Testing some scenarios in modified gravity

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Abstract

Modified gravity theories have emerged as an alternative for solving the dark energy and dark matter problems by introducing additional degrees of freedom inside the standard gravitational theory. These extra degrees of freedom might be problematic at the moment of analyzing some scenarios where they can reproduce some pathologies, generally absent inside the standard formulation of gravity (General Relativity). In this document, we analyze two modifications of gravity. The first one corresponds to the de-Rham-Gabadadze-Tolley (dRGT) theory of non-linear massive gravity which is a candidate for solving the dark energy problem. In this theory we study the perturbative behavior of the Schwarzschild de-Sitter (S-dS) solution with one free parameter satisfying $\beta = \alpha^2$. We find that the linear perturbation equations become identical to those for the vacuum Einstein theory when they are expressed in terms of the gauge-invariant variables. This implies that this black hole is stable in the dRGT theory as far as the spacetime structure is concerned in contrast to the case of the bi-Schwarzschild solution in the bi-metric theory. However, we have also found a pathological feature that the general solution to the perturbation equations contain a single arbitrary function of spacetime coordinates. This implies a degeneracy of dynamics in the Stückelberg field sector at the linear perturbation level in this background. Physical significance of this degeneracy depends on how the Stückelberg fields couple to the observable ones. As dRGT is not able to reproduce dark matter effects, we analyzed a second model. It corresponds to one of the proposed non-local models of gravity. The model was used before in order to recreate screening effects for the cosmological constant (Λ) value. Here we analyze the possibility of reproducing dark matter effects in that formulation. Although the model in the weak-field approximation (in static coordinates) can reproduce the field equations in agreement with the AQUAL Lagrangian, the solutions are scale dependent and cannot reproduce the observed dark matter dynamics in agreement with the Modified Newtonian Dynamics (MOND) proposed by Milgrom.

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Chapter 1

Introduction

The alternative theories of gravity have emerged as possible explanations for the observed accelerated expansion of the universe [1]. In other cases, they also attempt to explain the dark matter effects by introducing new degrees of freedom like scalar or vector components in addition to the tensorial one [2, 3]. These models of course do not solve the problem completely since they introduce new parameters which anyway we have to tune in order to agree with the observations. For now it seems that every single observation agrees with the fact that the Einstein theory of gravity with the addition of a cosmological constant (Λ) and dark matter is the appropriate one and no evidence for new degrees of freedom have been observed at all in the current experiments [1, 4, 5]. However, it has been difficult to find a natural explanation for the observed small value of Λ in comparison with what is expected from the zero point quantum fluctuations if we have an ultraviolet (UV) cut-off at the Planck scale [1]. It is at this point that the alternative theories of gravity might be important in order to provide possible scenarios for solving naturally the observed value of Λ. Among the most popular suggestions as alternatives to gravity, we have Massive Gravity theories, Modified Gravity (MOG) and Non-local gravity theories [2, 3, 6, 7, 8, 9]. All of them have something in common, namely, the introduction of new degrees of freedom. It is however possible to introduce non-localities without new degrees of freedom as suggested by Deser and colleagues if the non-localities come from quantum fluctuations or from another effect [6]. In this document we focus principally in two models. The first one is the dRGT non-linear massive gravity theory which is a candidate for solving the dark energy problem [10, 11]. The second model, perhaps not so attractive as dRGT, corresponds to one of the many proposals for introducing non-localities inside the Einstein-Hilbert action [8, 9]. In this particular model, the non-localities are equivalent to the introduction of new degrees of freedom. If dRGT is supposed to be a real theory of nature, it must be consistent with all the observed features. In particular, it must be consistent with the 'observed' existence of astrophysical black holes. In many cases, this requirement leads to non-trivial constraints. For example, it was recently claimed that the bi-Schwarzschild solution is unstable against a spherically symmetric perturbation in the bi-metric theory of gravity [12]. Motivated by this, the stability of the Schwarzschild-de Sitter black hole was analyzed in the framework of the linear massive gravity theory by Brito, Cardoso and Pani[14, 13]. They found that the black hole is unstable generically, but becomes stable when the mass of the graviton takes the particular value $m^2 = 2\Lambda/3$. In this case, the theory is inside the regime of partially massless gravity, where the Vainshtein mechanism seems to be unnecessary since the vDVZ discontinuity does not appear anymore[15]. However, it has been demonstrated that the partially massless theories of gravity have several problems of consistency [16]. In the present paper, we analyze the stability of the Schwarzschild-de Sitter solution in the framework of the non-linear dRGT massive theory of gravity. We do not introduce the cosmological constant as an extra parameter of the theory, but instead, we utilize the fact that the Schwarzschild-de Sitter black hole is an exact solution to the non-linear dRGT theory if the parameters $\alpha = 1 + 3\alpha_3$ and $\beta = 3(\alpha_3 + 4\alpha_4)$ of the theory satisfy the relation $\beta = \alpha^2$. For this parameter choice, the mass term of the theory behaves exactly as the cosmological constant term in the Einstein theory for a spherically symmetric geometry as pointed out by Berezhiani et al[17]. We exhaust all Schwarzschild-de Sitter-type solutions to the non-linear dRGT theory in the unitary gauge for the Stückelberg fields assuming $\beta = \alpha^2$. We find a family of solutions that are gauge equivalent to the standard Schwarzschild-de Sitter solution if we neglect the non-trivial transformation of the Stückelberg fields. In the massive gravity theory, they should be regarded as different solutions because if the metric are put into the standard Schwarzschild-de Sitter form, the Stückelberg fields behave differently. The solution obtained in [17] is one solution in this family that is regular at the future horizon. There exists no solution that is regular both at the future and the past horizons. We consider linear perturbations of this background solution in the framework of the nonlinear dRGT theory only assuming the parameter relation $\beta = \alpha^2$. Hence, we generally expect to obtain perturbation equations that are different from those in the Einstein theory with the cosmological constant. In fact, we do if we do not impose the constraint coming from the Bianchi identity on the mass term. However, when we impose that constraint, the extra terms are required to vanish. Hence, we obtain the perturbation equations that are identical to those in the Einstein theory with a cosmological constant and some additional constraints on the metric perturbation variables that correspond to the gauge-dependent parts in the Einstein theory. From this result and the Birkhoff theorem for the Einstein theory, we can easily find the general solution to the perturbation equations and deduce the stability of the black hole against linear perturbations concerning the spacetime structure. However, we also find that this general solution contains an arbitrary function of the spacetime coordinates that reduces to a part of the gauge transformation freedom in the absence of the Stückelberg fields. In the gauge in which the background metric takes the standard Schwarzschild-de Sitter form, this freedom goes to the Stückelberg fields. Hence, we cannot determine the behavior of the fields only by initial data. Along with this general argument, we point out that the general solution to the vector-type perturbation equations contains a family of stationary modes that correspond the rotation of a black hole in the Einstein theory. Regarding the non-local model of gravity, the extra degrees of freedom inside this model might modify the observed galactic dynamics. Then it is natural to ask whether or not some of these models are able to reproduce dark matter effects. The model which recreates the observed galactic dynamics with one fit parameter is MOND, originally proposed by Milgrom [18, 19, 20]. In this paper I take an already suggested non-local gravity model

which introduce two scalar fields (one non-dynamical) in order to create an screening effect of the Λ value and I then compare it with the MOND results in agreement with the AQUAL-like equation $\nabla \cdot \left(\mu\left(\frac{|\nabla\Phi|}{a_0}\right)\nabla\Phi\right) = 4\pi G\rho$ [21] in order to reproduce the dark matter effects at least for the observed galaxy rotation curves. The AQUAL equation provides the appropriate predictions for the extragalactic phenomenology even if the AQUAL model itself cannot be realistic since it provides unphysical results [22, 23, 24]. It is however already known that an appropriate Relativistic version of MOND must reproduce the AQUAL equation in the weak field approximation [21].
In this document we find that a non-local model of gravity can reproduce the same AQUAL equations in agreement with MOND, but not the same dynamics since the interpolating function parameter μ in this model is a scale-dependent quantity. This is the case because μ depends on the potential ϕ rather than on the acceleration $\nabla\phi$. I am not concerned about the origin of the non-localities in this paper [7]. The thesis is organized as follows: In chapter (2), I review the weak field approximation of the Einstein-Hilbert action and then analyze the propagation of gravitational waves inside this formalism [25, 26]. I use this chapter as an introduction for the formalism of massive gravity. Basically the Einstein-Hilbert action expanded up to second order corresponds (in massive gravity language) to the most basic potential without interaction between the fiducial metric and the dynamical one. The obtained potential has however undesirable properties as they are explained in [27]. In chapter (3), I make a review of the linear formulation of massive gravity known as the Fierz-Pauli theory. I then explain the first pathology found inside these formulations, namely, the vDVZ discontinuity. In chapter (4), I make a brief review of the non-linear formulation of massive gravity and then I analyze the Schwarzschild de-Sitter solution. After then the black hole stability is studied. Finally I make a brief comparison with the bi-gravity formulation. In chapter (5), I introduce another alternative for modifying gravity and then I analyze the possibility for finding dark matter effects inside the formulation. In chapter (6), I conclude.

Chapter 2

Weak field approximation with Λ

There is usually the question of whether the cosmological constant in the weak field approximation corresponds to a graviton mass. This is not the case as it is explained in [27]. It is very well known that the most general covariant action including Λ is [28]

$$S_{EH} = \int d^4x\sqrt{-g}(R - 2\Lambda). \tag{2.1}$$

The field equations obtained from this action become

$$R_{\mu\nu} = -\Lambda g_{\mu\nu}. \tag{2.2}$$

These equations are just equivalent to the standard Einstein equations in vacuum and given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \Lambda g_{\mu\nu}, \tag{2.3}$$

as can be easily demonstrated. In weak field approximation, eq. (2.2) becomes [29]

$$\Box h_{\mu\nu} - \partial^\lambda\partial_\mu h_{\lambda\nu} - \partial^\lambda\partial_\nu h_{\lambda\mu} + \partial_\mu\partial_\nu h = -2\Lambda g_{\mu\nu}. \tag{2.4}$$

These equations are equivalent to

$$\Box h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\sigma\partial_\nu h^\sigma{}_\mu - \eta_{\mu\nu}(\Box h - \partial_\sigma\partial_\alpha h^{\sigma\alpha}) - 2\Lambda\left(\eta_{\mu\nu} - h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h\right) = 0. \tag{2.5}$$

The previous results can be derived directly from the Einstein-Hilbert action expanded up to second order [30]

$$\begin{aligned}\mathcal{L} = -2\Lambda\left(1 + \frac{1}{2}h - \frac{1}{4}h_{\alpha\beta}h^{\alpha\beta} + \frac{1}{8}hh\right) - \frac{1}{4}\partial_\nu h_{\alpha\beta}\partial^\nu h^{\alpha\beta} + \frac{1}{4}\partial_\mu h\partial^\mu h - \frac{1}{2}\partial_\beta h\partial_\mu h^{\beta\mu} \\ + \frac{1}{2}\partial_\alpha h_{\nu\beta}\partial^\nu h^{\alpha\beta}.\end{aligned} \tag{2.6}$$

We can rewrite the action defined in eq. (2.1) as

$$S=\int d^4x\sqrt{-g}\left(R+U_1\right), \tag{2.7}$$

where I introduce the effective potential term U_1 inside this action. In fact, the simplest potential is given by the cosmological constant itself. In the action (2.7), the kinetic term of the graviton field is contained inside $\sqrt{-q}R$ and represents contributions of the form

$$\sqrt{-g}R\to\partial h\partial h. \tag{2.8}$$

The second term is the simplest possible potential and it can be expanded as

$$\sqrt{-g}=1+\frac{h}{2}+\frac{h^2}{8}-\frac{h_{\mu\nu}^2}{4}+..., \tag{2.9}$$

up to some multiplicative factor related to Λ. This previous analysis, is related to a spin 2 field propagating over some background. In the simplest case, the background is selected to be the Minkowskian one. As the action (2.7) provides the dynamics of a spin 2 particle, then the term $\sqrt{-g}U_1$ cannot be considered as a massive term. The only possibility for introducing a massive term is by providing a second metric inside the formalism. This is true for the non-linear case. For the linear case, the second metric cannot be noticed since it is simply given by the Minkowskian background. In fact, the Fierz-Pauli action is a special case of the non-linear version of massive gravity as I will explain in forthcoming sections. The result (2.9) is obtained in detail in appendix (A).

2.1 Propagation of gravitational waves in an asymptotically de-Sitter space

The results of the previous section can be used if we want to analyze the propagation of gravitational waves in an asymptotically de-Sitter space [25, 26]. If we use the weak field approximation, the metric can be written as $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ [29], where $\eta_{\mu\nu}$ is the Minkowski metric. The field equations in the weak field approximation and with a positive Λ, after imposing the de-Donder gauge $\partial^\mu h_{\mu\nu}=\frac{1}{2}\partial_\nu h$ become [25, 26, 31]

$$\Box h_{\mu\nu}=-16\pi GS_{\mu\nu}-2\Lambda\eta_{\mu\nu}, \tag{2.10}$$

where we ignore terms of order Λh and the source term $S_{\mu\nu}=T_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}T$ is inckuded. It is just the trace reversed version of the energy-momentum tensor. The full set of solutions for $h_{\mu\nu}$ can be expressed as

$$h_{00}=e_{00}(\vec{r},\omega)e^{ikx}+c.c-\Lambda t^2,\qquad h_{0i}=e_{0i}(\vec{r},\omega)e^{ikx}+c.c+\frac{2}{3}\Lambda tx_i, \tag{2.11}$$
$$h_{ij}=e_{ij}(\vec{r},\omega)e^{ikx}+c.c+\Lambda t^2\delta_{ij}+\frac{1}{3}\Lambda\epsilon_{ij},$$

where $\epsilon_{ij} = x_i x_j$ for $i \neq j$ and 0 otherwise. The full solution of course respects the de-Donder condition $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h$. From the full solution, it can be observed that the particular one corresponds to the de-Sitter background. This is the solution obtained for the graviton in the massless limit ($m \to 0$) as can be shown from the results obtained in [32]. Additionally, in agreement with [25, 26], the energy-momentum (tensor) carried by the gravitational waves with a positive Λ, is given by

$$\hat{t}_{\mu\nu} = t_{\mu\nu} - \frac{1}{8\pi G}\Lambda h_{\mu\nu}, \tag{2.12}$$

where $t_{\mu\nu}$ is [29]

$$t_{\mu\nu} \equiv \frac{1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - R^{(1)}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}R^{(1)}\right), \tag{2.13}$$

and it is obtained from the second-order contributions to the Einstein's equations given by

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} - \Lambda\eta_{\mu\nu} = -8\pi G(T_{\mu\nu} + \hat{t}_{\mu\nu}). \tag{2.14}$$

If we expand up to second order in h, then $t_{\mu\nu}$ as it is defined in eq. (2.13) is explicitly

$$t_{\mu\nu} = \frac{1}{8\pi G}\left(-\frac{1}{2}h_{\mu\nu}R^{(1)} + \frac{1}{2}\eta_{\mu\nu}h^{\sigma\rho}R^{(1)}_{\sigma\rho} + R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\rho}R^{(2)}_{\sigma\rho}\right) + O(h^3). \tag{2.15}$$

If we take into account that the first order Ricci scalar is given by $R^{(1)} = -\eta_{\mu\nu}\Lambda$, then the Poynting vector corresponding to the background solution becomes [25, 26]

$$\hat{t}_{0i} = \frac{1}{8\pi G}\left(\frac{10}{9}\Lambda^2 t x_i\right), \tag{2.16}$$

where the definition for the second-order Ricci tensor has been used and it is given by

$$\begin{aligned} R^{(2)}_{\mu\kappa} = -\frac{1}{2}h^{\lambda\nu}(\partial_\kappa\partial_\mu h_{\lambda\nu} - \partial_\kappa\partial_\lambda h_{\mu\nu} - \partial_\nu\partial_\mu h_{\lambda\kappa} + \partial_\nu\partial_\lambda h_{\mu\kappa}) + \frac{1}{4}(2\partial_\nu h^\nu{}_\sigma - \partial_\sigma h)(\partial_\kappa h^\sigma{}_\mu + \partial_\mu h^\sigma{}_\kappa - \partial^\sigma h_{\mu\kappa}) \\ - \frac{1}{4}(\partial_\lambda h_{\sigma\kappa} + \partial_\kappa h_{\sigma\lambda} - \partial_\sigma h_{\lambda\kappa})(\partial^\lambda h^\sigma{}_\mu + \partial_\mu h^{\sigma\lambda} - \partial^\sigma h^\lambda{}_\mu). \end{aligned} \tag{2.17}$$

If we now assume a wave moving along the z-direction, then the relevant quantity for us is (remember that after averaging the contributions from $h\Lambda \to 0$ in agreement with [25, 26])

$$< t^{03} > = < t^{03} >_{wave} + < t^{03} >_\Lambda . \tag{2.18}$$

In appendix (B), I explain the justification for the linearity of the superposition when we take the average. The averages are performed over regions of the size of a wavelength and times of the length of a period of the wave [33]. Note, the subscript "wave" refers to the standard contribution without Λ. The critical distance [25, 26], is obtained as $< t^{03} >= 0$. After some calculation, it is given by

$$L_{crit} = \frac{6\sqrt{2}\pi f\hat{h}}{\sqrt{5}} r_\Lambda^2, \tag{2.19}$$

which depends on the frequency and the amplitude of the wave. L_{crit} could in principle be of any order of magnitude. But it becomes the background scale r_Λ after taking into account the condition $r_\Lambda \approx \frac{\lambda}{h}$ [34]. The wave cannot propagate to a distance larger than L_{crit}, it means that the inhomogeneities (in this case GW) eventually disappear and the space goes asymptotically to the de-Sitter one. This is in agreement with deeper studies performed in [35, 36, 37] where the Cosmic No-hair Conjecture (CNC) in the presence of GWs was confirmed.

2.2 Polarization analysis

In general, any plane wave solution can be expressed in the form

$$h_{\mu\nu} = e_{\mu\nu} exp(ik_\sigma x^\sigma) + e^*_{\mu\nu} exp(-ik_\sigma x^\sigma), \tag{2.20}$$

where $e_{\mu\nu}$ is the polarization tensor [29]. The homogeneous solution (in vacuum) of eq. (2.10), has to satisfy

$$k_\alpha k^\alpha = 0. \tag{2.21}$$

This means that the massless graviton propagates at the light velocity. The de-Donder condition $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h$ in momentum space becomes

$$k_\mu e^\mu{}_\nu = \frac{1}{2} k_\nu e, \tag{2.22}$$

where e is the trace for the polarization tensor. The polarization tensor $e_{\mu\nu}$ must be symmetric because $h_{\mu\nu}$ is already symmetric. Then in principle we have 10 degrees of freedom. However eq. (2.22) represents four relations between the components of the polarization tensor, so we only would have 6 independent components. From these six components, only two of them represent physical degrees of freedom [29]. If in a local transformation of coordinates, we select

$$\epsilon^\mu(x) = i\epsilon^\mu exp(ik_\sigma x^\sigma) - i\epsilon^{\mu *} exp(-ik_\sigma x^\sigma). \tag{2.23}$$

Then, under local gauge transformations, the plane wave solution (2.20) becomes

$$\begin{aligned} h'_{\mu\nu} = {} & e_{\mu\nu} exp(ik_\sigma x^\sigma) + e^*_{\mu\nu} exp(-ik_\sigma x^\sigma) + \\ & \epsilon_\mu k_\nu exp(ik_\sigma x^\sigma) + \epsilon^*_\mu k_\nu exp(-ik_\sigma x^\sigma) + \epsilon_\nu k_\nu exp(ik_\sigma x^\sigma) + \epsilon^*_\nu k_\nu exp(-ik_\sigma x^\sigma). \end{aligned} \tag{2.24}$$

Equivalently

$$h'_{\mu\nu} = e'_{\mu\nu} exp(ik_\sigma x^\sigma) + e'^*_{\mu\nu} exp(-ik_\sigma x^\sigma), \tag{2.25}$$

with $e'_{\mu\nu} = e_{\mu\nu} + k_\mu \epsilon_\nu + k_\nu \epsilon_\mu$. Then we can conclude that $e'_{\mu\nu}$ and $e_{\mu\nu}$ represent the same physical situation for arbitrary values of the four parameters ϵ_μ [29]. At the end only two degrees of freedom are relevant for the polarization tensor. Consider for example a wave traveling in the +z-direction, with wave vector given by

$$k^\mu \rightarrow (k, 0, 0, k), \tag{2.26}$$

with $k^3 = k^0 = k > 0$ as in the case of a wave moving at the light velocity and as a consequence satisfying the condition (2.21). If we use the de-Donder condition in momentum space given in (2.22), then

$$k^3 e_{3\nu} + k^0 e_{0\nu} = \frac{1}{2} k_\nu ({e^0}_0 + {e^1}_1 + {e^2}_2 + {e^3}_3), \tag{2.27}$$

where we have taken into account that the only non-vanishing components of the wave vector are $k^3 = k^0 = k$. For $\nu = 1$, we have

$$k^3 e_{31} + k^0 e_{01} = 0 = e_{31} + e_{01}. \tag{2.28}$$

Since $k^3 = k^0$; for $\nu = 2$, then

$$k^3 e_{32} + k^0 e_{02} = 0 = e_{32} + e_{02}. \tag{2.29}$$

When $\nu = 0$, we have

$$k^3 e_{30} + k^0 e_{00} = \frac{1}{2} k_0 ({e^0}_0 + {e^1}_1 + {e^2}_2 + {e^3}_3). \tag{2.30}$$

This is equivalent to

$$k^3 e_{30} + k^0 e_{00} = -\frac{1}{2} k^0 (-e_{00} + e_{11} + e_{22} + e_{33}). \tag{2.31}$$

And from from eq. (2.26), we get

$$-e_{30} - e_{00} = \frac{1}{2} (-e_{00} + e_{11} + e_{22} + e_{33}). \tag{2.32}$$

Finally, if $\nu = 3$, we have

$$k^3 e_{33} + k^0 e_{03} = \frac{1}{2} k^3 (-e_{00} + e_{11} + e_{22} + e_{33}). \tag{2.33}$$

And by the same arguments as before we get

$$e_{33} + e_{03} = \frac{1}{2} (-e_{00} + e_{11} + e_{22} + e_{33}). \tag{2.34}$$

We can find the physical degrees of freedom by performing a rotation of the system. From the previous equations, we have the partial results

$$e_{01} = -e_{31}, \quad e_{02} = -e_{32}, \quad e_{03} = -\frac{1}{2}(e_{33} + e_{00}), \quad e_{22} = -e_{11}. \tag{2.35}$$

Under local gauge transformation, the six components of $e_{\mu\nu}$ transform in agreement with

$$\begin{aligned} e'_{11} = e_{11}, \quad e'_{12} = e_{12}, \\ e'_{13} = e_{13} + k\epsilon_1, \quad e'_{23} = e_{23} + k\epsilon_2, \\ e'_{33} = e_{33} + 2k\epsilon_3, \quad e'_{00} = e_{00} - 2k\epsilon_0. \end{aligned} \tag{2.36}$$

Thus only e_{11} and e_{12} have absolute physical meaning, since the other components can be sent to zero with an appropriate choice of k. If we select the following values for ϵ as a function of k

$$\epsilon_1 = -\frac{e_{13}}{k}, \quad \epsilon_2 = -\frac{e_{23}}{k}, \quad \epsilon_3 = -\frac{e_{33}}{2k}, \quad \epsilon_0 = \frac{e_{00}}{2k}. \tag{2.37}$$

Then the unphysical components vanish. The spin-2 behavior is obtained if we select the appropriate linear combination of the tensor components e_{11} and e_{12}. The rotation around the z-axis is just a special case of a Lorentz transformation given in the following form

$$\begin{aligned} {R_1}^1 = cos\theta, \quad {R_1}^2 = sin\theta, \\ {R_2}^1 = -sin\theta, \quad {R_2}^2 = cos\theta, \\ {R_3}^3 = {R_0}^0 = 1, \quad other \ \ {R_\mu}^\nu = 0. \end{aligned} \tag{2.38}$$

As ${R_\mu}^\nu k_\nu = k_\mu$, then the only effect of a Lorentz transformation is to convert the polarization tensors $e_{\mu\nu}$ into

$$e'_{\mu\nu} = {R_\mu}^\rho {R_\nu}^\sigma e_{\rho\sigma}. \tag{2.39}$$

The non-vanishing components of the polarization tensor, are in agreement with eqns. (2.35) and (2.36); e_{11}, e_{12} and $e_{22} = -e_{11}$, then from eq. (2.39), we get

$$e'_{11} = {R_1}^\rho {R_1}^\sigma e_{\rho\sigma}. \tag{2.40}$$

Summing repeated indices, we get

$$e'_{11} = {R_1}^1 {R_1}^1 e_{11} + 2{R_1}^1 {R_1}^2 e_{12} + {R_1}^2 {R_1}^2 e_{22}. \tag{2.41}$$

Eq. (2.38) inside this previous result gives

$$e'_{11} = (cos^2\theta - sin^2\theta)e_{11} + 2cos\theta sin\theta e_{12}. \tag{2.42}$$

Doing the same process for e_{12}, we have

$$e'_{12} = {R_1}^\rho {R_2}^\sigma e_{\rho\sigma}. \tag{2.43}$$

Again if we sum over repeated indices, we get

$$e'_{12} = {R_1}^1{R_2}^1 e_{11} + {R_1}^1{R_2}^2 e_{12} + {R_1}^2{R_2}^1 e_{21} + {R_1}^2{R_2}^2 e_{22}. \tag{2.44}$$

Eq. (2.38) inside this expression and we get

$$e'_{12} = (cos^2\theta - sin^2\theta)e_{12} - 2cos\theta sin\theta e_{11}. \tag{2.45}$$

If we define the following linear combination

$$e_\pm \equiv e_{11} \mp ie_{12}, \tag{2.46}$$

then under the transformations (2.42) and (2.45), $e_\pm$ transforms as

$$e'_\pm = e'_{11} \mp ie'_{12} = (cos^2\theta - sin^2\theta)e_{11} + 2cos\theta sin\theta e_{12} \mp i((cos^2\theta - sin^2\theta)e_{12} - 2cos\theta sin\theta e_{11}). \tag{2.47}$$

Regrouping terms, this quantity becomes

$$e'_\pm = (cos^2\theta - sin^2\theta + 2isin\theta cos\theta)e_{11} \mp i(cos^2\theta - sin^2\theta + 2isin\theta cos\theta)e_{12}. \tag{2.48}$$

By factorizing common terms, we get

$$e'_\pm = (cos^2\theta - sin^2\theta + 2isin\theta cos\theta)(e_{11} \mp ie_{12}). \tag{2.49}$$

From definition (2.46), we get

$$e'_\pm = (cos^2\theta - sin^2\theta + 2isin\theta cos\theta)e_\pm = (cos\theta \pm isin\theta)^2 e_\pm. \tag{2.50}$$

Or equivalently

$$e'_\pm = exp(\pm 2i\theta)e_\pm. \tag{2.51}$$

In general, we know that any plane wave that transforms by a rotation of any angle θ about the direction of propagation into

$$\Phi = e^{in\theta}\Phi, \tag{2.52}$$

is said to have helicity n [29]. The helicity for the gravitational wave field is in agreement with the result (2.51) is $n = \pm 2$. Then the gravitational field is a spin-2 field.

2.3 The Λ gauge

The Λ gauge was already explained in [31]. In such a case, Λ is not a source anymore, but its effects will appear as gauge ones. Independent of the selected gauge, we expect to have 2 physical polarization tensors and we expect to obtain the same L_{crit} already found previously. By proving this, we show that the background effects are gauge-independent as it must be.

