PARABOLIC AND ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH LOCALIZED NONLINEARITY

WENBIAO WANG

MARCH 2014

PARABOLIC AND ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH LOCALIZED NONLINEARITY

A dissertation submitted to

THE GRADUATE SCHOOL OF ENGINEERING SCIENCE

OSAKA UNIVERSITY

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY IN SCIENCE

WENBIAO WANG

MARCH 2014

Abstract

Behavior of solutions of nonlinear partial differential equations is delicately influenced by the nonlinear terms. Such phenomenon is of interest in the theory of partial differential equations, in the context of mathematical modeling, as well as in the area of mathematical physics. In this paper, particularly interested in the effect of a compactly supported coefficient added on the nonlinear terms, we study the behavior of positive solutions to porous medium equation with localized reaction, and semilinear elliptic equation with localized nonlinearity.

The first half mainly deals with the critical exponents concerning the large-time behavior of positive solutions to porous medium equation with localized reaction in multi-dimensional space. We concluded our results with two main theorems – the two-dimensional case and the higher dimensional case. Especially for the latter one, namely, when the space dimension is not less than three, we clarified the relationship between the behavior of nonnegative solutions and the exponents contained by the diffusion and reaction terms of the equation. In addition, for further discussion of the support of blow-up solutions, a property concerning the support of solutions is also proved.

In the second half, to continue to study the effect of the localized nonlinearity on the behavior of solutions to partial differential equations, we studied the role of a localized coefficient in a priori estimate for positive solutions to the semilinear elliptic equation. For the semilinear elliptic equation without localized nonlinearity, the existence of an a priori bound for all positive solutions is a well-known result. However, we discovered that under the influence of the localized nonlinearity, certain conditions should be imposed to guarantee the existence of the a priori bound. In our two main theorems, we respectively obtained two types of such conditions for the existence of the a priori bound. Furthermore, for future work, we suggested possible improvement of the result, and presented a corresponding semilinear parabolic problem where our arguments and techniques may be applicable.

Contents

Chapter 1

Introduction

In the first half of this paper, we consider the Cauchy problem

$$\begin{cases} u_t = \Delta(u^m) + a(x)u^p, & (x,t) \in \mathbf{R}^n \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.0.1}$$

where integers $m > 1, p > 0$, the cut-off function $a(x) \geq 0$, the initial function $u_0(x)$ is continuous and nonnegative but not identical with zero, and both $a(x)$ and $u_0(x)$ are compactly supported.

The motivation for the following study lies in [7], which discovered the relationship between the behavior of nonnegative solutions to the problem and the exponents m and p when the space dimension $n = 1$. The result is that for

$$\begin{cases} u_t = (u^m)_{xx} + a(x)u^p, & (x,t) \in \mathbf{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{1.0.2}$$

whether the solutions blow up or not depends on m and p, as shown below:
(i)If $0 < p \leq \frac{m+1}{2}$, then all the solutions to (1.0.2) are globally defined;
(ii)If $\frac{m+1}{2} < p \leq m+1$, then all the solutions to (1.0.2) blow up in finite time;
(iii)If $p > m+1$, then both global solutions and blow-up solutions to (1.0.2) exist.

We call $p_0 = \frac{m+1}{2}$ and $p_C = m+1$ the *critical exponent for global existence* and the *Fujita exponent*, respectively. Based on these results, we hope to understand the two exponents for the multi-dimensional case.

For the Cauchy problem of the porous medium equation (PME, for short in the following) possessing reaction term without the cut-off function $a(x)$ in the space dimension higher than one, namely the case $a(x) \equiv 1$, Deng-Levine [3], Galaktionov-Vazquez [8] and Levine [15] have studied the role of exponents in blow-up problems: it has been discovered that $p_0 = 1$ and $p_C = m + \frac{2}{n}$. Levine and Sacks [16] have discussed the relationship between the reaction term and the behavior of solutions more generally; Pinsky [17] has made a relevant study of the semilinear heat equation with localized reaction. However, just as what is mentioned in [7], the relevant multidimensional problem for (1.0.1) is still "a subject of a future work". We have partially solved the problem([14]), and obtained the following results.

When the space dimension is not less than three, there are similar results to the one-dimensional case:

Theorem 1 *When* $n \geq 3$,
(i)If $0 < p < m$ *, then all the solutions to* (1.0.1) *are globally defined;*
(ii)If $p = m$*, then the solutions to* (1.0.1) *blow up in finite time or not depending on the form of the cut-off function* $a(x)$ *and the size of the initial data* $u_0(x)$*;*
(iii)If $p > m$*, then the solutions to* (1.0.1) *blow up in finite time or not depending on the size of the initial data* $u_0(x)$*.*

For the two-dimensional case, what can be discovered presently is the following.

Theorem 2 *When* $n = 2$,
(i)If $0 < p \leq \frac{m+1}{2}$*, then all the solutions to* (1.0.1) *are globally defined;*
(ii)If $p = m$*, then the solutions to* (1.0.1) *blow up in finite time, provided the cut-off function* $a(x)$ *and the size of the initial data* $u_0(x)$*satisfy proper conditions;*
(iii)If $p > m$*, then the solutions to* (1.0.1) *blow up in finite time, provided the size of the initial data* $u_0(x)$ *satisfies a proper condition.*

For the case $\frac{m+1}{2} < p < m$, we have not obtained any result yet, and whether there exist global solutions for the case $p \geq m$ is still unknown.

Additionally, in numerical computation of PME, the localization method is often employed. For the purpose of the numerical simulation on *supp* a, we must consider whether the solutions blow up on *supp* a.

At present, we have obtained a result concerning the support of a solution and the support of the cut-off function. In particular, we prove that whether a solution blows up or not, the intersection of its support and the support of the cut-off function will be non-empty at some time:

Theorem 3 *There exists* $t \in (0, \infty)$ *such that supp* $u(\cdot, t) \cap$ *supp* $a \neq \emptyset$*.*

In the latter half, we continue to investigate the effect of the localized nonlinearity on the behavior of solutions to partial differential equations. In particular, we study the role of such $a(x)$ in a priori bounds for positive solutions to the semilinear elliptic equation

$$-\Delta u = a(x)u^p \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.0.3}$$

where $p > 1$, $\Omega \subset \mathbf{R}^n$ is a bounded domain with C^2 boundary $\partial\Omega$, and

$$0 \leq a = a(x) \in C_0(\Omega), \quad a \not\equiv 0. \tag{1.0.4}$$

Here $a \in C_0(\Omega)$ means that $a = a(x)$ is a continuous function with its support contained in Ω.

If $a(x) \equiv 1$, or, more generally, $a(x)$ is continuous and strictly positive in $\overline{\Omega}$, Gidas-Spruck [10] and de Figueiredo-Lions-Nussbaum [6] have obtained a famous result that there exists an a priori bound for all positive solutions which guarantees actual existence of the solutions, provided that

$$1 < p < \frac{n+2}{n-2}. \tag{1.0.5}$$

We show that this property still holds for (1.0.3) if we reduce the nonlinearity to some extent or impose some assumptions on $a(x)$.

Theorem 4 *Let*

$$1 < p < \frac{n}{n-2}. \tag{1.0.6}$$

Then there exists $C = C(\Omega, a(x), p)$ *such that*

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

for any solution $u = u(x)$ *to* (1.0.3).

Theorem 5 *Let* p *be in* (1.0.5), *and* $a = a(x)$ *in* (1.0.4) *satisfy*

$$\begin{cases} (i) & \omega = \{x \in \Omega | a(x) > 0\} \text{ is star-shaped with } C^1 \text{ boundary} \\ (ii) & a \in C^1(\overline{\omega}) \\ (iii) & \frac{\partial a}{\partial \nu} < 0 \text{ on } \partial\omega \end{cases} \tag{1.0.7}$$

where ν *denotes the outer unit normal vector. Then there exists* $C = C(\Omega, a(x), p)$ *such that*

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

for any solution $u = u(x)$ *to* (1.0.3).

Chapter 2

Porous medium equation with localized reaction

2.1 Preliminaries

As a basis for the proof, we must properly define the weak solution to problem (1.0.1) that locally exists and is unique. The most usual way is to define it using integration by parts, and then prove its existence and uniqueness. This is convenient for standard PME, i.e. $u_t = \Delta(u^m)$, whereas for the PME with localized reaction, it is difficult to prove the existence and uniqueness. Another feasible method is the application of analytical semigroup and interpolation space to semilinear PDE, as in [19]. However in this section, we realize the construction of the proper solution by employing the extension of monotonic semigroups, exactly the same method used by Galaktionov-Vazquez in [9].

To start with, let X be an ordered topological space of functions $\Omega \to \bar{\mathbf{R}}_+$, where Ω is an open subset of $\mathbf{R}^n$, $\bar{\mathbf{R}}_+ = [0,\infty) \cup \{\infty\}$; B be a subspace of X which approximates X in a certain way, as explained below; and S_t be a semigroup acting in B. Now we need to extend S_t to act on X. For this purpose, we have to make the following assumptions:
(S1)S_t is order-preserving;
(S2)S_t is continuous and X-closed with respect to monotonic, increasing convergence(m.i.c. for short in the following).

In the second place, we consider a family of "approximation" operators $\{P_n : X \to B\}_{n\in\mathbf{N}}$ satisfying the following conditions:
(P1)$\{P_n\}$ is ordered: for every $u \in X$ and $n > m$, $P_n u \geq P_m u$ holds;
(P2)P_n is continuous under m.i.c.;
(P3)As $n \to \infty$, we have $P_n u \to u, u \in X$.

Next, we define the *extension* of S_t: for every $u \in X$ and $t > 0$, we put

$$T_t u = \lim_{n\to\infty} S_t P_n u.$$

Proposition 2.1 *T_t is a semigroup in X that extends S_t and is continuous under m.i.c.. The limit in the above expression is independent of the approximation sequence $\{P_n\}$ satisfying conditions (P1)-(P3).*

For our application, we assume X to be the space of nonnegative, measurable functions $\mathbf{R}^n \to \bar{\mathbf{R}}_+$, and B is chosen so that the equation

$$u_t = \Delta(u^m) + f(u), \quad m > 0$$

generates a semigroup S_t in B that satisfies (S1) and (S2). We have to assume the function f to be Lipschitz continuous so that S_t will be well-defined in B. Finally, the operator P_n can be any of the usual cut-off operators that produce bounded functions.

