + y'(t) &\le \delta^2 c_0^2 \varepsilon^\alpha \left(1 + \frac{\alpha}{2}\right)(1+t)^{-2\gamma} y(t)^{1+\alpha} \\
&\quad + \delta c_0 \varepsilon^{\alpha/2} (1+t)^{-\gamma} y(t)^{1+\alpha/2} \\
&= c_0 (1+t)^{-\gamma} y(t)^{1+\alpha} \cdot \delta \left(\delta c_0 \varepsilon^\alpha \left(1 + \frac{\alpha}{2}\right)(1+t)^{-\gamma} + \varepsilon^{\alpha/2} y(t)^{-\alpha/2} \right) \\
&\le c_0 (1+t)^{-\gamma} y(t)^{1+\alpha} \cdot \delta \left(\delta c_0 \varepsilon^\alpha \left(1 + \frac{\alpha}{2}\right) + 1 \right) \\
&\le c_0 (1+t)^{-\gamma} y(t)^{1+\alpha}.
\end{aligned}$$

Here we used that $\varepsilon^{\alpha/2} \le y(t)^{\alpha/2}$, which follows from $y(0) = \varepsilon$ and $y'(t) \ge 0$, and (2.3.20). Noting that $y'(0) = \delta c_0 \varepsilon^{1+\alpha} < F'(0)$, we can apply Lemma 2.12 with $k(t) = F(t), h(t) = y(t), a(t) = 1, b(t) = c_0(1+t)^{-\gamma}$ and obtain

$$F'(t) > y'(t).$$

In particular, we have $F(t) \ge y(t)$ and hence, $F(t)$ must blow up until the time $t = T_0$.

Next, we prove (2.3.19). We assume that $\varepsilon > 0$ satisfies $\varepsilon \le 1$ and $c_0\varepsilon^{1+\alpha} < F'(0)$. We take $\delta > 0$ so that

$$\delta \left(\delta c_0 \left(1 + \frac{\alpha}{2}\right) + 1 \right) \le 1.$$

In particular, it follows that $\delta < 1$. We note that in this case δ is independent of ε. Moreover, by the way of choosing δ and ε above, the conditions (2.3.20) and (2.3.21) are still true. Therefore, we can repeat the above argument and prove that $F(t)$ blows up until the time T_0 defined by (2.3.22). Furthermore, noting that δ is independent of ε, we have a simple estimate for T_0:

$$T_0 \le \begin{cases} C\varepsilon^{-\alpha/(\gamma-1)} & (\gamma < 1), \\ e^{C\varepsilon^{-\alpha}} & (\gamma = 1) \end{cases}$$

with some constant $C > 0$ depending only on α, c_0, γ. This completes the proof. □

REMARK 2.4. *For a constant $L \ge 1$, the same result as Proposition 2.13 holds for the differential inequality*

$$F''(t) + F'(t) \ge c_0 (L+t)^{-\gamma} |F(t)|^\alpha F(t).$$

This is immediately confirmed by noting $(L+t)^{-\gamma} \ge L^{-\gamma}(1+t)^{-\gamma}$.

By using Proposition 2.13, we can obtain a blow-up result for the semilinear damped wave equation (2.3.15).

THEOREM 2.14. *Let $(u_0, u_1) \in C_0^2(\mathbf{R}^n) \times C_0^1(\mathbf{R}^n)$ be initial data and let $u \in C^2([0,T) \times \mathbf{R}^n)$ be a classical solution of the Cauchy problem* (2.3.15) *with some $T > 0$. Moreover, we assume that $1 < p \le 1 + 1/n$ and*

$$\int_{\mathbf{R}^n} u_j(x)dx \ge 0 \quad (j = 0, 1), \qquad \int_{\mathbf{R}^n} (u_0(x) + u_1(x))dx > 0.$$

Then u cannot exist globally. Furthermore, if $\varepsilon > 0$ is sufficiently small, then the lifespan

$$T_\varepsilon := \sup\{T \in (0,\infty) \mid \sup_{x \in \mathbf{R}^n} |u(t,x)| < +\infty \ (t \in (0,T))\}$$

of u is estimated as

$$T_\varepsilon \le \begin{cases} C\varepsilon^{-1/\kappa} & (1 < p < 1 + 1/n), \\ e^{C\varepsilon^{-(p-1)}} & (p = 1 + 1/n) \end{cases} \tag{2.3.23}$$

with

$$\kappa = \frac{1}{p-1} - n$$

and some constant $C > 0$ independent of ε.

PROOF. We first note that there exists some constant $L > 0$ such that

$$\mathrm{supp}\,(u_0, u_1) \subset \{x \in \mathbf{R}^n \mid |x| \le L\}$$

and hence, $\mathrm{supp}\, u(t) \subset \{x \in \mathbf{R}^n \mid |x| \le L + t\}$. We define

$$F(t) := \int_{\mathbf{R}^n} u(t,x)dx.$$

By the Hölder inequality, we have

$$F(t) \le \left(\int_{|x| \le L+t} dx \right)^{(p-1)/p} \left(\int_{\mathbf{R}^n} |u(t,x)|^p dx \right)^{1/p} \le C(L+t)^{n(p-1)/p} \|u(t)\|_{L^p}.$$

Therefore, integrating the equation (2.3.15), we deduce that

$$F''(t) + F'(t) = \|u(t)\|_{L^p}^p \ge C(L+t)^{-n(p-1)} |F(t)|^p.$$

Hence we can apply Proposition 2.13 with $\alpha = p - 1, \gamma = n(p-1)$ and obtain the desired conclusion. □

By using Proposition (2.13), we can also obtain the nonexistence of global solutions for (2.3.15) for $1 < p \le 1 + 2/n$ when $n \le 3$. The following argument is due to Li and Zhou [**53**] and Nishihara [**79**].

We first show a blow-up of solutions to an integral inequality.

LEMMA 2.15. *Let $t_0 > 0$ and let $v = v(t) \in C([t_0, \infty))$ be a nonnegative function satisfying the integral inequality*

(2.3.24)

$$v(t) \ge C_1 + \begin{cases} C_2 \displaystyle\int_{t_0}^t (t-s)s^{-\gamma} v(s)^p ds & (t_0 \le s \le t_0 + 2), \\ C_2 \left(\displaystyle\int_{t-2}^t (t-s)s^{-\gamma} v(s)^p ds + 2 \int_{t_0}^{t-2} s^{-\gamma} v(s)^p ds \right) & (t \ge t_0 + 2), \end{cases}$$

where $\gamma \in [0,1], p > 1, C_1 > 0, C_2 > 0$. Then $v(t)$ must blow up in finite time. Moreover, if $C_1 = \varepsilon > 0$ is sufficiently small, then the lifespan

$$T_\varepsilon = \sup\{t \ge t_0 \mid v(t) < +\infty\}$$

is estimated as

$$T_\varepsilon \le \begin{cases} C\varepsilon^{-(p-1)/(1-\gamma)} & (0 \le \gamma < 1), \\ e^{C\varepsilon^{-(p-1)}} & (\gamma = 1). \end{cases} \tag{2.3.25}$$

Proof. Let $0 < C_1' < C_1$ and let $J(t)$ be the solution of the integral equation (2.3.26)

$$J(t) = C_1' + \begin{cases} C_2 \displaystyle\int_{t_0}^{t} (t-s)s^{-\gamma} J(s)^p ds & (t_0 \le t \le t_0 + 2), \\ C_2 \left(\displaystyle\int_{t-2}^{t} (t-s)s^{-\gamma} J(s)^p ds + 2 \int_{t_0}^{t-2} s^{-\gamma} J(s)^p ds \right) & (t \ge t_0 + 2). \end{cases}$$

We first note that such a solution $J(t)$ exists on some interval $[t_0, T]$. Indeed, we can rewrite the equation (2.3.26) as

$$J(t) = C_1' + C_2 \int_{t_0}^{t} \Phi(t, s, J(s)) ds$$

with

$$\Phi(t, s, J) = \min\{t - s, 2\} s^{-\gamma} J^p.$$

It is obvious that Φ is locally Lipschitz continuous and we can construct the solution by Picard's iteration argument. Next, we claim that $J(t) < v(t)$. In fact, if this claim fails, then there exists a time $t_* > t_0$ such that

$$\begin{cases} J(t_*) = v(t_*), \\ J(t) < v(t) \qquad (t_0 \le t < t_*). \end{cases}$$

However, by (2.3.24) and (2.3.26), we deduce that

$$\begin{aligned} v(t_*) &\ge C_1 + C_2 \left(\int_{t_*-2}^{t_*} (t-s)s^{-\gamma} v(s)^p ds + 2 \int_{t_0}^{t_*-2} s^{-\gamma} v(s)^p ds \right) \\ &> C_1' + C_2 \left(\int_{t_*-2}^{t_*} (t-s)s^{-\gamma} J(s)^p ds + 2 \int_{t_0}^{t_*-2} s^{-\gamma} J(s)^p ds \right) \\ &= J(t_*), \end{aligned}$$

which leads to a contradiction. Hence $J(t) < v(t)$.

Let us prove that $J(t)$ blows up in finite time. When $t \ge t_0 + 2$, we calculate

$$\begin{aligned} J'(t) &= C_2 \int_{t-2}^{t} s^{-\gamma} J(s)^p ds, \\ J''(t) &= C_2 \left(t^{-\gamma} J(t)^p - (t-2)^{-\gamma} J(t-2)^p \right). \end{aligned}$$

In view of $J(t_0) = C_1' > 0$, we can see that $J(t) > 0$ and $J'(t) > 0$. Thus, using a large parameter $\mu > 0$, we can deduce that

$$\begin{aligned} &J''(t) + \mu J'(t) \\ &= C_2 \left(t^{-\gamma} J(t)^p - (t-2)^{-\gamma} J(t-2)^p + \mu \int_{t-2}^{t} s^{-\gamma} J(s)^p ds \right) \\ &\ge C_2 \left(t^{-\gamma} J(t)^p + \left(2\mu t^{-\gamma} - (t-2)^{-\gamma} \right) J(t-2)^p \right) \\ &\ge C_2 t^{-\gamma} J(t)^p. \end{aligned}$$

Now we put $I(t) = J(\mu^{-1}t)$. Then it follows that

$$\begin{aligned} I''(t) + I'(t) &= \mu^{-2}J'(\mu^{-1}t) + \mu^{-1}J'(\mu^{-1}t) \\ &= \mu^{-2}\left(J''(\mu^{-1}t) + \mu J'(\mu^{-1}t)\right) \\ &\geq \mu^{-2}C_2(\mu^{-1}t)^{-\gamma}J(\mu^{-1}t)^p \\ &= C_2\mu^{\gamma-2}t^{-\gamma}J(\mu^{-1}t)^p, \end{aligned}$$

and hence,

$$I''(t) + I'(t) \geq C_0(1+t)^{-\gamma}I(t)^p$$

holds for $t \geq \mu(t_0+2)$. Therefore, we can apply Proposition 2.13 and obtain that $I(t)$ must blow up in finite time. By noting $I(t) = J(\mu^{-1}t)$ and $J(t) < v(t)$, it follows that $v(t)$ must also blow up in finite time.

When $C_1 = \varepsilon$ is sufficiently small, we put $C_1' = \varepsilon/2$. Then we can deduce that the lifespan $T_\varepsilon(I)$ of $I(t)$ is estimated as

$$T_\varepsilon(I) \leq \begin{cases} C\varepsilon^{-(p-1)/(1-\gamma)} & (0 \leq \gamma < 1), \\ e^{C\varepsilon^{-(p-1)}} & (\gamma = 1). \end{cases}$$

Therefore, the lifespan $T_\varepsilon(v)$ of $v(t)$ is estimated as

$$T_\varepsilon(v) \leq \begin{cases} C\mu^{-1}\varepsilon^{-(p-1)/(1-\gamma)} & (0 \leq \gamma < 1), \\ \mu^{-1}e^{C\varepsilon^{-(p-1)}} & (\gamma = 1). \end{cases}.$$

This completes the proof. □

Let us turn to the Cauchy problem of the semilinear damped wave equation (2.3.15). Let the initial data (u_0, u_1) belong to $C_0^\infty(\mathbf{R}^n) \cap C_0^\infty(\mathbf{R}^n)$ and we consider classical solutions. We recall that a function u is a classical solution of the Cauchy problem (2.3.15) if u has the initial data $u(0,x) = \varepsilon u_0(x), u_t(0,x) = \varepsilon u_1(x)$ and satisfies the equation

$$u_{tt} - \Delta u + u_t = |u|^p$$

at each point $(t,x) \in (0,T) \times \mathbf{R}^n$ with some $T > 0$ (see Section 9.4.3).

We note that a classical solution u of (2.3.15) also becomes a mild solution of (2.3.15) (see Proposition 9.17 and note that u is also strong solution defined in Section 9.4.3, since in this case $(u_0, u_1) \in H^2 \times H^1$ holds). Here we recall the definition of the mild solution of (2.3.15). We say that a function u is a mild solution of the Cauchy problem (2.3.15) if u satisfies the integral equation

$$u(t,x) = \varepsilon\partial_t(S_n(t)u_0(x)) + \varepsilon S_n(t)(u_0+u_1)(x) + \int_0^t S_n(t-s)|u(s,x)|^p ds, \tag{2.3.27}$$

where $S_n(t)$ denotes the solution operator which is expressed in Proposition 2.2. By Proposition 2.2, for $n = 1, 2, 3$ we have

$$\begin{aligned} S_1(t)g(x) &= \frac{e^{-t/2}}{2}\int_{|x-y|\leq t} I_0\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)g(y)dy, \\ S_2(t)g(x) &= \frac{e^{-t/2}}{2\pi}\int_{|x-y|\leq t} \frac{\cosh\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)}{\sqrt{t^2-|x-y|^2}}g(y)dy, \\ S_3(t)g(x) &= \frac{e^{-t/2}}{4\pi t}\partial_t\int_{|x-y|\leq t} I_0\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)g(y)dy. \end{aligned}$$

THEOREM 2.16 (Li and Zhou [**53**], Nishihara [**79**]). *Let $n \leq 3$ and $1 < p \leq 1 + 2/n$. We assume that the initial data (u_0, u_1) are in $C_0^\infty(\mathbf{R}^n)$ and*

$$u_0, u_1 \geq 0, \quad \int_{\mathbf{R}^n} (u_0 + u_1)(x)dx > 0.$$

Then the classical solution u of (2.3.15) *does not exist globally. Moreover, there exists a small $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the lifespan T_ε of u is estimated as*

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-1/\kappa} & (1 < p < 1 + 2/n), \\ e^{C\varepsilon^{-(p-1)}} & (p = 1 + 2/n) \end{cases}$$

with

$$\kappa = \frac{1}{p-1} - \frac{n}{2}.$$

In order to prove this theorem, we employ the following two lemmas

LEMMA 2.17. *Let $n \leq 3, \varepsilon > 0$ and $g(x) \in C_0^\infty(\mathbf{R}^n)$ satisfy $g(x) \geq 0$ and*

$$\int_{\mathbf{R}^n} g(x)dx > 0.$$

Then there exist a constant $C_0 > 0$ and a time $t_0 > 0$ such that for all $|x| \leq \sqrt{t}$ we have

$$S_n(t)(\varepsilon g(x)) \geq C_0 \varepsilon t^{-n/2}$$

for $t \geq t_0$.

Next, we describe an estimate for nonlinear terms. More generally, we consider $F(t, x) \in C([0, \infty) \times \mathbf{R}^n)$ and let u be a function satisfying $u \in C^2([0, \infty) \times \mathbf{R}^n)$ and the identity

$$u(t, x) = \int_0^t S_n(t-s)F(s, x)ds. \tag{2.3.28}$$

LEMMA 2.18. *Let $n \leq 3$ and let $F(t, x)$ be a nonnegative continuous function. We assume that a function $u \in C^2([0, \infty) \times \mathbf{R}^n)$ satisfies the identity* (2.3.28). *Let $t_0 \geq 1$. Then there exists a constant $C > 0$ such that for all $|x| \leq \sqrt{t}$ we have*
(2.3.29)

$$t^{n/2}u(t, x) \geq \begin{cases} C\displaystyle\int_{t_0}^t (t-s)G(s)ds & (t_0 \leq t \leq t_0 + 2), \\ C\left(\displaystyle\int_{t-2}^t (t-s)G(s)ds + 2\int_{t_0}^{t-2} G(s)ds\right) & (t \geq t_0 + 2), \end{cases}$$

where

$$G(s) = \inf_{|y| \leq \sqrt{s}} \left(s^{n/2}F(s, y)\right).$$

We postpone the proof of the above lemmas and give a proof of Theorem 2.16.

PROOF OF THEOREM 2.16. First, we note that the classical solution u of (2.3.15) also satisfies the integral equation (2.3.27). Moreover, by Proposition 2.1, we have $\|\partial_t(S_n(t)u_0)\|_{L^\infty} \leq C(1+t)^{-n/2-1}$. By the assumption on the data and Lemma 2.17, we see that

$$\varepsilon S_n(t)(u_0 + u_1) \geq C_0 \varepsilon t^{-n/2} \quad (t \geq t_0)$$

holds with sufficiently large $t_0 \geq 1$. Hence, retaking t_0 larger if needed, we obtain

$$\varepsilon \partial_t (S_n(t) u_0) + \varepsilon S_n(t)(u_0 + u_1) \geq C_1 \varepsilon t^{-n/2}$$

for $t \geq t_0$ with some constant $C_1 > 0$. Combining this with Lemma 2.18 with $F(t,x) = |u(t,x)|^p$, we can see that there exists a constant $C_2 > 0$ such that

$$\begin{aligned} &t^{n/2} u(t,x) \geq C_1 \varepsilon \\ &+ C_2 \begin{cases} \displaystyle\int_{t_0}^{t} (t-s) s^{-n(p-1)/2} V(s)^p ds & (t_0 \leq t \leq t_0 + 2), \\ \displaystyle\left(\int_{t-2}^{t} (t-s) s^{-n(p-1)/2} V(s)^p ds + 2 \int_{t_0}^{t-2} s^{-n(p-1)/2} V(s)^p ds \right) & (t \geq t_0 + 2) \end{cases} \end{aligned}$$

for $|x| \leq \sqrt{t}$, where

$$V(s) = \inf_{|y| \leq \sqrt{s}} \left(s^{n/2} |u(s,y)| \right).$$

Hence,

$$\begin{aligned} &V(t) \geq C_1 \varepsilon \\ &+ C_2 \begin{cases} \displaystyle\int_{t_0}^{t} (t-s) s^{-n(p-1)/2} V(s)^p ds & (t_0 \leq t \leq t_0 + 2), \\ \displaystyle\left(\int_{t-2}^{t} (t-s) s^{-n(p-1)/2} V(s)^p ds + 2 \int_{t_0}^{t-2} s^{-n(p-1)/2} V(s)^p ds \right) & (t \geq t_0 + 2). \end{cases} \end{aligned}$$

Therefore, we can apply Lemma 2.15 and obtain the desired conclusion. □

Proof of Lemma 2.17. We assume that $\operatorname{supp} g \subset \{x \in \mathbf{R}^n \mid |x| \leq R\}$ with some $R > 0$. Then for $|x| \leq \sqrt{t}$ and $y \in \operatorname{supp} g$, we have $|x-y| \leq |x| + |y| \leq \sqrt{t} + R$. Noting this, we have

$$t^2 \geq t^2 - |x-y|^2 \geq t^2 - (\sqrt{t} + R)^2 \geq \frac{1}{2} t^2$$

for sufficiently large t. When $n = 1$, by using the asymptotic expansion of the modified Bessel function (see Lemma 9.5)

$$I_0(s) = \frac{1}{\sqrt{2\pi s}} e^s \left(1 + O(s^{-1})\right),$$

we can see that

$$\begin{aligned} S_1(t)(\varepsilon g(x)) &= \frac{e^{-t/2}}{2} \int_{|x-y| \leq t} \frac{1}{\pi^{1/2} (t^2 - |x-y|^2)^{1/4}} e^{-\frac{1}{2}\sqrt{t^2 - |x-y|^2}} \\ &\quad \times \left(1 + O\left(\frac{1}{\sqrt{t^2 - |x-y|^2}} \right) \right) \varepsilon g(y) dy \\ &\geq t^{-1/2} \frac{1}{4\sqrt{\pi}} \int_{|x-y| \leq t} e^{-\frac{1}{2}(t - \sqrt{t^2 - |x-y|^2})} \varepsilon g(y) dy \\ &\geq t^{-1/2} \frac{1}{4e\sqrt{\pi}} \varepsilon \int_{\mathbf{R}^n} g(y) dy \\ &= C_0 \varepsilon t^{-1/2} \end{aligned}$$

for sufficiently large t. Here we have used that

$$1+O\left(\frac{1}{\sqrt{t^2-|x-y|^2}}\right)=1+O(t^{-1})\geq\frac{1}{2}$$

is true for large t and that

$$e^{-\frac{1}{2}(t-\sqrt{t^2-|x-y|^2})}=e^{-\frac{1}{2}\frac{|x-y|^2}{t+\sqrt{t^2-|x-y|^2}}}\geq e^{-\frac{|x-y|^2}{2t}}\geq e^{-\frac{(\sqrt{t}+R)^2}{2t}}\geq e^{-1}$$

holds for large t and $|x|\leq\sqrt{t}, |y|\leq R$.

The case of $n=3$ is proved by the almost same way. Indeed, by the decomposition (2.2.16), we can write S_3 as

$$S_3(t)(\varepsilon g(x))=e^{-t/2}W_3(t)(\varepsilon g(x))+J_3(t)(\varepsilon g(x)),$$

where

$$W_3(t)(\varepsilon g(x))=\frac{1}{4\pi t}\int_{|x-y|=t}\varepsilon g(y)dS_y$$

and

$$J_3(t)(\varepsilon g(x))=\frac{e^{-t/2}}{8\pi}\int_{|x-y|\leq t}\frac{I_1(\frac{1}{2}\sqrt{t^2-|x-y|^2})}{\sqrt{t^2-|x-y|^2}}\varepsilon g(y)dy.$$

We note that if t is sufficiently large and $|x|\leq\sqrt{t}, |y|\leq R$, then $W_3(t)(\varepsilon g(x))=0$. Therefore, we have to prove only the estimate for $J_0(t)(\varepsilon g(x))$. By noting that

$$I_1(s)=\frac{1}{\sqrt{2\pi s}}e^s\left(1+O(s^{-1})\right),$$

the desired estimate can be proved by the same way to the case of $n=1$. Thus, we omit the detail.

When $n=2$, as before, we note that

$$e^{-t/2}\cosh\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)\geq\frac{1}{2}e^{-\frac{1}{2}(t-\sqrt{t^2-|x-y|^2})}\geq\frac{1}{2e}$$

for sufficiently large t and $|x|\leq\sqrt{t}, |y|\leq R$. Hence, we can immediately obtain

$$S_2(t)(\varepsilon g(x))\geq\frac{1}{4e\pi}t^{-1}\int_{\mathbf{R}^n}\varepsilon g(y)dy=C_0\varepsilon t^{-1},$$

which completes the proof. □

Proof of Lemma 2.18. For the sake of convenience, we put

$$k_n(t,r)=\begin{cases}\frac{1}{2}e^{-t/2}I_0\left(\frac{1}{2}\sqrt{t^2-r^2}\right) & (n=1),\\ \frac{1}{2\pi}e^{-t/2}\frac{\cosh(\frac{1}{2}\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}} & (n=2),\\ \frac{1}{4\pi}e^{-t/2}\left(\frac{1}{t}\delta_{r=t}+\frac{1}{8\pi}\frac{I_1(\frac{1}{2}\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}}\right) & (n=3),\end{cases}$$

where $\delta_{r=t}$ denotes the Dirac measure on the point $r = t$. Let $t_0 \ge 1$. We first assume $t \ge t_0 + 2$. Using the above notation, we can calculate

$$
\begin{aligned}
u(t,x) &= \int_0^t S_n(t-s)F(s,x)ds \\
&= \int_0^t \int_{|x-y|\le t-s} k_n(t-s,|x-y|)F(s,y)dyds \\
&\ge \int_{t-2}^t \frac{G(s)}{s^{n/2}} \left(\int_{\substack{|x-y|\le t-s \\ |y|\le\sqrt{s}}} k_n(t-s,|x-y|)dy \right) ds \\
&\quad + \int_{t_0}^{t-2} \frac{G(s)}{s^{n/2}} \left(\int_{\substack{|x-y|\le\sqrt{t-s} \\ |y|\le\sqrt{s}}} k_n(t-s,|x-y|)dy \right) ds.
\end{aligned}
\tag{2.3.30}
$$

We note that the integral region $\{y \in \mathbf{R}^n \mid |x-y| \le t-s,\ |y| \le \sqrt{s}\}$ of the first integral includes a ball with the diameter $(t-s)/2$ if $t \ge 1, |x| \le \sqrt{t}$. In fact,

$$
\sqrt{s} - (|x| - (t-s)) \ge t - s + \sqrt{s} - \sqrt{t} = t - s - \frac{t-s}{\sqrt{t}+\sqrt{s}} \ge \frac{t-s}{2}.
$$

In a similar way, we can also prove that the integral region $\{y \in \mathbf{R}^n \mid |x - y| \le \sqrt{t-s},\ |y| \le \sqrt{s}\}$ of the second integral includes a ball with the diameter $\sqrt{t-s}\sqrt{s}/(2\sqrt{t})$ if $|x| \le \sqrt{t}$. Indeed,

$$
\begin{aligned}
\sqrt{s} - (|x| - \sqrt{t-s}) &\ge \sqrt{s} - \sqrt{t} + \sqrt{t-s} = \sqrt{t-s} - \frac{t-s}{\sqrt{t}+\sqrt{s}} \\
&= \sqrt{t-s}\left(\frac{\sqrt{t}+\sqrt{s}-\sqrt{t-s}}{\sqrt{t}+\sqrt{s}} \right) \ge \frac{\sqrt{t-s}\sqrt{s}}{\sqrt{t}+\sqrt{s}} \ge \frac{\sqrt{t-s}\sqrt{s}}{2\sqrt{t}}.
\end{aligned}
$$

Next, we find estimates of $k_n(t-s,|x-y|)$ from below. We first treat the case of $n = 1$. When $0 \le t - s \le 2$, noting that $I_0(s) \ge 1$, we see that

$$
\begin{aligned}
k_1(t-s,|x-y|) &= \frac{1}{2}e^{-(t-s)/2} I_0\left(\frac{1}{2}\sqrt{(t-s)^2 - |x-y|^2} \right) \\
&\ge \frac{1}{2e}.
\end{aligned}
$$

On the other hand, when $t - s \ge 2$ and $|x - y| \le \sqrt{t-s}$, noting that

$$
I_0(s) \ge cs^{-1/2}e^s
$$

holds for any $s \ge 1/2$ with some constant $c > 0$, we can see that

$$
\begin{aligned}
k_1(t-s,|x-y|) &= \frac{1}{2}e^{-(t-s)/2} I_0\left(\frac{1}{2}\sqrt{(t-s)^2 - |x-y|^2} \right) \\
&\ge \frac{c}{2}\left(\frac{1}{2}\sqrt{(t-s)^2 - |x-y|^2} \right)^{-1/2} e^{-\frac{1}{2}((t-s)-\sqrt{(t-s)^2-|x-y|^2})} \\
&\ge \frac{c}{2}(t-s)^{-1/2} e^{-\frac{|x-y|^2}{2(t-s)}} \\
&\ge \frac{c}{2e^{1/2}}(t-s)^{-1/2}.
\end{aligned}
$$

Combining these estimates with (2.3.30), we obtain

$$\begin{aligned}
u(t,x) &\geq C\int_{t-2}^{t}\frac{G(s)}{\sqrt{s}}\left(\int_{\substack{|x-y|\leq t-s\\ |y|\leq\sqrt{s}}}dy\right)ds\\
&\quad + C\int_{t_0}^{t-2}\frac{G(s)}{\sqrt{s}\sqrt{t-s}}\left(\int_{\substack{|x-y|\leq\sqrt{t-s}\\ |y|\leq\sqrt{s}}}dy\right)ds\\
&\geq C\int_{t-2}^{t}(t-s)\frac{G(s)}{\sqrt{s}}ds + C\int_{t_0}^{t-2}\frac{G(s)}{\sqrt{t}}ds\\
&\geq Ct^{-1/2}\left(\int_{t-2}^{t}(t-s)G(s)ds + \int_{t_0}^{t-2}G(s)ds\right)
\end{aligned}$$

and hence, we obtain the desired estimate in the case of $n=1$ and $t\geq t_0+2$. When $t_0\leq t\leq t_0+2$, instead of (2.3.30), we obtain

$$u(t,x)\geq\int_{t_0}^{t}\frac{G(s)}{s^{1/2}}\left(\int_{\substack{|x-y|\leq t-s\\ |y|\leq\sqrt{s}}}k_1(t-s,|x-y|)dy\right)ds.$$

As before, by noting that the integral region includes a ball with radius $(t-s)/2$ and that

$$k_1(t-s,|x-y|)\geq\frac{1}{2e},$$

we can deduce that

$$u(t,x)\geq Ct^{-1/2}\int_{t_0}^{t}(t-s)G(s)ds,$$

which shows the assertion on the lemma in the case of $n=1$.

We turn to the case of $n=2$. We first assume that $t\geq t_0+2$. When $0\leq t-s\leq 2$, noting that $\cosh(r)\geq 1$, we have

$$\begin{aligned}
k_2(t-s,|x-y|) &= \frac{1}{2\pi}e^{-(t-s)/2}\frac{\cosh(\frac{1}{2}\sqrt{(t-s)^2-|x-y|^2})}{\sqrt{(t-s)^2-|x-y|^2}}\\
&\geq\frac{1}{2e\pi}(t-s)^{-1}.
\end{aligned}$$

When $t-s\geq 2$ and $|x-y|\leq\sqrt{t-s}$, we can deduce that

$$\begin{aligned}
k_2(t-s,|x-y|) &\geq\frac{1}{2\pi}(t-s)^{-1}e^{-\frac{1}{2}(t-s-\sqrt{(t-s)^2-|x-y|^2})}\\
&\geq\frac{1}{2\pi}(t-s)^{-1}e^{-\frac{|x-y|^2}{2(t-s)}}\\
&\geq C(t-s)^{-1}.
\end{aligned}$$

Using these estimates and (2.3.30), we have

$$\begin{aligned}
u(t,x) &\geq C\int_{t-2}^{t}\frac{G(s)}{s(t-s)}\left(\int_{\substack{|x-y|\leq t-s\\ |y|\leq\sqrt{s}}}dy\right)ds\\
&\quad+C\int_{t_0}^{t-2}\frac{G(s)}{s(t-s)}\left(\int_{\substack{|x-y|\leq\sqrt{t-s}\\ |y|\leq\sqrt{s}}}dy\right)ds\\
&\geq C\int_{t-2}^{t}(t-s)\frac{G(s)}{s}ds+C\int_{t_0}^{t-2}\frac{G(s)}{t}ds\\
&\geq Ct^{-1}\left(\int_{t-2}^{t}(t-s)G(s)ds+\int_{t_0}^{t-2}G(s)ds\right).
\end{aligned}$$

In a similar way, we can see that if $t_0\leq t\leq t_0+2$, then it follows that

$$u(t,x)\geq Ct^{-1}\int_{t_0}^{t}(t-s)G(s)ds$$

for $|x|\leq\sqrt{t}$. This proves the conclusion of the lemma in the case of $n=2$.

Finally, we treat the case of $n=3$. We first assume that $t\geq t_0+2$. We deduce that

$$\begin{aligned}
u(t,x) &\geq C\int_{t-2}^{t}\frac{G(s)}{s^{3/2}}e^{-(t-s)/2}\frac{1}{t-s}\left(\int_{\substack{|x-y|=t-s\\ |y|\leq\sqrt{s}}}dS_y\right)ds\\
&\quad+C\int_{t_0}^{t-2}\frac{G(s)}{s^{3/2}}\int_{\substack{|x-y|\leq\sqrt{t-s}\\ |y|\leq\sqrt{s}}}e^{-(t-s)/2}\frac{I_1(\frac{1}{2}\sqrt{(t-s)^2-|x-y|^2})}{\sqrt{(t-s)^2-|x-y|^2}}dyds\\
&\geq C\int_{t-2}^{t}(t-s)\frac{G(s)}{s^{3/2}}ds\\
&\quad+C\int_{t_0}^{t-2}\frac{G(s)}{s^{3/2}(t-s)^{3/2}}\int_{\substack{|x-y|\leq\sqrt{t-s}\\ |y|\leq\sqrt{s}}}e^{-\frac{1}{2}(t-s-\sqrt{(t-s)^2-|x-y|^2})}dyds\\
&\geq C\int_{t-2}^{t}(t-s)\frac{G(s)}{s^{3/2}}ds\\
&\quad+C\int_{t_0}^{t-2}\frac{G(s)}{s^{3/2}(t-s)^{3/2}}\int_{\substack{|x-y|\leq\sqrt{t-s}\\ |y|\leq\sqrt{s}}}dyds\\
&\geq Ct^{-3/2}\left(\int_{t-2}^{t}(t-s)G(s)ds+\int_{t_0}^{t-2}G(s)ds\right),
\end{aligned}$$

which indicates the desired estimate. When $t_0\leq t\leq t_0+2$, in a similar way, we obtain

$$u(t,x)\geq Ct^{-3/2}\int_{t_0}^{t}(t-s)G(s)ds$$

instead of the above estimate. This completes the proof. □

2.3.4. Blow-up by test function method. In this subsection, we prove a blow-up result for a semilinear damped wave equation by using a test function method developed by Zhang [**122**]. We consider the Cauchy problem for the semilinear damped wave equation

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n. \end{cases} \tag{2.3.31}$$

We shall prove that if p is critical or subcritical

$$p \le p_c = 1 + \frac{2}{n},$$

then, in general, there is no global-in-time solution even if the initial data is sufficiently small. The argument introduced in this subsection is applicable for any space dimensions. However, we cannot obtain estimates of the lifespan of solutions in the critical case $p = p_c$ (for subcritical cases, we discuss about estimates of the lifespan in Chapter 7). We shall prove the following:

THEOREM 2.19. *If* $1 < p \le 1 + 2/n$ *and* $(u_0, u_1) \in (C^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)) \times (C^1(\mathbf{R}^n) \cap L^1(\mathbf{R}^n))$ *satisfies*

$$\int_{\mathbf{R}^n} (u_0(x) + u_1(x))dx > 0,$$

then the classical solution of (2.3.31) *does not exist globally.*

PROOF. Suppose that u is a global classical solution of (2.3.31). We define test functions

$$\phi(x) = \begin{cases} 1 & (|x| \le 1/2) \\ \dfrac{\exp(-1/(1-|x|^2))}{\exp(-1/(|x|^2-1/4)) + \exp(-1/(1-|x|^2))} & (1/2 < |x| < 1), \\ 0 & (|x| \ge 1), \end{cases}$$

$$\eta(t) = \begin{cases} 1 & (0 \le t \le 1/2), \\ \dfrac{\exp(-1/(1-t^2))}{\exp(-1/(t^2-1/4)) + \exp(-1/(1-t^2))} & (1/2 < t < 1), \\ 0 & (t \ge 1). \end{cases}$$

It is obvious that $\phi \in C_0^\infty(\mathbf{R}^n), \eta \in C_0^\infty([0,\infty))$. We also see that

$$|\eta'(t)| \lesssim \eta(t)^{1/p}, \quad |\eta''(t)| \lesssim \eta(t)^{1/p}, \quad |\Delta\phi(x)| \lesssim \phi(x)^{1/p}.$$

Indeed, let q, r satisfy $1/p + 1/q = 1, 1/p + 2/r = 1$ and let $\mu = \eta^{1/q}, \nu = \eta^{1/r}$. Then we have

$$|\eta'| = |(\mu^q)'| = |q\mu^{q-1}\mu'| \lesssim \mu^{q-1} = \eta^{1/p}$$

and

$$|\eta''| = |(\nu^r)'| \lesssim |\nu''|\nu^{r-1} + |\nu'|^2\nu^{r-2} \lesssim \nu^{r-2} = \eta^{1/p}.$$

The estimate for $\Delta\phi$ can be proved in the same way. Let $R > 1$ be a large parameter and let

$$\psi_R(t,x) = \eta(t/R^2)\phi(x/R).$$

We also define

$$I_R = \int_0^\infty \int_{\mathbf{R}^n} |u|^p \psi_R dxdt.$$

We note that by the Lebesgue dominated convergence theorem, it follows that

$$\int_{\mathbf{R}^n} (u_0(x) + u_1(x))\phi(x/R)dx > 0$$

for sufficiently large $R > 0$. Then by the equation, integration by parts and the Hölder inequality, we can calculate

$$\begin{aligned} I_R &= \int_0^\infty \int_{\mathbf{R}^n} (u_{tt} - \Delta u + u_t)\psi_R dxdt \\ &= \int_0^\infty \int_{\mathbf{R}^n} u(\partial_t^2 - \Delta - \partial_t)\psi_R dxdt - \int_{\mathbf{R}^n} (u_0(x) + u_1(x))\phi(x/R)dx \\ &\le \int_0^\infty \int_{\mathbf{R}^n} |u|(|\partial_t^2 \psi_R| + |\Delta\psi_R| + |\partial_t \psi_R|)dxdt \equiv K_1 + K_2 + K_3. \end{aligned}$$

We put

$$\hat{I}_R := \int_{R^2/2}^{R^2} \int_{|x|<R} |u|^p \psi_R(t,x)dxdt, \quad \tilde{I}_R := \int_0^{R^2} \int_{R/2<|x|<R} |u|^p \psi_R(t,x)dxdt.$$

By noting that

$$\begin{aligned} &\operatorname{supp} \partial_t \psi_R \subset \{(t,x) \in [0,\infty) \times \mathbf{R}^n \mid R^2/2 \le t \le R^2,\ |x| \le R\}, \\ &\operatorname{supp} \Delta\psi_R \subset \{(t,x) \in [0,\infty) \times \mathbf{R}^n \mid 0 \le t \le R^2,\ R/2 \le |x| \le R\} \end{aligned}$$

and the estimates

$$|\partial_t^2 \psi_R| \le CR^{-4}\psi_R(t,x)^{1/p}, \quad |\Delta\psi_R| \le CR^{-2}\psi_R(t,x)^{1/p}$$

and

$$|\partial_t \psi_R| \le CR^{-2}\psi_R(t,x)^{1/p}$$

hold, it follows from the Hölder inequality that

$$\begin{aligned} K_1 + K_3 &\le CR^{-2} \int_{R^2/2}^{R^2} \int_{|x|<R} |u|\psi_R^{1/p} dxdt \\ &\le CR^{-2} \left(\int_{R^2/2}^{R^2} \int_{|x|<R} |u|^p \psi_R dxdt \right)^{1/p} \left(\int_{R^2/2}^{R^2} \int_{|x|<R} dxdt \right)^{1/q} \\ &\le CR^{-2+(n+2)/q} \hat{I}_R^{1/p}, \end{aligned}$$

where q denotes the Hölder conjugate of p, that is $1/p + 1/q = 1$. In a similar way, we have

$$K_2 \le CR^{-2+(n+2)/q} \tilde{I}_R^{1/p}.$$

Combining the above two estimates, one can obtain

$$I_R \le CR^{-2+(n+2)/q} \left(\hat{I}_R^{1/p} + \tilde{I}_R^{1/p} \right). \tag{2.3.32}$$

In particular, using a trivial inequality $\hat{I}_R \le I_R, \tilde{I}_R \le I_R$, we deduce that

$$I_R \le CR^{-2+(n+2)/q} I_R^{1/p}$$

and hence,

$$I_R^{1-1/p} \le CR^{-2+(n+2)/q}. \tag{2.3.33}$$

We first consider the subcritical case $1 < p < 1 + 2/n$. Noting that $p < 1 + 2/n$ if and only if $-2 + (n+2)/q < 0$, the right-hand side of the above inequality tends to 0 as $R \to +\infty$. In particular, I_R is bounded when $R \to +\infty$ and this implies $u \in L^p((0,\infty) \times \mathbf{R}^n)$ and $\lim_{R\to\infty} I_R = \|u\|^p_{L^p((0,\infty)\times\mathbf{R}^n)}$. However, by using the above estimate again, it follows that $\lim_{R\to\infty} I_R = 0$. This means $u = 0$, which contradicts $(u_0, u_1) \neq 0$.

In the critical case $p = 1 + 2/n$, we can obtain only the boundedness of I_R from (2.3.33). However, $u \in L^p((0,\infty) \times \mathbf{R}^n)$ and $\lim_{R\to\infty} I_R = \|u\|^p_{L^p((0,\infty)\times\mathbf{R}^n)}$ is still true. From this, we can see that

$$\lim_{R\to\infty} \hat{I}_R = 0, \quad \lim_{R\to\infty} \tilde{I}_R = 0.$$

This and the estimate (2.3.32) imply

$$\lim_{R\to\infty} I_R = 0,$$

which shows $u \equiv 0$ and contradicts $(u_0, u_1) \neq 0$ again. □

CHAPTER 3

On diffusion phenomena for the linear wave equation with space-dependent damping

3.1. Introduction and results

In this chapter, we consider the asymptotic behavior of solutions to the wave equation with space-dependent damping:

$$\begin{cases} u_{tt} - \Delta u + a(x)u_t = 0, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^n. \end{cases} \tag{3.1.1}$$

Here $u = u(t,x)$ is real-valued unknown function and $a(x) = \langle x \rangle^{-\alpha} := (1+|x|^2)^{-\alpha/2}$ with $0 \le \alpha < 1$. For simplicity, we assume that

$$(u_0, u_1) \in C_0^\infty(\mathbf{R}^n), \quad \operatorname{supp}(u_0, u_1) \subset \{x \in \mathbf{R}^n \mid |x| \le L\} \tag{3.1.2}$$

with some $L > 0$. We also consider the Cauchy problem of the corresponding heat equation started at the time $\tau \ge 0$:

$$\begin{cases} a(x)v_t - \Delta v = 0, & (t,x) \in (\tau,\infty) \times \mathbf{R}^n, \\ v(\tau,x) = v_\tau(x), & x \in \mathbf{R}^n \end{cases} \tag{3.1.3}$$

with initial data $v_\tau(x) \in C_0^\infty(\mathbf{R}^n)$.

When $a(x) = 1$, as we mentioned in Section 1.3, the asymptotic profile of the solution of (3.1.1) is given by the solution of (3.1.3) with the initial data $v_0 = u_0 + u_1$ in several senses (see [**23, 27, 58, 73, 78, 121**] and see also [**3, 32, 97**] for abstract setting). On the other hand, Wirth [**117**] considered the wave equation with time-dependent damping

$$u_{tt} - \Delta u + b(t)u_t = 0.$$

He proved that if the damping is effective, that is, roughly speaking, $tb(t) \to +\infty$ as $t \to +\infty$ and $b(t)^{-1} \notin L^1((0,\infty))$, then the solution is asymptotically equivalent to that of the corresponding heat equation

$$b(t)v_t - \Delta v = 0$$

(see also [**120**] for abstract setting). We also mention that Ikehata, Todorova and Yordanov [**38**] recently proved the diffusion phenomenon for strongly damped wave equations.

Recently, Nishiyama [**88**] proved the diffusion phenomenon for abstract damped wave equations. His result includes space-dependent damping which does not decay near infinity. Due to the authors knowledge, there are no results on the asymptotic profile of solutions for decaying potential cases as (3.1.1). The difficulty is that we cannot use the Fourier transform for (3.1.1) as the previous results.

We also refer the reader to [**18, 19, 21, 20, 33, 40, 80, 84**] for the asymptotic profile for semilinear problems.

Todorova and Yordanov [**107**] obtained the following L^2-estimate for (3.1.1):

$$\|u(t,\cdot)\|_{L^2} \le C(1+t)^{-\frac{n-2\alpha}{2(2-\alpha)}+\varepsilon}, \tag{3.1.4}$$

where $\varepsilon > 0$ is arbitrary small number (see also [**37**] for the case $\alpha = 1$). They proved the above estimate by using a weighted energy method. Following their method, we can also deduce an L^2-estimate for (3.1.3) without any loss $\varepsilon > 0$:

$$\|v(t,\cdot)\|_{L^2} \le C(1+t)^{-\frac{n-2\alpha}{2(2-\alpha)}}. \tag{3.1.5}$$

We will give a proof of this inequality in the next section.

It seems that the decay rate $\frac{n-2\alpha}{2(2-\alpha)}$ is optimal. Because, the function

$$G(t,x) = t^{-\frac{n-\alpha}{2-\alpha}} e^{-\frac{|x|^{2-\alpha}}{(2-\alpha)^2 t}}$$

formally satisfies the equation

$$|x|^{-\alpha} v_t - \Delta v = 0$$

and

$$\|G(t,\cdot)\|_{L^2} = Ct^{-\frac{n-2\alpha}{2(2-\alpha)}}$$

with some constant $C > 0$. Indeed,

$$\begin{aligned}
\|G(t,\cdot)\|_{L^2}^2 &= t^{-\frac{2(n-\alpha)}{2-\alpha}} \int_{\mathbf{R}^n} e^{-\frac{2|x|^{2-\alpha}}{(2-\alpha)^2 t}} dx \\
&= Ct^{-\frac{2(n-\alpha)}{2-\alpha}+\frac{n}{2-\alpha}} \int_{\mathbf{R}^n} e^{-\frac{2|y|^{2-\alpha}}{(2-\alpha)^2}} dy \\
&= Ct^{-\frac{n-2\alpha}{2-\alpha}}.
\end{aligned}$$

We denote the solution operator of (3.1.3) by $E(t-\tau)$, that is, $v(t,x) = E(t-\tau)v_\tau(x)$ gives the solution of (3.1.3). It is known that $E(t-\tau)$ is a 0-th order pseudodifferential operator having the symbol

$$e(t-\tau,x,\xi) = e^{-\frac{|\xi|^2}{a(x)}(t-\tau)} + r_0(t-\tau,x,\xi)$$

with a remainder term r_0 (see Kumano-go [**49**]). The main result of this chapter is the following.

THEOREM 3.1. *Let $n \ge 1$ and let u be a solution of* (3.1.1) *with initial data* (u_0,u_1) *satisfying* (3.1.2). *Then we have*

$$\left\| u(t,\cdot) - E(t)\left[u_0 + \frac{1}{a(\cdot)}u_1\right](\cdot)\right\|_{L^2} = o(t^{-\frac{n-2\alpha}{2(2-\alpha)}}) \tag{3.1.6}$$

as $t \to +\infty$.

REMARK 3.1. *(i)Our proof needs the compactness of the support of the data. However, this assumption may be removed by using the energy concentration lemma (see Lemma 3.4); but we do not pursue that here.*

(ii)Combining the above theorem and the estimate (3.1.5)*, we can remove the loss of decay rate ε from the estimate* (3.1.4) *by Todorova and Yordanov.*

The crucial point of the proof of Theorem 1 is the following weighted energy estimates for higher order derivatives of solutions to (3.1.1). Let

$$\psi(t,x) = A\frac{\langle x\rangle^{2-\alpha}}{1+t}, \quad A := \frac{1}{(2-\alpha)^2(2+\delta)} \tag{3.1.7}$$

with small $\delta > 0$. We also put

$$I_0 = \int_{\mathbf{R}^n} e^{2\psi(0,x)}(u_0(x)^2 + |\nabla u_0(x)|^2 + |u_1(x)|^2)dx,$$
$$I_1 = \int_{\mathbf{R}^n} e^{2\psi(0,x)}(u_{tt}(0,x)^2 + |\nabla u_t(0,x)|^2)dx + I_0,$$
$$I_2 = \int_{\mathbf{R}^n} e^{2\psi(0,x)}(u_{ttt}(0,x)^2 + |\nabla u_{tt}(0,x)|^2)dx + I_1$$

and by inductively

$$I_k = \int_{\mathbf{R}^n} e^{2\psi(0,x)}(|\partial_t^{k+1}u(0,x)|^2 + |\nabla\partial_t^k u(0,x)|^2)dx + I_{k-1}.$$

Then, we can obtain weighted energy estimates for any order of derivatives:

THEOREM 3.2 (weighted energy estimates for higher order derivatives). *For any small $\varepsilon > 0$, there is some $\delta > 0$ such that the following estimates hold: For any integer $k \geq 0$, there exists some constant $C > 0$ such that for a solution u of* (3.1.1) *with initial data satisfying* (3.1.2), *we have*

$$(1+t)^{\frac{n-\alpha}{2-\alpha}+2k-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(t,x)}a(x)|\partial_t^k u(t,x)|^2dx \leq CI_k, \tag{3.1.8}$$

$$(1+t)^{\frac{n-\alpha}{2-\alpha}+2k+1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(t,x)}|\nabla\partial_t^k u(t,x)|^2dx \leq CI_k. \tag{3.1.9}$$

In particular, we use the following estimates for the proof of Theorem 3.1.

LEMMA 3.3. *For any small $\varepsilon > 0$, there are some constants $\delta > 0$ and $C > 0$ such that the following estimates hold:*

(i) *For a solution u of* (3.1.1), *we have*

$$(1+t)^{\frac{n-\alpha}{2-\alpha}+2-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(t,x)}a(x)u_t(t,x)^2dx \leq CI_1, \tag{3.1.10}$$

$$(1+t)^{\frac{n-\alpha}{2-\alpha}+3-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(t,x)}|\nabla u_t(t,x)|^2dx \leq CI_1, \tag{3.1.11}$$

$$(1+t)^{\frac{n-\alpha}{2-\alpha}+4-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(t,x)}a(x)u_{tt}(t,x)^2dx \leq CI_2, \tag{3.1.12}$$

$$(1+t)^{\frac{n-\alpha}{2-\alpha}+6-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(t,x)}a(x)u_{ttt}(t,x)^2dx \leq CI_3. \tag{3.1.13}$$

(ii) *For a solution v of* (3.1.3), *we have*

$$\int_{\mathbf{R}^n} a(x)|v(t,x)|^2dx \leq \int_{\mathbf{R}^n} a(x)|v_\tau(x)|^2dx, \tag{3.1.14}$$

$$\begin{aligned}(1+t-\tau)^{\frac{n-\alpha}{2-\alpha}+2-\varepsilon}&\int_{\mathbf{R}^n} a(x)|v_t(t,x)|^2dx \\ \leq &\int_{\mathbf{R}^n} a(x)^{-1}|\Delta v_\tau(x)|^2dx \\ &+ C(1+\tau)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}|\nabla v_\tau(x)|^2dx \\ &+ C(1+\tau)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}a(x)v_\tau(x)^2dx.\end{aligned} \tag{3.1.15}$$

We also use an energy concentration lemma.

LEMMA 3.4 (exponential decay outside parabolic regions). *For any small* $\varepsilon > 0$, *there is some* $\delta > 0$ *such that the following holds: let*

$$0 < \rho < 1 - \alpha, \quad 0 < \mu < 2A$$

and

$$\Omega_\rho(t) := \{x \in \mathbf{R}^n \mid \langle x \rangle^{2-\alpha} \geq (1+t)^{1+\rho}\}.$$

We also assume that v *is a solution of* (3.1.3) *with* $\tau = 0$. *Then we have*

$$\int_{\Omega_\rho(t)} v(t,x)^2 dx \leq C(1+t)^{\frac{\alpha}{2-\alpha}} e^{-(2A-\mu)(1+t)^\rho} \int_{\mathbf{R}^n} e^{2\psi(0,x)} a(x) v_0(x)^2 dx, \tag{3.1.16}$$

where $C > 0$ *is a constant depending on* δ, ρ *and* μ.

In the next section, we introduce a basic weighted energy method. This method was originally developed by Todorova and Yordanov [**106**] and refined by themselves [**107**] and Nishihara [**81**] to fit for the space-dependent damping. In Section 3.3, we prove Lemmas 3.3 and 3.4 and Theorem 3.2 by using the basic weighted energy estimates obtained in Section 3.2. In the final section, we give a proof of the main theorem.

3.2. Basic weighted energy estimates

In this section, we first give a proof of the estimate (3.1.5) for solutions to (3.1.3). Let

$$\psi_0(t,x) = \frac{\langle x \rangle^{2-\alpha}}{2(2-\alpha)^2(1+t)}.$$

We prove the following:

PROPOSITION 3.5.

$$(1+t)^{\frac{n-\alpha}{2-\alpha}} \int_{\mathbf{R}^n} e^{2\psi_0(t,x)} a(x) v(t,x)^2 dx \leq \int_{\mathbf{R}^n} e^{2\psi_0(0,x)} a(x) v_0(x)^2 dx.$$

PROOF. A straightforward calculation gives

$$-\partial_t \psi_0(t,x) = \frac{\langle x \rangle^{2-\alpha}}{2(2-\alpha)^2(1+t)^2}, \tag{3.2.1}$$

$$\nabla \psi_0(t,x) = \frac{\langle x \rangle^{-\alpha} x}{2(2-\alpha)(1+t)}, \tag{3.2.2}$$

$$\begin{aligned} \Delta \psi_0 &= \frac{n\langle x \rangle^{-\alpha}}{2(2-\alpha)(1+t)} - \frac{\alpha \langle x \rangle^{-\alpha-2}|x|^2}{2(2-\alpha)(1+t)} \\ &= \frac{(n-\alpha)\langle x \rangle^{-\alpha}}{2(2-\alpha)(1+t)} + \frac{\alpha \langle x \rangle^{-\alpha-2}}{2(2-\alpha)(1+t)} \\ &\geq \frac{(n-\alpha)\langle x \rangle^{-\alpha}}{2(2-\alpha)(1+t)}. \end{aligned} \tag{3.2.3}$$

By (3.2.1) and (3.2.2), we can easily obtain

$$-\partial_t \psi_0(t,x) a(x) \geq 2|\nabla \psi_0(t,x)|^2. \tag{3.2.4}$$

Multiplying the equation (3.1.3) by $e^{2\psi_0} v$, we have

$$\partial_t \left[\frac{e^{2\psi_0}}{2} a(x) v^2 \right] - \nabla \cdot \left(e^{2\psi_0} v \nabla v \right) + e^{2\psi_0} \{(-\partial_t \psi_0) a(x) v^2 + |\nabla v|^2 + 2\nabla \psi_0 \cdot v \nabla v\} = 0.$$

Noting that

$$\begin{aligned}2e^{2\psi_0}\nabla\psi_0\cdot v\nabla v &= 4e^{2\psi_0}\nabla\psi_0\cdot v\nabla v - 2e^{2\psi_0}\nabla\psi_0\cdot v\nabla v\\ &= 4e^{2\psi_0}\nabla\psi_0\cdot v\nabla v - \nabla\cdot\big(e^{2\psi_0}(\nabla\psi_0)v^2\big)\\ &\quad + 2e^{2\psi_0}|\nabla\psi_0|^2v^2 + e^{2\psi_0}(\Delta\psi_0)v^2,\end{aligned}$$

we obtain

$$\begin{aligned}&\partial_t\left[\frac{e^{2\psi_0}}{2}a(x)v^2\right] - \nabla\cdot\big(e^{2\psi_0}(v\nabla v + (\nabla\psi_0)v^2)\big)\\ &\quad + e^{2\psi_0}\{|\nabla v|^2 + 4\nabla\psi_0\cdot(v\nabla v) + ((-\partial_t\psi_0)a(x) + 2|\nabla\psi_0|^2)v^2 + (\Delta\psi_0)v^2\} = 0.\end{aligned}$$

By using (3.2.4), it follows that

$$\begin{aligned}&|\nabla v|^2 + 4\nabla\psi_0\cdot(v\nabla v) + ((-\partial_t\psi_0)a(x) + 2|\nabla\psi_0|^2)v^2\\ &\geq |\nabla v|^2 + 4\nabla\psi_0\cdot(v\nabla v) + 4|\nabla\psi_0|^2v^2\\ &= |\nabla v + 2(\nabla_0\psi)v|^2 \geq 0,\end{aligned}$$

and hence,

$$\partial_t\left[\frac{e^{2\psi_0}}{2}a(x)v^2\right] - \nabla\cdot\big(e^{2\psi_0}(v\nabla v + \nabla\psi_0 v^2)\big) + e^{2\psi_0}(\Delta\psi_0)v^2 \leq 0.$$

Integrating the above inequality, we have

$$\frac{d}{dt}\int_{\mathbf{R}^n}\frac{e^{2\psi_0}}{2}a(x)v^2dx + \int_{\mathbf{R}^n}e^{2\psi_0}(\Delta\psi_0)v^2dx \leq 0.$$

Multiplying this by $(1+t)^{\frac{n-\alpha}{2-\alpha}}$, we deduce that

$$\begin{aligned}&\frac{d}{dt}\left[(1+t)^{\frac{n-\alpha}{2-\alpha}}\int_{\mathbf{R}^n}\frac{e^{2\psi_0}}{2}a(x)v^2dx\right]\\ &\quad - \frac{n-\alpha}{2(2-\alpha)}(1+t)^{\frac{n-\alpha}{2-\alpha}-1}\int_{\mathbf{R}^n}e^{2\psi_0}a(x)v^2dx\\ &\quad + (1+t)^{\frac{n-\alpha}{2-\alpha}}\int_{\mathbf{R}^n}e^{2\psi_0}(\Delta\psi_0)v^2dx \leq 0.\end{aligned}$$

By using (3.2.3), one can obtain

$$\frac{d}{dt}\left[(1+t)^{\frac{n-\alpha}{2-\alpha}}\int_{\mathbf{R}^n}\frac{e^{2\psi_0}}{2}a(x)v^2dx\right] \leq 0.$$

Integrating this over $[0,t]$, we obtain the desired estimate. □

Using the estimate in Proposition 3.5, we can immediately show (3.1.5). Indeed, noting

$$e^{2\psi_0(t,x)}a(x) = e^{\frac{\langle x\rangle^{2-\alpha}}{(2-\alpha)^2(1+t)}}\left(\frac{\langle x\rangle^{2-\alpha}}{(1+t)}\right)^{-\alpha/(2-\alpha)}(1+t)^{-\alpha/(2-\alpha)} \geq C(1+t)^{-\alpha/(2-\alpha)},$$

we can see that

$$(1+t)^{\frac{n-2\alpha}{2-\alpha}}\int_{\mathbf{R}^n}v(t,x)^2dx \leq C\int_{\mathbf{R}^n}\frac{e^{2\psi_0(0,x)}}{2}a(x)v_0(x)^2dx,$$

which proves (3.1.5).

Next, we give estimates for a weighted L^2-norm

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)} a(x) u(t,x)^2 dx$$

and a weighted energy

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)} (u_t(t,x)^2 + |\nabla u(t,x)|^2) dx$$

for solutions to (3.1.1). The following estimate was essentially already obtained by Nishihara [**81**]. However, for the sake of completeness, we shall give a proof in this section.