2.3.1 The equations of motion in the Λ gauge

The field equations in the Λ gauge can be obtained if we introduce the gauge condition $\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h = \Lambda x_\nu$ inside the full weak field version of the Einstein's equations [25, 26, 31]. In this gauge, the equations simplify to

$$\Box h_{\mu\nu} = 0, \tag{2.53}$$

where the solutions for this equation must be obtained after taking into account the gauge condition. They were already found in [31] and repeated here for clarity

$$h_{\mu\nu} = e_{\mu\nu} e^{ikx} + c.c + \frac{\Lambda}{18}\left(4x_\mu x_\nu - \eta_{\mu\nu} x^2\right). \tag{2.54}$$

The first part of this solution is the plane wave contribution. The field equations (eqn. 2.53) in momentum space show the fact the the graviton propagates at the light velocity $k^\mu k_\mu$. The gauge condition, on the other hand, can be written partially in momentum space and is given by

$$k_\mu e^\mu{}_\nu = \frac{1}{2} k_\nu e + \Lambda x_\nu. \tag{2.55}$$

If $h_{\mu\nu}$ does not satisfy the Λ gauge condition, then it is always possible to find some $h'_{\mu\nu}$ that does by performing the appropriate coordinate transformations with [1]

$$\Box \epsilon_\nu \equiv \partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h. \tag{2.56}$$

If the Λ gauge condition is satisfied, then the previous equations can only be reduced to [25, 31]

$$\Box \epsilon_\nu = \Lambda x_\nu. \tag{2.57}$$

The homogeneous solutions for these equations are the standard ones already given in [1]

$$\epsilon^\mu(x) = i\epsilon^\mu e^{ikx} - i\epsilon^{\mu *} e^{-ikx}. \tag{2.58}$$

The particular solutions are

$$\begin{gathered} \epsilon^0_\Lambda = at^3 + br^2 t, \\ \epsilon^i_\Lambda = ct^2 x^i + d(x^i)^3 + ex^i((x^j)^2 + (x^k)^2). \end{gathered}$$

The constants must satisfy the conditions $-a + b = \frac{\Lambda}{6}$ and $-c + 6d + 4e = \Lambda$ in agreement with equation (2.57). We will write the particular solutions as $\epsilon^\mu_\Lambda(x)$. Then, the infinitesimal parameters for the coordinate transformations can be written as

$$\epsilon^\mu(x) = i\epsilon^\mu e^{ikx} - i\epsilon^{\mu *} e^{-ikx} + \epsilon^\mu_\Lambda(x). \tag{2.59}$$

2.3.2 Polarizations in the Λ gauge

If we assume that the wave is propagating along the z direction, namely, $k^1 = k^2 = 0$ and $k^3 = k^0 = k > 0$ [1]; then the relations among the polarizations components can be obtained from eq. (2.55) as

$$e_{01} = -e_{31} + \left(\frac{\Lambda}{\omega}\right) x, \quad e_{02} = -e_{32} + \left(\frac{\Lambda}{\omega}\right) y, \quad e_{03} = -\frac{1}{2}(e_{33} + e_{00}) + \left(\frac{\Lambda}{\omega}\right) z. \tag{2.60}$$

The previous equations correspond to the Λ gauge condition written in the form (2.55) for $\nu = 1, 2, 3$ respectively. For the case $\nu = 0$, we get

$$e_{03} = -\frac{1}{2}(e_{33} + e_{00}) - \left(\frac{\Lambda}{\omega}\right) t. \tag{2.61}$$

If we sum this result with the one obtained in eq. (2.60), we obtain

$$e_{03} = -\frac{1}{2}(e_{33} + e_{00}) - \left(\frac{\Lambda}{\omega}\right) (z - t), \tag{2.62}$$

for a wave traveling along the z-direction. The relevant coordinates are z and t. As the graviton must travel along a light cone, then the assumption $z = t$ is valid and the previous equations just become to be the same as the standard ones obtained in [1]

$$e_{01} = -e_{31}, \quad e_{02} = -e_{32}, \quad e_{03} = -\frac{1}{2}(e_{33} + e_{00}). \tag{2.63}$$

If additionally we analyze the transformations for the polarization tensors [1], then the independent components transform as

$$e'_{11} = e_{11} - 2\left(ct^2 + 3dx^2 + e(y^2 + z^2)\right), \tag{2.64}$$

$$e'_{12} = e_{12} - 4exy, \tag{2.65}$$

$$e'_{13} = e_{13} + k\epsilon_1 - 4exz, \tag{2.66}$$

$$e'_{23} = e_{23} + k\epsilon_2 - 4eyz, \tag{2.67}$$

$$e'_{33} = e_{33} + 2k\epsilon_3 - 2\left(ct^2 + 3dz^2 + e(y^2 + x^2)\right), \tag{2.68}$$

$$e'_{00} = e_{00} - 2k\epsilon_0 - 2\left(3at^2 + br^2\right). \tag{2.69}$$

Observing these expressions for the 6 independent polarization tensors, we might believe that there is a problem here because we are expecting only 2 of them to have physical relevance.

However, if $z = t$ and if additionally we ignore the coordinates x and y since the wave is propagating along z, the previous transformations just become

$$e'_{11} = e_{11} - 2z^2 \left(c + e\right), \tag{2.70}$$

$$e'_{12} = e_{12}, \tag{2.71}$$

$$e'_{13} = e_{13} + k\epsilon_1, \tag{2.72}$$

$$e'_{23} = e_{23} + k\epsilon_2, \tag{2.73}$$

$$e'_{33} = e_{33} + 2k\epsilon_3 - 2z^2 \left(c + 3d\right), \tag{2.74}$$

$$e'_{00} = e_{00} - 2k\epsilon_0 - 2z^2 \left(3a + b\right). \tag{2.75}$$

The conditions $-a+b = \frac{\Lambda}{6}$ and $-c+6d+4e = \Lambda$ must be satisfied. Here we want to keep e_{11} as a physically relevant component for the polarization tensor in agreement with [1]. Then the following additional conditions must be imposed

$$c = -e, \quad -5c + 6d = \Lambda, \tag{2.76}$$

$$c = -3d, \quad d = \frac{\Lambda}{21},$$

$$b = -3a, \quad a = -\frac{\Lambda}{24}.$$

Then the physical relevant components are

$$e'_{11} = e_{11}, \quad e'_{12} = e_{12}. \tag{2.77}$$

The behavior under rotations is still [1]

$$e'_{\pm} = exp(\pm 2i\theta)e_{\pm}, \tag{2.78}$$

with $e_{\pm} = e_{11} \mp ie_{12}$. This is what we were expecting. This result is in agreement with that obtained in [38] where it was found that Λ does not affect the polarization of the GW during its propagation. In [38] Λ only provides an isotropic contribution to the geodesic deviation equation given by

$$\ddot{Z}_1 = \ddot{Z}_2 = \ddot{Z}_3 = \frac{\Lambda}{3} Z_1. \tag{2.79}$$

Here Z is the geodesic deviation coordinate in agreement with [38]. Then if we have a group of particles creating a circle. They will move isotropically keeping the initial shape of the circle. In other words, the polarization is not affected by the presence of Λ. This is valid for Minkowski, de-Sitter and Anti de-Sitter spaces [38].

2.3.3 Power and critical distance in the Λ gauge

In [25, 26], the radiation flux of a gravitational wave when it propagates in an asymptotically de-Sitter space was calculated by taking Λ as an additional source of radiation (de-Donder gauge). Here we want to explore if the same critical distance L_{crit} can be obtained when we take Λ as a gauge effect. We can perform the same calculations as in [26], but this time the first order scalar curvature is $R^{(1)} = 0$ in agreement with the eqn. (2.53) [25]. The effective gravitational Poynting vector in the Λ gauge is then given by

$$\hat{t}_{0i} = \frac{1}{8\pi G}\left(R^{(2)}_{0i} - \Lambda h_{0i}\right) + O(h^3). \tag{2.80}$$

This result differs from the one obtained in [26] only in the first order contribution of the Ricci tensor ($R^{(1)}_{\mu\nu}$) which is zero in the present case. However, this difference is compensated by the gauge effect. If we replace the solutions given in eqn. (2.54), after some standard calculations and after taking an average over a large region of spacetime, we obtain the same result $L_{crit} \approx (f\hat{h})r_\Lambda^2$ as the one obtained in [26], repeated in eqn. (2.19) for clarity. If we introduce the condition $r_\Lambda \approx \frac{\lambda}{h}$ [34], when combined with the critical distance just found previously, implies

$$L_{crit} \approx r_\Lambda, \tag{2.81}$$

and then the so called critical distance found in [26] becomes the background curvature scale [34] and it helps us to evaluate the validity of the GW approach. Here however we interpret this result as a consequence of the power decay rate as the GW propagates inside the background. In other words, the background absorbs the energy of the wave as it propagates. This is in agreement with the CNC since it demonstrates the tendency for the inhomogeneities to be dissipated at large times (distances). It shows the tendency for the space to go asymptotically to the de-Sitter one. In reference [35] the equations for the evolution of GWs in an asymptotically de-Sitter space were written as in the asymptotically flat case but including a viscosity term and a decaying term which is in complete agreement with the present formalism and interpretation. The formalism developed here is however very simple and can be easily extended to massive gravity theories. These previous results are in agreement with the predictions for the stochastic background of gravitational waves as has been explained in [25, 39]. In this thesis I will not focus in such problems but it is important to remark that the future discovery of stochastic background of gravitational waves, would open a new window in physics since it would provide information of the universe when it was at the age 10^{-36} seconds. This can give some ideas about the physics at very high energies [33].

Chapter 3

Linear massive gravity: The Fierz Pauli theory

The idea of introducing mass to the graviton is not new. It was proposed since the beginning of General Relativity [40, 41]. The simplest theory for a non-self interacting graviton is the Fierz-Pauli theory [42]. Fierz and Pauli discovered that it is possible to reproduce a ghost-free theory of massive gravitons by introducing a mass term in the action of the form

$$\mathcal{L}_{mass} = b_1 h_{\mu\nu}^2 + b_2 h^2 = m^2 (h_{\mu\nu}^2 - h^2), \tag{3.1}$$

where $b_1 = -b_2$. Only this combination of coefficients provides a ghost free version of the theory, any other combination corresponds to a ghost [41, 42]. It is well known that the action (2.7) is invariant under the following gauge transformation

$$\delta_g h_{\mu\nu} = \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu. \tag{3.2}$$

However, the mass term in the Fierz-Pauli action given in (3.1) violates this symmetry [41]. Then the Fierz-Pauli action describes a massive graviton. Later we will see how this action can be extended to the non-linear case. For now, we can analyze that only the combination of coefficients given in (3.1) provides the appropriate ghost-free action. The full action for the Fierz-Pauli theory is

$$\mathcal{L} = -\frac{1}{2}\partial_\nu h_{\alpha\beta}\partial^\nu h^{\alpha\beta} + \frac{1}{2}\partial_\mu h \partial^\mu h - \partial_\beta h \partial_\mu h^{\beta\mu} + \partial_\alpha h_{\nu\beta}\partial^\nu h^{\alpha\beta} - \frac{1}{2}m^2 (h^{\mu\nu}h_{\mu\nu} - h^2). \tag{3.3}$$

If we set the massive term like $-\frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - (1-a)h^2)$. By applying the Euler-Lagrange formalism, we obtain the following field equations

$$\frac{\delta S}{\delta h^{\mu\nu}} = \Box h_{\mu\nu} - \partial_\lambda \partial_\mu h^\lambda{}_\nu - \partial_\lambda \partial_\nu h^\lambda{}_\mu + \partial_\mu \partial_\nu h + \eta_{\mu\nu}\left(\partial_\lambda \partial_\sigma h^{\lambda\sigma} - \Box h\right) - m^2 \left(h_{\mu\nu} - \eta_{\mu\nu}(1-a)h\right) = 0. \tag{3.4}$$

For $a = 0$, we recover the Fierz-Pauli tuning. For $a \neq 0$, it is possible to observe that we have a ghost. If we take the divergence of the previous equation, we get the constraint

$$\partial^\mu h_{\mu\nu} - (1-a)\partial_\nu h = 0. \tag{3.5}$$

For $a = 0$, this condition after taking its divergence, is equivalent to the vanishing Ricci scalar $R = 0$. If we introduce this condition inside eq. (3.4), then we get

$$\Box h_{\mu\nu} - (1-2a)\partial_\mu\partial_\nu h - a\eta_{\mu\nu}\Box h - m^2(h_{\mu\nu} - \eta_{\mu\nu}(1-a)h) = 0. \tag{3.6}$$

If we take the trace of this equation, we get

$$-2a\Box h - m^2(-3+4a)h = 0. \tag{3.7}$$

For $a = 0$, this result is the traceless condition $h = 0$ and eq. (3.5) would imply in such a case that $\partial^\mu h_{\mu\nu} = 0$. If we replace eq. (3.7) inside eq. (3.6), then

$$(\Box - m^2)h_{\mu\nu} - (1-2a)\partial_\mu\partial_\nu h + \frac{m^2}{2}\eta_{\mu\nu}(-3+4a)h + m^2(\eta_{\mu\nu}(1-a)h) = 0. \tag{3.8}$$

If $a \neq 0$, the condition $h = 0$ is not obtained. Then we can only use the four constraints obtained from 3.5. In such a case, we have 6 degrees of freedom propagating and then we have a ghost. If $a = 0$, the previous set of equations simplify to

$$(\Box - m^2)h_{\mu\nu} = 0, \qquad \partial^\mu h_{\mu\nu} = 0, \qquad h = 0. \tag{3.9}$$

The second and third equations provide a total of five constraints. As the tensor $h_{\mu\nu}$ has a total of ten components, then there are only five degrees of freedom propagating and there is no ghost for this particular case.

3.1 Another way for counting the degrees of freedom

We can rewrite the Lagrangian (3.3) in terms of canonical variables. The canonical momenta are given by

$$\pi_{ij} = \frac{\delta\mathcal{L}}{\delta\dot{h}_{ij}} = \dot{h}_{ij} - \dot{h}_{kk}\delta_{ij} - 2\partial_{(i}h_{j)0} + 2\partial_k h_{0k}\delta_{ij}. \tag{3.10}$$

If we invert for the velocities, we have

$$\dot{h}_{ij} = \pi_{ij} - \frac{1}{2}\pi_{kk}\delta_{ij} + 2\partial_{(i}h_{j)0}. \tag{3.11}$$

Then the Fierz-Pauli action written in terms of these Hamiltonian variables, becomes

$$\mathcal{L} = \pi_{ij}\dot{h}_{ij} - H + 2h_{0i}(\partial_j\pi_{ij}) + m^2h_{0i} + h_{00}(\vec{\nabla}^2h_{ij} - \partial_i\partial_j h_{ij} - m^2h_{ii}), \tag{3.12}$$

where

$$H = \frac{1}{2}\pi_{ij}^2 - \frac{1}{4}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} + \partial_i h_{ij}\partial_j h_{kk} - \frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \frac{1}{2}m^2(h_{ij}^2 - h_{ii}^2). \quad (3.13)$$

If we consider the case $m = 0$, the components h_{0i} and h_{00} are Lagrange multipliers, enforcing some constraints [41]. It is direct to demonstrate that in fact there are only two degrees of freedom propagating for this case. For $m \neq 0$, h_{0i} is not a Lagrange multiplier anymore. Its equation of motion is

$$h_{0i} = -\frac{1}{m^2}\partial_j \pi_{ij}. \quad (3.14)$$

After plugging back this result to the action (3.12), we obtain the Lagrangian

$$\mathcal{L} = \pi_{ij}\dot{h}_{ij} - H + h_{00}\left(\vec{\nabla}^2 h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii}\right). \quad (3.15)$$

This time, with H defined as

$$\begin{aligned} H = {} & \frac{1}{2}\pi_{ij}^2 - \frac{1}{4}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} + \partial_i h_{ij}\partial_j h_{kk} \\ & -\frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \frac{1}{2}m^2(h_{ij}^2 - h_{ii}^2) + \frac{1}{m^2}(\partial_j \pi_{ij})^2. \end{aligned} \quad (3.16)$$

Although the component h_{00} remains a Lagrange multiplier, the Hamiltonian is not a first class as in the case with $m = 0$ [41]. In fact, we have a secondary constraint from the Poisson bracket with the Hamiltonian $H = \int d^4xH$. The resulting set of two constraints is second class and then we do not have gauge freedom anymore [41].

3.2 The vDVZ discontinuity

It is well known that the Fierz-Pauli theory has a pathology. Namely, if we set the mass of the graviton to zero ($m = 0$), then we do not recover a massless theory as it is supposed to happen. Instead, we recover a massless graviton plus an interaction term representing the coupling between the scalar component and the trace of the energy-momentum tensor [41, 43] if we have a source term $T_{\mu\nu}$. This result cannot be observed in vacuum. It can only be obtained if there is a source. In other words, we have to include the energy-momentum tensor. In the previous section, all the field equations were analyzed in vacuum, then apparently the theory is not sick at this level. The source term included in the Lagrangian is given by

$$\mathcal{L}_{matter} \backsim 2M_{pl}^{-2}T_{\mu\nu}h^{\mu\nu}. \quad (3.17)$$

By adding this previous matter Lagrangian to the vacuum action given in eq. (3.3), the equations of motions become of the form

$$-\varepsilon_{\mu\nu} = -m^2(h_{\mu\nu} - h\eta_{\mu\nu}) + 2M_{pl}^{-2}T_{\mu\nu}, \quad (3.18)$$

with $\varepsilon_{\mu\nu}$ defined in agreement with the kinetic contribution given in eq. (3.4) [32, 43]. If the Bianchi identity is assumed and if we also have conservation of the energy-momentum $\partial^\mu T_{\mu\nu} = 0$, then the following constraint is obtained

$$\partial^\rho h_{\rho\mu} = \partial_\mu h. \tag{3.19}$$

If we take the divergence of this expression, again this is just the vanishing Ricci scalar. In fact, the condition is explicitly

$$\partial^\nu \partial^\mu h_{\mu\nu} - \Box h = 0. \tag{3.20}$$

If we take the trace of eq. (3.18) and taking into account the previous condition (3.20), then we get

$$h = -\frac{2}{3}\frac{T}{m^2 M_{pl}^2}. \tag{3.21}$$

Then even in the presence of a source term, the trace can be determined algebraically and it cannot propagate. It is non-dynamical. If we introduce the results (3.19) and (3.21) inside the field equations (3.18), then we get [43]

$$-(\Box - m^2)h_{\mu\nu} = \frac{2}{M_{pl}^2}\left(T_{\mu\nu} - \frac{1}{3}T\eta_{\mu\nu}\right) + \frac{2}{3}\frac{\partial_\mu \partial_\nu T}{m^2 M_{pl}^2}. \tag{3.22}$$

We can compute the graviton propagator by expanding the graviton field $h_{\mu\nu}$ in a Fourier expansion as

$$h_{\mu\nu}(x^\mu) = \frac{1}{(2\pi)^4}\int d^4k e^{ik_\sigma x^\sigma}\tilde{h}(k^\mu). \tag{3.23}$$

If we make an analogous decomposition for the energy-momentum tensor, then we can find the propagator from the definition

$$\tilde{h}_{\mu\nu} = D^{(m\neq 0)}_{\mu\nu\rho\sigma}\frac{\tilde{T}^{\rho\sigma}}{M_{pl}^2}. \tag{3.24}$$

If we replace the Fourier expansions inside eq. (3.22). The explicit expression for the propagator is

$$D^{(m\neq 0)}_{\mu\nu\rho\sigma} = \frac{1}{k^2 + m^2}\left(\eta_{\rho\mu}\eta_{\sigma\nu} + \eta_{\rho\nu}\eta_{\sigma\mu} - \frac{2}{3}\eta_{\rho\sigma}\eta_{\mu\nu} - \frac{2}{3}\eta_{\rho\sigma}\frac{k_\mu k_\nu}{m^2}\right). \tag{3.25}$$

If we want to find the propagator for the massless case, we have to start from the equation

$$-\varepsilon_{\mu\nu} = 2M_{pl}^{-2}T_{\mu\nu}. \tag{3.26}$$

In such a case, we have a gauge freedom and we can impose the De-Donder gauge condition $\partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu h$. Then the field equations become

$$\Box h_{\mu\nu} = -\frac{2}{M_{pl}^2}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right). \tag{3.27}$$

Note that the field equations are just the same obtained in eq. (2.10) if we set $\Lambda = 0$. If we replace the same Fourier expansion, then we get the massless propagator

$$D^{(m=0)}_{\mu\nu\rho\sigma} = \frac{1}{k^2}\left(\eta_{\rho\mu}\eta_{\sigma\nu} + \eta_{\rho\nu}\eta_{\sigma\mu} - \eta_{\rho\sigma}\eta_{\mu\nu}\right). \tag{3.28}$$

Note that there is a difference between the coefficients of the third term of the propagator of the massive graviton given in eq. (3.25) and the analogous term for the massless propagator given in eq. (3.28) even if the involved terms are independent of the mass of the graviton. The difference marks the origin of the vDVZ discontinuity which was originally found in [44]. The term proportional to $k_\mu k_\nu$ in eq. (3.25) does not appear in the final expression for the amplitude because it vanishes after contraction with the energy-momentum tensor. This is a consequence of the energy-momentum conservation in momentum space. We can compute the formal tree level amplitude of two conserved currents. The amplitude can be defined as [43, 45]

$$A = M_{pl}^2 \int d^4x S^{\mu\nu}(x) h_{\mu\nu}(T)(x), \tag{3.29}$$

where $h_{\mu\nu}(T)$ is the tree level graviton field generated by a conserved source $T_{\mu\nu}$. It is well known that the graviton field is given by

$$h_{\mu\nu}(T)(x) = M_{pl}^{-2}\int d^4x' D_{\mu\nu\rho\sigma}(x-x')T^{\rho\sigma}(x'), \tag{3.30}$$

where $D_{\mu\nu\rho\sigma}(x-x')$ can be taken as the propagator for both, the massive or massless graviton as they have been defined before. If we introduce eq. (3.30) inside (3.29) and using respectively (3.25) and (3.28), then we get

$$A^{m=0} = \int d^4k \frac{2}{k^2}\left(\tilde{S}^{\mu\nu}\tilde{T}_{\mu\nu} - \frac{1}{2}\tilde{S}\tilde{T}\right), \tag{3.31}$$

$$A^{m\neq 0} = \int d^4k \frac{2}{k^2}\left(\tilde{S}^{\mu\nu}\tilde{T}_{\mu\nu} - \frac{1}{3}\tilde{S}\tilde{T}\right), \tag{3.32}$$

where I have used the large momentum approximation $k^2 >> m^2$ and I use the result $k_\mu S^{\mu\nu} = 0$, which is a consequence of the energy-momentum conservation expressed in momentum space. Now we can consider two non-relativistic currents with $\tilde{T}^\mu{}_\nu \propto (\tilde{M}_1, 0, 0, 0)$ and $\tilde{S}^\mu{}_\nu \propto (\tilde{M}_2, 0, 0, 0)$. If we assume that the two sources are separated at a distance smaller than the Compton wavelength, then the amplitude for the exchange of a massive graviton is given by

$$A^{(m\neq 0)} = \frac{4}{3}A^{(m=0)} = \frac{4}{3}\int d^4k \frac{\tilde{M}_1\tilde{M}_2}{k^2}. \tag{3.33}$$

Then even if the graviton mass takes very small values, the amplitude of the massive graviton remains different with respect to the massless one.

3.3 The Stückelberg trick in vectorial theories

There is another way to obtain the vDVZ discontinuity. If we use the Stückelberg trick [46]. First, let's show how the trick works in a vector example. In such a case, the Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + A_\mu J^\mu. \tag{3.34}$$

The field equations are

$$\partial_\gamma F^{\gamma\sigma} = m^2 A^\sigma - J^\sigma. \tag{3.35}$$

If we take the divergence of this expression, then we get the constraint

$$\partial_\gamma A^\gamma = \frac{\partial\gamma J^\gamma}{m^2}. \tag{3.36}$$

If we replace this constraint inside eq. (3.35), then we get

$$(\Box - m^2)A^\sigma = -J^\sigma - \frac{1}{m^2}\partial^\sigma\partial_\gamma J^\gamma. \tag{3.37}$$

If we expand the field and the source terms in a Fourier series, then we can compute the propagator. In such a case, the eq. (3.37) becomes

$$\tilde{A}_\mu = \frac{1}{k^2+m^2}\left(\eta_{\mu\gamma} - \frac{1}{m^2}k_\mu k_\gamma\right)\tilde{J}^\gamma. \tag{3.38}$$

Then it is clear that the propagator is $\tilde{D}_{\mu\gamma} \propto \frac{1}{k^2+m^2}\left(\eta_{\mu\gamma} - \frac{1}{m^2}k_\mu k_\gamma\right)$ as expected. We can now take the limit $m \to 0$ in the Lagrangian (3.34) and we realize immediately that the limit only exist for a conserved source. There is however a discontinuity when the massless limit is taken because we only have two degrees of freedom instead of three in such a case [43]. We can introduce redundant variables in order to restore the gauge invariance and the continuity in the number of degrees of freedom if we introduce a scalar ϕ by doing the replacement

$$A_\mu \to A_\mu + \partial_\mu\phi. \tag{3.39}$$

Note that the tensor $F_{\mu\nu}$ remains unchanged under this transformation. Then only the mass term changes. We must emphasize that this is not a change of variables nor a gauge transformation (although it looks like one). The new Lagrangian then becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2(A_\mu + \partial_\mu\phi)^2 + A_\mu J^\mu - \phi\partial_\mu J^\mu, \tag{3.40}$$

where the coupling to the source has been integrated by parts. The new action, has the gauge symmetry

$$\delta_g A_\mu = \partial_\mu\Lambda \qquad \delta_g\phi = -\Lambda. \tag{3.41}$$

In the unitary gauge, we have $\phi = 0$. In such a case, we just recover the original action and the field equation for the scalar field ϕ just becomes equivalent to the divergence condition of the field equations corresponding to the vector fields A^μ. This can be observed if we compute explicitly the field equations given by

$$(\Box - m^2)A^\nu = (-\partial^\nu\Box + m^2\partial^\nu)\phi - J^\nu + \frac{1}{m^2}\partial^\nu\partial_\gamma J^\gamma, \tag{3.42}$$

for the field A^ν and

$$\Box\phi = -\partial_\sigma A^\sigma + \frac{1}{m^2}\partial_\sigma J^\sigma, \tag{3.43}$$

for the scalar field ϕ. In the unitary gauge, eq. (3.43) is obtained from the divergence of eq. (3.42). The new Lagrangian (3.40) now has a gauge symmetry. It is however physically equivalent to the action represented by eq. (3.34). We can normalize the action (3.40) by defining $\phi \to m^{-1}\phi$ in order to normalize the kinetic term and then the Lagrangian becomes [43]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu - mA_\mu\partial^\mu\phi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + A_\mu J^\mu - \frac{1}{m}\phi\partial_\mu J^\mu, \tag{3.44}$$

and now the gauge symmetry becomes

$$\delta_g A_\mu = \partial_\mu\Lambda, \qquad \delta_g\phi = -m\Lambda. \tag{3.45}$$

If we consider the limit $m \to 0$, which is valid if the current is conserved; then the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + A_\mu J^\mu, \tag{3.46}$$

and the gauge symmetry is

$$\delta_g A_\mu = \partial_\mu\Lambda \qquad \delta_g\phi = 0. \tag{3.47}$$

In this case, we can see that the massless limit preserves the number of degrees of freedom. Two of the 3 degrees of freedom go to the vector and the remaining one goes to the scalar. As the vector is now decoupled from the scalar, then we have basically two free massless fields, one vectorial and another scalar. The scalar degree of freedom does not disappear because we have re-scaled it as m^{-1}.