This construction possesses generality and applies to the case when the reaction term involves the space invariable, i.e. $f(x,u)$, if only it is Lipschitz continuous. For $f(x,u) = a(x)u^p$, which corresponds to problem (1.0.1), let $g(u) = u^p$, and, without loss of generality, let the cut-off function $a(x)$ be the characteristic function of the closed ball $\bar{B}(0,L)$: $a(x) = \chi_{\bar{B}(0,L)}(x), L > 0$. Perform the following approximation to f: Define

$$a_j(x) = \begin{cases} 1, & |x| \le L - 2^{-j}L, \\ 2^j L^{-1}(L - |x|), & L - 2^{-j}L < |x| < L, \\ 0, & |x| \ge L; \end{cases}$$

$$g_j(u) = \begin{cases} 2^j u, & 0 \le u < 2^{-\frac{j}{1-p}}, \\ u^p, & 2^{-\frac{j}{1-p}} \le u < j, \\ j^p, & u \ge j; \end{cases} \quad (if\ \ 0 < p < 1),$$

$$g_j(u) = \begin{cases} u^p, & 0 \le u < j \\ j^p, & u \ge j \end{cases} \quad (if\quad p \ge 1);$$

then $\{a_j\}$ and $\{g_j\}$ are both nonnegative, monotonically increasing, and Lipschitz continuous sequences, and meanwhile it holds that $a_j(x) \to a(x)$ for any $x \in \mathbf{R}^n$ while $g_j(u) \to g(u)$ for any $u \in [0,\infty)$. Then let

$$f_j(x,u) = a_j(x)g_j(u), \quad (x,u) \in \mathbf{R}^n \times [0,\infty).$$

We can easily prove that $\{f_j\}$ is a sequence of increasing, nonnegative, and Lipschitz continuous functions that satisfy

$$f_j(x,u) \to f(x,u), \quad (x,u) \in \mathbf{R}^n \times [0,\infty).$$

Now let S_t^j be the semigroup generated by the equation

$$u_t = \Delta(u^m) + f_j(x, u)$$

acting on B, and T_t^j be its extension to X as constructed above. Thus, for every $u \in X$, we define

$$T_t u = \lim_{j\to\infty} T_t^j u = \lim_{j\to\infty} \lim_{k\to\infty} S_t^j(P_k u).$$

Another natural definition performs the two approximation processes at the same time:

$$U_t u = \lim_{j\to\infty} S_t^j(P_j u).$$

Proposition 2.2 *The above two definitions are equivalent and provide a semigroup in X that is continuous under m.i.c.. The limit is independent of the approximation sequences $\{P_j\}$ and $\{f_j\}$. Furthermore, we have the equivalent definition*

$$T_t u = \lim_{j,k\to\infty} S_t^j P_k u.$$

This extended semigroup is called the *limit semigroup*. For the initial function $u_0 \in X$, the function $u : \Omega \times [0, \infty) \to \bar{\mathbf{R}}_+$ defined by

$$u(x, t) = T_t u_0(x)$$

is called the *proper solution* of problem (1.0.1).

Proposition 2.3 *The proper solutions satisfy the standard comparison theorem with respect to the data. In addition, the proper solution is minimal with respect to any kind of weak solution of the problem that satisfies the maximum theorem with respect to bounded weak solutions.*

2.2 Proof of Theorems 1 and 2

The above discussion has ensured the local existence and uniqueness of the proper solution to (1.0.1), and the practicality of the weak comparison theorem for $u \in L^\infty(\mathbf{R}^n) \cap H_0^1(\mathbf{R}^n)$. The essential method in the proofs of this section is the method of comparison, namely, to construct a globally defined supersolution in order to prove the global existence, or to construct a blow-up subsolution to prove that the solutions blow up. In the following we assume that the space dimension $n > 1$, unless otherwise stated.

Now we begin with a simple proof.

Lemma 2.1 *If $0 < p \leq 1$, all the solutions to problem* (1.0.1) *are global.*

Proof. From the conditions concerning $a(x)$ and $u_0(x)$, we naturally assume that $|a(x)| \le M, 0 \le u_0(x) \le C$, where M and C are positive numbers. Thus the solution to the Cauchy problem

$$\begin{cases} u_t = Mu^p, & (x,t) \in \mathbf{R}^n \times (0,T), \\ u(x,0) = C, & x \in \mathbf{R}^n, \end{cases}$$

namely,

$$u(x,t) = \begin{cases} [C^{1-p} + (1-p)Mt]^{\frac{1}{1-p}}, & 0 < p < 1, \\ Ce^{Mt}, & p = 1 \end{cases}$$

is a supersolution to problem (1.0.1). Hence when $0 < p \le 1$, the solution of (1.0.1) is defined for all $t \in (0, \infty)$, and therefore (1.0.1) has only global solutions. This proves Lemma 2.1. □

Put another way, there may exist blow-up solutions to (1.0.1) only if $p > 1$.

In the following, we shall prove two lemmas regarding the existence of global solution to (1.0.1) when $n \ge 3$. We shall employ the Liouville property of semilinear elliptic equation ([11] and [20], p.223, 1A).

Proposition 2.4 *When $n \ge 3$, if $q \ge 2^* - 1 = \frac{n+2}{n-2}$, then the equation*

$$\Delta U + U^q = 0, \quad x \in \mathbf{R}^n, \tag{2.2.1}$$

has a solution $U \in C^\infty(\mathbf{R}^n)$ with $U(x) > 0, x \in \mathbf{R}^n$.

Thus we have: when $n \ge 3$, if $p > m$ and $q \ge max\{\frac{p}{m}, \frac{n+2}{n-2}\}$, then there exists a positive solution $U(x)$ to equation (2.2.1). Since $a(x)$ is compactly supported, we can take a constant $\lambda > 0$ large enough to guarantee that

$$a(x) \le \lambda^{p-m} U^{q-\frac{p}{m}}, \quad x \in \mathbf{R}^n \tag{2.2.2}$$

Lemma 2.2 *Let $n \ge 3$. Assume $p > m$ and $q \ge max\{\frac{p}{m}, \frac{n+2}{n-2}\}$. If the initial data satisfies*

$$u_0(x) \le \lambda^{-1} U^{\frac{1}{m}}, \quad x \in \mathbf{R}^n, \tag{2.2.3}$$

where $U \in C^\infty(\mathbf{R}^n)$ is a positive solution to (2.2.1) and λ is a large enough positive constant satisfying (2.2.2), then the solutions of (1.0.1) are global-in-time.

Proof. Let $\phi_\lambda(x) = \lambda^{-1}[U(x)]^{\frac{1}{m}}$. We shall prove that the stationary solution $\phi_\lambda(x)$ is a supersolution to (1.0.1). Substituting its expression into the equation in (1.0.1), we have

$$\Delta(\phi_\lambda^m) + \lambda^{qm-m} \phi_\lambda^{qm} = 0.$$

It follows from (2.2.2) that

$$a(x) \le \lambda^{p-m}(\lambda^m \phi_\lambda^m)^{q-\frac{p}{m}} = \lambda^{qm-m}\phi_\lambda^{qm-p}, \quad x \in \mathbf{R}^n. \tag{2.2.4}$$

Thus,

$$(\phi_\lambda)_t = 0 = \Delta(\phi_\lambda^m) + \lambda^{qm-m}\phi_\lambda^{qm-p}\phi_\lambda^p \ge \Delta(\phi_\lambda^m) + a(x)\phi_\lambda^p.$$

Moreover, by (2.2.3),

$$\phi_\lambda(x) = \lambda^{-1}U^{\frac{1}{m}} \ge u_0(x), \quad x \in \mathbf{R}^n.$$

Therefore, $\phi_\lambda(x)$ is a supersolution to (1.0.1) and thus all the solutions to (1.0.1) are globally defined. This proves Lemma 2.2. □

Lemma 2.3 *When $n \ge 3$, if $p < m$, then all the solutions to* (1.0.1) *are global.*

Proof. At this time $\frac{n+2}{n-2} > 1 > \frac{p}{m}$, and thus by Proposition 2.4, (2.2.1) has a positive C^∞ solution $U(x)$ if $q \ge \frac{n+2}{n-2}$. Since $p - m < 0$, for any continuous and compactly supported initial function $u_0(x)$ and compactly supported cut-off function $a(x)$, there exits a small enough positive constant λ satisfying both (2.2.2) and (2.2.3). Therefore, it is proved exactly the same way as before that $\phi_\lambda(x) = \lambda^{-1}[U(x)]^{\frac{1}{m}}$ is a supersolution to (1.0.1). This completes the proof of Lemma 2.3. □

In the above, we have proved, by constructing the global supersolution to problem (1.0.1), that when the space dimension $n \ge 3$, if $p > m$ and the size of the initial data is "small", then (1.0.1) has only global solutions; while if $p < m$, then all the solutions are globally defined, disregarding the choice of the initial function.

Next, we shall explain that all the solutions to problem (1.0.1) are global if $1 < p \le \frac{m+1}{2}$, again by constructing a global supersolution. While this construction is realized with the help of a result in [7]:

Proposition 2.5 *If $0 < p \le \frac{m+1}{2}$, then all the solutions to* (1.0.2) *are global.*

Lemma 2.4 *If $1 < p \le \frac{m+1}{2}$, then every solution to* (1.0.1) *is global.*

Proof. Since $a(x)$ is a nonnegative and compactly supported function, we assume, without loss of generality, that $0 \le a(x) \le 1$. Proposition 2.5 shows that in the case when the space dimension is one, the solutions to the Cauchy problem

$$\begin{cases} w_t = (w^m)_{xx} + \chi_{[-L,L]}w^p, & (x,t) \in \mathbf{R} \times (0,T), \\ w(x,0) = \phi(x), & x \in \mathbf{R}, \end{cases}$$

are globally defined.

When $n > 1$, we define

$$\begin{array}{ll} \tilde{u}(x,t) = w(x_1,t), & x = (x_1,...,x_n), t \in (0,T), \\ \tilde{\chi}_{[-L,L]}(x) = \chi_{[-L,L]}(x_1), \tilde{\phi}(x) = \phi(x_1), & x = (x_1,...,x_n), \end{array}$$

then $\tilde{u}$ is a global solution to the following problem:

$$\begin{cases} \tilde{u}_t = \Delta(\tilde{u}^m) + \tilde{\chi}_{[-L,L]}\tilde{u}^p, & (x,t) \in \mathbf{R}^n \times (0,T), \\ \tilde{u}(x,0) = \tilde{\phi}(x), & x \in \mathbf{R}^n, \end{cases} \tag{2.2.5}$$

Since $u_0(x)$ is compactly supported, it also has compact support in x_1. Thus we can choose ϕ, such that $supp\ u_0 \subset supp\ \tilde{\phi}$ and $\sup u_0 \le \tilde{\phi}(x)$ for $x \in supp\ u_0$, which imply that $\tilde{u}$ is a supersolution to (1.0.1). Consequently, problem (1.0.1) has only global solutions. This concludes the proof of Lemma 2.4. □

Then, we shall use the energy method (originally from [2]) to show that when $p > m$, problem (1.0.1) has blow-up solutions if the initial function $u_0(x)$ satisfies some given condition.

Define the energy function

$$E(t) = \frac{1}{2}\int_{\mathbf{R}^n} |\nabla(u^m)|^2 dx - \frac{m}{p+m}\int_{\mathbf{R}^n} a(x)u^{p+m}dx.$$

Lemma 2.5 *When $p > m$, if there exists $t_0 > 0$ such that $E(t_0) < 0$, then all the solutions to* (1.0.1) *blow up in finite time.*

Proof. Since $u \in L^{p+m}(\mathbf{R}^n) \cap H_0^1(\mathbf{R}^n)$, by the definition of $E(t)$ we have

$$\begin{aligned} E'(t) &= \int_{\mathbf{R}^n} \langle \nabla(u^m), \nabla(u^m)_t\rangle dx - m\int_{\mathbf{R}^n} a(x)u^{p+m-1}u_t dx \\ &= -\int_{\mathbf{R}^n} \Delta(u^m)mu^{m-1}u_t dx - m\int_{\mathbf{R}^n} a(x)u^{p+m-1}u_t dx \\ &= m\int_{\mathbf{R}^n} u^{m-1}u_t(-\Delta(u^m) - a(x)u^p)dx \\ &= -m\int_{\mathbf{R}^n} u^{m-1}|u_t|^2 dx \le 0, \end{aligned}$$

which shows that $E(t)$ is decreasing. Thus $E(t) \le E(t_0) < 0$ for any $t \ge t_0$.