PROPOSITION 3.6 (Basic weighted energy estimates). *For any small* $\varepsilon > 0$, *there is some* $\delta > 0$ *having the following property: let* u *be a solution of* (3.1.1) *with initial data* (u_0, u_1) *satisfying* (3.1.2). *Then we have*

$$\begin{aligned}
&(1+t)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon} \int_{\mathbf{R}^n} e^{2\psi(t,x)} (u_t(t,x)^2 + |\nabla u(t,x)|^2) dx \\
&\quad + (1+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon} \int_{\mathbf{R}^n} e^{2\psi(t,x)} a(x) u(t,x)^2 dx \\
&\quad + \int_0^t \Big\{ (1+\tau)^{\frac{n-\alpha}{2-\alpha}-\varepsilon} \int_{\mathbf{R}^n} e^{2\psi(\tau,x)} (u_t^2 + |\nabla u|^2)(\tau,x) dx \\
&\qquad + (1+\tau)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon} \int_{\mathbf{R}^n} e^{2\psi(\tau,x)} (-\psi_t(\tau,x)) (u_t^2 + |\nabla u|^2)(\tau,x) dx \\
&\qquad + (t_0+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon} \int_{\mathbf{R}^n} e^{2\psi(\tau,x)} |\nabla \psi(\tau,x)|^2 u(\tau,x)^2 dx \\
&\qquad + (1+\tau)^{\frac{n-\alpha}{2-\alpha}-1-\varepsilon} \int_{\mathbf{R}^n} e^{2\psi(\tau,x)} a(x) u(\tau,x)^2 dx \\
&\qquad + (1+\tau)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon} \int_{\mathbf{R}^n} e^{2\psi(\tau,x)} a(x) u_t(\tau,x)^2 dx \Big\} d\tau \\
&\leq C I_0.
\end{aligned}$$

PROOF. From (3.1.7), it is easy to see that

$$-\psi_t = \frac{1}{1+t}\psi, \tag{3.2.5}$$

$$\nabla \psi = A \frac{(2-\alpha)\langle x\rangle^{-\alpha} x}{1+t}, \tag{3.2.6}$$

$$\begin{aligned}
\Delta \psi &= A(2-\alpha)(n-\alpha)\frac{\langle x\rangle^{-\alpha}}{1+t} + A(2-\alpha)\alpha \frac{\langle x\rangle^{-2-\alpha}}{1+t} \\
&\geq \frac{n-\alpha}{(2-\alpha)(2+\delta)} \frac{a(x)}{1+t} \\
&=: \left(\frac{n-\alpha}{2(2-\alpha)} - \delta_1 \right) \frac{a(x)}{1+t}.
\end{aligned} \tag{3.2.7}$$

Here and after, δ_i $(i = 1, 2, \ldots)$ denote positive constants depending only on δ such that

$$\delta_i \to 0^+ \quad \text{as} \quad \delta \to 0^+.$$

We also have

$$(-\psi_t)a(x) = A\frac{\langle x\rangle^{2-2\alpha}}{(1+t)^2} \geq \frac{1}{(2-\alpha)^2 A}A^2(2-\alpha)^2\frac{\langle x\rangle^{-2\alpha}|x|^2}{(1+t)^2} = (2+\delta)|\nabla\psi|^2. \tag{3.2.8}$$

By multiplying (3.1.1) by $e^{2\psi}u_t$, it follows that

$$\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)\right] - \nabla\cdot(e^{2\psi}u_t\nabla u) + e^{2\psi}\left(a(x) - \frac{|\nabla\psi|^2}{-\psi_t} - \psi_t\right)u_t^2 + \underbrace{\frac{e^{2\psi}}{-\psi_t}|\psi_t\nabla u - u_t\nabla\psi|^2}_{T_1} = 0. \tag{3.2.9}$$

Using the Schwarz inequality and (3.2.8), we can calculate

$$\begin{aligned} T_1 &= \frac{e^{2\psi}}{-\psi_t}(\psi_t^2|\nabla u|^2 - 2\psi_t u_t\nabla u\cdot\nabla\psi + u_t^2|\nabla\psi|^2) \\ &\geq \frac{e^{2\psi}}{-\psi_t}\left(\frac{1}{5}\psi_t^2|\nabla u|^2 - \frac{1}{4}u_t^2|\nabla\psi|^2\right) \\ &\geq e^{2\psi}\left(\frac{1}{5}(-\psi_t)|\nabla u|^2 - \frac{a(x)}{4(2+\delta)}u_t^2\right). \end{aligned}$$

This inequality and (3.2.8) lead to

$$\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)\right] - \nabla\cdot(e^{2\psi}u_t\nabla u) + e^{2\psi}\left\{\left(\frac{1}{4}a(x) - \psi_t\right)u_t^2 + \frac{-\psi_t}{5}|\nabla u|^2\right\} \leq 0. \tag{3.2.10}$$

Integrating over $\mathbf{R}^n$, we obtain

$$\frac{d}{dt}\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)dx + \int_{\mathbf{R}^n}e^{2\psi}\left\{\left(\frac{1}{4}a(x) - \psi_t\right)u_t^2 + \frac{-\psi_t}{5}|\nabla u|^2\right\}dx \leq 0. \tag{3.2.11}$$

On the other hand, we multiply (3.1.1) by $e^{2\psi}u$ and have

$$\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t + \frac{a(x)}{2}u^2\right)\right] - \nabla\cdot(e^{2\psi}u\nabla u) + e^{2\psi}\{|\nabla u|^2 + (-\psi_t)a(x)u^2 + \underbrace{2u\nabla\psi\cdot\nabla u}_{T_2} - 2\psi_t uu_t - u_t^2\} = 0. \tag{3.2.12}$$

We can rewrite the term $e^{2\psi}T_2$ as

$$\begin{aligned} e^{2\psi}T_2 &= 4e^{2\psi}u\nabla\psi\cdot\nabla u - 2e^{2\psi}u\nabla\psi\cdot\nabla u \\ &= 4e^{2\psi}u\nabla\psi\cdot\nabla u - \nabla\cdot(e^{2\psi}u^2\nabla\psi) + 2e^{2\psi}u^2|\nabla\psi|^2 + e^{2\psi}(\Delta\psi)u^2. \end{aligned}$$

By substituting this and using (3.2.7), it follows from (3.2.12) that

$$\begin{aligned}(3.2.13)\qquad &\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t+\frac{a(x)}{2}u^2\right)\right]-\nabla\cdot(e^{2\psi}(u\nabla u+u^2\nabla\psi))\\ &+e^{2\psi}\Big\{\underbrace{|\nabla u|^2+4u\nabla u\cdot\nabla\psi+((-\psi_t)a(x)+2|\nabla\psi|^2)u^2}_{T_3}\\ &+\left(\frac{n-\alpha}{2-\alpha}-2\delta_1\right)\frac{a(x)}{2(1+t)}u^2-2\psi_t uu_t-u_t^2\Big\}\le 0.\end{aligned}$$

By the Schwarz inequality, we can estimate the term T_3 as

$$\begin{aligned}T_3&=|\nabla u|^2+4u\nabla u\cdot\nabla\psi\\ &\quad+\left\{\left(1-\frac{\delta}{3}\right)(-\psi_t)a(x)+2|\nabla\psi|^2\right\}u^2+\frac{\delta}{3}(-\psi_t)a(x)u^2\\ &\ge|\nabla u|^2+4u\nabla u\cdot\nabla\psi\\ &\quad+\left(4+\frac{\delta}{3}-\frac{\delta^2}{3}\right)|\nabla\psi|^2u^2+\frac{\delta}{3}(-\psi_t)a(x)u^2\\ &=\left(1-\frac{4}{4+\delta_2}\right)|\nabla u|^2+\delta_2|\nabla\psi|^2u^2\\ &\quad+\left|\frac{2}{\sqrt{4+\delta_2}}\nabla u+\sqrt{4+\delta_2}u\nabla\psi\right|^2+\frac{\delta}{3}(-\psi_t)a(x)u^2\\ &\ge\delta_3(|\nabla u|^2+|\nabla\psi|^2u^2)+\frac{\delta}{3}(-\psi_t)a(x)u^2.\end{aligned}$$

Substituting this into (3.2.13), we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t+\frac{a(x)}{2}u^2\right)\right]-\nabla\cdot(e^{2\psi}(u\nabla u+u^2\nabla\psi))\\ &\quad+e^{2\psi}\delta_3|\nabla u|^2\\ &\quad+e^{2\psi}\left(\delta_3|\nabla\psi|^2+\frac{\delta}{3}(-\psi_t)a(x)+\left(\frac{n-\alpha}{2-\alpha}-2\delta_1\right)\frac{a(x)}{2(1+t)}\right)u^2\\ &\quad+e^{2\psi}(-2\psi_t uu_t-u_t^2)\\ &\le 0.\end{aligned}$$

Integration over $\mathbf{R}^n$ leads to

$$\begin{aligned}(3.2.14)\quad &\frac{d}{dt}\int_{\mathbf{R}^n}e^{2\psi}\left(uu_t+\frac{a(x)}{2}u^2\right)dx+\int_{\mathbf{R}^n}e^{2\psi}\delta_3|\nabla u|^2dx\\ &+\int_{\mathbf{R}^n}e^{2\psi}\left(\delta_3|\nabla\psi|^2+\frac{\delta}{3}(-\psi_t)a(x)+\left(\frac{n-\alpha}{2-\alpha}-2\delta_1\right)\frac{a(x)}{2(1+t)}\right)u^2dx\\ &+\int_{\mathbf{R}^n}e^{2\psi}(-2\psi_t uu_t-u_t^2)dx\\ &\le 0.\end{aligned}$$

By multiplying (3.2.11) by $(t_0+t)^\alpha$ with large parameter t_0, which determined later, we obtain

$$
\begin{aligned}
(3.2.15)\quad & \frac{d}{dt}\left[(t_0+t)^\alpha \int_{\mathbf{R}^n} \frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)dx\right] \\
& -\frac{\alpha}{2}(t_0+t)^{\alpha-1}\int_{\mathbf{R}^n} e^{2\psi}(u_t^2+|\nabla u|^2)dx \\
& +\int_{\mathbf{R}^n} e^{2\psi}\left\{\left(\frac{1}{4}+(t_0+t)^\alpha(-\psi_t)\right)u_t^2+(t_0+t)^\alpha\frac{-\psi_t}{5}|\nabla u|^2\right\}dx \\
& \le 0,
\end{aligned}
$$

provided that t_0 is sufficiently large, since $a(x) \ge \langle t+L\rangle^{-\alpha}$ holds by the finite propagation speed property. Let ν be a positive small parameter depending on δ. By (3.2.15)+ν(3.2.14), we have

(3.2.16)

$$
\begin{aligned}
& \frac{d}{dt}\left[\int_{\mathbf{R}^n} e^{2\psi}\left\{\frac{(t_0+t)^\alpha}{2}(u_t^2+|\nabla u|^2)+\nu\left(uu_t+\frac{a(x)}{2}u^2\right)\right\}dx\right] \\
& +\int_{\mathbf{R}^n} e^{2\psi}\left(\frac{1}{4}-\nu-\frac{\alpha}{2}(t_0+t)^{\alpha-1}+(t_0+t)^\alpha(-\psi_t)\right)u_t^2dx \\
& +\int_{\mathbf{R}^n} e^{2\psi}\left(\nu\delta_3-\frac{\alpha}{2}(t_0+t)^{\alpha-1}+(t_0+t)^\alpha\frac{-\psi_t}{5}\right)|\nabla u|^2dx \\
& +\nu\int_{\mathbf{R}^n} e^{2\psi}\left(\delta_3|\nabla\psi|^2+\frac{\delta}{3}(-\psi_t)a(x)+\left(\frac{n-\alpha}{2-\alpha}-2\delta_1\right)\frac{a(x)}{2(1+t)}\right)u^2dx \\
& +\nu\int_{\mathbf{R}^n} e^{2\psi}2(-\psi_t)uu_tdx \\
& \le 0.
\end{aligned}
$$

By the Schwarz inequality, the last term of the left-hand side of the above inequality can be estimated as

$$
\begin{aligned}
|2(-\psi_t)uu_t| &\le \frac{\delta}{3}(-\psi_t)(t_0+t)^{-\alpha}u^2+\frac{3}{\delta}(-\psi_t)(t_0+t)^\alpha u_t^2 \\
&\le \frac{\delta}{3}(-\psi_t)a(x)u^2+\frac{3}{\delta}(-\psi_t)(t_0+t)^\alpha u_t^2.
\end{aligned}
$$

Therefore, it follows from (3.2.16) that

$$
\begin{aligned}
(3.2.17)\quad & \frac{d}{dt}\left[\int_{\mathbf{R}^n} e^{2\psi}\left\{\frac{(t_0+t)^\alpha}{2}(u_t^2+|\nabla u|^2)+\nu\left(uu_t+\frac{a(x)}{2}u^2\right)\right\}dx\right] \\
& +\int_{\mathbf{R}^n} e^{2\psi}\left(\frac{1}{4}-\nu-\frac{\alpha}{2}(t_0+t)^{\alpha-1}+\left(1-\frac{3\nu}{\delta}\right)(t_0+t)^\alpha(-\psi_t)\right)u_t^2dx \\
& +\int_{\mathbf{R}^n} e^{2\psi}\left(\nu\delta_3-\frac{\alpha}{2}(t_0+t)^{\alpha-1}+(t_0+t)^\alpha\frac{-\psi_t}{5}\right)|\nabla u|^2dx \\
& +\nu\int_{\mathbf{R}^n} e^{2\psi}\left(\delta_3|\nabla\psi|^2+\left(\frac{n-\alpha}{2-\alpha}-2\delta_1\right)\frac{a(x)}{2(1+t)}\right)u^2dx \\
& \le 0.
\end{aligned}
$$

Now we choose the parameters ν and t_0 such that

$$\frac{1}{4}-\nu-\frac{\alpha}{2}(t_0+t)^{\alpha-1}\geq c_0,\quad 1-\frac{3\nu}{\delta}\geq c_0,$$
$$\nu\delta_3-\frac{\alpha}{2}(t_0+t)^{\alpha-1}\geq c_0,\quad \frac{1}{5}\geq c_0$$

hold for some constant $c_0>0$. This is possible because we first determine ν sufficiently small depending on δ and then we choose t_0 sufficiently large depending on ν. Consequently, we obtain

$$\begin{aligned}(3.2.18)\qquad &\frac{d}{dt}\left[\int_{\mathbf{R}^n}e^{2\psi}\left\{\frac{(t_0+t)^\alpha}{2}(u_t^2+|\nabla u|^2)+\nu\left(uu_t+\frac{a(x)}{2}u^2\right)\right\}dx\right]\\ &+c_0\int_{\mathbf{R}^n}e^{2\psi}(1+(t_0+t)^\alpha(-\psi_t))(u_t^2+|\nabla u|^2)dx\\ &+\nu\int_{\mathbf{R}^n}e^{2\psi}\left(\delta_3|\nabla\psi|^2+\left(\frac{n-\alpha}{2-\alpha}-2\delta_1\right)\frac{a(x)}{2(1+t)}\right)u^2dx\\ &\leq 0.\end{aligned}$$

We put

$$\tilde{E}_1(t)=\int_{\mathbf{R}^n}e^{2\psi}\left\{\frac{(t_0+t)^\alpha}{2}(u_t^2+|\nabla u|^2)+\nu\left(uu_t+\frac{a(x)}{2}u^2\right)\right\}dx,$$
$$E_{1,\psi}(t)=\int_{\mathbf{R}^n}e^{2\psi}(1+(t_0+t)^\alpha(-\psi_t))(u_t^2+|\nabla u|^2)dx$$
$$\tilde{H}_1(t)=\nu\int_{\mathbf{R}^n}e^{2\psi}\left(\delta_3|\nabla\psi|^2+\left(\frac{n-\alpha}{2-\alpha}-2\delta_1\right)\frac{a(x)}{2(1+t)}\right)u^2dx.$$

Then we can rewrite (3.2.18) as

$$(3.2.19)\qquad \frac{d}{dt}\tilde{E}_1(t)+c_0E_{1,\psi}(t)+\tilde{H}_1(t)\leq 0.$$

Take arbitrary $\varepsilon>0$ and we determine δ so that $\varepsilon=3\delta_1$. By multiplying (3.2.19) by $(t_0+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}$, we have

$$\begin{aligned}&\frac{d}{dt}[(t_0+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\tilde{E}_1(t)]-\left(\frac{n-\alpha}{2-\alpha}-\varepsilon\right)(t_0+t)^{\frac{n-\alpha}{2-\alpha}-1-\varepsilon}\tilde{E}_1(t)\\ &\quad+c_0(t_0+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}E_{1,\psi}(t)+(t_0+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\tilde{H}_1(t)\\ &\leq 0.\end{aligned}$$

Since

$$|\nu uu_t|\leq\frac{\nu\delta_4}{2}a(x)u^2+\frac{\nu}{2\delta_4}(t_0+t)^\alpha u_t^2,$$

we estimate

$$\tilde{E}_1(t)\leq\int_{\mathbf{R}^n}e^{2\psi}\left(\left(1+\frac{\nu}{\delta_4}\right)\frac{(t_0+t)^\alpha}{2}u_t^2+\frac{(t_0+t)^\alpha}{2}|\nabla u|^2+\nu(1+\delta_4)\frac{a(x)}{2}u^2\right)dx.$$

Choosing δ_4 sufficiently small and then t_0 sufficiently large so that

$$\left(\frac{n-\alpha}{2-\alpha}-2\delta_1\right)-\left(\frac{n-\alpha}{2-\alpha}-3\delta_1\right)(1+\delta_4)\geq c_1,$$
$$c_0-\frac{1}{2}\left(1+\frac{\nu}{\delta_4}\right)(t_0+t)^{\alpha-1}\geq c_1$$

with some $c_1 > 0$, we have

$$\begin{aligned}&\frac{d}{dt}[(t_0+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\tilde{E}_1(t)]\\&\quad + c_1(t_0+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}E_{1,\psi}(t) + (t_0+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}H_1(t)\\&\leq 0,\end{aligned}$$

where

$$H_1(t) = \nu\int_{\mathbf{R}^n} e^{2\psi}\left(\delta_3|\nabla\psi|^2 + c_1\frac{a(x)}{2(1+t)}\right)u^2dx.$$

Integrating over $[0,t]$, one can obtain

$$(t_0+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\tilde{E}_1(t) + \int_0^t (t_0+\tau)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}(c_1E_{1,\psi}(\tau)+H_1(\tau))d\tau \leq \tilde{E}_1(0).$$

We also put

$$E_1(t) := \int_{\mathbf{R}^n} e^{2\psi}\{(t_0+t)^{\alpha}(u_t^2+|\nabla u|^2) + a(x)u^2\}dx.$$

Then it is easy to see that $\tilde{E}_1(t) \sim E_1(t)$ and $E_1(0) \lesssim I_0$. From this, we have

$$(t_0+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}E_1(t) + \int_0^t (t_0+\tau)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}(E_{1,\psi}(\tau)+H_1(\tau))d\tau \leq CI_0. \tag{3.2.20}$$

To reach the conclusion of the proposition, we multiply (3.2.11) by $(t_0+t)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}$ and obtain

$$\begin{aligned}&\frac{d}{dt}\left[(t_0+t)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)dx\right]\\&-\left(\frac{n-\alpha}{2-\alpha}+1-\varepsilon\right)(t_0+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)dx\\&+(t_0+t)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi}\left\{\left(\frac{1}{4}a(x)-\psi_t\right)u_t^2+\frac{-\psi_t}{5}|\nabla u|^2\right\}dx \leq 0.\end{aligned}$$

By integrating over $[0,t]$, it holds that

$$\begin{aligned}&(t_0+t)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)dx\\&\quad-\left(\frac{n-\alpha}{2-\alpha}+1-\varepsilon\right)\int_0^t(t_0+\tau)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)dxd\tau\\&\quad+\int_0^t(t_0+\tau)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi}\left\{\left(\frac{1}{4}a(x)-\psi_t\right)u_t^2+\frac{-\psi_t}{5}|\nabla u|^2\right\}dxd\tau\\&\leq CI_0.\end{aligned} \tag{3.2.21}$$

Taking (3.2.20)+η(3.2.21) with small parameter $\eta > 0$ satisfying

$$1-\frac{\eta}{2}\left(\frac{n-\alpha}{2-\alpha}+1-\varepsilon\right) > 0,$$

we can see that

$$
\begin{aligned}
&(t_0+t)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(t,x)}(u_t(t,x)^2+|\nabla u(t,x)|^2)dx\\
&\quad+(t_0+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(t,x)}a(x)u(t,x)^2dx\\
&\quad+\int_0^t \Big\{(t_0+\tau)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}(u_t^2+|\nabla u|^2)(\tau,x)dx\\
&\qquad+(t_0+\tau)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}(-\psi_t(\tau,x))(u_t^2+|\nabla u|^2)(\tau,x)dx\\
&\qquad+(t_0+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}|\nabla\psi(\tau,x)|^2u(\tau,x)^2dx\\
&\qquad+(t_0+\tau)^{\frac{n-\alpha}{2-\alpha}-1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}a(x)u(\tau,x)^2dx\\
&\qquad+(t_0+\tau)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}a(x)u_t(\tau,x)^2dx\Big\}d\tau\\
&\le CI_0.
\end{aligned}
$$

Finally, we note that $(t_0+t)\sim(1+t)$ and obtain the conclusion. □

3.3. Weighted energy estimates for higher order derivatives

In this section, we give a proof of Lemmas 3.3 and 3.4 and Theorem 3.2. We first prove (3.1.10). Differentiating (3.1.1) with respect to t, we obtain

$$
u_{ttt}-\Delta u_t+a(x)u_{tt}=0. \tag{3.3.1}
$$

We apply the weighted energy method again. First, by Proposition 3.6, we have

$$
\int_0^t (t_0+\tau)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}a(x)u_t(\tau,x)^2dxd\tau\le CI_0. \tag{3.3.2}
$$

Multiplying (3.3.1) by $e^{2\psi}u_{tt}$ and $e^{2\psi}u_t$, and the same argument as the derivation of (3.2.19), we can obtain

$$
\frac{d}{dt}\tilde{E}_2(t)+c_0E_{2,\psi}(t)+\tilde{H}_2(t)\le 0, \tag{3.3.3}
$$

where

$$
\begin{aligned}
\tilde{E}_2(t)&=\int_{\mathbf{R}^n} e^{2\psi}\left\{\frac{(t_0+t)^\alpha}{2}(u_{tt}^2+|\nabla u_t|^2)+\nu\left(u_tu_{tt}+\frac{a(x)}{2}u_t^2\right)\right\}dx,\\
E_{2,\psi}(t)&=\int_{\mathbf{R}^n} e^{2\psi}(1+(t_0+t)^\alpha(-\psi_t))(u_{tt}^2+|\nabla u_t|^2)dx,\\
\tilde{H}_2(t)&=\nu\int_{\mathbf{R}^n} e^{2\psi}\left(\delta_3|\nabla\psi|^2+\left(\frac{n-\alpha}{2-\alpha}-2\delta_1\right)\frac{a(x)}{2(1+t)}\right)u_t^2dx.
\end{aligned}
$$

Multiplying (3.3.3) by $(t_0+t)^{\frac{n-\alpha}{2-\alpha}+2-\varepsilon}$ and retaking t_0 larger, we have

$$\begin{aligned}
&\frac{d}{dt}[(t_0+t)^{\frac{n-\alpha}{2-\alpha}+2-\varepsilon}\tilde{E}_2(t)] \\
&\quad + c_1(t_0+t)^{\frac{n-\alpha}{2-\alpha}+2-\varepsilon}E_{2,\psi}(t) \\
&\quad + \nu\delta_3(t_0+t)^{\frac{n-\alpha}{2-\alpha}+2-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}|\nabla\psi|^2 u_t^2 dx \\
&\le C(t_0+t)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}a(x)u_t^2 dx
\end{aligned}$$

with some $c_1>0$. By (3.3.2), integrating over $[0,t]$ and noting $\tilde{E}_2(t)\sim E_2(t)$, where

$$E_2(t)=\int_{\mathbf{R}^n} e^{2\psi}\left\{(t_0+t)^{\alpha}(u_{tt}^2+|\nabla u_t|^2)+a(x)u_t^2\right\}dx,$$

it follows that

$$\begin{aligned}
&(t_0+t)^{\frac{n-\alpha}{2-\alpha}+2-\varepsilon}E_2(t) \\
&\quad + \int_0^t (t_0+\tau)^{\frac{n-\alpha}{2-\alpha}+2-\varepsilon}E_{2,\psi}(\tau)d\tau \\
&\quad + \int_0^t (t_0+\tau)^{\frac{n-\alpha}{2-\alpha}+2-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}|\nabla\psi|^2 u_t^2 dxd\tau \\
&\le CI_1.
\end{aligned}\tag{3.3.4}$$

In particular, we can see that (3.1.10) holds. Furthermore, we obtain

$$\int_0^t (t_0+\tau)^{\frac{n-\alpha}{2-\alpha}+2-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}(u_{tt}^2+|\nabla u_t|^2)dxd\tau \le CI_1.\tag{3.3.5}$$

Using this, we can prove (3.1.11). Indeed, by the same argument as proving (3.2.11), we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}(u_{tt}^2+|\nabla u_t|^2)dx \\
&\quad + \int_{\mathbf{R}^n} e^{2\psi}\left\{\left(\frac{1}{4}a(x)-\psi_t\right)u_{tt}^2+\frac{-\psi_t}{5}|\nabla u_t|^2\right\}dx \le 0.
\end{aligned}\tag{3.3.6}$$

Multiplying (3.3.6) by $(t_0+t)^{\frac{n-\alpha}{2-\alpha}+3-\varepsilon}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[(t_0+t)^{\frac{n-\alpha}{2-\alpha}+3-\varepsilon}\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}(u_{tt}^2+|\nabla u_t|^2)dx\right] \\
&\quad + (t_0+t)^{\frac{n-\alpha}{2-\alpha}+3-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}\left\{a(x)u_{tt}^2+(-\psi_t)(u_{tt}^2+|\nabla u_t|^2)\right\}dx \\
&\le C(t_0+t)^{\frac{n-\alpha}{2-\alpha}+2-\varepsilon}\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}(u_{tt}^2+|\nabla u_t|^2)dx.
\end{aligned}$$

Integration over the interval $[0,t]$ and the estimate (3.3.5) imply

$$\begin{aligned}
&(t_0+t)^{\frac{n-\alpha}{2-\alpha}+3-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}(u_{tt}^2+|\nabla u_t|^2)dx \\
&\quad + \int_0^t (t_0+\tau)^{\frac{n-\alpha}{2-\alpha}+3-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}\left\{a(x)u_{tt}^2+(-\psi_t)(u_{tt}^2+|\nabla u_t|^2)\right\}dxd\tau \\
&\le CI_1.
\end{aligned}\tag{3.3.7}$$

In particular, we have (3.1.11) and

$$\int_0^t (t_0+\tau)^{\frac{n-\alpha}{2-\alpha}+3-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}a(x)u_{tt}(\tau,x)^2dxd\tau \le CI_1, \tag{3.3.8}$$

which will be used to obtain (3.1.12) and (3.1.13).

To prove (3.1.12) and (3.1.13), we differentiate (3.3.1) again and have

$$u_{tttt} - \Delta u_{tt} + a(x)u_{ttt} = 0. \tag{3.3.9}$$

Using (3.3.8) instead of (3.3.2) and by the same argument as above, we can prove instead of (3.3.4) that

$$\begin{aligned}(t_0+t)^{\frac{n-\alpha}{2-\alpha}+4-\varepsilon}&E_3(t)\\ &+\int_0^t (t_0+\tau)^{\frac{n-\alpha}{2-\alpha}+4-\varepsilon}E_{3,\psi}(\tau)d\tau\\ &+\int_0^t (t_0+t)^{\frac{n-\alpha}{2-\alpha}+4-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}|\nabla\psi|^2u_{tt}^2dxd\tau\\ &\le CI_2,\end{aligned} \tag{3.3.10}$$

where

$$E_3(t) = \int_{\mathbf{R}^n} e^{2\psi}\left\{(t_0+t)^\alpha(u_{ttt}^2+|\nabla u_{tt}|^2)+a(x)u_{tt}^2\right\}dx,$$

$$E_{3,\psi}(t) = \int_{\mathbf{R}^n} e^{2\psi}(1+(t_0+t)^\alpha(-\psi_t))(u_{ttt}^2+|\nabla u_{tt}|^2)dx.$$

In particular, we obtain (3.1.12). Moreover, by the same argument as deriving (3.3.7), one can obtain

$$\begin{aligned}(t_0+t)^{\frac{n-\alpha}{2-\alpha}+5-\varepsilon}&\int_{\mathbf{R}^n} e^{2\psi}(u_{ttt}^2+|\nabla u_{tt}|^2)dx\\ &+\int_0^t (t_0+\tau)^{\frac{n-\alpha}{2-\alpha}+5-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}\left\{a(x)u_{ttt}^2+(-\psi_t)(u_{ttt}^2+|\nabla u_{tt}|^2)\right\}dxd\tau\\ &\le CI_3.\end{aligned} \tag{3.3.11}$$

In particular, we have

$$\int_0^t (t_0+\tau)^{\frac{n-\alpha}{2-\alpha}+5-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}a(x)u_{ttt}^2dxd\tau \le CI_3. \tag{3.3.12}$$

Using (3.3.12) instead of (3.3.8) again, we can prove (3.1.13). Furthermore, we can continue the argument starting at (3.3.8) and obtaining (3.3.12) as much as we want. Therefore, we can obtain the conclusion of Theorem 3.2.

Finally, we prove (3.1.14) and (3.1.15). Multiplying (3.1.3) by v, we have

$$\frac{\partial}{\partial t}\left[\frac{a(x)}{2}v^2\right] - \nabla\cdot(v\nabla v) + |\nabla v|^2 = 0.$$

Integrating over $\mathbf{R}^n$, one can obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^n} a(x)v(t,x)^2dx + \int_{\mathbf{R}^n}|\nabla v(t,x)|^2dx = 0.$$

Thus, we have

$$\frac{1}{2}\int_{\mathbf{R}^n} a(x)v(t,x)^2dx + \int_\tau^t\int_{\mathbf{R}^n}|\nabla v(s,x)|^2dxds = \frac{1}{2}\int_{\mathbf{R}^n} a(x)v_\tau(x)^2dx,$$

which implies (3.1.14). To prove (3.1.15), we apply a similar argument to (3.1.3) as in the previous section. We first multiply (3.1.3) by $e^{2\psi}v_t$ and have

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}|\nabla v|^2\right]-\nabla\cdot(e^{2\psi}v_t\nabla v)\\&+e^{2\psi}\left(a(x)-\frac{|\nabla\psi|^2}{-\psi_t}\right)v_t^2+\underbrace{\frac{e^{2\psi}}{-\psi_t}|\psi_t\nabla v-v_t\nabla\psi|^2}_{T_1}=0.\end{aligned}$$

By (3.2.8) and

$$T_1\geq e^{2\psi}\left(\frac{1}{5}(-\psi_t)|\nabla v|^2-\frac{a(x)}{4(2+\delta)}v_t^2\right),$$

it follows that

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}|\nabla v|^2\right]-\nabla\cdot(e^{2\psi}v_t\nabla v)\\&+e^{2\psi}\left(\frac{1}{4}a(x)v_t^2+\frac{1}{5}(-\psi_t)|\nabla v|^2\right)\leq 0.\end{aligned}$$

Integrating over $\mathbf{R}^n$, we obtain

$$\frac{d}{dt}\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}|\nabla v|^2dx+\int_{\mathbf{R}^n}e^{2\psi}\left(\frac{1}{4}a(x)v_t^2+\frac{1}{5}(-\psi_t)|\nabla v|^2\right)dx\leq 0. \tag{3.3.13}$$

On the other hand, by multiplying (3.1.3) by $e^{2\psi}v$, it follows that

$$\begin{aligned}&\frac{\partial}{\partial t}\left[e^{2\psi}\frac{a(x)}{2}v^2\right]-\nabla\cdot(e^{2\psi}v\nabla v)\\&+e^{2\psi}\{|\nabla v|^2+2v\nabla\psi\cdot\nabla v+(-\psi_t)a(x)v^2\}=0.\end{aligned}$$

By the same argument as the derivation of (3.2.14), we can see that

$$\begin{aligned}&\frac{d}{dt}\left[\int_{\mathbf{R}^n}e^{2\psi}\frac{a(x)}{2}v^2dx\right]\\&\quad+\int_{\mathbf{R}^n}e^{2\psi}\left(\delta_3(|\nabla v|^2+|\nabla\psi|^2v^2)+\frac{\delta}{3}(-\psi_t)a(x)v^2\right.\\&\qquad\left.+\left(\frac{n-\alpha}{2-\alpha}-2\delta_1\right)\frac{a(x)}{2(1+t)}v^2\right)dx\\&\leq 0.\end{aligned} \tag{3.3.14}$$

Taking $\nu(1+t)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}$ (3.3.13)+$(1+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}$(3.3.14) with small parameter $\nu>0$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\nu(1+t)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}|\nabla v|^2dx+(1+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi}\frac{a(x)}{2}v^2dx\right]\\
&\quad+\left(\delta_3-\frac{\nu}{2}\left(\frac{n-\alpha}{2-\alpha}+1-\varepsilon\right)\right)(1+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi}|\nabla v|^2dx\\
&\quad+\nu(1+t)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi}\left(\frac{1}{4}a(x)v_t^2+\frac{(-\psi_t)}{5}|\nabla v|^2\right)dx\\
&\quad+(1+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi}\left(\delta_3|\nabla\psi|^2v^2+\frac{\delta}{3}(-\psi_t)a(x)v^2\right)dx\\
&\quad+(\varepsilon-2\delta_1)(1+t)^{\frac{n-\alpha}{2-\alpha}-1-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi}\frac{a(x)}{2}v^2dx\\
&\le 0.
\end{aligned}$$

We determine δ so that $\varepsilon=3\delta_1$ holds. Then we take ν sufficiently small. Integrating the above inequality over $[\tau,t]$, we have

$$\begin{aligned}
(3.3.15)\qquad &(1+t)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}|\nabla v|^2dx+(1+t)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi}\frac{a(x)}{2}v^2dx\\
&\quad+\int_\tau^t(1+s)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi}|\nabla v|^2dxds\\
&\quad+\int_\tau^t(1+s)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi}(a(x)v_t^2+(-\psi_t)|\nabla v|^2)dxds\\
&\quad+\int_\tau^t(1+s)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi}\left(|\nabla\psi|^2v^2+(-\psi_t)a(x)v^2\right)dxds\\
&\quad+\int_\tau^t(1+s)^{\frac{n-\alpha}{2-\alpha}-1-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi}a(x)v^2dxds\\
&\le C(1+\tau)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi(\tau,x)}|\nabla v_\tau(x)|^2dx\\
&\quad+C(1+\tau)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi(\tau,x)}a(x)v_\tau(x)^2dx.
\end{aligned}$$

In particular, it follows that

$$\begin{aligned}
(3.3.16)\qquad &\int_\tau^t(1+s)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi(s,x)}a(x)v_t(s,x)^2dxds\\
&\quad\le C(1+\tau)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi(\tau,x)}|\nabla v_\tau(x)|^2dx\\
&\quad+C(1+\tau)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi(\tau,x)}a(x)v_\tau(x)^2dx.
\end{aligned}$$

Using this estimate, we prove (3.1.15). We differentiate (3.1.3) with respect to t and have

$$a(x)v_{tt}-\Delta v_t=0.$$

Multiplying this by v_t, we obtain

$$\frac{\partial}{\partial t}\left[\frac{a(x)}{2}v_t^2\right]-\nabla\cdot(v_t\nabla v_t)+|\nabla v_t|^2=0.$$

Integrating over $\mathbf{R}^n$, one can see that

$$\frac{d}{dt}\int_{\mathbf{R}^n} a(x)v_t(t,x)^2dx \le 0.$$

Moreover, taking into account (3.3.16), multiplying this by $(1+t-\tau)^{\frac{n-\alpha}{2-\alpha}+2-\varepsilon}$ with $0 \le \tau \le t$, we have

$$\begin{aligned}
&\frac{d}{dt}\left[(1+t-\tau)^{\frac{n-\alpha}{2-\alpha}+2-\varepsilon}\int_{\mathbf{R}^n} a(x)v_t(t,x)^2dx\right]\\
&\quad\le C(1+t-\tau)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n} a(x)v_t(t,x)^2dx\\
&\quad\le C(1+t)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n} a(x)v_t(t,x)^2dx.
\end{aligned}$$

Integrating over $[\tau,t]$, and using (3.3.16), we can conclude that

$$\begin{aligned}
&(1+t-\tau)^{\frac{n-\alpha}{2-\alpha}+2-\varepsilon}\int_{\mathbf{R}^n} a(x)v_t(t,x)^2dx\\
&\le \int_{\mathbf{R}^n} a(x)v_t(\tau,x)^2dx\\
&\quad + C\int_\tau^t (1+s)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n} a(x)v_t(s,x)^2dxds\\
&\le \int_{\mathbf{R}^n} a(x)^{-1}|\Delta v_\tau(x)|^2dx\\
&\quad + C(1+\tau)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}|\nabla v_\tau(x)|^2dx\\
&\quad + C(1+\tau)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}a(x)v_\tau(x)^2dx.
\end{aligned}$$

Here we note that

$$a(x)v_t(\tau,x) = \Delta v_\tau(x),$$

since v satisfies (3.1.3). Thus, we obtain (3.1.15).

Proof of Lemma 1.4. By (3.3.14), we have

$$\frac{d}{dt}\left[\int_{\mathbf{R}^n} e^{2\psi}\frac{a(x)}{2}v^2dx\right] \le 0.$$

This shows

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)}\frac{a(x)}{2}v(t,x)^2dx \le \int_{\mathbf{R}^n} e^{2\psi(0,x)}\frac{a(x)}{2}v(0,x)^2dx. \tag{3.3.17}$$

Let $0<\rho<1-\alpha$, $0<\mu<2A$ and

$$\Omega_\rho(t) := \{x\in\mathbf{R}^n \mid \langle x\rangle^{2-\alpha} \ge (1+t)^{1+\rho}\}.$$

A simple calculation implies

$$e^{2\psi}a(x) \ge c(1+t)^{-\frac{\alpha}{2-\alpha}}e^{(2A-\mu)\frac{\langle x\rangle^{2-\alpha}}{1+t}}.$$

By noting that

$$\frac{\langle x\rangle^{2-\alpha}}{1+t} \ge (1+t)^\rho$$

on $\Omega_\rho(t)$ and (3.3.17), it follows that

$$\begin{aligned}
&(1+t)^{-\frac{\alpha}{2-\alpha}}\int_{\Omega_\rho(t)} e^{(2A-\mu)(1+t)^\rho}v(t,x)^2dx\\
&\le C(1+t)^{-\frac{\alpha}{2-\alpha}}\int_{\Omega_\rho(t)} e^{(2A-\mu)\frac{\langle x\rangle^{2-\alpha}}{1+t}}v(t,x)^2dx\\
&\le C\int_{\mathbf{R}^n} e^{2\psi(t,x)}a(x)v(t,x)^2dx\\
&\le C\int_{\mathbf{R}^n} e^{2\psi(0,x)}a(x)v(0,x)^2dx.
\end{aligned}$$

Thus, we obtain

$$\int_{\Omega_\rho(t)} v(t,x)^2dx \le C(1+t)^{\frac{\alpha}{2-\alpha}}e^{-(2A-\mu)(1+t)^\rho}\int_{\mathbf{R}^n} e^{2\psi(0,x)}a(x)v(0,x)^2dx.$$

This proves Lemma 3.4.

□

3.4. Proof of the main theorem

The equation (3.1.1) can be expressed as

$$u_t - a(x)^{-1}\Delta u = -a(x)^{-1}u_{tt}.$$

By the Duhamel principle, we can write

$$u(t,x) = E(t)u_0(x) - \int_0^t E(t-\tau)[a(\cdot)^{-1}u_{tt}(\tau,\cdot)](x)d\tau. \tag{3.4.1}$$

By the integration by parts, we have

$$\begin{aligned}
-\int_0^t E(t-\tau)[a(\cdot)^{-1}u_{tt}(\tau,\cdot)]d\tau = &-\int_{t/2}^t E(t-\tau)[a(\cdot)^{-1}u_{tt}(\tau,\cdot)]d\tau\\
&- E(t/2)[a^{-1}u_t(t/2)] + E(t)[a^{-1}u_1]\\
&- \int_0^{t/2}\frac{\partial E}{\partial t}(t-\tau)[a^{-1}u_t(\tau)]d\tau,
\end{aligned}$$

where $\frac{\partial E}{\partial t}(t-\tau)$ is the pseudodifferential operator with the symbol $\frac{\partial e}{\partial t}(t-\tau,x,\xi)$, that is, $\frac{\partial E}{\partial t}(t-\tau)v_\tau$ denotes the derivative of $E(t-\tau)v_\tau$ with respect to t. Therefore, we obtain

$$\begin{aligned}
u(t) - E(t)[u_0 + a^{-1}u_1] = &-\int_{t/2}^t E(t-\tau)[a(\cdot)^{-1}u_{tt}(\tau,\cdot)]d\tau\\
&- E(t/2)[a^{-1}u_t(t/2)]\\
&- \int_0^{t/2}\frac{\partial E}{\partial t}(t-\tau)[a^{-1}u_t(\tau)]d\tau
\end{aligned} \tag{3.4.2}$$

and it suffices to prove that the each term of the right-hand side is $o(t^{-\frac{n-2\alpha}{2(2-\alpha)}})$ in the L^2-sense. First, by the finite propagation speed property, we have $u(t,x) = \chi(t,x)u(t,x)$ with the characteristic function $\chi(t,x)$ of the region $\{(t,x)\in(0,\infty)\times\mathbf{R}^n \mid |x| < t+L\}$. Moreover, by Lemma 3.4, the L^2-norm of $(1-\chi(t,x))E(t)[u_0 +$

$a^{-1}u_1]$ decays exponentially. Thus, by multiplying (3.4.2) by $\chi(t,x)$, it suffices to estimate the terms

$$K_1 := \chi(t,x)\int_{t/2}^{t} E(t-\tau)[a(\cdot)^{-1}u_{tt}(\tau,\cdot)]d\tau, \quad K_2 := \chi(t,x)E(t/2)[a^{-1}u_t(t/2)]$$

and

$$K_3 := \chi(t,x)\int_0^{t/2} \frac{\partial E}{\partial t}(t-\tau)[a^{-1}u_t(\tau)]d\tau.$$

We first estimate K_1. By (3.1.14) and (3.1.12), we have

$$\begin{aligned}
\|K_1\|_{L^2} &= \left\|\chi(t)\int_{t/2}^{t} E(t-\tau)[a^{-1}u_{tt}(\tau)]d\tau\right\|_{L^2} \\
&= \left\|\chi(t)\frac{1}{\sqrt{a}}\int_{t/2}^{t} \sqrt{a}E(t-\tau)[a^{-1}u_{tt}(\tau)]d\tau\right\|_{L^2} \\
&\lesssim (1+t)^{\alpha/2}\int_{t/2}^{t} \|\sqrt{a}E(t-\tau)[a^{-1}u_{tt}(\tau)]\|_{L^2}d\tau \\
&\le (1+t)^{\alpha/2}\int_{t/2}^{t} \|\sqrt{a}a^{-1}u_{tt}(\tau)\|_{L^2}d\tau \\
&\lesssim (1+t)^{3\alpha/2}\int_{t/2}^{t} \|\sqrt{a}u_{tt}(\tau)\|_{L^2}d\tau \\
&\lesssim (1+t)^{3\alpha/2-\frac{n-\alpha}{2(2-\alpha)}-1-\varepsilon/2} = o(t^{-\frac{n-2\alpha}{2(2-\alpha)}}),
\end{aligned}$$

provided that $\varepsilon > 0$ is taken sufficiently small. Because, it is true that

$$\frac{3}{2}\alpha - \frac{n-\alpha}{2(2-\alpha)} - 1 < -\frac{n-2\alpha}{2(2-\alpha)} \tag{3.4.3}$$

holds if $0 \le \alpha < 1$.

We can estimate K_2 by a similar way. Using (3.1.14), (3.1.10) and (3.4.3), we obtain

$$\begin{aligned}
\|K_2\|_{L^2} &= \left\|\chi(t)\frac{1}{\sqrt{a}}\sqrt{a}E(t/2)[a^{-1}u_t(t/2)]\right\|_{L^2} \\
&\lesssim (1+t)^{\alpha/2}\left\|\sqrt{a}E(t/2)[a^{-1}u_t(t/2)]\right\|_{L^2} \\
&\lesssim (1+t)^{\alpha/2}\left\|\sqrt{a}a^{-1}u_t(t/2)\right\|_{L^2} \\
&\lesssim (1+t)^{3\alpha/2}\left\|\sqrt{a}u_t(t/2)\right\|_{L^2} \\
&\lesssim (1+t)^{3\alpha/2-\frac{n-\alpha}{2(2-\alpha)}-1+\varepsilon/2} = o(t^{-\frac{n-2\alpha}{2(2-\alpha)}}).
\end{aligned}$$

Finally, we estimate K_3. By (3.1.15), we have

$$\begin{aligned}\|K_3\|_{L^2} &= \left\|\chi(t)\int_0^{t/2}\frac{\partial E}{\partial t}(t-\tau)[a^{-1}u_t(\tau)]d\tau\right\|_{L^2}\\ &= \left\|\chi(t)\frac{1}{\sqrt{a}}\int_0^{t/2}\sqrt{a}\frac{\partial E}{\partial t}(t-\tau)[a^{-1}u_t(\tau)]d\tau\right\|_{L^2}\\ &\lesssim (1+t)^{\alpha/2}\int_0^{t/2}\left\|\sqrt{a}\frac{\partial E}{\partial t}(t-\tau)[a^{-1}u_t(\tau)]\right\|_{L^2}d\tau\\ &\lesssim (1+t)^{\alpha/2}\int_0^{t/2}(1+t-\tau)^{-\frac{n-\alpha}{2(2-\alpha)}-1+\varepsilon/2}(J_1+J_2+J_3)d\tau\\ &\lesssim (1+t)^{\alpha/2-\frac{n-\alpha}{2(2-\alpha)}-1+\varepsilon/2}\int_0^{t/2}(J_1+J_2+J_3)d\tau,\end{aligned}$$

where

$$\begin{aligned}J_1^2 &= \int_{\mathbf{R}^n} a(x)^{-1}|\Delta(a(x)^{-1}u_t(\tau,x))|^2dx,\\ J_2^2 &= (1+\tau)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}|\nabla(a(x)^{-1}u_t(\tau,x))|^2dx\end{aligned}$$

and

$$J_3^2 = (1+\tau)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}a(x)|a(x)^{-1}u_t(\tau,x)|^2dx.$$

By noting

$$\alpha-\frac{n-\alpha}{2(2-\alpha)}-\frac{1}{2}<-\frac{n-2\alpha}{2(2-\alpha)} \tag{3.4.4}$$

if $0\le\alpha<1$, it is only necessary to prove

$$\int_0^{t/2}J_kd\tau\le C(1+t)^{\frac{\alpha+1}{2}} \tag{3.4.5}$$

for $k=1,2,3$ with some constant $C>0$.

Now we prove (3.4.5). We first estimate J_3. By (3.1.10) and the finite propagation speed property again, we can estimate

$$\begin{aligned}J_3^2 &\lesssim (1+\tau)^{\frac{n-\alpha}{2-\alpha}-\varepsilon}(1+\tau)^{2\alpha}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}a(x)u_t(\tau,x)^2dx\\ &\lesssim (1+\tau)^{2\alpha-2}.\end{aligned}$$

By a simple calculation, we can see that

$$\int_0^{t/2}J_3d\tau\lesssim\int_0^{t/2}(1+\tau)^{\alpha-1}d\tau\lesssim(1+t)^{\frac{\alpha+1}{2}}.$$

Next, we estimate J_2. Noting

$$\nabla(a^{-1}u_t)=\nabla(a^{-1})u_t+a^{-1}\nabla u_t$$

and $|\nabla(a^{-1})|\lesssim\langle x\rangle^{\alpha-1}$, we have

$$J_2^2\lesssim(1+\tau)^{\frac{n-\alpha}{2-\alpha}+1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}(|\langle x\rangle^{\alpha-1}u_t(\tau,x)|^2+|\langle x\rangle^{\alpha}\nabla u_t(\tau,x)|^2)dx.$$

By (3.1.10) and (3.1.12), we obtain

$$\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}|\langle x\rangle^{\alpha-1}u_t(\tau,x)|^2dx \lesssim (1+\tau)^{\alpha}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}a(x)u_t(\tau,x)^2dx$$
$$\lesssim (1+\tau)^{\alpha-\frac{n-\alpha}{2-\alpha}-2+\varepsilon}$$

and

$$\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}|\langle x\rangle^{\alpha}\nabla u_t(\tau,x)|^2dx \lesssim (1+\tau)^{2\alpha}\int_{\mathbf{R}^n} e^{2\psi(\tau,x)}|\nabla u_t(\tau,x)|^2dx$$
$$\lesssim (1+\tau)^{2\alpha-\frac{n-\alpha}{2-\alpha}-3+\varepsilon}.$$

Therefore, it holds that

$$\int_0^{t/2} J_2 d\tau \lesssim (1+t)^{\frac{\alpha+1}{2}}.$$

Finally, we estimate J_1. Noting

$$\Delta(a^{-1}u_t) = \Delta(a^{-1})u_t + 2\nabla(a^{-1})\cdot\nabla u_t + a^{-1}\Delta u_t,$$

we further divide J_1 into three parts:

$$J_1^2 \lesssim \int_{\mathbf{R}^n} a^{-1}|\Delta(a^{-1})u_t|^2dx + \int_{\mathbf{R}^n} a^{-1}|\nabla(a^{-1})|^2|\nabla u_t|^2dx + \int_{\mathbf{R}^n} a^{-1}|a^{-1}\Delta u_t|^2dx$$
$$\equiv J_{11}^2 + J_{12}^2 + J_{13}^2.$$

By noting $|\Delta(a^{-1})| \lesssim \langle x\rangle^{\alpha-2}$ and (3.1.10), we have

$$J_{11}^2 \lesssim \int_{\mathbf{R}^n} \langle x\rangle^{4\alpha-4}a(x)u_t(\tau,x)^2dx$$
$$\lesssim (1+\tau)^{-\frac{n-\alpha}{2-\alpha}-2+\varepsilon}.$$

Therefore, we immediately obtain

$$\int_0^{t/2} J_{11}d\tau \lesssim 1,$$

provided that ε is sufficiently small.

Next, we estimate J_{12}.

$$J_{12}^2 \lesssim (1+\tau)^{\alpha}\int_{\mathbf{R}^n} |\nabla u_t(\tau,x)|^2dx \lesssim (1+\tau)^{\alpha-\frac{n-\alpha}{2-\alpha}-3+\varepsilon}$$

and hence

$$\int_0^{t/2} J_{12}d\tau \lesssim 1.$$

Since u_t also satisfies (3.1.1), we can rewrite

$$\Delta u_t = u_{ttt} - au_{tt}.$$

Therefore, we have

$$J_{13}^2 \lesssim \int_{\mathbf{R}^n} a(x)^{-4}a(x)|u_{ttt}(\tau,x)|^2dx + \int_{\mathbf{R}^n} a(x)^{-2}a(x)|u_{tt}(\tau,x)|^2dx$$
$$\lesssim (1+\tau)^{4\alpha-\frac{n-\alpha}{2-\alpha}-6+\varepsilon} + (1+\tau)^{2\alpha-\frac{n-\alpha}{2-\alpha}-4+\varepsilon}.$$

These estimates imply

$$\int_0^{t/2} J_1 d\tau \lesssim 1.$$

This completes the proof.

CHAPTER 4

Small data global existence for the semilinear wave equation with damping depending on time and space variables

4.1. Introduction and results

In this chapter, we consider the Cauchy problem for the semilinear wave equation with damping depending on time and space variables

$$
(4.1.1) \qquad \begin{cases} u_{tt} - \Delta u + a(x)b(t)u_t = f(u), & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^n, \end{cases}
$$

where

$$
a(x) = a_0 \langle x \rangle^{-\alpha}, \quad b(t) = (1+t)^{-\beta}, \quad \text{with } a_0 > 0, \alpha, \beta \ge 0, \alpha + \beta < 1
$$

and $\langle x \rangle = \sqrt{1+|x|^2}$. Here u is a real-valued unknown function and $(u_0, u_1) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$. We note that u_0 and u_1 need not be compactly supported. The nonlinear term $f(u)$ is given by

$$
f(u) = \pm |u|^p \quad \text{or} \quad |u|^{p-1}u
$$

and the power p satisfies

$$
1 < p \le \frac{n}{n-2} \quad (n \ge 3), \quad 1 < p < \infty \quad (n = 1, 2).
$$

Our aim is to determine the critical exponent p_c, which is a number defined by the following property:

If $p_c < p$, all small data solutions of (4.1.1) are global; if $1 < p \le p_c$, the time-local solution cannot be extended time-globally even for small data in general.

In view of the diffusive phenomenon, it is expected that the critical exponent of (4.1.1) is given by

$$
p_c = 1 + \frac{2}{n-\alpha}.
$$

Because for the corresponding heat equation

$$
-\Delta v + a(x)b(t)v_t = |v|^p,
$$

the critical exponent is actually given by the above one. In this chapter we shall prove the existence of global solutions with small data when $p > 1 + 2/(n-\alpha)$. However, it is still open whether there exists a blow-up solution when $1 < p \le 1 + 2/(n-\alpha)$.

For the constant coefficient case $\alpha = \beta = 0$, as we mentioned in Section 1.3 and Chapter 2, there are many results for determining the critical exponent. For space-dependent damping case $\beta = 0$, the critical exponent p_c is determined as $1 + 2/(n-\alpha)$ by Ikehata, Todorova and Yordanov [**36**]. On the other hand, for the time-dependent damping $\alpha = 0$, Lin, Nishihara and Zhai [**56**] proved that the critical exponent is $1+2/n$. D'Abbicco, Lucente and Reissig [**8**] extended the result of [**56**] to more general time-dependent effective damping and improved the decay rate of the energy of solutions.

To state our results, we define an auxiliary function

$$\psi(t,x) := A\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}} \tag{4.1.2}$$

with

$$A = \frac{(1+\beta)a_0}{(2-\alpha)^2(2+\delta)}, \quad \delta > 0 \tag{4.1.3}$$

This type of weight function is introduced by Todorova and Yordanov [**105, 106**] and arranged by several authors (see [**35, 107, 54**]). We have the following result:

THEOREM 4.1. *If*

$$1+\frac{2}{n-\alpha} < p \le \frac{n}{n-2} \ (n\ge 3), \quad 1+\frac{2}{n-\alpha} < p < \infty \ (n=1,2),$$

then there exists a small positive number $\delta_0 > 0$ *such that for any* $0<\delta\le\delta_0$ *the following holds: If*

$$I_0^2 := \int_{\mathbf{R}^n} e^{2\psi(0,x)}(u_1(x)^2 + |\nabla u_0(x)|^2 + |u_0(x)|^2)dx$$

is sufficiently small, then there exists a unique solution $u \in C([0,\infty); H^1(\mathbf{R}^n)) \cap C^1([0,\infty); L^2(\mathbf{R}^n))$ *to* (4.1.1) *satisfying*

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)}u(t,x)^2dx \le C_\delta I_0^2(1+t)^{-(1+\beta)\frac{n-2\alpha}{2-\alpha}+\varepsilon}, \tag{4.1.4}$$

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)}(u_t(t,x)^2 + |\nabla u(t,x)|^2)dx \le C_\delta I_0^2(1+t)^{-(1+\beta)(\frac{n-\alpha}{2-\alpha}+1)+\varepsilon},$$

where

$$\varepsilon = \varepsilon(\delta) := \frac{3(1+\beta)(n-\alpha)}{2(2-\alpha)(2+\delta)}\delta \tag{4.1.5}$$

and C_δ *is a constant depending on* δ.

REMARK 4.1. *A similar result was obtained by Khader* [**46**] *independently.*

As a consequence of the main theorem, we have an exponential decay estimate outside a parabolic region.

COROLLARY 4.2. *If*

$$1+\frac{2}{n-\alpha} < p \le \frac{n}{n-2} \ (n\ge 3), \quad 1+\frac{2}{n-\alpha} < p < \infty \ (n=1,2),$$

then there exists a small positive number $\delta_0 > 0$ *such that for any* $0<\delta\le\delta_0$ *the following holds: Take* ρ *and* μ *so small that*

$$0<\rho<1-\alpha-\beta, \quad \text{and} \quad 0<\mu<2A,$$

and put

$$\Omega_\rho(t) := \{x\in\mathbf{R}^n; \langle x\rangle^{2-\alpha} \ge (1+t)^{1+\beta+\rho}\}.$$

Then, for the global solution u in Theorem 4.1, we have

$$\int_{\Omega_\rho(t)} (u_t(t,x)^2 + |\nabla u(t,x)|^2 + u(t,x)^2)dx \\ \leq C_{\delta,\rho,\mu} I_0^2 (1+t)^{-\frac{(1+\beta)(n-2\alpha)}{2-\alpha}+\varepsilon} e^{-(2A-\mu)(1+t)^\rho},$$

here ε is defined by (4.1.5) *and $C_{\delta,\rho,\mu}$ is a constant depending on δ, ρ and μ.*

Namely, the decay rate of solution in the region $\Omega_\rho(t)$ is exponential. We note that the support of $u(t)$ and the region $\Omega_\rho(t)$ can intersect even if the data are compactly supported. This phenomenon was first discovered by Todorova and Yordanov [**106**]. We can interpret this result as follows: The support of the solution is strongly suppressed by damping, so that the solution is concentrated in the parabolic region much smaller than the light cone.