3.4 Graviton Stückelberg and origin of the vDVZ discontinuity

The massive Lagrangian of a graviton at the linear level becomes

$$\mathcal{L} = \mathcal{L}_{m=0} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \kappa h_{\mu\nu}T^{\mu\nu}, \tag{3.48}$$

which is in agreement with the expression (3.3). If we want to preserve the gauge symmetry $\delta_g h_{\mu\nu} = \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu$, then we introduce the Stückelberg field pattern

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu. \tag{3.49}$$

This redefinition can only change the mass term of the action (3.48). The new action becomes

$$\mathcal{L} = \mathcal{L}_{m=0} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) - \frac{1}{2}m^2 F_{\mu\nu}F^{\mu\nu} - 2m^2(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu) + \kappa h_{\mu\nu}T^{\mu\nu} - 2\kappa A_\mu \partial_\nu T^{\mu\nu}, \tag{3.50}$$

where the definition $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ has been introduced. The new action (3.50) is invariant under the following transformation

$$\delta_g h_{\mu\nu} = \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu, \qquad \delta_g A_\mu = -\zeta_\mu. \tag{3.51}$$

We can find the field equations from the action (3.50). They are

$$\begin{aligned}\frac{\delta S}{\delta h^{\mu\nu}} = \Box h_{\mu\nu} - \partial_\lambda \partial_\mu h^\lambda{}_\nu - \partial_\lambda \partial_\nu h^\lambda{}_\mu + \partial_\mu \partial_\nu h + \eta_{\mu\nu}\left(\partial_\lambda \partial_\sigma h^{\lambda\sigma} - \Box h\right)& \\ -m^2(h_{\mu\nu} - \eta_{\mu\nu}h) - 2m^2(\partial_\mu A_\nu - \eta_{\mu\nu}\partial_\gamma A^\gamma) + \kappa T_{\mu\nu} = 0,&\end{aligned} \tag{3.52}$$

for the variations with respect to $h_{\mu\nu}$. And

$$\partial_\mu F^{\mu\nu} + m^2(\partial_\mu h^{\mu\nu} - \partial^\nu h) = \kappa \partial_\mu T^{\mu\nu}, \tag{3.53}$$

for the variations with respect to A_μ. The gauge $A_\mu = 0$ recovers the original action. In such a case, it is easy to verify that eq. (3.53) becomes equivalent to the Bianchi identity obtained from the divergence of eq. (3.52). Following the same procedure as in the vectorial case, we can rescale the field $A_\mu \to m^{-1}A_\mu$. In such a case, the action (3.50) becomes

$$\mathcal{L} = \mathcal{L}_{m=0} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2m(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu) + \kappa h_{\mu\nu}T^{\mu\nu} - \frac{2\kappa}{m}A_\mu \partial_\nu T^{\mu\nu}. \tag{3.54}$$

In this case, if we take the limit $m \to 0$, then we have a tensor field and a vector field. Both of them are decoupled each other and they represent a massless graviton and a massless photon for a total of 4 degrees of freedom. Then at this point, the massless limit loses one degree of freedom. However, we can introduce a scalar gauge symmetry by introducing another Stückelberg field ϕ

$$A_\mu \to A_\mu + \partial_\mu \phi. \tag{3.55}$$

With this new field, the action (3.50) becomes

$$\mathcal{L} = \mathcal{L}_{m=0} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) - \frac{1}{2}m^2 F_{\mu\nu}F^{\mu\nu} - 2m^2(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu)$$
$$-2m^2(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial^2\phi) + \kappa h_{\mu\nu}T^{\mu\nu} - 2\kappa A_\mu \partial_\nu T^{\mu\nu} + 2\kappa\phi\partial_\mu\partial_\nu T^{\mu\nu}. \tag{3.56}$$

Now we have two gauge symmetries, given by

$$\delta_g h_{\mu\nu} = \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu, \qquad \delta_g A_\mu = -\zeta_\mu. \tag{3.57}$$

$$\delta_g A_\mu = \partial_\mu \Lambda, \qquad \delta_g \phi = -\Lambda. \tag{3.58}$$

If we fix the gauge $\phi = 0$, then we recover the Lagrangian (3.50). The equations of motion in this case become

$$\Box h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + \partial_\mu\partial_\nu h + \eta_{\mu\nu}\left(\partial_\lambda\partial_\sigma h^{\lambda\sigma} - \Box h\right)$$
$$-m^2(h_{\mu\nu} - \eta_{\mu\nu}h) - 2m^2(\partial_\mu A_\nu - \eta_{\mu\nu}\partial_\gamma A^\gamma)$$
$$-m^2(\partial_\mu\partial_\nu\phi - \eta_{\mu\nu}\partial^2\phi) + \kappa T_{\mu\nu} = 0, \tag{3.59}$$

for the variation with respect to $h_{\mu\nu}$. The equations corresponding to the variation with respect to A_μ are just the same as those obtained in eq. (3.53). On the other hand, the equation related to the variation of the scalar field ϕ is given by

$$\partial_\mu\partial_\nu h^{\mu\nu} - \Box h = \frac{\kappa\partial_\mu\partial_\nu T^{\mu\nu}}{m^2}. \tag{3.60}$$

If the source term $T_{\mu\nu}$ is conserved, then the previous expression is equivalent to the vanishing of the Ricci scalar R. If we rescale $A_\mu \to (1/m)A_\mu$ and $\phi \to (1/m^2)\phi$, then the Lagrangian (3.56) becomes

$$\mathcal{L} = \mathcal{L}_{m=0} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2m(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu)$$
$$-2(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial^2\phi) + \kappa h_{\mu\nu}T^{\mu\nu} - \frac{2}{m}\kappa A_\mu\partial_\nu T^{\mu\nu} + \frac{2}{m^2}\kappa\phi\partial_\mu\partial_\nu T^{\mu\nu}, \tag{3.61}$$

and the gauge transformations become

$$\delta_g h_{\mu\nu} = \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu, \qquad \delta_g A_\mu = -m\zeta_\mu.$$
$$\delta_g A_\mu = \partial_\mu \Lambda, \qquad \delta_g \phi = -m\Lambda. \tag{3.62}$$

One factor of m is absorbed inside the gauge parameter Λ. If we assume the source to be conserved and then we take the limit $m \to 0$, then the Lagrangian takes the form

$$\mathcal{L} = \mathcal{L}_{m=0} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial^2\phi) + \kappa h_{\mu\nu}T^{\mu\nu}. \tag{3.63}$$

In this limit, we observe that we have 5 degrees of freedom but the scalar is kinematically mixed with the tensor. The vector field is completely decoupled. If we perform a linear conformal transformation by defining

$$h_{\mu\nu} = h'_{\mu\nu} + \phi\eta_{\mu\nu}. \tag{3.64}$$

Then the massless graviton Lagrangian in eq. (3.63) becomes

$$\mathcal{L}_{m=0} = \mathcal{L}_{m=0}(h') + 2\left((\partial_\mu\phi)(\partial^\mu h') - (\partial_\beta\phi)(\partial_\mu h'^{\beta\mu}) + \frac{3}{2}(\partial_\mu\phi)(\partial^\mu\phi)\right). \tag{3.65}$$

Then the Lagrangian (3.63) is

$$\mathcal{L} = \mathcal{L}_{m=0}(h') - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 3(\partial_\mu\phi)(\partial^\mu\phi) + \kappa h'_{\mu\nu}T^{\mu\nu} + \kappa\phi T, \tag{3.66}$$

and the gauge transformations become

$$\begin{aligned} \delta_g h'_{\mu\nu} &= \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu, \qquad \delta_g A_\mu = 0. \\ \delta_g A_\mu &= \partial_\mu\Lambda, \qquad\qquad \delta_g\phi = 0. \end{aligned} \tag{3.67}$$

This massless limit is soft in the sense that now we have 5 degrees of freedom, 2 in the massless graviton field, other two in the massless photon field and the remaining one in the massless scalar. Note however that there is a coupling between the scalar component and the tensorial one in the Lagrangian (3.66) even in the massless limit. Although this coupling disappears for the case of photons, it will appear for ordinary matter. This coupling marks the origin of the vDVZ discontinuity. This discontinuity is in fact a pathology due to the linear approach. It is necessary to go to the non-linear level in order to recover the continuity of the theory at the massless limit.

3.4.1 Graviton Stückelberg and the propagating ghost

Starting this chapter I explained that for a massive term given by $-\frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - (1-a)h^2)$, there is a ghost for $a \neq 0$. Here I will show that by using the Stückelberg trick, it is possible to derive the reason for the absence of a ghost when $a = 0$. If we follow the same procedure as in the first part of the chapter, then for the transformation given by eq. (3.49), the massive term defined previously for $a \neq 0$ becomes

$$-\frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - (1-a)h^2) - \frac{1}{2}m^2F_{\mu\nu}F^{\mu\nu} - 2m^2(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu) - 2m^2a((\partial_\mu A^\mu)^2 + h\partial_\mu A^\mu). \tag{3.68}$$

If again, we make the transformation suggested in eq. (3.55), then the massive term will contain the following additional terms with respect to the result (3.56)

$$-2m^2a\left((\partial_\mu A^\mu)^2 + 2\partial_\mu A^\mu\Box\phi + (\Box\phi)^2 + h\partial_\mu A^\mu + h\Box\phi\right). \tag{3.69}$$

Note the presence of the term $(\Box\phi)^2$. It represents a higher derivative contribution and it implies the existence of a ghost. Only if $a = 0$, this term disappears. This result is in agreement with the analysis performed in [48].

Chapter 4

The non-linear formulation of massive gravity

Based in the arguments of the previous section, it seems that we should reject any theory of massive gravity since it would be in disagreement with solar system observations due to the vDVZ discontinuity [43, 44, 45]. However, Vainshtein realized that non-linearities can restore the continuity of the theory [49]. It was however demonstrated that non-linearities can activate the so-called Boulware-Deser ghost [43, 50]. It took time for this pathology to be solved with the Λ_3 version of massive gravity developed by de-Rham, Gabadadze and Tolley [10, 11] (dRGT). The dRGT theory is ghost-free. The fact that the theory is ghost free, was first analyzed at the decoupling limit [10] and after outside the decoupling limit [11]. The first work demonstrating that the theory is fully ghost-free for any background was performed by Hassan et. al [51]. The generalization of the Fierz-Pauli action to the non-linear case considers the dynamical metric $g_{\mu\nu}$ and the same kinetic terms of General Relativity. Then the same kinetic part of the action (2.1) remains and we need a mass term [41, 43]. The mass term is not necessarily linear. Of course, we expect that the mass term goes to the Fierz-Pauli one after quadratic expansions around a flat background and working in weak field approximation. In order to get a massive term in a non-linear theory, we have to introduce an auxiliary metric ($f_{\mu\nu}$) [43], otherwise, we would have simply two propagating degrees of freedom. The extra metric is normally taken to be flat (unitary gauge) and nondynamical. With a second metric coming to the scenario, we will have a new term in the action $S_{int}(f, g)$, which represents the interaction of the two metrics. This action is taken such that i). the theory remains covariant under diffeomorphisms (common to the two metrics). ii). The theory has a flat solution of the field equations for $g_{\mu\nu}$. iii). When one expands $g_{\mu\nu}$ to second order around the canonical Minkowski metric $\eta_{\mu\nu}$ as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and additionally, we let the fiducial metric $f_{\mu\nu}$ to take the Minkowskian form (Unitary gauge), then the potential at quadratic order for $h_{\mu\nu}$ takes the Fierz-Pauli form given by eq. (3.3). In principle, there are many possible interaction terms satisfying these conditions. Among the possibilities, we have [27, 52]

$$S_{int} = -\frac{1}{8}m^2 M_{pl}^2 \int d^4x\sqrt{-f}H_{\mu\nu}H_{\sigma\tau}\left(f^{\mu\sigma}f^{\nu\tau} - f^{\mu\nu}f^{\sigma\tau}\right), \tag{4.1}$$

or equivalently

$$S_{int} = -\frac{1}{8}m^2 M_{pl}^2 \int d^4x\sqrt{-f}H_{\mu\nu}H_{\sigma\tau}\left(g^{\mu\sigma}g^{\nu\tau} - g^{\mu\nu}g^{\sigma\tau}\right), \tag{4.2}$$

where $H_{\mu\nu}$ is the covariantization of the field $h_{\mu\nu}$ defined previously. This field is defined as $H_{\mu\nu} = g_{\mu\nu} - f_{\mu\nu}$ [43]. Independent of the explicit form of the interaction term S_{int}, it has the form [53]

$$S_{int} = -\frac{1}{8}m^2 M_{pl}^2 \int d^4x\sqrt{-g}V^{(a)}(g^{-1}f), \tag{4.3}$$

where $\sqrt{-g}V^{(a)}(g^{-1}f)$ is a suitable "potential" density associated with the scalar function $V^{(a)}$. We can observe that the metrics interact in a non-derivative way and that the theory has to be invariant under diffeomorphism [53]. The diffeomorphism invariance as usual is expressed as transformations acting on the metrics as follows

$$g_{\mu\nu} = \partial_\mu x'^\sigma(x)\partial_\nu x'^\tau(x)g'_{\sigma\tau}(x'(x)), \tag{4.4}$$

$$f_{\mu\nu} = \partial_\mu x'^\sigma(x)\partial_\nu x'^\tau(x)f'_{\sigma\tau}(x'(x)). \tag{4.5}$$

Under these transformations, $V^{(a)}$ transforms as a scalar. When matter is introduced, it is usually assumed to be minimally coupled to the dynamical metric $g_{\mu\nu}$ and the total action is normally taken as

$$S = \int d^4x\sqrt{-g}\left(\frac{M_{pl}^2}{2}R + L(g)\right) + S_{int}(f,g). \tag{4.6}$$

The possible non-vanishing cosmological constant is included inside the interaction term $S_{int}(f,g)$. In fact, if there are no interaction terms, the eq. (4.6) just reduces to the Einstein-Hilbert action already shown in eq. (2.1). The decoupling limit for an action of the form (4.6) is in general defined as [10, 11]

$$\mathcal{L} = M_{pl}^2\sqrt{-g}R - \frac{M_{pl}^2 m^2}{4}\sqrt{-g}\left(U_2(g,H) + U_3(g,H) + U_4(g,H) + U_5(g,H) + ...\right), \tag{4.7}$$

where the U_i terms denote the interaction at ith order in $H_{\mu\nu}$

$$U_2(g,H) = H_{\mu\nu}^2 - H^2, \tag{4.8}$$

$$U_3(g,H) = c_1 H_{\mu\nu}^3 + c_2 H H_{\mu\nu}^2 + c_3 H^3, \tag{4.9}$$

$$U_4(g,H) = d_1 H^4_{\mu\nu} + d_2 H H^3_{\mu\nu} + d_3 H^2_{\mu\nu} H^2_{\alpha\beta} + d_4 H^2 H^2_{\mu\nu} + d_5 H^4, \tag{4.10}$$

$$U_5(g,H) = f_1 H^5_{\mu\nu} + f_2 H H^4_{\mu\nu} + f_3 H^2 H^3_{\mu\nu} + f_4 H^2_{\alpha\beta} H^3_{\mu\nu} + f_5 H (H^2_{\mu\nu})^2 + f_6 H^3 H^2_{\mu\nu} + f_7 H^5, \tag{4.11}$$

where the index contractions are performed using the inverse metric $g^{\mu\nu}$. For example $H = g^{\mu\nu} H_{\mu\nu}$ and $H^2_{\mu\nu} = g^{\mu\nu} g^{\alpha\beta} H_{\mu\alpha} H_{\nu\beta}$ denote traces with respect to the dynamical metric. The coefficients c_i, d_i and f_i are arbitrary but in the ghost-free version of massive gravity, they are selected such that any ghost is absent. In [10], the authors obtained a ghost-free version up to quintic order. In [11], the authors proved that the theory is ghost-free at all orders in the decoupling limit. Finally in [51], the authors proved that the theory is ghost-free even outside the decoupling limit. The tensor $H_{\mu\nu}$ is defined as

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{pl}} = H_{\mu\nu} + \eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b, \tag{4.12}$$

where $a, b = 0, 1, 2, 3$ and $\eta_{ab} = diag(-1, 1, 1, 1)$. $H_{\mu\nu}$ behaves as a tensor if the fields ϕ^a transform as scalars under a change of coordinates. The definition (4.12) gives the form for the covariantization of the perturbation $h_{\mu\nu}$. The purpose is to restore the gauge invariance by introducing redundant variables. If we define

$$\phi^a = (x^\alpha - \pi^\alpha) \delta^a{}_\alpha, \tag{4.13}$$

then we get

$$H_{\mu\nu} = \frac{h_{\mu\nu}}{M_{pl}} + \partial_\mu \pi_\nu + \partial_\nu \pi_\mu - \eta_{\alpha\beta} \partial_\mu \pi^\alpha \partial_\nu \pi^\beta. \tag{4.14}$$

The index for π^μ is raised or lowered by using the Minkowskian metric $\eta_{\mu\nu}$. If we use the substitution $\pi_\alpha = \partial_\alpha \pi / \Lambda_3^3$ with the effective field scale Λ_3, then we get

$$H_{\mu\nu} = \frac{h_{\mu\nu}}{M_{pl}} + \frac{2}{M_{pl} m^2} \Pi_{\mu\nu} - \frac{1}{M_{pl}^2 m^4} \Pi^2_{\mu\nu}, \tag{4.15}$$

where $\Pi_{\mu\nu} = \partial_\mu \partial_\nu \pi$ and $\Pi^2_{\mu\nu} = \eta^{\alpha\beta} \Pi_{\mu\alpha} \Pi_{\beta\nu}$. The traces are defined as $[\Pi^2] = \Pi^{\mu\nu} \Pi_{\mu\nu}$ and $[\Pi]^2 = \Pi^\mu{}_\mu \Pi^\nu{}_\nu$. It is possible to define [11]

$$Q^\mu{}_\nu(g,H) = \delta^\mu{}_\nu - \sqrt{\delta^\mu{}_\nu - H^\mu{}_\nu}. \tag{4.16}$$

With this definition, we can demonstrate that

$$Q_{\mu\nu}(g,H)_{h_{\mu\nu}=0} = \frac{\Pi_{\mu\nu}}{M_{pl} m^2}. \tag{4.17}$$

This is obtained if we take into account that $H_{\mu\nu h=0} = \frac{2}{M_{pl} m^2} \Pi_{\mu\nu} - \frac{1}{M_{pl}^2 m^4} \Pi^2_{\mu\nu}$ from eq. (4.12) and then replacing it inside eq. (4.16). In what follows, we will use square brackets

$[\Pi] = \eta^{\mu\nu}\Pi_{\mu\nu}$ for the traces with respect to the Minkowskian metric. And we will use $\Pi_n = Tr(\Pi^n)$ or $\Pi^n = (Tr(\Pi))^n$ for the traces with respect to the dynamical metric $g_{\mu\nu}$. The consistency of the Fierz-Pauli action term $h^2 - h_{\mu\nu}^2$ is related to the fact that the Lagrangian term

$$\mathcal{L}_{der}^{(2)} = [\Pi]^2 - [\Pi^2], \tag{4.18}$$

is a total derivative. Inside the non-linear formulation of massive gravity, it has been demonstrated that additionally the terms

$$\mathcal{L}_{der}^{(3)} = [\Pi]^3 - 3[\Pi][\Pi^2] + 3[\Pi^3], \tag{4.19}$$

$$\mathcal{L}_{der}^{(4)} = [\Pi]^4 - 6[\Pi^2][\Pi]^2 + 8[\Pi^3][\Pi] + 3[\Pi^2]^2 - 6[\Pi^4], \tag{4.20}$$

are also total derivatives. The key point of dRGT theory is that the coefficients for the potential terms given from eq. (4.8) until (4.11) are tuned such that the total derivative term interactions defined in eqns. (4.18), (4.19) and (4.20) are obtained [10]. The expansions (4.18) until (4.20) correspond to the interaction terms up to quartic order. It has been demonstrated by de-Rham and Gabadadze [10], that the higher-order combinations satisfy $L^n = 0$ for any $n \geq 5$. After tuning the coefficients of the potential such that the total derivative conditions are satisfied, all the interactions that arise at an energy scale lower than Λ_3 disappear. As a consequence of this, the decoupling limit in this theory is considered as

$$m \to 0, \qquad M_{pl} \to \infty, \qquad with \quad \Lambda_3 \equiv (m^2 M_{pl})^{1/3} \quad fixed. \tag{4.21}$$

In [10], the coefficients for the potential were selected such that the previous conditions are satisfied up to quintic order. If we want to ensure that no ghost appears outside the decoupling limit, then it is possible to extend the second and higher order terms (4.18), (4.19) and (4.20) away from the condition $h_{\mu\nu} \to 0$ [11]. This is equivalent to replace the terms Π by the corresponding Q terms defined previously in eq. (4.16). Then the action becomes

$$\mathcal{L} = \frac{M_{pl}^2}{2}\sqrt{-g}\left(R + m^2 U(g, H)\right), \tag{4.22}$$

where the potential terms for $U(g, H)$ will be defined in the next section inside the Λ_3 version of massive gravity.

4.1 Formulation of the dRGT

In the standard formalism of the dRGT theory, the action is given by [10, 11]

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R + m^2 U(g, \phi)), \tag{4.23}$$

with the effective potential depending on two free parameters as

$$U(g,\phi) = U_2 + \alpha_3 U_3 + \alpha_4 U_4, \tag{4.24}$$

where

$$U_2 = Q^2 - Q_2, \tag{4.25}$$

$$U_3 = Q^3 - 3QQ_2 + 2Q_3, \tag{4.26}$$

$$U_4 = Q^4 - 6Q^2 Q_2 + 8QQ_3 + 3Q_2^2 - 6Q_4, \tag{4.27}$$

$$Q = Q_1, \qquad Q_n = Tr(Q^n)^{\mu}{}_{\nu}, \tag{4.28}$$

$$Q^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - M^{\mu}{}_{\nu}, \tag{4.29}$$

$$(M^2)^{\mu}{}_{\nu} = g^{\mu\alpha} f_{\alpha\nu}, \tag{4.30}$$

$$f_{\mu\nu} = \eta_{ab}\partial_\mu \phi^a \partial_\nu \phi^b. \tag{4.31}$$

We can the compute the field equations as follows

$$G_{\mu\nu} = -m^2 X_{\mu\nu}, \tag{4.32}$$

where

$$X_{\mu\nu} = \frac{\delta U}{\delta g^{\mu\nu}} - \frac{1}{2} U g_{\mu\nu}. \tag{4.33}$$

Here $f_{\mu\nu}$ is the fiducial metric and Q is the trace of the matrix $Q^{\mu}{}_{\nu}$ with respect to the dynamical metric. The potential (U) defined above is unique. It is impossible to add polynomial terms without introducing a ghost [10, 11]. If we want to prove that eq. (4.32) is consistent with the definition of the tensor $X_{\mu\nu}$ as given in eq. (4.33); we have to demonstrate that $X_{\mu\nu}$ is symmetric and divergence-free. The first requirement is automatically satisfied because $Q_{\mu\nu}$ is a symmetric tensor. In contrast, the second requirement leads to constraints on $g_{\mu\nu}$ and /or ϕ^a. To see this, we use the diffeomorphism invariance of the mass terms of the action

$$\int d^4x' \sqrt{-g'} U(g', \phi') = \int d^4x \sqrt{-g} U(g, \phi). \tag{4.34}$$

For an infinitesimal coordinate transformation

$$\delta x = \zeta^\mu, \qquad \delta g_{\mu\nu} = -2\nabla_{(\mu}\zeta_{\nu)}, \qquad \delta\phi = -\zeta^\mu \partial_\mu \phi, \tag{4.35}$$

this equation leads to

$$0 = \int d^4x\sqrt{-g}\left(-m^2\nabla_\nu X^{\mu\nu}\zeta_\mu - \frac{\delta U}{\delta\phi}\nabla_\mu\phi\zeta^\mu\right). \tag{4.36}$$

Hence we obtain

$$m^2\nabla_\nu X^\nu{}_\mu = -\partial_\mu\phi^a\frac{\delta U}{\delta\phi^a} = \partial_\mu\phi^a\nabla_\nu\left(\frac{\delta U}{\partial(\partial_\nu\phi^a)}\right). \tag{4.37}$$

If the field equations (4.32) holds, then the left-hand side of this equation should vanish due to the Bianchi identity. Because $\partial_\mu\phi^a$ is regular matrix, this constraint is equivalent to the Euler equation for the Stückelberg field

$$\nabla_\mu\left(\frac{\partial U}{\partial(\partial_\mu\phi^a)}\right) = 0. \tag{4.38}$$

4.2 The Schwarzschild-de Sitter solution in Einstein gravity

The Schwarzschild-de Sitter solution in Einstein gravity corresponds to the vacuum solutions for the field equations given in (2.3). The solution in static coordinates is given by a metric of the form [54]

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2d\Omega_2^2, \tag{4.39}$$

where

$$V(r) = 1 - \frac{r_s}{r} - \frac{1}{3}\frac{r^2}{r_\Lambda^2}, \qquad r_s = 2G_NM, \qquad r_\Lambda = \frac{1}{\sqrt{\Lambda}}. \tag{4.40}$$

The event horizons for this metric can be calculated by using the standard condition [1]

$$g^{rr}(r_c) = 0. \tag{4.41}$$

After solving this equation, the two event horizons are given by [54]

$$r_{CH} = -2r_\Lambda cos\left(\frac{1}{3}\left(cos^{-1}\left(\frac{3r_s}{2r_\Lambda}\right) + 2\pi\right)\right), \tag{4.42}$$
$$r_{BH} = -2r_\Lambda cos\left(\frac{1}{3}\left(cos^{-1}\left(\frac{3r_s}{2r_\Lambda}\right) + 4\pi\right)\right),$$

where r_{CH} corresponds to the cosmological horizon and r_{BH} corresponds to the black hole event horizon. The equations (4.42) show that the maximum mass for a black hole in an universe with a positive cosmological constant Λ is given by

$$M_{max} = \frac{1}{3}\frac{m_{pl}^2}{m_\Lambda}, \tag{4.43}$$

where m_{pl} corresponds to the Planck mass and $m_\Lambda = \sqrt{\Lambda}$. If the mass of a black hole is larger than the value (4.43), then there is no radiation process at all and we only have a naked singularity. M_{max} is however of the order of magnitude of the mass of the universe if we take the observed value for Λ [55]. As $M = M_{max}$, the two event horizons take the same value $\left(r_{BH} = r_{CH} = r_\Lambda = \frac{1}{\sqrt{\Lambda}}\right)$, they are degenerate and there is no net radiation due to the thermodynamic equilibrium established. For degenerate horizons, the Schwarzschild-like coordinates given by the expression (4.39) and (4.40) are not valid anymore [56, 57]. As Bousso and Hawking have explained before [56, 57], as $M \to M_{max}$, then $V(r) \to 0$ between the two horizons (BH and Cosmological). In such a case we need a new coordinate system. In agreement with Ginsparg and Perry [58], we can write

$$9G^2M^2\Lambda = 1 - 3\epsilon^2, \qquad 0 \leq \epsilon \ll 1, \tag{4.44}$$

where the degenerate case (where the two horizons become the same) corresponds to $\epsilon \to 0$. We must then define the new radial and the new time coordinates to be

$$t = \frac{1}{\epsilon\sqrt{\Lambda}}\psi, \qquad r = \frac{1}{\sqrt{\Lambda}}\left(1 - \epsilon cos\chi - \frac{1}{6}\epsilon^2\right). \tag{4.45}$$

In these coordinates, the black hole horizon corresponds to $\chi = 0$ and the cosmological horizon to $\chi = \pi$ [56, 57]. The metric then becomes

$$ds^2 = -r_\Lambda^2\left(1 + \frac{2}{3}\epsilon cos\chi\right) sin^2\chi d\psi^2 + r_\Lambda^2\left(1 - \frac{2}{3}\epsilon cos\chi\right) d\chi^2 + r_\Lambda^2(1 - 2\epsilon cos\chi)d\Omega_2^2. \tag{4.46}$$

This metric has been expanded up to first order in ϵ. Eq. (4.46) is of course the appropriate metric to be used as the mass of the black hole is near to its maximum value given by (4.43). The result (4.46) is obtained after doing the transformations for the metric in the form

$$g_{\mu\nu} = \frac{\partial x^a}{\partial x^\mu}\frac{\partial x^b}{\partial x^\nu}g_{ab}. \tag{4.47}$$

It is easy to verify that the previous result is reproduced after introducing the coordinates (4.45) and taking the positive root in eq. (4.44). The negative root of eq. (4.44) is unphysical and then ignored.