Now we define another energy function

$$M(t) = \frac{1}{m+1}\int_0^1 \int_{\mathbf{R}^n} u^{m+1}(x,s)dxds.$$

For any $t \ge t_0$, we have $M(t) > 0$,

$$M'(t) = \frac{1}{m+1}\int_{\mathbf{R}^n} u^{m+1}(x,t)dx > 0,$$

and

$$M''(t) = \int_{\mathbf{R}^n} u^{m+1} u_t dx = \int_{\mathbf{R}^n} u^m (\Delta(u^m) + a(x)u^p) dx$$

$$= -\int_{\mathbf{R}^n} |\nabla(u^m)|^2 dx + \int_{\mathbf{R}^n} a(x) u^{p+m} dx$$

$$= \frac{p+m}{m}\left[-\frac{1}{2}\int_{\mathbf{R}^n} |\nabla(u^m)|^2 dx + \frac{m}{p+m}\int_{\mathbf{R}^n} a(x) u^{p+m} dx\right]$$

$$- \left(1 - \frac{p+m}{2m}\right)\int_{\mathbf{R}^n} |\nabla(u^m)|^2 dx.$$

From $p > m$ it follows that $-\left(1 - \frac{p+m}{2m}\right) > 0$, thus the inequality

$$M''(t) \geq \frac{p+m}{m}(-E(t)) > \frac{p+m}{m}(-E(t_0)) > 0 \tag{2.2.6}$$

holds; namely, for any $t \geq t_0$, we have $M''(t) > C > 0$, where C is a certain positive constant. Since $M'(t_0) > 0$ and $M(t_0) > 0$, we obtain, after integrating the above expression over the interval $[t_0, t]$, that $M'(t) \geq Ct + C_1$, where C_1 is a constant. Therefore, $M'(t) \to \infty$ as $t \to \infty$.

In the following we shall prove that there exists $T < \infty$ such that $M(t) \to \infty$ as $t \to T$.

Notice that $E'(t)$ can also be written as

$$E'(t) = -\frac{4m}{(m+1)^2}\int_{\mathbf{R}^n} |(u^{\frac{m+1}{2}})_t|^2 dx,$$

and thus

$$E(t) - E(t_0) = \int_{t_0}^t E'(s) ds = -\frac{4m}{(m+1)^2}\int_{t_0}^t \int_{\mathbf{R}^n} |(u^{\frac{m+1}{2}})_s|^2 dx ds.$$

Since $E(t_0) < 0$, we have

$$E(t) < -\frac{4m}{(m+1)^2}\int_{t_0}^t \int_{\mathbf{R}^n} |(u^{\frac{m+1}{2}})_s|^2 dx ds.$$

Thus, by (2.2.6),

$$M''(t) > \frac{4(p+m)}{(m+1)^2}\int_{t_0}^t \int_{\mathbf{R}^n} |(u^{\frac{m+1}{2}})_s|^2 dx ds.$$

Multiplying the above expression with the defining expression of $M(t)$, we have

$$M(t)M''(t) > \frac{4(p+m)}{(m+1)^3}\int_{t_0}^t \int_{\mathbf{R}^n} (u^{\frac{m+1}{2}})^2 dx ds \int_{t_0}^t \int_{\mathbf{R}^n} |(u^{\frac{m+1}{2}})_s|^2 dx ds.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} M(t)M''(t) &> \frac{4(p+m)}{(m+1)^3}\left[\int_{t_0}^{t}\int_{\mathbf{R}^n} u^{\frac{m+1}{2}}(u^{\frac{m+1}{2}})_s dxds\right]^2 \\ &= \frac{p+m}{m+1}\left[\frac{1}{m+1}\int_{\mathbf{R}^n} u^{m+1}(x,t)dx - \frac{1}{m+1}\int_{\mathbf{R}^n} u^{m+1}(x,t_0)dx\right]^2 \\ &= \frac{p+m}{m+1}[M'(t)-M'(t_0)]^2. \end{aligned}$$

Due to the fact that $M'(t) \to \infty$ in the limit $t \to \infty$ and since $\frac{p+m}{m+1} > 1$ for $p > m$, there exists a constant $\alpha > 0$ such that

$$M(t)M''(t) \geq (1+\alpha)M'^2(t), \tag{2.2.7}$$

when t is large enough. This is equivalent to the proposition that $M^{-\alpha}$ is a concave function. (In fact, $M^{-\alpha}$ is concave if and only if

$$\begin{aligned} &(M^{-\alpha})'' = (-\alpha M^{-\alpha-1}M')' \\ = \; &(-\alpha)(-\alpha-1)M^{-\alpha-2}(M')^2 + (-\alpha)M^{-\alpha-1}M'' \leq 0, \end{aligned}$$

which is equivalent to (2.2.7).)

Since $M(t) \geq 0$, there exists $0 < T < \infty$ such that $M^{-\alpha}(T) = 0$, and therefore $M(t) \to \infty$ as $t \to T$.

Furthermore, we claim that $M''(t) \to \infty$ as $t \to T$.

Assume the opposite, namely, that $M''(t)$ is bounded in $[0,T]$. Let $M''(t) \leq 2C_1$ for $t \in [0,T]$, where C_1 is a positive constant. Integrating the inequality twice over the interval $[0,T]$, we obtain that $M(t) \leq C_1 t^2 + C_2 t + C_3, t \in [0,T]$, where C_2 and C_3 are constants. The right side is bounded in $[0,T]$, and so is $M(t)$, which contradicts that $M(t) \to \infty$ as $t \to T$. Hence $M''(t) \to \infty$ in the limit $t \to T$.

At the last stage we prove that $u(x,t)$ blows up. By (1.0.1),

$$\frac{d}{dt}\left(\frac{1}{m+1}u^{m+1}\right) = u^m u_t = u^m \Delta(u^m) + a(x)u^{p+m},$$

and thus

$$\begin{aligned}
M''(t) &= \frac{d}{dt}\left(\frac{1}{m+1}\int_{\mathbf{R}^n} u^{m+1}(x,t)dx\right) \\
&= \int_{\mathbf{R}^n} \frac{d}{dt}\left(\frac{1}{m+1}u^{m+1}(x,t)\right)dx \\
&= \int_{\mathbf{R}^n} [u^m\Delta(u^m)+a(x)u^{p+m}]dx \\
&= -\int_{\mathbf{R}^n} |\nabla(u^m)|^2dx + \int_{\mathbf{R}^n} a(x)u^{p+m}dx \\
&\leq \int_{\mathbf{R}^n} a(x)u^{p+m}dx \leq meas(supp\ a)\|u(\cdot,t)\|_\infty^{p+m},
\end{aligned}$$

and consequently $u(x,t)$ blows up in the sense of L^∞-norm. This proves Lemma 2.5. □

From Lemmas 2.2 and 2.5, we can conclude that when the space dimension $n \geq 3$ and $p > m$, whether the solutions to problem (1.0.1) blow up or not depends on the initial data: when the size is "small", all the solutions are global; on the contrary, when it is "large", the solutions blow up.

Finally, we consider the case $p = m$.

When $n \geq 3$, Proposition 2.4 and the proof of Lemma 2.2 apply in this case. Let $q \geq \frac{n+2}{n-2}$ and $U(x)$ be a positive solution to equation (2.2.1). If $a(x)$ and $u_0(x)$ satisfy

$$a(x) \leq U^{q-1}, \quad x \in \mathbf{R}^n, \tag{2.2.8}$$

$$u_0(x) \leq U^{\frac{1}{m}}, \quad x \in \mathbf{R}^n \tag{2.2.9}$$

then all the solutions to (1.0.1) are global. Hence the following lemma:

Lemma 2.6 *When $n \geq 3$ and $p = m$, all the solutions to* (1.0.1) *are globally defined if $a(x)$ and $u_0(x)$ satisfy* (2.2.8) *and* (2.2.9) *respectively.*

At last, we shall prove that when the cut-off function $a(x)$ satisfies some certain condition, (1.0.1) has only blow-up solutions.

Lemma 2.7 *When $n \geq 2$ and $p = m$, all the solutions to* (1.0.1) *blow up if $a(x)$ satisfies*

$$a(x) \geq \delta > 0, \quad x \in B_R(0)(R \gg 1),$$

where δ is a constant such that $\delta > \lambda_R$, and λ_R is the first eigenvalue of $-\Delta$ in ball $B_R(0)$.

Proof. By the assumption,

$$\begin{cases} -\Delta\phi = \lambda_R\phi, & x \in B_R(0), \\ \phi(x) = 0, & x \in \partial B_R(0), \end{cases} \tag{2.2.10}$$

where ϕ is the first eigenfunction corresponding to λ_R such that $\|\phi\| = 1$.

Let

$$E(t) = \int_{B_R(0)} u(x,t)\phi(x)dx,$$

so that

$$\begin{aligned} E'(t) &= \int_{B_R(0)} (\Delta(u^m) + a(x)u^m)\phi(x)dx \\ &= \int_{B_R(0)} \Delta\phi u^m dx + \int_{\partial B_R(0)} \frac{\partial u^m}{\partial n}\phi dS \\ &- \int_{\partial B_R(0)} u^m \frac{\partial \phi}{\partial n} dS + \int_{B_R(0)} a u^m \phi dx. \end{aligned}$$

Consider the right side. By Hopf's lemma, $\int_{\partial B_R(0)} u^m \frac{\partial \phi}{\partial n} dS < 0$. Combined with $a(x) \geq \delta$ and (2.2.10), we have

$$E'(t) > (\delta - \lambda_R) \int_{B_R(0)} \phi u^m dx.$$

After integrating it over $[0, t]$,

$$E(t) \geq E(0) + (\delta - \lambda_R) \int_0^t \int_{B_R(0)} \phi u^m dx ds.$$

Notice that

$$\begin{aligned} E(t) &= \int_{B_R(0)} u\phi dx = \int_{B_R(0)} u\phi^{\frac{1}{m}}\phi^{\frac{m-1}{m}} dx \\ &\leq \left(\int_{B_R(0)} u^m \phi dx\right)^{\frac{1}{m}} \left(\int_{B_R(0)} \phi dx\right)^{\frac{m-1}{m}}, \end{aligned}$$

or

$$E^m(t) \leq \left(\int_{B_R(0)} u^m \phi dx\right) \|\phi\|_1^{m-1} = \int_{B_R(0)} u^m \phi dx.$$

Thus, we have

$$E(t) \geq E(0) + (\delta - \lambda_R) \int_0^t E^m(s) ds.$$

Since $E(0) > 0$ and $\delta - \lambda_R > 0$, $E(t)$ blows up in finite time since $m > 1$. While according to the definition,

$$E(t) = \int_{B_R(0)} u\phi dx \leq \|u\|_\infty \|\phi\|_1 = \|u\|_\infty.$$

Consequently there exists $T < \infty$ such that $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to T$. This proves Lemma 2.7. □

The above discussion has explained that the case $p = m$ differs from the case $p > m$ ($n \geq 3$), in that whether the solutions to (1.0.1) blow up or not depends not only on the size of initial data, but also on the form of the cut-off function.