4.2. A priori estimate

To prove Theorem 4.1, we use a multiplier method which was originally developed by Todorova and Yordanov [**105, 106**]. We first describe the local existence:

PROPOSITION 4.3. *Let*

$$1 < p \leq \frac{n}{n-2} \ (n \geq 3), \quad 1 < p < \infty \ (n = 1, 2)$$

and $\delta > 0$. If $I_0^2 < +\infty$, then there exists $T^* \in (0, +\infty]$ depending on I_0^2 such that the Cauchy problem (4.1.1) *has a unique solution $u \in C([0,T^*); H^1(\mathbf{R}^n)) \cap C^1([0,T^*); L^2(\mathbf{R}^n))$ satisfying*

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)}(u_t^2 + |\nabla u|^2 + u^2)(t,x)dx < +\infty$$

for all $t \in (0, T^*)$. Moreover, if $T^* < +\infty$ then we have

$$\liminf_{t\to T^*} \int_{\mathbf{R}^n} e^{2\psi(t,x)}(u_t^2 + |\nabla u|^2 + u^2)(t,x)dx = +\infty.$$

We give a proof of this proposition in Appendix (see Proposition 9.21). We prove a priori estimate for the following functional:

$$M(t) := \sup_{0\leq\tau<t} \left\{ (1+\tau)^{B+1-\varepsilon} \int_{\mathbf{R}^n} e^{2\psi(\tau,x)}(u_t^2 + |\nabla u|^2)(\tau,x)dx \right. \\ \left. + (1+\tau)^{B-\varepsilon} \int_{\mathbf{R}^n} e^{2\psi(\tau,x)} a(x)b(\tau)u(\tau,x)^2 dx \right\}, \tag{4.2.1}$$

where

$$B := \frac{(1+\beta)(n-\alpha)}{2-\alpha} + \beta$$

and ε is given by (4.1.5). Theorem 4.1 is an immediate conclusion of Proposition 4.3 and the following a priori estimate:

PROPOSITION 4.4. *Let*

$$1 + \frac{2}{n-\alpha} < p \leq \frac{n+2}{n-2} \ (n \geq 3), \quad 1 + \frac{2}{n-\alpha} < p < \infty \ (n = 1, 2).$$

Then there exist constants $\delta_0 > 0$ such that for any $0 < \delta \le \delta_0$, the following holds: If $I_0^2 < +\infty$, ε is defined by (4.1.5) and $u \in C([0,T); H^1(\mathbf{R}^n)) \cap C^1([0,T); L^2(\mathbf{R}^n))$ is a solution of the Cauchy problem (4.1.1) for some $T > 0$ such that

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)}(u^2 + |\nabla u|^2 + u_t^2)(t,x)dx < +\infty$$

for all $0 \le t < T$, then it follows that

$$M(t) + L(t) \le CI_0^2 + CM(t)^{p+1},$$

where

$$\begin{aligned} L(t) = \int_0^t \Big\{ &(1+\tau)^{B-\varepsilon} \int_{\mathbf{R}^n} e^{2\psi(\tau,x)}(u_t^2 + |\nabla u|^2)(\tau,x)dx \\ &+ (1+\tau)^{B+1-\varepsilon} \int_{\mathbf{R}^n} e^{2\psi(\tau,x)}(-\psi_t(\tau,x))(u_t^2 + |\nabla u|^2)(\tau,x)dx \\ &+ (1+\tau)^{B-1-\varepsilon} \int_{\mathbf{R}^n} e^{2\psi(\tau,x)} a(x)b(\tau)u(\tau,x)^2 dx \\ &+ (1+\tau)^{B+1-\varepsilon} \int_{\mathbf{R}^n} e^{2\psi(\tau,x)} a(x)b(\tau)u_t(\tau,x)^2 dx \Big\} d\tau \end{aligned}$$

and $C = C(p,\varepsilon)$ is a positive constant depending on p, ε.

4.2.1. Proof of Proposition 4.4.

Proof of Proposition 4.4. From (4.1.2) and (4.1.3), it is easy to see that

$$-\psi_t = \frac{1+\beta}{1+t}\psi, \tag{4.2.2}$$

$$\nabla\psi = A\frac{(2-\alpha)\langle x\rangle^{-\alpha}x}{(1+t)^{1+\beta}}, \tag{4.2.3}$$

$$\begin{aligned} \Delta\psi &= A(2-\alpha)(n-\alpha)\frac{\langle x\rangle^{-\alpha}}{(1+t)^{1+\beta}} + A(2-\alpha)\alpha\frac{\langle x\rangle^{-2-\alpha}}{(1+t)^{1+\beta}} \\ &\ge \frac{(1+\beta)(n-\alpha)}{(2-\alpha)(2+\delta)}\frac{a(x)b(t)}{1+t} \\ &=: \left(\frac{(1+\beta)(n-\alpha)}{2(2-\alpha)} - \delta_1\right)\frac{a(x)b(t)}{1+t}. \end{aligned} \tag{4.2.4}$$

Here and after, $\delta_i (i = 1, 2, \dots)$ is a positive constant depending only on δ such that

$$\delta_i \to 0 \quad \text{as} \quad \delta \to 0.$$

We also have

$$\begin{aligned} (-\psi_t)a(x)b(t) &= Aa_0(1+\beta)\frac{\langle x\rangle^{2-2\alpha}}{(1+t)^{2+2\beta}} \\ &\ge \frac{a_0(1+\beta)}{(2-\alpha)^2 A}A^2(2-\alpha)^2\frac{\langle x\rangle^{-2\alpha}|x|^2}{(1+t)^{2+2\beta}} \\ &= (2+\delta)|\nabla\psi|^2. \end{aligned} \tag{4.2.5}$$

By multiplying (4.1.1) by $e^{2\psi}u_t$, it follows that

$$\begin{aligned}(4.2.6)\qquad &\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)\right]-\nabla\cdot(e^{2\psi}u_t\nabla u)\\ &+e^{2\psi}\left(a(x)b(t)-\frac{|\nabla\psi|^2}{-\psi_t}-\psi_t\right)u_t^2+\underbrace{\frac{e^{2\psi}}{-\psi_t}|\psi_t\nabla u-u_t\nabla\psi|^2}_{T_1}\\ &=\frac{\partial}{\partial t}\left[e^{2\psi}F(u)\right]+2e^{2\psi}(-\psi_t)F(u),\end{aligned}$$

where F is the primitive of f satisfying $F(0)=0$, namely $F'(u)=f(u)$, for example,

$$F(u)=\begin{cases}\pm\frac{1}{p+1}|u|^pu & \text{if } f(u)=\pm|u|^p,\\ \frac{1}{p+1}|u|^{p+1} & \text{if } f(u)=|u|^{p-1}u.\end{cases}$$

Using the Schwarz inequality and (4.2.5), we can calculate

$$\begin{aligned}T_1&=\frac{e^{2\psi}}{-\psi_t}(\psi_t^2|\nabla u|^2-2\psi_tu_t\nabla u\cdot\nabla\psi+u_t^2|\nabla\psi|^2)\\ &\geq\frac{e^{2\psi}}{-\psi_t}\left(\frac{1}{5}\psi_t^2|\nabla u|^2-\frac{1}{4}u_t^2|\nabla\psi|^2\right)\\ &\geq e^{2\psi}\left(\frac{1}{5}(-\psi_t)|\nabla u|^2-\frac{a(x)b(t)}{4(2+\delta)}u_t^2\right).\end{aligned}$$

From this and (4.2.5), we obtain

$$\begin{aligned}(4.2.7)\qquad &\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)\right]-\nabla\cdot(e^{2\psi}u_t\nabla u)\\ &+e^{2\psi}\left\{\left(\frac{1}{4}a(x)b(t)-\psi_t\right)u_t^2+\frac{-\psi_t}{5}|\nabla u|^2\right\}\\ &\leq\frac{\partial}{\partial t}\left[e^{2\psi}F(u)\right]+2e^{2\psi}(-\psi_t)F(u).\end{aligned}$$

By multiplying (4.2.7) by $(t_0+t)^{B+1-\varepsilon}$, here $t_0\geq 1$ is determined later, it follows that

$$\begin{aligned}(4.2.8)\qquad &\frac{\partial}{\partial t}\left[(t_0+t)^{B+1-\varepsilon}\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)\right]\\ &-(B+1-\varepsilon)(t_0+t)^{B-\varepsilon}\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)\\ &-\nabla\cdot((t_0+t)^{B+1-\varepsilon}e^{2\psi}u_t\nabla u)\\ &+e^{2\psi}(t_0+t)^{B+1-\varepsilon}\left\{\left(\frac{1}{4}a(x)b(t)-\psi_t\right)u_t^2+\frac{-\psi_t}{5}|\nabla u|^2\right\}\\ &\leq\frac{\partial}{\partial t}\left[(t_0+t)^{B+1-\varepsilon}e^{2\psi}F(u)\right]-(B+1-\varepsilon)(t_0+t)^{B-\varepsilon}e^{2\psi}F(u)\\ &+2(t_0+t)^{B+1-\varepsilon}e^{2\psi}(-\psi_t)F(u).\end{aligned}$$

We put

$$E(t) := \int_{\mathbf{R}^n} e^{2\psi}(u_t^2 + |\nabla u|^2)dx, \quad E_\psi(t) := \int_{\mathbf{R}^n} e^{2\psi}(-\psi_t)(u_t^2 + |\nabla u|^2)dx,$$
$$J(t; g) := \int_{\mathbf{R}^n} e^{2\psi} g dx, \qquad J_\psi(t; g) := \int_{\mathbf{R}^n} e^{2\psi}(-\psi_t) g dx.$$

Integrating (4.2.8) over the whole space, we have

$$\begin{aligned}
(4.2.9)\qquad & \frac{1}{2}\frac{d}{dt}\left[(t_0+t)^{B+1-\varepsilon}E(t)\right] - \frac{1}{2}(B+1-\varepsilon)(t_0+t)^{B-\varepsilon}E(t) \\
&+ \frac{1}{4}(t_0+t)^{B+1-\varepsilon}J(t, a(x)b(t)u_t^2) + \frac{1}{5}(t_0+t)^{B+1-\varepsilon}E_\psi(t) \\
&\le \frac{d}{dt}\left[(t_0+t)^{B+1-\varepsilon}\int e^{2\psi}F(u)dx\right] \\
&+ C(t_0+t)^{B+1-\varepsilon}J_\psi(t; |u|^{p+1}) + C(t_0+t)^{B-\varepsilon}J(t; |u|^{p+1}).
\end{aligned}$$

Noting that $e^{2\psi(t,x)}u_t(t,x)\nabla u(t,x) \in L^1(\mathbf{R}^n)$, we can use the divergence theorem and (4.2.9) is valid. Therefore, we integrate (4.2.9) on the interval $[0,t]$ and obtain the estimate for $(t_0+t)^{B+1-\varepsilon}E(t)$, which is the first term of $M(t)$:

$$\begin{aligned}
(4.2.10)\quad & (t_0+t)^{B+1-\varepsilon}E(t) - C\int_0^t (t_0+\tau)^{B-\varepsilon}E(\tau)d\tau \\
&+ \int_0^t (t_0+\tau)^{B+1-\varepsilon}J(\tau; a(x)b(t)u_t^2) + (t_0+\tau)^{B+1-\varepsilon}E_\psi(\tau)d\tau \\
&\le CI_0^2 + C(t_0+t)^{B+1-\varepsilon}J(t; |u|^{p+1}) \\
&+ C\int_0^t (t_0+\tau)^{B+1-\varepsilon}J_\psi(\tau; |u|^{p+1})d\tau \\
&+ C\int_0^t (t_0+t)^{B-\varepsilon}J(\tau; |u|^{p+1})d\tau.
\end{aligned}$$

In order to complete a priori estimate, however, we have to manage the second term of the inequality above whose sign is negative, and we also have to estimate the second term of $M(t)$. The following argument, which is little more complicated, can settle both of these problems.

First, we multiply (4.1.1) by $e^{2\psi}u$ and have

(4.2.11)

$$\begin{aligned}
& \frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t + \frac{a(x)b(t)}{2}u^2\right)\right] - \nabla\cdot(e^{2\psi}u\nabla u) \\
&+ e^{2\psi}\left\{|\nabla u|^2 + \left(-\psi_t + \frac{\beta}{2(1+t)}\right)a(x)b(t)u^2 + \underbrace{2u\nabla\psi\cdot\nabla u}_{T_2} - 2\psi_t uu_t - u_t^2\right\} \\
&= e^{2\psi}uf(u).
\end{aligned}$$

We calculate

$$\begin{aligned}
e^{2\psi}T_2 &= 4e^{2\psi}u\nabla\psi\cdot\nabla u - 2e^{2\psi}u\nabla\psi\cdot\nabla u \\
&= 4e^{2\psi}u\nabla\psi\cdot\nabla u - \nabla\cdot(e^{2\psi}u^2\nabla\psi) + 2e^{2\psi}u^2|\nabla\psi|^2 + e^{2\psi}(\Delta\psi)u^2
\end{aligned}$$

and by (4.2.4) we can rewrite (4.2.11) to

$$\begin{aligned}(4.2.12)\qquad &\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t + \frac{a(x)b(t)}{2}u^2\right)\right] - \nabla\cdot(e^{2\psi}(u\nabla u + u^2\nabla\psi))\\ &+ e^{2\psi}\Big\{\underbrace{|\nabla u|^2 + 4u\nabla u\cdot\nabla\psi + (-\psi_t)a(x)b(t) + 2|\nabla\psi|^2)u^2}_{T_3}\\ &+ (B-2\delta_1)\frac{a(x)b(t)}{2(1+t)}u^2 - 2\psi_t uu_t - u_t^2\Big\} \le e^{2\psi}uf(u).\end{aligned}$$

It follows from (4.2.5) that

$$\begin{aligned}T_3 &= |\nabla u|^2 + 4u\nabla u\cdot\nabla\psi\\ &\quad + \left\{\left(1-\frac{\delta}{3}\right)(-\psi_t)a(x)b(t) + 2|\nabla\psi|^2\right\}u^2 + \frac{\delta}{3}(-\psi_t)a(x)b(t)u^2\\ &\ge |\nabla u|^2 + 4u\nabla u\cdot\nabla\psi\\ &\quad + \left(4+\frac{\delta}{3}-\frac{\delta^2}{3}\right)|\nabla\psi|^2u^2 + \frac{\delta}{3}(-\psi_t)a(x)b(t)u^2\\ &= \left(1-\frac{4}{4+\delta_2}\right)|\nabla u|^2 + \delta_2|\nabla\psi|^2u^2\\ &\quad + \left|\frac{2}{\sqrt{4+\delta_2}}\nabla u + \sqrt{4+\delta_2}u\nabla\psi\right|^2 + \frac{\delta}{3}(-\psi_t)a(x)b(t)u^2\\ &\ge \delta_3(|\nabla u|^2 + |\nabla\psi|^2u^2) + \frac{\delta}{3}(-\psi_t)a(x)b(t)u^2,\end{aligned}$$

where

$$\delta_2 := \frac{\delta}{6} - \frac{\delta^2}{6},\quad \delta_3 := \min(1-\frac{4}{4+\delta_2}, \delta_2).$$

Thus, we obtain

$$\begin{aligned}(4.2.13)\qquad &\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t + \frac{a(x)b(t)}{2}u^2\right)\right] - \nabla\cdot(e^{2\psi}(u\nabla u + u^2\nabla\psi))\\ &+ e^{2\psi}\delta_3|\nabla u|^2\\ &+ e^{2\psi}\left(\delta_3|\nabla\psi|^2 + \frac{\delta}{3}(-\psi_t)a(x)b(t) + (B-2\delta_1)\frac{a(x)b(t)}{2(1+t)}\right)u^2\\ &+ e^{2\psi}(-2\psi_t uu_t - u_t^2)\\ &\le e^{2\psi}uf(u).\end{aligned}$$

Following Lin, Nishihara and Zhai [**55**], related to the size of $1+|x|^2$ and the size of $(1+t)^2$, we divide the space $\mathbf{R}^n$ into two disjoint zones $\Omega(t;K,t_0)$ and $\Omega^c(t;K,t_0)$, where

$$\Omega = \Omega(t;K,t_0) := \{x\in\mathbf{R}^n \mid (t_0+t)^2 \ge K + |x|^2\},$$

and $\Omega^c = \Omega^c(t;K,t_0) := \mathbf{R}^n\setminus\Omega(t;K,t_0)$ with $K\ge 1$ determined later. Since $a(x)b(t)\ge a_0(t_0+t)^{-(\alpha+\beta)}$ in the domain Ω, we multiply (4.2.7) by $(t_0+t)^{\alpha+\beta}$ and obtain

$$
\begin{aligned}
(4.2.14)\qquad & \frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(t_0+t)^{\alpha+\beta}(u_t^2+|\nabla u|^2)\right]-\nabla\cdot(e^{2\psi}(t_0+t)^{\alpha+\beta}u_t\nabla u)\\
&+e^{2\psi}\left[\left(\frac{a_0}{4}-\frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}\right)+(t_0+t)^{\alpha+\beta}(-\psi_t)\right]u_t^2\\
&+e^{2\psi}\left[\frac{-\psi_t}{5}(t_0+t)^{\alpha+\beta}-\frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}\right]|\nabla u|^2\\
&\le\frac{\partial}{\partial t}[(t_0+t)^{\alpha+\beta}e^{2\psi}F(u)]-\frac{\alpha+\beta}{(t_0+t)^{1-\alpha-\beta}}e^{2\psi}F(u)\\
&+2(t_0+t)^{\alpha+\beta}e^{2\psi}(-\psi_t)F(u).
\end{aligned}
$$

Let ν be a small positive number depends on δ, which will be chosen later. By (4.2.14)+ν(4.2.13), we have

$$
\begin{aligned}
(4.2.15)\qquad & \frac{\partial}{\partial t}\left[e^{2\psi}\left(\frac{(t_0+t)^{\alpha+\beta}}{2}u_t^2+\nu uu_t+\frac{\nu a(x)b(t)}{2}u^2+\frac{(t_0+t)^{\alpha+\beta}}{2}|\nabla u|^2\right)\right]\\
&-\nabla\cdot(e^{2\psi}(t_0+t)^{\alpha+\beta}u_t\nabla u+\nu e^{2\psi}(u\nabla u+u^2\nabla\psi))\\
&+e^{2\psi}\left[\left(\frac{a_0}{4}-\frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}-\nu\right)+(t_0+t)^{\alpha+\beta}(-\psi_t)\right]u_t^2\\
&+e^{2\psi}\left[\nu\delta_3-\frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}+\frac{-\psi_t}{5}(t_0+t)^{\alpha+\beta}\right]|\nabla u|^2\\
&+e^{2\psi}\nu\left[\delta_3|\nabla\psi|^2+\frac{\delta}{3}(-\psi_t)a(x)b(t)+(B-2\delta_1)\frac{a(x)b(t)}{2(1+t)}\right]u^2\\
&+2\nu e^{2\psi}(-\psi_t)uu_t\\
&\le\frac{\partial}{\partial t}[(t_0+t)^{\alpha+\beta}e^{2\psi}F(u)]-\frac{\alpha+\beta}{(t_0+t)^{1-\alpha-\beta}}e^{2\psi}F(u)\\
&+2(t_0+t)^{\alpha+\beta}e^{2\psi}(-\psi_t)F(u)+\nu e^{2\psi}uf(u).
\end{aligned}
$$

By the Schwarz inequality, the last term of the left-hand side of the above inequality can be estimated as

$$
\begin{aligned}
|2\nu(-\psi_t)uu_t| &\le \frac{\nu a_0\delta}{3}(-\psi_t)(t_0+t)^{-(\alpha+\beta)}u^2+\frac{3\nu}{a_0\delta}(-\psi_t)(t_0+t)^{\alpha+\beta}u_t^2\\
&\le\frac{\nu\delta}{3}(-\psi_t)a(x)b(t)u^2+\frac{3\nu}{a_0\delta}(-\psi_t)(t_0+t)^{\alpha+\beta}u_t^2
\end{aligned}
$$

in $\Omega(t;K,t_0)$. Thus, we have

$$
\begin{aligned}
(4.2.16)\qquad & \frac{\partial}{\partial t}\left[e^{2\psi}\left(\frac{(t_0+t)^{\alpha+\beta}}{2}u_t^2+\nu u u_t+\frac{\nu a(x)b(t)}{2}u^2+\frac{(t_0+t)^{\alpha+\beta}}{2}|\nabla u|^2\right)\right]\\
& -\nabla\cdot\left(e^{2\psi}(t_0+t)^{\alpha+\beta}u_t\nabla u+\nu e^{2\psi}(u\nabla u+u^2\nabla\psi)\right)\\
& +e^{2\psi}\left[\left(\frac{a_0}{4}-\frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}-\nu\right)\right.\\
& \quad\left.+\left(1-\frac{3\nu}{a_0\delta}\right)(t_0+t)^{\alpha+\beta}(-\psi_t)\right]u_t^2\\
& +e^{2\psi}\left[\nu\delta_3-\frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}+\frac{-\psi_t}{5}(t_0+t)^{\alpha+\beta}\right]|\nabla u|^2\\
& +e^{2\psi}\nu\left[\delta_3|\nabla\psi|^2+(B-2\delta_1)\frac{a(x)b(t)}{2(1+t)}\right]u^2\\
\le\ & \frac{\partial}{\partial t}[(t_0+t)^{\alpha+\beta}e^{2\psi}F(u)]-\frac{\alpha+\beta}{(t_0+t)^{1-\alpha-\beta}}e^{2\psi}F(u)\\
& +2(t_0+t)^{\alpha+\beta}e^{2\psi}(-\psi_t)F(u)+\nu e^{2\psi}uf(u).
\end{aligned}
$$

Now we choose the parameters ν and t_0 such that

$$
\begin{aligned}
\frac{a_0}{4}-\frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}-\nu\ge c_0,\quad 1-\frac{3\nu}{a_0\delta}\ge c_0,\\
\nu\delta_3-\frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}\ge c_0,\quad \nu\delta_3\ge c_0,\quad \frac{1}{5}\ge c_0,
\end{aligned}
$$

hold for some constant $c_0>0$. This is possible because we first determine ν sufficiently small depending on δ and then we choose t_0 sufficiently large depending on ν. Therefore, integrating (4.2.16) over $\Omega(t;K,t_0)$, we obtain the following energy inequality:

$$
(4.2.17)\qquad \frac{d}{dt}\overline{E}_\psi(t;\Omega(t;K,t_0))-N_1(t)-M_1(t)+H_\psi(t;\Omega(t;K,t_0))\le P_1,
$$

where

$$
\begin{aligned}
\overline{E}_\psi(t;\Omega) &= \overline{E}_\psi(t;\Omega(t;K,t_0))\\
&:= \int_\Omega e^{2\psi}\left(\frac{(t_0+t)^{\alpha+\beta}}{2}u_t^2+\nu u u_t+\frac{\nu a(x)b(t)}{2}u^2+\frac{(t_0+t)^{\alpha+\beta}}{2}|\nabla u|^2\right)dx,
\end{aligned}
$$

$$
\begin{aligned}
N_1(t) := \int_{\mathbf{S}^{n-1}} & e^{2\psi}\left(\frac{(t_0+t)^{\alpha+\beta}}{2}u_t^2+\nu u u_t+\frac{\nu a(x)b(t)}{2}u^2\right.\\
&\left.\left.+\frac{(t_0+t)^{\alpha+\beta}}{2}|\nabla u|^2\right)\right|_{|x|=\sqrt{(t_0+t)^2-K}}\\
&\times[(t_0+t)^2-K]^{(n-1)/2}d\theta\cdot\frac{d}{dt}\sqrt{(t_0+t)^2-K},
\end{aligned}
$$

$$
M_1(t) := \int_{\partial\Omega}\left(e^{2\psi}(t_0+t)^{\alpha+\beta}u_t\nabla u+\nu e^{2\psi}(u\nabla u+u^2\nabla\psi)\right)\cdot\vec{n}dS,
$$

$$
\begin{aligned}
H_\psi(t;\Omega) &= H_\psi(t;\Omega(t;K,t_0)) \\
&:= c_0\int_\Omega e^{2\psi}(1+(t_0+t)^{\alpha+\beta}(-\psi_t))(u_t^2+|\nabla u|^2)dx \\
&\quad + \nu(B-2\delta_1)\int_\Omega \frac{e^{2\psi}a(x)b(t)}{2(1+t)}u^2dx,
\end{aligned}
$$

$$
\begin{aligned}
P_1 := &\frac{d}{dt}\left[(t_0+t)^{\alpha+\beta}\int_\Omega e^{2\psi}F(u)dx\right] \\
&- \int_{\mathbf{S}^{n-1}} (t_0+t)^{\alpha+\beta}e^{2\psi}F(u)\Big|_{|x|=\sqrt{(t_0+t)^2-K}} \\
&\times [(t_0+t)^2-K]^{(n-1)/2}d\theta\cdot\frac{d}{dt}\sqrt{(t_0+t)^2-K} \\
&+ C\int_\Omega e^{2\psi}(1+(t_0+t)^{\alpha+\beta}(-\psi_t))|u|^{p+1}dx.
\end{aligned}
$$

Here $\vec{n}$ denotes the unit outer normal vector of $\partial\Omega$. We note that by $\nu\le a_0/4$ and

$$
\begin{aligned}
|\nu u u_t| &\le \frac{\nu a_0}{4}(t_0+t)^{-(\alpha+\beta)}u^2 + \frac{\nu(t_0+t)^{\alpha+\beta}}{a_0}u_t^2 \\
&\le \frac{\nu a(x)b(t)}{4}u^2 + \frac{(t_0+t)^{\alpha+\beta}}{4}u_t^2
\end{aligned}
$$

in $\Omega(t;K,t_0)$, it follows that

$$
\begin{aligned}
&c\int_\Omega e^{2\psi}(t_0+t)^{\alpha+\beta}(u_t^2+|\nabla u|^2)dx + c\int_\Omega e^{2\psi}a(x)b(t)u^2dx \\
&\le \overline{E}_\psi(t;\Omega(t;K,t_0)) \\
&\le C\int_\Omega e^{2\psi}(t_0+t)^{\alpha+\beta}(u_t^2+|\nabla u|^2)dx + C\int_\Omega e^{2\psi}a(x)b(t)u^2dx
\end{aligned}
$$

for some constants $c>0$ and $C>0$.

Next, we derive an energy inequality in the domain $\Omega^c(t;K,t_0)$. We use the notation

$$\langle x\rangle_K := (K+|x|^2)^{1/2}.$$

Since $a(x)b(t)\ge a_0\langle x\rangle_K^{-(\alpha+\beta)}$ in $\Omega^c(t,;K,t_0)$, we multiply (4.2.7) by $\langle x\rangle_K^{\alpha+\beta}$ and obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}\langle x\rangle_K^{\alpha+\beta}(u_t^2+|\nabla u|^2)\right] - \nabla\cdot(e^{2\psi}\langle x\rangle_K^{\alpha+\beta}u_t\nabla u) \\
&\quad + e^{2\psi}\left(\frac{a_0}{4}+(-\psi_t)\langle x\rangle_K^{\alpha+\beta}\right)u_t^2 + \frac{1}{5}e^{2\psi}(-\psi_t)\langle x\rangle_K^{\alpha+\beta}|\nabla u|^2 \\
&\quad + (\alpha+\beta)e^{2\psi}\langle x\rangle_K^{\alpha+\beta-2}x\cdot u_t\nabla u \\
&\le \frac{\partial}{\partial t}[e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)] + 2e^{2\psi}\langle x\rangle_K^{\alpha+\beta}(-\psi_t)F(u).
\end{aligned} \tag{4.2.18}
$$

By (4.2.14)+$\hat{\nu}\times$(4.2.13), here $\hat{\nu}$ is a small positive parameter determined later, it follows that

$$
\begin{aligned}
(4.2.19)\quad & \frac{\partial}{\partial t}\left[e^{2\psi}\left(\frac{\langle x\rangle_K^{\alpha+\beta}}{2}u_t^2+\hat{\nu}uu_t+\frac{\hat{\nu}a(x)b(t)}{2}u^2+\frac{\langle x\rangle_K^{\alpha+\beta}}{2}|\nabla u|^2\right)\right] \\
&\quad -\nabla\cdot\left(e^{2\psi}\langle x\rangle_K^{\alpha+\beta}u_t\nabla u+\hat{\nu}e^{2\psi}(u\nabla u+u^2\nabla\psi)\right) \\
&\quad +e^{2\psi}\left[\frac{a_0}{4}-\hat{\nu}+(-\psi_t)\langle x\rangle_K^{\alpha+\beta}\right]u_t^2+e^{2\psi}\left[\hat{\nu}\delta_3+\frac{-\psi_t}{5}\langle x\rangle_K^{\alpha+\beta}\right]|\nabla u|^2 \\
&\quad +e^{2\psi}\hat{\nu}\left[\delta_3|\nabla\psi|^2+\frac{\delta}{3}(-\psi_t)a(x)b(t)+(B-2\delta_1)\frac{a(x)b(t)}{2(1+t)}\right]u^2 \\
&\quad +e^{2\psi}\underbrace{\left[(\alpha+\beta)\langle x\rangle_K^{\alpha+\beta-2}x\cdot u_t\nabla u-2\hat{\nu}\psi_t uu_t\right]}_{T_4} \\
&\le \frac{\partial}{\partial t}\left[e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)\right]+2e^{2\psi}\langle x\rangle_K^{\alpha+\beta}(-\psi_t)F(u)+\hat{\nu}e^{2\psi}uf(u).
\end{aligned}
$$

By the Schwarz inequality, the terms T_4 can be estimated as

$$
\begin{aligned}
|(\alpha+\beta)\langle x\rangle_K^{\alpha+\beta-2}x\cdot u_t\nabla u| &\le (\alpha+\beta)\langle x\rangle_K^{\alpha+\beta-1}|u_t\nabla u| \\
&\le \frac{\hat{\nu}\delta_3}{2}|\nabla u|^2+\frac{(\alpha+\beta)^2}{2\hat{\nu}\delta_3K^{2(1-\alpha-\beta)}}u_t^2,
\end{aligned}
$$

$$
\begin{aligned}
|2\hat{\nu}(-\psi_t)uu_t| &\le \frac{\hat{\nu}a_0\delta}{3}(-\psi_t)\langle x\rangle_K^{-(\alpha+\beta)}u^2+\frac{3\hat{\nu}}{a_0\delta}(-\psi_t)\langle x\rangle_K^{\alpha+\beta}u_t^2 \\
&\le \frac{\hat{\nu}\delta}{3}(-\psi_t)a(x)b(t)u^2+\frac{3\hat{\nu}}{a_0\delta}(-\psi_t)\langle x\rangle_K^{\alpha+\beta}u_t^2.
\end{aligned}
$$

From this we can rewrite (4.2.19) as

$$
\begin{aligned}
(4.2.20)\quad & \frac{\partial}{\partial t}\left[e^{2\psi}\left(\frac{\langle x\rangle_K^{\alpha+\beta}}{2}u_t^2+\hat{\nu}uu_t+\frac{\hat{\nu}a(x)b(t)}{2}u^2+\frac{\langle x\rangle_K^{\alpha+\beta}}{2}|\nabla u|^2\right)\right] \\
&\quad -\nabla\cdot\left(e^{2\psi}\langle x\rangle_K^{\alpha+\beta}u_t\nabla u+\hat{\nu}e^{2\psi}(u\nabla u+u^2\nabla\psi)\right) \\
&\quad +e^{2\psi}\left[\left(\frac{a_0}{4}-\hat{\nu}-\frac{(\alpha+\beta)^2}{2\hat{\nu}\delta_3K^{2(1-\alpha-\beta)}}\right)+\left(1-\frac{3\hat{\nu}}{a_0\delta}\right)(-\psi_t)\langle x\rangle_K^{\alpha+\beta}\right]u_t^2 \\
&\quad +e^{2\psi}\left[\frac{\hat{\nu}\delta_3}{2}+\frac{-\psi_t}{5}\langle x\rangle_K^{\alpha+\beta}\right]|\nabla u|^2 \\
&\quad +e^{2\psi}\hat{\nu}\left[\delta_3|\nabla\psi|^2+(B-2\delta_1)\frac{a(x)b(t)}{2(1+t)}\right]u^2 \\
&\le \frac{\partial}{\partial t}\left[e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)\right]+2e^{2\psi}\langle x\rangle_K^{\alpha+\beta}(-\psi_t)F(u)+\hat{\nu}e^{2\psi}uf(u).
\end{aligned}
$$

Now we choose the parameters $\hat{\nu}$ and K in the same manner as before. Indeed taking $\hat{\nu}$ sufficiently small depending on δ and then choosing K sufficiently large depending on $\hat{\nu}$, we can obtain

$$
\frac{a_0}{4}-\hat{\nu}-\frac{(\alpha+\beta)^2}{2\hat{\nu}\delta_3K^{2(1-\alpha-\beta)}}\ge c_1,\quad 1-\frac{3\hat{\nu}}{a_0\delta}\ge c_1,\quad \hat{\nu}\delta_3\ge c_1,\quad \frac{1}{5}\ge c_1
$$

for some constant $c_1 > 0$. Consequently, By integrating (4.2.20) on Ω^c, the energy inequality on Ω^c follows:

$$\frac{d}{dt}\overline{E}_\psi(t;\Omega^c(t;K,t_0)) + N_2(t) + M_2(t) + H_\psi(t;\Omega^c(t;K,t_0)) \le P_2, \tag{4.2.21}$$

where

$$\begin{aligned}
\overline{E}_\psi(t;\Omega^c) &= \overline{E}_\psi(t;\Omega^c(t;K,t_0))\\
&:= \int_{\Omega^c} e^{2\psi}\left(\frac{\langle x\rangle_K^{\alpha+\beta}}{2}u_t^2 + \hat{\nu}uu_t + \frac{\hat{\nu}a(x)b(t)}{2}u^2 + \frac{\langle x\rangle_K^{\alpha+\beta}}{2}|\nabla u|^2\right)dx,
\end{aligned}$$

$$\begin{aligned}
N_2(t) :=& \int_{\mathbf{S}^{n-1}} e^{2\psi}\left(\frac{\langle x\rangle_K^{\alpha+\beta}}{2}u_t^2 + \hat{\nu}uu_t + \frac{\hat{\nu}a(x)b(t)}{2}u^2\right.\\
&\left.+\frac{\langle x\rangle_K^{\alpha+\beta}}{2}|\nabla u|^2\right)\Big|_{|x|=\sqrt{(t_0+t)^2-K}}\\
&\times[(t_0+t)^2-K]^{(n-1)/2}d\theta\cdot\frac{d}{dt}\sqrt{(t_0+t)^2-K},
\end{aligned}$$

$$M_2(t) := \int_{\partial\Omega^c}(e^{2\psi}\langle x\rangle_K^{\alpha+\beta}u_t\nabla u + \hat{\nu}e^{2\psi}(u\nabla u + u^2\nabla\psi))\cdot\vec{n}dS,$$

$$\begin{aligned}
H_\psi(t;\Omega^c) &= H_\psi(t;\Omega^c(t;K,t_0))\\
&:= c_1\int_\Omega e^{2\psi}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t))(u_t^2+|\nabla u|^2)dx\\
&\quad + \hat{\nu}(B-2\delta_1)\int_{\Omega^c}\frac{e^{2\psi}a(x)b(t)}{2(1+t)}u^2dx,
\end{aligned}$$

$$\begin{aligned}
P_2 :=& \frac{d}{dt}\left[\int_{\Omega^c} e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)dx\right]\\
&+\int_{\mathbf{S}^{n-1}}\langle x\rangle_K^{\alpha+\beta}e^{2\psi}F(u)\Big|_{|x|=\sqrt{(t_0+t)^2-K}}\\
&\times[(t_0+t)^2-K]^{(n-1)/2}d\theta\cdot\frac{d}{dt}\sqrt{(t_0+t)^2-K}\\
&+C\int_{\Omega^c} e^{2\psi}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t))|u|^{p+1}dx.
\end{aligned}$$

In a similar way as for the case in Ω, we note that

$$\begin{aligned}
&c\int_{\Omega^c} e^{2\psi}(t_0+t)^{\alpha+\beta}(u_t^2+|\nabla u|^2)dx + c\int_{\Omega^c} e^{2\psi}a(x)b(t)u^2dx\\
&\le \overline{E}_\psi(t;\Omega^c(t;K,t_0))\\
&\le C\int_{\Omega^c} e^{2\psi}(t_0+t)^{\alpha+\beta}(u_t^2+|\nabla u|^2)dx + C\int_{\Omega^c} e^{2\psi}a(x)b(t)u^2dx
\end{aligned}$$

for some constants $c > 0$ and $C > 0$.

We add the energy inequalities on Ω and Ω^c. We note that replacing ν and $\hat{\nu}$ by $\nu_0 := \min\{\nu,\hat{\nu}\}$, we can still have the inequalities (4.2.17) and (4.2.21), provided that we retake t_0 and K larger.

By ((4.2.17)+(4.2.21)) $\times (t_0+t)^{B-\varepsilon}$, we have

$$\begin{aligned}(4.2.22)\qquad &\frac{d}{dt}[(t_0+t)^{B-\varepsilon}(\overline{E}_\psi(t;\Omega)+\overline{E}_\psi(t;\Omega^c))]\\ &-\underbrace{(B-\varepsilon)(t_0+t)^{B-1-\varepsilon}(\overline{E}_\psi(t;\Omega)+\overline{E}_\psi(t;\Omega^c))}_{T_5}\\ &+\underbrace{(t_0+t)^{B-\varepsilon}(H_\psi(t;\Omega)+H_\psi(t;\Omega^c))}_{T_6}\\ &\le (t_0+t)^{B-\varepsilon}(P_1+P_2),\end{aligned}$$

here we note that

$$N_1(t)=N_2(t),\quad M_1(t)=M_2(t)$$

on $\partial\Omega$. Since

$$|\nu_0 uu_t|\le \frac{\nu_0\delta_4}{2}a(x)b(t)u^2+\frac{\nu_0}{2\delta_4 a_0}(t_0+t)^{\alpha+\beta}u_t^2$$

on Ω and

$$|\nu_0 uu_t|\le \frac{\nu_0\delta_4}{2}a(x)b(t)u^2+\frac{\nu_0}{2\delta_4 a_0}\langle x\rangle_K^{\alpha+\beta}u_t^2$$

on Ω^c, here δ_4 is chosen later, we have

$$(4.2.23)\qquad -T_5+T_6\ge (t_0+t)^{B-\varepsilon}I_1+(t_0+t)^{B-\varepsilon}I_2,$$

where

$$\begin{aligned}I_1 :=& \int_\Omega e^{2\psi}\left\{\frac{c_0}{2}(1+(t_0+t)^{\alpha+\beta}(-\psi_t)) -\frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)(t_0+t)^{\alpha+\beta}\right\}u_t^2\\ &+e^{2\psi}\left\{\frac{c_0}{2}(1+(t_0+t)^{\alpha+\beta}(-\psi_t))-\frac{B-\varepsilon}{2(t_0+t)}(t_0+t)^{\alpha+\beta}\right\}|\nabla u|^2dx\\ &+\int_{\Omega^c} e^{2\psi}\left\{\frac{c_1}{2}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t))-\frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)\langle x\rangle_K^{\alpha+\beta}\right\}u_t^2\\ &+e^{2\psi}\left\{\frac{c_1}{2}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t))-\frac{B-\varepsilon}{2(t_0+t)}\langle x\rangle_K^{\alpha+\beta}\right\}|\nabla u|^2dx\\ =:& I_{11}+I_{12},\end{aligned}$$

$$\begin{aligned}I_2 :=& \nu_0(B-2\delta_1-(1+\delta_4)(B-\varepsilon))\left(\int_\Omega+\int_{\Omega^c}\right)e^{2\psi}\frac{a(x)b(t)}{2(1+t)}u^2dx\\ &+\frac{c_2}{2}\int_{\mathbf{R}^n}e^{2\psi}(u_t^2+|\nabla u|^2)dx,\end{aligned}$$

and $c_2:=\min(c_0,c_1)$. Recall the definition of ε and δ_1 (i.e. (4.1.5) and (4.2.4)). A simple calculation shows $\varepsilon=3\delta_1$. Choosing δ_4 sufficiently small depending on ε, we have

$$(t_0+t)^{B-\varepsilon}I_2\ge c_3(t_0+t)^{B-1-\varepsilon}\int_{\mathbf{R}^n}e^{2\psi}a(x)b(t)u^2dx+\frac{c_2}{2}(t_0+t)^{B-\varepsilon}E(t)$$

for some constant $c_3>0$. Next, we prove that $I_1\ge 0$. By noting that $\alpha+\beta<1$, it is easy to see that $I_{11}\ge 0$ if we retake t_0 larger depending on c_0,ν_0 and δ_4. To estimate I_{12}, we further divide the region Ω^c into

$$\Omega^c(t;K,t_0)=(\Omega^c(t;K,t_0)\cap\Sigma_L)\cup(\Omega^c(t;K,t_0)\cap\Sigma_L^c),$$

where

$$\Sigma_L:=\{x\in\mathbf{R}^n;\langle x\rangle^{2-\alpha}\le L(1+t)^{1+\beta}\},\quad \Sigma_L^c:=\mathbf{R}^n\setminus\Sigma_L$$

with $L \gg 1$ determined later. First, since

$$K + |x|^2 \le K(1+|x|^2) \le KL^{2/(2-\alpha)}(1+t)^{2(1+\beta)/(2-\alpha)}$$

on $\Omega^c \cap \Sigma_L$, we have

$$\begin{aligned}
&\frac{c_1}{2}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t)) - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)\langle x\rangle_K^{\alpha+\beta} \\
&\ge \frac{c_1}{2} - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)K^{(\alpha+\beta)/2}L^{(\alpha+\beta)/(2-\alpha)}(1+t)^{\frac{(1+\beta)(\alpha+\beta)}{2-\alpha}}.
\end{aligned}$$

We note that $-1+\frac{(1+\beta)(\alpha+\beta)}{2-\alpha} < 0$ by $\alpha+\beta<1$. Thus, we obtain

$$\frac{c_1}{2} - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)K^{(\alpha+\beta)/2}L^{(\alpha+\beta)/(2-\alpha)}(1+t)^{\frac{(1+\beta)(\alpha+\beta)}{2-\alpha}} \ge 0$$

for large t_0 depending on L and K. Secondly, on $\Omega^c \cap \Sigma_L^c$, we have

$$\begin{aligned}
&\frac{c_1}{2}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t)) - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)\langle x\rangle_K^{\alpha+\beta} \\
&\ge \left\{\frac{c_1}{2}(1+\beta)\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{2+\beta}} - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)\right\}\langle x\rangle_K^{\alpha+\beta} \\
&\ge \left\{\frac{c_1}{2}(1+\beta)\frac{L}{1+t} - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)\right\}\langle x\rangle_K^{\alpha+\beta}.
\end{aligned}$$

Therefore one can obtain $I_{12} \ge 0$, provided that $L \ge \frac{B-\varepsilon}{c_1(1+\beta)}(1+\frac{2\nu_0}{\delta_4 a_0})$. Consequently, we have $I_1 \ge 0$. By (4.2.23) and that we mentioned above, it follows that

$$-T_5 + T_6 \ge c_3(t_0+t)^{B-1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}a(x)b(t)u^2dx + \frac{c_2}{2}(t_0+t)^{B-\varepsilon}E(t).$$

Therefore, by (4.2.22), we have

$$\begin{aligned}
&\frac{d}{dt}[(t_0+t)^{B-\varepsilon}(\overline{E}_\psi(t;\Omega)+\overline{E}_\psi(t;\Omega^c)] + \frac{c_2}{2}(t_0+t)^{B-\varepsilon}E(t) \\
&\quad + c_3(t_0+t)^{B-1-\varepsilon}J(t;a(x)b(t)u^2) \\
&\le (t_0+t)^{B-\varepsilon}(P_1+P_2).
\end{aligned} \tag{4.2.24}$$

Integrating (4.2.24) on the interval $[0,t]$, one can obtain the energy inequality on the whole space:

$$\begin{aligned}
&(t_0+t)^{B-\varepsilon}(\overline{E}_\psi(t;\Omega)+\overline{E}_\psi(t;\Omega^c)) + \frac{c_2}{2}\int_0^t (t_0+\tau)^{B-\varepsilon}E(\tau)d\tau \\
&\quad + c_3\int_0^t (t_0+\tau)^{B-1-\varepsilon}J(\tau;a(x)b(\tau)u^2)d\tau \\
&\le CI_0^2 + \int_0^t (t_0+\tau)^{B-\varepsilon}(P_1+P_2)d\tau,
\end{aligned} \tag{4.2.25}$$

where

$$I_0^2 := \int_{\mathbf{R}^n} e^{2\psi(0,x)}(u_1^2+|\nabla u_0|^2+|u_0|^2)dx.$$

By (4.2.25) $+\mu\times$ (4.2.10), here μ is a small positive parameter determined later, it follows that

$$
\begin{aligned}
(4.2.26)\quad & (t_0+t)^{B-\varepsilon}\overline{E}_\psi(t;\Omega)+(t_0+t)^{B-\varepsilon}\overline{E}_\psi(t;\Omega^c)+\mu(t_0+t)^{B+1-\varepsilon}E(t)\\
&+\int_0^t \frac{c_2}{2}(t_0+\tau)^{B-\varepsilon}E(\tau)-\mu C(t_0+\tau)^{B-\varepsilon}E(\tau)d\tau\\
&+c_3\int_0^t (t_0+\tau)^{B-1-\varepsilon}J(\tau;a(x)b(\tau)u^2)d\tau\\
&+\mu\int_0^t (t_0+\tau)^{B+1-\varepsilon}J(\tau;a(x)b(\tau)u_t^2)+(t_0+\tau)^{B+1-\varepsilon}E_\psi(\tau)d\tau\\
\le{}& CI_0^2+P\\
&+C(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1})\\
&+C\int_0^t (t_0+\tau)^{B+1-\varepsilon}J_\psi(\tau;|u|^{p+1})d\tau\\
&+C\int_0^t (t_0+\tau)^{B-\varepsilon}J(\tau;|u|^{p+1})d\tau,
\end{aligned}
$$

where

$$P=\int_0^t (t_0+\tau)^{B-\varepsilon}(P_1+P_2)d\tau.$$

Now we choose μ sufficiently small, then we can rewrite (4.2.26) as

$$
\begin{aligned}
(4.2.27)\quad & (t_0+t)^{B+1-\varepsilon}E(t)+(t_0+t)^{B-\varepsilon}J(t;a(x)b(t)u^2)\\
&+\int_0^t \Big\{(t_0+\tau)^{B-\varepsilon}E(\tau)+(t_0+\tau)^{B+1-\varepsilon}E_\psi(\tau)\\
&+(t_0+\tau)^{B+1-\varepsilon}J(\tau;a(x)b(\tau)u_t^2)\Big\}d\tau\\
\le{}& CI_0^2+P+C(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1})\\
&+C\int_0^t (t_0+\tau)^{B+1-\varepsilon}J_\psi(\tau;|u|^{p+1})d\tau\\
&+C\int_0^t (t_0+\tau)^{B-\varepsilon}J(\tau;|u|^{p+1})d\tau.
\end{aligned}
$$

We shall estimate the right-hand side of (4.2.27). We first estimate the term

$$(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1}).$$

Applying the Gagliardo-Nirenberg inequality (see Lemma 9.10 in Appendix), we have

$$
\begin{aligned}
(4.2.28)\quad J(t;|u|^{p+1})\le{}& C\left(\int_{\mathbf{R}^n} e^{\frac{4}{p+1}\psi}u^2dx\right)^{(1-\sigma)(p+1)/2}\\
&\times\left(\int_{\mathbf{R}^n} e^{\frac{4}{p+1}\psi}|\nabla\psi|^2u^2dx+\int_{\mathbf{R}^n} e^{\frac{4}{p+1}\psi}|\nabla u|^2dx\right)^{\sigma(p+1)/2}
\end{aligned}
$$

with $\sigma = \frac{n(p-1)}{2(p+1)}$. Here we note that $\sigma \in [0,1]$ if $1 < p \le \frac{n+2}{n-2}$ $(n = 3)$, $1 < p < \infty$ $(n = 1, 2)$. We see that

$$\begin{aligned} e^{\frac{4}{p+1}\psi}u^2 &= (e^{2\psi}a(x)b(t)u^2)a(x)^{-1}b(t)^{-1}e^{\left(\frac{4}{p+1}-2\right)\psi} \\ &\le C(e^{2\psi}a(x)b(t)u^2)\left[\left(\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}\right)^{\frac{\alpha}{2-\alpha}} e^{\left(\frac{4}{p+1}-2\right)\psi}\right] \\ &\qquad \times (1+t)^{\beta+\alpha(1+\beta)/(2-\alpha)} \\ &\le C(1+t)^{\beta+\alpha(1+\beta)/(2-\alpha)}e^{2\psi}a(x)b(t)u^2 \end{aligned}$$

and

$$\begin{aligned} e^{\frac{4}{(p+1)}\psi}|\nabla\psi|^2u^2 &\le C\frac{\langle x\rangle^{2-2\alpha}}{(1+t)^{2+2\beta}}e^{\frac{1}{2}\left(\frac{4}{p+1}-2\right)\psi}e^{\frac{1}{2}\left(\frac{4}{p+1}-2\right)\psi}e^{2\psi}u^2 \\ &\le Ce^{\frac{1}{2}\left(\frac{4}{p+1}-2\right)\psi}e^{2\psi}\left[\left(\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}\right)^{\frac{2-2\alpha}{2-\alpha}} e^{\frac{1}{2}\left(\frac{4}{p+1}-2\right)\psi}\right] \\ &\qquad \times (1+t)^{-2(1+\beta)+(1+\beta)(2-2\alpha)/(2-\alpha)}u^2 \\ &\le C(1+t)^{-2(1+\beta)/(2-\alpha)}e^{\frac{1}{2}\left(\frac{4}{p+1}-2\right)\psi}e^{2\psi}u^2 \\ &\le C(1+t)^{-2(1+\beta)/(2-\alpha)}(1+t)^{\beta+\alpha(1+\beta)/(2-\alpha)}e^{2\psi}a(x)b(t)u^2 \\ &= C(1+t)^{-1}e^{2\psi}a(x)b(t)u^2. \end{aligned}$$

Using them, we can estimate (4.2.28) as

$$\begin{aligned} J(t;|u|^{p+1}) &\le C(1+t)^{[\beta+(1+\beta)\alpha/(2-\alpha)](1-\sigma)(p+1)/2}J(t;a(x)b(t)u^2)^{(1-\sigma)(p+1)/2} \\ &\qquad \times [(1+t)^{-1}J(t;a(x)b(t)u^2)+E(t)]^{\sigma(p+1)/2} \end{aligned}$$

and hence

$$(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1}) \le C(t_0+t)^{\gamma}M(t)^{(p+1)/2},$$

where

$$\begin{aligned} \gamma = B+1-\varepsilon+\left[\beta+(1+\beta)\frac{\alpha}{2-\alpha}\right]\frac{1-\sigma}{2}(p+1)-\frac{\sigma}{2}(p+1) \\ -(B-\varepsilon)\frac{p+1}{2} \end{aligned} \tag{4.2.29}$$

By a simple calculation it follows that if

$$p > 1+\frac{2}{n-\alpha},$$

then by taking ε sufficiently small (i.e. δ sufficiently small) γ is negative. We note that

$$J_\psi(t;|u|^{p+1}) = \int_{\mathbf{R}^n} e^{2\psi}(-\psi_t)|u|^{p+1}dx \le \frac{C}{1+t}\int_{\mathbf{R}^n} e^{(2+\rho)\psi}|u|^{p+1}dx,$$

where ρ is a sufficiently small positive number. Therefore, we can estimate the terms

$$\int_0^t (t_0+\tau)^{B+1-\varepsilon}J_\psi(\tau;|u|^{p+1})d\tau \quad \text{and} \quad \int_0^t (t_0+\tau)^{B-\varepsilon}J(\tau;|u|^{p+1})d\tau$$

in the same manner as before. Noting that

$$
\begin{aligned}
P_1 + P_2 = \frac{d}{dt}\left[(t_0+t)^{\alpha+\beta}\int_\Omega e^{2\psi}F(u)dx + \int_{\Omega^c} e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)dx\right] \\
+ C\int_\Omega e^{2\psi}(1+(t_0+t)^{\alpha+\beta}(-\psi_t))|u|^{p+1}dx \\
+ C\int_{\Omega^c} e^{2\psi}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t))|u|^{p+1}dx,
\end{aligned}
$$

we have

$$
\begin{aligned}
P &= \int_0^t (t_0+\tau)^{B-\varepsilon}(P_1+P_2)d\tau \\
&\le CI_0^2 + C(t_0+t)^{B-\varepsilon}\int_\Omega e^{2\psi}(t_0+t)^{\alpha+\beta}F(u)dx \\
&\quad + C(t_0+t)^{B-\varepsilon}\int_{\Omega^c} e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)dx \\
&\quad + C\int_0^t (t_0+\tau)^{B-1-\varepsilon}\int_\Omega e^{2\psi}(t_0+\tau)^{\alpha+\beta}F(u)dxd\tau \\
&\quad + C\int_0^t (t_0+\tau)^{B-1-\varepsilon}\int_{\Omega^c} e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)dxd\tau \\
&\quad + C\int_0^t (t_0+\tau)^{B-\varepsilon}\int_\Omega e^{2\psi}(1+(t_0+\tau)^{\alpha+\beta}(-\psi_t))|u|^{p+1}dxd\tau \\
&\quad + C\int_0^t (t_0+\tau)^{B-\varepsilon}\int_{\Omega^c} e^{2\psi}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t))|u|^{p+1}dxd\tau.
\end{aligned}
$$

We calculate

$$
\begin{aligned}
e^{2\psi}\langle x\rangle_K^{\alpha+\beta} &= e^{2A\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}}\langle x\rangle_K^{\alpha+\beta} \\
&\le Ce^{2A\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}}\left(\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}\right)^{\frac{\alpha+\beta}{2-\alpha}}(1+t)^{\frac{(\alpha+\beta)(1+\beta)}{2-\alpha}} \\
&\le Ce^{(2+\rho)\psi}(1+t)^{\frac{(\alpha+\beta)(1+\beta)}{2-\alpha}}
\end{aligned}
$$

for small $\rho > 0$. Noting that $\frac{(\alpha+\beta)(1+\beta)}{2-\alpha} < 1$ and taking ρ sufficiently small, we can estimate the terms P in the same manner as estimating $(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1})$. Consequently, by (4.2.27), we have a priori estimate for $M(t)$:

$$
M(t) + L(t) \le CI_0^2 + CM(t)^{(p+1)/2}, \tag{4.2.30}
$$

where

$$
\begin{aligned}
L(t) = \int_0^t \Big\{&(t_0+\tau)^{B-\varepsilon}E(\tau) + (t_0+\tau)^{B+1-\varepsilon}E_\psi(\tau) \\
&+ (t_0+\tau)^{B-1-\varepsilon}J(\tau;a(x)b(\tau)u^2) + (t_0+\tau)^{B+1-\varepsilon}J(\tau;a(x)b(\tau)u_t^2)\Big\}d\tau.
\end{aligned}
$$

□

4.2.2. Proof of Theorem 4.1 and Corollary 4.2.

Proof of Theorem 4.1. In (4.2.30), we may assume that $C \geq 1$. We take I_0 sufficiently small so that it holds that

$$2CI_0^2 > CI_0^2 + C(2CI_0^2)^{p+1}, \tag{4.2.31}$$

that is,

$$2C(2CI_0^2)^p < 1.$$

Let M_1, M_2 $(M_1 < M_2)$ be the positive roots of the identity

$$M = CI_0^2 + CM^{p+1}.$$

Here we note that M_1, M_2 exist because of (4.2.31). By the continuity of $M(t)+L(t)$, we obtain

$$M(t) + L(t) \leq M_1$$

for all $t \geq 0$. Because, first, it is obvious that $M(0) + L(0) = M(0) = I_0^2 \leq CI_0^2$ and secondly, if $M_1 < M(t_0) + L(t_0) < M_2$ holds for some t_0, then we have

$$M(t_0) + L(t_0) > CI_0^2 + C(M(t_0) + L(t_0))^{p+1},$$

which contradicts (4.2.30). Thus, $M(t)$ is bounded and we have $T^* = +\infty$, where T^* is the maximal existence time of the local solution as in Proposition 4.3.