4.3 The Schwarzschild de-Sitter (SdS) solution in dRGT theory

If we want the SdS solution to be a solution for the field equations in massive gravity, the tensor $X_{\mu\nu}$ should be a constant multiple of $g_{\mu\nu}$ and given by [17]

$$m^2X_{\mu\nu} = \Lambda g_{\mu\nu}, \tag{4.48}$$

with $X^\mu{}_\nu$ given by

$$
\begin{aligned}
g^{\mu\alpha}X_{\alpha\nu} = -Q - \frac{1}{2}(Q^2 - Q_2) + (1+Q)Q^\mu{}_\nu - (Q^2)^\mu{}_\nu + \frac{\alpha_3}{2}\{3(Q_2 - Q^2) \\
-Q^3 + 3QQ_2 - 2Q_3 + 3(2Q + Q^2 - Q_2)Q^\mu{}_\nu - 6(1+Q)(Q^2)^\mu{}_\nu + 6(Q^3)^\mu{}_\nu\} \\
+\alpha_4\{-2Q^3 + 6QQ_2 - 4Q_3 + 6(Q^2 - Q_2)Q^\mu{}_\nu - 12Q(Q^2)^\mu{}_\nu + 12(Q^3)^\mu{}_\nu\}.
\end{aligned}
\tag{4.49}
$$

If the field equations satisfy the condition (4.48), then we have a set of solutions for the Einstein equations with Λ. The different solutions are not however necessarily equivalent even if they are connected by diffeomorphism transformation. This is due to the fact that the Stückelberg fields are not necessarily related by the same transformation. Later in this document we will see that this apparently harmless issue, becomes highly pathological for the black hole physics in this formulation. In agreement with the definition (4.33). If the parameters satisfy

$$12\alpha_4 = 1 + 3\alpha_3 + 9\alpha_3^2, \tag{4.50}$$

then any metric of the form

$$ds^2 = g_{tt}dt^2 + 2g_{tr}dtdr + g_{rr}dr^2 + r^2 S_0^2 d\Omega_2^2, \tag{4.51}$$

will satisfy the condition (4.48) with

$$S_0 = \frac{3\alpha_3 + 1}{3\alpha_3 + 2}. \tag{4.52}$$

Note that $S_0 \neq 1$ for any value of the parameter α_3, this means that the cosmological constant defined in unitary gauge and given by

$$\Lambda = m^2 \frac{1 - S_0}{S_0} = \frac{m^2}{3\alpha_3 + 1}, \tag{4.53}$$

is different from zero. Although in principle the metric is arbitrary, it can be translated to the standard S-dS form after some coordinate (gauge) transformations. If we work in unitary gauge, all the degrees of freedom will be inside the dynamical metric. The Stückelberg fields ϕ^a are given by

$$\phi^0 = t, \quad \phi^1 = x = r\cos\theta, \quad \phi^2 = y = r\sin\theta\cos\phi, \quad \phi^3 = z = r\sin\theta\sin\phi. \tag{4.54}$$

In this gauge, the fiducial metric $f_{\mu\nu}$ is

$$f_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{4.55}$$

Hence, for the metric (4.51), the matrix M^2 defined in (4.30), is given by

$$(M^2) = (g^* f_*) = \begin{pmatrix} -g^{tt} & g^{tr} & 0 & 0 \\ -g^{tr} & g^{rr} & 0 & 0 \\ 0 & 0 & \frac{1}{S^2} & 0 \\ 0 & 0 & 0 & \frac{1}{S^2} \end{pmatrix}. \tag{4.56}$$

We define the root square of this matrix by using the expression (4.29) and defining [59]

$$Q^\mu{}_\nu = \begin{pmatrix} a & c & 0 & 0 \\ -c & b & 0 & 0 \\ 0 & 0 & 1-\frac{1}{S} & 0 \\ 0 & 0 & 0 & 1-\frac{1}{S} \end{pmatrix}. \tag{4.57}$$

Here

$$1 - a = \frac{1}{M_1}(-g^{tt} + (-g_{(2)})^{-1/2}), \tag{4.58}$$

$$c = -\frac{g^{tr}}{M_1}, \tag{4.59}$$

$$1 - b = \frac{1}{M_1}(g^{rr} + (-g_{(2)})^{-1/2}), \tag{4.60}$$

$$M_1 = (-g_{(2)})^{-1/2} \left(-g_{tt} + g_{rr} + 2(-g_{(2)})^{1/2}\right)^{1/2}, \tag{4.61}$$

$$g_{(2)} = g_{tt} - g_{tr}^2. \tag{4.62}$$

In terms of $M_{\mu\nu}$, we get

$$M_{tt} = \frac{1}{M_1}(-1 + g_{tt}(-g_{(2)})^{-1/2}), \tag{4.63}$$

$$M_{tr} = \frac{(-g_{(2)})^{-1/2}}{M_1} g_{tr}, \tag{4.64}$$

$$M_{rr} = \frac{1}{M_1}(1 + g_{rr}(-g_{(2)})^{-1/2}), \tag{4.65}$$

$$M_{ij} = \frac{1}{S} g_{ij}. \tag{4.66}$$

We can also express $g_{\mu\nu}$ in terms of the components of $Q^\mu{}_\nu$ as follows

$$g_{tt} = -\frac{(1-b)^2 - c^2}{[(1-a)(1-b) + c^2]^2}, \tag{4.67}$$

$$g_{rr} = \frac{(1-a)^2 - c^2}{[(1-a)(1-b) + c^2]^2}, \tag{4.68}$$

$$g_{tr} = -\frac{c(2-a-b)}{[(1-a)(1-b)+c^2]^2}, \tag{4.69}$$

$$g_{\theta\theta} = r^2 S^2, \tag{4.70}$$

$$g_{\phi\phi} = r^2 S^2 sin^2\theta, \tag{4.71}$$

$$(-g_{(2)})^{-1/2} = c^2 + (1-a)(1-b). \tag{4.72}$$

If we introduce the definition of $Q^{\mu}{}_{\nu}$ given by eq. (4.57) inside eq. (4.49), we obtain

$$X^t{}_t = -bF_3 - (F_1+1)\frac{(S-1)}{S}, \tag{4.73}$$

$$X^t{}_r = cF_3, \tag{4.74}$$

$$X^t{}_t - X^r{}_r = (a-b)F_3, \tag{4.75}$$

$$X^t{}_t - X^\theta{}_\theta = F_1\left(a - 1 + \frac{1}{S}\right) + F_2\left(ab + c^2 - b\frac{(S-1)}{S}\right), \tag{4.76}$$

where we define

$$F_1 = 3\alpha_3 + 2 - \frac{(1+3\alpha_3)}{S}, \tag{4.77}$$

$$F_2 = 1 + 6\alpha_3 + 12\alpha_4 - \frac{3(4\alpha_4+\alpha_3)}{S}, \tag{4.78}$$

$$F_3 = F_1 + \frac{(S-1)}{S}F_2. \tag{4.79}$$

If the following conditions are satisfied

$$F_1 = F_2 = 0, \quad S = \frac{3\alpha_3+1}{3\alpha_3+2}, \quad 12\alpha_4 = 1 + 3\alpha_3 + 9\alpha_3^2. \tag{4.80}$$

Then the conditions (4.50) and (4.52) are satisfied. As a consequence, $X^{\mu}{}_{\nu}$ becomes multiple of the unit matrix

$$X^{\mu}{}_{\nu} = \frac{1-S}{S}\delta^{\mu}{}_{\nu}. \tag{4.81}$$

This is true independent of the form of $a(t,r)$, $b(t,r)$ or $c(t,r)$ and as a consequence of the background metric. If we require the metric (4.51) to be a solution of the field equations (4.32) with (4.50), owing to the Birkhoff theorem for the Einstein vacuum system, it must be isomorphic to the Schwarzschild-de Sitter solution in the standard form for which $g_{tt} = -f(r)$, $g_{tr} = 0$ and $g_{rr} = 1/f(r)$ with $f(r) = 1 - 2M/r - \Lambda r^2/3$. The above result means that g_{tt}, g_{tr} and g_{rr} obtained from this standard form by arbitrary change of time coordinate $t \to T(t, r)$ also satisfies the field equations (4.32). Because we have already fixed the spacetime coordinates by the unitary gauge condition (4.54), these solutions obtained from the standard form by fixing the Stückelberg fields and applying the coordinate transformation only to the metric should be regarded to be inequivalent mutually [59]. In appendix (C), we exhaust all the other possible solutions as follows. The solution with a flat metric. The metric form should be that of (C.3) with S given by one of the values in (C.4). No constraint on the parameters α and β is required. **Solution SdS-I**: The Schwarzschild-de Sitter type solution discussed in this section. The cosmological constant is given by $\Lambda = m^2/\alpha$, and the metric is given by (C.9) with $S = \alpha/(1+\alpha)$. The parameters are constrained as $\beta = \alpha^2$, but the function $T_0(t, r)$ can be arbitrary. **Solution SdS-II**: The Schwarzschild-de Sitter type solution whose metric is given by (C.9) with constant S given by (C.13) and the cosmological constant (C.14). The parameters α and β are weakly constrained as $\beta < \alpha^2$, but the function T_0 is constrained to those satisfying (C.12). In the unitary gauge, if we introduce all the degrees of freedom inside the dynamical metric, the Schwarzschild de-Sitter (SdS-I) metric becomes

$$ds^2 = -\mu^2 f(Sr)dt^2 - 2\mu h'(r) f(Sr) dt dr + \frac{S^2 - (h'(r))^2 f^2(Sr)}{f(Sr)} dr^2 + S^2 r^2 d\Omega_2^2. \tag{4.82}$$

This metric corresponds to the rescaling $r \to Sr$ with a constant S and introducing the coordinate transformation

$$t \to \mu t + h(r), \tag{4.83}$$

inside a metric of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2. \tag{4.84}$$

From the metric (4.82) and the metric components defined previously in (4.67), (4.68) and (4.69), together with the determinant condition (4.72), we get

$$c^2 + (1-a)(1-b) = \frac{1}{\mu S}, \tag{4.85}$$

$$(1-b)^2 - c^2 = -\frac{1}{S^2} f(Sr), \tag{4.86}$$

$$(1-a)^2 - c^2 = \frac{S^2 - (h')^2 f(Sr)}{\mu^2 S^2}, \tag{4.87}$$

$$c(2-a-b) = \frac{h'(r) f(Sr)}{\mu S^2}. \tag{4.88}$$

From these equations, we can observe that only the condition (4.85) is invariant under coordinate transformations. It corresponds to the determinant of the matrix $M^{\mu}{}_{\nu} = (1-Q)^{\mu}{}_{\nu}$ in agreement with the result (4.57).

4.4 Gauge invariant formulation for Black Hole perturbations

In this section we introduce some notions for working with the gauge invariant perturbation approach as has been formulated previously in [60, 61]. We start from a background metric given by

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = g_{ab}(y)dy^a dy^b + r^2(y)d\sigma_n^2, \tag{4.89}$$

where g_{ab} is the metric of a two-dimensional spacetime D^2 and

$$d\sigma_n^2 = \gamma_{ij}dx^i dx^j, \tag{4.90}$$

is the metric with a constant sectional curvature K on a unit two-sphere S^2. The internal metric γ_{ij} in S^2 is proportional to the $i-j$ components of the Ricci tensor

$$\hat{R}_{ij} = (n-1)K\gamma_{ij}, \tag{4.91}$$

for some constant K. If S^2 is maximally symmetric, then the constant K corresponds to the sectional curvature of S^2 and we can normalize $K = 0, \pm 1$. In general, we assume that S^2 is complete since it describes the cross section of the event horizon. Here I will take $K = 1$ and $n = 2$. We denote the covariant derivative, the connection coefficients, and the curvature tensors for the different spaces as follows

$$\nabla_{\mu}, \qquad \Gamma^{\mu}_{\nu\lambda}, \qquad R_{\mu\nu\lambda\beta}, \tag{4.92}$$

for the four-dimensional spacetime (whole),

$$D_a, \qquad {}^m\Gamma^a_{bc}, \qquad {}^mR_{abcd}, \tag{4.93}$$

for the subspace with metric g_{ab}. And finally

$$\hat{D}_i, \qquad \hat{\Gamma}^i_{jk}, \qquad \hat{R}_{ijkl} = K(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}), \tag{4.94}$$

for the bi-dimensional subspace with internal metric γ_{ij}. The geometric structure of the background, requires a background energy-momentum tensor given by

$$T_{ab} = T_{ab}(y), \qquad T_{ai} = 0, \qquad T^i{}_j = P\delta^i{}_j. \tag{4.95}$$

Here P is a scalar field on D^2. In this manuscript, we will be working with a static background metric.

4.4.1 Tensorial decomposition of perturbations

We classify the perturbation variables into three different types according to their tensorial behavior on the subspace S^2 in such a way that the different type of perturbations get decoupled. After we can introduce the harmonic tensors on S^2 such that each type of Einstein equations reduce to a set of equations for the subspace D^2. The perturbation variables can be classified in agreement with the decomposition theorems as follows:
i). If (S^2, γ_{ij}) is a compact Riemannian manifold. Then any dual vector field on S^2 can be uniquely decomposed as

$$v_i = V_i + \hat{D}_i S, \tag{4.96}$$

where $\hat{D}^i V_i = 0$. Here V_i and S are respectively, the vector and scalar-type components of the dual vector v_i.
ii). For a compact Riemannian Einstein space (S^2, γ_{ij}) with curvature tensor $\hat{R}_{ij} = c\gamma_{ij}$ for some constant c. Any second rank symmetric tensor t_{ij} can be uniquely decomposed as

$$t_{ij} = 2\hat{D}_{(i}t^{(1)}_{j)} + t_L\gamma_{ij} + \hat{L}_{ij}t_T, \tag{4.97}$$

$$\hat{L}_{ij} := \hat{D}_i\hat{D}_j - \frac{1}{2}\gamma_{ij}\hat{\Delta}, \tag{4.98}$$

where $\hat{D}^i t^{(1)}_i = 0$ and $t_L = t^m{}_m/n$ and $\hat{\Delta}$ is the Laplace-Beltrami operator. We refer to $t^{(1)}_i$ and (t_T, t_L) as the vector and scalar-type components for t_{ij} respectively. If we consider perturbations $h_{\mu\nu}$ over the full-spacetime with metric $g_{\mu\nu}dx^\mu dx^\nu$, then we can decompose the perturbation as

$$h_{\mu\nu}dx^\mu dx^\nu = h_{ab}dy^a dy^b + 2h_{ai}dy^a dz^i + h_{ij}dz^i dz^j. \tag{4.99}$$

Here the components h_{ab} are purely scalar with respect to transformations over S^2. If we apply the decomposition theorems (i) and (ii) explained above over the vectorial h_{ai} and tensorial components h_{ij}, then we get

$$h_{ai} = \hat{D}_i h_a + h^{(1)}_{ai}, \tag{4.100}$$

$$h_{ij} = 2\hat{D}_{(i}h^{(1)}_{j)} + h_L\gamma_{ij} + \hat{L}_{ij}h^{(0)}_T, \tag{4.101}$$

where,

$$\hat{D}^i h^{(1)}_{ai} = 0, \qquad \hat{D}^i h^{(1)}_{T\ i} = 0. \tag{4.102}$$

The vector part of $h_{\mu\nu}$ is $(h^{(1)}_{ai}, h^{(1)}_{T\ i})$, and the scalar part consists of $(h_{ab}, h_a, h_L, h^{(0)}_T)$. Note that it is possible to add an arbitrary function of y to h_a in eq. (4.100). This corresponds to the exceptional mode $l = 0$ (S-mode) in the harmonic expansion to be explained later. In eq. (4.101), $h^{(0)}_T$ is uniquely determined up to functions belonging to the Kernel of the operator $\hat{L}_{ij}$, which consists of the S-mode ($l = 0$) and the $l = 1$ modes in the harmonic expansion. Similarly, $h^{(1)}_{Ti}$ is unique up to a combination of the Killing vector of S^2 with arbitrary functions of y as coefficients. This corresponds to the exceptional mode with $l = 1$ in the harmonic expansion. These exceptional modes are spurious as the S-mode for h_a and should be discarded in physical arguments. With this understanding, the scalar components (h_{ab}, h_a, h_L, h_T) of the metric perturbation $h_{\mu\nu}$ describe *the scalar perturbation*, and the vector components $(h^{(1)}_{ai}, h^{(1)}_{T\,i})$ describe *the vector perturbation*. In a similar way, we can decompose the energy-momentum perturbations as follows

$$\delta T^a_{\ i} = \hat{D}_i \delta T^a + \delta T^{a(1)}_{\ i}, \tag{4.103}$$

$$\delta T^i_{\ j} = \delta T^{(1)i}_{\ j} + \delta P \delta_{ij} + \hat{L}_{ij} \delta T^{(0)}_T, \tag{4.104}$$

with $\hat{D}^i \delta T^{(1)}_{ai} = 0 = \hat{D}^i \delta T^{(1)}_{T\ i}$. Hence, the scalar and vector components of the perturbation of the energy-momentum tensor consist of $(\delta T_{ab}, \delta T^a, \delta P, \delta T^{(0)}_T)$ and $(\delta T^{(1)a}_i, \delta T^{(1)i}_j)$, respectively. There exist spurious exceptional modes in δT^a and $\delta T^{(0)}_T$ as in the metric perturbation decomposition.

4.4.2 Gauge invariant variables

The Einstein equations are invariant under the gauge transformation generated by an (infinitesimal) vector field ζ^M. After performing a gauge transformation, the perturbation variable $h_{\mu\nu}$ and its transform $h_{\mu\nu} - \pounds_\zeta g_{\mu\nu}$, both represent the same physical situation. Then we have an ambigüity since there are infinite perturbation variables representing the same. If we want to remove the redundancy, we can construct a gauge-invariant set of variables and then use them as a basis for the perturbation equations. This automatically extract the physical degrees of freedom related to the perturbations.
First, let's write the gauge transformations for the components of $h_{\mu\nu}$ as follows

$$h_{ab} \to h_{ab} - D_a \zeta_b - D_b \zeta_a, \tag{4.105}$$

$$h_{ai} \to h_{ai} - r^2 D_a \left(\frac{\zeta_i}{r^2} \right) - \hat{D}_i \zeta_a, \tag{4.106}$$

$$h_{ij} \to h_{ij} - 2\hat{D}_{(i}\zeta_{j)} - 2r(D^a r)\zeta_a \gamma_{ij}. \tag{4.107}$$

In a similar way, the gauge transformations for the perturbation of the energy-momentum tensor components become

$$\delta T_{ab} \to \delta T_{ab} - \zeta^c D_c T_{ab} - T_{ac} D_b \zeta^c - T_{bc} D_a \zeta^c, \tag{4.108}$$

$$\delta T_{ai} \to \delta T_{ai} - T_{ab} \hat{D}_i \zeta^b - r^2 P D_a(\zeta_i / r^2), \tag{4.109}$$

$$\delta T_{ij} \to \delta T_{ij} - \zeta^a D_a(r^2 P)\gamma_{ij} - P(\hat{D}_i\zeta_j + \hat{D}_j\zeta_i). \tag{4.110}$$

From the decomposition theorem, the generators ζ^M are separated into vector and scalar components as

$$\zeta_a = T_a, \qquad \zeta_i = V_i + \hat{D}_i S, \tag{4.111}$$

where $\gamma^{ij}\hat{D}_i V_j = 0$. For vector perturbations, the gauge transformation of the metric perturbation is given as

$$h^{(1)}_{ai} \to h^{(1)}_{ai} - r^2 D_a\left(\frac{V_i}{r^2}\right), \tag{4.112}$$

$$h^{(1)}_{Ti} \to h^{(1)}_{Ti} - V_i. \tag{4.113}$$

Then we can define the gauge invariant basis

$$F^{(1)}_{ai} = h^{(1)}_{ai} - r^2 D_a\left(\frac{h^{(1)}_{Ti}}{r^2}\right). \tag{4.114}$$

In a similar way, we can define gauge invariant variables for the gauge transformations of the components of the energy-momentum tensor

$$\tau^{(1)a}_{\ \ i} := \delta T^{(1)a}_{\ \ i}, \tag{4.115}$$

$$\tau^{i(1)}_{\ j} := \delta T^{(1)i}_{\ \ j}. \tag{4.116}$$

Then any vector invariant variable can be expressed as a linear combination of $(F^{(1)}_{ai}, \tau^{(1)}_{ai}, \tau^{(1)}_{ij})$ and their derivatives. Then we can define the perturbations $(h^{(1)}_{ai}, \delta T^{(1)}_{ai}, \delta T^{(1)}_{Tj})$ in terms of the above defined invariants and $h^{(1)}_{Ti}$. Under gauge transformations, $h^{(1)}_{Ti}$ just transforms like ζ_i, then if we define this variable in terms of the invariant basis, we are automatically specifying the gauge. For scalar perturbations, the gauge transformation law is given by

$$h_{ab} \to h_{ab} - 2D_{(a}T_{b)}, \tag{4.117}$$

$$h_a \to h_a - T_a - r^2 D_a\left(\frac{S}{r^2}\right), \tag{4.118}$$

$$h_L \to h_L - 2r(D^a r)T_a - \hat{\Delta}S, \tag{4.119}$$

$$h_T \to h_T - 2S. \tag{4.120}$$

We can define $X_M = (X_a, X_i = \hat{D}_i X_L)$ as

$$X_a := -h_a + \frac{r^2}{2} D_a \left(\frac{h_T}{r^2}\right), \qquad X_L := -\frac{h_T}{2}, \tag{4.121}$$

and X_M just transforms like $X_M \to X_M + \zeta_M$, and then

$$(X_a, X_L) \to (X_a + T_a, X_L + S). \tag{4.122}$$

We can now define the following set of gauge invariant variables

$$F^{(0)}_{ab} = h_{ab} + 2D_{(a}X_{b)}, \tag{4.123}$$

$$F^{(0)} = h_L + 2r(D^a r)X_a + \hat{\Delta} X_L. \tag{4.124}$$

For the exceptional modes, these are not gauge invariants. For the case of matter perturbations, we have

$$\Sigma^{(0)}_{ab} := \delta T_{ab} + X^c D_c T_{ab} + T_{ac} D_b X^c + T_{bc} D_a X^c, \tag{4.125}$$

$$\Sigma^{a(0)}_i := \hat{D}_i \delta T^a + T^a{}_b \hat{D}_i X^b - P \hat{D}_i X^a, \tag{4.126}$$

$$\Sigma^{(0)}_L := \delta P + X^a D_a P, \tag{4.127}$$

$$\Pi^{(0)} = \delta T^{(0)}_T. \tag{4.128}$$

For the exceptional modes, all or some of these variables are not gauge invariants. Further, for the S-modes, $\Sigma^{(0)}_{ai}$ and $\Pi^{(0)}_{ij}$ do not exist, and for the exceptional modes with $l = 1$, $\Pi^{(0)}_{ij}$ does not exist. As in the case of metric perturbations, any scalar gauge invariant can be expressed as a combination of the variables $(F^{(0)}_{ab}, F^{(0)}, \Sigma^{(0)}_{ab}, \Sigma^{(0)}_{ai}, \Sigma^{(0)}, \Pi^{(0)}_{ij})$ and their derivatives. In general, we can express the metric and matter perturbations in terms of the above set of gauge invariants and in terms of X_M. Writing X_M in terms of the gauge invariant variables is equivalent to fixing gauge. In the next sections we will use the gauge invariant approach explained here by expanding the corresponding variables in terms of Harmonic tensors as it is explained in [60, 61].

4.4.3 Harmonic expansions

In practical arguments, it is often more convenient to use the harmonic expansions for perturbation variables and their gauge-invariant combinations. We also use it in the subsequent sections. Here we give some expressions for scalar and vector harmonic expansions relevant to the analysis in our paper, but more details can be found in [60, 61]. First, in order to expand vector perturbations, we use the irreducible harmonic vectors defined by the eigenvalue problem

$$\hat{\Delta} V_i = -k_v^2 V_i, \quad \hat{D}_i V^i = 0. \tag{4.129}$$

For S^2, the eigenvalue k_v^2 is given by

$$k_v^2 = l(l+1) - 1, \quad l = 1, 2, ... \tag{4.130}$$

Note that V_i is proportional to $\epsilon_{ij}\hat{D}^j S$ where S is some scalar harmonics with the same l. The lowest mode with $l = 1$ is exceptional because it can be shown to be a Killing vector field on S^2 and satisfies

$$V_{ij} := -\frac{1}{k_v}\hat{D}_{(i}V_{j)} = 0. \tag{4.131}$$

The basic variables for vector perturbations can be expanded in terms of the vector-type harmonic basis as

$$h_{ai}^{(1)} = r f_a V_i, \qquad h_{Ti}^{(1)} = -\frac{r^2}{k_v} H_T V_i, \tag{4.132}$$

and correspondingly, the gauge-invariant variables are expanded as

$$F_{ai}^{(1)} = r F_a V_i, \qquad \tau_i^{(1)a} = r\tau^a V_i, \qquad \tau_j^{(1)i} = \tau_T V_j^i, \tag{4.133}$$

for the case of generic modes satisfying $m_V := k_v^2 - 1 = (l+2)(l-1) > 0$, where the indices of the harmonic tensors are lowered and raised by γ_{ij}. Here and in the following, we omit the index for the harmonic basis and the corresponding summation symbols for simplicity. For the exceptional modes with $m_V = 0$, i.e. $l = 1$, there is only one gauge-invariant

$$F_{ab\,i}^{(1)} = r F_{ab}^{(1)} V_i, \qquad F_{ab}^{(1)} = r D_a\left(\frac{F_b}{r}\right) - r D_b\left(\frac{F_a}{r}\right). \tag{4.134}$$

For scalar perturbations, we use a basis for the scalar harmonic functions satisfying the eigenvalue problem

$$\hat{\Delta} S = -k_s^2 S, \qquad k_s^2 = l(l+1), l = 0, 1, 2, ..., \tag{4.135}$$

and the associated vector and tensors defined by

$$S_i = -\frac{1}{k_s}\hat{D}_i S, \quad S_{ij} = \frac{1}{k_s^2}\hat{L}_{ij} S. \tag{4.136}$$

In terms of these harmonic tensors, the perturbation variables for scalar perturbations can be expanded as

$$\begin{aligned}
h_{ab} &= f_{ab} S, \quad h_a = -\frac{r}{k_s} f_a S, \\
h_L &= 2r^2 H_L S, \quad h_T = 2\frac{r^2}{k_s^2} H_T S, \\
\delta T_{ab} &= \tau_{ab} S, \quad \delta T^a = -\frac{r}{k_s}\tau^a S, \\
\delta P &= \tau_L S, \quad \delta T_T^{(0)} = \frac{r^2}{k_s^2}\tau_T S,
\end{aligned} \tag{4.137}$$

and the corresponding gauge-invariant variables are

$$\begin{aligned} F_{ab}^{(0)} &= F_{ab}S, \quad F^{(0)} = 2r^2 FS, \\ \Sigma_{ab}^{(0)} &= \Sigma_{ab}S, \quad \Sigma_i^{(0)a} = r\Sigma^a S_i, \\ \Sigma_L^{(0)} &= \Sigma_L S, \quad \Pi^{(0)} = \frac{r^2}{k_s^2}\tau_T S. \end{aligned} \tag{4.138}$$

For exceptional modes; τ_T does not exist for the $l = 0$ and $l = 1$ modes, and Σ_a does not exist for the $l = 0$ modes.