These lemmas complete the proof of Theorems 1 and 2.

2.3 Proof of Theorem 3

Case 1. Let the solution to (1.0.1) be globally defined, that is, $T = \infty$. The Cauchy problem

$$\begin{cases} u_t = \Delta(u^m), & (x,t) \in \mathbf{R}^n \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{2.3.1}$$

has a self-similar solution $u_S(x,t)$ with constant energy (i.e. the Barenblatt solution, [20], p.19-21):

$$u_S(x,t) = t^{-\frac{n}{n\sigma+2}} \left[\frac{\sigma}{2(n\sigma+2)} \left(\eta_0^2 - |x|^2 t^{-\beta} \right)_+ \right]^{\frac{1}{\sigma}},$$

where $\sigma = m-1, \beta = \frac{2}{n\sigma+2}$,

$$\eta_0 = \eta_0(E_0) = \left\{ \pi^{-\frac{n}{2}} \left[\frac{2(n\sigma+2)}{\sigma} \right]^{\frac{1}{\sigma}} \frac{\Gamma(\frac{n}{2}+1+\frac{1}{\sigma})}{\Gamma(\frac{1}{\sigma}+1)} E_0 \right\}^{\frac{\sigma}{n\sigma+2}}$$

and $E_0 = \int_{\mathbf{R}^n} u(x,t)dx$ is a fixed positive constant chosen beforehand.

Since the maximum point and the size of support of $u_S(x,t)$ are proportional to $t^{-\frac{n}{n\sigma+2}} \eta_0^{\frac{2}{m-1}}$ and $\eta_0 t^{\frac{\beta}{2}}$ respectively, we can appropriately choose η_0 such that $u_S(x,t_0) < u_0(x)$ for some $t_0 > 0$. Then let t_0 be the initial time, and $\tilde{u}_S(x,t)$ be a Barenblatt solution to the Cauchy problem corresponding to (2.3.1) starting at t_0. We can choose $\tilde{u}_S(x,t)$ as a subsolution to (1.0.1).

Since $\eta_0 t^{\frac{\beta}{2}} \to \infty$ as $t \to \infty$, *supp* $\tilde{u}_S(\cdot,t)$ expands as time passes. While for the support of the solution to (1.0.1), we have *supp* $u(\cdot,t) \supset$ *supp* $\tilde{u}_S(\cdot,t)$. Hence there exists $t \in (0,\infty)$ such that *supp* $u(\cdot,t) \cap$ *supp* $a \neq \emptyset$.

Case 2. Let the solution blow up in finite time $T < \infty$. By the definition of blow-up set:

$$B(u) = \left\{ x | ^\exists x_n \to x, t_n \to T^-, s.t. \lim_{n\to\infty} u(x_n,t_n) = \infty \right\},$$

there exists some t in the neighborhood of T such that $B(u) \subset$ *supp* $u(\cdot,t)$. Thus it suffices to prove *supp* $a \cap B(u) \neq \emptyset$ in the following.

Assume the opposite, namely, that $u(x,t)$ does not blow up on *supp a*. Since *supp a* is compact and $u \in H_0^1(\mathbf{R}^n)$, $u(x,t)$ is uniformly bounded on *supp a* within its time of existence, that is, there exists a constant $M > 0$ such that

$$\sup_{x \in supp\ a} |u(x,t)| < M$$

for any $t \in (0,T)$. Additionally, since $u(x,t)$ is a blow-up solution, by Lemma 2.1, we have $p > 1$, and thus u^{p-1} is also uniformly bounded on *supp a*. Now let $|au^{p-1}| \leq M_1$ (M_1 is a positive constant), and we obtain

$$u_t \leq \Delta(u^m) + M_1 u, \quad (x,t) \in \mathbf{R}^n \times (0,T). \tag{2.3.2}$$

After the transformation

$$v(x,t) = e^{-M_1 t} u(x,t),$$

inequality (2.3.2) takes the form

$$v_t \leq e^{(m-1)M_1 t} \Delta(v^m).$$

For $t \in (0,T), e^{(m-1)M_1 t}$ is bounded: $e^{(m-1)M_1 t} \leq C$ (C is a positive constant), and thus the above inequality can be further written as

$$v_t \leq C\Delta(v^m).$$

Again perform the transformation

$$\tilde{v}(x,t) = v(x, \frac{t}{C}),$$

and we finally obtain

$$\tilde{v}_t \leq \Delta(\tilde{v}^m), \quad (x,t) \in \mathbf{R}^n \times (0,T).$$

Notice that $\tilde{v}(x,t)$ has the initial data $\tilde{v}(x,0) = u_0(0)$.

It follows that $\tilde{v}$ is a subsolution to problem (2.3.1). Therefore, as in Case 1, we can choose an appropriate Barenblatt solution $u_S(x,t)$ such that $u_S(x,0) \geq u_0(x), x \in \mathbf{R}^n$. By comparison, we have

$$\tilde{v}(x,t) \leq u_S(x,t), \quad (x,t) \in \mathbf{R}^n \times (0,T),$$

which contradicts the assumption that $\tilde{v}(x,t)$ blows up as $t \to T^-$. This concludes the proof of Theorem 3. □

Chapter 3

Semilinear elliptic equation with localized nonlinearity

We take the solution $u \in H_0^1(\Omega)$ to (1.0.3) with $a(x)$ and p satisfying (1.0.4) and (1.0.5). The standard elliptic regularity then guarantees $u \in W^{2,q}(\Omega)$ for any $q > 1$. For simplicity, we assume $n \geq 3$.

3.1 Proof of Theorem 4

First, we apply Kaplan's method ([21]) to obtain some a priori bound.

Lemma 3.1 *If $a = a(x)$ satisfies* (1.0.4)*, the value*

$$\lambda_1 = \inf\{\|\nabla\phi\|_{L^2(\Omega)}^2 | \phi \in H_0^1(\Omega), \int_\Omega a(x)|\phi(x)|^2 dx = 1\} > 0$$

is attained by $\phi_1 = \phi_1(x)$ satisfying

$$-\Delta\phi_1 = \lambda_1 a(x)\phi_1, \ \phi_1 > 0 \ \textit{in } \Omega, \quad \phi_1 = 0 \ \textit{on } \partial\Omega. \tag{3.1.1}$$

Proof . We show first that λ_1 is attained at some $\phi \in H_0^1(\Omega)$.

To begin with, take a minimizing sequence $\{\phi_k\} \subset H_0^1(\Omega)$ satisfying

$$\int_\Omega a(x)|\phi_k(x)|^2 dx = 1, \text{ and } \|\nabla\phi\|_{L^2(\Omega)}^2 \to \lambda_1 \text{ as } k \to \infty. \tag{3.1.2}$$

Thus $\{\nabla\phi_k\}$ is bounded in $L^2(\Omega)$, and so is $\{\phi_k\}$, by Poincaré's inequality. Therefore, $\{\phi_k\}$ is bounded in $H^1(\Omega)$. By the weak compactness of reflexive

Banach space, there exists a subsequence $\{\phi_{k_j}\}_{j=1}^\infty \subset \{\phi_k\}_{k=1}^\infty$ (we still denote it as $\{\phi_k\}_{k=1}^\infty$ in the following) and $\phi \in H^1(\Omega)$, such that

$$\phi_k \rightharpoonup \phi \text{ in } H^1(\Omega).$$

Furthermore, $\phi = 0$ on $\partial\Omega$ in the trace sense, so

$$\phi \in H_0^1(\Omega).$$

Then, since $\{\phi_k\}$ is bounded in $H_0^1(\Omega)$, by Rellich-Kondrachov Compactness Theorem, $\{\phi_k\}$ has a convergent subsequence in $L^2(\Omega)$. Thus, we have

$$\phi_k \to \phi \text{ in } L^2(\Omega).$$

Since

$$\begin{aligned} \left|\int_\Omega a(|\phi|^2 - |\phi_k|^2)dx\right| &\le \|a\|_{L^\infty(\Omega)} \int_\Omega |\phi - \phi_k||\phi + \phi_k|dx \\ \le\ & \|a\|_{L^\infty(\Omega)}\|\phi - \phi_k\|_{L^2(\Omega)}(\|\phi\|_{L^2(\Omega)} + \|\phi_k\|_{L^2(\Omega)}) \to 0 \text{ as } k \to \infty, \end{aligned}$$

by (3.1.2),

$$\int_\Omega a|\phi|^2 dx = \int_\Omega a|\phi_k|^2 dx + \int_\Omega a(|\phi|^2 - |\phi_k|^2)dx = 1.$$

Thus, by the definition of λ_1,

$$\lambda_1 \le \|\nabla\phi\|_{L^2(\Omega)}^2. \tag{3.1.3}$$

On the other hand, since $\|\nabla \cdot \|_{L^2(\Omega)}$ is a norm of $H_0^1(\Omega)$, by the weak lower semi-continuity of norms in Banach space, together with (3.1.2), we have

$$\|\nabla\phi\|_{L^2(\Omega)}^2 \le \lim\inf_{k\to\infty} \|\nabla\phi_k\|_{L^2(\Omega)}^2 = \lambda_1.$$

This and (3.1.3) yield

$$\lambda_1 = \|\nabla\phi\|_{L^2(\Omega)}^2.$$

Finally, since

$$\|\nabla|\phi|\|_{L^2(\Omega)} = \|\nabla\phi\|_{L^2(\Omega)},$$

letting $\phi_1 = |\phi|$, we conclude that λ_1 is attained at $\phi_1 \in H_0^1(\Omega)$.

We show next that the minimizer ϕ_1 is indeed a solution of (3.1.1).