We note that

$$e^{2\psi} a(x) b(t) \geq c(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha}-\beta}$$

with some constant $c > 0$. Then we have

$$\int_{\mathbf{R}^n} e^{2\psi} a(x) b(t) u^2 dx \geq c(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha}-\beta} \int_{\mathbf{R}^n} u^2 dx. \tag{4.2.32}$$

This implies the decay estimate of global solution (4.1.4) and completes the proof of Theorem 4.1. □

Proof of Corollary 4.2. In a similar way to derive (4.2.32), we have

$$\int_{\mathbf{R}^n} e^{2\psi} a(x) b(t) u^2 dx \geq c(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha}-\beta} \int_{\mathbf{R}^n} e^{(2A-\mu)\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}} u^2 dx.$$

By noting that

$$\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}} \geq (1+t)^\rho$$

on $\Omega_\rho(t)$ and Theorem 4.1, it follows that

$$\begin{aligned}
&(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha}-\beta} \int_{\Omega_\rho(t)} e^{(2A-\mu)(1+t)^\rho} (u_t^2 + |\nabla u|^2 + u^2) dx \\
&\leq C(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha}-\beta} \int_{\Omega_\rho(t)} e^{(2A-\mu)\frac{\langle x\rangle^{2-\alpha}}{(1+t)^\beta}} (u_t^2 + |\nabla u|^2 + u^2) dx \\
&\leq C \int_{\mathbf{R}^n} e^{2\psi} (u_t^2 + |\nabla u|^2 + a(x) b(t) u^2) dx \\
&\leq C(1+t)^{-B+\varepsilon}.
\end{aligned}$$

Thus, we obtain

$$\int_{\Omega_\rho(t)} (u_t^2 + |\nabla u|^2 + u^2) dx \leq C(1+t)^{-\frac{(1+\beta)(n-2\alpha)}{2-\alpha}+\varepsilon} e^{-(2A-\mu)(1+t)^\rho}.$$

This proves Corollary 4.2. □

4.3. Estimates of the lifespan from below

The proof of Theorem 4.1 gives an estimate of the lifespan from below when $1 < p \le 1 + 2/(n-\alpha)$. We will discuss on estimates from above in Chapter 7. In this section, we consider the Cauchy problem (4.1.1) with the initial data $\lambda(u_0, u_1)$ instead of (u_0, u_1), where (u_0, u_1) is fixed and $\lambda > 0$ denotes the amplitude of the data. We define the lifespan of the local solution by

$$T_\lambda := \sup\{\, T \in (0,\infty]; \text{there exists a unique solution } u \in X(T)\,\},$$

where $X(T) = C([0,T); H^1(\mathbf{R}^n)) \cap C^1([0,T); L^2(\mathbf{R}^n))$. Then we have

Proposition 4.5. *Let*

$$1 < p \le 1 + \frac{2}{n-\alpha},$$

$(u_0, u_1) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ *satisfy* $I_0^2 < +\infty$ *and let* $\tilde{\varepsilon}$ *be any positive number. We assume that* $\alpha, \beta \in [0,1)$ *and* $\alpha + \beta < 1$. *Then there exists a constant* $C = C(\tilde{\varepsilon}, n, p, \alpha, \beta, u_0, u_1) > 0$ *such that for any* $\lambda > 0$, *we have*

$$C\lambda^{-1/\kappa + \tilde{\varepsilon}} \le T_\lambda,$$

where

$$\kappa = \frac{2(1+\beta)}{2-\alpha}\left(\frac{1}{p-1} - \frac{n-\alpha}{2}\right).$$

Proof. From the proof of (4.2.30), we can obtain

$$M(t) \le \lambda^2 C I_0^2 + C(1+t)^\gamma M(t)^{(p+1)/2},$$

where γ is defined by (4.2.29) and $C \ge 1$. We also note that

$$M(0) = \lambda^2 I_0^2 \le \lambda^2 C I_0^2.$$

Let T be the first time such that $M(t) = 2\lambda^2 C I_0^2$. Namely $M(T) = 2\lambda^2 C I_0^2$ and $M(t) < 2\lambda^2 C I_0^2$ are valid for any $0 \le t < T$. Then we have

$$2\lambda^2 C I_0^2 \le \lambda^2 C I_0^2 + C(1+T)^\gamma (2\lambda^2 C I_0^2)^{(p+1)/2}$$

and hence

$$C\lambda^{-(p-1)} \le (1+T)^\gamma.$$

Since $1 < p \le 2/(n-\alpha)$, we can see that $\gamma > 0$. Therefore, we obtain

$$C\lambda^{-(p-1)/\gamma} \le T < T_\lambda.$$

Now we calculate γ. Noting $\sigma = \frac{n(p-1)}{2(p+1)}$, $1-\sigma = \frac{1}{2(p+1)}((2-n)p+(n+2))$, $B = \frac{(1+\beta)(n-\alpha)}{2-\alpha} + \beta$, we can rewrite γ as

$$\begin{aligned}
\gamma &= \frac{(1+\beta)(n-\alpha)}{2-\alpha} + (1+\beta) - \varepsilon + \left[\beta + \frac{(1+\beta)\alpha}{2-\alpha}\right] \frac{(2-n)p+(n+2)}{4} \\
&\quad - \frac{n(p-1)}{4} - \left(\frac{(1+\beta)(n-\alpha)}{2-\alpha} + \beta - \varepsilon\right) \frac{p+1}{2} \\
&= (p-1)\left\{\left(\beta + \frac{(1+\beta)\alpha}{2-\alpha}\right)\frac{2-n}{4} - \frac{n}{4} - \frac{1}{2}\left(\frac{(1+\beta)(n-\alpha)}{2-\alpha} + \beta - \varepsilon\right)\right\} \\
&\quad + 1 + \beta + \frac{(1+\beta)\alpha}{2-\alpha} \\
&= (p-1)\left\{\frac{(1+\beta)\alpha}{2-\alpha} - \frac{n}{2}(1+\beta)\left(1 + \frac{\alpha}{2-\alpha}\right) + \frac{\varepsilon}{2}\right\} + \frac{2(1+\beta)}{2-\alpha} \\
&= (p-1)\left\{-\frac{(1+\beta)(n-\alpha)}{2-\alpha} + \frac{2(1+\beta)}{2-\alpha}\frac{1}{p-1} + \frac{\varepsilon}{2}\right\}.
\end{aligned}$$

This implies

$$-\frac{p-1}{\gamma} = -\frac{1}{\kappa} + \tilde{\varepsilon}$$

with $\tilde{\varepsilon}$ determined from ε. □

CHAPTER 5

Critical exponent for the semilinear wave equation with scale-invariant damping

5.1. Introduction and results

We consider the Cauchy problem for the semilinear damped wave equation

$$
\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{1+t} u_t = |u|^p, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n, \end{cases} \tag{5.1.1}
$$

where $\mu > 0$, $(u_0,u_1) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ and $1 < p \le \frac{n}{n-2}$ $(n \ge 3), 1 < p < \infty$ $(n = 1,2)$. Our aim is to determine the critical exponent p_c, which is the number defined by the following property:

If $p_c < p$, all small data solutions of (5.1.1) are global; if $1 < p \le p_c$, the time-local solution cannot be extended time-globally for some data regardless of smallness.

We note that the linear part of (5.1.1) is invariant with respect to the hyperbolic scaling

$$
\tilde{u}(t,x) = u(\lambda(1+t) - 1, \lambda x).
$$

In this case the asymptotic behavior of solutions is very delicate. It is known that the size of the damping term μ plays an essential role. The damping term $\mu/(1+t)$ is known as the borderline between the *effective* and *non-effective* dissipation, here *effective* means that the solution behaves like that of the corresponding parabolic equation, and *non-effective* means that the solution behaves like that of the free wave equation.

Concretely, for the linear damped wave equation

$$
u_{tt} - \Delta u + (1+t)^{-\beta} u_t = 0, \tag{5.1.2}
$$

if $-1 < \beta < 1$, then the solution u has the same L^p-L^q decay rates as those of the solution of the corresponding heat equation

$$
-\Delta v + (1+t)^{-\beta} v_t = 0. \tag{5.1.3}
$$

Moreover, if $-1/3 < \beta < 1$, then the lower frequency part of the solution u of (5.1.2) is asymptotically equivalent to that of a solution v of (5.1.3) in the L^2-sense (see [**117**]). This is called the *local diffusion phenomenon.* Wirth [**115**] also proved that when $\beta \le -1/3$, u is for each frequency asymptotically equivalent to that of v. This is called the *global diffusion phenomenon.* On the other hand, if $\beta > 1$, then the asymptotic profile of the solution of (5.1.2) is given by that of the free wave equation $\Box w = 0$ (see [**116, 118**]). We mention that Wirth treated more general time-dependent damping terms and we refer the reader to [**115, 116, 117, 118**] for detail.

Wirth [**114**] considered the linear problem

$$(5.1.4)\qquad \begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{1+t} u_t = 0, \\ (u, u_t)(0,x) = (u_0, u_1)(x). \end{cases}$$

He proved several L^p-L^q estimates for the solutions to (5.1.4). For example, if $\mu > 1$ it follows that

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{\max\{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{\mu}{2}, -n\left(\frac{1}{p}-\frac{1}{q}\right)\}} (\|u_0\|_{H_p^s} + \|u_1\|_{H_p^{s-1}}),$$

$$\|(u_t, \nabla u)(t,\cdot)\|_{L^q} \lesssim (1+t)^{\max\{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{\mu}{2}, -n\left(\frac{1}{p}-\frac{1}{q}\right)-1\}} (\|u_0\|_{H_p^{s+1}} + \|u_1\|_{H_p^s}),$$

where $1 < p \le 2$, $1/p + 1/q = 1$ and $s = n(1/p - 1/q)$. This shows that if μ is sufficiently large, then the solution behaves like that of the corresponding heat equation

$$(5.1.5)\qquad \frac{\mu}{1+t} v_t - \Delta v = 0$$

as $t \to \infty$, and if μ is sufficiently small, then the solution behaves like that of the free wave equation in the above sense.

The Gauss kernel of (5.1.5) is given by

$$G_\mu(t,x) = \left(\frac{\mu}{2\pi((1+t)^2 - 1)} \right)^{\frac{n}{2}} e^{-\frac{\mu|x|^2}{2((1+t)^2-1)}}.$$

We can obtain the L^p-L^q estimates of the solution of (5.1.5). In fact, it follows that

$$\|v(t,\cdot)\|_{L^q} \lesssim (1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \|v(0,\cdot)\|_{L^p}$$

for $1 \le p \le q \le \infty$. In particular, taking $q = 2$ and $p = 1$, we have

$$\|v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2}} \|v(0,\cdot)\|_{L^1}.$$

From the viewpoint of the diffusion phenomenon, we expect that the same type estimate holds for the solution of (5.1.4) when μ is large. To state our results, we introduce an auxiliary function

$$\psi(t,x) := \frac{a|x|^2}{(1+t)^2}, \quad a = \frac{\mu}{2(2+\delta)}$$

with a positive parameter δ. We have the following linear estimate:

THEOREM 5.1. *For any $\varepsilon > 0$, there exist constants $\delta > 0$ and $\mu_0 > 1$ having the following property: If $\mu \ge \mu_0$ and (u_0, u_1) satisfy*

$$I_0^2 := \int_{\mathbf{R}^n} e^{2\psi(0,x)} (u_0(x)^2 + |\nabla u_0(x)|^2 + u_1(x)^2) dx < +\infty,$$

then the solution of (5.1.4) *satisfies*

$$(5.1.6)\qquad \int_{\mathbf{R}^n} e^{2\psi} u(t,x)^2 dx \le C I_0^2 (1+t)^{-n+\varepsilon},$$

$$(5.1.7)\qquad \int_{\mathbf{R}^n} e^{2\psi} (u_t(t,x)^2 + |\nabla u(t,x)|^2) dx \le C I_0^2 (1+t)^{-n-2+\varepsilon}$$

for $t \ge 0$, where $C = C_{\mu,\varepsilon}$ is a positive constant depending on μ, ε.

REMARK 5.1. *The constant* μ_0 *depends on* ε. *The relation is*

$$\mu_0 \sim \varepsilon^{-2}.$$

Therefore, as ε *is smaller,* μ_0 *must be larger.*

We also consider the critical exponent problem for (5.1.1). For the corresponding heat equation (5.1.5) with nonlinear term $|u|^p$, the critical exponent is given by the Fujita critical exponent:

$$p_F := 1 + \frac{2}{n}.$$

Thus, we can expect that the critical exponent of (5.1.1) is also given by p_F if μ is sufficiently large.

We write $X(T) = C([0,T); H^1(\mathbf{R}^n)) \cap C^1([0,T); L^2(\mathbf{R}^n))$ for $T \in (0,\infty]$. We first recall a local existence result, which is proved in Appendix (see Proposition 9.21).

PROPOSITION 5.2. *Let*

$$1 < p \le \frac{n}{n-2} \ (n \ge 3), \quad 1 < p < \infty \ (n = 1, 2)$$

and $\delta > 0$. *If* $I_0^2 < +\infty$, *then there exists* $T^* \in (0, +\infty]$ *depending on* I_0^2 *such that the Cauchy problem* (5.1.1) *has a unique solution* $u \in X(T^*)$ *satisfying*

$$\int_{\mathbf{R}^n} e^{\psi(t,x)} (u_t^2 + |\nabla u|^2 + u^2)(t,x) dx < +\infty$$

for all $t \in (0, T^*)$. *Moreover, if* $T^* < +\infty$ *then we have*

$$\liminf_{t \to T^*} \int_{\mathbf{R}^n} e^{\psi(t,x)} (u_t^2 + |\nabla u|^2 + u^2)(t,x) dx = +\infty.$$

Our main result is the following:

THEOREM 5.3. *Let* $p_F < p \le n/(n-2)$ $(n \ge 3)$, $p_F < p < \infty$ $(n = 1, 2)$ *and* $0 < \varepsilon < 2n(p - p_F)/(p-1)$. *Then there exist constants* $\delta > 0$ *and* $\mu_0 > 1$ *having the following property: if* $\mu \ge \mu_0$ *and*

$$I_0^2 = \int_{\mathbf{R}^n} e^{2\psi(0,x)} (u_0^2 + |\nabla u_0|^2 + u_1^2) dx$$

is sufficiently small, then there exists a unique solution $u \in C([0,\infty); H^1(\mathbf{R}^n)) \cap C^1([0,\infty); L^2(\mathbf{R}^n))$ *of* (5.1.1) *satisfying*

$$\int_{\mathbf{R}^n} e^{2\psi} u^2 dx \le C_{\mu,\varepsilon} I_0^2 (1+t)^{-n+\varepsilon}, \tag{5.1.8}$$

$$\int_{\mathbf{R}^n} e^{2\psi} (u_t^2 + |\nabla u|^2) dx \le C_{\mu,\varepsilon} I_0^2 (1+t)^{-n-2+\varepsilon} \tag{5.1.9}$$

for $t \ge 0$, *where* $C_{\mu,\varepsilon}$ *is a positive constant depending on* μ *and* ε.

REMARK 5.2. *As before, we note that* μ_0 *depends on* ε. *The relation is*

$$\mu_0 \sim \varepsilon^{-2} \sim (p - p_F)^{-2}.$$

Therefore, as p *is closer to* p_F, μ_0 *must be larger. Thus, we can expect that* ε *can be removed and the same result holds for much smaller* μ. *Recently, D'Abbicco* [**5**] *improved the global existence result for* $\mu \ge n+2$ *and* $p > p_F$. *He also obtained the decay rates of the solution without any loss* ε.

We prove Theorem 1.2 by a weighted energy method developed by [**106**]. Lin, Nishihara and Zhai [**56**] refined this method to fit the damping term $b(t) = (1+t)^{-\beta}$ with $-1 < \beta < 1$. They used the property $\beta < 1$ and so we cannot apply their method directly to our problem (5.1.1). Therefore, we need a further modification. Instead of the property $\beta < 1$, we assume that μ is sufficiently large and modify the parameters used in the calculation.

REMARK 5.3. *We can also treat other nonlinear terms, for example* $-|u|^p, |u|^{p-1}u$.

We also have a blow-up result when $\mu > 1$ and $1 < p \le p_F$. We first introduce the definition of a weak solution.

DEFINITION 5.4. *Let* $T \in (0, \infty]$. *We say that* $u \in X(T)$ *is a (weak) solution of the Cauchy problem* (5.1.1) *on the interval* $[0, T)$ *if it holds that*

$$\begin{aligned} &\int_{[0,T)\times\mathbf{R}^n} u(t,x)(\partial_t^2\psi(t,x) - \partial_x^2\psi(t,x) - \partial_t\left(\frac{\mu}{1+t}\psi(t,x)\right)\Big) dxdt \\ &= \int_{\mathbf{R}^n} \{(\mu u_0(x) + u_1(x))\psi(0,x) - u_0(x)\partial_t\psi(0,x)\}\, dx \\ &\quad + \int_{[0,T)\times\mathbf{R}^n} |u(t,x)|^p \psi(t,x) dxdt \end{aligned} \tag{5.1.10}$$

for any $\psi \in C_0^\infty([0,T) \times \mathbf{R})$ *(see also Section 9.4.3). In particular, when* $T = \infty$, *we call* u *a global solution.*

THEOREM 5.5. *Let* $u_0, u_1 \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$, $1 < p \le p_F$ *and* $\mu > 1$. *Moreover, we assume that*

$$\liminf_{R\to\infty} \int_{|x|<R} (\mu - 1)u_0 + u_1 dx > 0. \tag{5.1.11}$$

Then there is no global solution for (5.1.1), *that is,*

$$\lim_{t\to T^*-0} \|(u, u_t)\|_{H^1\times L^2} = +\infty$$

holds for some $T^* \in (0, \infty)$.

REMARK 5.4. *(i)If*

$$(\mu - 1)u_0 + u_1 \in L^1(\mathbf{R}^n) \quad \textit{and} \quad \int_{\mathbf{R}^n} (\mu - 1)u_0 + u_1 dx > 0,$$

then the condition (5.1.11) *holds. (ii)Theorem 5.5 is essentially included in a recent work by D'Abbicco and Lucente* [**7**]. *In this paper we shall give a much simpler proof.*

One of our novelty is blow-up results for non-effective damping cases. We also obtain blow-up results in the case $0 < \mu \le 1$.

THEOREM 5.6. *Let* $0 < \mu \le 1$ *and*

$$1 < p \le 1 + \frac{2}{n + (\mu - 1)}.$$

We also assume $(u_0, u_1) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ *and*

$$\liminf_{R\to\infty} \int_{|x|<R} u_1(x) dx > 0. \tag{5.1.12}$$

Then there is no global solution for (5.1.1)*, that is,*

$$\lim_{t\to T^*-0} \|(u,u_t)\|_{H^1\times L^2} = +\infty$$

holds for some $T^* \in (0,\infty)$.

Remark 5.5. *(i)If*

$$u_1 \in L^1(\mathbf{R}^n) \quad \textit{and} \quad \int_{\mathbf{R}^n} u_1(x)dx > 0,$$

then the condition (5.1.12) *holds. (ii)In Theorem 5.6, we do not put any assumption on the data* u_0*, and the blow-up result even holds for some* $p \ge p_F$*. We can interpret this phenomena as that the equation* (5.1.1) *loses the parabolic structure and recover the hyperbolic structure if* μ *is sufficiently small.*

We prove this theorem by a test-function method. To transform the equation into divergence form, we follow [**56**] and transform the equation into divergence form by multiplying an appropriate function (see also [**7**]). In the same way of the proof of Theorem 5.6, we can treat the damping terms $(1+t)^{-\beta}$ with $\beta > 1$ (see Remark 5.6).

5.2. Proof of global existence

By Proposition 5.2, it suffices to proof the boundedness of $H^1(\mathbf{R}^n)\times L^2(\mathbf{R}^n)$ norm of solutions. We prove a priori estimate for the following functional:

$$M(t) := \sup_{0\le\tau\le t}\left\{(1+\tau)^{n+2-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}(u_t^2+|\nabla u|^2)dx + (1+\tau)^{n-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}u^2dx\right\}.$$

We shall prove the following a priori estimate for $M(t)$:

Proposition 5.7. *Let*

$$p_F < p \le (n+2)/(n-2)\ (n\ge 3), \quad p_F < p < \infty\ (n=1,2)$$

and $0 < \varepsilon < 2n(p-p_F)/(p-1)$*. Then there exist constants* $\delta > 0$ *and* $\mu_0 > 1$ *having the following property: if* $\mu \ge \mu_0$*,* (u_0,u_1) *satisfies* $I_0^2 < +\infty$ *and* $u \in X(T)$ *is a solution of the Cauchy problem* (5.1.1) *for some* $T > 0$ *such that*

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)}(u^2+|\nabla u|^2+u_t^2)dx < +\infty$$

for all $0 \le t < T$*, then it follows that*

$$M(t) + L(t) \le CI_0^2 + CM(t)^{p+1} \tag{5.2.1}$$

for any $0\le t<T$*, where*

$$L(t) := \int_0^t (1+\tau)^{n+1-\varepsilon}\left\{E(\tau) + (1+\tau)J_\psi(\tau;(u_t^2+|\nabla u|^2))\right\}d\tau.$$

and $C = C(p,\varepsilon,\mu)$ *is a positive constant depending only on* p,ε,μ.

Proof. We put $b(t) = \frac{\mu}{1+t}$ and $f(u) = |u|^p$. By a simple calculation, we have

$$-\psi_t = \frac{2}{1+t}\psi, \quad \nabla\psi = \frac{2ax}{(1+t)^2}, \quad \frac{|\nabla\psi|^2}{-\psi_t} = \frac{b(t)}{2+\delta},$$

and

$$\Delta\psi = \frac{n}{2+\delta}\frac{b(t)}{1+t} =: \left(\frac{n}{2}-\delta_1\right)\frac{b(t)}{1+t}.$$

Here and after, δ_i $(i = 1, 2, \dots)$ denote a positive constant depending only on δ such that

$$\delta_i \to 0 \quad \text{as} \quad \delta \to 0.$$

Multiplying (5.1.1) by $e^{2\psi}u_t$, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)\right] - \nabla\cdot(e^{2\psi}u_t\nabla u) \\
&\quad + e^{2\psi}\left(b(t)-\frac{|\nabla\psi|^2}{-\psi_t}-\psi_t\right)u_t^2 + \underbrace{\frac{e^{2\psi}}{-\psi_t}|\psi_t\nabla u - u_t\nabla\psi|^2}_{T_1} \\
&= \frac{\partial}{\partial t}\left[e^{2\psi}F(u)\right] + 2e^{2\psi}(-\psi_t)F(u),
\end{aligned} \tag{5.2.2}$$

where F is the primitive of f satisfying $F(0) = 0$. Using the Schwarz inequality, we can calculate

$$T_1 \ge e^{2\psi}\left(\frac{1}{5}(-\psi_t)|\nabla u|^2 - \frac{b(t)}{4(2+\delta)}u_t^2\right).$$

From this and integrating (5.2.2), we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)dx + \int_{\mathbf{R}^n}e^{2\psi}\left\{\left(\frac{b(t)}{4}-\psi_t\right)u_t^2+\frac{-\psi_t}{5}|\nabla u|^2\right\}dx \\
&\qquad \le \frac{d}{dt}\int_{\mathbf{R}^n}e^{2\psi}F(u)dx + 2\int_{\mathbf{R}^n}e^{2\psi}(-\psi_t)F(u)dx.
\end{aligned} \tag{5.2.3}$$

Here we note that $e^{2\psi(t,x)}u_t(t,x)\nabla u(t,x) \in L^1(\mathbf{R}^n)$ and so we can use the divergence theorem and (5.2.3) is valid. On the other hand, by multiplying (5.1.1) by $e^{2\psi}u$, it follows that

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t+\frac{b(t)}{2}u^2\right)\right] - \nabla\cdot(e^{2\psi}u\nabla u) \\
&\quad + e^{2\psi}\Big\{|\nabla u|^2 + \left(-\psi_t+\frac{1}{2(1+t)}\right)b(t)u^2 + \underbrace{2u\nabla\psi\cdot\nabla u}_{T_2} - 2\psi_t uu_t - u_t^2\Big\} \\
&= e^{2\psi}uf(u).
\end{aligned}$$

We calculate

$$\begin{aligned}
e^{2\psi}T_2 &= 4e^{2\psi}u\nabla\psi\cdot\nabla u - 2e^{2\psi}u\nabla\psi\cdot\nabla u \\
&= 4e^{2\psi}u\nabla\psi\cdot\nabla u - \nabla\cdot(e^{2\psi}u^2\nabla\psi) + 2e^{2\psi}u^2|\nabla\psi|^2 + e^{2\psi}(\Delta\psi)u^2
\end{aligned}$$

and have

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t+\frac{b(t)}{2}u^2\right)\right] - \nabla\cdot(e^{2\psi}(u\nabla u + u^2\nabla\psi)) \\
&\quad + e^{2\psi}\Big\{\underbrace{|\nabla u|^2 + 4u\nabla u\cdot\nabla\psi + ((-\psi_t)b(t)+2|\nabla\psi|^2)u^2}_{T_3} \\
&\quad + (n+1-2\delta_1)\frac{b(t)}{2(1+t)}u^2 - 2\psi_t uu_t - u_t^2\Big\} = e^{2\psi}uf(u).
\end{aligned} \tag{5.2.4}$$

The term T_3 is estimated as follows:

$$\begin{aligned} T_3 &= \left(1 - \frac{4}{4+\delta/2}\right)|\nabla u|^2 + \frac{\delta}{2}|\nabla\psi|^2 u^2 + \left|\frac{2}{\sqrt{4+\delta/2}}\nabla u + \sqrt{4+\delta/2}\nabla\psi\right|^2 \\ &\geq \delta_2(|\nabla u|^2 + b(t)(-\psi_t)u^2). \end{aligned}$$

Thus, we can rewrite (5.2.4) as

$$\begin{aligned} &\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t + \frac{b(t)}{2}u^2\right)\right] - \nabla\cdot(e^{2\psi}(u\nabla u + u^2\nabla\psi)) \\ &\quad + e^{2\psi}\left\{\delta_2(|\nabla u|^2 + b(t)(-\psi_t)u^2) + (n+1-2\delta_1)\frac{b(t)}{2(1+t)}u^2 - 2\psi_t uu_t - u_t^2\right\} \\ &\leq e^{2\psi}uf(u). \end{aligned}$$

Integrating the above inequality and then multiplying by a large parameter ν and adding $(1+t)\times$ (5.2.3), we obtain

$$\begin{aligned} &\frac{d}{dt}\left[\int_{\mathbf{R}^n} e^{2\psi}\left\{\frac{1+t}{2}(u_t^2 + |\nabla u|^2) + \nu uu_t + \frac{\nu b(t)}{2}u^2\right\}dx\right] \\ &\quad + \int_{\mathbf{R}^n} e^{2\psi}\Big\{\Big(\underbrace{\frac{\mu}{4} - \nu - \frac{1}{2}}_{T_4} + (-\psi_t)(1+t)\Big)u_t^2 + \Big(\underbrace{\nu\delta_2 - \frac{1}{2}}_{T_5} + \frac{(-\psi_t)(1+t)}{5}\Big)|\nabla u|^2 \\ &\quad + \nu\delta_2 b(t)(-\psi_t)u^2 + (n+1-2\delta_1)\frac{\nu b(t)}{2(1+t)}u^2 + \underbrace{2\nu(-\psi_t)uu_t}_{T_6}\Big\}dx \\ &\leq \frac{d}{dt}\left[(1+t)\int_{\mathbf{R}^n} e^{2\psi}F(u)dx\right] + C\int_{\mathbf{R}^n} e^{2\psi}(1+(1+t)(-\psi_t))|u|^{p+1}dx. \end{aligned}$$

We put a condition for μ and ν as

$$\frac{\mu}{4} - \nu - \frac{1}{2} > 0, \tag{5.2.5}$$

$$\nu\delta_2 - \frac{1}{2} > 0. \tag{5.2.6}$$

Then the terms T_4 and T_5 are positive. Using the Schwarz inequality, we can estimate T_6 as

$$|T_6| \leq \frac{1}{2}(-\psi_t)(1+t)u_t^2 + \frac{2\nu^2}{1+t}(-\psi_t)u^2.$$

Now we put another condition

$$\mu \geq \frac{2\nu}{\delta_2}. \tag{5.2.7}$$

Then we obtain the following estimate:

$$\begin{aligned} &\frac{d}{dt}\hat{E}(t) + H(t) + \frac{1}{5}(1+t)J_\psi(t;(u_t^2 + |\nabla u|^2)) + (n+1-2\delta_1)\frac{\nu b(t)}{2(1+t)}J(t;u^2) \\ &\leq \frac{d}{dt}[(1+t)J(t;F(u))] + C(J(t;|u|^{p+1}) + (1+t)J_\psi(t;|u|^{p+1})), \end{aligned} \tag{5.2.8}$$

where

$$\hat{E}(t) := \int_{\mathbf{R}^n} e^{2\psi}\left\{\frac{1+t}{2}(u_t^2 + |\nabla u|^2) + \nu uu_t + \frac{\nu b(t)}{2}u^2\right\}dx,$$

$$H(t) = \int_{\mathbf{R}^n} e^{2\psi} \left\{ \left(\frac{\mu}{4} - \nu - \frac{1}{2} \right) u_t^2 + \left(\nu\delta_2 - \frac{1}{2} \right) |\nabla u|^2 \right\} dx,$$
$$J(t;u) = \int_{\mathbf{R}^n} e^{2\psi} u dx, \quad J_\psi(t;u) = \int_{\mathbf{R}^n} e^{2\psi}(-\psi_t) u dx.$$

Multiplying (5.2.8) by $(1+t)^{n+1-\varepsilon}$, we have

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^{n+1-\varepsilon}\hat{E}(t)] - \underbrace{(n+1-\varepsilon)(1+t)^{n-\varepsilon}\hat{E}(t)}_{T_7} \\
&\quad + (1+t)^{n+1-\varepsilon} H(t) + \frac{1}{5}(1+t)^{n+2-\varepsilon} J_\psi(t; (u_t^2 + |\nabla u|^2)) \\
&\quad + (n+1-2\delta_1)(1+t)^{n+1-\varepsilon} \frac{\nu b(t)}{2(1+t)} J(t;u^2) \\
&\leq \frac{d}{dt}[(1+t)^{n+2-\varepsilon} J(t; F(u))] \\
&\quad + C((1+t)^{n+1-\varepsilon} J(t; |u|^{p+1}) + (1+t)^{n+2-\varepsilon} J_\psi(t; |u|^{p+1})).
\end{aligned}$$

Now we estimate the bad term T_7. First, by the Schwarz inequality, one can obtain

$$|\nu u u_t| \leq \frac{\nu}{4\delta_3 b(t)} u_t^2 + \delta_3 \nu b(t) u^2,$$

where δ_3 is determined later. From this, T_7 is estimated as

$$\begin{aligned}
T_7 &\leq (n+1-\varepsilon)(1+t)^{n-\varepsilon} \\
&\quad \times \int_{\mathbf{R}^n} e^{2\psi} \left\{ \left(\frac{1+t}{2} + \frac{\nu(1+t)}{4\delta_3 \mu} \right) u_t^2 + \frac{1+t}{2} |\nabla u|^2 + \frac{\nu b(t)}{2}(1+2\delta_3) u^2 \right\} dx.
\end{aligned}$$

We strengthen the conditions (5.2.5) and (5.2.6) as

$$\frac{\mu}{4} - \nu - \frac{1}{2} - (n+1-\varepsilon)\left(\frac{1}{2} + \frac{\nu}{4\delta_3\mu} \right) > 0, \tag{5.2.9}$$

$$\nu\delta_2 - \frac{1}{2}(n+2-\varepsilon) > 0. \tag{5.2.10}$$

Moreover, we take $\varepsilon = 3\delta_1$ and then choose δ_3 such that

$$(n+1-2\delta_1) - (n+1-3\delta_1)(1+2\delta_3) > 0.$$

Under these conditions, we can estimate T_7 and obtain

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^{n+1-\varepsilon}\hat{E}(t)] \\
&\quad + c(1+t)^{n+1-\varepsilon} E(t) + c(1+t)^{n+2-\varepsilon} J_\psi(t; (u_t^2 + |\nabla u|^2)) \\
&\leq \frac{d}{dt}[(1+t)^{n+2-\varepsilon} J(t; F(u))] \\
&\quad + C((1+t)^{n+1-\varepsilon} J(t; |u|^{p+1}) + (1+t)^{n+2-\varepsilon} J_\psi(t; |u|^{p+1}))
\end{aligned}$$

with some $c > 0$, where

$$E(t) := \int_{\mathbf{R}^n} e^{2\psi}(u_t^2 + |\nabla u|^2) dx.$$

By integrating the above inequality, it follows that

$$\begin{aligned}&(1+t)^{n+1-\varepsilon}\hat{E}(t)\\&\quad+c\int_0^t(1+\tau)^{n+1-\varepsilon}\left\{E(\tau)+(1+\tau)J_\psi(\tau;(u_t^2+|\nabla u|^2))\right\}d\tau\\&\le CI_0^2+(1+t)^{n+2-\varepsilon}J(t;|u|^{p+1})\\&\quad+C\int_0^t(1+\tau)^{n+1-\varepsilon}\left\{J(\tau;|u|^{p+1})+(1+\tau)J_\psi(\tau;|u|^{p+1}))\right\}d\tau.\end{aligned}$$

By a simple calculation, we have

$$(1+t)E(t)+\frac{1}{1+t}J(t;u^2)\le C\hat{E}(t).$$

Thus, we obtain

$$\begin{aligned}(5.2.11)\quad&(1+t)^{n+2-\varepsilon}E(t)+(1+t)^{n-\varepsilon}J(t;u^2)\\&\quad+c\int_0^t(1+\tau)^{n+1-\varepsilon}\left\{E(\tau)+(1+\tau)J_\psi(\tau;(u_t^2+|\nabla u|^2))\right\}d\tau\\&\le CI_0^2+(1+t)^{n+2-\varepsilon}J(t;|u|^{p+1})\\&\quad+C\int_0^t(1+\tau)^{n+1-\varepsilon}\left\{J(\tau;|u|^{p+1})+(1+\tau)J_\psi(\tau;|u|^{p+1}))\right\}d\tau.\end{aligned}$$

Now we turn to estimate the nonlinear terms. Noting that

$$J(t;|u|^{p+1})=\int_{\mathbf{R}^n}\left|e^{\frac{2}{p+1}\psi}u\right|^{p+1}dx$$

and

$$\nabla(e^{\frac{2}{p+1}\psi}u)=\frac{2}{p+1}e^{\frac{2}{p+1}\psi}(\nabla\psi)u+e^{\frac{2}{p+1}\psi}\nabla u,$$

we apply Lemma 9.10 to $J(t;|u|^{p+1})$ and have

$$\begin{aligned}J(t;|u|^{p+1})\le{}&C\left(\int_{\mathbf{R}^n}e^{\frac{4}{p+1}\psi}u^2dx\right)^{\frac{1-\sigma}{2}(p+1)}\\&\times\left(\int_{\mathbf{R}^n}e^{\frac{4}{p+1}\psi}|\nabla\psi|^2u^2dx+\int_{\mathbf{R}^n}e^{\frac{4}{p+1}\psi}|\nabla u|^2dx\right)^{\frac{\sigma}{2}(p+1)},\end{aligned}$$

where $\sigma=\frac{n(p-1)}{2(p+1)}$. We note that

$$e^{\frac{4}{p+1}\psi}|\nabla\psi|^2=\frac{4a^2|x|^2}{(1+t)^4}e^{\frac{4}{p+1}\psi}\le C\frac{1}{(1+t)^2}e^{2\psi}$$

and obtain

$$\begin{aligned}J(t;|u|^{p+1})\le{}&C\left(\int_{\mathbf{R}^n}e^{2\psi}u^2dx\right)^{\frac{1-\sigma}{2}(p+1)}\\&\times\left(\frac{1}{(1+t)^2}\int_{\mathbf{R}^n}e^{2\psi}u^2dx+\int_{\mathbf{R}^n}e^{2\psi}|\nabla u|^2dx\right)^{\frac{\sigma}{2}(p+1)}.\end{aligned}$$

Therefore, we can estimate

$$\begin{aligned}(1+t)^{n+2-\varepsilon}J(t;|u|^{p+1}) &\le C(1+t)^{n+2-\varepsilon}\{(1+t)^{-(n-\varepsilon)}M(t)\}^{\frac{1-\sigma}{2}(p+1)}\\ &\quad\times\{(1+t)^{-(n+2-\varepsilon)}M(t)\}^{\frac{\sigma}{2}(p+1)}\\ &= C(1+t)^{\gamma}M(t)^{p+1}\end{aligned}$$

with

$$\begin{aligned}\gamma &= n+2-\varepsilon-(n-\varepsilon)\frac{1-\sigma}{2}(p+1)-(n+2-\varepsilon)\frac{\sigma}{2}(p+1)\\ &= n+2-\varepsilon-(n-\varepsilon)\frac{p+1}{2}-\sigma(p+1)\\ &= n+2-\frac{n(p+1)}{2}-\frac{n(p-1)}{2}+\frac{\varepsilon(p-1)}{2}\\ &= -n\,(p-p_F)+\frac{\varepsilon(p-1)}{2}\end{aligned}$$

In a similar way, we can estimate the other nonlinear terms and one can see that

$$M(t)+L(t)\le CI_0^2+C(1+t)^{\gamma}M(t)^{p+1}, \tag{5.2.12}$$

where

$$L(t):=\int_0^t(1+\tau)^{n+1-\varepsilon}\left\{E(\tau)+(1+\tau)J_{\psi}(\tau;(u_t^2+|\nabla u|^2))\right\}d\tau.$$

Hence, if

$$\varepsilon<\frac{2n\,(p-p_F)}{p-1}, \tag{5.2.13}$$

then $\gamma<0$ and we have (5.2.1). □

The rest of the proof of Theorem 5.3 is the same as in Section 4.2.2 and so we omit the detail.

5.3. Proof of blow-up

In this section we first give a proof of Theorem 5.5. We use a method by Lin, Nishihara and Zhai [**56**] to transform (5.1.1) into divergence form and then a test-function method by Qi S. Zhang [**122**].

Let $\mu>1$. We multiply (5.1.1) by a positive function $g(t)\in C^2([0,\infty))$ and obtain

$$(gu)_{tt}-\Delta(gu)-(g'u)_t+(-g'+gb)u_t=g|u|^p.$$

We now choose $g(t)$ as the solution of the Cauchy problem for the ordinary differential equation

$$\begin{cases}-g'(t)+g(t)b(t)=1, & t>0,\\ g(0)=\frac{1}{\mu-1}.\end{cases} \tag{5.3.1}$$

The solution $g(t)$ is explicitly given by

$$g(t)=\frac{1}{\mu-1}(1+t).$$

Thus, we obtain the equation in divergence form

$$(gu)_{tt}-\Delta(gu)-(g'u)_t+u_t=g|u|^p. \tag{5.3.2}$$

Next, we apply a test function method. We first introduce test functions as in Section 1.4.2:

$$\phi(x) = \begin{cases} 1 & (|x| \le 1/2) \\ \dfrac{\exp(-1/(1-|x|^2))}{\exp(-1/(|x|^2-1/4)) + \exp(-1/(1-|x|^2))} & (1/2 < |x| < 1), \\ 0 & (|x| \ge 1), \end{cases}$$

$$\eta(t) = \begin{cases} 1 & (0 \le t \le 1/2), \\ \dfrac{\exp(-1/(1-t^2))}{\exp(-1/(t^2-1/4)) + \exp(-1/(1-t^2))} & (1/2 < t < 1), \\ 0 & (t \ge 1). \end{cases}$$

It is obvious that $\phi \in C_0^\infty(\mathbf{R}^n), \eta \in C_0^\infty([0,\infty))$. We also see that

$$|\eta'(t)| \lesssim \eta(t)^{1/p}, \quad |\eta''(t)| \lesssim \eta(t)^{1/p}, \quad |\Delta\phi(x)| \lesssim \phi(x)^{1/p}.$$

Indeed, we put q, r so that $1/p + 1/q = 1, 1/p + 2/r = 1$ and let $\mu = \eta^{1/q}, \nu = \eta^{1/r}$. Then we obtain

$$\begin{aligned} |\eta'(t)| &= |(\mu^q)'| = |q\mu^{q-1}\mu'| \lesssim \mu^{q-1} = \eta^{1/p}, \\ |\eta''(t)| &= |(\nu^r)''| \lesssim |\nu''|\nu^{r-1} + |\nu'|^2\nu^{r-2} \lesssim \nu^{r-2} = \eta^{1/p}. \end{aligned}$$

The assertion for ϕ can be proved in the same way. Let R be a large parameter in $(0,\infty)$. We define the test function

$$\psi_{\tau,R}(t,x) := \eta_\tau(t)\phi_R(x) := \eta\left(\frac{t}{\tau}\right)\phi\left(\frac{x}{R}\right).$$

Let q be the dual of p, that is $q = \frac{p}{p-1}$. Suppose that u is a global weak solution of (5.1.1)) with initial data (u_0, u_1) satisfying (5.1.11). We define

$$I_{\tau,R} := \int_0^\tau \int_{B_R} g(t)|u(t,x)|^p \psi_{\tau,R}(t,x)dxdt,$$

where τ, R are parameter in the intervals $[\tau_0,\infty), [R_0,\infty)$, respectively and $B_R = \{|x| < R\}$. We will appropriately chose $\tau_0 \ge 1, R_0 > 0$ later. According to the idea of the transformation the equation into divergence form (5.3.2), in the definition of the weak solution (5.1.10), we substitute $g(t)\psi_{\tau,R}(t,x)$ into $\psi(t,x)$ and obtain

$$\begin{aligned} I_{\tau,R} &= -\frac{1}{\mu-1}\int_{B_R} ((\mu-1)u_0 + u_1)\phi_R dx \\ &\quad + \int_0^\tau \int_{B_R} gu\partial_t^2(\psi_{\tau,R})dxdt + \int_0^\tau \int_{B_R} (g'u - u)\partial_t(\psi_{\tau,R})dxdt \\ &\quad - \int_0^\tau \int_{B_R} gu\Delta(\psi_{\tau,R})dxdt \\ &=: -\frac{1}{\mu-1}\int_{B_R} ((\mu-1)u_0 + u_1)\phi_R dx + K_1 + K_2 + K_3. \end{aligned}$$

By the assumption on the data, there exists $R_0 > 0$ such that

$$\int_{B_R} ((\mu-1)u_0 + u_1)\phi_R dx > 0$$

for $R > R_0$ and so

$$I_R < K_1 + K_2 + K_3.$$

We first estimate K_3. By the definition of test functions and the Hölder inequality, we have

$$\begin{aligned}|K_3| &\lesssim R^{-2}\int_0^\tau\int_{B_R\setminus B_{R/2}} g(t)|u|\Delta\psi_{\tau,R}dxdt\\ &\lesssim R^{-2}\left(\int_0^\tau\int_{B_R\setminus B_{R/2}} g(t)|u|^p\psi_{\tau,R}(t,x)dxdt\right)^{1/p}\left(\int_0^\tau\int_{B_R\setminus B_{R/2}} g(t)dxdt\right)^{1/q}\\ &\lesssim \tau^{2/q}R^{n/q-2}\tilde{I}_{\tau,R}^{1/p},\end{aligned}$$

where

$$\tilde{I}_{\tau,R} := \int_0^\tau\int_{B_R\setminus B_{R/2}} g(t)|u|^p\psi_{\tau,R}(t,x)dxdt.$$

In a similar way, we can estimate K_1 and K_2 as

$$|K_1|+|K_2| \lesssim \tau^{2/q-2}R^{\frac{n}{q}}\hat{I}_{\tau,R}^{1/p},\quad \hat{I}_{\tau,R} := \int_{\tau/2}^\tau\int_{B_R} g(t)|u|^p\psi_{\tau,R}(t,x)dxdt.$$

Hence, we obtain

$$I_{\tau,R} \lesssim \tau^{2/q}R^{n/q-2}\tilde{I}_{\tau,R}^{1/p} + \tau^{2/q-2}R^{n/q}\hat{I}_{\tau,R}^{1/p}, \tag{5.3.3}$$

in particular $I_{\tau,R}^{1-1/p} \lesssim R^{\frac{n+2}{q}-2}$. We put $\tau_0 = \max\{R_0, 1\}$ and $\tau = R$. Then we have

$$I_{\tau,\tau}^{1-1/p} \lesssim \tau^{\frac{n+2}{q}-2}.$$

We note that $1 < p < p_F$ if and only if $\frac{n+2}{q} - 2 < 0$. Therefore, if $1 < p < p_F$, by letting $\tau \to \infty$ we have $I_{\tau,\tau} \to 0$ and, hence $u \equiv 0$. Therefore, by the definition of the solution (5.1.10), we have

$$\int_{\mathbf{R}^n} ((\mu-1)u_0 + u_1)\phi_R(x)dx = 0$$

for any $R > 0$, which contradicts the assumption on the data. If $p = p_F$, we have only $I_{\tau,\tau} \le C$ with some constant C independent of τ. This implies that $g(t)|u|^p$ is integrable on $(0,\infty)\times\mathbf{R}^n$ and, hence

$$\lim_{\tau\to\infty}(\tilde{I}_{\tau,\tau} + \hat{I}_{\tau,\tau}) = 0.$$

By (5.3.3), we obtain $\lim_{\tau\to\infty} I_{\tau,\tau} = 0$. Therefore, u must be 0. This also leads to a contradiction.

Proof of Theorem 3.6. The proof is almost the same as above. Let $0 < \mu \le 1$. Instead of (5.3.1), we consider the ordinary differential equation we consider the ordinary differential equation

$$-g'(t) + g(t)b(t) = 0 \tag{5.3.4}$$

with $g(0) > 0$. We can easily solve this and have

$$g(t) = g(0)(1+t)^\mu.$$

Then we have

$$(gu)_{tt} - \Delta(gu) - (g'u)_t = g|u|^p. \tag{5.3.5}$$

Using the same test function $\psi_{\tau,R}(t,x)$ as above, we can calculate

$$I_{\tau,R} := \int_0^\tau \int_{B_R} g(t)|u|^p \psi_{\tau,R} dxdt = -\int_{B_R} g(0) u_1 \phi_R dx + \sum_{k=1}^3 J_k,$$

where

$$\begin{aligned} J_1 &= \int_0^\tau \int_{B_R} gu\partial_t^2(\psi_{\tau,R})dxdt, \\ J_2 &= \int_0^\tau \int_{B_R} g'u\partial_t(\psi_{\tau,R})dxdt, \\ J_3 &= -\int_0^\tau \int_{B_R} gu\Delta(\psi_{\tau,R})dxdt. \end{aligned}$$

We note that the term of u_0 vanishes and so we put the assumption only for u_1. We first estimate J_2. Noting $g'(t) = \mu g(0)(1+t)^{\mu-1}$, we have

$$|J_2| \lesssim \frac{1}{R} \int_{\tau/2}^\tau \int_{B_R} (1+t)^{\mu-1} |u| \partial_t \psi_{\tau,R} dxdt.$$

By noting that $(1+t)^{\mu-1} \sim g(t)^{1/p}(1+t)^{\mu/q-1}$ and the Hölder inequality, it follows that

$$\begin{aligned} |J_2| &\lesssim \frac{1}{R} \left(\int_{\tau/2}^\tau g|u|^p \psi_R dxdt \right)^{1/p} \left(\int_{\tau/2}^\tau \int_{B_R} (1+t)^{\mu-q} dxdt \right)^{1/q} \\ &\lesssim \frac{1}{R} \hat{I}_{\tau,R}^{1/p} (1+\tau)^{(\mu-q)/q} \left(\int_{\tau/2}^\tau \int_{B_R} dxdt \right)^{1/q} \\ &\lesssim \tau^{(\mu-q+1)/q} R^{-1+n/q} \hat{I}_{\tau,R}^{1/p}, \end{aligned}$$

where $\hat{I}_R$ is defined as before. In the same way, we can estimate J_1 and J_3 as

$$|J_1| + |J_3| \lesssim \tau^{-2+(\mu+1)/q} R^{n/q} \hat{I}_{\tau,R}^{1/p} + \tau^{(\mu+1)/q} R^{-2+n/q} \tilde{I}_{\tau,R}^{1/p},$$

where $\tilde{I}_{\tau,R}$ is the same as before. We put $\tau_0 = \max\{R_0, 1\}, \tau = R$. Then by (5.1.12) and the above estimates, we have

$$I_{\tau,\tau} \lesssim \tau^{-2+(n+\mu+1)/q} (\tilde{I}_{\tau,\tau}^{1/p} + \hat{I}_{\tau,\tau}^{1/p}).$$

We note that

$$-2 + (n+\mu+1)/q \le 0 \iff p \le 1 + \frac{2}{n+(\mu-1)}.$$

The rest of the proof is same as before. □

REMARK 5.6. *We can apply the proof of Theorem 1.4 to the wave equation with non-effective damping terms*

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = |u|^p, \\ (u, u_t)(0,x) = (u_0, u_1)(x), \end{cases}$$

where

$$b(t) = (1+t)^{-\beta}$$

with $\beta > 1$. We can easily solve (5.3.4) *and have*

$$g(t) = g(0) \exp\left(\frac{1}{-\beta+1}((1+t)^{-\beta+1} - 1) \right).$$

We note that $g(t) \sim 1$. The same argument implies that if

$$1 < p \le 1 + \frac{2}{n-1}, \quad \liminf_{R \to \infty} \int_{B_R} u_1 dx > 0,$$

then there is no global solution ($1 < p < \infty$ when $n = 1$). We note that the exponent $1 + 2/(n-1)$ is greater than the Fujita exponent.

We also mention that if $b(t)$ is nonnegative and satisfies

$$\lim_{t \to \infty} tb(t) = 0,$$

then we can obtain the nonexistence of global solutions for $1 < p < 1 + 2/(n-1)$. This shows that when the damping is non-effective, the equation loses the parabolic structure even in the nonlinear cases. One can expect that the critical exponent is given by the well-known Strauss critical exponent. However, this problem is completely open due to the author's knowledge.

CHAPTER 6

Blow-up of solutions to the one-dimensional semilinear wave equation with damping depending on time and space variables

6.1. Introduction and results

In this chapter, we consider the Cauchy problem of the one-dimensional semilinear damped wave equation

$$
\begin{cases} u_{tt} - u_{xx} + a(t,x)u_t = |u|^p, & (t,x) \in (0,\infty) \times \mathbf{R}, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}, \end{cases} \tag{6.1.1}
$$

where $u = u(t,x)$ is real-valued unknown and $p > 1$. We assume that $a(t,x)$ is a nonnegative smooth function satisfying

$$
|\partial_t^\alpha \partial_x^\beta a(t,x)| \le \frac{\delta}{(1+t)^{k+\alpha}} \quad (\alpha, \beta = 0, 1) \tag{6.1.2}
$$

with some $k > 1$ and small $\delta > 0$. This assumption means that the damping is non-effective. Therefore, it is expected that the critical exponent of (6.1.1) agrees with that of the wave equation

$$
w_{tt} - w_{xx} = |w|^p.
$$

Kato [**42**] proved that the critical exponent of the wave equation is given by $+\infty$. More precisely, he proved that if the initial data has compact support and satisfies $\int_{\mathbf{R}} w_t(0,x)dx > 0$, then for any $1 < p < \infty$, the local-in-time solution blows up in finite time. We will introduce a corresponding blow-up result for (6.1.1).

Our proof is based on the test function method. In order to apply the test function method, we follow an idea by Lin, Nishihara and Zhai [**56**] and transform the equation (6.1.1) into divergence form. Multiplying (6.1.1) by a positive function $g = g(t,x)$, we have

$$
(gu)_{tt} - (gu)_{xx} + 2(g_x u)_x + ((-2g_t + ga)u)_t + (g_{tt} - g_{xx} - (ga)_t)u = g|u|^p. \tag{6.1.3}
$$

Thus, if g satisfies

$$
g_{tt} - g_{xx} - (ga)_t = 0, \tag{6.1.4}
$$

then (6.1.3) becomes divergence form and we can apply the test function method. We will find a solution g of (6.1.4) having the form

$$
g(t,x) = 1 + h(t,x), \tag{6.1.5}
$$

where h has small amplitude, more precisely, $|h(t,x)| \le \theta$ with some $\theta \in (0,1)$. This ensures the positivity of g and so the nonlinearity $g|u|^p$. Then h must satisfy

$$
h_{tt} - h_{xx} - a(t,x)h_t - a_t(t,x)(1+h) = 0. \tag{6.1.6}
$$

We can find a classical solution h of (6.1.6) having the desired property by the method of characteristics.

LEMMA 6.1. *Let $\theta \in (0,1)$ and $k > 1$. Then there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ the following holds: if a satisfies* (6.1.2), *then there exists a solution $h \in C^2([0,\infty) \times \mathbf{R})$ of* (6.1.6) *satisfying*

$$|h(t,x)| \le \frac{\theta}{(1+t)^{k-1}}, \quad |\partial_t^\alpha \partial_x^\beta h(t,x)| \le \frac{C}{(1+t)^k} \quad (\alpha+\beta=1) \tag{6.1.7}$$

for all $(t,x) \in [0,\infty) \times \mathbf{R}$ with some constant $C > 0$.

Using this h, we can obtain a blow-up result for (6.1.1). To state our result precisely, we define the solution of (6.1.1). Let $T \in (0, \infty]$. We say that $u \in X(T) = C([0,T); H^1(\mathbf{R})) \cap C^1([0,T); L^2(\mathbf{R}))$ is a (weak) solution of the Cauchy problem (6.1.1) on the interval $[0,T)$ if it holds that

$$\begin{aligned}
&\int_{[0,T)\times\mathbf{R}} u(t,x)(\partial_t^2\psi(t,x) - \partial_x^2\psi(t,x) - \partial_t(a(t,x)\psi(t,x)))dxdt \\
&= \int_{\mathbf{R}} \{(a(0,x)u_0(x) + u_1(x))\psi(0,x) - u_0(x)\partial_t\psi(0,x)\}\, dx \\
&\quad + \int_{[0,T)\times\mathbf{R}} |u(t,x)|^p \psi(t,x) dxdt
\end{aligned} \tag{6.1.8}$$

for any $\psi \in C_0^2([0,T) \times \mathbf{R})$ (see also Section 9.4.3). In particular, when $T = \infty$, we call u a global solution.

We first recall a local existence result:

PROPOSITION 6.2. *Let $1 < p < \infty$ and $(u_0, u_1) \in H^1(\mathbf{R}) \times L^2(\mathbf{R})$. Then there exist $T^* \in (0, +\infty]$ and a unique solution $u \in X(T^*)$. Moreover, if $T^* < +\infty$, then it follows that*

$$\liminf_{t \to T^*-0} \|(u, u_t)(t)\|_{H^1 \times L^2} = +\infty.$$

For the proof, see Proposition 9.21. We put an assumption on the data

$$\liminf_{R\to\infty} \int_{-R}^{R} ((-g_t(0,x) + g(0,x)a(0,x))u_0(x) + g(0,x)u_1(x))dx > 0, \tag{6.1.9}$$

where g is defined by (6.1.5) with h in Lemma 6.1. Our main result is the following.

THEOREM 6.3. *Let $1 < p < \infty$. Under the same situation as Lemma 6.1, let $(u_0, u_1) \in H^1(\mathbf{R}) \times L^2(\mathbf{R})$ satisfy* (6.1.9). *Then the local solution u of* (6.1.1) *blows up in finite time, that is, $\lim_{t\to T^*-0} \|(u, u_t)(0)\|_{H^1\times L^2} = +\infty$ holds for some $T^* \in (0, +\infty)$.*

REMARK 6.1. *(i) If the initial data (u_0, u_1) satisfies that $u_0 = 0, u_1 \ge 0$ and $\int_{\mathbf{R}} u_1(x)dx > 0$, then the condition* (6.1.9) *is fulfilled. (ii) For Lemma 6.1, our method does not work in higher dimensional cases $n \ge 2$ and we have no idea to find an appropriate solution g of* (6.1.4). *(iii) We expect that the assumption on the smallness of δ is removable.*

We can also treat other types of damping. We give three examples. These examples have the form $a(t,x) = b(t) + d(t,x)$, here $b(t)$ and $d(t,x)$ denote the main term and a perturbation term, respectively.

The first example is the case that $a(t,x)$ is a perturbation of $b(t) = 2/(1+t)$, that is, a is given by

$$a(t,x) = \frac{2}{1+t} + d(t,x) \tag{6.1.10}$$

and $d(t,x)$ is a smooth function satisfying (6.1.2) with $\delta \le 2$. Note that this condition implies $a(t,x) \ge 0$.

In this case by putting

$$g(t,x) = (1+t)(1+h(t,x)), \tag{6.1.11}$$

the equation (6.1.6) becomes

$$h_{tt} - h_{xx} - d(t,x)h_t - \left(\frac{d(t,x)}{1+t} + d_t(t,x)\right)(1+h) = 0. \tag{6.1.12}$$

In the same way as in the proof of Lemma 1.1, we can obtain a solution h of (6.1.12):

LEMMA 6.4. *Let $\theta \in (0,1)$ and $k > 1$. Then there exists $\delta_0 > 0$ such that for all $\delta \in (0,\delta_0)$ the following holds: if d satisfies* (6.1.2) *and a is given by* (6.1.10), *then there exists a solution $h \in C^2([0,\infty) \times \mathbf{R})$ of* (6.1.12) *satisfying*

$$|h(t,x)| \le \frac{\theta}{(1+t)^{k-1}}, \quad |\partial_t^\alpha \partial_x^\beta h(t,x)| \le \frac{C}{(1+t)^k} \quad (\alpha+\beta=1) \tag{6.1.13}$$

for all $(t,x) \in [0,\infty) \times \mathbf{R}$ with some constant $C > 0$.

Using this h, we can apply the test function method and obtain a blow-up result.

THEOREM 6.5. *Let $1 < p \le 3$. Under the same situation as Lemma 6.4, let $(u_0,u_1) \in H^1(\mathbf{R}) \times L^2(\mathbf{R})$ satisfy* (6.1.9). *Then the local solution u of* (6.1.1) *blows up in finite time.*

The second example is

$$a(t,x) = b(t) + d(t,x) \tag{6.1.14}$$

with smooth nonnegative function $b(t)$ satisfying

$$b(t) \le \frac{\mu}{1+t} \tag{6.1.15}$$

for some $\mu > 0$ and a smooth function $d(t,x)$ satisfying (6.1.2) with $\delta \le \mu$. In this case, by putting

$$g(t,x) = f(t)(1+h(t,x)), \tag{6.1.16}$$

we have

$$h_{tt} - h_{xx} + \left(2\frac{f'}{f} - b - d\right)h_t + \left(\frac{f''}{f} - \frac{f'}{f}b - b' - \frac{f'}{f}d - d_t\right)(1+h) = 0.$$

The bad terms are

$$\frac{f''}{f} - \frac{f'}{f}b - b'$$

and hence, we choose the function $f(t)$ so that

$$\frac{f''}{f} - \frac{f'}{f}b - b' = 0. \tag{6.1.17}$$

Thus, putting

$$f(t) = \exp\left(\int_0^t b(s)ds\right), \tag{6.1.18}$$

we obtain the equation

$$h_{tt} - h_{xx} + (b - d)\,h_t - (bd + d_t)\,(1 + h) = 0. \tag{6.1.19}$$

In a similar way to Lemma 6.1 with some technical argument, we can find an appropriate solution h of (6.1.19).

LEMMA 6.6. *Let $\theta \in (0,1), \mu > 0$ and $k > \max\{1,\mu\}$. Then there exists $\delta_0 > 0$ such that for all $\delta \in (0,\delta_0)$ the following holds: if b and d satisfy* (6.1.15) *and* (6.1.2)*, respectively, and a is given by* (6.1.14)*, then there exists a solution $h \in C^2([0,\infty) \times \mathbf{R})$ of* (6.1.19) *satisfying*

$$|h(t,x)| \le \frac{\theta}{(1+t)^{k-1}}, \quad |\partial_t^\alpha \partial_x^\beta h(t,x)| \le \frac{C}{(1+t)^k} \quad (\alpha+\beta=1) \tag{6.1.20}$$

for all $(t,x) \in [0,\infty) \times \mathbf{R}$ with some constant $C > 0$.

This lemma and the test function method imply

THEOREM 6.7. *Let $1 < p \le 1 + 2/\mu$. Under the same situation as Lemma 6.6, let $(u_0,u_1) \in H^1(\mathbf{R}) \times L^2(\mathbf{R})$ satisfy* (6.1.9)*. Then the local solution u of* (6.1.1) *blows up in finite time.*

REMARK 6.2. *(i)When $b(t) = \mu/(1+t)$ with $\mu = 2$, Theorem 6.5 is better than Theorem 6.7.*

(ii) We can choose the function $f(t)$ as

$$f(t) = \int_0^t \exp\left(\int_s^t b(\tau)d\tau\right) ds$$

instead of (6.1.18)*. Then the equation* (6.1.17) *also holds. However, the behavior of $f(t)$ above is worse than* (6.1.18) *and we can obtain only weak result.*

The third example is the form

$$a(t,x) = b(t) + d(t,x)$$

with non-effective $b(t)$ and a perturbation term $d(t,x)$. The function $b(t)$ is assumed to be smooth, nonnegative and satisfies (6.1.15) with $\mu > 0$. Moreover, we assume that

$$\lim_{t\to\infty} tb(t) = 0. \tag{6.1.21}$$

A typical example of $b(t)$ is

$$b(t) = \frac{1}{(e+t)\log(e+t)}.$$

From the assumption (6.1.21), it follows for any $\varepsilon > 0$, there exists $t_0 > 0$ such that

$$b(t) \le \frac{\varepsilon}{1+t}$$

holds for $t \ge t_0$. Thus, we have

$$\exp\left(\int_0^t b(s)ds\right) \le \exp\left(\int_0^{t_0} b(s)ds\right) \exp\left(\int_{t_0}^t \frac{\varepsilon}{1+s}ds\right) \le C_\varepsilon(1+t)^\varepsilon \tag{6.1.22}$$

if $t \ge t_0$. Noting this, we can prove a blow-up result for any $1 < p < \infty$.