4.5 Perturbation analysis in the dRGT formalism

4.5.1 Background

The perturbation analysis inside the bimetric formalism revealed that the black hole reproduces the Gregory-Laflamme instability under spherical perturbations [13, 14, 62], the exception is the case where the mass of the graviton (in the massive metric) takes some specific value in relation to Λ (Partially massless gravity regime). However, it has been recently confirmed that the black holes are stable when the fiducial metric takes the Minkowskian form even inside the bi-metric formulation [63]. This result is clearly inspired in the original derivation [59], where the authors followed the standard formulation for the non-linear massive gravity theory with only one dynamical metric. It is possible to keep the gravitational degrees of freedom inside the dynamical metric instead of distributing them between the dynamical metric and the Stückelberg fields as has been done in [17]. This means that basically we will work in the unitary gauge. In the unitary gauge, the fiducial metric takes the form (4.55). The dynamical spherically symmetric metric defined in eq. (4.82), is equivalent to

$$g_{\mu\nu}dx^\mu dx^\nu = -f(S_0 r)(\partial_t h(r,t)dt + \partial_r h(r,t)dr)^2 + \frac{S_0^2}{f(S_0 r)}dr^2 + S_0^2 r^2 d\Omega_2^2, \tag{4.139}$$

where in this case we have a gauge function $h(r,t)$ depending on both, t and r and $\mu = 1$ without loss of generality. The coordinate transformation (4.83) in this case is extended to

$$dt \to \partial_t h(t,r)dt + \partial_r h(r,t)dr \qquad dr \to S_0 dr. \tag{4.140}$$

If $f(r) = 1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2$, then the previous metric is equivalent to the Schwarzschild de-Sitter one (as has been explained previously) after some coordinate transformations. The fiducial and the dynamical metric after rescaling in unitary gauge become

$$f_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \frac{dr^2}{S_0^2} + \frac{r^2}{S_0^2}d\Omega_2^2, \tag{4.141}$$

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -f(r)(\partial_t h(r,t)dt + \partial_r h(r,t)dr)^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \tag{4.142}$$
$$= g_{ab}dy^a dy^b + r^2 d\Omega_2^2. \tag{4.143}$$

Rescaling the fiducial metric in the form given by eq. (4.141) is equivalent to rescale the Stückelberg fields like

$$\phi^t = t, \quad \phi^x = \frac{rcos\theta}{S_0}, \quad \phi^y = \frac{rsin\theta cos\phi}{S_0}, \quad \phi^z = \frac{rsin\theta sin\psi}{S_0}, \tag{4.144}$$

where θ and ψ are the zenithal and azimuthal angles for the spherical symmetry. The above r-coordinate rescaling also affects the Q matrix. Because the dRGT theory has general covariance, $(g^* f_*) = (g^{\mu\alpha} f_{\alpha\nu})$ transforms as

$$g^* f_* \to T^{-1} g^* f_* T; \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/S_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.145}$$

Because the mixed tensor Q should behave exactly as $g^* f_*$ under a coordinate transformation, the r-rescaling transforms Q from the old value Q' to

$$Q^{\mu}{}_{\nu} = T^{-1} Q'^{\mu}{}_{\nu} T = \begin{pmatrix} a & \frac{c}{S_0} & 0 & 0 \\ -S_0 c & b & 0 & 0 \\ 0 & 0 & 1 - \frac{1}{S_0} & 0 \\ 0 & 0 & 0 & 1 - \frac{1}{S_0} \end{pmatrix}. \tag{4.146}$$

Note that due to the r-rescaling, the expression for g_{ab} in terms of a, b, and c is modified as follows

$$\begin{aligned} g_{tt} &= -\frac{(1-b)^2 - c^2}{[(1-a)(1-b)+c^2]^2}, \\ S_0^2 g_{rr} &= \frac{(1-a)^2 - c^2}{[(1-a)(1-b)+c^2]^2}, \\ S_0 g_{tr} &= -\frac{c(2-a-b)}{[(1-a)(1-b)+c^2]^2}, \\ S_0^{-1}(-g_{(2)})^{-1/2} &= c^2 + (1-a)(1-b). \end{aligned} \tag{4.147}$$

Similarly, a, b and c are expressed in terms of the new metric g_{ab} as

$$\begin{aligned} 1 - a &= \frac{1}{\underline{\mathrm{M}}_1}(-S_0 g^{tt} + (-g_{(2)})^{-1/2}), \\ c &= -\frac{g^{tr}}{\underline{\mathrm{M}}_1}, \\ 1 - b &= \frac{1}{\underline{\mathrm{M}}_1}(S_0^{-1} g^{rr} + (-g_{(2)})^{-1/2}), \end{aligned} \tag{4.148}$$

with

$$\underset{\sim}{\mathrm{M}}_1 = (-g_{(2)})^{-1/2}\left(-g_{tt} + S_0^2 g_{rr} + 2S_0(-g_{(2)})^{1/2}\right)^{1/2},$$
$$g_{(2)} = g_{tt}g_{rr} - g_{tr}^2. \tag{4.149}$$

4.5.2 Perturbation of $X^\mu{}_\nu$

We can calculate the perturbation of the tensor $X^\mu{}_\nu$ by using the definitions

$$h_{ab} = f_{ab}(t,r)Y, \qquad h_{ai} = r f_a(t,r)Y_i, \qquad h_{ij} = 2r^2[H_L\gamma_{ij}Y + H_T Y_{ij}], \tag{4.150}$$

where Y, Y_i and Y_{ij} represent the corresponding tensors for either the scalar or vector harmonics. We can simplify the expression (4.49) as follows

$$X^\mu{}_\nu = \chi_0\delta^\mu{}_\nu + \chi_1 Q^\mu{}_\nu + \chi_2(Q^2)^\mu{}_\nu + \chi_3(Q^3)^\mu{}_\nu, \tag{4.151}$$

where

$$\chi_0 = -\frac{\beta}{3}Q_3 + \frac{\alpha+\beta Q}{2}Q_2 - Q - \frac{\alpha}{2}Q^2 - \frac{\beta}{6}Q^3, \tag{4.152}$$

$$\chi_1 = 1 + \alpha Q + \frac{\beta}{2}(Q^2 - Q_2), \tag{4.153}$$

$$\chi_2 = -\alpha - \beta Q, \tag{4.154}$$

$$\chi_3 = \beta, \tag{4.155}$$

and we define

$$\alpha = 1 + 3\alpha_3, \qquad \beta = 3(\alpha_3 + 4\alpha_4). \tag{4.156}$$

Then the perturbation of the matrix $X^\mu{}_\nu$ is determined by

$$\delta X^\mu{}_\nu = \delta\chi_0\delta^\mu{}_\nu + \delta\chi_1 Q^\mu{}_\nu + \delta\chi_2(Q^2)^\mu{}_\nu + \delta\chi_3(Q^3)^\mu{}_\nu + \chi_1\delta Q^\mu{}_\nu + \chi_2\delta(Q^2)^\mu{}_\nu + \chi_3\delta(Q^3)^\mu{}_\nu. \tag{4.157}$$

Here the perturbations $\delta\chi_n$ are linear combinations of δQ_n (where Q_n is defined in eq. (4.28)), namely, the perturbations of the traces of the powers of matrices $(Q^n)^\mu{}_\nu$. δQ_n can be calculated as

$$\delta Q_n = \frac{n}{2}Tr[h^\mu{}_\gamma(Q^{n-1})^\gamma{}_\beta(1-Q)^\beta{}_\omega], \tag{4.158}$$

and the perturbations for the matrices $Q^\mu{}_\nu$ can be determined from the equations

$$(\delta Q^\mu{}_\sigma)(Q^\sigma{}_\nu - \delta^\sigma{}_\nu) + (Q^\mu{}_\sigma - \delta^\mu{}_\sigma)\delta Q^\sigma{}_\nu = -\delta(M^2)^\mu{}_\nu = -(g^{\mu\gamma}h_{\gamma\beta}(M^2)^\beta{}_\nu), \tag{4.159}$$

where we have used $M^2 = g^* f_*$ and then $\delta(M^2) = (\delta g^*) f_*$ if all the gravitational degrees of freedom are inside the dynamical metric. Otherwise $\delta(M^2) = (\delta g^*) f_* + g^*(\delta f_*)$. However there is no loss of generality by assuming that all the degrees of freedom are inside the dynamical metric since it is always possible to make a coordinate transformation in order to go back to the unitary gauge if we start from any gauge. In solving eq. (4.159), it is important to note that the background metric g and the matrix $M = g^* f_*$ are direct sum of two dimensional sub matrices

$$g = g_{(1)}(t, r) \oplus g_{(2)}(\theta, \phi), \tag{4.160}$$

$$M = M_{(1)} \oplus M_{(2)}. \tag{4.161}$$

Because the calculations $\delta Q^a{}_b$, $\delta Q^a{}_i$ and $\delta Q^i{}_j$ decouple from each other, except for the calculation of δQ_n, namely, the perturbation of the traces, in appendix (D), we have calculated explicitly the components for δQ_1, δQ_2 and δQ_3. We concentrate in the family of solutions satisfying $\beta = \alpha^2$. That this condition is special, was first reported by Th. Nieuwenhuizen [64]. Additionally, we impose the determinant condition (4.85), which is invariant under any coordinate transformation including rescaling. First, the angular part $\delta Q^i{}_j$ can be easily calculated because $I_2 - Q^i{}_j = (1/S_0) I_2$. Then

$$\begin{aligned}\delta X^i{}_j &= \delta\chi_0 \delta^i{}_j + \delta\chi_1(1 - S_0^{-1})\delta^i{}_j \\ &\quad + \delta\chi_2(1 - S_0^{-1})^2 \delta^i{}_j + \delta\chi_3(1 - S_0^{-1})^3 \delta^i{}_j + \delta Q^i{}_j[\chi_1 + 2\chi_2(1 - S_0^{-1}) + 3\chi_3(1 - S_0^{-1})^2],\end{aligned} \tag{4.162}$$

where $\delta\chi_3 = 0$ and the angular components for $\delta Q^i{}_j$ can be found from eq. (4.159). They are

$$\delta Q^i{}_j = \frac{H_L Y \delta^i{}_j + H_T Y^i{}_j}{S_0}, \tag{4.163}$$

thus

$$\delta X^i{}_j = \omega(r)(H_L \delta^i{}_j - H_T Y^i{}_j), \tag{4.164}$$

where

$$\omega(r) = \frac{1+\alpha}{\alpha}\{\beta(c^2 + ab) + \alpha(a + b) + 1\}. \tag{4.165}$$

Next, for the $r-t$ components, again solving the set of equations generated by the expression (4.159), we get

$$\delta Q^a{}_b = \frac{1}{2(2 - a - b)} \left(h^a{}_c (M^2)^c{}_b + (1 - Q)^a{}_c f^{cd} h_{dh} (1 - Q)^h{}_b \right). \tag{4.166}$$

From eq. (4.157), the $a - b$ components for the perturbation of $X^\mu{}_\nu$ are

$$\delta X^a{}_b = \delta\chi_0\delta^a{}_b + \delta\chi_1 Q^a{}_b + \delta\chi_2 (Q^2)^a{}_b + \chi_1\delta Q^a{}_b + \chi_2\delta(Q^2)^a{}_b + \chi_3\delta(Q^3)^a{}_b. \tag{4.167}$$

If we replace the previous background results together with eq. (4.166), then

$$\delta X^a{}_b = 0. \tag{4.168}$$

Finally, for the $a-i$ components, we know that $(Q^n)^a{}_i = 0$ as can be verified from the background matrix (4.146). Then the $a-i$ components of the perturbation of $X^\mu{}_\nu$ are

$$\delta X^a{}_i = \chi_1\delta Q^a{}_i + \chi_2\delta(Q^2)^a{}_i + \chi_3\delta(Q^3)^a{}_i. \tag{4.169}$$

Here

$$\delta(Q^2)^a{}_i = \left(1-\frac{1}{S_0}\right)\delta Q^a{}_i + Q^a{}_b\delta Q^b{}_i, \tag{4.170}$$

$$\delta(Q^3)^a{}_i = \left(1-\frac{1}{S_0}\right)^2\delta Q^a{}_i + \left(1-\frac{1}{S_0}\right)Q^a{}_b\delta Q^b{}_i + (Q^2)^a{}_b\delta Q^b{}_i. \tag{4.171}$$

The components $\delta Q^a{}_i$ can be found again by solving the general equation (4.159), this time given by

$$[(1+1/S_0)\delta^a{}_b - Q^a{}_b]\delta Q^a{}_i = \frac{r}{S_0^2} f^a Y. \tag{4.172}$$

Then eq. (4.169) becomes

$$\delta X^a{}_i = \frac{1}{S_0^2}(\{\chi_1 + (1-1/S_0)\chi_2 + (1-1/S_0)^2\chi_3\}\delta^a{}_b + \{\chi_2 + (1-1/S_0)\chi_3\}Q^a{}_b \tag{4.173}$$

$$+\chi_3(Q^2)^a{}_b) \times ([1+1/S_0 - Q]^{-1})^b{}_c f^c Y_i. \tag{4.174}$$

If we insert all the background values for $Q^\mu{}_\nu$, then we obtain

$$\delta X^a{}_i = 0 \tag{4.175}$$

4.5.3 Vector perturbations

For vector perturbations, we can use the following harmonic expansion

$$h_{ab} = 0, \quad h_{ai} = r f_a Y_i, \quad h_{ij} = 2r^2 H_T Y_{ij}. \tag{4.176}$$

Similarly, a vector perturbation of the energy-momentum tensor

$$\kappa^2\tau^\mu{}_\nu := \kappa^2\delta T^\mu{}_\nu = -m^2\delta X^\mu{}_\nu, \tag{4.177}$$

has the harmonic expansion [60, 61]

$$\tau^a{}_b = 0, \quad \tau^a{}_i = r\tau^a Y_i, \quad \tau^i{}_j = \tau_T Y^i{}_j, \tag{4.178}$$

where τ^a and τ_T are gauge-invariant. From the calculations of the previous section, we get

$$\tau^a = 0, \tag{4.179}$$

$$\kappa^2 \tau_T = m^2 \omega(r) H_T. \tag{4.180}$$

These source terms have to satisfy the Bianchi identities, which for a vector perturbation reduce to [60, 61]

$$D_a(r^3\tau^a) + \frac{(l+2)(l-1)}{2[l(l+1)-1]^{1/2}} r^2 \tau_T = 0, \quad \rightarrow (l-1)\omega(r) H_T = 0. \tag{4.181}$$

Because $\omega(r) \neq 0$ for the family of solutions defined by $\beta = \alpha^2$, it follows that $H_T = 0$ for $l \geq 2$. Hence the perturbations are just identical to those for the vacuum Einstein gravity. Then the general solution for the perturbations is

$$f_a = F_a, \qquad H_T = 0, \tag{4.182}$$

where F_a is the gauge invariant variable satisfying the perturbed vacuum Einstein equations. It is obtained when we perform the Harmonic function expansions for the variable (4.114) as $F^{(1)}_{ai} = rF_a V_i$ [60, 61]. We can also write the Einstein equations as

$$D_a(r^3 F^{(1)}) - m_V r \epsilon_{ab} F^b = -2\kappa^2 r^3 \epsilon_{ab} \tau^b = 0, \tag{4.183}$$

$$k_V D_a(rF^a) = -\kappa^2 \tau_T = 0, \tag{4.184}$$

with $F^{(1)}$ defined as

$$F^{(1)} = \epsilon^{ab} r D_a \left(\frac{F_b}{r} \right). \tag{4.185}$$

The solutions are then stable for vector perturbations. For the exceptional mode $l = 1$, H_T does not exist and then F_a is not gauge invariant and transforms for $\zeta^a = 0$, $\zeta^i = LV^i$ as

$$\delta_g F_a = -rD_a L. \tag{4.186}$$

We know that the general solution for $l = 1$ in the Einstein case is a linear combination of this gauge mode and the rotational perturbation corresponding to the angular momentum component in the Kerr metric [61]. Hence, the general solution in the present case is given by

$$f_a = -rD_a L - \frac{2aM}{r} \partial_a T_0(t, r). \tag{4.187}$$

In particular, this shows that the dRGT theory admits a rotational black hole solution at the linear perturbation level.

4.5.4 Scalar perturbations

For scalar perturbations we take the full set of perturbations for the metric given in eq. (4.150). We also take the full set of perturbations for the source terms given by

$$\delta T_{ab} = \tau_{ab} Y, \quad \delta T^a{}_i = r\tau^a Y_i, \quad \delta T^i{}_j = \delta P \delta^i{}_j Y + \tau_T Y^i{}_j. \tag{4.188}$$

Then

$$\delta X_{ab} = \delta g_{ac} X^c{}_b + g_{ac} \delta X^c{}_b = \frac{\Lambda}{m^2} f_{ab} Y. \tag{4.189}$$

Hence, the perturbations of the effective energy-momentum tensor are given by

$$\tau_{ab} = -\Lambda f_{ab}, \tag{4.190}$$

$$\kappa^2 \tau^a = 0, \tag{4.191}$$

$$\kappa^2 \delta P = -m^2 \omega(r) H_L, \tag{4.192}$$

$$\kappa^2 \tau_T = m^2 \omega(r) H_T. \tag{4.193}$$

The corresponding standard gauge invariant variables are

$$\kappa^2 \Sigma_{ab} = \kappa^2 \tau_{ab} - 2\Lambda D_{(a} X_{b)} = -\Lambda F_{ab}, \tag{4.194}$$

$$\Sigma_a = \tau_a = 0, \tag{4.195}$$

$$\kappa^2 \Sigma_L = -m^2 \omega H_L, \tag{4.196}$$

and τ_T. These should satisfy the conservation laws

$$\frac{1}{r^3} D_a (r^3 \Sigma^a) - \frac{k}{r} \Sigma_L + \frac{k^2 - 2}{2kr} \tau_T = 0, \tag{4.197}$$

$$\frac{1}{r^2} D_b [r^2 (\Sigma^b_a + \Lambda F^b_a)] + \frac{k}{r} \Sigma_a - 2\frac{D_a r}{r} \Sigma_L = 0. \tag{4.198}$$

These reduce to

$$-2l(1+l) H_L = (l+2)(l-1) H_T, \qquad (l \geq 1), \tag{4.199}$$

$$H_L = 0. \tag{4.200}$$

Hence for all modes including the cases $l = 0, 1$ for which H_T does not exist, we obtain the constraint $H_L = H_T = 0$, and the perturbation equations are identical to those for the vacuum Einstein system with Λ, which has the structure

$$E_{ab} = 2\kappa^2\Sigma_{ab} = 0, \tag{4.201}$$
$$E^a = 2\kappa^2\Sigma^a = 0, \tag{4.202}$$
$$E_L = 2\kappa^2\Sigma_L = 0, \tag{4.203}$$
$$-\frac{k_s^2}{r^2}F_a^a = 2\kappa^2\tau_T = 0, \tag{4.204}$$

where E_{ab}, E_a and E_L are tensors written as differential linear combinations of the gauge-invariants F_{ab} and F. In particular, no instability occurs. The general solution for $l \geq 2$ is expressed in terms of gauge invariant quantities satisfying the perturbation equations for the vacuum Einstein system with Λ as

$$f^r = kF, \tag{4.205}$$

$$f_{ab} = F_{ab} - \frac{1}{k}[D_a(rf_b) + D_b(rf_a)], \qquad (k > 0), \tag{4.206}$$

$$H_L = H_T = 0, \tag{4.207}$$

where $f^t(t,r)$ is left as an arbitrary function. This corresponds to the freedom associated with infinitesimal coordinate transformation $\delta t = T^t S$, $\delta r = 0$ and $\delta z^i = 0$

$$\delta_g f_{ab} = -D_aT_b - D_bT_a, \qquad \delta_g f_a = \frac{k}{r}T_a, \qquad \delta_g H_L = \delta_g H_T = 0. \tag{4.208}$$

The exceptional modes with $l = 0, 1$ should be treated separately with care. First, for the S-mode with $l = 0$, the variables f_a and H_T do not exists. Hence

$$F_{ab} = f_{ab}, \qquad H_L = 0. \tag{4.209}$$

In this case, F_{ab} is not gauge invariant and transforms as

$$\delta_g f_{ab} = -D_aT_b - D_bT_a, \tag{4.210}$$

$$\delta_g H_L = -\frac{1}{r}T^r = 0. \tag{4.211}$$

The residual gauge freedom is represented by $\delta t = T^t(t,r)$. This result is in consistence with the existence of the degeneracy represented by the single function $T_0(t,r)$ in the background solution. Because the solution satisfies the Einstein's equations, from the Birkhoff theorem, we know that the general solution is a linear combination of the above gauge transformation from the background solution and the perturbation corresponding to the variation of the mass parameter in the background metric

$$f_{tt} = \delta M \partial_M g_{tt} = \frac{2\delta M}{r}, \tag{4.212}$$

$$f_{rr} = \delta M \partial_M g_{rr} = \frac{2\delta M}{r}\left(\frac{1}{f^2} + (h')^2\right) - f\delta M \partial_M (h'^2), \tag{4.213}$$

$$f_{tr} = \delta M \partial_M g_{tr} = \frac{2\delta M}{r} h' - f\delta M \partial_M h'. \tag{4.214}$$

Next, for the $l = 1$ mode, there is no H_T function, but now we have a non-zero f^a. However, due to the absence of H_T, then the functions F and F_{ab} are not gauge invariant, and they transform under the coordinate transformation $\delta y^a = T^a S$ and $\delta z^i = L(t,r) S^i$ as

$$\delta_g F = -\frac{k}{2} L - \frac{r}{k} g^{ra} D_a L, \tag{4.215}$$

$$\delta_g F_{ab} = -\frac{1}{k}[D_a(r^2 D_b L) + D_b(r^2 D_a L)]. \tag{4.216}$$

The function L is restricted by the condition $H_L = 0$ as

$$\delta_g H_L = -\frac{k}{2} L - \frac{1}{r} T^r = 0. \tag{4.217}$$

Because we know that the corresponding solutions with $l = 1$ to the vacuum Einstein system are exhausted by (F, F_{ab}) obtained from the trivial solution (0,0) by the above gauge transformation [60], the general solution to our perturbation with $l = 1$ is given by the gauge modes

$$f_{ab} = -D_a T_b - D_b T_a, \tag{4.218}$$

$$f_a = -r D_a L + \frac{k}{r} T_a, \tag{4.219}$$

$$H_L = 0, \tag{4.220}$$

where

$$L = -\frac{2}{kr} T^r. \tag{4.221}$$

4.6 Gauge invariant formulation for the perturbation formalism in the dRGT

The dRGT theory is completely covariant if the Stückelberg field is treated as a dynamical one. It is possible to write a gauge invariant formulation by introducing new gauge invariant variables for the perturbation of the Stückelberg field ϕ^α. If we define

$$\sigma^\alpha = \delta\phi^\alpha, \tag{4.222}$$

its gauge transformation under the coordinate transformation $\delta_g x^\mu = \zeta^\mu$ is given by

$$\delta_g \sigma^\alpha = -\ell_\zeta \phi^\alpha = -\zeta^\mu \partial_\mu \phi^\alpha. \tag{4.223}$$

In unitary gauge, in agreement with eq. (4.144), the Stückelberg fields are defined as

$$\phi^t = \frac{t}{\mu}, \quad \phi^r = \frac{r}{S_0}, \quad \phi^\theta = \frac{\theta}{S_0}, \quad \phi^\varphi = \frac{\varphi}{S_0}. \tag{4.224}$$

Then for $\delta_g y^a = T^a$, $\delta_g z^i = LY^i$, σ^a transforms as

$$\delta_g \sigma^t = -\frac{T^t}{\mu}, \qquad \delta_g \sigma^r = -\frac{T^r}{S_0}, \tag{4.225}$$

$$\delta_g \sigma_T = -\frac{L}{S_0}, \tag{4.226}$$

where

$$\sigma^i = \sigma_T Y^i. \tag{4.227}$$

Vector perturbations:

For vector perturbations, we have

$$\sigma^a = 0, \qquad \sigma^i = \sigma_T V^i. \tag{4.228}$$

From

$$\delta_g f_a = -r D_a L, \qquad \delta_g H_T = kL, \tag{4.229}$$

it is possible then to construct the gauge invariant variable

$$\hat{\sigma}_T = \sigma_T + \frac{1}{kS_0} H_T, \tag{4.230}$$

for generic modes with $l \geq 2$, in addition to the standard gauge-invariant variable for the metric

$$F_a = f_a + \frac{r}{k} D_a H_T. \tag{4.231}$$

Then the source term for the massive gravity equation $(\delta X^\mu{}_\nu)$ can be expressed in terms of the gauge invariant variable as

$$\tau^a = 0, \qquad \kappa^2 \tau_T = m^2 \omega(r) k S_0 \hat{\sigma}_T. \tag{4.232}$$

In terms of the gauge invariant $\hat{\sigma}_T$, our previous results are expressed as

$$\hat{\sigma}_T = 0, \qquad (l \geq 2). \tag{4.233}$$

This implies that the dynamical degree of freedom of the Stückelberg field is completely suppressed and the perturbation of the metric behaves exactly in the same way as in the Einstein gravity case. For the exceptional modes with $l = 1$, we only have a single gauge invariant quantity defined as

$$\hat{F}_a = f_a - S_0 r \partial_a(\sigma_T). \tag{4.234}$$

Our analysis showed that for $l = 1$, the general solution for F_a is given by

$$\hat{F}_a = -r D_a L - \frac{2\alpha M}{r} \partial_a T_0, \tag{4.235}$$

where $L(r,t)$ is an arbitrary function and α is an arbitrary constant corresponding to the angular momentum parameter. Thus, a functional degeneracy appears.

Scalar perturbations:

In this case, we adopt the gauge invariant variables for σ^α defined by

$$\hat{\sigma}^t = \sigma^t + \frac{X^t}{\mu}, \tag{4.236}$$

$$\hat{\sigma}^r = \sigma^r + \frac{X^r}{S_0}, \tag{4.237}$$

$$\hat{\sigma}_T = \sigma_T + \frac{1}{kS_0} H_T. \tag{4.238}$$

In this case, then the source terms corresponding to $\delta X^\mu{}_\nu$ are expressed as

$$\kappa^2 \Sigma_{ab} = -\Lambda F_{ab}, \tag{4.239}$$

$$\kappa^2 \Sigma_a = 0, \tag{4.240}$$

$$\kappa^2 \Sigma_L = m^2 \omega(r) \left(\frac{kS_0}{2} \hat{\sigma}_T + \frac{S_0}{r} (D_a r) \hat{\sigma}^a - F \right), \tag{4.241}$$

$$\kappa^2 \tau_T = m^2 \omega(r) k S_0 \hat{\sigma}_T. \tag{4.242}$$

From the calculations of the previous sections, we found that these source terms should vanish. From eqns. (4.239), (4.240), (4.241) and (4.242), we find the following constraints

$$\hat{\sigma}_T = 0, \qquad \hat{\sigma}^r = \frac{r}{S_0} F, \tag{4.243}$$

but $\sigma^t(\hat{t}, r)$ can be an arbitrary function. Hence, the functional degeneracy appears even for generic modes.

Exceptional modes

For the exceptional modes with $l = 1$, the variables F, F_{ab}, $\hat{\sigma}^a$, $\hat{\sigma}_T$ are not gauge invariants anymore since H_T is not defined in this case. F and F_{ab} transform as has been shown in eqns. (4.215) and (4.216). Additionally, $\hat{\sigma}^a$ and $\hat{\sigma}_T$ transform as

$$\delta_g \hat{\sigma}^t = -\frac{r^2}{\mu k} D^t L, \tag{4.244}$$

$$\delta_g \hat{\sigma}^r = -\frac{r^2}{S_0 k} D^r L, \tag{4.245}$$

$$\delta_g \hat{\sigma}_T = -\frac{L}{S_0}. \tag{4.246}$$

In this case however, we can construct the following set of gauge invariants

$$\hat{F} = F - \frac{S_0 r}{k} D^r(\hat{\sigma}_T) - \frac{k S_0}{2}\hat{\sigma}_T, \tag{4.247}$$

$$\hat{F}_{ab} = F_{ab} - \frac{S_0}{k}\left(D_a(r^2 D_b(\hat{\sigma}_T)) + D_b\left(r^2 D_a(\hat{\sigma}_T)\right)\right), \tag{4.248}$$

$$\tilde{\sigma}^t = \hat{\sigma}^t - \frac{S_0 r^2}{\mu k} D^t(\hat{\sigma}_T), \tag{4.249}$$

$$\tilde{\sigma}^r = \hat{\sigma}^r - \frac{r^2}{k} D^r(\hat{\sigma}_T). \tag{4.250}$$

The perturbation equations for these variables are obtained by the replacements

$$F \to \hat{F}, \qquad F_{ab} \to \hat{F}_{ab}, \qquad \hat{\sigma}^a \to \tilde{\sigma}^a, \qquad \hat{\sigma}_T \to 0. \tag{4.251}$$

The general solution is then expressed in terms of these variables as

$$\hat{F} = -\frac{k}{2} L - \frac{r}{k} D^r L, \tag{4.252}$$

$$\hat{F}_{ab} = -\frac{1}{k}\left(D_a(r^2 D_b L) + D_b(r^2 D_a L)\right), \tag{4.253}$$

$$\tilde{\sigma}^a = 0, \tag{4.254}$$

where $L(r,t)$ is an arbitrary function. For the exceptional modes with $l = 0$, from the gauge transformation formula

$$\delta_g f_{ab} = -D_a T_b - D_b T_a, \tag{4.255}$$

$$\delta_g H_L = -\frac{1}{r} T^r, \tag{4.256}$$

we can construct the gauge invariants from f_{ab}, H_L and σ_a as follows

$$\hat{F}_{ab} = f_{ab} - D_a\bar{\sigma}_b - D_b\bar{\sigma}_a, \tag{4.257}$$

$$\hat{F} = H_L - \frac{S_0}{r}\sigma^r, \tag{4.258}$$

where we have defined

$$\bar{\sigma}^t = \mu\sigma^t, \qquad \bar{\sigma}^r = S_0\sigma^r. \tag{4.259}$$

We have demonstrated that the general solution for $l = 0$ can be expressed in terms of these gauge invariants as

$$\hat{F}_{tt} = \frac{2\delta M}{r} + 2f\dot{T}^t, \tag{4.260}$$

$$\hat{F}_{rr} = \frac{2\delta M}{r}\left(\frac{1}{f^2} + (h')^2\right) - f\delta M\partial_M(h'^2) + 2h'f\partial_r T^t, \tag{4.261}$$

$$\hat{F}_{tr} = \frac{2\delta M}{r}h' - f\delta M\partial_M h' + f(h'\dot{T}^t + \partial_r T^t) - f'T^t, \tag{4.262}$$

where T^t is an arbitrary function of t and r, and δM is an arbitrary constant corresponding to the mass variation.