What we have obtained so far can be rewritten as (see [5], p. 463-464)

$$I[\phi_1] = \frac{\lambda_1}{2} = \min_{\phi \in \mathscr{A}} I[\phi],$$

where the energy functional

$$I[\phi] = \frac{1}{2}\|\nabla\phi\|^2_{L^2(\Omega)},$$

and the admissible class

$$\mathscr{A} = \{\phi \in H^1_0(\Omega) | J[\phi] = 0\},$$

in which the functional of the side condition is

$$J[\phi] = \int_\Omega G(\phi(x), a(x))dx, \text{ with } G(\phi, a) = a|\phi|^2 - \frac{1}{|\Omega|},$$

namely,

$$\int_\Omega a(x)|\phi(x)|^2 dx = 1. \tag{3.1.4}$$

By the principle of Lagrange multiplier, there exists a real number λ such that ϕ_1 is a weak solution of the boundary value problem

$$-\Delta\phi_1(x) = \lambda\frac{\partial G}{\partial\phi}(\phi_1(x), a(x)) = 2\lambda a(x)\phi_1(x) \text{ in } \Omega, \quad \phi_1(x) = 0 \text{ on } \partial\Omega,$$

which, together with the side condition (3.1.4) and the definition of λ_1, yields that

$$2\lambda = 2\lambda\int_\Omega a|\phi|^2 dx = \int_\Omega \nabla\phi_1 \cdot \nabla\phi_1 dx = \lambda_1.$$

Hence, ϕ_1 solves

$$-\Delta\phi_1 = \lambda_1 a(x)\phi_1, \ \phi_1 \geq 0 \text{ in } \Omega, \quad \phi_1 = 0 \text{ on } \partial\Omega. \tag{3.1.5}$$

If there exists $x_0 \in \Omega$ such that $\phi_1(x_0) = 0$, then

$$-\phi_1(x_0) = \sup_\Omega(-\phi_1).$$

Since (3.1.5) implies $\Delta(-\phi_1) \geq 0$, by the strong maximum principle, $-\phi_1$ must be constant in Ω. By making use of (3.1.5) again, we have $\phi_1 = 0$, which contradicts the side condition (3.1.4). Therefore, $\phi_1 > 0$ in Ω, and consequently solves (3.1.1). $\square$

Henceforth, by replacing $\phi_1(x)$ by $\frac{1}{\int_\Omega a(x)\phi_1(x)dx}\phi_1(x)$, we normalize the above $\phi_1 = \phi_1(x) > 0$ by

$$\int_\Omega a(x)\phi_1(x)dx = 1.$$

Lemma 3.2 *If $a = a(x)$ satisfies* (1.0.4)*, then each $1 \leq q^* < \frac{n}{n-2}$ admits $C = C(\Omega, a(x), p, q^*)$ such that*

$$\|u\|_{L^{q^*}(\Omega)} \leq C \tag{3.1.6}$$

*for any solution $u \in H_0^1(\Omega)$ to (*1.0.3*).*

Proof . By Lemma 3.1 and the convexity of function $g(y) = y^p$ for $p > 1$, we apply Jensen's inequality to obtain

$$\begin{aligned} \left(\int_\Omega a(x)\phi_1 u dx\right)^p &\leq \int_\Omega a(x)u^p\phi_1 dx = \int_\Omega (-\Delta u)\phi_1 dx \\ = & \int_\Omega (-\Delta\phi_1)u dx = \lambda_1 \int_\Omega a(x)\phi_1 u dx, \end{aligned}$$

from which it follows that

$$\int_\Omega a(x)u\phi_1 dx \leq \lambda_1^{\frac{1}{p-1}},$$

and furthermore

$$\int_\Omega a(x)u^p\phi_1 dx = \lambda_1 \int_\Omega a(x)\phi_1 u dx \leq \lambda_1^{\frac{p}{p-1}}.$$

Since $\phi_1 > 0$ within Ω and $\omega \subset\subset \Omega$,

$$\phi_1 \geq^\exists \delta > 0, \text{ in } \omega.$$

Thus,

$$\lambda_1^{\frac{p}{p-1}} \geq \int_\omega a(x)u^p\phi_1 dx \geq \delta \int_\omega a(x)u^p dx = \delta \int_\Omega a(x)u^p dx,$$

namely,

$$\|\Delta u\|_{L^1(\Omega)} = \int_\Omega a(x)u^p dx \leq \delta^{-1}\lambda_1^{\frac{p}{p-1}}.$$

Applying Brezis-Strauss L^1 estimate (Lemma 23 in [1]):

$$\|u\|_{W^{1,q}(\Omega)} \leq^\exists C(q)(\|\Delta u\|_{L^1(\Omega)} + \|u\|_{L^1(\partial\Omega)}), \quad 1 \leq^\forall q < \tfrac{n}{n-1},$$

together with the boundary condition, we obtain

$$\|u\|_{W^{1,q}(\Omega)} \leq^\exists C_1(\Omega, a(x), p, q), \quad 1 \leq^\forall q < \tfrac{n}{n-1}.$$

Then, it follows from Sobolev's inequality that

$$\|u\|_{L^{q^*}(\Omega)} \leq^{\exists} C_2(\Omega, a(x), p, q),$$

where $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n} \in [\frac{n}{n-1}, \frac{n}{n-2})$, namely, (3.1.6) holds for any $q^* \in [1, \frac{n}{n-2})$. $\square$

Now we are ready to prove Theorem 4 using a standard bootstrap argument based on (3.1.6).

Proof of Theorem 4. Considering (1.0.6), we choose a constant $s > 1$ satisfying both

$$s < \frac{n}{2}, \tag{3.1.7}$$

$$ps < \frac{n}{n-2}. \tag{3.1.8}$$

Notice that (3.1.7) is vacuous when $n \geq 4$, since in that case $p > 1$ and (3.1.8) imply $s < \frac{n}{n-2}\frac{1}{p} < \frac{n}{n-2}\frac{n-2}{n} < \frac{n}{n-2}\frac{n-2}{2} = \frac{n}{2}$. By (3.1.6) and $a \in C_0(\Omega)$, it holds that

$$\|au^p\|_{L^s(\Omega)} \leq^{\exists} C_1(p, s, \Omega, a(x)). \tag{3.1.9}$$

Then, replacing q^* with s in (3.1.6) since $s < \frac{n}{n-2}$ by (3.1.8), we apply the elliptic L^s estimate (see [13]), the compactness of Ω, and (3.1.9) to obtain

$$\|u\|_{W^{2,s}(\Omega)} \leq^{\exists} C_2(p, s, \Omega, a(x)).$$

Since $2 < \frac{n}{s}$ by (3.1.7), the Sobolev's inequality implies

$$\|u\|_{L^{q_1^*}(\Omega)} \leq^{\exists} C_3(p, q_1^*, \Omega, a(x)), \;\; \frac{1}{q_1^*} = \frac{1}{s} - \frac{2}{n}.$$

Notice that $\frac{1}{q_1^*} < 1 - \frac{2}{n} < \frac{1}{q^*}$, namely, $q_1^* > q^*$.

Next, by (3.1.8),

$$\frac{q_1^*}{p} > \frac{ns}{n-2s}\frac{n-2}{2} > s,$$

so there exists a constant $s_1 > s$ such that

$$s_1 < \frac{n}{2} \text{ and } ps_1 \leq q_1^*. \tag{3.1.10}$$

In a similar fashion, we can show

$$\|u\|_{L^{q_2^*}(\Omega)} \leq^{\exists} C_4(p, q_2^*, \Omega, a(x)), \;\; \frac{1}{q_2^*} = \frac{1}{s_1} - \frac{2}{n}. \tag{3.1.11}$$

By (3.1.10), $\frac{1}{q_2^*} < \frac{1}{s} - \frac{2}{n} = \frac{1}{q_1^*}$, namely, $q_2^* > q_1^*$.

Iterating this process, we can find a strictly increasing sequence $\{q_i^*\}$ such that

$$\|u\|_{L^{q_i^*}(\Omega)} \leq^{\exists} C_5(p, q_i^*, \Omega, a(x)).$$

Thus, for all $r > n$,

$$\|u\|_{L^r(\Omega)}, \|au^p\|_{L^r(\Omega)} \leq^{\exists} C_6(p, r, \Omega, a(x)).$$

Then the elliptic L^r estimate and the compactness of Ω lead to

$$\|u\|_{W^{2,r}(\Omega)} \leq^{\exists} C_7(p, r, \Omega, a(x)),$$

and Morrey's inequality with $\gamma = 1 - \frac{n}{r}$ implies

$$\|u\|_{C^{1,\gamma}(\Omega)} \leq^{\exists} C_8(p, \gamma, \Omega, a(x)).$$

Consequently, by the definition of Hölder norm and the arbitrariness of $r > n$, we obtain the desired a priori bound for u. □

Theorem 4 can also be proved by blow-up analysis.

To start with, the harmonic function theory implies the following lemma.

Lemma 3.3 *Under the assumption of* (1.0.4)*, it holds that*

$$\|u\|_{L^\infty(\Omega)} = \|u\|_{L^\infty(\omega)} \tag{3.1.12}$$

*for any solution $u \in H_0^1(\Omega)$ to (*1.0.3*).*

Proof . Since u is harmonic in $\Omega \setminus \omega$, it follows that

$$\|u\|_{L^\infty(\Omega\setminus\omega)} = \|u\|_{L^\infty(\partial\Omega\cup\partial\omega)}$$

from the maximum principle. Hence we obtain (3.1.12) by the zero boundary condition. □

The local a priori estimate in ω, on the other hand, is obtained as in ([10]).

Lemma 3.4 *If p and $a = a(x)$ satisfy* (1.0.4) *and* (1.0.5)*, then any compact set $K \subset \omega$ admits $C = C(K, \Omega, a(x), p)$ such that*

$$\|u\|_{L^\infty(K)} \leq C$$

for any solution $u \in H_0^1(\Omega)$ to (1.0.3)*.*

Proof . Assuming the contrary, and take a sequence of solutions $\{u_k\}$ to (1.0.3) satisfying

$$\|u_k\|_\infty = u_k(x_k) \to +\infty, \quad k \to \infty. \tag{3.1.13}$$

with $x_k \in K$. Passing to a subsequence, we obtain

$$x_k \to x_\infty, \quad k \to \infty.$$

For the rescaled solution

$$\tilde{u}_k(x) = \mu_k^{\frac{2}{p-1}} u_k(\mu_k x + x_k)$$

with $\mu_k > 0$ defined by

$$\mu_k^{\frac{2}{p-1}} u_k(x_k) = \mu_k^{\frac{2}{p-1}} \|u_k\|_{L^\infty(\Omega)} = 1,$$

it holds that

$$\mu_k \to 0 \quad as \quad k \to \infty \tag{3.1.14}$$

and

$$-\Delta \tilde{u}_k = a(\mu_k x + x_k)\tilde{u}_k^p, \quad 0 \le \tilde{u}_k \le \tilde{u}_k(0) = 1$$

in $B_{\frac{d}{\mu_k}}(0)$, where $d = \frac{1}{2} dist(x_\infty, \partial\Omega)$. Passing to a subsequence again, by elliptic L^r estimate, Morrey's inequality, the Arzelà-Ascoli theorem, and then a diagonal argument ([10]), we obtain

$$\begin{aligned} \tilde{u}_k &\rightharpoonup u_\infty \text{ in } W^{2,r}_{loc}(\mathbf{R}^n), \quad 1 < \forall q < \infty, \\ \tilde{u}_k &\to u_\infty \text{ in } C^{1,\beta}_{loc}(\mathbf{R}^n), \quad \beta = 1 - \tfrac{n}{r}, \end{aligned}$$

with u_∞ satisfying

$$-\Delta u_\infty = a(x_\infty)u_\infty^p, \quad 0 \le u_\infty \le u_\infty(0) = 1 \text{ in } \mathbf{R}^n. \tag{3.1.15}$$

Since $x_\infty \in K \subset \omega$, we have $a(x_\infty) > 0$ and the Liouville property proven by ([11]) guarantees that there is no such u_∞ in the case of (1.0.5). □

Another proof of Theorem 4. If the global a priori estimate to (1.0.3) fails, we have (3.1.13) with $x_k \in \omega$ by Lemma 3.3. Passing to a subsequence we have

$$x_k \to x_\infty \in \overline{\omega}, \quad as \quad k \to \infty$$

and then $x_\infty \in \omega$ is impossible by Lemma 3.4 under the assumption of (1.0.4) and (1.0.5).