THEOREM 6.8. *Let $1 < p < \infty$. Under the same situation as Lemma 6.6, we assume the additional assumption* (6.1.21). *Let* $(u_0, u_1) \in H^1(\mathbf{R}) \times L^2(\mathbf{R})$ *satisfy* (6.1.9). *Then the local solution u of* (6.1.1) *blows up in finite time.*

6.2. Existence of multiplier function

In this section, we construct a solution of (6.1.6) by the method of characteristics. First, we diagonalize (6.1.6). Put

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} h_t + h_x \\ h_t - h_x \end{pmatrix}.$$

Then v_1 and v_2 satisfy

$$\partial_t v_1 = \partial_x v_1 + \frac{a(t,x)}{2}(v_1 + v_2) + a_t(t,x)(1+h) \tag{6.2.1}$$

and

$$\partial_t v_2 = -\partial_x v_2 + \frac{a(t,x)}{2}(v_1 + v_2) + a_t(t,x)(1+h), \tag{6.2.2}$$

respectively. We can rewrite (6.2.1)-(6.2.2) as

$$\begin{aligned} \partial_t(v_1(t, x-t)) = & \frac{a(t,x-t)}{2}(v_1(t,x-t) + v_2(t,x-t)) \\ & + a_t(t,x-t)(1+h(t,x-t)), \\ \partial_t(v_2(t, x+t)) = & \frac{a(t,x+t)}{2}(v_1(t,x+t) + v_2(t,x+t)) \\ & + a_t(t,x+t)(1+h(t,x+t)). \end{aligned}$$

We seek solutions satisfying $\lim_{t\to+\infty}(v_1, v_2) = 0$ uniformly in x. Integrating the above identities over $[t, \infty)$ and changing variables, one has a system of integral equations

$$v_1(t,x) = -\int_t^\infty \left\{\frac{a}{2}(v_1+v_2) + a_t(1+h)\right\}(s, x+t-s)ds, \tag{6.2.3}$$

$$v_2(t,x) = -\int_t^\infty \left\{\frac{a}{2}(v_1+v_2) + a_t(1+h)\right\}(s, x-(t-s))ds. \tag{6.2.4}$$

Next, we construct solutions to (6.2.3), (6.2.4) by an iteration argument in an appropriate Banach space. We define a function space Y. We say $V = (v_1, v_2, h) \in Y$ if $V \in (C([0,\infty)\times\mathbf{R}))^3$, V is differentiable with respect to x for all $(t,x) \in [0,\infty)\times\mathbf{R}$, $\partial_x V \in (C([0,\infty)\times\mathbf{R}))^3$, and $\|V\|_Y = \|(v_1, v_2, h)\|_Y < +\infty$, where

$$\begin{aligned} & \|(v_1, v_2, h)\|_Y \\ & \quad = \sup_{t\in[0,\infty)} \left\{(1+t)^k\|v_1(t)\|_{\mathcal{B}^1} + (1+t)^k\|v_2(t)\|_{\mathcal{B}^1} + (1+t)^{k-1}\|h(t)\|_{\mathcal{B}^1}\right\}, \\ & \|h(t)\|_{\mathcal{B}^1} = \|h(t,\cdot)\|_\infty + \|\partial_x h(t,\cdot)\|_\infty, \quad \|h(t)\|_\infty = \sup_{x\in\mathbf{R}}|h(t,x)|. \end{aligned}$$

Then Y is a Banach space with norm $\|V\|_Y$. Let $\theta \in (0,1)$ and let

$$\begin{aligned} K_\theta = & \{(v_1, v_2, h) \in Y \mid \sup_{t\in[0,\infty)} (1+t)^k\|v_1(t)\|_\infty \le \theta, \\ & \sup_{t\in[0,\infty)} (1+t)^k\|v_2(t)\|_\infty \le \theta, \quad \sup_{t\in[0,\infty)} (1+t)^{k-1}\|h(t)\|_\infty \le \theta\}. \end{aligned}$$

Take $(v_1^{(0)}, v_2^{(0)}, h^{(0)}) \in K_\theta$ arbitrarily and define $V^{(n)} = (v_1^{(n)}, v_2^{(n)}, h^{(n)})$ inductively by

(6.2.5)

$$\begin{aligned} v_1^{(n)}(t,x) &= -\int_t^\infty \left\{ \frac{a}{2}(v_1^{(n-1)} + v_2^{(n-1)}) + a_t(1+h^{(n-1)}) \right\}(s, x+t-s))ds, \\ v_2^{(n)}(t,x) &= -\int_t^\infty \left\{ \frac{a}{2}(v_1^{(n-1)} + v_2^{(n-1)}) + a_t(1+h^{(n-1)}) \right\}(s, x-(t-s))ds, \\ h^{(n)}(t,x) &= -\frac{1}{2}\int_t^\infty (v_1^{(n)} + v_2^{(n)})(s,x)ds. \end{aligned}$$

The following proposition shows that if the coefficient of damping term δ is sufficiently small, then $\{V^{(n)}\}_{n=0}^\infty$ is a Cauchy sequence.

Proposition 6.9. *Let $k > 1$ and $\theta \in (0,1)$. Then there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$ the following holds: if a satisfies* (6.1.2)*, then $\{V^{(n)}\}_{n=0}^\infty \in K_\theta$ for all n and $\{V^{(n)}\}_{n=0}^\infty$ is a Cauchy sequence with respect to the norm $\|\cdot\|_Y$.*

Proof. First, we prove that if $V^{(n-1)} \in K_\theta$, then $V^{(n)} \in K_\theta$. Assume $V^{(n-1)} \in K_\theta$. It is obvious that $V^{(n)} \in (C([0,\infty) \times \mathbf{R}))^3$. We have

$$\begin{aligned} |v_1^{(n)}(t,x)| &\le \int_t^\infty \left\{ \frac{|a|}{2}(|v_1^{(n-1)} + |v_2^{(n-1)}|) + |a_t|(1+|h^{(n-1)}|) \right\}(s, x+t-s)ds \\ &\le \int_t^\infty \frac{\theta\delta}{(1+s)^{2k}} + \frac{\delta}{(1+s)^{k+1}}(1+\theta(1+s)^{-(k-1)})ds \\ &= \delta C_1(1+t)^{-k} \end{aligned}$$

with some $C_1 > 0$. Hereafter, C_j $(j = 1, 2, \dots)$ denotes a constant depending only on k, θ. Moreover, differentiating under the integral sign, we obtain

$$\begin{aligned} \partial_x v_1^{(n)}(t,x) = -\int_t^\infty &\left\{ \frac{a_x}{2}(v_1^{(n-1)} + v_2^{(n-1)}) + \frac{a}{2}(\partial_x v_1^{(n-1)} + \partial_x v_2^{(n-1)}) \right. \\ &\left. + a_{tx}(1+h^{(n-1)}) + a_t h_x^{(n-1)} \right\}(s, x+t-s)ds. \end{aligned}$$

This implies $\partial_x v_1^{(n)} \in C([0,\infty) \times \mathbf{R})$ and

$$|\partial_x v_1^{(n)}(t,x)| \le \delta C_2(1+t)^{-k}.$$

We can also obtain the same estimates for $v_2^{(n)}$. By differentiating under the integral sign again, we have

$$\partial_x h^{(n)}(t,x) = -\frac{1}{2}\int_t^\infty (\partial_x v_1^{(n)} + \partial_x v_2^{(n)})(s,x)ds.$$

Thus, we can see that

$$\begin{aligned} |h^{(n)}(t,x)| &\le \delta C_1 \int_t^\infty \frac{ds}{(1+s)^k} = \delta C_3(1+t)^{-(k-1)}, \\ |\partial_x h^{(n)}(t,x)| &\le \delta C_2 \int_t^\infty \frac{ds}{(1+t)^k} = \delta C_4(1+t)^{-(k-1)}. \end{aligned}$$

The above estimates show $V^{(n)} \in Y$. Moreover, taking δ_0 so small that $\delta_0 \max\{C_1, C_3\} \le \theta$, we have $V^{(n)} \in K_\theta$ for all $\delta \in (0, \delta_0]$.

Next, we prove that $\{V^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_Y$. It follows that

$$\begin{aligned}|v_1^{(n)}(t,x)-v_1^{(n-1)}(t,x)| &\le \int_t^\infty \left\{\frac{\delta}{2(1+s)^k}(|v_1^{(n-1)}-v_1^{(n-2)}|+|v_2^{(n-1)}-v_2^{(n-2)}|)\right.\\ &\qquad \left.+\frac{\delta}{(1+s)^{k+1}}|h^{(n-1)}-h^{(n-2)}|\right\}(s,x+t-s)ds\\ &\le \delta C_5(1+t)^{-(2k-1)}\|V^{(n-1)}-V^{(n-2)}\|_Y.\end{aligned}$$

In the same way, we have

$$|\partial_x v_1^{(n)}(t,x)-\partial_x v_1^{(n-1)}(t,x)| \le \delta C_6(1+t)^{-(2k-1)}\|V^{(n-1)}-V^{(n-2)}\|_Y.$$

and the same estimates are true for $v_2^{(n)}-v_2^{(n-1)}$. We also obtain

$$\begin{aligned}|h^{(n)}(t,x)-h^{(n-1)}(t,x)| &\le \delta C_7(1+t)^{-2(k-1)}\|V^{(n-1)}-V^{(n-2)}\|_Y,\\ |\partial_x h^{(n)}(t,x)-\partial_x h^{(n-1)}(t,x)| &\le \delta C_8(1+t)^{-2(k-1)}\|V^{(n-1)}-V^{(n-2)}\|_Y.\end{aligned}$$

Consequently, taking δ_0 smaller so that $r=\delta_0(2C_5+2C_6+C_7+C_8)<1$, we have

$$\|V^{(n)}-V^{(n-1)}\|_Y \le r\|V^{(n-1)}-V^{(n-2)}\|_Y,$$

which shows that $\{V^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence. □

Proof of Lemma 6.1. By the above proposition, $\{V^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence and converges to some element $(v_1,v_2,h)\in K_\theta$. Therefore, (v_1,v_2,h) satisfies the integral equation (6.2.3), (6.2.4). By noting the differentiability with respect to t of the right-hand side of (6.2.3) (6.2.4), we have $v_1,v_2\in C^1([0,\infty)\times\mathbf{R})$. Differentiating (6.2.3) and (6.2.4), one can see that v_1,v_2 satisfy the differential equation (6.2.1), (6.2.2), respectively. By the equation of h, we also have

$$\partial_t h(t,x)=\frac{1}{2}(v_1(t,x)+v_2(t,x)),\quad \partial_x h(t,x)=\frac{1}{2}(v_1(t,x)-v_2(t,x)). \tag{6.2.6}$$

Thus, $h\in C^2([0,\infty)\times\mathbf{R})$ and h is a classical solution of (6.1.6). By noting $(v_1,v_2,h)\in K_\theta$ and (6.2.6), the estimate (6.1.7) is obvious. □

6.3. Proof of blow-up

In this section, we give a proof of Theorem 6.3. By Lemma 6.1, there exists h satisfying (6.1.6). Thus, (6.1.4) holds for g given by (6.1.5) and we can transform the equation (6.1.1) into divergence form

$$(gu)_{tt}-(gu)_{xx}+2(g_x u)_x+((-2g_t+ga)u)_t=g|u|^p. \tag{6.3.1}$$

We apply the test function method to (6.3.1). Since g is defined by (6.1.5) and h satisfies (6.1.7), we have

$$C^{-1}\le g(t,x)\le C,\quad |g_t(t,x)|\le \frac{C}{(1+t)^k},\quad |g_x(t,x)|\le \frac{C}{(1+t)^k} \tag{6.3.2}$$

with some constant $C > 0$. We define test functions

$$\phi(x) = \begin{cases} 1 & (|x| \le 1/2) \\ \dfrac{\exp(-1/(1-x^2))}{\exp(-1/(x^2-1/4)) + \exp(-1/(1-x^2))} & (1/2 < |x| < 1), \\ 0 & (|x| \ge 1), \end{cases}$$

$$\eta(t) = \begin{cases} 1 & (0 \le t \le 1/2), \\ \dfrac{\exp(-1/(1-t^2))}{\exp(-1/(t^2-1/4)) + \exp(-1/(1-t^2))} & (1/2 < t < 1), \\ 0 & (t \ge 1). \end{cases}$$

It is obvious that $\phi \in C_0^\infty(\mathbf{R}), \eta \in C_0^\infty([0,\infty))$. We also see that

$$\begin{aligned} |\phi'(x)| &\lesssim \phi(x)^{1/p}, \quad |\phi''(x)| \lesssim \phi(x)^{1/p}, \\ |\eta'(t)| &\lesssim \eta(t)^{1/p}, \quad |\eta''(t)| \lesssim \eta(t)^{1/p}. \end{aligned} \tag{6.3.3}$$

Indeed, let q, r satisfy $1/p + 1/q = 1, 1/p + 2/r = 1$ and let $\mu = \phi^{1/q}, \nu = \phi^{1/r}$. Then we have

$$|\phi'| = |(\mu^q)'| = |q\mu^{q-1}\mu'| \lesssim \mu^{q-1} = \phi^{1/p}$$

and

$$|\phi''| = |(\nu^r)'| \lesssim |\nu''|\nu^{r-1} + |\nu'|^2\nu^{r-2} \lesssim \nu^{r-2} = \phi^{1/p}.$$

The assertion for η can be proved by the same way. To prove Theorem 6.3, we use a contradiction argument. Suppose $u \in X(\infty)$ is a global solution to (6.1.1) with initial data (u_0, u_1) satisfying (6.1.9). Let τ, R be parameters such that $\tau \in (\tau_0, \infty), R \in (R_0, \infty)$, where $\tau_0 \ge 1, R_0 > 0$ are defined later. We put

$$\begin{aligned} \eta_\tau(t) = \eta(t/\tau), \quad \phi_R(x) = \phi(x/R), \\ \psi_{\tau,R}(t,x) = \eta_\tau(t)\phi_R(x) \end{aligned}$$

and

$$\begin{aligned} I_{\tau,R} &:= \int_0^\tau \int_{-R}^R g|u|^p \psi_{\tau,R} dx dt, \\ J_R &:= \int_{-R}^R ((-g_t(0,x) + g(0,x)a(0,x))u_0(x) + g(0,x)u_1(x)) \phi_R(x) dx, \end{aligned}$$

Substituting the test function $g(t,x)\psi_{\tau,R}(t,x)$ into the definition of solution (6.1.8), we see that

$$\begin{aligned} I_{\tau,R} + J_R = \int_0^\tau \int_{-R}^R &\big(gu\partial_t^2\psi_{\tau,R} - gu\partial_x^2\psi_{\tau,R} - 2(g_x u)\partial_x\psi_{\tau,R} \\ &-(-2g_t + ga)u\partial_t\psi_{\tau,R}\big) dx dt =: K_1 + K_2 + K_3 + K_4. \end{aligned}$$

Next, we estimate the terms $K_1, \dots, K_4$. Let q be the dual of p, that is $q = p/(p-1)$. By using the Hölder inequality and (6.3.2), (6.3.3), it follows that

$$\begin{aligned} K_1 &\le \tau^{-2} \int_0^\tau \int_{-R}^R |gu||\eta''(t/\tau)|\phi_R(x) dx dt \\ &\lesssim \tau^{-2} I_{\tau,R}^{1/p} \left(\int_{-R}^R \left(\int_0^\tau g(t,x) dt \right) \phi_R(x) dx \right)^{1/q} \\ &\lesssim \tau^{-2+1/q} R^{1/q} I_{\tau,R}^{1/p}, \end{aligned}$$

$$
\begin{aligned}
K_2 &\le R^{-2}\int_0^\tau \int_{-R}^R |gu||\phi''(x/R)|\eta_\tau(t)dxdt \\
&\lesssim R^{-2} I_{\tau,R}^{1/p}\left(\int_{-R}^R \left(\int_0^\tau g(t,x)\eta_\tau(t)dt\right)dx\right)^{1/q} \\
&\lesssim \tau^{1/q}R^{-2+1/q} I_{\tau,R}^{1/p},
\end{aligned}
$$

$$
\begin{aligned}
K_3 &\le R^{-1}\int_0^\tau \int_{-R}^R |g_x u||\phi'(x/R)|\eta_\tau dxdt \\
&\lesssim R^{-1} I_{\tau,R}^{1/p}\left(\int_{-R}^R \left(\int_0^\tau (1+t)^{-qk}dt\right)dx\right)^{1/q} \\
&\lesssim R^{-1+1/q} I_{\tau,R}^{1/p}.
\end{aligned}
$$

Finally, we estimate K_4. Noting that $\operatorname{supp}\eta'(t) \subset [1/2,1]$, we have

$$
\begin{aligned}
K_4 &\le \tau^{-1}\int_0^\tau \int_{-R}^R (2|g_t| + |ga|)|u||\eta'(t/\tau)|\phi_R dxdt \\
&\lesssim \tau^{-1} I_{\tau,R}^{1/p}\left(\int_{-R}^R \left(\int_{\tau/2}^\tau (1+t)^{-kq}dt\right)\phi_R(x)dx\right)^{1/q} \\
&\lesssim \tau^{-1-k+1/q}R^{1/q} I_{\tau,R}^{1/p}.
\end{aligned}
$$

Therefore, putting

$$D(\tau,R) := \tau^{-2+1/q}R^{1/q} + \tau^{1/q}R^{-2+1/q} + R^{-1+1/q},$$

we obtain

$$I_{\tau,R} + J_R \le CD(\tau,R) I_{\tau,R}^{1/p}. \tag{6.3.4}$$

By the assumption on the data (6.1.9), there exists $R_0 > 0$ such that $J_R > 0$ holds for $R \ge R_0$. This implies

$$I_{\tau,R} \le CD(\tau,R)^q.$$

Putting $\tau_0 = R_0$ and $R = \tau$, we have

$$I_{\tau,\tau} \le C\tau^{-1+1/q} \tag{6.3.5}$$

for $\tau \ge \tau_0$. In particular, $I_{\tau,\tau} \le C$ with some $C > 0$ and hence, $g|u|^p \in L^1([0,\infty)\times\mathbf{R})$ and $\lim_{\tau\to+\infty} I_{\tau,\tau} = \|g|u|^p\|_{L^1([0,\infty)\times\mathbf{R})}$. Moreover, since $-1+1/q < 0$, by letting $\tau \to +\infty$, the right-hand side of (6.3.5) tend to 0. This gives $\|g|u|^p\|_{L^1([0,\infty)\times\mathbf{R})} = 0$, that is $u \equiv 0$. However, in view of (6.1.8), it contradicts $(u_0,u_1) \ne 0$. This completes the proof.

6.4. The case of perturbation of $\frac{2}{1+t}$

In this section, we give a proof of Theorem 6.5. Note that we can prove Lemma 6.4 by the same argument as the proof of Lemma 6.1. The only difference between the proofs of Lemmas 6.1 and 6.4 is that of the coefficients of (6.1.6) and (6.1.12). However, by the assumption on $d(t,x)$, it follows that

$$|d(t,x)| \le \frac{\delta}{(1+t)^k}, \quad \left|\frac{d(t,x)}{1+t} + d_t(t,x)\right| \le \frac{2\delta}{(1+t)^{k+1}}.$$

Using this estimate, we can prove Lemma 6.4 in the same way as Section 6.2 and hence, we omit the detailed proof.

Now we prove Theorem 6.5. By (6.1.13) and (6.1.11), we have

$$g \sim (1+t), \quad |g_t| \lesssim 1, \quad |g_x| \lesssim (1+t)^{-k+1}. \tag{6.4.1}$$

We use the same notation as in Section 6.3 and suppose that u is a global solution. The main difference with the previous section lies in the estimate of the terms $K_1, \dots, K_4$. In this case, we have

$$\begin{aligned} K_1 &\le \tau^{-2} \int_0^\tau \int_{-R}^R |gu||\eta''(t/\tau)|\phi_R(x) dx dt \\ &\lesssim \tau^{-2} I_{\tau,R}^{1/p} \left(\int_{-R}^R \left(\int_0^\tau g(t,x) dt \right) \phi_R(x) dx \right)^{1/q} \\ &\lesssim \tau^{-2+2/q} R^{1/q} I_{\tau,R}^{1/p}, \end{aligned}$$

$$\begin{aligned} K_2 &\le R^{-2} \int_0^\tau \int_{-R}^R |gu||\phi''(x/R)|\eta_\tau(t) dx dt \\ &\lesssim R^{-2} I_{\tau,R}^{1/p} \left(\int_{-R}^R \left(\int_0^\tau g(t,x)\eta_\tau(t) dt \right) dx \right)^{1/q} \\ &\lesssim \tau^{2/q} R^{-2+1/q} I_{\tau,R}^{1/p}, \end{aligned}$$

$$\begin{aligned} K_3 &\le R^{-1} \int_0^\tau \int_{-R}^R |g_x u||\phi'(x/R)|\eta_\tau dx dt \\ &\le R^{-1} \int_0^\tau \int_{-R}^R |g^{1/p} u||g^{-1/p} g_x||\phi'(x/R)|\eta_\tau dx dt \\ &\lesssim R^{-1} I_{\tau,R}^{1/p} \left(\int_{-R}^R \left(\int_0^\tau (1+t)^{-q(1/p+k-1)} dt \right) dx \right)^{1/q} \\ &\lesssim F(\tau) R^{-1+1/q} I_{\tau,R}^{1/p} \end{aligned}$$

and

$$\begin{aligned} K_4 &\le \tau^{-1} \int_0^\tau \int_{-R}^R (2|g_t| + |ga|)|u||\eta'(t/\tau)|\phi_R dx dt \\ &\le \tau^{-1} \int_0^\tau \int_{-R}^R |g^{1/p} u| g^{-1/p} |\eta'(t/\tau)|\phi_R dx dt \\ &\lesssim \tau^{-1} I_{\tau,R}^{1/p} \left(\int_{-R}^R \left(\int_{\tau/2}^\tau (1+t)^{-q/p} dt \right) \phi_R(x) dx \right)^{1/q} \\ &\lesssim \tau^{-1-1/p+1/q} R^{1/q} I_{\tau,R}^{1/p}, \end{aligned}$$

where

$$F(\tau) = \begin{cases} 1 & (-q(1/p+k-1) < -1), \\ (\log \tau)^{1/q} & (-q(1/p+k-1) = -1), \\ \tau^{-(1/p+k-1)+1/q} & (-q(1/p+k-1) > -1) \end{cases} \tag{6.4.2}$$

and for the estimate of K_4, we have used that $\operatorname{supp}\eta'(t) \subset [1/2, 1]$ and (6.4.1). Let

$$D(\tau, R) := \tau^{-2+2/q}R^{1/q} + \tau^{2/q}R^{-2+1/q} + F(\tau)R^{-1+1/q}. \tag{6.4.3}$$

We note that the powers of each term of $D(\tau, \tau)$ are negative if $1 < p < 3$. Thus, by putting $R = \tau$ and the same argument as the previous section, we can lead to a contradiction.

When $p = 3$, we need a certain modification of the above argument. We put

$$I'_{\tau,R} = \int_{\tau/2}^{\tau}\int_{-R}^{R} g|u|^p\psi_{\tau,R}dxdt, \quad I''_{\tau,R} = \int_0^{\tau}\int_{R/2<|x|<R} g|u|^p\psi_{\tau,R}dxdt.$$

Then we can improve the estimates of $K_1, \ldots, K_4$ as

$$\begin{aligned}
K_1 &\le \tau^{-2+2/q}R^{1/q}(I'_{\tau,R})^{1/p},\\
K_2 &\le \tau^{2/q}R^{-2+1/q}(I''_{\tau,R})^{1/p},\\
K_3 &\le F(\tau)R^{-1+1/q}(I''_{\tau,R})^{1/p},\\
K_4 &\le \tau^{-1-1/p+1/q}R^{1/q}(I'_{\tau,R})^{1/p}.
\end{aligned}$$

Thus, we have

$$I_{\tau,R} \le C\left(\tau^{-2+2/q}R^{1/q}(I'_{\tau,R})^{1/p} + (\tau^{2/q}R^{-2+1/q} + F(\tau)R^{-1+1/q})(I''_{\tau,R})^{1/p}\right).$$

Substituting $p = 3$ and $R = \tau$, we obtain

$$I_{\tau,\tau} \le C((I'_{\tau,\tau})^{1/3} + (I''_{\tau,\tau})^{1/3}). \tag{6.4.4}$$

In particular, we see that $I_{\tau,\tau} \le C$ with some constant $C > 0$, since $I'_{\tau,\tau} \le I_{\tau,\tau}$ and $I''_{\tau,\tau} \le I_{\tau,\tau}$. Hence $g|u|^3 \in L^1([0,\infty) \times \mathbf{R})$ and $\lim_{\tau\to\infty} I_{\tau,\tau} = \|g|u|^3\|_{L^1([0,\infty)\times\mathbf{R})}$. However, by noting the integral region of $I'_{\tau,\tau}, I''_{\tau,\tau}$, we can see that the integrability of $g|u|^3$ shows

$$\lim_{\tau\to\infty} I'_{\tau,\tau} = 0, \quad \lim_{\tau\to\infty} I''_{\tau,\tau} = 0.$$

Therefore, turning back to (6.4.4), we obtain $\lim_{\tau\to\infty} I_{\tau,\tau} = 0$. This implies $u \equiv 0$. In view of (6.1.8), this contradicts $(u_0, u_1) \ne 0$. This completes the proof.

6.5. Proof of Lemma 6.6

In this section, we give a proof of Lemma 6.6. We modify the argument in Section 6.2 and look for an appropriate solution by the following iteration:

$$\begin{aligned}
v_1^{(n)}(t,x) &= \int_t^{\infty}\left\{\frac{1}{2}(b-d)(v_1^{(n-1)} + v_2^{(n-1)})\right.\\
&\qquad \left. -(bd + d_t)(1 + h^{(n-1)})\right\}(s, x+t-s)ds,\\
v_2^{(n)}(t,x) &= \int_t^{\infty}\left\{\frac{1}{2}(b-d)(v_1^{(n-1)} + v_2^{(n-1)})\right.\\
&\qquad \left. -(bd + d_t)(1 + h^{(n-1)})\right\}(s, x-(t-s))ds,\\
h^{(n)}(t,x) &= -\frac{1}{2}\int_t^{\infty}(v_1^{(n)} + v_2^{(n)})(s,x)ds.
\end{aligned} \tag{6.5.1}$$

We modify the definition of function space Y in Section 6.2 as follows. We say $V = (v_1, v_2, h) \in Y$ if $V \in (C([0,\infty) \times \mathbf{R}))^3$, V is differentiable with respect to x for all $(t,x) \in [0,\infty) \times \mathbf{R}$, $\partial_x V \in (C([0,\infty)\times\mathbf{R}))^3$, and $\|V\|_Y = \|(v_1,v_2,h)\|_Y < +\infty$, where

$$\begin{aligned}\|(v_1,v_2,h)\|_Y \\ = \sup_{t\in[0,\infty)} \left\{\lambda(1+t)^k\|v_1(t)\|_{\mathcal{B}^1} + \lambda(1+t)^k\|v_2(t)\|_{\mathcal{B}^1} + (1+t)^{k-1}\|h(t)\|_{\mathcal{B}^1}\right\},\end{aligned}$$

with a large parameter λ fixed later. We put

$$\begin{aligned}K_\theta :=&\{(v_1,v_2,h)\in Y \mid \sup_{t\in[0,\infty)}(1+t)^k\|v_1(t)\|_\infty \le \theta', \\ &\sup_{t\in[0,\infty)}(1+t)^k\|v_2(t)\|_\infty \le \theta', \quad \sup_{t\in[0,\infty)}(1+t)^{k-1}\|h(t)\|_\infty \le \theta\},\end{aligned} \tag{6.5.2}$$

where $\theta' := \theta\min\{k-1,1\}$. We take $(v_1^{(0)}, v_2^{(0)}, h^{(0)}) \in K_\theta$ arbitrarily and define $V^{(n)} = (v_1^{(n)}, v_2^{(n)}, h^{(n)})$ inductively by (6.5.1). Now we prove that $\{V^{(n)}\}_{n=0}^\infty$ is a Cauchy sequence in K_θ for sufficiently large λ and small δ.

PROPOSITION 6.10. *If $k > \max\{1,\mu\}$, then there exist λ and δ_0 having the following property: if $\delta \in (0,\delta_0]$, then $\{V^{(n)}\}_{n=0}^\infty$ is a Cauchy sequence in K_θ with respect to the norm $\|\cdot\|_Y$.*

PROOF. We first show that if $V^{(n-1)} \in K_\theta$, then $V^{(n)} \in K_\theta$. Using (6.1.15), we calculate

$$|v_1^{(n)}(t,x)| \le \left(\frac{\mu}{k}\theta' + \delta C\right)(1+t)^{-k}.$$

In view of $k > \mu$, by taking δ sufficiently small, we obtain

$$(1+t)^k|v_1^{(n)}(t,x)| \le \theta'.$$

By the same way, we also have $(1+t)^k|v_2^{(n)}(t,x)| \le \theta'$. Noting that $\theta'/(k-1) \le \theta$, we obtain

$$|h^{(n)}(t,x)| \le \int_t^\infty \frac{\theta'}{(1+s)^k}ds \le \theta(1+t)^{-(k-1)}.$$

By differentiating under the integral sign and noting that $V^{(n-1)} \in Y$, we have

$$(1+t)^k|\partial_x v_1^{(n)}(t,x)| \le C, \quad (1+t)^k|\partial_x v_2^{(n)}(t,x)| \le C, \quad (1+t)^{k-1}|\partial_x h^{(n)}(t,x)| \le C$$

with some constant $C > 0$. Therefore we have $V^{(n)} \in K_\theta$.

Next, we prove that $\{V^{(n)}\}_{n=0}^\infty$ is a Cauchy sequence. By a straightforward calculation, we can estimate

$$\begin{aligned}&\sum_{\alpha=0,1}\sum_{j=1,2}|\partial_x^\alpha v_j^{(n)}(t,x) - \partial_x^\alpha v_j^{(n)}(t,x)| \\ &\le \int_t^\infty \frac{\mu}{1+s}\sum_{\alpha=0,1}\sum_{j=1,2}\|\partial_x^\alpha v_j^{(n-1)} - \partial_x^\alpha v_j^{(n-2)}\|_\infty ds \\ &\quad + \delta C\int_t^\infty \frac{1}{(1+s)^k}\sum_{\alpha=0,1}\sum_{j=1,2}\|\partial_x^\alpha v_j^{(n-1)} - \partial_x^\alpha v_j^{(n-2)}\|_\infty ds \\ &\quad + \delta C\int_t^\infty \frac{1}{(1+s)^{k+1}}\sum_{\alpha=0,1}\|\partial_x^\alpha h^{(n-1)} - \partial_x^\alpha h^{(n-2)}\|_\infty ds.\end{aligned}$$

Since $k > 1$, this implies

$$\lambda(1+t)^k \sum_{\alpha=0,1} \sum_{j=1,2} |\partial_x^\alpha v_j^{(n)}(t,x) - \partial_x^\alpha v_j^{(n-1)}(t,x)| \le \left(\frac{\mu}{k} + \delta\lambda C\right) \|V^{(n-1)} - V^{(n-2)}\|_Y.$$

Using this, we can estimate the difference of $h^{(n)}$ and $h^{(n-1)}$ as

$$\begin{aligned}(1+t)^{k-1} &\sum_{\alpha=0,1} |\partial_x^\alpha h^{(n)}(t,x) - \partial_x^\alpha h^{(n-1)}(t,x)| \\ &\le \frac{1}{2\lambda(k-1)} \left(\frac{\mu}{k} + \delta\lambda C\right) \|V^{(n)} - V^{(n-1)}\|_Y.\end{aligned}$$

Adding the two inequalities above, we obtain

$$\|V^{(n)} - V^{(n-1)}\|_Y \le \left(1 + \frac{1}{2\lambda(k-1)}\right) \left(\frac{\mu}{k} + \delta\lambda C\right) \|V^{(n-1)} - V^{(n-2)}\|_Y.$$

Thus, by taking λ sufficiently large and then δ sufficiently small, we obtain

$$\|V^{(n)} - V^{(n-1)}\|_Y \le r \|V^{(n-1)} - V^{(n-2)}\|_Y$$

with some $0 < r < 1$. This completes the proof. □

By Proposition 6.10, we can complete the proof of Lemma 6.6 by the same argument as in the proof of Lemma 6.1.

6.6. Proof of Theorems 6.7 and 6.9

We first give a proof of Theorem 6.7. By Lemma 6.6, we find a solution g of (6.1.4) satisfying

$$1 \lesssim g \lesssim (1+t)^\mu, \quad \|g_t(t)\|_\infty \lesssim (1+t)^{\mu-1}, \quad \|g_x(t)\|_\infty \lesssim (1+t)^{\mu-k}. \tag{6.6.1}$$

Moreover, by the definition of g (6.1.16) with f defined by (6.1.18), we can calculate

$$\begin{aligned} g_x(t,x) &= f(t) h_x(t,x), \\ g_t(t,x) &= f'(t)(1 + h(t,x)) + f(t) h_t(t,x) \\ &= b(t)(1 + h(t,x)) + f(t) h_t(t,x), \end{aligned}$$

and hence,

$$\frac{g_x}{g} = \frac{h_x}{1+h}, \quad \frac{g_t}{g} = b(t) + \frac{h_t}{1+h}.$$

The estimate (6.1.13) implies

$$\left|\frac{g_x}{g}\right| \le C(1+t)^{-k}, \quad \left|\frac{g_t}{g}\right| \le C(1+t)^{-1}. \tag{6.6.2}$$

In what follows, we use the same notation as in Section 6.3. Suppose that u is a global solution. Using the estimates (6.6.1) and (6.6.2), we can obtain

$$\begin{aligned} K_1 &\le \tau^{-2} \int_0^\tau \int_{-R}^R |gu| |\eta''(t/\tau)| \phi_R(x) dx dt \\ &\lesssim \tau^{-2} I_{\tau,R}^{1/p} \left(\int_{-R}^R \left(\int_0^\tau g(t,x) dt \right) \phi_R(x) dx \right)^{1/q} \\ &\lesssim \tau^{-2+(\mu+1)/q} R^{1/q} I_{\tau,R}^{1/p}, \end{aligned}$$

$$
\begin{aligned}
K_2 &\le R^{-2} \int_0^\tau \int_{-R}^R |gu||\phi''(x/R)|\eta_\tau(t)dxdt \\
&\lesssim R^{-2} I_{\tau,R}^{1/p} \left(\int_{-R}^R \left(\int_0^\tau g(t,x)\eta_\tau(t)dt \right) dx \right)^{1/q} \\
&\lesssim \tau^{(\mu+1)/q} R^{-2+1/q} I_{\tau,R}^{1/p},
\end{aligned}
$$

$$
\begin{aligned}
K_3 &\le R^{-1} \int_0^\tau \int_{-R}^R |g_x u||\phi'(x/R)|\eta_\tau dxdt \\
&= R^{-1} \int_0^\tau \int_{-R}^R |u|g \left| \frac{g_x}{g} \right| |\phi'(x/R)|\eta_\tau dxdt \\
&\lesssim R^{-1} \int_0^\tau \int_{-R}^R g^{1/p}|u|g^{1/q}|h_x||\phi'(x/R)|\eta_\tau dxdt \\
&\lesssim R^{-1} I_{\tau,R}^{1/p} \left(\int_{-R}^R \left(\int_0^\tau (1+t)^{-q(k-\mu/q)} dt \right) dx \right)^{1/q} \\
&\lesssim G(\tau) R^{-1+1/q} I_{\tau,R}^{1/p}
\end{aligned}
$$

and

$$
\begin{aligned}
K_4 &\le \tau^{-1} \int_0^\tau \int_{-R}^R (2|g_t| + |ga|)|u||\eta'(t/\tau)|\phi_R dxdt \\
&\le \tau^{-1} \int_0^\tau \int_{-R}^R |u|g \left(\left| \frac{g_t}{g} \right| + a \right) |\eta'(t/\tau)|\phi_R dxdt \\
&\lesssim \tau^{-1} I_{\tau,R}^{1/p} \left(\int_{-R}^R \left(\int_{\tau/2}^\tau (1+t)^{\mu-q} dt \right) \phi_R(x) dx \right)^{1/q} \\
&\lesssim \tau^{-2+(\mu+1)/q} R^{1/q} I_{\tau,R}^{1/p},
\end{aligned}
$$

where

$$
G(\tau) = \begin{cases} 1 & (\mu - kq < -1), \\ (\log \tau)^{1/q} & (\mu - kq = -1), \\ \tau^{-k+(\mu+1)/q} & (\mu - kq > -1). \end{cases}
$$

In this case we put

$$
D(\tau, R) := \tau^{-2+(\mu+1)/q} R^{1/q} + \tau^{(\mu+1)/q} R^{-2+1/q} + G(\tau) R^{-1+1/q}. \tag{6.6.3}
$$

We note that the powers of each term of $D(\tau,\tau)$ do not exceed 0 if $1 < p \le 1 + 2/\mu$. Thus, by putting $R = \tau$ and the same argument as Sections 6.3 and 6.4, we can lead to a contradiction and complete the proof of Theorem 6.7.

Let us turn to the proof of Theorem 6.8. We first recall that

$$
g(t,x) = f(t)(1 + h(t,x)), \quad f(t) = \exp\left(\int_0^t b(s)ds \right)
$$

and hence, the estimates (6.1.15), (6.1.22) and Lemma 6.6 imply

$$1 \lesssim g(t,x) \lesssim (1+t)^{\varepsilon},$$
$$|g_t(t,x)| \le |f'(t)(1+h(t,x)) + f(t)h_t(t,x)| \lesssim b(t)f(t) \lesssim (1+t)^{-1+\varepsilon},$$
$$|g_x(t,x)| \le |f(t)h_x(t,x)| \lesssim (1+t)^{-k+\varepsilon}.$$

Using these estimates, we can see that

$$\begin{aligned} K_1 &\le \tau^{-2}\int_0^\tau \int_{-R}^R |gu||\eta''(t/\tau)|\phi_R(x)dxdt \\ &\lesssim \tau^{-2} I_{\tau,R}^{1/p}\left(\int_{-R}^R\left(\int_0^\tau g(t,x)dt\right)\phi_R(x)dx\right)^{1/q} \\ &\lesssim \tau^{-2+1/q+\varepsilon/q}R^{1/q}I_{\tau,R}^{1/p}, \end{aligned}$$

$$\begin{aligned} K_2 &\le R^{-2}\int_0^\tau \int_{-R}^R |gu||\phi''(x/R)|\eta_\tau(t)dxdt \\ &\lesssim R^{-2} I_{\tau,R}^{1/p}\left(\int_{-R}^R\left(\int_0^\tau g(t,x)\eta_\tau(t)dt\right)dx\right)^{1/q} \\ &\lesssim \tau^{1/q+\varepsilon/q}R^{-2+1/q}I_{\tau,R}^{1/p}, \end{aligned}$$

$$\begin{aligned} K_3 &\le R^{-1}\int_0^\tau \int_{-R}^R |g_x u||\phi'(x/R)|\eta_\tau dxdt \\ &\lesssim R^{-1} I_{\tau,R}^{1/p}\left(\int_{-R}^R\left(\int_0^\tau (1+t)^{-q(k-\varepsilon)}dt\right)dx\right)^{1/q} \\ &\lesssim R^{-1+1/q}I_{\tau,R}^{1/p}, \end{aligned}$$

provided that ε taken so small that $q(k-\varepsilon) > 1$. Finally, noting that $\operatorname{supp}\eta'(t) \subset [1/2,1]$ and $|ga| \lesssim f(t)b(t) \lesssim (1+t)^{-1+\varepsilon}$, we have

$$\begin{aligned} K_4 &\le \tau^{-1}\int_0^\tau \int_{-R}^R (2|g_t| + |ga|)|u||\eta'(t/\tau)|\phi_R dxdt \\ &\lesssim \tau^{-1} I_{\tau,R}^{1/p}\left(\int_{-R}^R\left(\int_{\tau/2}^\tau (1+t)^{-q(1-\varepsilon)}dt\right)\phi_R(x)dx\right)^{1/q} \\ &\lesssim \tau^{-2+1/q+\varepsilon/q}R^{1/q}I_{\tau,R}^{1/p}. \end{aligned}$$

Therefore, we put

$$D(\tau,R) = \tau^{-2+1/q+\varepsilon/q}R^{1/q} + \tau^{1/q+\varepsilon/q}R^{-2+1/q} + R^{-1+1/q}$$

and have

$$I_{\tau,R} + J_R \le CD(\tau,R)I_{\tau,R}^{1/p}.$$

Then the same argument as before shows

$$I_{\tau,\tau} \lesssim D(\tau,\tau)^{1/q} \lesssim \tau^{-1+1/q+\varepsilon/q}.$$

Now we choose ε so small that $-1+1/q+\varepsilon/q < 0$. Then it follows that

$$\lim_{\tau\to\infty} I_{\tau,\tau} = 0,$$

which yields $u \equiv 0$ and a contradiction.

CHAPTER 7

Estimates of the lifespan

7.1. Effective damping cases

In this chapter, we consider the semilinear damped wave equation

$$u_{tt} - \Delta u + a(t,x)u_t = |u|^p, \quad (t,x) \in (0,\infty) \times \mathbf{R}^n, \tag{7.1.1}$$

with the initial condition

$$(u,u_t)(0,x) = \varepsilon(u_0,u_1)(x), \quad x \in \mathbf{R}^n, \tag{7.1.2}$$

where $u = u(t,x)$ is a real-valued unknown function of (t,x), $1 < p$, $(u_0,u_1) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ and ε is a positive small parameter. The coefficient of the damping term is given by

$$a(t,x) = \langle x\rangle^{-\alpha}(1+t)^{-\beta}$$

with $\alpha \in [0,1)$, $\beta \in (-1,1)$ and $\alpha\beta = 0$. Here $\langle x\rangle$ denotes $\sqrt{1+|x|^2}$. The condition $\alpha\beta = 0$ means that a depends on only one of t or x.

Our aim is to obtain an upper bound of the lifespan of solutions to (7.1.1).

In the constant coefficient case $\alpha = \beta = 0$, when $n = 1,2$, Li and Zhou [**53**] obtained the sharp upper bound of the maximal existence time of classical solutions:

$$T_\varepsilon \le \begin{cases} \exp(C\varepsilon^{-2/n}), & \text{if } p = 1+2/n, \\ C\varepsilon^{-1/\kappa}, & \text{if } 1 < p < 1+2/n, \end{cases} \tag{7.1.3}$$

where $C = C(n,p,u_0,u_1) > 0$ and $\kappa = 1/(p-1) - n/2$ for the data $u_0, u_1 \in C_0^\infty(\mathbf{R}^n)$ satisfying $\int(u_0+u_1)dx > 0$. Nishihara [**79**] extended this result to $n = 3$ by using the explicit formula of the solution to the linear part of (7.1.1) with initial data $(0,u_1)$:

$$u(t,x) = e^{-t/2}W_3(t)u_1 + J_3(t)u_1.$$

Here $W_3(t)u_1$ is the solution of the wave equation $\Box u = 0$ with initial data $(0,u_1)$ and $J_3(t)u_1$ behaves like a solution of the heat equation $-\Delta v + v_t = 0$. However, both the methods of [**53**] and [**79**] do not work in higher dimensional cases $n \ge 4$, because they used the positivity of the fundamental solution of the free wave equation, which is valid only in the case $n \le 3$. We shall extend both of the results to $n \ge 4$ in subcritical cases $1 < p < 1+2/n$ by using an another method.

For the case $\alpha \in [0,1), \beta = 0$, Ikehata, Todorova and Yordanov [**36**] determined the critical exponent for (7.1.1) as $p_c = 1 + 2/(n-\alpha)$, which also agrees with that of the corresponding heat equation $-\Delta v + \langle x\rangle^{-\alpha}v_t = |v|^p$. Here we emphasize that in this case there are no results about upper estimates for the lifespan. It will be given in this chapter.

Next, for the case $\beta \in (-1,1), \alpha = 0$, Nishihara [**83**] and Lin, Nishihara and Zhai [**56**] proved $p_c = 1 + 2/n$, which is also same as that of the heat equation $-\Delta v + (1+t)^{-\beta}v_t = |v|^p$. On the other hand, upper estimates of the lifespan have not been well studied. Nishihara [**83**] obtained a similar result of [**53, 79**]: let $n \geq 1, \beta \geq 0$ and (u_0, u_1) satisfy $\int_{\mathbf{R}^n} u_i(x)dx \geq 0$ $(i = 0, 1)$, $\int_{\mathbf{R}^n}(u_0+u_1)(x)dx > 0$. Then there exists a constant $C > 0$ such that

$$T_\varepsilon \leq \begin{cases} e^{C\varepsilon^{-(1+\beta)/n}}, & \text{if } p = 1 + (1+\beta)/n, \\ C\varepsilon^{-1/\hat{\kappa}}, & \text{if } 1 + 2\beta/n \leq p < 1 + (1+\beta)/n, \end{cases}$$

where $\hat{\kappa} = (1+\beta)/(p-1) - n$. We note that the rate $\hat{\kappa}$ is not optimal, because it is not same as that of the corresponding heat equation. Moreover, there are no results for $1 + (1+\beta)/n < p \leq 1 + 2/n$. We will improve the above result for all $1 < p < 1 + 2/n$ and give the sharp upper estimate.

To state our results, we define the solution of (7.1.1). Let $T \in (0, \infty]$. We say that $u \in X(T) := C([0,T); H^1(\mathbf{R}^n)) \cap C^1([0,T); L^2(\mathbf{R}^n))$ is a (weak) solution of (7.1.1) with initial data (7.1.2) on the interval $[0, T)$ if the identity

$$\begin{aligned} &\int_{[0,T)\times\mathbf{R}^n} u(t,x)(\partial_t^2\psi(t,x) - \Delta\psi(t,x) - \partial_t(a(t,x)\psi(t,x)))dxdt \\ &= \varepsilon \int_{\mathbf{R}^n} \{(a(0,x)u_0(x) + u_1(x))\psi(0,x) - u_0(x)\partial_t\psi(0,x)\}\, dx \\ &\quad + \int_{[0,T)\times\mathbf{R}^n} |u(t,x)|^p\psi(t,x)dxdt \end{aligned} \tag{7.1.4}$$

holds for any $\psi \in C_0^\infty([0,T) \times \mathbf{R}^n)$ (see also Section 9.4.3). We also define the lifespan for the local solution of (7.1.1)-(7.1.2) by

$$T_\varepsilon := \sup\{\, T \in (0,\infty]; \text{there exists a unique solution } u \in X(T) \text{ of (7.1.1)-(7.1.2)} \,\}.$$

We first describe the local existence result.

Proposition 7.1. *Let $\alpha \geq 0, \beta \in \mathbf{R}, 1 < p \leq n/(n-2)$ $(n \geq 3)$, $1 < p < \infty$ $(n = 1, 2)$, $\varepsilon > 0$ and $(u_0, u_1) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$. Then $T_\varepsilon > 0$, that is, there exists a unique solution $u \in X(T_\varepsilon)$ to (7.1.1)-(7.1.2). Moreover, if $T_\varepsilon < +\infty$, then it follows that*

$$\lim_{t\to T_\varepsilon - 0} \|(u, u_t)(t,\cdot)\|_{H^1\times L^2} = +\infty.$$

For the proof, see Proposition 9.21.

Next, we state our main result, which gives an upper bound of T_ε.

Theorem 7.2. *Let $\alpha \in [0,1), \beta \in (-1,1), \alpha\beta = 0$ and let $1 < p < 1+2/(n-\alpha)$. We assume that the initial data $(u_0, u_1) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ satisfy*

$$\liminf_{R\to\infty} \int_{|x|<R} (\langle x\rangle^{-\alpha} B u_0(x) + u_1(x))dx > 0, \tag{7.1.5}$$

where

$$B = \left(\int_0^\infty e^{-\int_0^t (1+s)^{-\beta} ds} dt \right)^{-1}.$$

Then there exists $C > 0$ depending only on n, p, α, β and (u_0, u_1) such that T_ε is estimated as

$$T_\varepsilon \leq C \begin{cases} \varepsilon^{-1/\kappa} & \text{if } \ 1 + \alpha/(n-\alpha) < p < 1 + 2/(n-\alpha), \\ \varepsilon^{-(p-1)}(\log(\varepsilon^{-1}))^{p-1} & \text{if } \ \alpha > 0,\ p = 1 + \alpha/(n-\alpha), \\ \varepsilon^{-(p-1)} & \text{if } \ \alpha > 0,\ 1 < p < 1 + \alpha/(n-\alpha) \end{cases}$$

for any $\varepsilon \in (0,1]$, *where*

$$\kappa = \frac{2(1+\beta)}{2-\alpha}\left(\frac{1}{p-1} - \frac{n-\alpha}{2}\right).$$

REMARK 7.1. *Estimates of* T_ε *from below were obtained in Proposition 4.5 for* $\alpha \geq 0, \beta \geq 0$ *and by* [**56**] $\alpha = 0, \beta \in (-1,1)$. *The results can be expressed by the following table:*

	$\alpha = 0$	$\beta = 0$
p_c	$1 + \dfrac{2}{n}$	$1 + \dfrac{2}{n-\alpha}$
$T_\varepsilon \lesssim$	$\varepsilon^{-1/\kappa}$	$\begin{cases} \varepsilon^{-1/\kappa}, & \left(1 + \dfrac{\alpha}{n-\alpha} < p < 1 + \dfrac{2}{n-\alpha}\right) \\ \varepsilon^{-(p-1)}(\log(\varepsilon^{-1}))^{p-1}, & \left(p = 1 + \dfrac{\alpha}{n-\alpha}\right) \\ \varepsilon^{-(p-1)}, & \left(1 < p < 1 + \dfrac{\alpha}{n-\alpha}\right) \end{cases}$
$T_\varepsilon \gtrsim$	$\varepsilon^{-1/\kappa+\delta}$	$\varepsilon^{-1/\kappa+\delta}$
κ	$(1+\beta)\left(\dfrac{1}{p-1} - \dfrac{n}{2}\right)$	$\dfrac{2}{2-\alpha}\left(\dfrac{1}{p-1} - \dfrac{n-\alpha}{2}\right)$

Here δ *denotes any small positive number.*

REMARK 7.2. *The condition* (7.1.5) *holds if* $\langle x\rangle^{-\alpha} B u_0(x) + u_1(x) \in L^1(\mathbf{R}^n)$ *and*

$$\int_{\mathbf{R}^n} (\langle x\rangle^{-\alpha} B u_0(x) + u_1(x))dx > 0.$$

REMARK 7.3. *In view of Proposition 4.5, it is expected that the rate* κ *in Theorems 7.2 is sharp except for the case* $\alpha > 0, 1 < p \leq 1 + \alpha/(n-\alpha)$.

REMARK 7.4. *In Theorem 7.2, the explicit form of* $a = \langle x\rangle^{-\alpha}(1+t)^{-\beta}$ *is not necessary. Indeed, we can treat more general coefficients, for example,* $a(t,x) = a(x)$ *satisfying* $a \in C(\mathbf{R}^n)$ *and* $0 \leq a(x) \lesssim \langle x\rangle^{-\alpha}$, *or* $a(t,x) = a(t)$ *satisfying* $b \in C^1([0,\infty))$ *and* $b(t) \sim (1+t)^{-\beta}$.

REMARK 7.5. *The same conclusion of Theorem 2.3 is valid for the corresponding heat equation* $-\Delta v + a(t,x)v_t = |v|^p$ *in the same manner as our proof.*

Our proof is based on a test function method. However, the method of [**122**] was based on a contradiction argument and so upper estimates of the lifespan cannot be obtained. To avoid the contradiction argument, we use an idea by Kuiper [**48**]. He obtained an upper bound of the lifespan for some parabolic equations (see also [**102**]). We note that to treat the time-dependent damping case, we also use a transformation of equation by Lin, Nishihara and Zhai [**56**] (see also [**7**]).

7.2. Proof of the estimates of the lifespan

In the case $\beta \neq 0$, (7.1.1) is not divergence form and so we cannot apply the test function method. Therefore, we need to transform the equation (7.1.1) into divergence form. Let $g(t)$ be the solution of the ordinary differential equation

$$\begin{cases} -g'(t) + (1+t)^{-\beta} g(t) = 1, \\ g(0) = B^{-1}. \end{cases}$$

The solution $g(t)$ is explicitly given by

$$g(t) = e^{\int_0^t (1+s)^{-\beta} ds} \left(B^{-1} - \int_0^t e^{-\int_0^\tau (1+s)^{-\beta} ds} d\tau \right).$$

By the de l'Hôpital theorem, we have

$$\lim_{t\to\infty} (1+t)^{-\beta} g(t) = 1$$

and so $g(t) \sim (1+t)^\beta$. We note that $B = 1$ and $g(t) \equiv 1$ if $\beta = 0$. By the definition of $g(t)$, we also have $|g'(t)| \lesssim |(1+t)^{-\beta} g(t) - 1| \lesssim 1$. Multiplying the equation (7.1.1) by $g(t)$, we obtain the divergence form

$$(gu)_{tt} - \Delta(gu) - ((g'-1)\langle x\rangle^{-\alpha} u)_t = g|u|^p, \tag{7.2.1}$$

here we note that $\alpha\beta = 0$. Therefore, we can apply the test function method to (7.2.1).

As before, we define the following test functions:

$$\phi(x) := \begin{cases} 1 & (|x| \le 1/2) \\ \dfrac{\exp(-1/(1-|x|^2))}{\exp(-1/(|x|^2-1/4)) + \exp(-1/(1-|x|^2))} & (1/2 < |x| < 1), \\ 0 & (|x| \ge 1), \end{cases}$$

$$\eta(t) := \begin{cases} 1 & (0 \le t \le 1/2), \\ \dfrac{\exp(-1/(1-t^2))}{\exp(-1/(t^2-1/4)) + \exp(-1/(1-t^2))} & (1/2 < t < 1), \\ 0 & (t \ge 1). \end{cases}$$

It is obvious that $\phi \in C_0^\infty(\mathbf{R}^n), \eta \in C_0^\infty([0,\infty))$. Then we have

$$|\Delta\phi(x)| \le C\phi(x)^{1/p}, \quad |\eta'(t)| \le C\eta(t)^{1/p}, \quad |\eta''(t)| \le C\eta(t)^{1/p}. \tag{7.2.2}$$

Indeed, we put q, r so that $1/p + 1/q = 1, 1/p + 2/r = 1$ and let $\mu = \eta^{1/q}, \nu = \eta^{1/r}$. Then we obtain

$$|\eta'(t)| = |(\mu^q)'| = |q\mu^{q-1}\mu'| \lesssim \mu^{q-1} = \eta^{1/p},$$
$$|\eta''(t)| = |(\nu^r)''| \lesssim |\nu''|\nu^{r-1} + |\nu'|^2\nu^{r-2} \lesssim \nu^{r-2} = \eta^{1/p}.$$

The assertion for ϕ can be proved in the same way.