4.7 Black Holes in bi-gravity theories

The bi-gravity formalism emerges as an extension of massive gravity for the case where the fiducial metric becomes dynamical. In such a case, the action is [51, 65]

$$S = M_{pl}^2\int d^4x\sqrt{-g}\left(\frac{R_g}{2} + m^2U\right) + \frac{\kappa M_{pl}^2}{2}\int d^4x\sqrt{-f}R_f, \tag{4.263}$$

where R_g and R_f correspond to the Ricci scalars for both, the g and the f metrics respectively. κ is a dimensionless parameter used to distinguish the coupling constant for the f metric and U again represents the interaction of both metrics, it is given by the same expression given by eq. (4.24). As in the case of massive gravity, we can write the interaction term in terms of a matrix of the form

$$K^\mu{}_\nu = \delta^\mu{}_\nu - \sqrt{g^{\mu\gamma}f_{\gamma\nu}}. \tag{4.264}$$

If we vary the action (4.263) with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$, then we obtain the corresponding equations [12]

$$G_{\mu\nu} = m^2T^U_{\mu\nu}, \qquad \hat{G}_{\mu\nu} = \frac{\sqrt{-g}}{\sqrt{-f}}\frac{m^2}{\kappa}\hat{T}^U_{\mu\nu}, \tag{4.265}$$

where $G_{\mu\nu}$ and $\hat{G}_{\mu\nu}$ are the Einstein tensors for $g_{\mu\nu}$ and $f_{\mu\nu}$ respectively. And $T^U_{\mu\nu}$ with $\hat{T}^U_{\mu\nu}$ come from the interaction term U coming from the action (4.263). This term is given explicitly from eq. (4.24), where the potentials U_2, U_3 and U_4 are given by analogous expressions to (4.25), (4.26) and (4.27). We can rewrite these expressions here as

$$U_2 = \frac{1}{2!}\left((K)^2 - K^\mu{}_\nu K^\nu{}_\mu\right), \tag{4.266}$$

$$U_3 = -\frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\sigma}K^\mu{}_\alpha K^\nu{}_\beta K^\rho{}_\gamma, \tag{4.267}$$

$$U_4 = -\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}K^\mu{}_\alpha K^\mu{}_\beta K^\rho{}_\gamma K^\sigma{}_\delta, \tag{4.268}$$

where $\epsilon_{\mu\nu\rho\beta}$ is the Levi-Civita symbol. Then , $T^U_{\mu\nu}$ and $\hat{T}^U_{\mu\nu}$ are given explicitly by

$$T^U_{\mu\nu} = -g_{\mu\beta}(\delta^\beta{}_\alpha - K^\beta{}_\alpha)(K^\alpha{}_\nu - K\delta^\sigma{}_\nu) + O(K^3), \tag{4.269}$$

$$\hat{T}^U_{\mu\nu} = -T^U_{\mu\nu} + O(K^3). \tag{4.270}$$

Normally in bigravity formulations, the metrics are taken such that $g_{\mu\nu} = f_{\mu\nu}$. In such a case, all the K matrices are identically zero $K = 0$ as can be easily verified from expression (4.264). However, the perturbations for both metrics, are in general taken as different [12]. If $K = 0$ at the background level, then the vacuum solutions for bimetric become the same as in Einstein gravity. The simplest vacuum solution in bi-gravity is then the Schwarzschild metric given by

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} + r^2 d\Omega_2^2. \tag{4.271}$$

The perturbations over this background, are defined as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}, \qquad f_{\mu\nu} = f^{(0)}_{\mu\nu} + \tilde{h}_{\mu\nu}, \tag{4.272}$$

where $g^0_{\mu\nu} = f^{(0)}_{\mu\nu}$. If this is true, then $g^{\mu\alpha}f_{\alpha\nu} \approx \delta^\mu{}_\nu + (\tilde{h}^\mu{}_\nu - h^\mu{}_\nu)$ and then the K-matrix becomes

$$K^\mu{}_\nu \approx \frac{1}{2}h^{(-)\mu}{}_\nu + O(h^2), \tag{4.273}$$

where $h^{(-)\mu}{}_\nu = (h^\mu{}_\nu - \tilde{h^\mu}{}_\nu)$. If we perturb eqns. (4.265), then we get the following results for the first equation

$$\varepsilon^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} = m^2\delta T^U_{\mu\nu}, \tag{4.274}$$

with

$$\delta T^U_{\mu\nu} = -\frac{g_{\mu\beta}}{2}\delta^\beta{}_\alpha\left(h^{(-)\alpha}_{\ \nu} - h^{(-)}\delta^\alpha{}_\nu\right), \tag{4.275}$$

which is simplified to

$$\delta T^U_{\mu\nu} = -\frac{1}{2}\left(h^{(-)}_{\mu\nu} - h^{(-)} g_{\mu\nu}\right). \tag{4.276}$$

Additionally, for the second equation, we get

$$\varepsilon^{\alpha\beta}_{\mu\nu}\tilde{h}_{\alpha\beta} = \frac{m^2}{2\kappa}\left(h^{(-)}_{\mu\nu} - h^{(-)} g_{\mu\nu}\right). \tag{4.277}$$

Taking into account that $\delta T^U_{\mu\nu} = -\delta\hat{T}^U_{\mu\nu}$ in agreement with (4.270). The operator $\varepsilon^{\alpha\beta}_{\mu\nu}$ is given by

$$\varepsilon^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} = -\frac{1}{2}\left(\nabla_\mu\nabla_\nu h - \nabla_\nu\nabla_\sigma h^\sigma{}_\mu - \nabla_\mu\nabla_\sigma h^\sigma{}_\nu + \Box h_{\mu\nu} - g_{\mu\nu}\Box h + g_{\mu\nu}\nabla_\alpha\nabla_\beta h^{\alpha\beta} + 2R^\sigma{}_\mu{}^\lambda{}_\nu h_{\lambda\sigma}\right), \tag{4.278}$$

and it is just the linearized Einstein tensor around any background. In vacuum $R = 0 = R_{\mu\nu}$. It is possible to combine equations (4.274) and (4.277) such that

$$\varepsilon^{\alpha\beta}_{\mu\nu}h^{(-)}_{\alpha\beta} + \frac{m'^2}{2}(h^{(-)} - g_{\mu\nu}h^{(-)}) = 0, \tag{4.279}$$

$$\varepsilon^{\alpha\beta}_{\mu\nu}h^{(+)}_{\alpha\beta} = 0, \tag{4.280}$$

with $m' = m\sqrt{1+\frac{1}{\kappa}}$ and $h^{(+)} = h_{\mu\nu} + \kappa h_{\mu\nu}$. Then $h^{(-)}_{\mu\nu}$ is a massive field and $h^{(+)}_{\mu\nu}$ is a massless one. If we apply the divergence over eq. (4.279), then we get $\nabla^\nu h^{(-)}_{\mu\nu} = \nabla_\mu h^{(-)}$. Then substituting back these conditions in the trace of eq. (4.279), we get

$$\nabla^\mu h^{(-)}_{\mu\nu} = 0 = h^{(-)}. \tag{4.281}$$

Then the field equations (4.279) become

$$\Box h^{(-)}_{\mu\nu} + 2R^\sigma{}_\mu{}^\lambda{}_\nu h^{(-)}_{\lambda\sigma} = m'^2 h^{(-)}_{\mu\nu}. \tag{4.282}$$

The field $h^{(-)}_{\mu\nu}$ does not change under infinitesimal transformations, $\zeta^\mu = \delta x^\mu$. However, for $h^{(+)}_{\mu\nu}$, this change corresponds to the standard gauge transformation. Some authors claim that as the equations (4.282) look similar to the equations obtained from a 5-dimensional black string, then the Gregory-Laflamme instability should appear also in bi-gravity formulations [12, 62]. It was however found later by Cardoso et. al [13] that the instability disappears in the partially-massless limit. Here I will review briefly what is the meaning of this limit. Recently it was verified by Babichev that the black hole solutions are stable when the fiducial metric takes the Minkowskian form (flat) [63]. This work is inspired from the results obtained from the previous section [59].

The partially massless limit

The partially massless limit corresponds to some specific value taken by the graviton mass (for the massive field), such that the theory only propagates gravitons and photons and the scalar component disappears [66]. This limit only occurs if $m^2 = \frac{2}{3}\Lambda$, with m corresponding to the graviton mass. This specific value is known as the Higuchi limit [67]. For understanding the meaning of this, we can start from the massive action including cosmological constant Λ considered in [13]. The field equations are

$$\varepsilon^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} - \Lambda\left(h_{\mu\nu} - \frac{h}{2}g_{\mu\nu}\right) + \frac{m^2}{2}(h_{\mu\nu} - g_{\mu\nu}h) = 0. \tag{4.283}$$

The traces are taken with respect to the metric $g_{\mu\nu}$ which will be assumed to be the background metric. If we take the trace and divergence of the previous equation and then we use the following identity

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)h_{\alpha\beta} = R_{\alpha\omega\mu\nu}h^{\omega}{}_{\beta} + R_{\beta\omega\mu\nu}h_{\alpha}{}^{\omega}, \tag{4.284}$$

then we get the following set of equations

$$\begin{aligned} \Box h_{\mu\nu} + 2R_{\alpha\mu\beta\nu}h^{\alpha\beta} - m^2 h_{\mu\nu} &= 0, \\ \nabla^\mu h_{\mu\nu} &= 0, \\ \left(m^2 - \frac{2}{3}\Lambda\right)h &= 0. \end{aligned} \tag{4.285}$$

Higuchi demonstrated that for masses satisfying $m^2 < 2/3\Lambda$, the scalar component becomes a ghost. However, for masses larger than this value, all the dynamical components become physically relevant. The discontinuity point given by $m^2 = 2/3\Lambda$ is the very well known Higuchi limit [67] and it represents the partially massless regime. Cardoso and colleagues have demonstrated that in this limit, the black hole solutions in bi-gravity becomes stable [13]. Note that the partially massless regime requires the presence of the cosmological constant Λ. Inside the dRGT, it was demonstrated by de Rham that the theory expanded around de-Sitter becomes the best candidate for partially massless gravity theory [66].

Chapter 5

Modified theories and dark matter effects

In the previous chapter, we studied in detail a theory which is a candidate for solving the cosmological constant problem by introducing mass to the graviton. The extensions of this theory are known as bi-gravity formulations, where the fiducial metric also becomes dynamical. Within this extension, it has been recently proposed that dark matter effects can be found if both metrics are allowed to be coupled; to matter in the case of the physical metric, and to some kind of "twin" matter for the case of the fiducial metric [68]. Both metrics cannot be coupled to the ordinary matter since this would violate the equivalence principle [69]. Although the origin of dark matter is still a mystery, we understand perfectly its consequences in the galactic dynamics, clusters, etc. As well as its effects in the CMB and the weak gravitational lenses [70, 71, 72, 73, 74]. Some alternative theories have been proposed in order to explain what is observed. In particular the dRGT theory of massive gravity cannot reproduce dark matter effects. One of the reasons is that the modification of the Einstein-Hilbert action enters as non-derivative terms. It has been demonstrated by Deffayet et al., that in order to get the observed galactic dynamics we require to modify gravity such that first-order derivative terms appear in the action [75, 76]. This condition is not achieved by the dRGT or not even for some other attempts to modify gravity. In what follows I will show one of the attempts to modify gravity for large scales. In this case, the model fails to reproduce the very well known galactic dynamics which has been compactified in the expression proposed by Milgrom [18, 19, 20, 77] inside the so called Modified Newtonian Dynamics (MOND) theory. The model analyzed in what follows, can reproduce a similar expression in comparison with MOND but the dynamics is different due to the scale dependence of the interpolating function μ to be explained later. The appropriate galactic dynamics requires a scale-independent interpolating function μ. Before analyzing the dynamics of the model around a static-spherically symmetric solution, I will make a brief review of the MONdian formulation.

5.1 Lagrangian formulation of MOND

The modified dynamics assumption in agreement with Milgrom [78], can be based in the following set of minimal assumptions [21, 22]:
1). There exist a breakdown of Newtonian dynamics (second law and/or gravity) in the limit of small accelerations.
2). In this limit, the acceleration $\vec{a}$, of a test particle in the gravitating system is given by $\vec{a}\left(\frac{\vec{a}}{a_0}\right) \approx \vec{g}_N$, where $\vec{g}_N$ is the conventional gravitational field and a_0 is a constant with dimensions of acceleration.
3). The transition from the Newtonian regime to the small acceleration asymptotic region occurs within a range of order a_0 about a_0. The value of a_0 is of the same order of magnitude of cH_0. The original results obtained by Milgrom [78] can be described either of the following ways. A modification of the inertia

$$m\mu\left(\frac{a}{a_0}\right)\vec{a} = \vec{F}, \tag{5.1}$$

where $\vec{F}$ is an arbitrary static force assumed to depend on its sources in the conventional way, m is the gravitational mass of the accelerated particle and μ is the interpolating function which will be defined later. In the case of gravity, $\vec{F} = m\vec{g}_N$, where $\vec{g}_N = -\nabla\phi_N$ and ϕ_N is the gravitational potential deduced in the usual way from the Poisson equation. Alternatively, MOND in agreement with [78] can be described as a modification of gravity leaving the law of inertia ($m\vec{a} = \vec{F}$) intact. Then, $\vec{F} = m\vec{g}$ and $\vec{g}$ is a modified gravitational field derived from $\vec{g}_N$ using the relation

$$\mu(g/a_0)\vec{g} = \vec{g}_N. \tag{5.2}$$

If only gravitational forces were present, both formulations of MOND given by eqns. (5.1) and (5.2) would be equivalent. However, if we consider any force in general, then the previous formulations are not the same at all. The interpolating function $\mu(x)$ satisfies the following conditions

$$\mu(x) \approx 1 \quad if \quad x >> 1, \qquad \mu(x) \approx x \quad if \quad x << 1, \tag{5.3}$$

where the left side corresponds to the standard Newtonian limit. The function $\mu(x)$, can be defined in different ways. Here I will not be concerned with its definition but rather on its asymptotic behavior.
In cases of high symmetry (spherical, plane, or cylindrical), the gravitational field $\vec{g}$ as given by equation (5.2) is derivable from a scalar potential ϕ. However, in the most general cases this is not possible. As has been already explained by Milgrom in his paper [18, 19, 20], MOND cannot be considered as a theory, but only a successful phenomenological scheme for which an underlying theory can be constructed. One of the reasons is for example, that inside MOND theory there is no momentum conservation. In fact, momentum is only conserved approximately as far as the mass of the test body is much smaller than the source one [22].

Milgrom and Bekenstein have already derived a Lagrangian formulation for MOND, where μ and a_0 are introduced by hand. One of the purposes for constructing a fundamental theory which can contain MOND as a non-relativistic limit is to obtain the interpolating function $\mu(x)$ in terms of some fundamental quantities. This was one of the motivations for the work done in [7]. A Lagrangian formulation for MOND solves the momentum conservation problem associated typically to the original MOND version. A Lagrangian formulation also enables to calculate the dynamics of an arbitrary non-relativistic system [22].
There are two important assumptions inside the MOND theory, they are:
1). A composite particle (star or a cluster of stars) moving in an external field, say a galaxy, moves like a test particle according to MOND. Even if within the body, the relativistic accelerations are large. This assumption is possible as far as the mass of the test particle is much smaller than the mass of the galaxy.
2). When a system is accelerated as a whole in an external field, the internal dynamics of the system is affected by the presence of an external field (even when this field is constant without tidal forces). In particular, in the limit when the external (center of mass) acceleration of the system becomes much larger than the MOND scale a_0, the internal dynamics approaches to the Newtonian behavior even when the accelerations within the system are much smaller than a_0.
This second observation due to the Milgrom proposal is quite interesting and one of the motivations in [7] since it seems that the internal dynamics of the system can in principle be affected by the presence of some external field. This could in principle be done due to a non-local connection between the internal and external dynamics.
Normally Newtonian gravity is recovered at the non-relativistic regime of General Relativity (GR), and of a number of other relativistic theories of gravity. It is however, necessary to construct a new relativistic version of GR such that MOND could be recovered as a natural non-relativistic limit [22]. This new version could be constructed by considering two possibilities. 1). Additional degrees of freedom. However, dRGT is not able to do this. 2). Non-localities. If we choose to explain the dark matter effects by introducing non-localities, we must be able to explain the origin of such effects. In [7], the origin of non-localities is not relevant. However, some extra-degrees of freedom are introduced in order to recreate some "non-local" effect. This model was proposed by Nojiri and Odintsov some years ago [9]. Later the model was used by Sasaki and colleagues for reproducing screening effects for the cosmological constant for example. [8]
There are two very important reasons in order to construct a Relativistic version for MOND.
1). To help incorporate principles of MOND into the framework of modern theoretical physics.
2). To provide tools for investigating cosmology in light of MOND. [22]

5.2 The MOND field equations

In Newtonian gravity test bodies move with an acceleration equal to $\vec{g}_N = -\nabla\phi_N$, where ϕ_N is the Newtonian gravitational potential. It is determined by the Poisson equation $\nabla^2\phi_N = 4\pi G\rho$, where ρ is the mass density which produces ϕ_N. The Poisson equation may be derived from the Lagrangian

$$\mathcal{L}_N = -\int d^3r(\rho\phi_N + (8\pi G)^{-1}(\nabla\phi_N)^2). \tag{5.4}$$

Milgrom and Bekenstein suggested that in searching for a modification of this theory, we will want to retain the notion of a single potential ϕ_N from which acceleration derives. We want ϕ_N to be arbitrary up to some additive constant. The most general modification of L_N which yield these features is

$$\mathcal{L} = -\int d^3r \left(\rho\phi + (8\pi G)^{-1}a_0^2 f\left(\frac{(\nabla\phi)^2}{a_0^2}\right)\right), \tag{5.5}$$

where $f(x^2)$ is an arbitrary function. The scale of acceleration is necessary unless we are in the Newtonian case. If we perform the variation of $\mathcal{L}$ with respect to ϕ, with the variation of ϕ vanishing on the boundaries, we get

$$\vec{\nabla}\cdot\left(\mu\left(\frac{|\vec{\nabla}\phi|}{a_0}\right)\vec{\nabla}\phi\right) = 4\pi G\rho, \tag{5.6}$$

where $\mu(x) = f'(x^2)$. Eq. (5.6) is the equation determining the modified potential. A test particle is assumed to have acceleration $\vec{g} = -\vec{\nabla}\phi$. We supplement equation (5.6) by the boundary condition $|\vec{\nabla}\phi| \to 0$ as $r \to \infty$.
It is useful to write the field equation in terms of the modified Newtonian field $\vec{g}_N = -\vec{\nabla}\phi_N$, for the same mass distribution, which satisfies the Poisson equation. By eliminating ρ in equation (5.6), we get

$$\vec{\nabla}\cdot\left(\mu\left(\frac{|\vec{\nabla}\phi|}{a_0}\right)\vec{\nabla}\phi - \vec{\nabla}\phi_N\right) = 0. \tag{5.7}$$

The expression in parenthesis, must then be a curl, since its divergence is zero. Then we can write

$$\mu\left(\frac{g}{a_0}\right)\vec{g} = \vec{g}_N + \vec{\nabla}\times\vec{h}. \tag{5.8}$$

It has already been demonstrated by Bekenstein and Milgrom that the present theory satisfies the basic assumptions of MOND and that the curl term vanishes exactly for the spherically symmetric case and it vanishes at least as fast as $\frac{1}{r^3}$ at large distances from the source in general cases. In the next section I will consider the review of this point.

5.3 The field equations at large distances from the source

In agreement with [22], let's consider a bound density distribution of total mass M with the origin at the center of mass. Following the Bekenstein and Milgrom notation, let's define the vector field $\vec{u}$ as $\vec{u} \equiv \vec{\nabla}\phi_N - \mu\left(\frac{|\vec{\nabla}\phi|}{a_0}\right)\vec{\nabla}\phi$. For the reasons explained in the previous section, the vector $\vec{u}$, satisfies $\vec{\nabla}\cdot\vec{u} = 0$ and it vanishes at infinity. It is possible then to write $\vec{u}$ in terms of the vector potential $\vec{A}$

$$\vec{u} = \vec{\nabla}\times\vec{A}, \qquad \vec{A}(\vec{r}) = (4\pi)^{-1}\int \frac{\vec{\nabla}'\times\vec{u}(\vec{r'})}{|\vec{r}-\vec{r'}|}d^3r'. \tag{5.9}$$

The only term with an r^{-2} behavior at infinity which $\vec{u}$ can have is $\vec{u}^{(2)} = \vec{\nabla}\times(r^{-1}\vec{B}) = -r^{-3}\vec{r}\times\vec{B}$. Where $\vec{B} = (4\pi)^{-1}\int\vec{\nabla}'\times\vec{u}'d^3r'$, which is the lowest order in the multipole expansion (5.9).

If we can demonstrate that $\vec{B} = 0$, then we guarantee that the lowest contributing multipole term to $\vec{u}$ vanishes at least as fast as r^{-3}. In the limit of large r, we have

$$\mu\left(\frac{|\vec{\nabla}\phi|}{a_0}\right)\vec{\nabla}\phi = \vec{\nabla}\phi_N - \vec{u} = r^{-3}(GM\vec{r} + \vec{r}\times\vec{B}) + \vec{O}(r^{-3}). \tag{5.10}$$

Taking the absolute value, we get

$$\mu\left(\frac{|\vec{\nabla}\phi|}{a_0}\right)|\vec{\nabla}\phi| = (|\vec{\nabla}\phi_N|^2 + |\vec{u}|^2)^{1/2}. \tag{5.11}$$

This expression can be translated into

$$\mu\left(\frac{|\vec{\nabla}\phi|}{a_0}\right)|\vec{\nabla}\phi| = \left(\left(\frac{GM}{r^2}\right)^2 + \frac{B^2 sin^2}{r^4}\theta\right)^{1/2}. \tag{5.12}$$

As $r\to\infty$, the full MONDian regime operates and we can assume that $\mu\left(\frac{|\vec{\nabla}\phi|}{a_0}\right) = \frac{|\vec{\nabla}\phi|}{a_0}$. In such a case, the expression (5.12) becomes

$$|\vec{\nabla}\phi| = \frac{a_0^{1/2}}{r}\left(G^2M^2 + B^2 sin^2\theta\right)^{1/4}. \tag{5.13}$$

Assuming again the MONDian regime and replacing the previous expression inside of (5.10), we get

$$\vec{\nabla}\phi = a_0^{1/2}r^{-2}\frac{(GM\vec{r} + \vec{r}\times\vec{B})}{(G^2M^2 + B^2 sin^2\theta)^{1/4}} + \vec{O}(r^{-2}). \tag{5.14}$$

Here θ is the angle between $\vec{r}$ and $\vec{B}$ which we can take along the z-axis without lost of generality. Requiring the azimuthal component of $\vec{\nabla}\times(\vec{\nabla}\phi)$ to vanish, gives $\vec{B} = 0$ [22]. This means that $\vec{u}$ vanishes at large distances from a mass, at least as $\vec{O}(r^{-3})$. With this result, the equation (5.8) as $r\to\infty$, becomes

$$\mu\left(\frac{g}{a_0}\right)\vec{g} = \vec{g}_N + \vec{O}(r^{-3}), \tag{5.15}$$

consistent with the MOND predictions explained in eqns. (5.1) and (5.2). As $r \to \infty$, we get

$$\vec{g} \to -\frac{(GMa_0)^{1/2}}{r^2}\vec{r} + \vec{O}(r^{-2}). \tag{5.16}$$

In this limit, the potential becomes

$$\phi \to (GMa_0)^{1/2} ln\left(\frac{r}{r_0}\right) + O(r^{-1}), \tag{5.17}$$

where r_0 is an arbitrary radius. This potential leads to an asymptotically constant circular velocity $V_\infty = (GMa_0)^{1/4}$ as it is observed in the outskirts of spiral galaxies.

The field equation (5.6) is nonlinear and difficult to solve in general. However, in cases of high symmetry, the curl term in equation (5.8) vanishes identically and we have the exact result $\mu\left(\frac{g}{a_0}\right)\vec{g} = \vec{g}_N$ which is identical to equation (5.2). For systems for high degree of symmetry, then the solution for ϕ is straightforward and all the results obtained from the standard Newtonian theory can then be extended to the present formalism.

For example, the acceleration field at a distance r from the center in a spherical system depends only on the total mass $M(r)$, interior to r (in agreement with the Gauss' theorem), and in fact is given by $\mu\left(\frac{g}{a_0}\right)\vec{g} = -\frac{M(r)G\vec{r}}{r^3}$.

The field equation (5.6) is analogous to the equation for the electrostatic potential in a nonlinear isotropic medium in which the dielectric coefficient is a function of the electric field strength [22].

The field equation (5.6) is also equivalent to the stationary flow equations of an irrotational fluid which has a density $\hat{\rho} = \mu\left(\frac{|\vec{\nabla}\phi|}{a_0}\right)$, a negative pressure $\hat{P} = -\frac{1}{2}a_0^2 f\left(\frac{(\vec{\nabla}\phi)^2}{a_0^2}\right)$, flow velocity $\hat{\vec{v}} = \vec{\nabla}\phi$, and a source distribution $\hat{S}(\vec{r}) = 4\pi G\rho$. The fluid satisfies an equation of state $\hat{P}(\hat{\rho}) = -\frac{1}{2}a_0^2 f\left([(\mu^{-1}(\hat{\rho})]^2\right)$.

An equation of the same form as equation (5.6) has been studied to describe classical models of quark confinement using a very different form of the function μ at both, large and small values of its argument [79]. The conservation laws and other results related to the Lagrangian formulation of MOND can be found in [22]. In this manuscript I will omit such analysis.

5.4 A Non-local model for gravity

The non-local action suggested in [8] is given by

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}(R(1 + f(\Box^{-1}R)) - 2\Lambda) + l_{matter}(Q, g)\right), \tag{5.18}$$

where f is some function, $\Box$ is just the D'Alembertian for the scalar field, Λ is the cosmological constant which is supposed to be screened by the introduced non-localities and Q corresponds to the matter fields. We can rewrite the action by introducing two scalar fields ψ and ζ as follows [8]

$$
\begin{aligned}
S &= \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}(R(1+f(\psi)) - \zeta(\Box\psi - R) - 2\Lambda) + l_{matter}\right) \\
&= \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}(R(1+f(\psi)+\zeta) + g^{\mu\nu}\partial_\mu\zeta\partial_\nu\psi - 2\Lambda) + l_{matter}\right).
\end{aligned} \tag{5.19}
$$

If we vary the above action with respect to ζ, then $\Box\psi = R$ or $\psi = \Box^{-1}R$. The variation with respect to the metric is

$$
\begin{aligned}
0 = \frac{1}{2}g_{\mu\nu}\left(R(1+f(\psi)+\zeta) + g^{\alpha\beta}\partial_\alpha\zeta\partial_\beta\psi - 2\Lambda\right) - R_{\mu\nu}(1+f(\psi)+\zeta) \\
-\frac{1}{2}(\partial_\mu\zeta\partial_\nu\psi + \partial_\mu\psi\partial_\nu\zeta) - (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)(f(\psi)+\zeta) + \kappa^2 T_{\mu\nu},
\end{aligned} \tag{5.20}
$$

and the variation with respect to ψ gives

$$
0 = \Box\zeta - f'(\psi)R. \tag{5.21}
$$

The explicit solutions for the previous equations, can be found if we introduce a metric. In this manuscript, I will focus on spherically symmetric solutions.