Therefore, $x_\infty \in \partial\omega$. Hence u_∞ is harmonic in $\mathbf{R}^n$, which together with (3.1.15) yields

$$u_\infty \equiv 1, \quad in \quad \mathbf{R}^n,$$

by the strong maximum principle. Namely,

$$\tilde{u}_k \to 1 \text{ locally uniformly in } \mathbf{R}^n.$$

In particular, for any $q^* < \frac{n}{n-2}$,

$$\int_{|x|<1} \tilde{u}_k(x)^{q^*} dx \to |B_1(0)| > 0, \quad as \quad k \to \infty. \tag{3.1.16}$$

On the other hand, letting $x' = \mu_k x + x_k$, we have

$$\begin{aligned}
&\int_{|x|<1} \tilde{u}_k(x)^{q^*} dx = \int_{|x'-x_k|<\mu_k} \left(\mu_k^{\frac{2}{p-1}} u_k(x')\right)^{q^*} \mu_k^{-n} dx' \\
= \;& \mu_k^{\frac{2q^*}{p-1}-n} \int_{|x'-x_k|<\mu_k} u_k(x')^{q^*} dx' \le \mu_k^{\frac{2q^*}{p-1}-n} \int_\Omega u_k(x')^{q^*} dx' \\
= \;& \mu_k^{\frac{2q^*}{p-1}-n} \|u_k\|_{L^{q^*}(\Omega)} \le C_2 \mu_k^{\frac{2q^*}{p-1}-n}.
\end{aligned}$$

The last inequality is implied by (3.1.6). Now, notice the exponent in the above inequality. Taking $q^* = \frac{(p-1)n}{2}$, by (1.0.6), we have $q^* < \frac{n}{n-2}$ Therefore, by (3.1.14),

$$\int_{|x|<1} \tilde{u}_k(x)^{q^*} dx \to 0 \text{ as } k \to \infty,$$

which contradicts (3.1.16). □

3.2 Proof of Theorem 5

Proof of Theorem 5. We mainly follow the procedure of [6] and divide the proof into three steps.

Step 1. L^1 bound for ∇u in a neighborhood of $\partial\Omega$.

Firstly, since u is harmonic in $\Omega \setminus \omega$, by mean value theorem,

$$\|u\|_{L^\infty(K)} \le C_1(p, \Omega, a(x), K), \; {}^\forall K \subset\subset \Omega \setminus \omega. \tag{3.2.1}$$

In fact, $u \in C(\Omega \setminus \omega)$ is harmonic, if and only if for any ball $B = B_R(y) \subset\subset \Omega \setminus \omega$,

$$u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u dS,$$

where ω_n is the volume of unit ball in $\mathbf{R}^n$. Thus, for any $x \in \overline{K}$, taking $B_R(x)$ with radius

$$R = \frac{1}{2}\min\{dist(K, \partial\Omega), dist(K, \partial\omega)\},$$

we have

$$|u(x)| = \left| C_2(K) \int_{\partial B} u dS \right| \leq C_2(K)\|u\|_{L^1(\Omega)}, \ \forall x \in \overline{K}.$$

Recalling (3.1.6) with $q^* = 1$, we obtain (3.2.1).

Next, choosing

$$\Omega_1 = \{x \in \Omega | dist(x, \partial\Omega) < \frac{1}{2} dist(\omega, \partial\Omega)\},$$

it is clear that $\partial\Omega_1 = \partial\Omega \cup \partial[\Omega \setminus \Omega_1]$. Taking (3.2.1) and the boundary condition in (1.0.3) into consideration, we have

$$\|u\|_{L^\infty(\partial\Omega_1)} = \|u\|_{L^\infty(\partial[\Omega\setminus\Omega_1])} \leq^{\exists} C_3(p, \Omega, a(x)). \tag{3.2.2}$$

Then, by the strong maximum principle for harmonic functions,

$$\|u\|_{L^\infty(\overline{\Omega}_1)} = \|u\|_{L^\infty(\partial\Omega_1)} \leq C_3(p, \Omega, a(x)). \tag{3.2.3}$$

Finally, since $a(x)u(x)^p = 0$ in Ω_1 and (3.2.3), for any $s > n$, applying elliptic L^s estimate and the compactness of Ω_1,

$$\|u\|_{W^{2,s}(\Omega_1)} \leq^{\exists} C_4(s, \Omega)(\|u\|_{L^s(\Omega_1)} + \|a(\cdot)u(\cdot)^p\|_{L^s(\Omega_1)}) \leq C_5(p, s, \Omega, a(x)).$$

Then by Morrey's inequality,

$$\|u\|_{C^{1,\gamma}(\overline{\Omega_1})} \leq^{\exists} C_6(\gamma, \Omega_1)\|u\|_{W^{2,s}(\Omega_1)} \leq C_7(p, s, \Omega, a(x)), \ \gamma = 1 - \frac{n}{s}.$$

By the definition of Hölder norm and the arbitrariness of s, we obtain

$$\|\nabla u\|_{L^\infty(\Omega_1)} \leq C_8(p, \Omega, a(x)). \tag{3.2.4}$$

Step 2. Applying Pohozaev's identity (Lemma 1.1 in [6], originally from [18]) to obtain the a priori bound for ∇u in L^2.

Let

$$f(x,t) = a(x)t^p, \ F(x,t) = \int_0^t f(x,s)ds = \frac{1}{p+1}a(x)t^{p+1}, \ (x,t) \in \overline{\Omega} \times \mathbf{R}_+,$$

and

$$\frac{\partial F}{\partial x_i}(x,t) = \frac{1}{p+1}\frac{\partial a}{\partial x_i}(x)t^{p+1}, \ i = 1, ..., n, \ (x,t) \in \omega \times \mathbf{R}_+.$$

Then (1.0.3) is rewritten as

$$-\Delta u = f(x, u(x)) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

and the applicable version of Pohozaev identity is

$$\begin{aligned}
& \int_{\partial\Omega} x \cdot \iota(x)|\nabla u(x)|^2 dS \\
= \quad & 2n \int_\Omega F(x,u)dx + 2\sum_{i=1}^n \int_\omega x_i \frac{\partial F}{\partial x_i}(x,u)dx - (n-2)\int_\Omega f(x,u)udx \\
= \quad & \frac{2}{p+1}\int_\Omega nau^{p+1}dx + \frac{2}{p+1}\int_\omega (x\cdot\nabla a)u^{p+1}dx - \int_\Omega (n-2)au^{p+1}dx,
\end{aligned} \tag{3.2.5}$$

where $\iota = \iota(x) = (\iota_1(x), ..., \iota_n(x))^T$ denotes the unit outward normal to Ω at x.

The proof mostly follows [21] (p.9-10) but the integration that produces the second term on the right-hand side is tackled differently.

For notational convenience, we write

$$u_i = \frac{\partial u}{\partial x_i}, \; u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \; i, j = 1, ..., n.$$

Consider Gauss divergence formula

$$\int_\Omega \nabla \cdot \mathbf{b} dx = \int_{\partial\Omega} \mathbf{b} \cdot \iota dS, \tag{3.2.6}$$

where vector field

$$\mathbf{b} = \mathbf{b}(x) = (x \cdot \nabla u)\nabla u = \sum_{i=1}^n x_i u_i \; {}^t(u_1, ..., u_n).$$

The integrand on the left-hand side is

$$\begin{aligned}
\nabla \cdot \mathbf{b} \quad &= \quad \sum_{j=1}^n [(\sum_{i=1}^n x_i u_i)u_j]_j = \sum_{j,i}(\delta_{ij}u_iu_j + x_iu_{ij}u_j + x_iu_iu_{jj}) \\
&= \quad \sum_i u_i^2 + \sum_{i,j} x_iu_{ij}u_j + (\sum_i x_iu_i)(\sum_j u_{jj}) \\
&= \quad |\nabla u|^2 + \sum_{i,j} x_iu_{ij}u_j + (x\cdot\nabla u)\Delta u,
\end{aligned}$$

so we denote the left-hand side of (3.2.6) as

$$\int_\Omega \nabla \cdot \mathbf{b} dx = \int_\Omega |\nabla u|^2 dx + J_2 + J_3.$$

Let

$$I_{ij} = \int_\Omega x_i u_{ij} u_j dx.$$

$$\begin{aligned} I_{ij} &= \int_\Omega (u_j)_i x_i u_j dx = -\int_\Omega u_j (x_i u_j)_i dx + \int_{\partial\Omega} u_j x_i u_j \iota_i dS \\ &= \int_{\partial\Omega} x_i \iota_i u_j^2 dS - \int_\Omega u_j (u_j + x_i u_{ij}) dx = \int_{\partial\Omega} x_i \iota_i u_j^2 dS - \int_\Omega u_j^2 dx - I_{ij}, \end{aligned}$$

which implies

$$I_{ij} = \frac{1}{2}\int_{\partial\Omega} x_i \iota_i u_j^2 dS - \frac{1}{2}\int_\Omega u_j^2 dx.$$

Thus

$$J_2 = \sum_{i,j} I_{ij} = \frac{1}{2}\int_{\partial\Omega} x\cdot\iota |\nabla u|^2 dS - \frac{n}{2}\int_\Omega |\nabla u|^2 dx.$$

Now we proceed to calculate J_3. We should pay attention to the integral domain since some terms are only defined on ω.

$$\begin{aligned} J_3 &= \int_\Omega (x\cdot\nabla u)\Delta u dx = -\int_\Omega (x\cdot\nabla u) f(x,u) dx = -\int_\omega (x\cdot\nabla u) f(x,u) dx \\ &= -\sum_i \int_\omega x_i u_i f(x,u) dx = \sum_i \int_\omega x_i \left(\frac{\partial F}{\partial x_i}(x,u) - (F(x,u))_i\right) dx \\ &= \sum_i \int_\omega x_i \frac{\partial F}{\partial x_i}(x,u) dx - \int_\omega x\cdot\nabla F(x,u) dx \\ &= \sum_i \int_\omega x_i \frac{\partial F}{\partial x_i}(x,u) dx - \int_{\partial\omega} (x\cdot\nu) F(x,u) dS + \int_\omega nF(x,u) dx. \end{aligned}$$

Since $a(x) = 0$ on $\partial\omega$, $F(x,u(x)) = \frac{1}{p+1}a(x)u(x)^{p+1} = 0$ on $\partial\omega$. Thus,

$$J_3 = \sum_i \int_\omega x_i \frac{\partial F}{\partial x_i}(x,u) dx + \int_\Omega nF(x,u) dx$$

Therefore, the left-hand side of (3.2.6) is

$$\begin{aligned} \int_\Omega \nabla\cdot\mathbf{b} dx &= n\int_\Omega F(x,u) dx + \left(1-\frac{n}{2}\right)\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_{\partial\Omega} x\cdot\iota |\nabla u|^2 dS \\ &+ \sum_i \int_\omega x_i \frac{\partial F}{\partial x_i}(x,u) dx \end{aligned} \tag{3.2.7}$$

On the other hand, since

$$\nabla u(x) = \pm |\nabla u(x)| \iota(x), \text{ on } \Omega$$

by the zero boundary condition of (1.0.3), the right-hand side of (3.2.6) is

$$\int_{\partial\Omega} \mathbf{b} \cdot \iota dS = \int_{\partial\Omega} (x \cdot \nabla u)\nabla u \cdot \iota dS = \int_{\partial\Omega} x \cdot \iota |\nabla u|^2 dS. \tag{3.2.8}$$

Then (3.2.6), (3.2.7), (3.2.8) and

$$\int_\Omega |\nabla u|^2 dx = -\int_\Omega u \Delta u dx = \int_\Omega f(x,u) u dx$$

yield (3.2.5).