Let τ_0, R_0 be constants depending only on $n, \alpha, \beta, u_0, u_1$ and determined later. We note that if $T_\varepsilon \le \tau_0$, then it is obvious that $T_\varepsilon \le \tau_0 \varepsilon^{-1/\kappa}$ for any $\kappa > 0$ and $\varepsilon \in (0,1]$. Therefore, we may assume that $T_\varepsilon > \tau_0$. Let u be a solution on $[0, T_\varepsilon)$ and $\tau \in (\tau_0, T_\varepsilon), R \ge R_0$ parameters. We define

$$\psi_{\tau,R}(t,x) := \eta_\tau(t)\phi_R(x) := \eta(t/\tau)\phi(x/R)$$

and

$$
\begin{aligned}
I_{\tau,R} &:= \int_{[0,\tau)\times B_R} g(t)|u(t,x)|^p \psi_{\tau,R}(t,x)dxdt,\\
J_R &:= \varepsilon B^{-1}\int_{B_R} (\langle x\rangle^{-\alpha} Bu_0(x) + u_1(x))\phi_R(x)dx,
\end{aligned}
$$

where $B_R = \{|x| < R\}$. Since $\psi_{\tau,R} \in C_0^\infty([0,T_\varepsilon)\times \mathbf{R}^n)$ and u is a solution on $[0,T_\varepsilon)$, we have

$$
\begin{aligned}
I_{\tau,R} + J_R &= \int_{[0,\tau)\times B_R} g(t)u\partial_t^2\psi_{\tau,R}dxdt - \int_{[0,\tau)\times B_R} g(t)u\Delta\psi_{\tau,R}dxdt\\
&\quad + \int_{[0,\tau)\times B_R} (g'(t)-1)\langle x\rangle^{-\alpha} u\partial_t\psi_{\tau,R}dxdt\\
&=: K_1 + K_2 + K_3.
\end{aligned}
$$

Here we have used the property $\partial_t\psi(0,x) = 0$ and substituted the test function $g(t)\psi(t,x)$ into the definition of solution (7.1.4). We note that for the corresponding heat equation, we have the same decomposition without the term K_1 and so we can obtain the same conclusion (see Remark 7.5). We first estimate K_1. By the Hölder inequality and (7.2.2), we have

$$
\begin{aligned}
K_1 &\le \tau^{-2}\int_{[0,\tau)\times B_R} g(t)|u||\eta''(t/\tau)|\phi_R(x)dxdt\\
&\le C\tau^{-2}\int_{[\tau/2,\tau)\times B_R} g(t)|u|\eta_\tau(t)^{1/p}\phi_R(x)dxdt\\
&\le \tau^{-2} I_{\tau,R}^{1/p}\left(\int_{\tau/2}^{\tau} g(t)dt\cdot\int_{B_R}\phi_R(x)dx\right)^{1/q}\\
&\le C\tau^{-2+1/q}(1+\tau)^{\beta/q}R^{n/q}I_{\tau,R}^{1/p}.
\end{aligned}
\tag{7.2.3}
$$

Using (7.2.2) and a similar calculation, we obtain

$$
\begin{aligned}
K_2 &\le R^{-2}\int_{[0,\tau)\times B_R} g(t)|u||\Delta\phi(x/R)|\eta(t/\tau)dxdt\\
&\le CR^{-2}\int_{[0,\tau)\times B_R} g(t)|u||\phi(x/R)|^{1/p}\eta(t/\tau)dxdt\\
&\le CR^{-2}I_{\tau,R}^{1/p}\left(\int_0^{\tau} g(t)\eta(t/\tau)dt\cdot\int_{B_R} 1dx\right)^{1/q}\\
&\le C(1+\tau)^{(1+\beta)/q}R^{-2+n/q}I_{\tau,R}^{1/p}.
\end{aligned}
\tag{7.2.4}
$$

For K_3, using (7.2.2) and $|g'(t)-1| \lesssim C$, we have

$$
\begin{aligned}
K_3 &\le \tau^{-1}\int_{[0,\tau)\times B_R} \langle x\rangle^{-\alpha}|u||\eta'(t/\tau)|\phi_R(x)dxdt\\
&\le \tau^{-1}I_{\tau,R}^{1/p}\left(\int_{\tau/2}^{\tau} g(t)^{-q/p}dt\cdot\int_{B_R}\langle x\rangle^{-\alpha q}\phi_R(x)dx\right)^{1/q}\\
&\le C\tau^{-1+1/q}(1+\tau)^{-\beta/p}F_{p,\alpha}(R)I_{\tau,R}^{1/p},
\end{aligned}
\tag{7.2.5}
$$

where

$$F_{p,\alpha}(R) = \begin{cases} R^{-\alpha+n/q} & (\alpha q < n), \\ (\log(1+R))^{1/q} & (\alpha q = n), \\ 1 & (\alpha q > n). \end{cases}$$

Thus, putting

$$D(\tau, R) := \tau^{-(1+\beta)/p}(\tau^{-1+\beta}R^{n/q} + \tau^{1+\beta}R^{-2+n/q} + F_{p,\alpha}(R))$$

and combining this with the estimates (7.2.3)-(7.2.5), we have

$$J_R \le CD(\tau, R)I_{\tau,R}^{1/p} - I_{\tau,R}. \tag{7.2.6}$$

Now we use a fact that the inequality

$$ac^b - c \le (1-b)b^{b/(1-b)}a^{1/(1-b)}$$

holds for all $a > 0, 0 < b < 1, c \ge 0$. We can immediately prove it by considering the maximal value of the function $f(c) = ac^b - c$. From this and (7.2.6), we obtain

$$J_R \le CD(\tau, R)^q. \tag{7.2.7}$$

On the other hand, by the assumption on the data, there exist $C > 0$ and R_0 such that $J_R \ge C\varepsilon$ holds for all $R > R_0$. Combining this with (7.2.7), we have

$$\varepsilon \le CD(\tau, R)^q \tag{7.2.8}$$

for all $\tau \in (\tau_0, T_\varepsilon)$ and $R > R_0$. Now we define

$$\tau_0 := \max\{1, R_0^{(2-\alpha)/(1+\beta)}\},$$

and we substitute

$$R = \begin{cases} \tau^{(1+\beta)/(2-\alpha)} & (\alpha q < n), \\ \tau & (\alpha q \ge n) \end{cases} \tag{7.2.9}$$

into (7.2.8). Here we note that $R > R_0$ if R is given by (7.2.9). As was mentioned at the beginning of this section, we may assume that $\tau_0 < T_\varepsilon$. Finally, we have

$$\varepsilon \le C \begin{cases} \tau^{-\kappa} & (\alpha q < n), \\ \tau^{-1/(p-1)}\log(1+\tau) & (\alpha q = n), \\ \tau^{-1/(p-1)} & (\alpha q > n), \end{cases}$$

with

$$\kappa = \frac{2(1+\beta)}{2-\alpha}\left(\frac{1}{p-1} - \frac{n-\alpha}{2}\right).$$

We can rewrite this relation as

$$\tau \le C \begin{cases} \varepsilon^{-1/\kappa} & \text{if} \quad 1+\alpha/(n-\alpha) < p < 1+2/(n-\alpha), \\ \varepsilon^{-(p-1)}(\log(\varepsilon^{-1}))^{p-1} & \text{if} \quad \alpha > 0,\ p = 1+\alpha/(n-\alpha), \\ \varepsilon^{-(p-1)} & \text{if} \quad \alpha > 0,\ 1 < p < 1+\alpha/(n-\alpha). \end{cases}$$

Here we note that $\kappa > 0$ if and only if $1 < p < 1+2/(n-\alpha)$ and that $\alpha q = n$ is equivalent to $p = 1+\alpha/(n-\alpha)$. Since τ is arbitrary in (τ_0, T_ε), we can obtain the conclusion of the theorem.

7.3. Scale-invariant damping case

In this section, we consider

$$
\left\{
\begin{array}{ll}
u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u|^p & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\
(u, u_t)(0,x) = \varepsilon (u_0, u_1)(x) & x \in \mathbf{R}^n.
\end{array}
\right.
\tag{7.3.1}
$$

In Chapter 5, we proved that if $(u_0, u_1) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ and

$$
\text{“}\mu > 1,\ 1 < p \le 1 + 2/n \text{ and } \liminf_{R\to\infty} \int_{|x|<R} ((\mu-1)u_0 + u_1)(x)dx > 0\text{”} \tag{7.3.2}
$$

or

$$
\text{“}0 < \mu \le 1,\ 1 < p \le 1 + 2/(n+\mu-1) \text{ and } \liminf_{R\to\infty} \int_{|x|<R} u_1(x)dx > 0\text{”}, \tag{7.3.3}
$$

then the solution blows up in finite time. The aim of this section is to obtain an upper bound of the lifespan. In a similar way to the previous section, we have the following estimates:

Proposition 7.3. *We assume* (7.3.2) *or* (7.3.3) *with* $1 < p < 1 + 2/n$, $1 < p < 1 + 2/(n+\mu-1)$, *respectively. Then we have*

$$
T_\varepsilon \lesssim \varepsilon^{-1/\kappa}, \tag{7.3.4}
$$

where

$$
\kappa = \begin{cases}
2\left(\frac{1}{p-1} - \frac{n}{2}\right) & (\mu > 1, 1 < p < 1 + 2/n), \\
2\left(\frac{1}{p-1} - \frac{n+\mu-1}{2}\right) & (0 < \mu \le 1, 1 < p < 1 + 2/(n+\mu-1)).
\end{cases}
\tag{7.3.5}
$$

Proof. We shall show only a brief outline. First, we treat the case of (7.3.2). By the derivation of (5.3.3) and the argument in the proof of Theorem 7.2, we can easily deduce that

$$
\begin{aligned}
J_R &\le C(\tau^{2/q} R^{n/q-2} + \tau^{2/q-2} R^{n/q}) I_{\tau,R}^{1/p} - I_{\tau,R} \\
&\le C(\tau^{2/q} R^{n/q-2} + \tau^{2/q-2} R^{n/q})^q \\
&= C\tau^2 (R^{n-2q} + \tau^{-2q} R^n),
\end{aligned}
$$

where

$$
I_{\tau,R} = \int_0^\tau \int_{B_R} g(t) |u|^p \psi_{\tau,R}(t,x) dx dt,
$$

with $g(t) = \frac{1}{\mu-1}(1+t)$, ψ is the same test function as in the previous section and

$$
J_R = \varepsilon \int_{B_R} ((\mu-1)u_0 + u_1)\phi_R(x) dx.
$$

In the same way as in the previous section, using the assumption on the data and taking $R = \tau$, we can see that

$$
\varepsilon \le C\tau^{-\kappa}, \quad \kappa = 2(1/(p-1) - n/2)
$$

for $\tau \in (\tau_0, T_\varepsilon)$. This implies $T_\varepsilon \lesssim \varepsilon^{-1/\kappa}$. The case of (7.3.3) can be proved by the same way by noting that

$$
I_{\tau,\tau} + J_\tau \le C\tau^{-2+(n+\mu+1)/q} I_{\tau,\tau}^{1/p}.
$$

□

7.4. Time and space dependent damping case

We consider the 1-dimensional semilinear wave equation with damping depending on time and space variables

$$\begin{cases} u_{tt} - u_{xx} + a(t,x)u_t = |u|^p & (t,x) \in (0,\infty) \times \mathbf{R}, \\ (u,u_t)(0,x) = \varepsilon(u_0,u_1)(x) & x \in \mathbf{R}. \end{cases} \tag{7.4.1}$$

We assume that a satisfies (6.1.2), namely

$$|\partial_t^\alpha \partial_x^\beta a(t,x)| \le \frac{\delta}{(1+t)^{k+\alpha}} \quad (\alpha, \beta = 0, 1)$$

with some $k > 1$ and small $\delta > 0$. As we mentioned in the previous chapter, we can apply the test function method by using an appropriately multiplier $g(t,x) = 1 + h(t,x)$ defined by Lemma 6.1 and obtain a blow-up result. In this section we give an estimate of the lifespan. For the sake of simplicity, we treat only the case that a satisfies (6.1.2).

Proposition 7.4. *Let $1 < p < \infty$ and let $(u_0, u_1) \in H^1(\mathbf{R}) \times L^2(\mathbf{R})$ satisfy* (6.1.9) *with g defined by* (6.1.5) *and h in Lemma 6.1. Then T_ε is estimated as*

$$T_\varepsilon \le C\varepsilon^{-1/\kappa} \tag{7.4.2}$$

with some constant $C > 0$ and $\kappa = \frac{1}{p-1}(1+1/p)$.

Proof. We use the same notation as in the previous section. By (6.3.4), we have

$$I_{\tau,R} + J_R \le CD(\tau,R)I_{\tau,R}^{1/p}.$$

with

$$D(\tau,R) := \tau^{-2+1/q}R^{1/q} + \tau^{1/q}R^{-2+1/q} + R^{-1+1/q}$$

and

$$J_R = \varepsilon \int_{-R}^{R} \{(-g_t(0,x) + g(0,x)a(0,x))u_0(x) + g(0,x)u_1(x)\}\phi_R(x)dx.$$

By the assumption on the data, there exist some constants $R_0, c > 0$ such that $c\varepsilon \le J_R$ holds for $R \ge R_0$. Thus, we have

$$\begin{aligned} \varepsilon &\le CD(\tau,R)I_{\tau,R}^{1/p} - I_{\tau,R} \\ &\le CD(\tau,R)^q \end{aligned} \tag{7.4.3}$$

for all $\tau \in (\tau_0, T_\varepsilon), R \in (R_0, \infty)$. We put $R = \tau^\alpha$ with $\alpha > 0$ and $\tau_0 := \max\{1, R_0^{1/\alpha}\}$. Then we obtain

$$D(\tau,\tau^\alpha) \le \tau^{\max\{-1-1/p+(1-1/p)\alpha, 1-1/p+(-1-1/p)\alpha, -\alpha/p\}}. \tag{7.4.4}$$

Now we take $\alpha = 1 + 1/p$, which minimizes the power of τ in (7.4.4). From (7.4.3) we see that

$$\varepsilon \lesssim D(\tau, \tau^{1+1/p})^q \lesssim \tau^{-\frac{1}{p-1}(1+1/p)} = \tau^{-\kappa}.$$

Therefore, we have

$$\tau \le C\varepsilon^{-1/\kappa}.$$

Since τ is arbitrary in (τ_0, T_ε), it follows that

$$T_\varepsilon \le C\varepsilon^{-1/\kappa},$$

which completes the proof. □

CHAPTER 8

A remark on L^p-L^q estimates for solutions to the linear damped wave equation

8.1. Introduction

This chapter is devoted to L^p-L^q estimates for solutions to the linear damped wave equation

$$\left\{ \begin{array}{ll} u_{tt} - \Delta u + u_t = 0, & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n. \end{array} \right. \tag{8.1.1}$$

In Section 2.1.2, we introduce a solution representation formula for (8.1.1) and in Section 2.2.2, we prove the following L^p-L^q estimate in the three dimensional case:

$$\left\| u(t) - v(t) - e^{-t/2}\widetilde{\mathbf{W}}_3(t;u_0,u_1) \right\|_{L^p} \le Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}(\|u_0\|_{L^q} + \|u_1\|_{L^q}), \tag{8.1.2}$$

where $t > 0, 1 \le q \le p \le \infty$, $v(t)$ is the solution of the heat equation

$$v_t - \Delta v = 0, \quad v(0,x) = u_0(x) + u_1(x), \tag{8.1.3}$$

$\widetilde{\mathbf{W}}_3(t;u_0,u_1)$ is defined by

$$\widetilde{\mathbf{W}}_3(t;u_0,u_1) = \left(\frac{1}{2} + \frac{t}{8}\right) W(t)u_0 + \partial_t(W(t)u_0) + W(t)u_1$$

and $W(t)g$ denotes the solution of the free wave equation

$$w_{tt} - \Delta w = 0, \quad (w,w_t)(0,x) = (0,g)(x).$$

In this chapter, we extend the above estimate to any space dimension. In lower dimensional cases $n = 1,2,3$, the estimate (8.1.2) has been already obtained by Marcati and Nishihara [**58**], Hosono and Ogawa [**23**], Nishihara [**78**], respectively. For higher dimensional cases $n \ge 4$, Narazaki [**73**] proved the following estimates:

$$\|\mathcal{F}^{-1}\{\chi(\xi)(\hat{u}(t) - \hat{v}(t))\}\|_{L^p} \le C(1+t)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1+\varepsilon}(\|u_0\|_{L^q} + \|u_1\|_{L^q}), \tag{8.1.4}$$

where $1 \le q \le p \le \infty$, ε is an arbitrary small positive number, $C = C(p,q,\varepsilon)$ is a positive constant depending on p,q,ε, $\chi(\xi)$ is a compactly supported smooth radial function satisfying $\chi(\xi) = 1$ near $\xi = 0$ and $v(t,x)$ is the solution of the corresponding heat equation (8.1.3). Moreover, in the case where $1 < q < p < \infty, (p,q) = (2,2)$ or $(p,q) = (\infty,1)$, we may take $\varepsilon = 0$;

$$\|\mathcal{F}^{-1}\{(1-\chi(\xi))(\hat{u}(t) - e^{-t/2}(\mathbf{M}_0(t,\cdot)\hat{u_0} + \mathbf{M}_1(t,\cdot)\hat{u_1}))\}\|_{L^p} \le Ce^{-\delta t}\|g\|_{L^q}$$

for some $\delta > 0$, where $1 < q \le p < \infty$, $\chi(\xi)$ is as above, $C = C(p,q) > 0$ is a constant depending on p, q and

$$\mathbf{M}_1(t,\xi) = \frac{1}{\sqrt{|\xi|^2 - 1/4}} \Bigg(\sin(t|\xi|) \sum_{0 \le k < (n-1)/4} \frac{(-1)^k}{(2k)!} t^{2k} \Theta(\xi)^{2k} - \cos(t|\xi|) \sum_{0 \le k < (n-3)/4} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \Theta(\xi)^{2k+1} \Bigg),$$

$$\mathbf{M}_0(t,\xi) = \cos(t|\xi|) \sum_{0 \le k < (n+1)/4} \frac{(-1)^k}{(2k)!} t^{2k} \Theta(\xi)^{2k} + \sin(t|\xi|) \sum_{0 \le k < (n-1)/4} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \Theta(\xi)^{2k+1} + \frac{1}{2} \mathbf{M}_1(t,\xi)$$

with $\Theta(\xi) = |\xi| - \sqrt{|\xi|^2 - 1/4}$.

The aim of this chapter is to remove ε in the estimate (8.1.4) and give a simpler proof by using the solution representation formula.

The estimate (8.1.2) shows that the solution operator $S_n(t)$ of (8.1.1) is asymptotically expressed by

$$S_n(t) \sim e^{t\Delta} + e^{t/2} W_n(t)$$

when $n = 3$ (see [**58, 23**] for the cases $n = 1, 2$). We first give the corresponding decomposition for higher dimensional cases.

LEMMA 8.1. *(i) When $n \ge 3$ and and odd number, $S_n(t)$ is expressed as follows:*

$$(8.1.5) \quad \begin{aligned} S_n(t)g(x) &= J_n(t)g(x) + e^{-t/2}\mathbf{W}_n(t)g(x) \\ &= \frac{e^{-t/2}}{(n-2)!!|S^{n-1}|} \frac{1}{4^{\frac{n-1}{2}}} \int_{|x-y| \le t} k_{\frac{n-1}{2}}\left(\frac{1}{2}\sqrt{t^2 - |x-y|^2}\right) g(y)dy \\ &+ \frac{e^{-t/2}}{(n-2)!!|S^{n-1}|} \sum_{l=0}^{(n-3)/2} \frac{1}{8^l l!} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2-l} \left(\frac{1}{t}\int_{|x-y|=t} g(y)dS_y\right), \end{aligned}$$

where

$$k_l(s) = \frac{1}{2^l} \sum_{m=0}^{\infty} \frac{1}{m!(m+l)!} \left(\frac{s}{2}\right)^{2m} = s^{-l} I_l(s).$$

Moreover, $k_l(s)$ has the asymptotic expansion

$$(8.1.6) \quad k_l(s) = s^{-l} \frac{1}{\sqrt{2\pi s}} e^s \left(1 - \frac{(l-1/2)(l+1/2)}{2s} + O(s^{-2})\right)$$

as $s \to +\infty$.

(ii) When n is an even number, $S_n(t)$ is expressed as follows:

(8.1.7)

$$\begin{aligned} S_n(t)g &= J_n(t)g(x) + e^{-t/2}\mathbf{W}_n(t)g(x) \\ &= \frac{e^{-t/2}}{(n-1)!!|S^n|}\frac{1}{4^{\frac{n-2}{2}}}\int_{|x-y|\le t} k_{\frac{n}{2}}\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)g(y)dy \\ &+ \frac{e^{-t/2}}{(n-1)!!|S^n|}\sum_{l=0}^{(n-2)/2}\frac{1}{8^l l!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2-l}\int_{|x-y|\le t}\frac{1}{\frac{1}{2}\sqrt{t^2-|x-y|^2}}g(y)dy, \end{aligned}$$

where

$$k_l(s) = \sum_{m=0}^{\infty}\frac{(2(m+l)-1)!!}{(2(m+l))!(2m+1)!!}s^{2m+1}.$$

Moreover, $k_l(s)$ has the asymptotic expansion

$$k_l(s) = s^{-l}\frac{e^s}{2}\left(1+\frac{a_l}{s}+O(s^{-2})\right) \tag{8.1.8}$$

with some constant a_l.

Using the above lemma, we can obtain the following L^p-L^q estimate without any loss $\varepsilon > 0$. To state the result, we put

$$\widetilde{\mathbf{W}}_n(t;u_0,u_1) = \frac{1}{2}\mathbf{W}_n(t)u_0 + \mathbf{W}_n(t)u_1 + \partial_t(\mathbf{W}_n(t)u_0) + \widehat{\mathbf{W}}_n(t)u_0,$$

where

$$\widehat{\mathbf{W}}_n(t)u_0 = \frac{e^{-t/2}}{(n-2)!!|S^{n-1}|}\frac{1}{8^{\frac{n-1}{2}}(\frac{n-1}{2})!}\int_{|x-y|=t}u_0(y)dS_y$$

when $n \ge 3$ and is an odd number, and

$$\widehat{\mathbf{W}}_n(t)u_0 = \frac{e^{-t/2}}{(n-1)!!|S^n|}\frac{1}{8^{\frac{n}{2}}(\frac{n}{2})!}\int_{|x-y|\le t}\frac{1}{\frac{1}{2}\sqrt{t^2-|x-y|^2}}u_0(y)dy$$

when $n \ge 2$ and is an even number.

THEOREM 8.2. *Let u, v be the solutions to* (8.1.1), (8.1.3), *respectively. For $1 \le q \le p \le \infty$ and $t > 0$, we have*

$$\|u(t) - v(t) - e^{-t/2}\widetilde{\mathbf{W}}_n(t;u_0,u_1)\|_{L^p} \le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-1}(\|u_0\|_{L^q} + \|u_1\|_{L^q}).$$

8.2. Proof of Lemma 8.1

First, we prove in the case that $n \ge 3$ is an odd number. By Lemma 9.3, it is easy to see that

$$k_{l+1}(s) = \frac{k_l'(s)}{s}$$

and

$$k_l(0) = \frac{1}{2^l l!}.$$

By using the solution formula in Proposition 2.2, we have

$$
\begin{aligned}
S_n(t)g(x) &= \frac{e^{-t/2}}{(n-2)!!|S^{n-1}|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-1)/2}\int_{|x-y|\le t} I_0\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)g(y)dy\\
&= \frac{e^{-t/2}}{(n-2)!!|S^{n-1}|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2}\left(\frac{1}{t}\int_{|x-y|=t} g(y)dS_y\right)\\
&\quad+\frac{e^{-t/2}}{(n-2)!!|S^{n-1}|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2}\int_{|x-y|\le t}\frac{I_1\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)}{2\sqrt{t^2-|x-y|^2}}g(y)dy.
\end{aligned}
$$

Note that the second term of the right-hand side can be written as

$$
\frac{e^{-t/2}}{4(n-2)!!|S^{n-1}|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2}\int_{|x-y|\le t} k_1\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)g(y)dy.
$$

Since

$$
\frac{1}{t}\left(\frac{\partial}{\partial t}k_1\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)\right)=\frac{k_1'\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)}{2\sqrt{t^2-|x-y|^2}}=\frac{1}{4}k_2\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right),
$$

one can see that

$$
\begin{aligned}
&\frac{e^{-t/2}}{4(n-2)!!|S^{n-1}|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2}\int_{|x-y|\le t} k_1\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)g(y)dy\\
&= \frac{e^{-t/2}}{4(n-2)!!|S^{n-1}|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-5)/2}\left(\frac{k_1(0)}{t}\int_{|x-y|=t} g(y)dS_y\right)\\
&\quad+\frac{e^{-t/2}}{4^2(n-2)!!|S^{n-1}|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-5)/2}\int_{|x-y|\le t} k_2\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)g(y)dy.
\end{aligned}
$$

Continuing this argument, we can obtain the conclusion. The asymptotic expansion (8.1.6) is a direct consequence from that of the modified Bessel functions (see Lemma 9.5).

Next, we turn to the even dimensional cases. First note that a simple calculation shows

$$
k_1(s)=\frac{\cosh(s)-1}{s},
$$

$$
k_l(s)=\frac{k_{l-1}'(s)-k_{l-1}'(0)}{s}
$$

and

$$
k_l(0)=0,\quad k_l'(0)=\frac{(2l-1)!!}{(2l)!}.
$$

By using the solution formula in Proposition 2.2, we have

$$\begin{aligned}
u(t,x) &= \frac{2e^{-t/2}}{(n-1)!!|S^n|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\int_{|x-y|\le t}\frac{\cosh\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)}{\sqrt{t^2-|x-y|^2}}g(y)dy\\
&= \frac{e^{-t/2}}{(n-1)!!|S^n|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\int_{|x-y|\le t}\frac{1}{\frac{1}{2}\sqrt{t^2-|x-y|^2}}g(y)dy\\
&+ \frac{e^{-t/2}}{(n-1)!!|S^n|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\int_{|x-y|\le t}\frac{\cosh\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)-1}{\frac{1}{2}\sqrt{t^2-|x-y|^2}}g(y)dy.\\
&\equiv V_1+K_1.
\end{aligned}$$

Here we note that K_1 can be written as

$$K_1 = \frac{e^{-t/2}}{(n-1)!!|S^n|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2}\int_{|x-y|\le t}k_1\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)g(y)dy.$$

Using $k_1(0)=0$, we deduce that

$$\begin{aligned}
K_1 &= \frac{e^{-t/2}}{(n-1)!!|S^n|}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-4)/2}\int_{|x-y|\le t}\frac{k_1'\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)}{2\sqrt{t^2-|x-y|^2}}g(y)dy\\
&= \frac{e^{-t/2}}{(n-1)!!|S^n|}\frac{k_1'(0)}{4}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-4)/2}\int_{|x-y|\le t}\frac{1}{\frac{1}{2}\sqrt{t^2-|x-y|^2}}g(y)dy\\
&+ \frac{e^{-t/2}}{(n-1)!!|S^n|}\frac{1}{4}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-4)/2}\int_{|x-y|\le t}k_2\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)g(y)dy\\
&\equiv V_2+K_2.
\end{aligned}$$

In a similar way, we have

$$\begin{aligned}
K_2 &= \frac{e^{-t/2}}{(n-1)!!|S^n|}\frac{k_2'(0)}{4^2}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-6)/2}\int_{|x-y|\le t}\frac{1}{\frac{1}{2}\sqrt{t^2-|x-y|^2}}g(y)dy\\
&+ \frac{e^{-t/2}}{(n-1)!!|S^n|}\frac{1}{4^2}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-6)/2}\int_{|x-y|\le t}k_3\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)g(y)dy\\
&\equiv V_3+K_3
\end{aligned}$$

Continuing this argument, we obtain

$$V_l = \frac{e^{-t/2}}{(n-1)!!|S^n|}\frac{k_{l-1}'(0)}{4^{l-1}}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{n/2-l}\int_{|x-y|\le t}\frac{1}{\frac{1}{2}\sqrt{t^2-|x-y|^2}}g(y)dy.$$

Noting

$$\frac{k_l'(0)}{4^l} = \frac{(2l-1)!!}{4^l(2l)!} = \frac{1}{8^l l!},$$

we can reach the conclusion. The asymptotic expansion (8.1.8) immediately follows from that

$$k_1(s) = \frac{e^s}{2s}+\frac{e^{-s}}{2s}-\frac{1}{s}$$

and the definition of $k_l(s)$.

8.3. Proof of Theorem 8.2

We note that the solution of (8.1.1) is expressed as

$$u(t,x) = S_n(t)(u_0+u_1) + \partial_t(S_n(t)u_0). \tag{8.3.1}$$

By Lemma 8.1, we can decompose

$$S_n(t)g = J_n(t)g + e^{-t/2}\mathbf{W}_n(t)g.$$

Therefore, it follows that

$$\partial_t(S_n(t)g) = \partial_t(J_n(t)g) - \frac{1}{2}e^{-t/2}\mathbf{W}_n(t)g + e^{-t/2}\partial_t(\mathbf{W}_n(t)g). \tag{8.3.2}$$

When $n \geq 3$ and is an odd number, noting that $k_l(0) = (2^l l!)^{-1}$, we have

$$\begin{aligned}\partial_t(J_n(t)g) = &\frac{e^{-t/2}}{(n-2)!!|S^{n-1}|}\frac{1}{8^{\frac{n-1}{2}}(\frac{n-1}{2})!}\int_{|x-y|=t} g(y)dS_y \\ &+ \frac{1}{(n-2)!!|S^{n-1}|}\frac{1}{4^{\frac{n-1}{2}}} \\ &\times \int_{|x-y|\leq t} \partial_t\left(e^{-t/2}k_{\frac{n-1}{2}}\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)\right)g(y)dy \\ \equiv\ & e^{-t/2}\widehat{\mathbf{W}}_n(t)g + \tilde{J}_n(t)g.\end{aligned}$$

When $n \geq 2$ and an even number, noting $k_l(0) = 0$, we obtain

$$\partial_t(J_n(t)g) = \frac{1}{(n-1)!!|S^n|}\frac{1}{4^{\frac{n-2}{2}}}\int_{|x-y|\leq t}\partial_t\left(e^{-t/2}k_{\frac{n}{2}}\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)\right)g(y)dy.$$

By using $k_{l+1}(s) = (k_l'(s) - k_l'(0))/s$, we calculate

$$\begin{aligned}&\partial_t\left(e^{-t/2}k_{\frac{n}{2}}\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)\right) \\ &= -\frac{1}{2}e^{-t/2}k_{\frac{n}{2}}\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right) \\ &\quad + e^{-t/2}k'_{\frac{n}{2}}\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)\frac{t}{2\sqrt{t^2-|x-y|^2}} \\ &= -\frac{1}{2}e^{-t/2}k_{\frac{n}{2}}\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right) \\ &\quad + \frac{1}{4}e^{-t/2}tk_{\frac{n}{2}+1}\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right) \\ &\quad + \frac{1}{4}e^{-t/2}k'_{\frac{n}{2}}(0)\frac{t}{\frac{1}{2}\sqrt{t^2-|x-y|^2}}.\end{aligned}$$

Thus, noting $k_l'(0) = (2^l l!)^{-1}$, one can see that

$$\begin{aligned}
\partial_t(J_n(t)g) &= \frac{e^{-t/2}t}{(n-1)!!|S^n|}\frac{1}{8^{\frac{n}{2}}(\frac{n}{2})!}\int_{|x-y|\le t}\frac{1}{2\sqrt{t^2-|x-y|^2}}g(y)dy \\
&\quad + \frac{1}{(n-1)!!|S^n|}\frac{e^{-t/2}}{2}\int_{|x-y|\le t}\left(\frac{t}{2}k_{\frac{n}{2}+1}\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)\right. \\
&\quad \left. -k_{\frac{n}{2}}\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)\right)g(y)dy \\
&\equiv e^{-t/2}\widehat{\mathbf{W}}_n(t)g + \tilde{J}_n(t)g.
\end{aligned}$$

Substituting this into (8.3.2), we have

$$\partial_t(S_n(t)g) = \tilde{J}_n(t)g + e^{-t/2}\widehat{\mathbf{W}}_n(t)g - \frac{1}{2}\mathbf{W}_n(t)g + e^{-t/2}\partial_t(\mathbf{W}_n(t)g).$$

Combining this with (8.3.1), we can deduce that

$$\begin{aligned}
u(t,x) &= J_n(t)(u_0+u_1) + \tilde{J}_n(t)u_0 \\
&\quad + e^{-t/2}\left\{\frac{1}{2}\mathbf{W}_n(t)u_0 + \mathbf{W}_n(t)u_1 + \partial_t(\mathbf{W}_n(t)u_0) + \widehat{\mathbf{W}}_n(t)u_0\right\}.
\end{aligned}$$

Consequently, we have

$$u(t,x) - v(t,x) - e^{-t/2}\widetilde{\mathbf{W}}_n(t;u_0,u_1) = J_n(t)(u_0+u_1) - v(t,x) + \tilde{J}_n(t)u_0.$$

Therefore, it suffices to prove that

$$\|J_n(t)(u_0+u_1) - v(t)\|_{L^p} \le Ct^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}(\|u_0\|_{L^q} + \|u_1\|_{L^q}), \tag{8.3.3}$$

$$\|\tilde{J}_n(t)u_0\|_{L^p} \le Ct^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}\|u_0\|_{L^q}. \tag{8.3.4}$$

More generally, we shall prove the following estimates:

LEMMA 8.3. *For $1 \le q \le p \le \infty$, there exists a constant $C > 0$ such that*

$$\|J_n(t)g\|_{L^p} \le C(1+t)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|g\|_{L^q}, \tag{8.3.5}$$

$$\|\tilde{J}_n(t)g\|_{L^p} \le C(1+t)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}\|g\|_{L^q} \tag{8.3.6}$$

for $t \ge 0$ and

$$\|J_n(t)g - e^{t\Delta}g\|_{L^p} \le C(1+t)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}\|g\|_{L^q} \tag{8.3.7}$$

for $t > 0$.

PROOF. Let us prove (8.3.5) and (8.3.7) in the case that $n \ge 3$ and is an odd number. We assume that $t \ge 1$. Let $c_n = \frac{1}{4^{(n-1)/2}(n-2)!!|S^{n-1}|}$. We put $g = u_0 + u_1$ and divide

$$J_n(t)g - v(t,x) = X_1 + X_2 + X_3,$$

where

$$X_1 = \int_{\rho \le t^{(1+\varepsilon)/2}} \left(c_n e^{-t/2} k_{(n-1)/2}\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right) - \frac{e^{-\rho^2/(4t)}}{(4\pi t)^{n/2}} \right) g(y)dy,$$

$$X_2 = \int_{t^{(1+\varepsilon)/2} \le \rho \le t} \left(c_n e^{-t/2} k_{(n-1)/2}\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right) - \frac{e^{-\rho^2/(4t)}}{(4\pi t)^{n/2}} \right) g(y)dy,$$

$$X_3 = \int_{t\le\rho} \frac{e^{-\rho^2/(4t)}}{(4\pi t)^{n/2}} g(y)dy,$$

with $\rho = |x-y|$, $\varepsilon \in (0,1/2)$. By Lemma 9.9, we can estimate

$$\begin{aligned} \|X_3\|_{L^p} &\le C\left(\int_{t\le|y|} \frac{e^{-r|y|^2/(4t)}}{(4\pi t)^{rn/2}} dy\right)^{1/r} \|g\|_{L^q} \\ &\le Ce^{-t/8}\|g\|_{L^q}. \end{aligned}$$

For the estimate of X_2, We further divide the integral region into $\{y \in \mathbf{R}^n \mid \sqrt{t^2-1} \le |y| \le t\}$ and $\{y \in \mathbf{R}^n \mid t^{(1+\varepsilon)/2} \le |y| \le \sqrt{t^2-1}\}$. When $\sqrt{t^2-1} \le |y| \le t$, noting that $k_l(s)$ is bounded for $s \le 1/2$, we have

$$e^{-t/2} k_{(n-1)/2}\left(\frac{1}{2}\sqrt{t^2-|y|^2}\right) \le Ce^{-t/2}.$$

When $t^{(1+\varepsilon)/2} \le |y| \le \sqrt{t^2-1}$, using the asymptotic expansion (8.1.6) and $\sqrt{t^2-|y|^2} \ge 1$, we obtain

$$\begin{aligned} &e^{-t/2} k_{(n-1)/2}\left(\frac{1}{2}\sqrt{t^2-|y|^2}\right) \\ &\le Ce^{-t/2}(t^2-|y|^2)^{-n/2} e^{\frac{1}{2}\sqrt{t^2-|y|^2}} \left(1 + C\frac{1}{\sqrt{t^2-|y|^2}}\right) \\ &\le Ce^{-\frac{1}{2}(t-\sqrt{t^2-t^{1+\varepsilon}})} \\ &\le Ce^{-t^{1+\varepsilon}/(2(t+\sqrt{t^2-t^{1+\varepsilon}}))} \\ &\le Cte^{-t^\varepsilon/4}. \end{aligned}$$

Moreover, it is easy to see that

$$\frac{e^{-|y|^2/(4t)}}{(4\pi t)^{n/2}} \le Ce^{-t^\varepsilon/4}$$

for $t^{(1+\varepsilon)/2} \le |y| \le t$. Combining the above estimate, we can see that

$$\begin{aligned} \|X_2\|_{L^p} \le C &\left(\int_{t^{(1+\varepsilon)/2}\le|y|\le t} c_n^r e^{-rt/2} k_{(n-1)/2}\left(\frac{1}{2}\sqrt{t^2-|y|^2}\right)^r dy\right)^{1/r} \|g\|_{L^q} \\ &+ \left(\int_{t^{(1+\varepsilon)/2}\le|y|\le t} \frac{e^{-r|y|^2/(4t)}}{(4\pi t)^{rn/2}} dy\right)^{1/r} \|g\|_{L^q} \\ \le\ & Ce^{-ct^\varepsilon}\|g\|_{L^q} \end{aligned}$$

with some constant $c > 0$. Finally, we prove the estimate of X_1. By the asymptotic expansion (8.1.6), one has

$$
\begin{aligned}
&c_n e^{-t/2} k_{(n-1)/2}\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)\\
&= \frac{1}{(4\pi)^{n/2}}\frac{1}{(\sqrt{t^2-\rho^2})^{n/2}} e^{-t/2+\frac{1}{2}\sqrt{t^2-\rho^2}}\left(1-\frac{n(n-2)}{4\sqrt{t^2-\rho^2}}+O\left(\frac{1}{t^2-\rho^2}\right)\right).
\end{aligned}
$$

Therefore, it follows that

$$
X_1 = \frac{1}{(4\pi t)^{n/2}}\int_{\rho\le t^{(1+\varepsilon)/2}} e^{-\rho^2/(4t)} D(t,\rho) g(y)dy
$$

and hence,

$$
\|X_1\|_{L^p} \le \frac{C}{t^{n/2}}\left(\int_{|y|\le t^{(1+\varepsilon)/2}} e^{-r\rho^2/(4t)} D(t,|y|)^r dy\right)^{1/r}\|g\|_{L^q},
$$

where

$$
\begin{aligned}
D(t,\rho) = {} & e^{\rho^2/(4t)-t/2+\sqrt{t^2-\rho^2}/2}\left(\frac{t}{\sqrt{t^2-\rho^2}}\right)^{n/2}\\
&\times\left((1-\frac{n(n-2)}{4\sqrt{t^2-\rho^2}}+O\left(\frac{1}{t^2-\rho^2}\right)\right)-1
\end{aligned}
$$

Here we note that

$$
\frac{t}{\sqrt{t^2-|y|^2}} = \frac{1}{\sqrt{1-(|y|/t)^2}} = 1+\frac{1}{t}O\left(\frac{|y|^2}{t}\right), \tag{8.3.8}
$$

$$
\begin{aligned}
\frac{|y|^2}{4t}-\frac{t}{2}+\frac{1}{2}\sqrt{t^2-|y|^2} &= \frac{|y|^2}{4t}-\frac{|y|^2}{2(t+\sqrt{t^2-|y|^2})}\\
&= -\frac{|y|^2(t-\sqrt{t^2-|y|^2})}{4t(t+\sqrt{t^2-|y|^2})}\\
&= -\frac{|y|^4}{4t(t+\sqrt{t^2-|y|^2})^2}
\end{aligned}
$$

and

$$
e^{|y|^2/(4t)-t/2+\sqrt{t^2-|y|^2}/2} = 1+\frac{1}{t}O(\frac{|y|^4}{t^2}). \tag{8.3.9}
$$

Consequently, we obtain

$$
\begin{aligned}
D(t,|y|) &= \left(1+\frac{1}{t}O\left(\frac{|y|^4}{t^2}\right)\right)\left(1+\frac{1}{t}O\left(\frac{|y|^2}{t}\right)\right)^{n/2}\\
&\quad\times\left(1-\frac{n(n-2)}{4t}\left(1+\frac{1}{t}O\left(\frac{|y|^2}{t}\right)\right)+\frac{1}{t^2}\left(1+O\left(\frac{|y|^2}{t^2}\right)\right)\right)-1\\
&= \frac{1}{t}O\left(1+\frac{|y|^2}{t}+\cdots+\left(\frac{|y|^2}{t}\right)^N\right)
\end{aligned}
$$

for some large integer N. Finally, we have

$$\begin{aligned}\|X_1\|_{L^p} &\le \frac{C}{t^{n/2+1}}\left(\int_{|y|\le t^{(1+\varepsilon)/2}} e^{-r|y|^2/(4t)}\left(1+\frac{|y|^2}{t}+\cdots+\left(\frac{|y|^2}{t}\right)^N\right)^r dy\right)^{1/r} \\ &\quad\times \|g\|_{L^q} \\ &\le \frac{C}{t^{n/2+1}} t^{n/(2r)}\left(\int_{\mathbf{R}^n} e^{-r|z|^2}(1+|z|^2+\cdots+|z|^{2N})^r dy\right)^{1/r}\|g\|_{L^q} \\ &\le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-1}\|g\|_{L^q},\end{aligned}$$

which proves the estimate (8.3.7) for $t \ge 1$. Moreover, we recall the well-known fact

$$\|e^{t\Delta}g\|_{L^p} \le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|g\|_{L^q} \quad (t>0)$$

(see [**14**]). Using this we see that

$$\begin{aligned}\|J_n(t)g\|_{L^p} &\le \|J_n(t)g-e^{t\Delta}g\|_{L^p}+\|e^{t\Delta}g\|_{L^p} \\ &\le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|g\|_{L^q}.\end{aligned}$$

This shows (8.3.5) for $t \ge 1$. Next, we treat the case $0 \le t < 1$. We note that $k_{\frac{n-1}{2}}(s)$ is a real analytic function of s and hence, $k_{\frac{n-1}{2}}(s)$ is bounded for $0 \le s < 1/2$. Thus, using Lemma 9.9, we can deduce that

$$\begin{aligned}\|J_n(t)g\|_{L^p} &\le C\left(\int_{|y|\le t} k_{\frac{n-1}{2}}\left(\frac{1}{2}\sqrt{t^2-|y|^2}\right)^r dy\right)^{1/r}\|g\|_{L^q}, \\ &\le Ct^{n/r}\|g\|_{L^q}\end{aligned}$$

which yields (8.3.5) for $0 \le t < 1$. Furthermore, we have

$$\begin{aligned}\|J_n(t)g-e^{t\Delta}g\|_{L^p} &\le \|J_n(t)g\|_{L^p}+\|e^{t\Delta}g\|_{L^p} \\ &\le Ct^{n/r}\|g\|_{L^p}+Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|g\|_{L^q} \\ &\le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-1}\|g\|_{L^q}.\end{aligned}$$

This proves (8.3.7) for $0 < t < 1$.

Next, we show (8.3.6). As before, we first assume that $t \ge 1$ and put $c_n = \frac{1}{4^{(n-1)/2}(n-2)!!|S^{n-1}|}$, $\rho = |x-y|$ and $\varepsilon \in (0,1/2)$. We divide $\tilde{J}_n(t)g$ as

$$\begin{aligned}\tilde{J}_n(t) = c_n &\int_{\rho\le t^{(1+\varepsilon)/2}} \partial_t\left(e^{-t/2}k_{\frac{n-1}{2}}\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)\right)g(y)dy \\ &+ c_n\int_{t^{(1+\varepsilon)/2}\le\rho\le t} \partial_t\left(e^{-t/2}k_{\frac{n-1}{2}}\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)\right)g(y)dy.\end{aligned}$$

We calculate

$$\begin{aligned}&\partial_t\left(e^{-t/2}k_{\frac{n-1}{2}}\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)\right) \\ &\quad = e^{-t/2}\left\{-\frac{1}{2}k_{\frac{n-1}{2}}\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)+\frac{t}{2\sqrt{t^2-\rho^2}}k'_{\frac{n-1}{2}}\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)\right\} \\ &\quad = e^{-t/2}\left\{-\frac{1}{2}k_{\frac{n-1}{2}}\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)+\frac{t}{4}k_{\frac{n+1}{2}}\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)\right\}.\end{aligned}$$

From this and the Hausdorff-Young inequality (Lemma 9.9), it follows that

$$\|X_4\|_{L^p} \le C \Bigg(\int_{t^{(1+\varepsilon)/2} \le |y| \le t} e^{-rt/2} \times \left| -\frac{1}{2} k_{\frac{n-1}{2}} \left(\frac{1}{2}\sqrt{t^2-|y|^2} \right) + \frac{t}{4} k_{\frac{n+1}{2}} \left(\frac{1}{2}\sqrt{t^2-|y|^2} \right) \right|^r dy \Bigg)^{1/r} \times \|g\|_{L^q},$$

where $1 - 1/r = 1/q - 1/p$. We further divide the integral region into $\{y \in \mathbf{R}^n \mid \sqrt{t^2-1} \le |y| \le t\}$ and $\{y \in \mathbf{R}^n \mid t^{(1+\varepsilon)/2} \le |y| \le \sqrt{t^2-1}\}$. When $\sqrt{t^2-1} \le |y| \le t$, noting that $k_l(s)$ is bounded for $s \le 1/2$, we can see that

$$e^{-t/2} \left| -\frac{1}{2} k_{\frac{n-1}{2}} \left(\frac{1}{2}\sqrt{t^2-|y|^2} \right) + \frac{t}{4} k_{\frac{n+1}{2}} \left(\frac{1}{2}\sqrt{t^2-|y|^2} \right) \right| \le Cte^{-t/2}.$$

On the other hand, when $t^{(1+\varepsilon)/2} \le |y| \le \sqrt{t^2-1}$, using the asymptotic expansion (8.1.6) again and $\sqrt{t^2-|y|^2} \ge 1$, one can deduce that

$$\begin{aligned} & e^{-t/2} \left| -\frac{1}{2} k_{\frac{n-1}{2}} \left(\frac{1}{2}\sqrt{t^2-|y|^2} \right) + \frac{t}{4} k_{\frac{n+1}{2}} \left(\frac{1}{2}\sqrt{t^2-|y|^2} \right) \right| \\ & \le Cte^{-\frac{1}{2}(t-\sqrt{t^2-t^{1+\varepsilon}})} \\ & \le Cte^{-t^\varepsilon/4}. \end{aligned}$$

Therefore, we have

$$\|X_4\|_{L^p} \le Ce^{-ct^\varepsilon} \|g\|_{L^q}$$

with some constant $c > 0$. Now we turn to the estimate for X_5. By using the asymptotic expansion (8.1.6), (8.3.8) and (8.3.9) again, it can be seen that

$$\begin{aligned} & \partial_t \left(e^{-t/2} k_{\frac{n-1}{2}} \left(\frac{1}{2}\sqrt{t^2-\rho^2} \right) \right) \\ & = \frac{1}{2\sqrt{\pi}} e^{-\frac{\rho^2}{4t}} e^{\frac{\rho^2}{4t} - \frac{1}{2}(t-\sqrt{t^2-\rho^2})} 2^{\frac{n-1}{2}} \left\{ -(t^2-\rho^2)^{-\frac{n}{4}} + t(t^2-\rho^2)^{-\frac{n+2}{4}} \right\} \\ & \quad \times (1 + O(t^{-1})) \\ & = \frac{2^{\frac{n-3}{2}}}{\sqrt{\pi}} e^{-\frac{\rho^2}{4t}} e^{\frac{\rho^2}{4t} - \frac{1}{2}(t-\sqrt{t^2-\rho^2})} (t^2-\rho^2)^{-\frac{n}{4}} \left(\frac{t}{\sqrt{t^2-\rho^2}} - 1 \right) (1 + O(t^{-1})) \\ & = \frac{2^{\frac{n-3}{2}}}{\sqrt{\pi}} e^{-\frac{\rho^2}{4t}} \left(1 + \frac{1}{t} O\left(\frac{\rho^2}{t} \right) \right) t^{-n/2} \left(1 + \frac{1}{t} O\left(\frac{\rho^2}{t} \right) \right)^{n/2} \\ & \quad \times \frac{1}{t} O\left(\frac{\rho^2}{t} \right) \left(1 + O\left(\frac{1}{t} \right) \right) \\ & \le Ct^{-n/2-1} e^{-\frac{\rho^2}{4t}} \left(1 + \frac{\rho^2}{t} + \cdots + \left(\frac{\rho^2}{t} \right)^N \right) \end{aligned}$$

for $\rho \le t^{(1+\varepsilon)/2}$ with some large integer $N \in \mathbf{N}$. Consequently, we obtain

$$\begin{aligned}\|X_5\|_{L^p} &\le Ct^{-\frac{n}{2}-1}\left(\int_{|y|\le t^{(1+\varepsilon/2}} e^{-\frac{r|y|^2}{4t}}\left(1+\frac{|y|^2}{t}+\cdots+\left(\frac{|y|^2}{t}\right)^N\right)^r dy\right)^{1/r}\|g\|_{L^q}\\ &\le Ct^{-\frac{n}{2}-1-\frac{n}{2r}}\left(\int_{\mathbf{R}^n} e^{-\frac{r|z|^2}{4}}(1+|z|^2+\cdots+|z|^{2N})^r dz\right)^{1/r}\|g\|_{L^q}\\ &\le Ct^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|g\|_{L^q},\end{aligned}$$

which shows (8.3.6) for $t \ge 1$. Finally, we assume that $0 \le t < 1$. We note that

$$\tilde{J}_n(t)g = c_n\int_{|x-y|\le t}\tilde{k}_{\frac{n-1}{2}}\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)g(y)dy,$$

where

$$\tilde{k}_{\frac{n-1}{2}}\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right) := e^{-t/2}\left\{-\frac{1}{2}k_{\frac{n-1}{2}}\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)+\frac{t}{4}k_{\frac{n+1}{2}}\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)\right\}.$$

Since both $k_{\frac{n-1}{2}}(s)$ and $k_{\frac{n+1}{2}}(s)$ are real analytic functions of s, it follows that they are bounded for $0 \le s < 1/2$. This implies that

$$\begin{aligned}\|\tilde{J}_n(t)g\|_{L^p} &\le C\left(\int_{|y|\le t}\tilde{k}_{\frac{n-1}{2}}\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)^r dy\right)^{1/r}\|g\|_{L^q}\\ &\le Ct^{n/r}\|g\|_{L^q},\end{aligned}$$

which proves (8.3.6) for $0 \le t < 1$.

In the even dimensional cases, noting the asymptotic expansion

$$k_l(s) = \frac{e^s}{2s^l}\left(1+\frac{a_l}{s}+O(s^{-2})\right)$$

and using this instead of (8.1.6), we can obtain the desired estimate in the almost same way. □

CHAPTER 9

Appendix

9.1. Notation

We explain some notation and terminology used throughout this thesis. First, $\mathbf{N}, \mathbf{Z}, \mathbf{R}, \mathbf{C}$ denote the sets of natural numbers, integers, real numbers, complex numbers, respectively. We also write $\mathbf{Z}_{\geq 0} := \mathbf{N} \cup \{0\}$.

We note that the letter C indicates the generic constant, which may change from line to line. For $a \in \mathbf{R}$, we put $[a] := \max\{b \in \mathbf{Z} \mid b \leq a\}$. We sometimes use the symbols $n/[n-2]_+$ and $(n+2)/[n-2]_+$ to indicate the Gagliardo-Nirenberg exponent and Sobolev exponent, respectively. Their meanings are

$$\frac{n}{[n-2]_+} = \begin{cases} \frac{n}{n-2} & (n \geq 3), \\ \infty & (n = 1, 2), \end{cases} \qquad \frac{n+2}{[n-2]_+} = \begin{cases} \frac{n+2}{n-2} & (n \geq 3), \\ \infty & (n = 1, 2). \end{cases}$$

Let $n!$ denote the factorial of the nonnegative integer n, that is, $n! = n \cdot (n-1) \cdots 2 \cdot 1$. According to a useful convention, we define $0! = 1$. We also denote by $n!!$ the double factorial of n, that is, $n!! = n \cdot (n-2) \cdot (n-4) \cdots 1$.

Let $n, k \in \mathbf{Z}_{\geq 0}$ satisfy $k \leq n$. The symbol $\binom{n}{k}$ denotes the binomial coefficients, that is,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

We sometimes use the symbol of the generalized binomial coefficients. Let $\alpha \in \mathbf{R}$ and $k \in \mathbf{Z}_{\geq 0}$. We define

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

We also use the symbols $\lesssim$ and $\sim$. The relation $f \lesssim g$ stands for $f \leq Cg$ with some constant $C > 0$ and $f \sim g$ means that $f \lesssim g$ and $g \lesssim f$.

For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbf{R}^n$, we define the product $x \cdot y = \sum_{j=1}^n x_j y_j$, the absolute value $|x| = \sqrt{x \cdot x}$ and the bracket $\langle \cdot \rangle$ by

$$\langle x \rangle = \sqrt{1 + |x|^2}.$$

We sometimes use the Landau notation. Let $f(t)$ and $g(t)$ be two functions defined on a subset of $\mathbf{R}$. We write

$$f(t) = O(g(t)) \quad \text{as } t \to +\infty$$

if there exist constants $M > 0$ and $t_0 > 0$ such that $|f(t)| \leq M|g(t)|$ holds for all $t \geq t_0$. We also write

$$f(t) = o(g(t)) \quad \text{as } t \to +\infty$$

if for any $\varepsilon > 0$, there exists a constant $t_0 > 0$ such that $|f(t)| \leq \varepsilon|g(t)|$ holds for all $t \geq t_0$.

We write the partial derivatives $\partial_t = \partial/\partial t$, $\partial_{x_j} = \partial/\partial x_j$, $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$ and $\Delta = \sum_{j=1}^n \partial_{x_j}^2$, $\Box = \partial_t^2 - \Delta$. For simplicity, for a function $u(t,x)$ defined on a subset of $\mathbf{R} \times \mathbf{R}^n$, we also use the notation $u_t = \partial_t u$ and $u_{x_j} = \partial_{x_j} u$ $(j = 1, \ldots, n)$. We also use the symbol of the multi-index. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_{\geq 0}^n$ and a function $f : \mathbf{R}^n \to \mathbf{R}$, we write

$$\partial_x^\alpha f(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f(x)$$

and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Let $\hat{f}(\xi)$ or $\mathcal{F}[f](\xi)$ denote the Fourier transform of a function $f(x)$, that is,

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x) dx,$$

and we write the inverse Fourier transform of a function $\phi(\xi)$ by

$$\mathcal{F}^{-1}[\phi](x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \phi(\xi) d\xi.$$

For two functions f, g defined on $\mathbf{R}^n$, the notation $f * g$ stands for the convolution of f and g, that is,

$$f * g(x) = \int_{\mathbf{R}^n} f(x-y) g(y) dy.$$

Next, we explain the function spaces used in this thesis. Let $K \subset \mathbf{R}^n$. We define $C(K)$ by the space of continuous functions $f : K \to \mathbf{R}$. $C_0(K)$ denotes the space of continuous functions with compact support, where the support of a function $f : K \to \mathbf{R}$ is given by

$$\operatorname{supp} f = \overline{\{x \in K \mid f(x) \neq 0\}} \cap K.$$

Let Ω be an open set in $\mathbf{R}^n$. We denote $C^r(\Omega)$ by the space of r-times continuously differentiable functions defined on Ω. As above, we define $C_0^r(\Omega)$ by the subspace of $C^r(\Omega)$ whose elements have compact support, and $C^\infty(\Omega) = \cap_{r=0}^\infty C^r(\Omega)$, $C_0^\infty(\Omega) = \cap_{r=0}^\infty C_0^r(\Omega)$.

Let $L^p(\Omega)$ denote the Lebesgue space equipped with the norm

$$\|f\|_p = \|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}$$

for $1 < p < \infty$ and

$$\|f\|_\infty = \|f\|_{L^\infty(\Omega)} = \text{ess.sup}\{|f(x)| \mid x \in \Omega\}.$$

We denote the usual Sobolev space by $H^m(\Omega)$ for $m \in \mathbf{N}$, that is,

$$H^m(\Omega) = \{f \in L^2(\Omega) \mid \partial_x^\alpha f \in L^2(\Omega) \ (|\alpha| \leq m)\},$$

where the derivatives are in the sense of distribution. $H^m(\Omega)$ is Hilbert space with the inner product

$$(f, g)_{H^m} = \sum_{|\alpha| \leq m} (\partial_x^\alpha f, \partial_x^\alpha g)_{L^2}.$$

We define $H_0^1(\Omega)$ by the completion of $C_0^\infty(\Omega)$ with respect to the norm defined from the above inner product with $m = 1$. To express Sobolev spaces, we also use the notation $W^{m,p}(\Omega)$ $(1 \leq p \leq \infty)$:

$$W^{m,p}(\Omega) := \{f \in L^p(\Omega) \mid \partial_x^\alpha f \in L^p(\Omega) \ (|\alpha| \leq m)\}.$$

It is well known that $W^{m,p}(\Omega)$ is Banach space with the norm

$$\|f\|_{W^{m,p}} := \sum_{|\alpha|\leq m} \|\partial_x^\alpha f\|_{L^p}.$$

We sometimes use the symbol of product space norm. For two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, we write $\|(u,v)\|_{X\times Y} := \|u\|_X + \|v\|_Y$.

For an interval I and a Banach space X, we define $C^r(I;X)$ as the space whose element is an r-times continuously differentiable mapping from I to X with respect to the topology in X (if I is a semi-open or closed interval, the differential at the endpoint is interpreted as one-sided derivative).

9.2. Special functions

We give the definition and some properties of the special functions used in this thesis.

9.2.1. definition.

(i) Gamma function:

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$$

for $s > 0$.

(ii) Beta function:

$$B(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1}dt$$

for $r, s > 0$.

(iii) Modified Bessel function:

$$I_\nu(s) = \sum_{m=0}^\infty \frac{1}{m!\Gamma(m+\nu+1)}\left(\frac{s}{2}\right)^{2m+\nu}$$

for $\nu \in \mathbf{Z}_{\geq 0}$ and $s \geq 0$.

9.2.2. Properties.

Lemma 9.1. *For $r, s > 0$, we have*

(i) $\Gamma(s+1) = s\Gamma(s)$,
(ii) $\Gamma(1/2) = \sqrt{\pi}$,
(iii) $\Gamma(r)\Gamma(s) = \Gamma(r+s)B(r,s)$.