5.4.1 The ghost free condition

In [8], it was found that after a conformal transformation to the Einstein frame, we get

$$
\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \qquad \tilde{R} = \frac{1}{\Omega^2}(R - 6(\Box ln\Omega + g^{\mu\nu}\nabla_\mu ln\Omega\nabla_\nu ln\Omega)), \tag{5.22}
$$

$$
\Omega^2 = \frac{1}{1+f(\psi)+\zeta}, \tag{5.23}
$$

which gives an action given by [8]

$$
S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}(\hat{R} - 6g^{\mu\nu}\nabla_\mu\phi'\nabla_\nu\phi' + e^{2\phi'}g^{\mu\nu}\nabla_\mu\zeta\nabla_\nu\psi - 2e^{4\phi'}\Lambda) + e^{4\phi'}l_{matter}(Q;e^{2\phi'}g)\right), \tag{5.24}
$$

where

$$
\phi' \equiv ln\Omega = -\frac{1}{2}ln(1+f(\psi)+\zeta), \tag{5.25}
$$

and $\hat{R}$ is the resulting Ricci scalar after performing the transformation (5.22). The condition for gravity to have a normal sign is

$$
1 + f(\psi) + \zeta > 0. \tag{5.26}
$$

If ϕ' and ψ are considered to be the independent fields, then

$$\zeta = e^{-2\phi'} - (1 + f(\psi)). \tag{5.27}$$

In terms of the new set of independent variables, the action is

$$S = \int \sqrt{-g}\frac{1}{2\kappa^2}(R - 6\nabla^\mu\phi'\nabla_\mu\phi' - 2\nabla^\mu\phi'\nabla_\mu\psi - e^{2\phi'}f'(\psi)\nabla^\mu\psi\nabla_\mu\psi - 2e^{4\phi'}\Lambda) + e^{4\phi'}l_{matter}(Q; e^{2\phi'g}). \tag{5.28}$$

The ghost free condition is simply

$$f'(\psi) > \frac{1 + f(\psi) + \zeta}{6} > 0. \tag{5.29}$$

5.5 The Newtonian limit in the standard case in S-dS metric

In [8], the metric is assumed to be FLRW. In this case, as we are concerned with the dark matter effects and we want to compare with the MONDian case, it is reasonable to assume that the space corresponds to the Newton-Hooke one, which is just the Newtonian limit for the Schwarzschild-de Sitter metric. Explicitly in eqs. (5.20) and (5.21), I will assume the metric to be

$$ds^2 = -\left(1 - \frac{2GM}{r} - \frac{1}{3}\frac{r^2}{r_\Lambda^2}\right)dt^2 + \left(1 - \frac{2GM}{r} - \frac{1}{3}\frac{r^2}{r_\Lambda^2}\right)^{-1}dr^2 + r^2d\Omega^2, \tag{5.30}$$

with $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$. I will work under the condition $r_s << r << r_\Lambda$, where the weak field approximation is then justified. Then we have to satisfy the standard results

$$G_{00}^{(1)} \approx \Box g_{00} \approx R^{(1)} = 2R_{00}^{(1)}, \tag{5.31}$$

where the weak field approximation for the Ricci tensor is given by

$$R_{\mu\nu}^{(1)} \equiv \frac{1}{2}(\Box h_{\mu\nu} - \partial^\lambda\partial_\mu h_{\lambda\nu} - \partial^\lambda\partial_\nu h_{\lambda\mu} + \partial_\mu\partial_\nu h), \tag{5.32}$$

and then, the first order Einstein's equations become

$$R_{\mu\nu}^{(1)} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} + \eta_{\mu\nu}\Lambda = -8\pi G_N T_{\mu\nu}^{(1)}, \tag{5.33}$$

with the metric given by (5.30), then we get

$$\nabla^2 g_{00} = -8\pi G\rho + 2\Lambda. \tag{5.34}$$

If $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$, then $\nabla g_{00} \approx \nabla h_{00}$. Thus

$$\nabla^2 \phi = 4\pi G_N \rho - \Lambda, \tag{5.35}$$

with $h_{00} = -2\phi = h_{ij}$ and $T_{00} \approx \rho$. The spherical symmetry assumption taken from the metric (5.30) is important since it implies that the curl term in equation (5.8) can be ignored in agreement with the analysis performed in the previous section. The results of this section will be used in the field equations 5.20 and 5.21.

5.5.1 The weak field approximation in non-local gravity and its relation with MOND

In agreement with Bekenstein [21], at the Newtonian limit we have to satisfy an equation similar to the AQUAL one given already in eq. (5.6). We rewrite it as follows

$$\nabla^2 \phi \approx \mu^{-1}(\kappa^2 \rho - \Lambda) - \mu^{-1}(\nabla \phi).\nabla \left(\mu \left(\frac{|\nabla \phi|}{a_0} \right) \right). \tag{5.36}$$

For the Newtonian limit of the field equations (5.20), I will make the expansions up to second order in the potential ϕ. Even if the second order terms are most likely negligible, I will keep them in order to get a more accurate result. Take into account that the non-localities, represented by $f(\psi)+\zeta$ in eq. (5.20) reproduce an amplification of the non-linearities related to the space time curvature and it includes the second order contributions. This amplification will however depend on a parameter γ which will be defined later. The 0-0 component of eq. (5.20) is then given by

$$\begin{aligned} 0 \approx & -\frac{1}{2}\left(R^{(1)}(1+f(\psi)+\zeta) + \nabla_r \zeta \nabla_r \psi - 2\Lambda\right) - \phi R^{(1)}(1+f(\psi)+\zeta) \\ & -R^{(1)}_{00}(1+f(\psi)+\zeta) + (1+2\phi)\Box(f(\psi)+\zeta) + \nabla_0\nabla_0(f(\psi)+\zeta) - \kappa^2\rho, \end{aligned} \tag{5.37}$$

where we have used $T_{00} = -\rho$. In this approach, we neglect the time-dependence of the scalar fields. However we take into account the curvature effects through the Christoffel connections. We can write ζ in terms of $f(\psi)$ if we solve the equation (5.21). For that purpose, we assume an exponential solution like $f(\psi) = f_0 e^{\gamma\psi}$ as has been suggested in [8]. Then the following relations are true

$$\nabla_\mu f(\psi) = \gamma f(\psi)\nabla_\mu \psi, \qquad \Box f(\psi) = \gamma f(\psi)\Box\psi + \gamma^2 f(\psi)\nabla_\mu\psi\nabla^\mu\psi. \tag{5.38}$$

We can then find the the solution for eq. (5.21) if we expand both sides of the equation and then compare the same order of magnitude terms as follows

$$2\nabla\phi\cdot\nabla\zeta + \frac{2}{r}\frac{\partial\zeta}{\partial r} + \frac{\partial^2\zeta}{\partial r^2} \approx -\gamma f(-2\phi)\left(4\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{4}{r}\frac{\partial\phi}{\partial r} + 2\frac{\partial^2\phi}{\partial r^2}\right). \tag{5.39}$$

The solution for ζ is (ignoring second order contributions)

$$\zeta(\psi) \approx f(\psi), \tag{5.40}$$

where we have used the Lagrange multiplier condition $R = \Box\psi$. If we replace the solution for $\zeta(\psi)$, and taking into account eq. (5.21) and the Lagrange multiplier condition, then eq. (5.37) becomes

$$0 \approx -\frac{1}{2}\left(R^{(1)}(1+2f(\psi)) + \gamma f(\psi)(\nabla\psi)^2 - 2\Lambda\right) - \phi R^{(1)}(1+2f(\psi)) - R^{(1)}_{00}(1+2f(\psi)) \\ +2\gamma(1+2\phi)f(\psi)\Box\psi + 2\nabla_0\nabla_0 f(\psi) - \kappa^2\rho. \tag{5.41}$$

The Christoffel connection component is given by $\Gamma^r_{00} \approx \partial_r\phi = \nabla_r\phi$. I will take the spatial components of the Einstein's equations as given by the standard Newtonian approach as it is explained in the standard textbooks. In such a case, I will take the Ricci tensor and the curvature scalars as

$$R^{(1)} = 2R^{(1)}_{00} \approx \Box g_{00} = \Box h_{00} = -2\Box\phi \approx -4(\nabla\phi)\cdot(\nabla\phi) - 2\nabla^2_M\phi - 4\phi\nabla^2_M\phi. \tag{5.42}$$

Note that we are just rewriting the result (5.32) for the case of a static potential. Note also that in the standard Newtonian approach $\Box h_{00} = \nabla^2_M h_{00}$, where the subindex M makes reference to the Minkowskian case. However, in this case I consider the expansion up to second order and it includes the curvature effects obtained from the Christoffel connections. In principle, the scalar curvature and the Ricci tensor when expanded up to second order are given by

$$R \approx R^{(1)} + R^{(2)}, \qquad R_{\mu\nu} \approx R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}. \tag{5.43}$$

Up to first order, then the approximation $\psi = -2\phi$ is valid in agreement with eq. (5.42) and the Lagrange multiplier condition. On the other hand, eq. (5.41) expanded up to second order is equivalent to

$$\nabla^2_M\phi \approx \mu^{-1}(\kappa^2\rho - \Lambda) - \mu^{-1}\nabla\phi\cdot\nabla\phi(4\omega), \tag{5.44}$$

where $\mu(\phi)$ and ω are defined by

$$\mu(\phi) = 2(1+2f(\psi)(1+3\phi-2\gamma-8\phi\gamma)+3\phi), \qquad \omega = 1+2f(\psi)\left(1-\frac{3}{4}\gamma\right). \tag{5.45}$$

We can observe that eq. (5.44) has the same form of eq. (5.36) which describes the MONDian dynamics. There is however a fundamental difference since in the non-local model, μ defined in eq. (5.45) is just a function of the potential rather than a function of the acceleration ($|\nabla\phi|$) as in the case of the MONDian dynamics. For the weak field approximation, eqs. (5.45) are

$$\mu(\phi) \approx 2(1+2f_0(1+3\phi-2\gamma-8\phi\gamma)+3\phi-4\gamma f_0\phi(1-2\gamma)), \qquad \omega \approx 1+2f_0\left(1-\frac{3}{4}\gamma\right). \tag{5.46}$$

5.6 Explicit solutions for ϕ and μ

I will compute the explicit solutions for μ and ϕ in agreement with the equation (5.44). Then different regimes will be explored (different values for γ) and I will identify some special values for γ. It is simpler to start by solving μ. For that purpose we have to find the solutions for the following equations in agreement with the result (5.46) for the weak field approximation

$$\nabla\mu = -4\nabla\phi\left((-3+8\gamma)f_0 - \frac{3}{2} + 2f_0\gamma(1-2\gamma)\right), \tag{5.47}$$

$$\nabla^2\mu = -4\nabla^2\phi\left((-3+8\gamma)f_0 - \frac{3}{2} + 2f_0\gamma(1-2\gamma)\right). \tag{5.48}$$

Then we can write eq. (5.44) in terms of μ. In vacuum the result is

$$\mu\nabla^2_M\mu \approx C\nabla\mu\cdot\nabla\mu, \tag{5.49}$$

where we have defined C as

$$C = \frac{\omega}{\left((-3+8\gamma)f_0 - \frac{3}{2} + 2f_0\gamma(1-2\gamma)\right)}. \tag{5.50}$$

The general solution for μ is given by

$$\mu(r) = A\left(\frac{-1+C}{r} + B\right)^{\frac{1}{1-C}}. \tag{5.51}$$

Note that this solution is valid for $C \neq 1$. As $C = 1$, eq. (5.49) becomes

$$\mu\nabla^2_M\mu \approx \nabla\mu\cdot\nabla\mu. \tag{5.52}$$

The solution for this equation is

$$\mu(r) = De^{-E/r}. \tag{5.53}$$

5.6.1 Solutions for μ and ϕ for special values of γ

There are different possible solutions for ϕ and μ in agreement with the results obtained in the previous section. In vacuum and ignoring the cosmological constant Λ, we can write the equation (5.44) as follows

$$\mu\nabla^2_M\phi = C(\nabla\phi)\cdot(\nabla\mu), \tag{5.54}$$

where we have used the results obtained in eq. (5.47) and the definition (5.50). I will analyze some relevant results summarized in the following table

Table 5.1: Relevant values for C as a function of the parameter γ and ω

C	γ	ω
0	∞	$-\infty$
0	$-\infty$	∞
1	$\frac{23f_0-\sqrt{f_0}\sqrt{-160+209f_0}}{16f_0}$	$1-2f_0-\frac{3}{32}\left(23f_0-\sqrt{f_0}\sqrt{-160+209f_0}\right)$
1	$\frac{23f_0+\sqrt{f_0}\sqrt{-160+209f_0}}{16f_0}$	$1-2f_0-\frac{3}{32}\left(23f_0+\sqrt{f_0}\sqrt{-160+209f_0}\right)$
-1	$\frac{17f_0+\sqrt{f_0}\sqrt{-32+225f_0}}{16f_0}$	$1-2f_0-\frac{3}{32}\left(17f_0+\sqrt{f_0}\sqrt{-32+225f_0}\right)$
-1	$\frac{17f_0-\sqrt{f_0}\sqrt{-32+225f_0}}{16f_0}$	$1-2f_0-\frac{3}{32}\left(17f_0-\sqrt{f_0}\sqrt{-32+225f_0}\right)$
∞	$\frac{5f_0+\sqrt{-6f_0+13f_0^2}}{4f_0}$	$1-2f_0-\frac{3}{8}\left(5f_0+\sqrt{-6f_0+13f_0^2}\right)$
$-\infty$	$\frac{5f_0-\sqrt{-6f_0+13f_0^2}}{4f_0}$	$1-2f_0-\frac{3}{8}\left(5f_0-\sqrt{-6f_0+13f_0^2}\right)$
0	$\frac{2}{3f_0}+\frac{4}{3}$	0
Min	$\frac{8+16f_0-\sqrt{2}\sqrt{32+35f_0-58f_0^2}}{12f_0}$	$1-2f_0-\frac{3}{24}\left(8+16f_0-\sqrt{2}\sqrt{32+35f_0-58f_0^2}\right)$
Max	$\frac{8+16f_0+\sqrt{2}\sqrt{32+35f_0-58f_0^2}}{12f_0}$	$1-2f_0-\frac{3}{24}\left(8+16f_0+\sqrt{2}\sqrt{32+35f_0-58f_0^2}\right)$
$\frac{1}{2}$	$\frac{13f_0-\sqrt{-56f_0+57f_0^2}}{8f_0}$	$1-2f_0-\frac{3}{16}\left(13f_0-\sqrt{-56f_0+57f_0^2}\right)$
$\frac{1}{2}$	$\frac{13f_0+\sqrt{-56f_0+57f_0^2}}{8f_0}$	$1-2f_0-\frac{3}{16}\left(13f_0+\sqrt{-56f_0+57f_0^2}\right)$

.

Here Min and Max correspond to a local minimum and a local maximum respectively for the parameter C as can be easily verified. If we replace the result (5.51) inside the definition of μ given in eq. (5.45), we then obtain the solution for ϕ consistent with eq. (5.52). Up to first order, the result is

$$\phi = A(\gamma)\left(\frac{-1+C}{r}+B\right)^{\frac{1}{1-C}} - 2\left(\frac{1+2f_0(1-2\gamma)}{2(3+2f_0(3-10\gamma+4\gamma^2))}\right), \tag{5.55}$$

where $A(\gamma)$ is defined as

$$A(\gamma) = -\left(\frac{A}{4}\right)\frac{C}{\omega}. \tag{5.56}$$

The same result for the case $C=1$ is

$$\phi = A(\gamma)e^{-D/r} - 2\left(\frac{1+2f_0(1-2\gamma)}{2(3+2f_0(3-10\gamma+4\gamma^2))}\right), \tag{5.57}$$

where D is just another integration constant and ω has to be evaluated for the case $C=1$. The case $C=1$ corresponds to two different values for the parameter γ as can be seen from Table 5.1. The equations (5.55) and (5.57) can be rewritten in a compact form as

$$\phi = -\left(\frac{A}{4}\right)\left(\frac{C}{\omega}\right)\left(\frac{-1+C}{r}+B\right)^{\frac{1}{1-C}} + \frac{1}{2}\left(\frac{C}{\omega}\right)(1+2f_0(1-2\gamma)), \tag{5.58}$$

$$\phi = -\left(\frac{A}{4}\right)\left(\frac{1}{\omega}\right)e^{-D/r} + \frac{1}{2}\left(\frac{1}{\omega}\right)(1+2f_0(1-2\gamma)). \tag{5.59}$$

In both cases, the condition $\omega \neq 0$ is satisfied. If $\omega = 0$, then C can take three different values in agreement with the Table 5.1. The standard Newtonian behavior is recovered for the case $C = 0 = \omega$. There will be values of γ for which the potential ϕ will be attractive and other values for which it will be repulsive.

5.6.2 Special cases for different values of the parameter γ and the ghost-free condition

If we calculate the gradient of the potential ϕ. Without loss of generality, we can set $B = 0$ for the cases $C \neq -1$ since this constant does not affect the dynamics at large distances. If we calculate the gradient from eq. (5.55) as $B = 0$, we get

$$\nabla\phi = -\left(\frac{A}{4}\right)\left(\frac{C}{\omega}\right)\left(\frac{-1+C}{r}\right)^{\frac{C}{1-C}}\frac{1}{r^2}, \tag{5.60}$$

for $C \neq -1$. This potential can be attractive or repulsive depending of the value of the ratio $\frac{C}{\omega}$ and the relative sign of C with respect to -1 for the terms inside the parenthesis. On the other hand, the gradient for the case $C = 1$ is taken from eq. (5.59) and it is given by

$$\nabla\phi = -\left(\frac{A}{4}\right)\left(\frac{D}{\omega}\right)\left(\frac{e^{-D/r}}{r^2}\right), \tag{5.61}$$

which is attractive or repulsive depending on the sign of ω.

5.6.3 The relevant cases for the potential

If we take into account that dynamically the potential satisfies the condition $\vec{\nabla}\phi = \frac{v^2}{r}$, where v is the magnitude of the velocity, then a flat rotation curve for a galaxy can be reproduced only if $\nabla\phi \propto \frac{1}{r}$. But it seems that in this case the behavior is not reproduced for any value of the parameter C. From eq. (5.60), it is clear that the Newtonian behavior is reproduced as $C = 0$. For a well behaved solution, from the table 5.1, we can see that in such a case, $\omega = 0$. From eq. (5.50), the relation $\frac{C}{\omega}$, then becomes

$$\frac{C}{\omega} = \frac{1}{\frac{29}{9}f_0 - \frac{35}{18} - \frac{16}{9f_0}} = \frac{0}{0}, \tag{5.62}$$

where we have introduced the appropriate value for γ taken from the table 5.1. If we replace this condition inside eq. (5.60), we then obtain

$$\nabla\phi = -\left(\frac{A}{4}\right)\left(\frac{1}{\frac{29}{9}f_0 - \frac{35}{18} - \frac{16}{9f_0}}\right)\left(\frac{1}{r^2}\right) = -\frac{GM}{r^2}, \tag{5.63}$$

where we have imposed the Newtonian limit condition. We have to satisfy

$$GM = \left(\frac{A}{4}\right)\left(\frac{1}{-\frac{29}{9}f_0 + \frac{35}{18} + \frac{16}{9f_0}}\right). \tag{5.64}$$

Then equation (5.59), for the full potential becomes

$$\phi = -\frac{GM}{r} + \frac{15f_0}{32 - 29f_0}, \tag{5.65}$$

where we have replaced the appropriate values for the constant term in eq. (5.59). From the previous equations, it is clear that if we want to reproduce the appropriate Newtonian behavior, then the constant A has to satisfy

$$A = 4GM\left(-\frac{29}{9}f_0 + \frac{35}{18} + \frac{16}{9f_0}\right). \tag{5.66}$$

The remaining constant term is not important in order to obtain the Newtonian behavior. It is just a constant quantity which can be ignored for the computations.

5.6.4 The case $C = 1$

The case $C = 1$ is extremely relevant since it looks like a Yukawa potential. If we replace the appropriate value for γ and ω from the Table 5.1, then we can write the equation (5.61) like

$$\nabla\phi = -\left(\frac{A}{4}\right)\frac{D}{\left(1 - 2f_0 - \frac{3}{32}(23f_0 - \sqrt{f_0}\sqrt{-160 + 209f_0}\right)}\frac{e^{-D/r}}{r^2}, \tag{5.67}$$

where we have used the first value for ω corresponding to $C = 1$. For the second value of ω corresponding to $C = 1$, we can obtain the following result

$$\nabla\phi = -\left(\frac{A}{4}\right)\frac{D}{\left(1 - 2f_0 - \frac{3}{32}(23f_0 + \sqrt{f_0}\sqrt{-160 + 209f_0}\right)}\frac{e^{-D/r}}{r^2}. \tag{5.68}$$

The form of this solution suggests that the behavior of this potential is approximately Newtonian as the exponential factor tends to 1. The attractive or repulsive character of this solution depends on the values taken by f_0.

5.6.5 The case $C = -1$

The case $C = -1$ is perhaps the most interesting for our present purpose. This case is interesting since the field equation (5.54) becomes

$$\mu\nabla^2\phi = -\nabla\phi\cdot\nabla\mu. \tag{5.69}$$

This equation has can be written as

$$\nabla\cdot(\mu\nabla\phi) = 0. \tag{5.70}$$

In vacuum, this has the same structure as the equation (5.7). With the difference that in the non-local model the interpolating function μ is a function of the potential itself, rather than a function of its gradient as MOND suggests. The case $C = -1$ requires $B \neq 0$ and $B > \frac{2}{r}$ in eq. (5.55), otherwise the potential in such a case becomes complex.

5.7 A comparison with other models

The non-local model analyzed in this manuscript is able to reproduce the equation (5.44) with the definitions (5.45). This equation (after some arrangements) is similar to eq. (5.6) or (5.36) which is obtained from the AQUAL Lagrangian (5.5). However, the present model cannot reproduce the same dynamics due to MOND since the predicted interpolating function μ is a function of the potential (ϕ), rather than a function of the acceleration ($|\nabla\phi|$) as can be observed from eq. (5.45) and the fact that $f(\psi) = f_0 e^{\gamma\psi}$ with $\psi = -2\phi$. This previous relation is precisely the source of the problem for reproducing MOND appropriately. In [75], it has been demonstrated that in order to reproduce the MONDian dynamics, it is necessary to add to the Einstein-Hilbert action, a Lagrangian such that it cancels the quadratic parts of the action and also provides some additional terms whose variations are

$$\frac{c^2}{2a_0 r^2}((rb'(r))^2)' = \frac{8\pi G\rho}{c^4}, \tag{5.71}$$

$$\frac{c^2}{a_0 r^3}(krb'(r) - a(r))^2 = 0, \tag{5.72}$$

where $a(r)$ and $b(r)$ are given by

$$a(r) \equiv A(r) - 1, \qquad b(r) \equiv B(r) - 1, \tag{5.73}$$

with a static, spherically symmetric geometry defined by

$$ds^2 = -B(r)c^2dt^2 + A(r)dr^2 + r^2 d\Omega^2. \tag{5.74}$$

The key point in the work performed in [75] is to change how the potentials depend upon the source without changing how they depend each other. In fact, the relation between the linearized potentials is given by

$$a(r) \approx rb'(r). \tag{5.75}$$

This relation is necessary in order to reproduce the appropriate amount of weak lensing consistent with the data. In the standard formalism of General Relativity, the linearized potentials take the form

$$rb'(r) \approx \frac{2GM(r)}{c^2 r}, \tag{5.76}$$

but in the MONDian regime, the following relation has to be satisfied

$$rb'(r) \to \frac{2\sqrt{a_0 GM(r)}}{c^2}. \tag{5.77}$$

The MOND Lagrangian which cancels the quadratic terms of the Einstein-Hilbert action and also reproduces the results (5.71) and (5.72) is [75]

$$\begin{aligned}\mathcal{L}_{MOND} \to \frac{c^4}{16\pi G}&\left(rab'(r) - \frac{a^2(r)}{2} + O(h^3)\right) \\ &+ \frac{c^2}{a_0}\left(\frac{\alpha a^3(r)}{r} + \beta a^2(r)b'(r) + \gamma r a(r)b'^2(r) + \delta r^2 b'^3(r) + O(h^4)]\right).\end{aligned} \tag{5.78}$$

The first line of this Lagrangian, just cancels the Einstein-Hilbert action terms given by

$$\mathcal{L}_{EH} = \frac{c^4}{16\pi G}R\sqrt{-g} \to \frac{c^4}{16\pi G}\left(-ra(r)b'(r) + \frac{a^2(r)}{2} + O(h^3)\right), \tag{5.79}$$

where the right-hand side (after the arrow) is obtained after integration by parts and here we ignore total derivative terms. Note that for the total Lagrangian $\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_{MOND}$, the Einstein-Hilbert terms vanishes. It has been demonstrated in [75] that no local invariant Lagrangian can reproduce the cubic terms of the MONDian action (5.78). The reason is that the curvature tensor and its possible contractions, can only reproduce terms involving two derivatives acting on one or more weak fields in the following way [75]

$$(Curvature)^N \sim (h'')^N + O((h')^2(h'')^{N-1}). \tag{5.80}$$

On the other hand, the MOND corrections in eq. (5.78), involve powers of just one derivative acting on a single weak field like

$$\mathcal{L}_{MOND} \sim \frac{c^4 r^2}{16\pi G}\left((h')^2 + \frac{c^2}{a_0}(h')^3 + O(h^4)\right), \tag{5.81}$$

is in this part where the model proposed in [9] fails. In [7], it was explained that the non-localities enter through the function $f(\psi)$ with the Lagrange multiplier condition $R = \Box\psi$ (with $-2\phi = \psi$). In such a case, then the non-localities will enter as an algebraic expansion of the potentials. This can be seen in eq. (5.45) and the action (5.19) if we take into account that $f(\psi) = f_0 e^{\gamma\psi}$. If we want to reproduce the MONDian dynamics, one possibility is for example to expand the function $f(\psi)$ around the scale defined as the geometric average of

the gravitational radius and the inverse of the acceleration scale a_0 characteristic of MOND, the scale is $r_0 = \sqrt{\frac{GM}{a_0}}$. In such a case, we would get $f(\psi) \approx f_0 + f_0\gamma(\nabla\psi)_{r=r_0}$. But doing this expansion just breaks the nature of the model since in such a case, we are imposing by hand the scale at which the MONDian regime applies rather than obtaining it. The model proposed here cannot reproduce the form for the Lagrangian (5.78) or (5.81) in a natural way. In [75], the non-localities are used in order to reduce the number of derivatives for the weak fields and particular components for the curvature are selected by using a time-like 4-vector obtained from the gradient of the invariant volume of the past light cone (see [75] for details). The reduction of the number of derivatives is the appropriate such that the MONDian Lagrangians (5.78) or (5.81) with only a single derivative of the weak fields can be reproduced. Remember that the standard Einstein-Hilbert Lagrangian can only reproduce powers of two derivatives acting on weak fields. In the model analyzed in [7], the extra terms of the action do not have derivative terms (only algebraic relations are found).
Another attempt for reproducing MOND by using non-localities is done in [76]. In such a case, the model can reproduce galaxy rotation curves but not the observed gravitational lenses. The model in [76] proposes a Lagrangian given by

$$\mathcal{L} = \frac{c^4}{16\pi G}(R + c^{-4}a_0^2 F(c^4 a_0^{-2} g^{\mu\nu}\epsilon_{,\mu}\,\epsilon_{,\nu}\,))\sqrt{-g}, \tag{5.82}$$

where ϵ is the small potential as it is defined in [76]. The main point here is that an interpolating function $F(x)$ is introduced since the beginning and MOND is embedded inside this Lagrangian under the assumption that at small x the MONDian dynamics appear. The factor inside the interpolating function is a kinetic term for the small potential and it makes easier to recover the MONDian dynamics. This is the main difference with respect to the model proposed in this manuscript where, as has been said before, the non-localities are introduced as algebraic expansion of the potential. If we want to mimic in some sense the model suggested in [76], we would have to expand the function $f(\psi)$ around the neighborhood of some imposed scale as has been explained before in this section. The model in [76] also proposes the same relation (5.77) but it cannot reproduce the appropriate lenses without the dark matter assumption.