Now we return to the proof of Theorem 5. (3.2.5) is rewritten as

$$\begin{aligned} & \frac{p+1}{2} \int_{\partial\Omega} x \cdot \iota(x) |\nabla u(x)|^2 dS \\ = & \int_\omega \left[n a u^{p+1} + (x \cdot \nabla a) u^{p+1} - \frac{(n-2)(p+1)}{2} a u^{p+1} \right] dx \\ = & \int_\omega \left[n - \frac{(n-2)(p+1)}{2} + (x \cdot \nabla) \log a \right] a u^{p+1} dx \\ = & \int_\omega \left[\beta + \alpha(x) \right] a u^{p+1} dx, \end{aligned} \tag{3.2.9}$$

where

$$\beta = n - \frac{(n-2)(p+1)}{2} = p + 1 - \frac{n}{2}(p-1) > 0,$$

and

$$\alpha(x) = (x \cdot \nabla) \log a(x).$$

By (iii) of the assumption (1.0.7) and $a(x) = 0$ on $\partial\omega$, we have

$$\nabla a(x) = -|\nabla a(x)| \nu(x), \quad |\nabla a(x)| > 0 \quad \text{on } \partial\omega,$$

which together with (i) of (1.0.7) yields

$$x \cdot \nabla a(x) = -|\nabla a(x)| x \cdot \nu(x) < 0 \quad \text{on } \partial\omega.$$

Thus, since $\alpha(x) = \frac{x \cdot \nabla a(x)}{a(x)}$ in ω, by (ii) of (1.0.7), we obtain

$$\lim_{\delta \searrow 0} \sup_{\omega_\delta} \alpha(x) = -\infty,$$

where $\omega_\delta = \{x \in \omega | dist(x, \partial\omega) < \delta\}$ for $\delta > 0$. In particular, there exists $\delta > 0$ and $C_1 > 0$ such that

$$\beta + \alpha(x) \leq -C_1 \text{ in } \omega_\delta. \tag{3.2.10}$$

Moreover, for any $K \subset\subset \omega$, by Lemma 3.4,

$$\|u\|_{L^\infty(K)} \leq^{\exists} C_2(K). \tag{3.2.11}$$

Besides, by (ii) of (1.0.7),

$$\|\beta + \alpha(\cdot)\|_{L^\infty(K)} \leq^{\exists} C_3(K).$$

This, together with (3.2.11) and $a \in C_0(\Omega)$, implies that

$$\|[\beta + \alpha(\cdot)]a(\cdot)u(\cdot)^{p+1}\|_{L^\infty(K)} \leq^{\exists} C_4(K). \tag{3.2.12}$$

Now, taking $K = \omega \setminus \omega_\delta$, by (3.2.9), (3.2.10) and (3.2.12), we have

$$\frac{p+1}{2} \int_{\partial\Omega} x \cdot \nu(x) |\nabla u(x)|^2 dS \leq -C_1 \int_{\omega_\delta} a u^{p+1} dx + C_5. \tag{3.2.13}$$

While by (3.2.4) in the previous step,

$$\left| \frac{p+1}{2} \int_{\partial\Omega} x \cdot \nu(x) |\nabla u(x)|^2 dS \right| \leq C_6(p, \Omega) \int_{\partial\Omega} |\nabla u(x)|^2 dS \leq^{\exists} C_7(p, \Omega, a(x)),$$

so the left-hand side of (3.2.13) is bounded from below. Therefore,

$$\int_{\omega_\delta} a u^{p+1} dx \leq C_8(p, \Omega, a(x)).$$

By making use of (3.2.11) and $a \in C_0(\Omega)$ again,

$$\int_{\omega \setminus \omega_\delta} a u^{p+1} dx \leq C_9(p, \Omega, a(x)).$$

Hence

$$\int_\Omega |\nabla u|^2 dx = -\int_\Omega u \Delta u dx = \int_\Omega a u^{p+1} dx = \int_\omega a u^{p+1} dx \leq C_8 + C_9,$$

namely,

$$\|\nabla u\|_{L^2(\Omega)} \leq^{\exists} C_{10}(p, \Omega, a(x)). \tag{3.2.14}$$

Remark 1 *This is the only place we use the assumption* (1.0.7)*. Furthermore, from the proof we see that* (1.0.7) *can be weakened: it suffices to require*

$$\lim_{\delta \searrow 0} \sup_{\omega_\delta} (x \cdot \nabla) \log a(x) < -\left[n - \frac{(n-2)(p+1)}{2}\right], \tag{3.2.15}$$

from which (3.2.10) *follows immediately.*

Step 3. To conclude the a priori bound for u.

Notice that $f = f(x, u(x)) = a(x)u(x)^p$ satisfies:

$$f(x,\cdot) \text{ is bounded on } [0,L] \text{ for any } L > 0 \text{ uniformly in } x \in \overline{\Omega}, \tag{3.2.16}$$

$$\lim_{t \to +\infty} f(x,t)t^{-\sigma} = 0 \text{ uniformly in } x \in \overline{\Omega} \text{ for } \sigma = \frac{n+2}{n-2}. \tag{3.2.17}$$

We show that (3.2.14) implies the a priori bound for u.

For all $r \geq 1$, by the equation and its boundary condition (1.0.3),

$$\begin{aligned} &\int_\Omega f(x,u)u^r dx = -\int_\Omega (\Delta u)u^r dx = -\int_\Omega (\nabla \cdot \nabla u)u^r dx \\ = &\int_\Omega \nabla u \cdot \nabla(u^r) dx = r\int_\Omega |\nabla u|^2 u^{r-1} dx = \frac{4r}{(r+1)^2}\int_\Omega |\nabla(u^{\frac{r+1}{2}})|^2 dx. \end{aligned}$$

By (3.2.17) and (3.2.16), for all $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$f(x,t)t^r \leq \epsilon t^{r+\sigma} + C_\epsilon.$$

Thus,

$$\int_\Omega |\nabla(u^{\frac{r+1}{2}})|^2 dx = \frac{(r+1)^2}{4r}\int_\Omega f(x,u)u^r dx \leq C_0\epsilon \int_\Omega u^{r+\sigma} dx + C'_\epsilon.$$

Since by Sobolev's inequality,

$$\left(\int_\Omega u^q dx\right)^{\frac{n-2}{n}} = \|u^{q\frac{n-2}{2n}}\|^2_{L^{\frac{2n}{n-2}}(\Omega)} \leq C_1\int_\Omega |\nabla(u^{q\frac{n-2}{2n}})|^2 dx,$$

letting

$$q\frac{n-2}{2n} = \frac{r+1}{2}, \text{ or } q = \frac{n(r+1)}{n-2},$$

we have

$$\left(\int_\Omega u^q dx\right)^{\frac{n-2}{n}} \leq C_2\epsilon \int_\Omega u^{r+\sigma} dx + C''_\epsilon.$$

On the right-hand side, we note the exponent of u:

$$r+\sigma=(r+1)+(\sigma-1)=q\frac{n-2}{n}+\frac{2n}{n-2}\frac{2}{n},$$

and thus by Hölder's inequality,

$$\left(\int_\Omega u^q dx\right)^{\frac{n-2}{n}} \le C_2\epsilon\left(\int_\Omega u^q dx\right)^{\frac{n-2}{n}}\left(\int_\Omega u^{\frac{2n}{n-2}}dx\right)^{\frac{2}{n}}+C''_\epsilon. \tag{3.2.18}$$

Moreover, (3.2.14) and Sobolev's inequality imply

$$\int_\Omega u^{\frac{2n}{n-2}}dx \le C_3. \tag{3.2.19}$$

Therefore, choosing ϵ in (3.2.18) small enough, we obtain

$$\|u\|_{L^q(\Omega)} \le C_4(q),\ \text{for } q=\frac{n(r+1)}{n-2},\ {}^\forall r\ge 1,$$

and in particular,

$$\|u\|_{L^q(\Omega)} \le C_4(q),\ {}^\forall q\ge n. \tag{3.2.20}$$

Furthermore, by (3.2.17) and (3.2.16),

$$\|f(\cdot,u(\cdot))\|_{L^{\frac{q}{\sigma}}(\Omega)} \le C_5(q),\ {}^\forall q\ge n. \tag{3.2.21}$$

Thus for all $s>n$, by elliptic L^s estimate and the compactness of Ω,

$$\|u\|_{W^{2,s}(\Omega)} \le^{\exists} C_6(s,\Omega)(\|u\|_{L^s(\Omega)}+\|f(x,u)\|_{L^s(\Omega)}) \le C_7(s,\Omega),$$

and then by Morrey's inequality,

$$\|u\|_{C^{1,\gamma}(\overline{\Omega})} \le^{\exists} C_8(\gamma,\Omega)\|u\|_{W^{2,s}(\Omega)} \le C_9(\gamma,\Omega),\ \gamma=1-\frac{n}{s}.$$

Consequently, by the definition of Hölder norm, we obtain the a priori bound for u. □

3.3 Open problems

Based on the above results, we present here two open problems for future work.

Elliptic Problem. The first one concerns whether the assumption (1.0.7) on $a=a(x)$ in Theorem 5 can be further weakened.

Specifically, assuming $\partial\omega \in C^1$, (ii) and (iii) of (1.0.7), we have for any $x_0 \in \partial\omega$,

$$\nabla a(x_0) = -|\nabla a(x_0)|\nu_0,$$

so that by the Taylor expansion,

$$a(x) = -|\nabla a(x_0)|(x - x_0) \cdot \nu_0 + o(|x - x_0|), \text{ as } x \to x_0, x \in \omega, \tag{3.3.1}$$

where $\nu_0 = \nu(x_0)$ denotes the outer unit normal vector at x_0. In addition, we notice that

$$|\nabla a(x_0)| = -\frac{\partial a}{\partial \nu}(x_0) > 0.$$

We wonder whether (3.3.1), or, more generally, the assumption that for any $x_0 \in \partial\omega$ there exist $\alpha > 0$ and $m > 0$ such that

$$a(x) = -\alpha(x - x_0) \cdot \nu_0 |(x - x_0) \cdot \nu_0|^{m-1} + o(|x - x_0|^m), x \in \omega \to x_0 \tag{3.3.2}$$

suffices to admit the a priori bound

$$\|u\|_{L^\infty(\Omega)} \leq C = C(\Omega, a(x), p)$$

for any solution $u = u(x)$ to (1.0.3).