Proof. By the integration by parts, it follows that

$$\Gamma(s+1) = \int_0^\infty t^s e^{-t}dt = \left[-t^s e^{-t}\right]_{t=0}^{t=\infty} + s\int_0^\infty t^{s-1}e^{-t}dt = s\Gamma(s),$$

which proves (i). Next, we have

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right)^2 &= \left(\int_0^\infty t^{-1/2}e^{-t}dt\right)^2\\ &= \left(2\int_0^\infty e^{-u^2}du\right)^2\\ &= 4\left(\int_0^\infty e^{-u^2}du\right)\left(\int_0^\infty e^{-v^2}dv\right)\\ &= 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}dudv\\ &= 4\int_0^{\pi/2}\int_0^\infty e^{-\rho^2}\rho d\rho d\theta\\ &= -\pi\left[e^{-\rho^2}\right]_{\rho=0}^{\rho=\infty} = \pi.\end{aligned}$$

This shows (ii). Finally, we prove (iii). We see that

$$\begin{aligned}\Gamma(r)\Gamma(s) &= \left(\int_0^\infty t^{r-1}e^{-t}dt\right)\left(\int_0^\infty t^{s-1}e^{-t}dt\right)\\ &= 4\left(\int_0^\infty u^{2r-1}e^{-u^2}du\right)\left(\int_0^\infty v^{2s-1}e^{-v^2}dv\right)\\ &= 4\int_0^\infty\int_0^\infty u^{2r-1}v^{2s-1}e^{-(u^2+v^2)}dudv\\ &= \left(2\int_0^\infty \rho^{2r+2s-1}e^{-\rho^2}d\rho\right)\left(2\int_0^{\pi/2}(\cos\theta)^{2r-1}(\sin\theta)^{2s-1}d\theta\right)\\ &= \left(\int_0^\infty t^{r+s-1}e^{-t}dt\right)\left(\int_0^1 t^{r-1}(1-t)^{s-1}dt\right)\\ &= \Gamma(r+s)B(r,s).\end{aligned}$$

□

LEMMA 9.2. *Let $a > 0, c \geq 0$. Then, we have*

$$\int_{-a}^{a}\frac{e^{ct}}{\sqrt{a^2-t^2}}dt = \pi I_0(ca).$$

PROOF. First, we see that

$$\begin{aligned}\int_{-a}^{a}\frac{e^{ct}}{\sqrt{a^2-t^2}}dt &= \int_{-1}^{1}\frac{e^{cat}}{\sqrt{1-t^2}}dt\\ &= \int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\sum_{m=0}^{\infty}\frac{(ca)^m}{m!}t^m dt.\end{aligned}$$

By the dominated convergence theorem, the integral commutes with the sum. We also note that if m is odd, then the integrand becomes an odd function and so the value of its integral is 0. Therefore, it follows that

$$
\begin{aligned}
\int_{-a}^{a} \frac{e^{ct}}{\sqrt{a^2-t^2}}dt &= 2\sum_{m=0}^{\infty} \frac{(ca)^{2m}}{(2m)!}\int_0^1 \frac{t^{2m}}{\sqrt{1-t^2}}dt \\
&= \sum_{m=0}^{\infty} \frac{(ca)^{2m}}{(2m)!}\int_0^1 s^{m-1/2}(1-s)^{-1/2}ds \quad (\because t^2=s, 2tdt=ds) \\
&= \sum_{m=0}^{\infty} \frac{(ca)^{2m}}{(2m)!}B(m+1/2,1/2) \\
&= \sum_{m=0}^{\infty} \frac{(ca)^{2m}}{(2m)!}\frac{\Gamma(m+1/2)\Gamma(1/2)}{\Gamma(m+1)} \\
&= \sum_{m=0}^{\infty} \frac{(ca)^{2m}}{(2m)!}\frac{((m-1/2)(m-3/2)\cdots\sqrt{\pi})\sqrt{\pi}}{m!} \\
&= \sum_{m=0}^{\infty} \frac{(ca)^{2m}}{(2m)!}\frac{(2m-1)(2m-3)\cdots 3\cdot 1\cdot \pi}{2^m m!} \\
&= \pi\sum_{m=0}^{\infty} \frac{1}{(m!)^2}\left(\frac{ca}{2}\right)^{2m} = \pi I_0(ca),
\end{aligned}
$$

which shows the conclusion. □

Lemma 9.3 (derivatives).

$$
\begin{aligned}
&I_0'(s) = I_1(s), \quad I_1'(s) = I_0(s) - \frac{1}{s}I_1(s), \\
&s^{-1}\frac{d}{ds}\left(s^{-l}I_l(s)\right) = s^{-(l+1)}I_{l+1}(s).
\end{aligned}
$$

Proof. By the definition of $I_0(s)$, we have

$$
\begin{aligned}
I_0'(s) &= \sum_{m=1}^{\infty} \frac{2m}{(m!)^2}\frac{1}{2}\left(\frac{s}{2}\right)^{2m-1} \\
&= \sum_{m=1}^{\infty} \frac{1}{m!(m-1)!}\left(\frac{s}{2}\right)^{2m-1} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!(m+1)!}\left(\frac{s}{2}\right)^{2m+1} \\
&= I_1(s).
\end{aligned}
$$

From the definition of $I_1(s)$, we can also calculate

$$\begin{aligned}
I_1'(s) &= \sum_{m=0}^{\infty} \frac{2m+1}{m!(m+1)!}\frac{1}{2}\left(\frac{s}{2}\right)^{2m} \\
&= \sum_{m=0}^{\infty} \frac{(2m+2)-1}{m!(m+1)!}\frac{1}{2}\left(\frac{s}{2}\right)^{2m} \\
&= \sum_{m=0}^{\infty} \frac{1}{(m!)^2}\left(\frac{s}{2}\right)^{2m} - \frac{1}{s}\sum_{m=0}^{\infty} \frac{1}{m!(m+1)!}\left(\frac{s}{2}\right)^{2m+1} \\
&= I_0(s) - \frac{1}{s}I_1(s).
\end{aligned}$$

Moreover, it follows that

$$\begin{aligned}
s^{-1}\frac{d}{ds}\left(s^{-l}I_l(s)\right) &= (-l)s^{-(l+2)}I_l(s) + s^{-(l+1)}I_l'(s) \\
&= s^{-(l+1)}\left(-ls^{-1}I_l(s) + I_l'(s)\right) \\
&= s^{-(l+1)}\left(-\frac{l}{s}\sum_{m=0}^{\infty}\frac{1}{m!(m+l)!}\left(\frac{s}{2}\right)^{2m+l}\right. \\
&\qquad \left. + \sum_{m=0}^{\infty}\frac{2m+l}{m!(m+l)!}\frac{1}{2}\left(\frac{s}{2}\right)^{2m+l-1}\right) \\
&= s^{-(l+1)}\sum_{m=1}^{\infty}\frac{m}{m!(m+l)!}\left(\frac{s}{2}\right)^{2m+l-1} \\
&= s^{-(l+1)}\sum_{m=0}^{\infty}\frac{1}{m!(m+l+1)!}\left(\frac{s}{2}\right)^{2m+l+1} \\
&= s^{-(l+1)}I_{l+1}(s).
\end{aligned}$$

□

LEMMA 9.4 (integral representations). *For $\nu \in \mathbf{Z}_{\geq 0}, s \geq 0$, we have*

$$I_\nu(s) = \frac{1}{\sqrt{\pi}\Gamma(\nu+1/2)}\left(\frac{s}{2}\right)^{\nu}\int_0^{\pi} e^{s\cos\theta}(\sin\theta)^{2\nu}d\theta. \tag{9.2.1}$$

PROOF. Noting that

$$\int_0^{\pi}(\cos\theta)^j(\sin\theta)^{2\nu}d\theta = 0$$

if j is an odd number, we obtain

$$\begin{aligned}
\int_0^{\pi} e^{s\cos\theta}(\sin\theta)^{2\nu}d\theta &= \int_0^{\pi}\sum_{j=0}^{\infty}\frac{1}{j!}s^j(\cos\theta)^j(\sin\theta)^{2\nu}d\theta \\
&= \int_0^{\pi}\sum_{j=0}^{\infty}\frac{1}{(2j)!}s^{2j}(\cos\theta)^{2j}(\sin\theta)^{2\nu}d\theta \\
&= 2\sum_{j=0}^{\infty}\frac{s^{2j}}{(2j)!}\int_0^{\pi/2}(\cos\theta)^{2j}(\sin\theta)^{2\nu}d\theta.
\end{aligned}$$

We claim that

$$\int_0^{\pi/2} (\cos\theta)^{2j}(\sin\theta)^{2\nu}d\theta = \frac{1}{2}\frac{\Gamma(j+1/2)\Gamma(\nu+1/2)}{\Gamma(j+\nu+1)} \tag{9.2.2}$$

for $j,\nu \in \mathbf{Z}_{\geq 0}$. In fact, we calculate

$$\begin{aligned}\int_0^{\pi/2} (\cos\theta)^{2j}(\sin\theta)^{2\nu}d\theta &= \frac{1}{2}\int_0^1 t^{j-1/2}(1-t)^{\nu-1/2}dt \\ &= \frac{1}{2}B(j+1/2,\nu+1/2) \\ &= \frac{1}{2}\frac{\Gamma(j+1/2)\Gamma(\nu+1/2)}{\Gamma(j+\nu+1)},\end{aligned}$$

which proves (9.2.2). Here we have used the changing variable $t=\cos^2\theta$.

Using (9.2.2), we can see that

$$\int_0^{\pi} e^{s\cos\theta}(\sin\theta)^{2\nu}d\theta = \sum_{j=0}^{\infty}\frac{s^{2j}}{(2j)!}\frac{\Gamma(j+1/2)\Gamma(\nu+1/2)}{\Gamma(j+\nu+1)}.$$

By noting that

$$(2j)! = (2j)!!(2j-1)!! = 2^j j! 2^j \frac{1}{\sqrt{\pi}}\Gamma(j+1/2),$$

it follows that

$$\begin{aligned}&\frac{1}{\sqrt{\pi}\Gamma(\nu+1/2)}\left(\frac{s}{2}\right)^{\nu}\int_0^{\pi} e^{s\cos\theta}(\sin\theta)^{2\nu}d\theta \\ &= \frac{1}{\sqrt{\pi}\Gamma(\nu+1/2)}\left(\frac{s}{2}\right)^{\nu}\sum_{j=0}^{\infty}\frac{s^{2j}}{2^j j! 2^j \frac{1}{\sqrt{\pi}}\Gamma(j+1/2)}\frac{\Gamma(j+1/2)\Gamma(\nu+1/2)}{\Gamma(j+\nu+1)} \\ &= \sum_{j=0}^{\infty}\frac{1}{j!\Gamma(j+\nu+1)}\left(\frac{s}{2}\right)^{2j+\nu} = I_\nu(s).\end{aligned}$$

This shows (9.2.1). □

LEMMA 9.5 (asymptotic expansion).

$$\begin{aligned}I_\nu(s) = \frac{1}{\sqrt{2\pi s}}e^s\Big(1 &- \frac{(\nu-1/2)(\nu+1/2)}{2s} + \frac{(\nu-1/2)(\nu-3/2)(\nu+3/2)(\nu+1/2)}{2!2^2s^2} \\ &- \cdots + (-1)^k\frac{1}{k!2^ks^k}\prod_{j=1}^{k}(\nu-(j-1/2))(\nu+(j-1/2)) \\ &+ O(s^{-k-1})\Big) \quad (s\to+\infty)\end{aligned}$$

for $k\in\mathbf{Z}_{\geq 0}$.

PROOF. By Lemma 9.4, we can write

$$I_\nu(s) = \frac{e^s}{\sqrt{\pi}\Gamma(\nu+1/2)}\left(\frac{s}{2}\right)^{\nu}\int_0^{\pi} e^{s(\cos\theta-1)}(\sin\theta)^{2\nu}d\theta.$$

We change the variable as $s(1-\cos\theta) = t$. Noting that $\cos\theta = 1 - t/s$ and $\sin\theta = \sqrt{1-(1-t/s)^2}$, we obtain

$$\begin{aligned}\int_0^\pi e^{s(\cos\theta-1)}(\sin\theta)^{2\nu}d\theta &= \frac{1}{s}\int_0^{2s} e^{-t}\left(\sqrt{1-\left(1-\frac{t}{s}\right)^2}\right)^{2\nu-1} dt \\ &= \frac{1}{s}\int_0^{2s} e^{-t}\left(\sqrt{\frac{t}{s}}\sqrt{2-\frac{t}{s}}\right)^{2\nu-1} dt \\ &= \frac{1}{s^{\nu+1/2}}\int_0^{2s} e^{-t}t^{\nu-1/2}\left(2-\frac{t}{s}\right)^{\nu-1/2} dt.\end{aligned}$$

Hence, we see that

$$\begin{aligned}I_\nu(s) &= \frac{e^s}{\sqrt{\pi}\Gamma(\nu+1/2)}\left(\frac{s}{2}\right)^\nu \frac{1}{s^{\nu+1/2}}\int_0^{2s} e^{-t}t^{\nu-1/2}\left(2-\frac{t}{s}\right)^{\nu-1/2} dt \\ &= \frac{e^s}{\sqrt{2\pi s}}\frac{1}{\Gamma(\nu+1/2)}\int_0^{2s} e^{-t}t^{\nu-1/2}\left(1-\frac{t}{2s}\right)^{\nu-1/2} dt\end{aligned}$$

By the Taylor theorem, one has

$$\begin{aligned}\left(1-\frac{t}{2s}\right)^{\nu-1/2} &= \sum_{j=0}^{k-1}\binom{\nu-1/2}{j}\left(-\frac{t}{2s}\right)^j \\ &\quad + k\binom{\nu-1/2}{k}\left(-\frac{t}{2s}\right)^k\int_0^1(1-\tau)^{k-1}\left(1-\tau\frac{t}{2s}\right)^{\nu-1/2-k} d\tau.\end{aligned}$$

Therefore, we have

$$I_\nu(s) = \frac{e^s}{\sqrt{2\pi s}}\frac{1}{\Gamma(\nu+1/2)}\left\{\int_0^{2s} e^{-t}t^{\nu-1/2}\sum_{j=0}^{k-1}\binom{\nu-1/2}{j}\left(-\frac{t}{2s}\right)^j dt + R_k(\nu,s)\right\},$$

where

$$\begin{aligned}R_k(\nu,s) = \int_0^{2s} & e^{-t}t^{\nu-1/2}k\binom{\nu-1/2}{k}\left(-\frac{t}{2s}\right)^k \\ &\times\left(\int_0^1(1-\tau)^{k-1}\left(1-\tau\frac{t}{2s}\right)^{\nu-1/2-k} d\tau\right) dt.\end{aligned}$$

We note that

$$\begin{aligned}\int_0^{2s} e^{-t}t^{\nu-1/2+j}dt &= \int_0^\infty e^{-t}t^{\nu-1/2+j}dt - \int_{2s}^\infty e^{-t}t^{\nu-1/2+j}dt \\ &= \Gamma(\nu+1/2+j) + \tilde{R}_j(\nu,s)\end{aligned}$$

with

$$\tilde{R}_j(\nu,s) = -\int_{2s}^\infty e^{-t}t^{\nu-1/2+j}dt.$$

Moreover, using

$$\Gamma(\nu+1/2+j) = (\nu-1/2+j)(\nu-3/2+j)\cdots(\nu+1/2)\Gamma(\nu+1/2),$$

we can deduce that

$$
\begin{aligned}
I_\nu(s) &= \frac{e^s}{\sqrt{2\pi s}}\left\{\sum_{j=0}^{k-1}\binom{\nu-1/2}{j}(\nu-1/2+j)\cdots(\nu+1/2)\frac{(-1)^j}{(2s)^j}\right.\\
&\quad\left.+\frac{1}{\Gamma(\nu+1/2)}\left(\sum_{j=0}^{k-1}\tilde{R}_j(\nu,s)+R_k(\nu,s)\right)\right\}\\
&= \frac{e^s}{\sqrt{2\pi s}}\left\{\sum_{j=0}^{k-1}(-1)^j\frac{(\nu-1/2)\cdots(\nu-(j-1/2))(\nu+(j-1/2))\cdots(\nu+1/2)}{j!2^js^j}\right.\\
&\quad\left.+\frac{1}{\Gamma(\nu+1/2)}\left(\sum_{j=0}^{k-1}\tilde{R}_j(\nu,s)+R_k(\nu,s)\right)\right\}.
\end{aligned}
$$

Therefore, it suffices to prove that $\tilde{R}_j(\nu,s)=O(s^{-k})$ for $j=0,1,\ldots,k-1$ and $R_k(\nu,s)=O(s^{-k})$ as $s\to+\infty$.

The estimates of $\tilde{R}_j(\nu,s)$ is easy. Indeed, we have

$$
\int_{2s}^\infty e^{-t}t^{\nu-1/2+j}dt\le e^{-s/2}\int_{2s}^\infty e^{-t/2}t^{\nu-1/2+j}dt\le Ce^{-s/2}.
$$

However, the estimate of $R_k(\nu,s)$ is more complicated. By the definition of $R_k(\nu,s)$, we can compute

$$
\begin{aligned}
R_k(\nu,s) &= \int_0^{2s}e^{-t}t^{\nu-1/2}k\binom{\nu-1/2}{k}\left(-\frac{t}{2s}\right)^k\\
&\quad\times\left(\int_0^1(1-\tau)^{k-1}\left(1-\tau\frac{t}{2s}\right)^{\nu-1/2-k}d\tau\right)dt\\
&= \frac{C}{s^k}\int_0^1\left(\int_0^{2s}e^{-t}t^{\nu-1/2+k}\left(1-\tau\frac{t}{2s}\right)^{\nu-1/2-k}dt\right)(1-\tau)^{k-1}d\tau.
\end{aligned}
$$

Thus, it suffices to prove that

$$
\int_0^1\left(\int_0^{2s}e^{-t}t^{\nu-1/2+k}\left(1-\tau\frac{t}{2s}\right)^{\nu-1/2-k}dt\right)(1-\tau)^{k-1}d\tau\le C \tag{9.2.3}
$$

with some constant $C>0$. When $k<\nu$, then it follows that $\nu-1/2-k\ge 0$ and (9.2.3) is obvious. When $k\ge\nu$, to see (9.2.3), we divide the integral region. If $0\le\tau\le 1/2$ or $0\le t\le s$, then we have

$$
1-\tau\frac{t}{2s}\ge\frac{1}{2}
$$

and hence, we can immediately obtain the desired estimate. Thus, we may consider only the region in which it holds that $1/2\le\tau\le 1$ and $s\le t\le 2s$. In this region, we can calculate

$$\begin{aligned}
&\int_{1/2}^{1}\left(\int_{s}^{2s} e^{-t}t^{\nu-1/2+k}\left(1-\tau\frac{t}{2s}\right)^{\nu-1/2-k}dt\right)(1-\tau)^{k-1}d\tau\\
&\le Ce^{-s}s^{\nu-1/2+k}\int_{1/2}^{1}\left(\int_{s}^{2s}\left(1-\tau\frac{t}{2s}\right)^{\nu-1/2-k}dt\right)(1-\tau)^{k-1}d\tau\\
&= Ce^{-s}s^{\nu-1/2+k}\\
&\quad\times\int_{1/2}^{1}\left(-\frac{2s}{\tau}\frac{1}{\nu+1/2-k}\left[\left(1-\tau\frac{t}{2s}\right)^{\nu+1/2-k}\right]_{t=s}^{t=2s}\right)(1-\tau)^{k-1}d\tau\\
&\le Ce^{-s}s^{\nu+1/2+k}\begin{cases}\displaystyle\int_{1/2}^{1}(1-\tau)^{\nu-1/2}d\tau & (k>\nu),\\ \displaystyle\int_{1/2}^{1}(1-\tau)^{\nu-1}d\tau & (k=\nu \text{ and } \nu\ge 1)\end{cases}\\
&\le Ce^{-s}s^{\nu+1/2+k},
\end{aligned}$$

which completes the proof. □

9.3. Lemmas

LEMMA 9.6. *If $1/p+1/p'=1, 1\le p\le 2$, then it holds that*

$$\|\mathcal{F}[f]\|_{L^{p'}}\le C\|f\|_{L^p},\quad \|\mathcal{F}^{-1}[\phi]\|_{L^{p'}}\le C\|\phi\|_{L^p}$$

for $f,\phi\in L^p(\mathbf{R}^n)$, where C is a positive constant depending only on p,n.

The following lemma is elementary but very useful for proving time decay estimates to the solution of damped wave equations.

LEMMA 9.7. *Let $\delta>0, c>0$ and $k\ge 0$, then we have*

$$\int_0^\delta r^k e^{-cr^2t}dr\le C(1+t)^{-(k+1)/2},$$

$$\sup_{0\le r\le\delta} r^k e^{-cr^2t}\le C(1+t)^{-k/2}$$

for all $t\ge 0$, where C is a positive constant independent of t.

PROOF. When $0\le t\le 1$, the estimates above are obvious. Hence we may assume $t\ge 1$. By the changing variable $r^2t=s$, we obtain

$$\begin{aligned}
t^{(k+1)/2}\int_0^\delta r^k e^{-cr^2t}dr &= \int_0^\delta (r^2t)^{k/2}e^{-cr^2t}t^{1/2}dr\\
&= \frac{1}{2}\int_0^{\delta^2 t} s^{k/2}e^{-cs}s^{-1/2}ds\\
&\le \frac{1}{2}\int_0^\infty s^{(k-1)/2}e^{-cs}ds<+\infty.
\end{aligned}$$

In a similar way, we have

$$t^{k/2}r^k e^{-cr^2t}=(r^2t)^{k/2}e^{-cr^2t}\le \sup_{s\ge 0} s^k e^{-cs}<+\infty$$

and the proof is completed. □

LEMMA 9.8. *For $a, b > 0$, there exists a constant $C > 0$ such that the following holds:*

(i) *When* $\max(a, b) > 1$,

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b} ds \le C(1+t)^{-\min(a,b)}.$$

(ii) *When* $b > 0$,

$$\int_0^t (1+s)^{-a} e^{-b(t-s)} ds \le C(1+t)^{-a}.$$

PROOF. First, we prove (i). We may assume $a = \max(a, b) > 1$ without loss of generality. Then we have

$$\begin{aligned}
\int_0^t (1+t-s)^{-a}(1+s)^{-b} ds &= \int_0^{t/2} (1+t-s)^{-a}(1+s)^{-b} ds \\
&\quad + \int_{t/2}^t (1+t-s)^{-a}(1+s)^{-b} ds \\
&\le (1+t/2)^{-a} \int_0^{t/2} (1+s)^{-b} ds \\
&\quad + (1+t/2)^{-b} \int_{t/2}^t (1+t-s)^{-a} ds \\
&= (1+t/2)^{-a} \begin{cases} \dfrac{1}{1-b}((1+t/2)^{1-b} - 1) & (b \ne 1), \\ \log(1+t/2) & (b = 1) \end{cases} \\
&\quad + (1+t/2)^{-b} \frac{1}{a-1}(1 - (1+t/2)^{1-a}) \\
&\le C(1+t)^{-b}.
\end{aligned}$$

Next, we prove (ii). Let $c > \max(a, 1)$. Then, noting $e^{-b(t-s)} \le C(1+t-s)^{-c}$, we obtain

$$\int_0^t (1+s)^{-a} e^{-b(t-s)} ds \le C \int_0^t (1+s)^{-a}(1+t-s)^{-c} ds \le C(1+t)^{-a}$$

by (i). □

LEMMA 9.9 (Hausdorff-Young). *Let $1 \le p, q, r \le \infty$ satisfy $1/q - 1/p = 1 - 1/r$. Then the inequality*

$$\|f * g\|_{L^p} \le C\|f\|_{L^r}\|g\|_{L^q}$$

holds for some constant $C > 0$.

LEMMA 9.10 (Gagliardo-Nirenberg). *Let $p, q, r (1 \le p, q, r \le \infty)$ and $\sigma \in [0, 1]$ satisfy*

$$\frac{1}{p} = \sigma\left(\frac{1}{r} - \frac{1}{n}\right) + (1-\sigma)\frac{1}{q}$$

except for $p = \infty$ or $r = n$ when $n \ge 2$. Then for some constant $C = C(p, q, r, n) > 0$, the inequality

$$\|h\|_{L^p} \le C\|\nabla h\|_{L^r}^{\sigma}\|h\|_{L^q}^{1-\sigma} \qquad \text{for any} \quad h \in W^{1,r}(\mathbf{R}^n) \cap L^q(\mathbf{R}^n)$$

holds.

For the proof see for example [**96**].

LEMMA 9.11 (Gronwall). *Let $f(t), g(t), h(t)$ be continuous functions defined on an interval $[a,b)$. We assume that $h(t) \geq 0$ and*

$$f(t) \leq g(t) + \int_a^t h(s)f(s)ds$$

hold for all $t \in [a,b)$. Then it follows that

$$f(t) \leq g(t) + \int_a^t \exp\left(\int_s^t h(\theta)d\theta\right) h(s)g(s)ds$$

for $t \in [a,b)$.

PROOF. Let

$$F(t) = \int_a^t h(s)f(s)ds.$$

Then we have $F'(t) = h(t)f(t)$. By the assumption, we obtain

$$F'(t) - h(t)F(t) \leq h(t)g(t).$$

We can rewrite this as

$$\frac{d}{dt}\left[\exp\left(-\int_a^t h(\theta)d\theta\right)F(t)\right] \leq \exp\left(-\int_a^t h(\theta)d\theta\right)h(t)g(t)$$

and hence,

$$\begin{aligned}F(t) &\leq \exp\left(\int_a^t h(\theta)d\theta\right)\int_a^t \exp\left(-\int_a^s h(\theta)d\theta\right)h(s)g(s)ds \\ &\leq \int_a^t \exp\left(\int_s^t h(\theta)d\theta\right)h(s)g(s)ds.\end{aligned}$$

Therefore, we have

$$f(t) \leq g(t) + F(t) \leq g(t) + \int_a^t \exp\left(\int_s^t h(\theta)d\theta\right)h(s)g(s)ds.$$

□

LEMMA 9.12 (A Gronwall type lemma). *Let $f(t)$ be a real-valued continuous function on an interval $[a,b]$ and $g \in \mathbf{R}$. Assume $h(t)$ is a nonnegative continuous function on $[a,b]$ and*

$$\frac{1}{2}f^2(t) \leq \frac{1}{2}g^2 + \int_a^t h(\tau)f(\tau)d\tau$$

for any $t \in [a,b]$. Then the inequality

$$|f(t)| \leq |g| + \int_a^t h(\tau)d\tau$$

holds for any $t \in [a,b]$.

PROOF. We put

$$F_\varepsilon(t) := \frac{1}{2}(g^2 + \varepsilon^2) + \int_a^t h(\tau)|f(\tau)|d\tau$$

for $t \in [a,b]$ and $\varepsilon > 0$. Then $F_\varepsilon(t)$ is strictly positive and

$$\begin{aligned} F'_\varepsilon(t) &= h(t)|f(t)| \\ &\le \sqrt{2F_\varepsilon(t)}h(t) \end{aligned}$$

by the assumption. This means

$$\frac{F'_\varepsilon(t)}{\sqrt{2F_\varepsilon(t)}} \le h(t)$$

and integrating on $[a,t]$ leads to

$$\sqrt{2F_\varepsilon(t)} \le \sqrt{2F_\varepsilon(a)} + \int_a^t h(\tau)d\tau.$$

From

$$\sqrt{2F_\varepsilon(a)} \le \sqrt{g^2+\varepsilon^2} \le |g| + \varepsilon$$

and the assumption, we have

$$\begin{aligned} |f(t)| &\le \sqrt{2F_\varepsilon(t)} \\ &\le \sqrt{2F_\varepsilon(a)} + \int_a^t h(\tau)d\tau \\ &\le |g| + \varepsilon + \int_a^t h(\tau)d\tau. \end{aligned}$$

By $\varepsilon \to +0$, we obtain the desired estimate. □

9.4. Definition of solutions

In this section, we give definitions of solutions used in this thesis.

9.4.1. Linear homogeneous equations. We consider the linear homogeneous equation

$$\begin{cases} u_{tt} - \Delta u + a(t,x)u_t = 0, & (t,x) \in (0,\infty)\times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n, \end{cases} \tag{9.4.1}$$

where $u = u(t,x)$ is a real-valued unknown, $a(t,x)$ is a smooth, nonnegative function and bounded with respect to x, that is,

$$\sup_{x\in\mathbf{R}^n} |a(t,x)| < +\infty \tag{9.4.2}$$

holds for all $t \ge 0$.

(i) *Classical solution* : let $(u_0,u_1) \in C^2(\mathbf{R}^n)\times C^1(\mathbf{R}^n)$. We say that a function u is a classical solution of (9.4.1) if $u \in C^2([0,\infty)\times\mathbf{R}^n)$ and u has the initial data $u(0,x) = u_0(x), u_t(0,x) = u_1(x)$ for all $x \in \mathbf{R}^n$ and satisfies the equation (9.4.1) at each point $(t,x) \in (0,\infty)\times\mathbf{R}^n$. For example, if $m \in \mathbf{N}, m \ge [\frac{n}{2}]+1$ and $(u_0,u_1) \in C_0^{m+2}(\mathbf{R}^n)\times C_0^{m+1}(\mathbf{R}^n)$, then there exists a unique classical solution (see [**25**, Theorem 2.27] and use the Sobolev imbedding theorem).

(ii) *Strong solution* : let $(u_0,u_1) \in H^2\times H^1$. We say that a function u is a strong solution of (9.4.1) if

$$u \in \bigcap_{j=0}^{2} C^{2-j}([0,\infty);H^j),$$

and u has the initial data $u(0) = u_0, u_t(0) = u_1$ and satisfies the equation (9.4.1) in the sense of $L^2(\mathbf{R}^n)$. It is well known that for any $(u_0, u_1) \in H^2 \times H^1$, there exists a unique strong solution (see [**25**, Theorem 2.27]). We also note that if u is a classical solution and satisfies $u \in \cap_{j=0}^2 C^{2-j}([0,\infty); H^j)$, then u also becomes a strong solution. Conversely, if u is a strong solution and $u \in C^2([0,\infty) \times \mathbf{R}^n$, then u becomes a classical solution.

Let us denote by $R(t, s)$ the operator which maps the initial data $(u_0, u_1) \in H^2 \times H^1$ given at the time $s \geq 0$ to the strong solution $u(t) \in H^2$ at the time $t \geq s$. We also write $S(t, s)u_1 = R(t, s)(0, u_1)$ for a function $u_1 \in H^1$.

Remark 9.1. *When $a(t, x) \equiv 1$, we can write*

$$R(t, s)(u_0, u_1) = S(t, s)(u_0 + u_1) + \partial_t(S(t, s)u_0).$$

This immediately follows from that the right-hand side actually satisfies the initial condition and the equation (9.4.1) *and the uniqueness of the strong solution. Moreover, if a is independent of t, then $R(t, s)$ is determined by only the difference $t - s$. Thus, we can write as $R(t - s)$ instead of $R(t, s)$.*

Proposition 9.13. *Let u be a strong solution of* (9.4.1). *Then it follows that*

$$\int_{\mathbf{R}^n} (u_t(t,x)^2 + |\nabla u(t,x)|^2)dx \leq \int_{\mathbf{R}^n} (u_1(x)^2 + |\nabla u_0(x)|^2)dx \tag{9.4.3}$$

for all $t \geq 0$. Moreover, for any $T > 0$, we have

$$\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + T\|(\nabla u_0, u_1)\|_{L^2 \times L^2} \tag{9.4.4}$$

for all $0 \leq t \leq T$.

Proof. Multiplying the equation (9.4.1) by u_t, we have

$$\frac{1}{2}\frac{\partial}{\partial t}(u_t^2 + |\nabla u|^2) - \nabla \cdot (u_t \nabla u) + a(t,x)u_t^2 = 0.$$

Integrating over $\mathbf{R}^n$, we see that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^n} (u_t^2 + |\nabla u|^2)dx + \int_{\mathbf{R}^n} a(t,x)u_t^2 dx = 0.$$

Here we have used that $u \in \cap_{j=0}^2 C^{2-j}([0,\infty); H^j)$ and the divergence theorem with the fact $u_t \nabla u \in L^1(\mathbf{R}^n)$. Moreover, integrating the above identity over the interval $[0, t]$, we deduce that

$$\begin{aligned} \frac{1}{2}\int_{\mathbf{R}^n} (u_t(t,x)^2 + |\nabla u(t,x)|^2)dx + \int_0^t \int_{\mathbf{R}^n} a(s,x)u_t(s,x)^2 dxds \\ = \frac{1}{2}\int_{\mathbf{R}^n} (u_1(x)^2 + |\nabla u_0(x)|^2)dx. \end{aligned}$$

In particular, noting that $a(t, x) \geq 0$, we have (9.4.3).

Next, we prove (9.4.4). For fixed $T > 0$, we can calculate

$$u(t) = u_0 + \int_0^t u_t(s)ds$$

and

$$\begin{aligned}\|u(t)\|_{L^2} &\le \|u_0\|_{L^2} + \int_0^t \|u_t(s)\|_{L^2} ds \\ &\le \|u_0\|_{L^2} + \int_0^t \|(\nabla u_0, u_1)\|_{L^2\times L^2} ds \\ &\le \|u_0\|_{L^2} + T\|(\nabla u_0, u_1)\|_{L^2\times L^2}\end{aligned}$$

for all $0 \le t \le T$. This completes the proof $\square$

From Proposition 9.13, the operators $R(t,s)$ and $S(t,s)$ can be extended uniquely to the operators defined on $H^1 \times L^2$. Indeed, for any fixed $T > 0$, the estimates (9.4.3) and (9.4.4) show that

$$\|R(t,s)(u_0,u_1)\|_{H^1} + \|\partial_t(R(t,s)(u_0,u_1))\|_{L^2} \le C(1+T)\|(u_0,u_1)\|_{H^1\times L^2}$$

holds for $s \le t \le s+T$. This leads to that the operator $R(t,s)$ can be extended uniquely to a operator such that $R(t,s) : H^1 \times L^2 \to C([s,T]; H^1) \cap C^1([s,T]; L^2)$. Since T is arbitrary, we have an extension of the operator $R(t,s)$ such that $R(t,s) : H^1 \times L^2 \to C([s,\infty); H^1) \cap C^1([s,\infty); L^2)$.

9.4.2. Linear inhomogeneous equations. Let us consider

$$\text{(9.4.5)} \qquad \begin{cases} u_{tt} - \Delta u + a(t,x)u_t = F(t,x), & (t,x) \in (0,\infty)\times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^n, \end{cases}$$

where $a(t,x)$ is a smooth, nonnegative function satisfying (9.4.2) and $F(t,x)$ denotes a inhomogeneous term.

(i) *Classical solution* : let $(u_0,u_1) \in C^2(\mathbf{R}^n)\times C^1(\mathbf{R}^n)$ and $F(t,x) \in C([0,\infty)\times \mathbf{R}^n)$. We say that a function u is a classical solution of (9.4.5) if $u \in C^2([0,\infty)\times\mathbf{R}^n)$ and u has the initial data $u(0,x) = u_0(x), u_t(0,x) = u_1(x)$ and satisfies the equation (9.4.5) at each point $(t,x) \in (0,\infty)\times\mathbf{R}^n$. For example, if $m \in \mathbf{N}, m \ge [\frac{n}{2}] + 1$, $(u_0,u_1) \in C_0^{m+2}(\mathbf{R}^n) \times C_0^{m+1}(\mathbf{R}^n)$ and $F(t,x) \in \cap_{j=0}^m C^j([0,\infty); H^{m-j})$, then there exists a unique classical solution (see [**25**, Theorem 2.27] and use the Sobolev imbedding theorem).

(ii) *Strong solution* : let $(u_0,u_1) \in H^2 \times H^1$ and $F \in C([0,\infty); L^2)$. We say that a function u is a strong solution of (9.4.5) if

$$u \in \bigcap_{j=0}^{2} C^{2-j}([0,\infty); H^j)$$

and u has the initial data $u(0) = u_0, u_t(0) = u_1$ and satisfies the equation (9.4.5) in the sense of $L^2(\mathbf{R}^n)$. It is well known that if $(u_0,u_1) \in H^2 \times H^1$ and $F \in C^1([0,\infty); L^2)$, then there exists a unique strong solution (see [**25**, Theorem 2.27]).

(iii) *Mild solution* : let $(u_0,u_1) \in H^1\times L^2$ and $F \in C([0,\infty); L^2)$. We say that a function u is a mild solution of (9.4.5) if $u \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ and u has the initial data $u(0) = u_0, u_t(0) = u_1$ and satisfies the integral equation

$$\text{(9.4.6)} \qquad u(t) = R(t,0)(u_0,u_1) + \int_0^t S(t,s)F(s)ds$$

in the sense of $H^1(\mathbf{R}^n)$.

(iv) *Weak solution* : let $(u_0, u_1) \in H^1 \times L^2$ and $F \in C([0,\infty); L^2)$. We say that a function u is a weak solution of (9.4.5) if $u \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ and u has the initial data $u(0) = u_0, u_t(0) = u_1$ and satisfies the identity

$$
\begin{aligned}
(9.4.7) \qquad & \int_{[0,\infty)\times\mathbf{R}^n} u(\phi_{tt} - \Delta\phi - (a(t,x)\phi)_t)dxdt \\
&= \int_{\mathbf{R}^n} ((a(0,x)u_0(x) + u_1(x))\phi(0,x) - u_0(x)\phi_t(0,x))dx \\
&\quad + \int_{[0,\infty)\times\mathbf{R}^n} F(t,x)\phi dxdt
\end{aligned}
$$

for any $\phi \in C_0^\infty([0,\infty) \times \mathbf{R}^n)$.

REMARK 9.2. *We can define the weak solution for more general function spaces wider than $C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$. However, in this thesis we treat only weak solutions belonging to $C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$.*

PROPOSITION 9.14. *Let $(u_0, u_1) \in H^2 \times H^1$, $F \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ and u be the strong solution of (9.4.5). Then u also becomes a mild solution. Moreover, u satisfies the following energy estimates.*

$$
(9.4.8) \qquad \|(u_t, \nabla u)(t)\|_{L^2\times L^2} \le C\|(u_1, \nabla u_0)\|_{L^2\times L^2} + C\int_0^t \|F(s)\|_{L^2} ds,
$$

$$
(9.4.9) \qquad \|u(t)\|_{L^2} \le \|u_0\|_{L^2} + C\int_0^t \left(\|(u_1, \nabla u_0)\|_{L^2\times L^2} + \int_0^s \|F(\tau)\|_{L^2} d\tau \right) ds.
$$

PROOF. Let $(u_0, u_1) \in H^2 \times H^1$, $F \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ and u be the strong solution of (9.4.5). We put

$$
\tilde{u}(t) := R(t,0)(u_0, u_1) + \int_0^t S(t,s)F(s)ds.
$$

Then $\tilde{u} \in \cap_{j=0}^2 C^{2-j}([0,\infty); H^j)$ and it is easy to see that $\tilde{u}$ is a strong solution of (9.4.5). Therefore, by the uniqueness of the strong solution, we have $u = \tilde{u}$.

Next, we prove the estimates (9.4.8) and (9.4.9). By multiplying the equation (9.4.5) by u_t, it follows that

$$
\frac{1}{2}\frac{\partial}{\partial t}(u_t^2 + |\nabla u|^2) - \nabla\cdot(u_t\nabla u) + a(t,x)u_t^2 = F(t,x)u_t.
$$

Integrating over $\mathbf{R}^n$, we obtain

$$
\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^n}(u_t^2 + |\nabla u|^2)dx + \int_{\mathbf{R}^n} a(t,x)u_t^2 dx = \int_{\mathbf{R}^n} F(t,x)u_t dx.
$$

Here we have used the divergence theorem with the fact $u_t\nabla u \in L^1(\mathbf{R}^n)$. Integrating over $[0,t]$ and using the Cauchy-Schwarz inequality, we deduce that

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}^n}(u_t(t)^2 + |\nabla u(t)|^2)dx \\
&\quad \le \frac{1}{2}\int_{\mathbf{R}^n}(u_1^2 + |\nabla u_0|^2)dx + \int_0^t \|F(s)\|_{L^2}\|u_t(s)\|_{L^2} ds.
\end{aligned}
$$

Therefore, we can apply Lemma 9.12 by putting

$$f(t) := \left(\int_{\mathbf{R}^n} (u_t(t)^2 + |\nabla u(t)|^2) dx\right)^{1/2}, \quad g(t) := \left(\int_{\mathbf{R}^n} (u_1^2 + |\nabla u_0|^2) dx\right)^{1/2},$$
$$h(t) := \|F(t)\|_{L^2}$$

and have

$$\begin{aligned}\left(\int_{\mathbf{R}^n} (u_t(t)^2 + |\nabla u(t)|^2) dx\right)^{1/2} \\ \leq \left(\int_{\mathbf{R}^n} (u_1^2 + |\nabla u_0|^2) dx\right)^{1/2} + \int_0^t \|F(s)\|_{L^2} ds.\end{aligned}$$

In particular, we conclude that

$$\|(u_t, \nabla u)(t)\|_{L^2 \times L^2} \leq C\|(u_1, \nabla u_0)\|_{L^2 \times L^2} + C\int_0^t \|F(s)\|_{L^2} ds,$$

which proves (9.4.8). The estimate (9.4.9) is immediately obtained by using

$$u(t) = u_0 + \int_0^t u_t(s) ds.$$

Indeed, we have

$$\begin{aligned}\|u(t)\|_{L^2} &\leq \|u_0\|_{L^2} + \int_0^t \|u_t(s)\|_{L^2} ds \\ &\leq \|u_0\|_{L^2} + C\int_0^t \left(\|(u_1, \nabla u_0)\|_{L^2 \times L^2} + \int_0^s \|F(\tau)\|_{L^2} d\tau\right) ds.\end{aligned}$$

This completes the proof. □

Proposition 9.15. *Let* $(u_0, u_1) \in H^1 \times L^2$*,* $F \in C([0, \infty); L^2)$*. Then there exists a unique mild solution* u *of* (9.4.5)*. Moreover, the mild solution* u *satisfies the estimates* (9.4.8) *and* (9.4.9)*.*

Proof. Let $T_0 > 0$ an arbitrary number. We take sequences $\{(u_0^{(j)}, u_1^{(j)})\}_{j=1}^{\infty} \subset H^2 \times H^1$ and $\{F^{(j)}\}_{j=1}^{\infty} \subset C([0, T_0]; H^1) \cap C^1([0, T_0]; L^2)$ such that

$$(u_0^{(j)}, u_1^{(j)}) \to (u_0, u_1) \text{ in } H^1 \times L^2, \quad F^{(j)} \to F \text{ in } C([0, T_0]; L^2)$$

as $j \to \infty$. Let $u^{(j)}$ be the strong solution of (9.4.5) corresponding to the initial data $(u_0^{(j)}, u_1^{(j)})$ and the inhomogeneous term $F^{(j)}(t, x)$. Then by Proposition 9.14, it follows that $\{u^{(j)}\}_{j=1}^{\infty}$ is a Cauchy sequence in $C([0, T_0]; H^1) \cap C^1([0, T_0]; L^2)$. In fact, the difference $u^{(j)} - u^{(k)}$ is the strong solution of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + a(t, x) u_t = F^{(j)} - F^{(k)}, \\ (u, u_t)(0, x) = (u_0^{(j)} - u_0^{(k)}, u_1^{(j)} - u_1^{(k)}). \end{cases}$$

Therefore, we can apply Proposition 9.14 to $u^{(j)} - u^{(k)}$ and have

$$\begin{aligned}
&\|(\partial_t(u^{(j)} - u^{(k)}), (\nabla(u^{(j)} - u^{(k)})))(t)\|_{L^2\times L^2} \\
&\quad \le C\|(u_1^{(j)} - u_1^{(k)}, \nabla(u_0^{(j)} - u_0^{(k)}))\|_{L^2\times L^2} + CT_0 \sup_{s\in[0,T_0]} \|(F^{(j)} - F^{(k)})(s)\|_{L^2}, \\
&\|(u^{(j)} - u^{(k)})(t)\|_{L^2} \\
&\quad \le \|u_0^{(j)} - u_0^{(k)}\|_{L^2} \\
&\quad + CT\left(\|(u_1^{(j)} - u_1^{(k)}, \nabla(u_0^{(j)} - u_0^{(k)}))\|_{L^2\times L^2} + T_0 \sup_{s\in[0,T_0]} \|(F^{(j)} - F^{(k)})(s)\|_{L^2}\right).
\end{aligned}$$

This shows that $\{u^{(j)}\}_{j=1}^{\infty}$ is a Cauchy sequence in $C([0,T_0];H^1)\cap C^1([0,T_0];L^2)$. Since $T_0 > 0$ is arbitrary, we can define the limit

$$\lim_{j\to\infty} u^{(j)} = u \in C([0,\infty);H^1)\cap C^1([0,\infty);L^2).$$

Using Proposition 9.14 again, we see that each $u^{(j)}$ satisfies the integral equation

$$u^{(j)}(t) = R(t,0)(u_0^{(j)}, u_1^{(j)}) + \int_0^t S(t,s)F^{(j)}(s)ds.$$

Noting that $R(t,0)$ and $S(t,s)$ can be extended uniquely to the operators defined on $H^1 \times L^2$ and L^2, respectively (see Proposition 9.13) and letting $j \to \infty$, we obtain

$$u(t) = R(t,0)(u_0, u_1) + \int_0^t S(t,s)F(s)ds,$$

which indicates that u is a mild solution of (9.4.5).

The uniqueness of mild solutions is obvious. Indeed, if two functions u, v satisfy the integral equation (9.4.6), then we immediately have $u - v = 0$.

Finally, by Proposition 9.14, each strong solution $u^{(j)}$ constructed above satisfies the estimates (9.4.8) and (9.4.9) with $u_0^{(j)}, u_1^{(j)}, F^{(j)}$. Taking the limit $j \to \infty$, we can obtain the same estimates also hold for the mild solution u. □

Proposition 9.16. *Let* $(u_0, u_1) \in H^1\times L^2$ *and* $F \in C([0,\infty);L^2)$. *If a function* $u \in C([0,\infty);H^1)\cap C^1([0,\infty);L^2)$ *is a mild solution of* (9.4.5)*, then* u *also becomes a weak solution of* (9.4.5) *and vice versa.*

Proof. Let u be a mild solution of (9.4.5) with $(u_0, u_1) \in H^1 \times L^2$ and $F \in C([0,\infty);L^2)$. Let $\phi \in C_0^\infty([0,\infty) \times \mathbf{R}^n)$. Then there exists some $T_0 > 0$ such that $\operatorname{supp}\phi \subset [0,T_0) \times \mathbf{R}^n$. As in the proof of Proposition 9.15, we take sequences $\{(u_0^{(j)}, u_1^{(j)})\}_{j=1}^{\infty} \subset H^2 \times H^1$ and $\{F^{(j)}\}_{j=1}^{\infty} \subset C([0,T_0];H^1)\cap C^1([0,T_0];L^2)$ such that

$$(u_0^{(j)}, u_1^{(j)}) \to (u_0, u_1) \text{ in } H^1 \times L^2, \quad F^{(j)} \to F \text{ in } C([0,T_0];L^2)$$

as $j \to \infty$. Let $u^{(j)}$ be the corresponding strong solution of (9.4.5) to $u_0^{(j)}, u_1^{(j)}, F^{(j)}$. Then by the proof of Proposition 9.15, it follows that $\lim_{j\to\infty} u^{(j)} = u$. Since each $u^{(j)}$ satisfies the equation

$$u_{tt}^{(j)} - \Delta u^{(j)} + a(t,x)u_t^{(j)} = F^{(j)}$$

in the sense of $L^2(\mathbf{R}^n)$, multiplying the above equation by a test function $\phi(t,x) \in C_0^\infty([0,\infty)\times\mathbf{R}^n)$ and integrating by parts, we can easily see that

$$\begin{aligned}\int_{[0,\infty)\times\mathbf{R}^n} & u^{(j)}(\phi_{tt}-\Delta\phi-(a(t,x)\phi)_t)dxdt \\ &= \int_{\mathbf{R}^n}((a(0,x)u_0^{(j)}(x)+u_1^{(j)}(x))\phi(0,x)-u_0^{(j)}(x)\phi_t(0,x))dx \\ &\quad + \int_{[0,\infty)\times\mathbf{R}^n} F^{(j)}(t,x)\phi dxdt\end{aligned}$$

Thus, letting $j\to+\infty$, we deduce that u satisfies the identity (9.4.7) and becomes a weak solution.

Next, we prove that if $u \in C([0,\infty);H^1)\cap C^1([0,\infty);L^2)$ is a weak solution, then u also becomes a mild solution. Let $\tilde{u}$ be the mild solution of (9.4.5) with $(u_0,u_1)\in H^1\times L^2$ and $F\in C([0,\infty);L^2)$. Then by the argument before, $\tilde{u}$ is also a weak solution. Therefore, taking the difference of u and $\tilde{u}$, we have

$$\int_{[0,\infty)\times\mathbf{R}^n}(u-\tilde{u})(\phi_{tt}-\Delta\phi-(a(t,x)\phi)_t)dxdt = 0$$

for any $\phi\in C_0^\infty([0,\infty)\times\mathbf{R}^n)$. Let $\psi\in C_0^\infty((0,\infty)\times\mathbf{R}^n)$ be an arbitrary test function and let $T_0>0$ be sufficiently large number so that $\operatorname{supp}\psi\subset(0,T_0)\times\mathbf{R}^n$. We take ϕ as the classical solution of the inhomogeneous equation

$$\phi_{tt}-\Delta\phi-(a(t,x)\phi)_t = \psi(t,x)$$

having the data $\phi(T_0,x)=\phi_t(T_0,x)=0$ (the existence of this classical solution ϕ can be proved by the same way as that of (9.4.5), for example, see [**25**, Theorem 2.27]). By noting the finite propagation property for solutions to the above equation (see [**25**, Theorem 2.7]), it follows that $\phi\in C_0^\infty([0,\infty)\times\mathbf{R}^n)$. Then we obtain

$$\int_{[0,\infty)\times\mathbf{R}^n}(u-\tilde{u})\psi(t,x)dxdt = 0$$

for any test function $\psi\in C_0^\infty((0,\infty)\times\mathbf{R}^n)$, which yields $u=\tilde{u}$. □

9.4.3. Semilinear equations. In this subsection, we consider solutions of the semilinear damped wave equation

$$\begin{cases} u_{tt}-\Delta u+a(t,x)u_t = f(u), & (t,x)\in(0,\infty)\times\mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x\in\mathbf{R}^n, \end{cases} \tag{9.4.10}$$

where $a(t,x)$ is a smooth, nonnegative function satisfying (9.4.2) and $f(u)$ denotes a nonlinear term. We assume that $f(\cdot):\mathbf{R}\to\mathbf{R}$ is a C^1 map and satisfies $f(0)=0$ and

$$|f(u)-f(v)| \le C(|u|+|v|)^{p-1}|u-v| \tag{9.4.11}$$

$$|f'(u)-f'(v)| \le C\begin{cases}(|u|+|v|)^{p-2}|u-v| & (p>2), \\ |u-v|^{p-1} & (1<p\le 2)\end{cases}$$

with some constant $C>0$ and p. Typical examples are $f(u)=\pm|u|^{p-1}u, \pm|u|^p$. Moreover, we also assume that

$$1<p<\infty \ (n=1,2), \quad 1<p\le\frac{n}{n-2} \ (n\ge 3). \tag{9.4.12}$$

Under these assumptions, we note that by the Gagliardo-Nirenberg inequality (Lemma 9.10), it follows that

$$\|f(v)\|_{L^2} \le C\|v\|_{L^{2p}}^p \le C\|\nabla v\|_{L^2}^{\sigma p}\|v\|_{L^2}^{(1-\sigma)p} \le C\|v\|_{H^1}^p$$

with $\sigma = n(p-1)/(2p) \in [0,1]$. We also have

$$\begin{aligned}\|f(v(t)) - f(v(s))\|_{L^2} &\le C\|(v(t)+v(s))\|_{L^{2p}}^{p-1}\|v(t)-v(s)\|_{L^{2p}} \\ &\le C(\|v(t)\|_{H^1} + \|v(s)\|_{H^1})^{p-1}\|v(t)-v(s)\|_{H^1}.\end{aligned}$$

Here we used the Hölder inequality with $(p-1)/(2p)+1/(2p) = 1/2$. The inequalities above show that if $v \in C([0,T);H^1)$ for some $T>0$, then $f(v) \in C([0,T);L^2)$. Similarly, we can obtain

$$\begin{aligned}\|\partial_t f(v(t))\|_{L^2} &= \|f'(v(t))v_t(t)\|_{L^2} \le C\||v(t)|^{p-1}v_t(t)\|_{L^2} \\ &\le C\|v(t)\|_{L^{2p}}^{p-1}\|v_t(t)\|_{L^{2p}} \le C\|v(t)\|_{H^1}^{p-1}\|v_t(t)\|_{H^1}, \\ \|\nabla_x f(v(t))\|_{L^2} &= \|f'(v(t))\nabla_x v(t)\|_{L^2} \le C\||v(t)|^{p-1}\nabla_x v(t)\|_{L^2} \\ &\le C\|v(t)\|_{L^{2p}}^{p-1}\|\nabla_x v(t)\|_{L^{2p}} \le C\|v(t)\|_{H^2}^p.\end{aligned}$$

The above estimates indicates that if $v \in \cap_{j=0}^2 C^{2-j}([0,T);H^j)$, then $\partial_t f(v(t))$ and $\nabla_x f(v(t))$ belong to $L^2(\mathbf{R}^n)$. Furthermore, we can see that

$$\begin{aligned}&\|\partial_t f(v(t)) - \partial_t f(v(s))\|_{L^2} \\ &\quad= \|f'(v(t))v_t(t) - f'(v(s))v_t(s)\|_{L^2} \\ &\quad\le C\|v(t)+v(s)\|_{L^{2p}}^{p-1}\|v_t(t)-v_t(s)\|_{L^{2p}} \\ &\quad\quad + C\begin{cases}\|v(t)+v(s)\|_{L^{2p}}^{p-2}\|v_t(t)+v_t(s)\|_{L^{2p}}\|v(t)-v(s)\|_{L^{2p}} & (p>2), \\ \|v_t(t)+v_t(s)\|_{L^{2p}}\|v(t)-v(s)\|_{L^{2p}}^{p-1} & (1<p\le 2)\end{cases}\end{aligned}$$

and

$$\begin{aligned}&\|\nabla_x f(v(t)) - \nabla_x f(v(s))\|_{L^2} \\ &\quad= \|f'(v(t))\nabla_x v(t) - f'(v(s))\nabla_x v(s)\|_{L^2} \\ &\quad\le C\|v(t)+v(s)\|_{L^{2p}}^{p-1}\|\nabla_x v(t)-\nabla_x v(s)\|_{L^{2p}} \\ &\quad\quad + C\begin{cases}\|v(t)+v(s)\|_{L^{2p}}^{p-2}\|\nabla_x v(t)+\nabla_x v(s)\|_{L^{2p}}\|v(t)-v(s)\|_{L^{2p}} & (p>2), \\ \|\nabla_x v(t)+\nabla_x v(s)\|_{L^{2p}}\|v(t)-v(s)\|_{L^{2p}}^{p-1} & (1<p\le 2).\end{cases}\end{aligned}$$

The above estimates imply that if $v \in \cap_{j=0}^2 C^{2-j}([0,T);H^j)$, then

$$f(v) \in C([0,T);H^1) \cap C^1([0,T);L^2).$$

Let us define solutions to (9.4.10). For nonlinear equations, it is not always true that there exist global-in-time solutions. Therefore, we consider solution defined on an interval $[0,T)$ for $T>0$. We call such a solution local-in-time solution (or local solution) and if we can take $T=\infty$, then we call it global-in-time solution (or global solution).

(i) *Classical solution* : let $(u_0,u_1) \in C^2(\mathbf{R}^n)\times C^1(\mathbf{R}^n)$. We say that a function u is a classical solution of (9.4.10) if $u \in C^2([0,T)\times\mathbf{R}^n)$ and u has the initial data $u(0,x) = u_0(x), u_t(0,x) = u_1(x)$ and satisfies the equation (9.4.10) at each point $(t,x) \in [0,T)\times\mathbf{R}^n$. We note that by the condition (9.4.11), $f(u)$ is continuous if u is a continuous function.

(ii) *Strong solution* : let $(u_0, u_1) \in H^2 \times H^1$. We say that a function u is a strong solution of (9.4.10) if

$$u \in \bigcap_{j=0}^{2} C^{2-j}([0,T); H^j)$$

and u has the initial data $u(0) = u_0, u_t(0) = u_1$ and satisfies the equation (9.4.10) in the sense of $L^2(\mathbf{R}^n)$. We note that if $u \in \cap_{j=0}^2 C^{2-j}([0,T); H^j)$, then $f(u) \in \cap_{j=0}^1 C^{1-j}([0,T); H^j)$.

(iii) *Mild solution* : let $(u_0, u_1) \in H^1 \times L^2$. We say that a function u is a mild solution of (9.4.10) if

$$u \in C([0,T); H^1) \cap C^1([0,T); L^2)$$

and u has the initial data $u(0) = u_0, u_t(0) = u_1$ and satisfies the integral equation

$$u(t) = R(t,0)(u_0,u_1) + \int_0^t S(t,s) f(u(s))ds \tag{9.4.13}$$

in the sense of $H^1(\mathbf{R}^n)$. We note that if $u \in C([0,T); H^1)$, then $f(u) \in C([0,T); L^2)$.

(iv) *Weak solution* : let $(u_0, u_1) \in H^1 \times L^2$. We say that a function u is a weak solution of (9.4.10) if

$$u \in C([0,T); H^1) \cap C^1([0,T); L^2)$$

and u has the initial data $u(0) = u_0, u_t(0) = u_1$ and satisfies the identity

$$\begin{aligned} &\int_{[0,T)\times\mathbf{R}^n} u(\phi_{tt} - \Delta\phi - (a(t,x)\phi)_t) dxdt \\ &= \int_{\mathbf{R}^n} ((a(0,x)u_0(x) + u_1(x))\phi(0,x) - u_0(x)\phi_t(0,x))dx \\ &\quad + \int_{[0,T)\times\mathbf{R}^n} f(u)\phi dxdt \end{aligned} \tag{9.4.14}$$

for any $\phi \in C_0^\infty([0,T) \times \mathbf{R}^n)$.

REMARK 9.3. *(i) We can define weak solutions of* (9.4.10) *for more general function spaces including* $C([0,T); H^1) \cap C^1([0,T); L^2)$. *However, in this thesis, we treat only weak solutions belonging to* $C([0,T); H^1) \cap C^1([0,T); L^2)$.

(ii) We also note that in the above definition, we can replace $\phi \in C_0^\infty([0,T) \times \mathbf{R}^n)$ *with* $\phi \in C_0^2([0,T) \times \mathbf{R}^n)$. *Because, any function* $\phi \in C_0^2([0,T) \times \mathbf{R}^n)$ *can be uniformly approximated up to the second derivatives by some sequence of functions belonging to* $C_0^\infty([0,T) \times \mathbf{R}^n)$. *This remark is useful in Chapter 6.*

PROPOSITION 9.17. *Let* $(u_0, u_1) \in H^2 \times H^1$ *and let* u *be a strong solution of* (9.4.10). *Then* u *also becomes a mild solution. Moreover,* u *satisfies the following energy estimates.*

$$\|(u_t, \nabla u)(t)\|_{L^2\times L^2} \le C\|(u_1, \nabla u_0)\|_{L^2\times L^2} + C\int_0^t \|u(s)\|_{L^{2p}}^p ds, \tag{9.4.15}$$

$$\|u(t)\|_{L^2} \le \|u_0\|_{L^2} + C\int_0^t \left(\|(u_1, \nabla u_0)\|_{L^2\times L^2} + \int_0^s \|u(\tau)\|_{L^{2p}}^p d\tau \right) ds, \tag{9.4.16}$$

where C *is a positive constant independent of* u_0, u_1, T.