Chapter 6

Conclusions

In this thesis I started from the most basic formulation of gravity, namely, the Einstein-Hilbert action with cosmological constant expanded up to second order. From the massive gravity point of view, this corresponds to the most basic potential without any interaction between the fiducial and the dynamical metric. I then analyzed the propagation of gravitational waves inside this formalism. The purpose is to show the type of physics that can be analyzed in these formulations. The gravitational waves will be the most important discovery in the forthcoming decades, providing a new window to physics. The discovery of stochastic background of gravitational waves for example, will provide evidence for physics at the highest energies ever reached by any accelerator.
Later I analyzed the Fierz-Pauli theory for massive gravitons and after a brief explanation of the number of degrees of freedom, I explained the origin of the vDVZ discontinuity, the first pathology found inside massive gravity. This was in fact the first motivation for working around the non-linear theory of massive gravity. It was Vainshtein who suggested that the vDVZ discontinuity was a consequence of working at the linear level. He then found the Vainshtein mechanism as a non-linear effect able to screen the additional attractive force due to the coupling of the scalar component of the graviton with the trace of the energy-momentum tensor. Then the theory became satisfactory for the solar system test. Later Boulware and Deser discovered by using the ADM formulation [80] that the non-linearities activate a sixth mode inside the theory. This is the so-called Boulware-Deser ghost. One of the most difficult pathologies found in this formulation. It took many years to solve it. It was de-Rham and Gabadadze whose solved the problem initially at the decoupling limit up to quintic order, and later with the collaboration of Tolley at all orders. They basically tuned the coefficients of the potential in the action such that the terms able to reproduce a ghost when combined with other terms of the action coming from higher order contributions, become total derivatives. Then any evidence of a ghost will never appear in the equation of motions. This is the so-called dRGT theory. Later Hassan and colleagues were able to demonstrate that this tune is successful even outside decoupling limit. Many people then began to work around this subject and Hassan was able to propose an extension of the theory by allowing the fiducial metric to be dynamical. Remember that in its generic form, the non-linear formulation for massive gravity requires a fiducial metric in order to get the 5 degrees of freedom consistent

with a massive gravity formulation.
Later some black hole solutions were found inside this formulation and in bi-gravity. In bi-gravity some authors analyzed the stability of black holes and they found the Gregory-Laflamme instability. However, Cardoso and colleagues found that there is a point where the instability disappears. This happens when the graviton mass takes some specific value in agreement with the partially massless regime. The special point is known as the Higuchi limit. It has been recently confirmed by Babichev, inspired in the work showed here that the instability is completely absent (independent of the mass value) when the fiducial metric becomes Minkowski.
In this thesis, we analyzed (in collaboration with Prof. Kodama) the stability of black holes inside the dRGT formulation of massive gravity. This analysis is in fact more difficult than the one performed by Babichev, Cardoso et- al. The reason is that in bi-gravity formulations, the two metrics are normally taken to be the same (although their perturbations might be different) and this means that at the background level, everything looks like Einstein gravity and you can take the background solutions from General Relativity.
In our analysis, we first looked for the parameter relation for which the non-linear massive gravity theory admits the Schwarzschild-de Sitter black hole as an exact solution systematically. We found that when the parameters satisfy the relation $\beta = \alpha^2$, there exists a family of solutions parameterized by an arbitrary function $T_0(t, r)$, which are isomorphic to the Schwarzschild-de Sitter spacetime but are not equivalent if the configuration of the Stückelberg fields are taken into account. We next investigated the perturbative stability of this family of Schwarzschild-de Sitter-type black holes in the framework of the dRGT formulation of the non-linear massive gravity with $\beta = \alpha^2$. We found that the perturbative equations derived from the field equations of the dRGT theory become identical to the perturbations equation for the vacuum Einstein theory with cosmological constant if we take into account the consistency condition obtained from the field equations by the Bianchi identity. This consistency condition is essentially equivalent to the field equation for the Stückelberg field. This implies that the Schwarzschild-de Sitter black hole solution is stable in the non-linear massive gravity theory at least in the linear perturbation level, in contrast to the bi-Schwarzschild solution in the bi-metric theory. In spite of this stability result, we found a pathological feature of the black hole solution in the dRGT theory with the parameter relation $\beta = \alpha^2$; the general solution to the perturbation equations contains an arbitrary function of the spacetime coordinates. This implies that the predictability of dynamics is lost at least in the linear perturbation level around this black hole solution. This degeneracy can be removed by coordinate transformations if we neglect the Stückelberg fields. Hence, the pathology appears to come from the dynamics of the Stückelberg fields. Because the Schwarzschild-de Sitter black hole becomes an exact solution only when the higher-order mass terms exist, there is a possibility that this pathology might be removed in the non-linear level of perturbations.
Finally, I analyzed another attempt to modify gravity in order to explain the necessary ingredients which a theory requires if we want to reproduce dark matter effects. I selected a non-local model proposed by Nojiri, Odintsov and later used by Sasaki and colleagues. My intention was to reproduce the galactic dynamics in agreement with the compact expression provided by MOND (Modified Newtonian Dynamics). I found that the model cannot re-

produce the dynamics because the interpolating function μ is scale-dependent contrary with what is expected from MOND. The non-local model in the present manuscript can reproduce some additional attractive effects for some range of the parameter γ. The model cannot reproduce the MONDian dynamics without a strong tunning of the parameters. However, it can reproduce the AQUAL field equations with a scale-dependent interpolating function μ for some special case given by $C = -1$. That case however, requires the additional condition $B > \frac{2}{r}$ everywhere. The reproduction of the AQUAL equations is in agreement with Milgrom and Bekenstein, the first step for getting a Relativistic version of MOND. Every attempt in modifying gravity in order to reproduce the MONDian dynamics, must reproduce equations like the AQUAL Lagrangian explained before in this manuscript. Further research is needed in order to see whether or not is viable to reproduce the dark matter effects in agreement with non-localities. Another alternatives for the introduction of non-localities have been explored in [75] and [76]. In [75], the non-localities were able to reproduce the MONDian dynamics and the appropriate gravitational lenses. In the case of [76], the non-localities could reproduce the dynamics but not the lenses.

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Appendix A

Second order expansion for the Λ term

As I explained inside the text, the simplest possible potential keeping the same degrees of freedom as in the case of General Relativity is that equivalent to a cosmological constant term as has been derived in [30]

$$\sqrt[2]{\overline{g}} = \sqrt[2]{g det(\delta_\nu{}^\alpha + h_\nu{}^\alpha)}. \tag{A.1}$$

We must solve first the determinant of the right hand side; using then the general formula for the development of a determinant, we get

$$Det(T^\alpha_{\ \beta}) = [\mu\nu\rho\sigma] T^\mu{}_0 T^\nu{}_1 T^\rho{}_2 T^\sigma{}_3, \tag{A.2}$$

$$det(\delta_\nu{}^\alpha + h_\nu{}^\alpha) = \begin{vmatrix} 1 + h^0{}_0 & h^0{}_1 & h^0{}_2 & h^0{}_3 \\ h^1{}_0 & 1 + h^1{}_1 & h^1{}_2 & h^1{}_3 \\ h^2{}_0 & h^2{}_1 & 1 + h^2{}_2 & h^2{}_3 \\ h^3{}_0 & h^3{}_1 & h^3{}_2 & 1 + h^3{}_3 \end{vmatrix}. \tag{A.3}$$

If we develope in detail the determinant, we get

$$\begin{aligned}
det(\delta_\nu{}^\alpha + h_\nu{}^\alpha) \;=\; & [0123](1+h^0{}_0)(1+h^1{}_1)(1+h^2{}_2)(1+h^3{}_3) + [0213](1+h^0{}_0)(h^2{}_1)(h^1{}_2)(1+h^3{}_3) \\
& +[0132](1+h^0{}_0)(1+h^1{}_1)(h^3{}_2)(h^2{}_3) + [0231](1+h^0{}_0)(h^2{}_1)(h^3{}_2)(h^1{}_3) \\
& +[0321](1+h^0{}_0)(h^3{}_1)(1+h^2{}_2)(h^1{}_3) + [0312](1+h^0{}_0)(h^3{}_1)(h^1{}_2)(h^2{}_3) \\
& +[1023](h^1{}_0)(h^0{}_1)(1+h^2{}_2)(1+h^3{}_3) + [1032](h^1{}_0)(h^0{}_1)(h^3{}_2)(h^2{}_3) \\
& +[1230](h^1{}_0)(h^2{}_1)(h^3{}_2)(h^0{}_3) + [1203](h^1{}_0)(h^2{}_1)(h^0{}_2)(1+h^3{}_3) \\
& +[1302](h^1{}_0)(h^3{}_1)(h^0{}_2)(h^2{}_3) + [1320](h^1{}_0)(h^3{}_1)(1+h^2{}_2)(h^0{}_3) \\
& +[2013](h^2{}_0)(h^0{}_1)(h^1{}_2)(1+h^3{}_3) + [2031](h^2{}_0)(h^0{}_1)(h^3{}_2)(h^1{}_3) \\
& +[2130](h^2{}_0)(1+h^1{}_1)(h^3{}_2)(h^0{}_3) + [2103](h^2{}_0)(1+h^1{}_1)(h^0{}_2)(1+h^3{}_3) \\
& +[2301](h^2{}_0)(h^3{}_1)(h^0{}_2)(h^1{}_3) + [2310](h^2{}_0)(h^3{}_1)(h^1{}_2)(h^0{}_3) \\
& +[3012](h^3{}_0)(h^0{}_1)(h^1{}_2)(h^2{}_3) + [3021](h^3{}_0)(h^0{}_1)(1+h^2{}_2)(h^1{}_3) \\
& +[3120](h^3{}_0)(1+h^1{}_1)(1+h^2{}_2)(h^0{}_3) + [3210](h^3{}_0)(h^2{}_1)(h^1{}_2)(h^0{}_3) \\
& +[3201](h^3{}_0)(h^2{}_1)(h^0{}_2)(h^1{}_3) + [3102](h^3{}_0)(1+h^1{}_1)(h^0{}_2)(h^2{}_3)
\end{aligned} \tag{A.4}$$

Here $[\mu\nu\rho\beta]$ represents the Levi-Civita symbol, it will be 1 or -1 depending on the order of the indices; with that in mind and ignoring second order terms $O(h^2)$, we get

$$\begin{aligned}
det(\delta_\nu{}^\alpha + h_\nu{}^\alpha) \;=\; & 1 + h^0{}_0h^1{}_1 + h^0{}_0h^2{}_2 + h^0{}_0h^3{}_3 + h^1{}_1h^2{}_2 + h^1{}_1h^3{}_3 + h^2{}_2h^3{}_3 \\
& + h^0{}_0 + h^1{}_1 + h^2{}_2 + h^3{}_3 - h^2{}_1h^1{}_2 - h^3{}_2h^2{}_3 - h^3{}_1h^1{}_3 - h^1{}_0h^0{}_1 \\
& - h^2{}_0h^0{}_2 - h^3{}_0h^0{}_3
\end{aligned} \tag{A.5}$$

We must take now the root square of determinant; then

$$\begin{aligned}
\sqrt[2]{det(\delta_\nu{}^\alpha + h_\nu{}^\alpha)} \;=\; & (1 + h^0{}_0h^1{}_1 + h^0{}_0h^2{}_2 + h^0{}_0h^3{}_3 + h^1{}_1h^2{}_2 + h^1{}_1h^3{}_3 + h^2{}_2h^3{}_3 + h^0{}_0 + h^1{}_1 \\
& + h^2{}_2 + h^3{}_3 - h^2{}_1h^1{}_2 - h^3{}_2h^2{}_3 - h^3{}_1h^1{}_3 - h^1{}_0h^0{}_1 - h^2{}_0h^0{}_2 - h^3{}_0h^0{}_3)^{1/2} \\
& .
\end{aligned} \tag{A.6}$$

Executing now the expansion, we obtain

$$\sqrt[2]{det(\delta_\nu{}^\alpha + h_\nu{}^\alpha)} \;=\; 1 + \frac{1}{2}h - \frac{1}{4}h^\alpha{}_\beta h^\beta{}_\alpha + \frac{1}{4}hh - \frac{1}{8}\left(h - \frac{1}{2}h^\alpha{}_\beta h^\beta{}_\alpha + \frac{1}{2}hh\right)^2. \tag{A.7}$$

If we ignore higher order contributions, we get

$$\sqrt[2]{det(\delta_\nu{}^\alpha + h_\nu{}^\alpha)} \;=\; 1 + \frac{1}{2}h - \frac{1}{4}h^\alpha{}_\beta h^\beta{}_\alpha + \frac{1}{8}hh. \tag{A.8}$$

Then we know already that the cosmological action is taken as [30]

$$\mathcal{L}_{\Lambda} = \Lambda \sqrt[2]{g det(\delta_{\nu}{}^{\alpha} + h_{\nu}{}^{\alpha})}. \tag{A.9}$$

From the final two equations, we get

$$\mathcal{L}_{\Lambda} = \Lambda \left(1 + \frac{1}{2}h - \frac{1}{4}h^{\alpha}{}_{\beta}h^{\beta}{}_{\alpha} + \frac{1}{8}hh\right). \tag{A.10}$$

Appendix B

Poynting vector average over big regions

Here I will show that the Poynting vectors, corresponding to a gravitational wave source, and the one corresponding to the Λ contribution can be added after performing an average over a big enough region of spacetime. I focus here in the case of massless theory ($m = 0$). The full solutions of gravitational radiation are

$$h_{\mu\nu} = h_{\mu\nu\ M} + h_{\mu\nu\ \Lambda}, \tag{B.1}$$

where the subindex M means matter and the subindex Λ is related to the cosmological constant contribution. The solution can then be expressed as a superposition of plane waves due to matter and the background contribution given by

$$h_{\mu\nu} = e_{\mu\nu} exp(ik_\sigma x^\sigma) + e^*_{\mu\nu} exp(-ik_\sigma x^\sigma) + h_{\mu\nu\Lambda}. \tag{B.2}$$

For the sake of simplicity, I will omit the possible spatial dependence for the polarization tensors, that dependence is shown explicitly in the power definition in [29]. On the other hand, the dependence in frequency is not important at this moment since we can work with a single Fourier component and after the full solution is just a superposition of different modes. As far as the equations and solutions are linear, everything is safe. The different solutions obtained in section (2.1) are repeated here as

$$h_{00} = e_{00} exp(ik_\sigma x^\sigma) + e^*_{00} exp(-ik_\sigma x^\sigma) - \Lambda t^2, \tag{B.3}$$

$$h_{0i} = e_{0i} exp(ik_\sigma x^\sigma) + e^*_{0i} exp(-ik_\sigma x^\sigma) + \frac{2}{3}\Lambda t x_i, \tag{B.4}$$

$$h_{ij} = e_{ij} exp(ik_\sigma x^\sigma) + e^*_{ij} exp(-ik_\sigma x^\sigma) + \Lambda t^2 \delta_{ij} + \frac{1}{3}\Lambda \epsilon_{ij}, \tag{B.5}$$

where $\epsilon_{ij} = x_i x_j$ for $i \neq j$ and zero otherwise. The total Poynting vector is given by

$$\hat{t}_{0i} = t_{0i} - \frac{h_{0i}}{8\pi G}\Lambda. \tag{B.6}$$

The second term of this expression can be expanded as

$$\frac{h_{0i}}{8\pi G}\Lambda = \frac{1}{8\pi G}\left(e_{0i}exp(ik_\sigma x^\sigma) + e^*_{0i}exp(-ik_\sigma x^\sigma) + \frac{2}{3}\Lambda t x_i\right)\Lambda. \tag{B.7}$$

If we average over a big enough space-time region, the terms proportional to $exp(\pm ik_\sigma x^\sigma)$ disappear, then

$$\left\langle \frac{h_{0i}}{8\pi G}\Lambda \right\rangle = \frac{1}{12\pi G} < \Lambda^2 t x_i > . \tag{B.8}$$

For $< t_{0i} >$, we only have to worry for evaluating terms of the form

$$< 2\Lambda h_{0i} >= \left\langle 2\Lambda\left(e_{0i}exp(ik_\sigma x^\sigma) + e^*_{0i}exp(-ik_\sigma x^\sigma) + \frac{2}{3}\Lambda t x_i\right)\right\rangle = \frac{4}{3}\Lambda^2 < t x_i > . \tag{B.9}$$

The terms proportional to $exp(\pm ik_\sigma x^\sigma)$ are killed by the spacetime average. We have to prove that there are no interference terms between the cosmological constant solutions and the matter contributions in the second-order Ricci tensor given in (2.17); the 0-i component is given by

$$\begin{aligned} R^{(2)}_{0i} = -\frac{1}{2}h^{\lambda\nu}(\partial_i\partial_0 h_{\lambda\nu} - \partial_i\partial_\lambda h_{0\nu} - \partial_\nu\partial_0 h_{\lambda i} + \partial_\nu\partial_\lambda h_{0i}) + \frac{1}{4}(2\partial_\nu h^\nu{}_\sigma - \partial_\sigma h)(\partial_i h^\sigma{}_0 + \partial_0 h^\sigma{}_i - \partial^\sigma h_{0i}) \\ -\frac{1}{4}(\partial_\lambda h_{\sigma i} + \partial_i h_{\sigma\lambda} - \partial_\sigma h_{\lambda i})(\partial^\lambda h^\sigma{}_0 + \partial_0 h^{\sigma\lambda} - \partial^\sigma h^\lambda{}_0). \end{aligned} \tag{B.10}$$

After a moment of reflexion, the reader can convince him/herself that by replacing the full solutions given by eqns. (B.3), (B.4) and (B.5), inside eq. (B.10), gives us interference terms of the form

$$h_{\mu\nu\Lambda}\left(e_{\alpha\beta}exp(ik_\sigma x^\sigma)\right), \tag{B.11}$$

which average to zero for big enough regions of spacetime, as an example let's find the first term of the equation (B.10) as follows

$$R^{(2)A}_{0i} = -\frac{1}{2}h^{\lambda\nu}\partial_i\partial_0 h_{\lambda\nu} = -\frac{1}{2}\left(h^{00}\partial_i\partial_0 h_{00} + h^{ii}\partial_i\partial_0 h_{ii} + 2h^{0j}\partial_i\partial_0 h_{0j} + 2h^{kj}_{k\neq j}\partial_i\partial_0 h_{kj}\right). \tag{B.12}$$

Explicitly, this expression becomes

$$h^{\lambda\nu}\partial_i\partial_0 h_{\lambda\nu} = \left(e^{00}exp(ik_\sigma x^\sigma) + c.c - \Lambda t^2\right)\left(-e_{00}k_ik_0 exp(ik_\sigma x^\sigma) - c.c\right) + \left(e^{jj}exp(ik_\sigma x^\sigma) + c.c + \Lambda t^2\right) \times \left(-e_{jj}k_ik_0 exp(ik_\sigma x^\sigma) - c.c\right) + 2\left(e^{0j}exp(ik_\sigma x^\sigma) + c.c - \frac{2}{3}\Lambda t x_j\right)\left(-e_{0j}k_0k_i exp(ik_\sigma x^\sigma) - c.c + \frac{2}{3}\Lambda\delta_{ij}\right) + 2\left(e^{kj}exp(ik_\sigma x^\sigma) + c.c + \frac{1}{3}\Lambda x_k x_j\right)\left(-e_{kj}k_0k_i exp(ik_\sigma x^\sigma) - c.c\right). \quad \text{(B.13)}$$

If we develop the products and take the average over a big enough space-time region, then crossed terms between the polarization tensors proportional to $exp(\pm 2ik_\sigma x^\sigma)$ are obtained. These terms disappear after average. On the other hand, you can find crossed terms between polarization tensors and cosmological constant contributions as follows

$$\Lambda e_{00}k_ik_0 t^2 exp(ik_\sigma x^\sigma). \quad \text{(B.14)}$$

The average over space-time tells

$$\Lambda e_{00}k_ik_0 < t^2 exp(ik_\sigma x^\sigma) > . \quad \text{(B.15)}$$

The averaged term becomes

$$< t^2 exp(ik_\sigma x^\sigma) >= \frac{1}{VT}\int d^4x t^2 exp(ik_\sigma x^\sigma) = \frac{1}{VT}\int d^4x t^2 exp(ik_0x^0 + ik_jx^j), \quad \text{(B.16)}$$

where V is the three-volume and T is the time of average. Thus

$$< t^2 exp(ik_\sigma x^\sigma) >= \frac{1}{VT}\int dtdx_1dx_2dx_3 t^2 exp(ik_0t)exp(ik_1x^1)exp(ik_2x^2)exp(ik_3x^3) = 0. \quad \text{(B.17)}$$

Since the integrals involving x_1, x_2 and x_3 go to zero. It is simple to realize that every average term involving crossed terms will be of the form (B.17) or

$$\frac{1}{VT}\int d^4x x_1 t exp(ik_\sigma x^\sigma) = \frac{1}{VT}\int dtdx_1dx_2dx_3 t x_1 exp(ik_0t)exp(ik_1x^1)exp(ik_2x^2)exp(ik_3x^3) = 0, \quad \text{(B.18)}$$

which vanishes since the integrals involving x_2 and x_3 go to zero. Then the total Poynting vector is just linear after average.

Appendix C

The other parameter choices admitting the Schwarzschild-de Sitter solution

In this appendix, we exhaust all possible choices of the parameters in which X becomes a constant multiple of the unit matrix as $m^2 X^\mu_\nu = \Lambda\delta^\mu_\nu$ assuming that the spacetime metric takes the spherically symmetric form (4.51) and the Stückelberg fields satisfy the unitary gauge condition (4.54). We use the same notations as in Section (4.3).

i). $c = 0, a = b$. In this case, the condition

$$X^t{}_t - X^\theta{}_\theta = (F_1 + aF_2)\left(a - 1 + \frac{1}{S}\right) = 0, \tag{C.1}$$

implies $a = 1 - 1/S$ or $a = -F_1/F_2$, if we exclude the case $F_1 = F_2 = 0$ discussed in Section (4.3). Then, from $X^t{}_t = \Lambda/m^2 = const$, if follows that S is constant. Hence, the metric must represent a flat spacetime and $\Lambda = 0$. Because the metric (4.25) with constant coefficients has vanishing curvature if

$$g_{(2)} = S^2 g_{tt} \equiv S^2\{(1-b)^2 - c^2\} = 1, \tag{C.2}$$

in general, in the present case, we obtain the constraint $a = b = 1 - 1/S$. The corresponding flat metric should have the form

$$ds^2 = S^2(-dt^2 + dr^2 + r^2 d\Omega^2). \tag{C.3}$$

No constraint on α and β is required, but the value of S is restricted from the condition $X^t_t = 0$ to

$$S = 1,\ \frac{3\alpha + 2\beta \pm \sqrt{9\alpha^2 - 12\beta}}{2(3 + 3\alpha + \beta)}. \tag{C.4}$$

The case $a = -F_1/F_2$ can be included in this solution as a special case.
ii). $c = 0, a \neq b$. In this case, we obtain $F_3 = 0$ from $X^t{}_t = X^r{}_r$, hence S must be constant.

From this it follows that ${X^t}_t = (1/S - 1)(F_1 + 1) = const$. Next, from

$$0 = X^t_t - X^\theta_\theta = (F_1 + bF_2)(a - 1 + 1/S), \tag{C.5}$$

we obtain $a = 1 - 1/S$ or $b = -F_1/F_2$, if we exclude the case $F_1 = F_2 = 0$ discussed Section (4.3). Because $c = 0$, T_0 should be a function only of t from (4.139). Hence, if $a = 1 - 1/S = const$, the metric should be flat because g_{tt} is constant from (4.67), (4.68), (4.69), (4.70) and (4.71). Then, from (C.2), we obtain b=1-1/S=a, contradicting the assumption. Next, when $b = -F_1/F_2 = const$, we find that the metric is flat and a is constant again. From $X^t_t = -(F_1 + 1)(1 - 1/S) = 0$, we obtain two constraints $F_1 = -1, F_2 = S/(S - 1)$ because $S = 1$ leads to $F_3 = 1$. This means that $b = 1 - 1/S$. This leads to the contradiction due to the regularity condition (C.2). Thus, this case has no other solution than those discussed in Section (4.3).

iii $c \neq 0, a = b$. In this case, $F_3 = 0$ is required, and from ${X^t}_t = const$, it follows that S is constant. Then, from

$$0 = X^t_t - {X^\theta}_\theta = F_2 \left\{(a - 1 + 1/S)^2 + c^2\right\}, \tag{C.6}$$

we obtain $F_2 = 0$. Hence, this case is a special case of the case with $F_1 = F_2 = 0$ discussed in Section (4.3).

iv $c \neq 0, a \neq b$. Again, we obtain $F_3 = 0$ and $S = const$. If $F_2 = 0$, this case reduces to the class $F_1 = F_2 = 0$ discussed in Section (4.3). Next, when $F_2 \neq 0$, the constraint

$$0 = {X^t}_t - {X^\theta}_\theta = F_2 \left((a - 1 + 1/S)(b - 1 + 1/S) + c^2\right), \tag{C.7}$$

leads to

$$0 = c^2 + (1 - a - 1/S)(1 - b - 1/S) = c^2 + (1 - a)(1 - b) - \frac{2 - a - b}{S} + \frac{1}{S^2}. \tag{C.8}$$

This should be satisfied by a, b, c corresponding to the Schwarzschild-de Sitter metric

$$ds^2 = -f(Sr)dT_0(t, r)^2 + \frac{S^2 dr^2}{f(Sr)} + S^2 r^2 d\Omega^2. \tag{C.9}$$

From the general formula in Section (4.3), we obtain

$$c^2 + (1 - a)(1 - b) = \frac{1}{S|\dot{T}_0|}, \tag{C.10}$$

$$2 - a - b = M_1 = \frac{1}{S|\dot{T}_0|} \left(f\dot{T}_0^2 + \frac{S^2}{f} - f(T_0')^2 + 2S\dot{T}_0\right)^{1/2}, \tag{C.11}$$

where $\dot{T}_0 = \partial_t T_0$, $T_0' = \partial_r T_0$, and $f = f(Sr)$ is understood. Inserting these into the above constraint, we obtain

$$(T_0')^2 = \frac{1 - f(Sr)}{f(Sr)} \left(\frac{S^2}{f(Sr)} - \dot{T}_0^2\right). \tag{C.12}$$

Hence, in this case, no relation is imposed on α and β, but instead the gauge transformation function $T_0(t,r)$ is constrained. The value of S is determined by $F_3 = 0$ as

$$S = \frac{\alpha + \beta \pm \sqrt{\alpha^2 - \beta}}{1 + 2\alpha + \beta}, \tag{C.13}$$

and the corresponding cosmological constant is given by

$$\Lambda = -m^2 \left(1 - \frac{1}{S}\right) \left(2 + \alpha - \frac{\alpha}{S}\right). \tag{C.14}$$

The condition $\Lambda \neq 0$ is given by

$$\beta \neq \frac{3}{4}\alpha^2 \equiv \Lambda \neq 0. \tag{C.15}$$

Note that (C.12) has a solution for T_0 locally with respect to r at most in general. One exception is the solution

$$T_0 = St \pm \int^{Sr} \left(\frac{1}{f(u)} - 1\right) du. \tag{C.16}$$

Interestingly, this corresponds to a Finkelstein-type time coordinate which is regular at the future horizon or the past horizon.

Appendix D

Explicit forms for δQ_1, δQ_2 and δQ_3

$$\begin{aligned}\delta Q_1 = \frac{2(1+\alpha)}{\alpha} H_L Y + \frac{\mu^2}{2}\{(a-1)^3 - c^2(2a+b-3)\}h_{tt} \\ + \frac{\alpha^2}{2(1+\alpha)^2}\{-(b-1)^3 + c^2(a+2b-3)\}h_{rr} \\ + \frac{\alpha\mu}{(1+\alpha)} c\{c^2 - (a^2+b^2+ab-3a-3b+3)\}h_{tr},\end{aligned} \tag{D.1}$$

$$\begin{aligned}\delta Q_2 = -\frac{4(\alpha+1)}{\alpha^2} H_L Y + \mu^2\{c^4 - c^2(3a^2+b^2+2ab-6a-3b+3) + a(a-1)^3\}h_{tt} \\ + \frac{\alpha^2}{(1+\alpha)^2}\{-c^4 + c^2(a^2+3b^2+2ab-3a-6b+3) - b(b-1)^3\}h_{rr} \\ + \frac{2\mu\alpha}{1+\alpha} c\{c^2(2a+2b-3) - a^3 - b^3 - ab(a+b) + 3a^2 + 3b^2 + 3ab - 3a - 3b + 1\}h_{tr},\end{aligned} \tag{D.2}$$

$$\begin{aligned}\delta Q_3 = \frac{6(\alpha+1)}{\alpha^3} H_L Y + \frac{3\mu^2}{2}\{c^4(3a+2b-3) - c^2(4a^3+b^3+2ab^2+3a^2b-9a^2-3b^2 \\ +6a+3b-6ab-1) + a^2(a-1)^3\}h_{tt} \\ - \frac{3\alpha^2}{2(1+\alpha)^2}\{c^4(3b+2a-3) - c^2(4b^3+a^3+2ba^2+3b^2a-9b^2-3a^2 \\ +6b-6ab+3a-1) + b^2(b-1)^3\}h_{rr} \\ + \frac{3\mu\alpha}{1+\alpha} c\{-c^4 + c^2(3a^2+3b^2+4ab-6a-6b+3) \\ -a^4 - b^4 - ab^3 - a^3b - a^2b^2 + 3a^3 + 3b^3 + 3ab^2 + 3a^2b - 3a^2 - 3b^2 - 3ab + a + b\}h_{tr}.\end{aligned} \tag{D.3}$$

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