To study this problem, one may apply the blow-up analysis with the argument of Du-Li [4] in discussing the limiting points of the maximizing point sequence $\{x_n\}$.

Parabolic Problem. We have also attempted to study the initial boundary value problem for the corresponding semilinear parabolic equation

$$\begin{cases} u_t = \Delta u + a(x)u^p & \text{in } \Omega \times (0, T) \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0, T) \end{cases} \tag{3.3.3}$$

with $u_0 \in C(\overline{\Omega})$. We wonder if, similar to the case $a(x) \equiv 1$, studied by Giga [12], there exists an a priori bound for positive solutions, provided (1.0.5) and $a = a(x)$ satisfies some conditions.

Concretely speaking, we wish to show that under the same assumption in Theorem 5, or some weaker assumption such as (3.3.1) or (3.3.2), there exists $C = C(p, \Omega, a(x), \|u_0\|_{L^\infty(\Omega)})$ such that

$$\sup_{t \geq 0} \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C$$

for any solution $u = u(x, t)$ to (3.3.3) global-in-time.

A possible proof may follow the procedure of [12], but the limit of the converging sequence of points have to be discussed in a different manner. In fact, since the other arguments hold true in this case, the proof would be completed if only the same lemma in [12] could be proved.

Conjecture. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with C^2 boundary $\partial\Omega$, p be in (1.0.5), and $a = a(x)$ satisfy (1.0.7), or some weaker assumption such as (3.3.1) or (3.3.2). Let $u = u(x,t)$ be a strong solution to (3.3.3), and assume that there exists a finite constant $N > 0$ such that

$$\int_0^T \int_\Omega |u_t| dxdt < N, \tag{3.3.4}$$

and that

$$\sup_{\Omega \times (0,T)} u \text{ is attained in } \Omega \times (t_0, T) \tag{3.3.5}$$

where $t_0 > 0$. Then there is a constant A which depends only on N and t_0, and is independent of u, u_0 and T, such that

$$u(x,t) \le A \text{ in } Q = \Omega \times [0,T).$$

Chapter 4

Conclusion and comments

In the first half, we studied the critical exponent for global existence p_0 and the Fujita exponent p_C for the porous medium equation with localized reaction in multi-dimensional space. In two spatial dimension, despite some results obtained in Theorem 1, there are still problems left unsolved, while in the case $n \geq 3$, Theorem 2 elucidated the relationship between the behavior of nonnegative solutions and the exponents p and m. Additionally, in Theorem 3, we showed a property concerning the support of nonnegative solutions.

From these results we can observe that the construction of critical exponents is greatly different from the case when the reaction is not localized, namely, $a(x) \equiv 1$. Especially when $n \geq 3$, the critical exponent $p_0 = p_C = m$ is even unrelated to the spatial dimention n. Besides, we remark that in the case $n \geq 3$, all results concerning the existence of global solution are obtained thanks to the Liouville property of semilinear elliptic equation, since it enabled us to construct a global supersolution for comparison. This is also the reason why we failed to obtain a complete result for the case $n = 2$, since this property holds only in spatial dimensions higher than 2.

In the second half, we studied the role of such $a(x)$ in a priori estimate for positive solutions to the semilinear elliptic equation. Different from the case $a(x) \equiv 1$ where the existence of an a priori bound for all positive solutions is guaranteed, to obtain the a priori bound, we have to reduce the critical exponent, or to impose some assumptions on the localized coefficient $a(x)$, as stated in Theorem 4 and Theorem 5 respectively. For the latter result, as future work, we also suggested possible improvement in that the assumptions may be weakened. Finally, we presented an open problem concerning the a priori estimate for the corresponding semilinear parabolic equation.

The main complexity caused by the localized nonlinearity lies in the arguments conducted in the neighborhood of $\partial\omega$. Recall $\omega = \{x \in \Omega | a(x) > 0\}$. This compelled us to impose appropriate assumptions on $a(x)$ in order to prove the existence of the a priori bound. We also observe that throughout both halves of the dissertation, Liouville property of semilinear elliptic equation played a crucial role: it not only realized the construction of global supersolution in the first half, as stated above, but also served to engender the contradiction in the proof of Theorem 4 by blow-up analysis.

Bibliography

[1] H. Brezis and W.A. Strauss, *Semi-linear second order elliptic equations in L^1*, J. Math. Soc. Japan **25** (1973), 565-590.

[2] T. Cazenave and P.-L. Lions, *Solutions globales d'equations de la chaleur semi lineaires*, Comm. Partial Differential Equations **9** (1984), 955-978.

[3] K. Deng and H.A. Levine, *The role of critical exponents in blow-up theorems: the sequel*, J. Math. Anal. Appl. **243** (2000), 85-126.

[4] Y. Du and S. Li, *Nonlinear Liouville theorems and a priori estimates for indefinite superlinear elliptic equations*, Adv. Differential Equations **10 (8)** (2005), 841-860.

[5] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, 1998.

[6] D.G de Figueiredo, P.-L. Lions and R.D. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations*, J. Math. Pures et Appl., **61** (1982), 41-63.

[7] R. Ferreira, A. de Pablo and J.L. Vazquez, *Classification of blow-up with nonlinear diffusion and localized reaction*, J. Differential Equations **231** (2006), 195-211.

[8] V.A. Galaktionov and J.L. Vazquez, *The problem of blow-up in nonlinear parabolic equations*, Discrete Contin. Dynam. Systems **A 8** (2002), 399-433.

[9] V.A. Galaktionov and J.L. Vazquez, *Continuation of blowup solutions of nonlinear heat equations in several space dimensions*, Communications on Pure and Applied Mathematics **L** (1997), 1-67.

[10] B. Gidas and J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations*, Communications in Partial Differential Equations **6** (1981), 883-901.

[11] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981), 525-598.

[12] Y. Giga, *A bound for global solutions of semilinear heat equations*, Commun. Math. Phys. **103** (1986), 415-421.

[13] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order (Second Edition)*, Springer-Verlag, Berlin, 1983.

[14] X. Kang, W. Wang and X. Zhou, *Classification of solutions of porous medium equation with localized reaction in higher space dimensions*, Differential and Integral Equations **24** (2011), 909-922.

[15] H.A. Levine, *The role of critical exponents in blow-up problems*, SIAM Review **32** (1990), 262-288.

[16] H.A. Levine and P. Sacks, *Some existence and nonexistence theorems for solutions of degenerate parabolic equations*, J. Differential Equations **52** (1984), 135-161.

[17] R.G. Pinsky, *Existence and nonexistence of global solutions for* $u_t = \Delta u + a(x)u^p$ *in* $\mathbf{R}^d$, J. Differential Equations **133** (1997), 152-177.

[18] S.I. Pohozaev, *Eigenfunctions of the equation* $\Delta u + \lambda f(x) = 0$, Soviet Math. Dokl. **6** (1965), 1408-1411.

[19] P. Quittner and P. Souplet, *Superlinear Parabolic Problems (Blow-up, Global Existence and Steady States)*, Birkhauser, Basel, 2007.

[20] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov and A.P. Mikhailov, *Blow-up in Quasilinear Parabolic Equations* (English Translation), Walter de Gruyter, Berlin, 1995.

[21] T. Suzuki and Y. Ueoka, *A Lecture on Partial Differential Equations – An Introduction to Semilinear Elliptic Equations (in Japanese)*, Baifuukan, Tokyo, 2005.

Acknowledgment (in Japanese)

謝辞

本論文の作成にあたり、まず、指導教授である大阪大学基礎工学研究科鈴木貴教授に心から謝意を申し上げます。三年間余りにわたり、始終懇切丁寧にご指導を賜りました。鈴木先生の深遠な知識、研究への真摯な姿勢、温和な人柄、人に倦まず教え諭す精神にとても感銘を受けました。また、研究指導のほかに、私の生活を気にかけてくださり、奨学金や大学院のアルバイトの申請に際し推薦してくださったことも感謝しております。鈴木先生に博士後期課程学生として受け入れていただけたことに幸運を感じております。

次に、ご自身の時間を惜しまず、研究に多くの知恵を貸してくださった愛媛大学理工学研究科内藤雄基教授、福島大学共生システム理工学研究科石渡通徳準教授に深く感謝申し上げます。研究に関する困難の克服のための助言をいただき、大変勉強になりました。

また、修士時代の指導教授である武漢大学周小方先生、康肖松先生のご指導のおかげで、本論文で一部の研究成果を挙げることができ、心より厚くお礼申し上げております。

授業科目に関しては、多大なご指導をくださった大阪大学基礎工学研究科関根順教授、経済学研究科大西匡光教授をはじめとする先生方に心からお礼申し上げます。大阪大学大学院において、質の高い教育を受けることができました。

続いて、鈴木研究室の皆様、特に、元招聘教授の雄山真弓先生、助教の高橋亮先生、特任研究員の板野景子さん、元事務補佐員の井内裕子さん、元技術補佐員の千喜良誠一さん、現事務補佐員の笠川康子さん、修了生の田崎創平博士、林娟博士、村上尊広博士、肉孜買買提・馬合木提博士、吉岡貴史さん、小寺悠佑さん、森裕也さん、在籍の院生の張瀟さん、崔亮さん、元進学希望者の龔茜さんに大変お世話になりました。感謝の気持ちを申し上げます。それに、いつも笑顔で諸業務を担当してくださった東堤享子さん、髙橋智子さんをはじめとする基礎工学研究科数理事務室の皆様、大学院係の皆様、また、留学生相談室ご担当の田坂恵美子さんに謝意を申し上げます。

更に、日本で巡り合えた友人、特に、基礎工学研究科留学生支援ボランティアとして助けてくださった結城陽子さん、病気の際やその他の面でも手を差し伸べてくれた外国語学部前田悠輝君、煩を厭わず日本語の作文を添削してくれた法学部小林亮君、また、来日間もなく不安だった私に気さくに声をかけてくれた理学研究科白石勇貴さん、工学研究科宇賀治元君、修了生の松山拓馬さん、基礎工学研究科山阪司祐人君、修了生の藤井康人君、その他長い間お世話になった理学部神谷拓志君、文学部益田行人君、工学部生野祐輝君、八島雅史君に感謝の意を表したいと思います。加えて、三年間お世話になったドーミー五月ヶ丘の管理人の伊東養子さん、及び食事を作ってくださった古謝幸子さん、中島親江さんに改めて私の気持ちをお伝えしたいと思います。

最後になりましたが、これまで私を温かく応援し続けてくれた両親及び親友たちに改めて心から感謝いたします。

王　文彪

2013 年 11 月

Publications and Presentations

Journal Articles

[J1] Xiaosong Kang, Wenbiao Wang and Xiaofang Zhou, *Classification of solutions of porous medium equation with localized reaction in higher space dimensions*, Differential and Integral Equations, **24** (2011), 909-922.

Workshop Presentations

[P1] (2011.2.16) *Classification of solutions of porous medium equation with localized reaction*, RIMS Workshop, Kyoto, Japan.

[P2] (2013.1.31) *A priori bound for positive solutions to semilinear elliptic equation with localized nonlinearity*, RIMS Workshop, Kyoto, Japan.