PROOF. We first prove the estimates (9.4.15) and (9.4.16). This proof is almost same as that of Proposition 9.14. We multiply the equation (9.4.10) by u_t and have

$$\frac{1}{2}\frac{\partial}{\partial t}[u_t^2 + |\nabla u|^2] - \nabla \cdot (u_t \nabla u) + a(t,x)u_t^2 = f(u)u_t.$$

Integrating over $\mathbf{R}^n$, we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^n}(u_t(t,x)^2 + |\nabla u(t,x)|^2)dx + \int_{\mathbf{R}^n} a(t,x)u_t(t,x)^2 dx = \int_{\mathbf{R}^n} f(u)u_t(t,x)dx.$$

Omitting the second term of the left-hand side and integrating over the interval $[0,t]$, we can see that

$$\begin{aligned}&\frac{1}{2}\int_{\mathbf{R}^n}(u_t(t,x)^2 + |\nabla u(t,x)|^2)dx \\ &\quad \le \frac{1}{2}\int_{\mathbf{R}^n}(u_1(x)^2 + |\nabla u_0(x)|^2)dx + \int_0^t \int_{\mathbf{R}^n} f(u)u_t(s,x)dxds.\end{aligned}$$

By the Schwarz inequality, it follows that

$$\int_0^t \int_{\mathbf{R}^n} f(u)u_t(s,x)dxds \le \int_0^t \|f(u(s))\|_{L^2}\|u_t(s)\|_{L^2}ds$$

and hence,

$$\|(u_t,\nabla u)(t)\|_{L^2\times L^2}^2 \le C\|(u_1,\nabla u_0)\|_{L^2\times L^2}^2 + C\int_0^t \|f(u(s))\|_{L^2}\|u_t(s)\|_{L^2}ds.$$

Thus, we can apply Lemma 9.12 and have

$$\begin{aligned}\|(u_t,\nabla u)(t)\|_{L^2\times L^2} &\le C\|(u_1,\nabla u_0)\|_{L^2\times L^2} + C\int_0^t \|f(u(s))\|_{L^2}ds \\ &\le C\|(u_1,\nabla u_0)\|_{L^2\times L^2} + C\int_0^t \|u(s)\|_{L^{2p}}^p ds,\end{aligned}$$

which proves (9.4.15). By noting that

$$u(t) = u_0 + \int_0^t u_t(s)ds,$$

we can also see that

$$\begin{aligned}\|u(t)\|_{L^2} &\le \|u_0\|_{L^2} + \int_0^t \|u_t(s)\|_{L^2}ds \\ &\le \|u_0\|_{L^2} + C\int_0^t \left(\|(u_1,\nabla u_0)\|_{L^2\times L^2} + \int_0^s \|u(\tau)\|_{L^{2p}}^p d\tau\right)ds,\end{aligned}$$

which shows (9.4.16).

Let us prove that u becomes a mild solution. Let $T_0 \in (0,T)$. Then we have $\sup_{t\in[0,T_0]}\|(u,u_t)(t)\|_{H^1\times L^2} < +\infty$ and hence, $\sup_{t\in[0,T_0]}\|f(u(t))\|_{L^2} < +\infty$. Therefore, we can take a sequence $\{F^{(j)}\}_{j=1}^\infty \subset C([0,T_0];H^1)\cap C^1([0,T_0];L^2)$ such that

$$\lim_{j\to\infty} F^{(j)} = f(u) \text{ in } C([0,T_0];L^2).$$

Let $u^{(j)}$ be the strong solution of the linear inhomogeneous equation (9.4.5) with the initial data (u_0, u_1) and the inhomogeneous term $F^{(j)}$. Then, as in the proof of Proposition 9.15, we can prove that

$$\|(\partial_t(u^{(j)} - u^{(k)}), \nabla_x(u^{(j)} - u^{(k)}))(t)\|_{L^2 \times L^2} \le CT_0 \sup_{s \in [0,T_0]} \|(F^{(j)} - F^{(k)})(s)\|_{L^2},$$

$$\|(u^{(j)} - u^{(k)})(t)\|_{L^2} \le CT_0^2 \sup_{s \in [0,T_0]} \|(F^{(j)} - F^{(k)})(s)\|_{L^2}.$$

This shows that $\{u^{(j)}\}_{j=1}^{\infty}$ is a Cauchy sequence in $C([0,T_0]; H^1) \cap C^1([0,T_0]; L^2)$ and we denote the limit function by $\tilde{u}$. Moreover, by Proposition 9.14, each $u^{(j)}$ satisfies the integral equation

$$u^{(j)}(t) = R(t,0)(u_0, u_1) + \int_0^t S(t,s)F^{(j)}(s)ds.$$

Letting $j \to +\infty$, we obtain

$$\tilde{u}(t) = R(t,0)(u_0, u_1) + \int_0^t S(t,s)f(u(s))ds$$

for $t \in [0, T_0]$. Therefore, it suffices to prove that $u = \tilde{u}$. We note that $u^{(j)} - u$ satisfies the equation

$$v_{tt} - \Delta v + a(t,x)v_t = F^{(j)} - f(u)$$

in $L^2(\mathbf{R}^n)$ and have the initial data $(u^{(j)} - u)(0,x) = 0, (\partial_t(u^{(j)} - u))(0,x) = 0$. Hence, by the proof of the estimates (9.4.15) and (9.4.16), we can see that

$$\|(\partial_t(u^{(j)} - u), \nabla_x(u^{(j)} - u))(t)\|_{L^2 \times L^2} \le CT_0 \sup_{s \in [0,T_0]} \|(F^{(j)} - f(u))(s)\|_{L^2},$$

$$\|(u^{(j)} - u)(t)\|_{L^2} \le CT_0^2 \sup_{s \in [0,T_0]} \|(F^{(j)} - f(u))(s)\|_{L^2}.$$

This implies that $u^{(j)} \to u$ in $C([0,T_0]; H^1) \cap C^1([0,T_0]; L^2)$ and hence, $u = \tilde{u}$. Therefore, u satisfies the integral equation (9.4.13) for $t \in [0, T_0]$. Since T_0 is arbitrary in $[0,T)$, u becomes a mild solution of (9.4.10). □

PROPOSITION 9.18. *Let $(u_0, u_1) \in H^1 \times L^2$ and $u \in C([0,T); H^1) \cap C^1([0,T); L^2)$. If u is a mild solution of* (9.4.10)*, then u becomes a weak solution of* (9.4.10) *and vice versa.*

PROOF. Let u be a mild solution of (9.4.10) and let $\phi \in C_0^\infty([0,T) \times \mathbf{R}^n)$. Then there exists some $T_0 \in (0,T)$ such that $\operatorname{supp} \phi \subset [0,T_0) \times \mathbf{R}^n$. We take sequences $\{(u_0^{(j)}, u_1^{(j)})\}_{j=1}^{\infty} \subset H^2 \times H^1$ and $\{F^{(j)}\}_{j=1}^{\infty} \subset C([0,T_0]; H^1) \cap C^1([0,T_0]; L^2)$ such that

$$\lim_{j \to \infty} (u_0^{(j)}, u_1^{(j)}) = (u_0, u_1) \text{ in } H^1 \times L^2, \quad \lim_{j \to \infty} F^{(j)} = f(u) \text{ in } C([0,T_0]; L^2).$$

Let $u^{(j)}$ be the strong solution of the linear inhomogeneous equation (9.4.5) with the initial data $(u_0^{(j)}, u_1^{(j)})$ and the inhomogeneous term $F^{(j)}$. Then by Proposition 9.14, we have

$$u^{(j)}(t) = R(t,0)(u_0^{(j)}, u_1^{(j)}) + \int_0^t S(t,s)F^{(j)}(s)ds.$$

Since u is a mild solution of (9.4.10), we obtain

$$u^{(j)}(t) - u(t) = R(t,0)(u_0^{(j)} - u_0, u_1^{(j)} - u_1) + \int_0^t S(t,s)(F^{(j)}(s) - f(u(s)))ds$$

and hence, by using Proposition 9.13, we can see that

$$\begin{aligned}&\|(u^{(j)} - u, \partial_t(u^{(j)} - u))(t)\|_{H^1\times L^2}\\ &\le C(1+T_0)\left(\|(u_0^{(j)} - u_0, u_1^{(j)} - u_1)\|_{H^1\times L^2} + T_0 \sup_{s\in[0,T_0]} \|F^{(j)}(s) - f(u(s))\|_{L^2}\right).\end{aligned}$$

This implies that $\lim_{j\to\infty} u^{(j)} = u$ in $C([0,T_0];H^1)\cap C^1([0,T_0];L^2)$. Since each $u^{(j)}$ is a strong solution of (9.4.5), $u^{(j)}$ is also a weak solution of (9.4.5), that is,

$$\begin{aligned}&\int_{[0,\infty)\times\mathbf{R}^n} u^{(j)}(\phi_{tt} - \Delta\phi - (a(t,x)\phi)_t)dxdt\\ &\quad = \int_{\mathbf{R}^n} ((a(0,x)u_0^{(j)}(x) + u_1^{(j)}(x))\phi(0,x) - u_0^{(j)}(x)\phi_t(0,x))dx\\ &\qquad + \int_{[0,\infty)\times\mathbf{R}^n} F^{(j)}(t,x)\phi dxdt.\end{aligned}$$

Thus, letting $j\to+\infty$, we deduce that u satisfies the identity (9.4.14). Since ϕ is arbitrary test function, u is a weak solution of (9.4.10).

Next, we prove that a weak solution u of (9.4.10) becomes a mild solution. Let $T_0\in(0,T)$. As before, we take sequences $\{(u_0^{(j)},u_1^{(j)})\}_{j=1}^\infty\subset H^2\times H^1$ and $\{F^{(j)}\}_{j=1}^\infty\subset C([0,T_0];H^1)\cap C^1([0,T_0];L^2)$ such that

$$\lim_{j\to\infty}(u_0^{(j)},u_1^{(j)}) = (u_0,u_1) \text{ in } H^1\times L^2,\quad \lim_{j\to\infty} F^{(j)} = f(u) \text{ in } C([0,T_0];L^2)$$

and let $u^{(j)}$ be the strong solution of the linear inhomogeneous equation (9.4.5) with the initial data $(u_0^{(j)},u_1^{(j)})$ and the inhomogeneous term $F^{(j)}$. By the same argument as above, we can see that $\{u^{(j)}\}_{j=1}^\infty$ is a Cauchy sequence in $C([0,T_0];L^2)$ and we denote the limit function by $\tilde{u}$. Then $\tilde{u}$ satisfies the integral equation

$$\tilde{u}(t) = R(t,0)(u_0,u_1) + \int_0^t S(t,s)f(u(s))ds.$$

Therefore, it suffices to prove that $u=\tilde{u}$. Since each $u^{(j)}$ is a strong solution and u is a weak solution, we can deduce that

$$\begin{aligned}&\int_{[0,T_0)\times\mathbf{R}^n} (u^{(j)} - u)(\phi_{tt} - \Delta\phi - (a(t,x)\phi)_t)dxdt\\ &\quad = \int_{\mathbf{R}^n} ((a(0,x)(u_0^{(j)} - u_0) + (u_1^{(j)} - u_1))\phi(0,x) - (u_0^{(j)} - u_0)\phi_t(0,x))dx\\ &\qquad + \int_{[0,T_0)\times\mathbf{R}^n} (F^{(j)} - f(u))\phi dxdt\end{aligned}$$

holds for any $\phi\in C_0^\infty([0,T_0)\times\mathbf{R}^n)$. Letting $j\to+\infty$, we obtain

$$\int_{[0,T_0)\times\mathbf{R}^n} (\tilde{u} - u)(\phi_{tt} - \Delta\phi - (a(t,x)\phi)_t)dxdt = 0.$$

Let $\psi \in C_0^\infty((0,T_0)\times \mathbf{R}^n)$ and we take a function ϕ satisfying the equation

$$\phi_{tt} - \Delta\phi - (a(t,x)\phi)_t = \psi$$

and having the data $\phi(T_0,x) = \phi_t(T_0,x) = 0$. Then we see that

$$\int_{[0,T_0)\times \mathbf{R}^n} (\tilde{u} - u)\psi(t,x)dxdt = 0.$$

Noting that ψ is an arbitrary test function, we have $u = \tilde{u}$ a.e. $(0,T_0)\times \mathbf{R}^n$. Since T_0 is arbitrary in $(0,T)$, we conclude that $u = \tilde{u}$ a.e. $(0,T)\times \mathbf{R}^n$. □

9.5. Local existence

In this section we prove a local existence of the solution to

$$\begin{cases} u_{tt} - \Delta u + a(t,x)u_t = f(u) & (t,x) \in (0,\infty)\times \mathbf{R}^n, \\ (u,u_t)(0,x) = (u_0,u_1)(x) & x \in \mathbf{R}^n. \end{cases} \tag{9.5.1}$$

where $a(t,x)$ is a smooth nonnegative function and satisfies (9.4.2), that is,

$$\sup_{x\in \mathbf{R}^n} |a(t,x)| < +\infty$$

holds for all $t \ge 0$. We assume that $f(\cdot) : \mathbf{R} \to \mathbf{R}$ is a C^1 map and satisfies $f(0) = 0$ and (9.4.11), that is,

$$|f(u) - f(v)| \le C(|u| + |v|)^{p-1}|u - v| \tag{9.5.2}$$

$$|f'(u) - f'(v)| \le C \begin{cases} (|u| + |v|)^{p-2}|u - v| & (p > 2), \\ |u - v|^{p-1} & (1 < p \le 2) \end{cases}$$

with some constant $C > 0$ and $p > 1$.

We consider weak solutions of (9.5.1). Let $T > 0$ and $X(T) := C([0,T);H^1(\mathbf{R}^n))\cap C^1([0,T);L^2(\mathbf{R}^n))$. Recall that a function $u \in X(T)$ is a weak solution of the Cauchy problem (9.5.1) on the interval $[0,T)$ if it holds that

$$\begin{aligned} &\int_{[0,T)\times \mathbf{R}^n} u(t,x)(\partial_t^2\phi(t,x) - \Delta\phi(t,x) - \partial_t(a(t,x)\phi(t,x)))dxdt \\ &= \int_{\mathbf{R}^n} \{(a(0,x)u_0(x) + u_1(x))\phi(0,x) - u_0(x)\partial_t\phi(0,x)\}\, dx \\ &\quad + \int_{[0,T)\times \mathbf{R}^n} f(u)\phi(t,x)dxdt \end{aligned} \tag{9.5.3}$$

for any $\phi \in C_0^\infty([0,T)\times \mathbf{R}^n)$.

Let $\psi(t,x) \in C^1([0,\infty)\times \mathbf{R}^n)$ be a nonnegative function satisfying

(9.5.4)

$$\psi_t(t,x) \le 0, \quad |\nabla\psi(t,x)|^2 \le -\psi_t(t,x)a(t,x), \quad e^{(\frac{2}{p}-2)\psi(t,x)}|\nabla\psi(t,x)|^2 \le C$$

with some constant $C > 0$.

In the following, we consider the initial data (u_0,u_1) satisfying

$$I_0^2 := \int_{\mathbf{R}^n} e^{2\psi(0,x)}(u_0(x)^2 + |\nabla u_0(x)|^2 + u_1(x)^2)dx < +\infty. \tag{9.5.5}$$

We also put

$$I_\psi(t,u)^2 := \int_{\mathbf{R}^n} e^{2\psi(t,x)}(u(t,x)^2 + u_t(t,x)^2 + |\nabla u(t,x)|^2)dx.$$

We note that when $\psi \equiv 0$, (9.5.5) is equivalent to $(u_0, u_1) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$. We first note that such a function can be approximated by smooth and compactly supported functions.

LEMMA 9.19. *Let u_0, u_1 be a function satisfying* (9.5.5). *Then there exist sequences $\{(u_0^{(j)}, u_1^{(j)})\}_{j=1}^\infty \subset C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$ such that*

$$\lim_{j\to\infty} \int_{\mathbf{R}^n} e^{2\psi(0,x)}(|u_0^{(j)}(x) - u(x)|^2 + |\nabla(u_0^{(j)}(x) - u_0(x))|^2)dx = 0 \tag{9.5.6}$$

and

$$\lim_{j\to\infty} \int_{\mathbf{R}^n} e^{2\psi(0,x)}(u_1^{(j)}(x) - u_1(x))^2 dx = 0. \tag{9.5.7}$$

Moreover, let $T > 0$ and let $v \in C([0,T); H^1) \cap C^1([0,T); L^2)$ satisfy

$$\sup_{t\in[0,T)} I_\psi(t,v) \le R$$

with some $R > 0$. Then there exists a sequence $\{v^{(j)}\}_{j=1}^\infty \subset C_0^\infty((0,T) \times \mathbf{R}^n)$ such that

(9.5.8)

$$\lim_{j\to\infty} \int_0^T \left(\int_{\mathbf{R}^n} e^{2\psi(t,x)}(|v^{(j)}(t,x) - v(t,x)|^2 + |\nabla v^{(j)}(t,x) - \nabla v(t,x)|^2)dx \right)^q dt = 0$$

for any $q \in [1,\infty)$.

PROOF. Let $\chi(x) \in C_0^\infty(\mathbf{R}^n)$ be a cut-off function satisfying

$$\chi(0) = 1, 0 \le \chi(x) \le 1, \quad \operatorname{supp}\chi \subset \{x \in \mathbf{R}^n \mid |x| \le 1\}, \quad |\nabla\chi(x)| \le C_0$$

with some constant $C_0 > 0$. Let $\delta \in (0,1)$ and we define $\chi_\delta(x) := \chi(\delta x)$. Next, we define the Friedrichs mollifier. Let $\rho \in C_0^\infty(\mathbf{R}^n)$ be an another cut-off function such that

$$\rho(x) \ge 0, \quad \operatorname{supp}\rho \subset \{x \in \mathbf{R}^n \mid |x| \le 1\}, \quad \int_{\mathbf{R}^n} \rho(x)dx = 1$$

and we define $\rho_\varepsilon(x) := \varepsilon^{-n}\rho(\varepsilon^{-1}x)$ for $\varepsilon \in (0,1)$. We will prove that u can be approximated by $\rho_\varepsilon * (\chi_\delta u)$ with sufficiently small ε, δ.

We first prove (9.5.7). Let $\eta > 0$ be an arbitrary small number. By the Lebesgue dominated convergence theorem, we see that there exists a constant δ_0 such that

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)} \left|\chi_\delta(x)u_1(x) - u_1(x)\right|^2 dx < \frac{\eta}{4}$$

holds for any $\delta \in (0, \delta_0]$. Since $\operatorname{supp}\chi_{\delta_0} \subset \{x \in \mathbf{R}^n \mid |x| \le 1/\delta_0\}$, we can obtain

$$e^{2\psi(0,x)} \le C_{\delta_0}$$

for $|x| \le 1/\delta_0 + 1$ with some constant $C_{\delta_0} > 0$. Therefore, we can deduce that there exists a constant $\varepsilon_0 > 0$ such that

$$\begin{aligned}
&\int_{\mathbf{R}^n} e^{2\psi(0,x)} \left|\rho_\varepsilon * (\chi_{\delta_0}u_1) - \chi_{\delta_0}u_1\right|^2 dx \\
&\le C_{\delta_0} \int_{\mathbf{R}^n} \left|\rho_\varepsilon * (\chi_{\delta_0}u_1) - \chi_{\delta_0}u_1\right|^2 dx \\
&< \frac{\eta}{4}
\end{aligned}$$

is true for any $\varepsilon \in (0, \varepsilon_0]$. Thus, we have

$$\begin{aligned}
&\int_{\mathbf{R}^n} e^{2\psi(t,x)} \left|\rho_\varepsilon * (\chi_{\delta_0} u_1) - u_1\right|^2 dx \\
&\quad \le 2 \int_{\mathbf{R}^n} e^{2\psi(0,x)} \left|\rho_\varepsilon * (\chi_{\delta_0} u_1) - \chi_{\delta_0} u_1\right|^2 dx \\
&\qquad + 2 \int_{\mathbf{R}^n} e^{2\psi(0,x)} \left|\chi_\delta(x) u_1(x) - u_1(x)\right|^2 dx < \eta,
\end{aligned}$$

which proves (9.5.7).

Next, we prove (9.5.6). Let $\eta > 0$ be an arbitrary small number. As before, we can see that there exists a constant $\delta_0 > 0$ such that

$$\int_{\mathbf{R}^n} e^{2\psi(0,x)} \left|\chi_\delta(x) u_0(x) - u_0(x)\right|^2 dx < \frac{\eta}{8}$$

holds for any $\delta \in (0, \delta_0]$. On the other hand, noting that

$$|\nabla \chi_\delta(x)| = \delta |(\nabla \chi)(\delta x)| \le C_0 \delta,$$

we can deduce that

$$\begin{aligned}
&\int_{\mathbf{R}^n} e^{2\psi(0,x)} |\nabla(\chi_\delta u_0) - \nabla u_0|^2 dx \\
&\quad \le 2 \int_{\mathbf{R}^n} e^{2\psi(0,x)} \delta^2 C_0^2 u_0(x)^2 dx + 2 \int_{\mathbf{R}^n} e^{2\psi(0,x)} |\chi_\delta \nabla u_0 - \nabla u_0|^2 dx.
\end{aligned}$$

From this, it follows that there exists a constant $\delta_1 > 0$ such that

$$\int_{\mathbf{R}^n} e^{2\psi(0,x)} |\nabla(\chi_\delta u_0) - \nabla u_0|^2 dx < \frac{\eta}{8}$$

holds for any $\delta \in (0, \delta_1]$. We put $\delta_2 := \min\{\delta_0, \delta_1\}$. Then, there exists some constant $C_{\delta_2} > 0$ such that

$$e^{2\psi(0,x)} \le C_{\delta_2}$$

for any $|x| \le 1/\delta_2 + 1$. Therefore, we can conclude that there exists a small $\varepsilon_0 > 0$ such that

$$\begin{aligned}
&\int_{\mathbf{R}^n} e^{2\psi(0,x)} \left|\rho_\varepsilon * (\chi_{\delta_2} u_0) - \chi_{\delta_2} u_0\right|^2 dx \\
&\quad \le C_{\delta_2} \int_{\mathbf{R}^n} \left|\rho_\varepsilon * (\chi_{\delta_2} u_0) - \chi_{\delta_2} u_0\right|^2 dx < \frac{\eta}{8}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbf{R}^n} e^{2\psi(0,x)} \left|\rho_\varepsilon * (\nabla(\chi_{\delta_2} u_0)) - \nabla(\chi_{\delta_2} u_0)\right|^2 dx \\
&\quad \le C_{\delta_2} \int_{\mathbf{R}^n} \left|\rho_\varepsilon * (\nabla(\chi_{\delta_2} u_0)) - \nabla(\chi_{\delta_2} u_0)\right|^2 dx < \frac{\eta}{8}
\end{aligned}$$

for any $\varepsilon \in (0, \varepsilon_0]$. Finally, we have

$$\begin{aligned}
&\int_{\mathbf{R}^n} e^{2\psi(0,x)} \left(|\rho_\varepsilon * (\chi_{\delta_2} u_0) - u_0|^2 + |\nabla(\rho_\varepsilon * (\chi_{\delta_2} u_0)) - \nabla u_0|^2\right) dx \\
&\quad \le 2 \int_{\mathbf{R}^n} e^{2\psi(0,x)} \left|\rho_\varepsilon * (\chi_{\delta_2} u_0)(x) - \chi_{\delta_2} u_0(x)\right|^2 dx \\
&\qquad + 2 \int_{\mathbf{R}^n} e^{2\psi(0,x)} \left|\chi_{\delta_2}(x) u_0(x) - u_0(x)\right|^2 dx \\
&\qquad + 2 \int_{\mathbf{R}^n} e^{2\psi(0,x)} |\nabla(\chi_{\delta_2} u_0)(x) - \nabla u_0(x)|^2 dx \\
&\qquad + 2 \int_{\mathbf{R}^n} e^{2\psi(0,x)} \left|\rho_\varepsilon * (\nabla(\chi_{\delta_2} u_0))(x) - (\nabla \chi_{\delta_2} u_0)(x)\right|^2 dx < \eta,
\end{aligned}$$

which shows (9.5.6).

Let us turn to prove the latter assertion. Let $T > 0$ and let $v \in C([0,T); H^1) \cap C^1([0,T); L^2)$ satisfy $\sup_{t\in[0,T)} I_\psi(t,v) \le R$. Let $\chi_\delta(x)$ and ρ be functions as above and let $\sigma(t) \in C_0^\infty(\mathbf{R})$ such that

$$\sigma(t) \ge 0, \quad \operatorname{supp}\sigma \subset (-1,1), \quad \int_{\mathbf{R}} \sigma(t) dt = 1.$$

We define $\sigma_\varepsilon(t) := \varepsilon^{-1}\sigma(\varepsilon^{-1}t)$. Let $\mu_\gamma(t)$ with small $\gamma > 0$ be a cut-off function such that

$$\mu_\gamma(t) = 1 \ (t \in [\gamma, T-\gamma]), \quad \mu_\gamma(t) = 0 \ (t \in [0,\gamma) \cup (T-\gamma, T]).$$

We shall prove that the function

$$v_{\delta,\gamma,\varepsilon} := \sigma_\varepsilon * \rho_\varepsilon * (\mu_\gamma \chi_\delta v)(t,x) := \int_0^T \int_{\mathbf{R}^n} \sigma_\varepsilon(t-s)\rho_\varepsilon(x-y)(\mu_\gamma\chi_\delta v)(s,y) dy ds$$

for sufficiently small $\delta, \gamma, \varepsilon$ gives the desired approximation of $v(t,x)$. Let $\eta > 0$ be an arbitrary number. Let $q \in [1,\infty)$. By noting that $\sup_{t\in[0,T)} I_\psi(t,v) \le R$, in a similar way to deriving (9.5.6), we can see that there exist some γ_0, δ_0 such that

$$\begin{aligned}
\int_0^T \Bigg(\int_{\mathbf{R}^n} & e^{2\psi(t,x)} (|\mu_\gamma(t)\chi_\delta(x)v(t,x) - v(t,x)|^2 \\
&+ |\nabla_x(\mu_\gamma(t)\chi_\delta(x)v(t,x)) - \nabla_x v(t,x)|^2) dx \Bigg)^q dt < \frac{\eta}{2^{q+1}}
\end{aligned}$$

for all $\gamma \in (0,\gamma_0], \delta \in (0,\delta_0]$. Noting that

$$\operatorname{supp}(\mu_{\gamma_0}\chi_{\delta_0} v) \subset [\gamma, T-\gamma] \times \{x \in \mathbf{R}^n \mid |x| \le 1/\delta_0\},$$

we deduce that there exists some $\varepsilon_0 > 0$ such that $\sigma_\varepsilon * \rho_\varepsilon * (\mu_{\gamma_0}\chi_{\delta_0} v) \in C_0^\infty((0,T)\times \mathbf{R}^n)$ holds for any $\varepsilon \in (0,\varepsilon_0]$. We remark that

$$e^{2\psi(t,x)} \le C_{\gamma_0,\delta_0}$$

is valid for $(t,x) \in \operatorname{supp} \sigma_\varepsilon * \rho_{\varepsilon_0} * (\mu_{\gamma_0}\chi_{\delta_0} v)$ with some constant $C_{\gamma_0,\delta_0} > 0$. By using this, it follows that there exists some $\varepsilon_1 \in (0, \varepsilon_0]$ such that

$$\begin{aligned}
&\int_0^T \left(\int_{\mathbf{R}^n} e^{2\psi(t,x)} (|\sigma_\varepsilon * \rho_\varepsilon * (\mu_{\gamma_0}\chi_{\delta_0} v)(t,x) - (\mu_{\gamma_0}\chi_{\delta_0} v)(t,x)|^2 \right.\\
&\qquad \left. + |\nabla_x(\sigma_\varepsilon * \rho_\varepsilon * (\mu_{\gamma_0}\chi_{\delta_0} v))(t,x) - \nabla_x(\mu_{\gamma_0}\chi_{\delta_0} v)(t,x)|^2) dx \right)^q dt \\
&\le C^q_{\gamma_0,\delta_0} \int_0^T \left(\int_{\mathbf{R}^n} (|\sigma_\varepsilon * \rho_\varepsilon * (\mu_{\gamma_0}\chi_{\delta_0} v)(t,x) - (\mu_{\gamma_0}\chi_{\delta_0} v)(t,x)|^2 \right.\\
&\qquad\qquad \left. + |\nabla_x(\sigma_\varepsilon * \rho_\varepsilon * (\mu_{\gamma_0}\chi_{\delta_0} v))(t,x) - \nabla_x(\mu_{\gamma_0}\chi_{\delta_0} v)(t,x)|^2) dx \right)^q dt \\
&< \frac{\eta}{2^{q+1}}
\end{aligned}$$

for any $\varepsilon \in (0, \varepsilon_1]$. Consequently, we have

$$\begin{aligned}
&\int_0^T \left(\int_{\mathbf{R}^n} e^{2\psi(t,x)} (|(\sigma_{\varepsilon_1} * \rho_{\varepsilon_1} * (\mu_{\gamma_0}\chi_{\delta_0} v))(t,x) - v(t,x)|^2 \right.\\
&\qquad \left. + |\nabla_x(\sigma_{\varepsilon_1} * \rho_{\varepsilon_1} * (\mu_{\gamma_0}\chi_{\delta_0} v))(t,x) - \nabla_x v(t,x)|^2) dx \right)^q dt < \eta,
\end{aligned}$$

which completes the proof. □

Using the above lemma, we can also obtain an approximation of the nonlinear term $f(v)$.

LEMMA 9.20. *Let $T > 0$, $f : \mathbf{R} \to \mathbf{R}$ be a C^1-map satisfying* (9.5.2) *and let $v \in C([0,T); H^1) \cap C^1([0,T); L^2)$ satisfy $\sup_{t\in[0,T)} I_\psi(t,v) \le R$ with some $R > 0$. We take an approximation $\{v^{(j)}\}_{j=1}^\infty \subset C_0^\infty((0,T) \times \mathbf{R}^n)$ of v satisfying* (9.5.8). *Then it follows that*

$$\lim_{j\to\infty} \int_0^T \|e^{\psi(t)}(f(v^{(j)}) - f(v))\|_{L^2} dt = 0. \tag{9.5.9}$$

PROOF. We first note that the condition (9.5.4) implies that

$$e^{\frac{2}{p}\psi(t,x)} |\nabla\psi(t,x)|^2 \le C e^{2\psi(t,x)} \tag{9.5.10}$$

with some constant $C > 0$. Using this, we deduce that

$$\|e^{\frac{1}{p}\psi(t)} v\|_{H^1} \le C(\|e^{\psi(t)} v\|_{L^2} + \|e^{\psi(t)} \nabla v\|_{L^2}). \tag{9.5.11}$$

Moreover, by the convergence (9.5.8) with $q = 2(p-1)$ ($q = 1$ if $2(p-1) < 1$), we see that

$$\int_0^T (\|e^\psi(t) v^{(j)}\|_{L^2} + \|e^{\psi(t)} \nabla v^{(j)}\|_{L^2})^q dt \le C$$

with some constant $C = C(T, R, q) > 0$. Moreover, by the Gagliardo-Nirenberg inequality (see Lemma 9.10), we have $\|v\|_{L^{2p}} \le C\|v\|_{H^1}$. Therefore, we can calculate

$$\begin{aligned}
&\int_0^T \|e^{\psi(t)}(f(v^{(j)}) - f(v))\|_{L^2} dt \\
&\le C\int_0^T \|e^{\psi(t)}(|v^{(j)}| + |v|)^{p-1}(v^{(j)} - v)\|_{L^2} dt \\
&\le C\int_0^T \|e^{\frac{p-1}{p}\psi(t)}(|v^{(j)}| + |v|)^{p-1}\|_{L^{\frac{2p}{p-1}}} \|e^{\frac{1}{p}\psi(t)}(v^{(j)} - v)\|_{L^{2p}} dt \\
&\le C\int_0^T \|e^{\frac{1}{p}\psi(t)}(|v^{(j)}| + |v|)\|_{L^{2p}}^{p-1} \|e^{\frac{1}{p}\psi(t)}(v^{(j)} - v)\|_{L^{2p}} dt \\
&\le C\int_0^T (\|e^{\frac{1}{p}\psi(t)}v^{(j)}\|_{L^{2p}} + \|e^{\frac{1}{p}\psi(t)}v\|_{L^{2p}})^{p-1} \|e^{\frac{1}{p}\psi(t)}(v^{(j)} - v)\|_{L^{2p}} dt \\
&\le C\int_0^T (\|e^{\frac{1}{p}\psi(t)}v^{(j)}\|_{H^1} + \|e^{\frac{1}{p}\psi(t)}v\|_{H^1})^{p-1} \|e^{\frac{1}{p}\psi(t)}(v^{(j)} - v)\|_{H^1} dt \\
&\le C\int_0^T (\|e^{\psi(t)}v^{(j)}\|_{L^2} + \|e^{\psi(t)}\nabla v^{(j)}\|_{L^2} + \|e^{\psi(t)}v\|_{L^2} + \|e^{\psi(t)}\nabla v\|_{L^2})^{p-1} \\
&\qquad \times (\|e^{\psi(t)}(v^{(j)} - v)\|_{L^2} + \|e^{\psi(t)}(\nabla v^{(j)} - \nabla v)\|_{L^2}) dt.
\end{aligned}$$

By the Schwarz inequality, the right-hand side is estimated by

$$\begin{aligned}
&C\left(\int_0^T (\|e^{\psi(t)}v^{(j)}\|_{L^2} + \|e^{\psi(t)}\nabla v^{(j)}\|_{L^2} + \|e^{\psi(t)}v\|_{L^2} + \|e^{\psi(t)}\nabla v\|_{L^2})^{2(p-1)} dt\right)^{1/2} \\
&\qquad \times \left(\int_0^T (\|e^{\psi(t)}(v^{(j)} - v)\|_{L^2} + \|e^{\psi(t)}(\nabla v^{(j)} - \nabla v)\|_{L^2})^2 dt\right)^{1/2} \\
&\le C\left(\int_0^T (\|e^{\psi(t)}v^{(j)}\|_{L^2} + \|e^{\psi(t)}\nabla v^{(j)}\|_{L^2} + \|e^{\psi(t)}v\|_{L^2} + \|e^{\psi(t)}\nabla v\|_{L^2})^{q} dt\right)^{1/2} \\
&\qquad \times \left(\int_0^T (\|e^{\psi(t)}(v^{(j)} - v)\|_{L^2} + \|e^{\psi(t)}(\nabla v^{(j)} - \nabla v)\|_{L^2})^2 dt\right)^{1/2}
\end{aligned}$$

with some constant $C = C(T, R, q)$, where

$$q = \begin{cases} 2(p-1) & (2(p-1) \ge 1), \\ 1 & (2(p-1) < 1). \end{cases}$$

Thus, the boundedness of $v, v^{(j)}$, the Hölder inequality and (9.5.8) with $q = 1$ imply that

$$\begin{aligned}
&\int_0^T \|e^{\psi(t)}(f(v^{(j)}) - f(v))\|_{L^2} dt \\
&\le C \left(\int_0^T (\|e^{\psi(t)}(v^{(j)} - v)\|_{L^2} + \|e^{\psi(t)}(\nabla v^{(j)} - \nabla v)\|_{L^2})^2 dt \right)^{1/2} \\
&\le C \left(\int_0^T (\|e^{\psi(t)}(v^{(j)} - v)\|_{L^2}^2 + \|e^{\psi(t)}(\nabla v^{(j)} - \nabla v)\|_{L^2}^2) dt \right)^{1/2} \\
&\to 0
\end{aligned}$$

as $j \to +\infty$. □

Let us prove the existence of local solution for (9.5.1).

PROPOSITION 9.21. *Let f satisfy* (9.5.2) *with*

$$1 < p \le \frac{n}{n-2} \ (n \ge 3), \quad 1 < p < \infty \ (n = 1, 2)$$

and $(u_0, u_1), \psi$ satisfy (9.5.5), (9.5.4), *respectively. Then there exists $T^* \in (0, \infty]$ and a unique weak solution $u \in X(T^*)$ of* (9.5.1) *such that $I_\psi(t, u)$ is continuous on $t \in [0, T^*)$ and $I_\psi(t, u) < +\infty$ for $t \in [0, T^*)$. Moreover, if $T^* < +\infty$, then it follows that*

$$\lim_{t \to T^* - 0} \int_{\mathbf{R}^n} e^{2\psi(t,x)}(u(t,x)^2 + u_t(t,x)^2 + |\nabla u(t,x)|^2) dx = +\infty.$$

PROOF. We follow the argument of Ikehata and Tanizawa [**35**]. Let $T, R > 0$ and

$$B_{T,R} := \{v \in X(T) \mid I_\psi(v,t) \in C([0,T)), \ \|v\|_{X_\psi(T)} \le R\},$$

where

$$\|v\|_{X_\psi(T)} := \sup_{t \in [0,T)} I_\psi(v,t).$$

We note that the usual norm of $X(T)$

$$\|v\|_{X(T)} := \sup_{t \in [0,T)} \{\|v_t(t)\|_{L^2} + \|\nabla v(t)\|_{L^2} + \|v(t)\|_{L^2}\}$$

is smaller than or equal to the weighted norm $\|v\|_{X_\psi(T)}$. We define a mapping $\Phi : B_{T,R} \to X(T)$ such that $u(t,x) = (\Phi v)(t,x)$ is the unique solution to the linear equation

$$\begin{cases} u_{tt} - \Delta u + a(t,x) u_t = f(v) & (t,x) \in (0,T) \times \mathbf{R}^n, \\ (u, u_t)(0,x) = (u_0, u_1)(x) & x \in \mathbf{R}^n. \end{cases} \tag{9.5.12}$$

Here we note that by Propositions 9.15 and 9.16, there exists a unique mild (weak) solution for the above Cauchy problem.

By multiplying the equation (9.5.12) by $e^{2\psi} u_t$, we have

$$\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(u_t^2 + |\nabla u|^2)\right] - \nabla \cdot (e^{2\psi} u_t \nabla u) + e^{2\psi}\left(a(t,x) - \frac{|\nabla \psi|^2}{-\psi_t} - \psi_t\right) u_t^2 = e^{2\psi} f(v) u_t.$$

The assumption of ψ implies

$$\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(u_t^2 + |\nabla u|^2)\right] - \nabla \cdot (e^{2\psi} u_t \nabla u) \le e^{2\psi} f(v) u_t.$$

Integrating this inequality, we obtain

$$\frac{d}{dt}\int_{\mathbf{R}^n}\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)dx \le C\int_{\mathbf{R}^n} e^{2\psi}|v|^p u_t dx.$$

Therefore, we have

$$E(u;t) \le E(u;0) + C\int_0^t\int_{\mathbf{R}^n} e^{2\psi}|v|^p u_t dx d\tau, \tag{9.5.13}$$

here we used the notation

$$E(u;t) := \int_{\mathbf{R}^n} e^{2\psi(t,x)}(u_t(t,x)^2+|\nabla u(t,x)|^2)dx.$$

By the Schwarz inequality, one can find

$$E(u;t) \le E(u;0) + C\int_0^t \left(\int_{\mathbf{R}^n} e^{2\psi}|v|^{2p}dx\right)^{1/2} E(u;\tau)^{1/2}d\tau. \tag{9.5.14}$$

We apply Lemma 9.12 to (9.5.14) and obtain

$$E(u;t)^{1/2} \le E(u;0)^{1/2} + C\int_0^t \left(\int_{\mathbf{R}^n} e^{2\psi}|v|^{2p}dx\right)^{1/2} d\tau.$$

From Lemma 9.10, we have

$$\int_{\mathbf{R}^n} e^{2\psi}|v|^{2p}dx \le C\left(\int_{\mathbf{R}^n} e^{\frac{2}{p}\psi}|v|^2dx\right)^{p(1-\sigma)}\left(\int_{\mathbf{R}^n} e^{\frac{2}{p}\psi}(|\nabla\psi|^2v^2+|\nabla v|^2)dx\right)^{p\sigma}$$

with $\sigma = n(p-1)/(2p)$. We note that the condition $\sigma \le 1$ is equivalent to $p \le n/(n-2)$ when $n \ge 3$. By (9.5.4), we have

$$e^{\frac{p}{2}\psi}|\nabla\psi|^2 \le Ce^{2\psi}.$$

Therefore, it follows that

$$\begin{aligned} E(u;t)^{1/2} &\le E(u;0)^{1/2} + C\int_0^t \|e^{\psi}v\|_{L^2}^{p(1-\sigma)}(\|e^{\psi}\nabla v\|_{L^2}+\|e^{\psi}v\|_{L^2})^{p\sigma}d\tau \\ &\le E(u;0)^{1/2} + CTR^p, \end{aligned} \tag{9.5.15}$$

since $v \in B_{T,R}$. On the other hand, since

$$u = u_0 + \int_0^t u_t d\tau,$$

and $\psi_t \le 0$, we calculate

$$\begin{aligned} \|e^{\psi}u(t)\|_{L^2} &\le \|e^{\psi(0)}u_0\|_{L^2} + \int_0^t \|e^{\psi(\tau)}u_t(\tau)\|_{L^2}d\tau \\ &\le \|e^{\psi(0)}u_0\|_{L^2} + \int_0^t (E(u;0)^{1/2}+CTR^p)d\tau \\ &\le \|e^{\psi(0)}u_0\|_{L^2} + E(u;0)^{1/2}T + CT^2R^p. \end{aligned} \tag{9.5.16}$$

Here we give a remark on the justification of the above argument. We first assume that $(u_0,u_1) \in C_0^\infty(\mathbf{R}^n)\times C_0^\infty(\mathbf{R}^n)$. Then we note that the condition (9.5.5) automatically follows. Moreover, we assume that $v \in C_0^\infty((0,T)\times\mathbf{R}^n)$. Under these assumptions, as we mentioned in Section 9.4.3, we have

$$f(v) \in C([0,T);H^1)\cap C^1([0,T);L^2).$$

Thus, as we described in Section 9.4.2, it is known that there exists a unique strong solution

$$u \in \bigcap_{j=0}^{2} C^{2-j}([0,T); H^j)$$

(see [**25**, Theorem 2.25]). Moreover, by the finite propagation speed property (see [**25**, Theorem 2.7]), it holds that

$$\operatorname{supp} u(t,\cdot) \subset \{x \in \mathbf{R}^n \mid |x| \le K\}$$

for $t \in [0,T)$ with some $K > 0$ (K is determined from the size of the support of u_0, u_1, v). Therefore, for this strong solution u, all of the above calculations make sense. Moreover, the continuity of $\|(u,u_t)(t)\|_{H^1\times L^2}$ induces those of the weighted energy $E(u,t)$ and L^2-norm $\|e^{\psi(t)}u(t)\|_{L^2}$. For general $(u_0,u_1) \in H^1 \times L^2$ and $v \in B_{T,R}$, we take sequences $\{(u_0^{(j)}, u_1^{(j)})\}_{j=1}^{\infty} \subset C_0^{\infty}(\mathbf{R}^n) \times C_0^{\infty}(\mathbf{R}^n)$ and $\{v^{(j)}\}_{j=1}^{\infty} \subset C_0^{\infty}((0,T)\times\mathbf{R}^n)$ such that

$$\lim_{j\to\infty} \int_{\mathbf{R}^n} e^{2\psi(0,x)}(|u_0^{(j)} - u_0|^2 + |\nabla(u_0^{(j)} - u_0)|^2 + |u_1^{(j)} - u_1|^2)dx = 0,$$
$$\lim_{j\to\infty} \int_0^T \left(\int_{\mathbf{R}^n} e^{2\psi(t,x)}(|v^{(j)} - v|^2 + |\nabla(v^{(j)} - v)|^2)dx \right)^q dt = 0$$

for $q \in [1,\infty)$ (see Lemma 9.19). From Lemma 9.20, it also follows that

$$\lim_{j\to\infty} \int_0^T \|e^{\psi(t)}(f(v^{(j)}) - f(v))\|_{L^2} dt = 0.$$

Let $u^{(j)}$ be the corresponding strong solution to the initial data $(u_0^{(j)}, u_1^{(j)})$ and the inhomogeneous term $f(v^{(j)})$. By using the above estimates, we can prove that

$$\lim_{j\to\infty} \left(\sup_{t\in[0,T)} \int_{\mathbf{R}^n} e^{2\psi(t,x)} \left(|u^{(j)} - u|^2 + |\nabla(u^{(j)} - u)|^2 + |u_t^{(j)} - u_t|^2 \right) dx \right) = 0$$

and hence, the estimates (9.5.15) and (9.5.16) still hold for weak solution u. In fact, we first note that the difference $u^{(j)} - u^{(k)}$ satisfies the linear inhomogeneous equation

$$w_{tt} - \Delta w + a(t,x)w_t = f(v^{(j)}) - f(v^{(k)})$$

and has the initial data $w(0) = u_0^{(j)} - u_0^{(k)}$, $w_t(0) = u_1^{(j)} - u_1^{(k)}$. Modifying the derivation of the inequality (9.5.13), we can see that

$$\begin{aligned} E(u^{(j)} - u^{(k)}; t) &\le E(u^{(j)} - u^{(k)}; 0) \\ &\quad + C \int_0^t \int_{\mathbf{R}^n} e^{2\psi(s,x)} \left((f(v^{(j)}) - f(v^{(k)}))(u_t^{(j)} - u_t^{(k)}) \right) dx. \end{aligned}$$

By the Schwarz inequality, we obtain

$$\begin{aligned} &\int_0^t \int_{\mathbf{R}^n} e^{2\psi(s,x)} \left((f(v^{(j)}) - f(v^{(k)}))(u_t^{(j)} - u_t^{(k)}) \right) dx \\ &\le \int_0^t \left(\int_{\mathbf{R}^n} e^{2\psi} \left| f(v^{(j)}) - f(v^{(k)}) \right|^2 dx \right)^{1/2} \left(\int_{\mathbf{R}^n} e^{2\psi} \left| u_t^{(j)} - u_t^{(k)} \right|^2 dx \right)^{1/2} ds \\ &\le C \int_0^t \left(\int_{\mathbf{R}^n} e^{2\psi} \left| f(v^{(j)}) - f(v^{(k)}) \right|^2 dx \right)^{1/2} E(u^{(j)} - u^{(k)}; s)^{1/2} ds. \end{aligned}$$

Therefore, we can apply Lemma 9.12 and deduce that

$$\begin{aligned}E(u^{(j)}-u^{(k)};t)^{1/2} &\le CE(u^{(j)}-u^{(k)};0)^{1/2}\\&\quad + C\int_0^t \|e^{\psi(s)}(f(v^{(j)})-f(v^{(k)})\|_{L^2}ds\\&\le CE(u^{(j)}-u^{(k)};0)^{1/2}\\&\quad + C\int_0^T \|e^{\psi(s)}(f(v^{(j)})-f(v^{(k)})\|_{L^2}ds\\&\to 0\end{aligned}$$

uniformly in $t\in[0,T)$ as $j,k\to+\infty$. Moreover, from this, we have

$$\begin{aligned}&\|e^{\psi(t)}(u^{(j)}-u^{(k)})(t)\|_{L^2}\\&\le C\|e^{\psi(0)}(u_0^{(j)}-u_0^{(k)})\|_{L^2} + C\int_0^t \|e^{\psi(s)}(u_t^{(j)}-u_t^{(k)})(s)\|_{L^2}ds\\&\le C\|e^{\psi(0)}(u_0^{(j)}-u_0^{(k)})\|_{L^2} + CT\sup_{s\in[0,T)}\|e^{\psi(s)}(u_t^{(j)}-u_t^{(k)})(s)\|_{L^2}\\&\to 0\end{aligned}$$

uniformly in $t\in[0,T)$ as $j,k\to+\infty$. Therefore, $\{u^{(j)}\}_{j=1}^{\infty}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{X_\psi(T)}$ and we denote the limit function by $\tilde{u}$. Let us prove that $\tilde{u}=\Phi(v)$. Since each $u^{(j)}$ is also weak solution of (9.5.12) with the initial data $(u_0^{(j)},u_1^{(j)})$ and the inhomogeneous term $f(v^{(j)})$, that is,

$$\begin{aligned}&\int_{[0,T)\times\mathbf{R}^n} u^{(j)}(t,x)(\partial_t^2\phi(t,x)-\Delta\phi(t,x)-\partial_t(a(t,x)\phi(t,x)))dxdt\\&\quad=\int_{\mathbf{R}^n}\left\{(a(0,x)u_0^{(j)}(x)+u_1^{(j)}(x))\phi(0,x)-u_0^{(j)}(x)\partial_t\phi(0,x)\right\}dx\\&\quad+\int_{[0,T)\times\mathbf{R}^n} f(v^{(j)})\phi(t,x)dxdt\end{aligned}$$

is true for any $\phi\in C_0^\infty([0,T)\times\mathbf{R}^n)$. Letting $j\to+\infty$, we can see that $\tilde{u}$ is a weak solution of (9.5.12). However, by Propositions 9.15 and 9.16, the weak solution of (9.5.12) is unique. This implies that $\tilde{u}=\Phi(v)$.

Consequently, by the estimates (9.5.15) and (9.5.16), we have

$$\|e^{\psi}u_t(t)\|_{L^2}+\|e^{\psi}\nabla u(t)\|_{L^2}+\|e^{\psi}u(t)\|_{L^2}\le I_0+TE(u;0)^{1/2}+CT(1+T)R^p.$$

We take R satisfying

$$I_0<\frac{R}{2}$$

and then choose T sufficiently small so that

$$\frac{RT}{2}+CT(1+T)R^p<\frac{R}{2}.$$

Then we have $\|u\|_T<R$ and this proves that Φ is a mapping from $B_{T,R}$ to $B_{T,R}$.

Next, we show that Φ is contractive by taking T smaller. Let $v,\overline{v}\in B_{T,R}, u:=\Phi(v),\overline{u}:=\Phi(\overline{v})$ and $w:=u-\overline{u}$. By noting that w is the unique solution for the

Cauchy problem

$$\begin{cases} w_{tt} - \Delta w + a(t,x)w_t = f(v) - f(\overline{v}) \\ w(0,x) = 0, \quad w_t(0,x) = 0 \end{cases} \tag{9.5.17}$$

and (9.5.14) we have

$$E(w;t) \le \int_0^t \int_{\mathbf{R}^n} |f(v) - f(\overline{v})| w_t dx d\tau.$$

By the assumption on f, using the Schwarz inequality and the Hölder inequality, we obtain

$$\begin{aligned} E(w;t) &\le C \int_0^t \int_{\mathbf{R}^n} e^{2\psi} |v(\tau) - \overline{v}(\tau)| (|v(\tau) + |\overline{v}(\tau)|)^{p-1} |w_t(\tau)| dx d\tau \\ &\le C \int_0^t \left(\int_{\mathbf{R}^n} e^{2\psi} w_t^2(\tau) dx \right)^{1/2} \\ &\times \left(\int_{\mathbf{R}^n} e^{2\psi} |v(\tau) - \overline{v}(\tau)|^2 (|v(\tau)| + |\overline{v}(\tau)|)^{2(p-1)} dx \right)^{1/2} d\tau \\ &\le C \int_0^t E(w;\tau)^{1/2} \| e^{\frac{2}{p}\psi} (v - \overline{v}) \|_{L^{2p}} \| e^{\frac{2}{p}\psi} (|v(\tau)| + |\overline{v}(\tau)|) \|_{L^{2p}}^{p-1} d\tau. \end{aligned}$$

We use Lemma 9.12 again and have

$$E(u;t)^{1/2} \le C \int_0^t \| e^{\frac{2}{p}\psi} (v(\tau) - \overline{v}(\tau)) \|_{L^{2p}} (\| e^{\frac{2}{p}\psi} v(\tau) \|_{L^{2p}} + \| e^{\frac{2}{p}\psi} \overline{v} \|_{L^{2p}})^{p-1} d\tau.$$

By Lemma 9.10, the integrands in the right-hand side of the inequality above are estimated as

$$\begin{aligned} \| e^{\frac{2}{p}\psi} (v(\tau) - \overline{v}(\tau)) \|_{L^{2p}} &\le C \| e^{\frac{2}{p}\psi} (v(\tau) - \overline{v}(\tau)) \|_{L^2}^{1-\sigma} \| \nabla (e^{\frac{2}{p}\psi} (v(\tau) - \overline{v}(\tau))) \|_{L^2}^{\sigma} \\ &\le C \| v - \overline{v} \|_{X_\psi(T)}, \\ \| e^{\frac{2}{p}\psi} v(\tau) \|_{L^{2p}} &\le C \| e^{\frac{2}{p}\psi} v(\tau) \|_{L^2}^{1-\sigma} \| \nabla (e^{\frac{2}{p}\psi} v(\tau)) \|_{L^2}^{\sigma} \\ &\le C \| v \|_{X_\psi(T)}. \end{aligned}$$

Thus, we have

$$E(w;t)^{1/2} \le C \| v - \overline{v} \|_{X_\psi(T)} (2R)^{p-1} T.$$

On the other hand, since

$$\begin{aligned} \| e^{2\psi(t)} w(t) \|_{L^2} &\le \int_0^t \| e^{2\psi(\tau)} w_t(\tau) \|_{L^2} d\tau \\ &\le C(2R)^{p-1} T^2 \| v - \overline{v} \|_{X_\psi(T)}, \end{aligned}$$

we obtain

$$\| u - \overline{u} \|_{X_\psi(T)} \le C R^{p-1} T^2 \| v - \overline{v} \|_{X_\psi(T)}.$$

Now we choose T sufficiently small so that

$$C R^{p-1} T^2 < \frac{1}{2}.$$

Then we have

$$\| u - \overline{u} \|_{X_\psi(T)} \le \frac{1}{2} \| v - \overline{v} \|_{X_\psi(T)}, \tag{9.5.18}$$

which implies that Φ is a contraction mapping. At last we take the sequence $\{u^{(k)}\}_{k=0}^{\infty}$ such that

$$u^{(0)}(t,x) := u_0(x), \quad u_0 \in B_{T,R}$$
$$u^{(k)}(t,x) := (\Phi u^{(k-1)})(t,x), \quad k = 1, 2, \dots .$$

That is

$$\begin{cases} u_{tt}^{(k)} - \Delta u^{(k)} + a(t,x)u_t^{(k)} = |u^{(k-1)}|^p, \\ u^{(k)}(0,x) = u_0(x), \quad u_t^{(k)}(0,x) = u_1(x). \end{cases}$$

By (9.5.18) and the fact $\|v\|_{X(T)} \le \|v\|_{X_\psi(T)}$, $\{u^{(k)}\}$ is a Cauchy sequence in $X(T)$. Hence there exists $u \in X(T)$ satisfying

$$u^{(k)} \to u \quad \text{in} \quad C([0,T); H^1(\mathbf{R}^n)),$$
$$u_t^{(k)} \to u_t \quad \text{in} \quad C([0,T); L^2(\mathbf{R}^n))$$

as $k \to \infty$, and so, u becomes a weak solution to (9.5.1). Moreover, we can see that $u \in B_{T,R}$, that is

$$\sup_{t\in[0,T)} I_\psi(t,u) \le R. \tag{9.5.19}$$

Indeed, we take an arbitrary small number $\varepsilon > 0$ and a subsequence $\{u^{(k(j))}\}_{j=1}^{\infty}$ of $\{u^{(k)}\}_{k=1}^{\infty}$ such that

$$\|u^{(k(j))} - u^{(k(j-1))}\|_{X_\psi(T)} \le \varepsilon 2^{-j}.$$

By noting that

$$u = u^{(k(0))} + \sum_{j=1}^{\infty} (u^{(k(j))} - u^{(k(j-1))}),$$

we can deduce that

$$\|u\|_{X_\psi(T)} \le \|u^{(k(0))}\|_{X_\psi(T)} + \sum_{j=1}^{\infty} \|u^{(k(j))} - u^{(k(j-1))}\|_{X_\psi(T)} \le R + \varepsilon.$$

Since ε is arbitrary, we have $\|u\|_{X_\psi(T)} \le R$. Thus, we obtain (9.5.19).

Next, we prove the uniqueness of the solution of (9.5.1) (this is also obtained from the proof of Proposition 9.18). By the derivation of (9.5.19), if u and $\overline{u}$ are solutions of (9.5.18), it is easy to see that

$$\|u - \overline{u}\|_{X_\psi(T)} \le \frac{1}{2}\|u - \overline{u}\|_{X_\psi(T)}.$$

Hence the uniqueness of the solution is obtained.

Finally, if the lifespan of the solution

$$T^* := \sup\{T > 0 \mid u \in X(T) \text{ solves (9.5.1)}, \ \|u\|_T < \infty\}$$

is finite, then the weighted energy of the solution blows up at T^*:

$$\liminf_{t\to T^*} (\|e^\psi u_t(t)\|_{L^2} + \|e^\psi \nabla u(t)\|_{L^2} + \|e^\psi u(t)\|_{L^2}) = \infty.$$

Because, if

$$\liminf_{t\to T^*} (\|e^\psi u_t(t)\|_{L^2} + \|e^\psi \nabla u(t)\|_{L^2} + \|e^\psi u(t)\|_{L^2}) =: M < \infty,$$

then there exists a time sequence $\{t_m\}_{m\in\mathbf{N}}$ tending to T^* as $m \to \infty$ and such that

$$\sup_{m\in\mathbf{N}} (\|e^\psi u_t(t_m)\|_{L^2} + \|e^\psi \nabla u(t_m)\|_{L^2} + \|e^\psi u(t_m)\|_{L^2}) \le M + 1.$$

The argument before shows that there exists $T(M+1) > 0$ such that the solution $u(t)$ can be extended on the interval $[t_m, t_m + T(M+1)]$ for any m. By taking m sufficiently large so that $t_m \geq T^* - (1/2)T(M+1)$, the solution $u(t)$ can be extended on $[T^*, T^* + (1/2)T(M+1)]$. This contradicts the definition of T^*. Thus, we complete the proof. $\square$

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Acknowledgements

First of all, I would like to express the deepest appreciation to Professor Tatsuo Nishitani. He gave me helpful advice and warm encouragements. Without his guidance and help, this thesis would not have been possible.

I am deeply grateful to Professor Kenji Nishihara from Waseda University. He suggested many problems and ideas. Actually, some of the results described in this thesis are based on discussions with him.

Special thanks go to Professor Michael Reissig from Technische Universität Bergakademie Freiberg for his generous hospitality during my stay at Freiberg.

I would like to thank to my coauthors Mr. Masahiro Ikeda and Mr. Shigehiro Sakata. Mr. Ikeda told me a method for estimating the lifespan of solutions to damped wave equations. The result in Chapter 7 is due to the discussion with him. Mr. Sakata suggested the study of the geometric property of solutions to the damped wave equation. The result in Chapter 8 is based on the discussion with him.

I would also like to show my gratitude to Professors Akitaka Matsumura, Nakao Hayashi, Shin-ichi Doi, Soichiro Katayama, Hideaki Sunagawa, Naohito Tomita, Hideyuki Miura, Hisashi Nishiyama, Chunhua Li, Ryo Takahashi and Yohei Fujishima for their fruitful suggestions and generous support.

I am grateful to Professor Ryo Ikehata from Hiroshima University for giving me constructive comments and warm words of encouragement.

My appreciation also goes to the office administrators of the Department of Mathematics, Graduate School of Science, Osaka University for the excellent working condition. Thanks to their perfect work, I could concentrate on my study and complete this thesis.

I am also thankful to my colleagues Hideo Ueda, Yohei Tsutsui, Kiyotaka Suzaki, Ryosuke Mineyama, Akihiro Higashitani, Ryoichi Kase, Mai Fujita, Tomoyuki Niizato, Mamoru Okamoto, Hayato Miyazaki, Yuusuke Sugiyama, Tomonari Watanabe and Kyouhei Wakasa for their suggestions, constructive comments, mathematical discussion and warm encouragement.

Finally, I want to express my sincere thanks to my family for their permanent support.

Yuta Wakasugi

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