Expected utility maximization under incomplete information and with Cox-process observations

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Abstract

We consider the maximization problem of expected terminal utility. The underlying market model is a regime-switching diffusion model in which the regime is determined by an unobservable factor process forming a finite-state Markov process. The main novelty is due to the fact that prices are observed and the portfolio is rebalanced only at random times corresponding to a Cox process in which intensity is further driven by the unobserved Markovian factor process. This leads to a more realistic modeling for several practical situations, as in markets with liquidity restrictions; on the other hand, it considerably complicates the problem to such a degree that traditional methodologies cannot be directly applied. Furthermore, we provide a numerical scheme for these problems to numerically compute the value functions.

Contents

Introduction

In this thesis, we study a classical portfolio optimization problem, namely, the maximization of expected utility from terminal wealth. We assume that the dynamics of the prices at which one makes as investment, are of the usual diffusion type but have the following two peculiarities:

- the coefficients in the dynamics depend on an unobservable finite-state Markovian factor process θ_t (regime-switching model);
- the prices S_t^i of the risky assets, or equivalently, their log-values are observed only at doubly stochastic random times $\tau_0, \tau_1, \cdots$, for which the associated counting process forms a Cox process (e.g. [3], [10]) with an intensity $n(\theta_t)$ that depends on the same unobservable factor process θ_t.

Such models are relevant in financial applications for various reasons: regime-switching models, which are also relevant in various other applied areas, have been extensively employed in the financial literature, because they account for various stylized facts such as volatility clustering. On the other hand, random discrete time observations are more realistic in comparison to diffusion-type models since, especially on small time scales, prices do not vary continuously but rather change. These prices are observed only at random times in reaction to trading or the arrival of significant new information, and it is reasonable to assume that the intensity of price changes depends on the same factors that specify the regime for price evolution (e.g. [7], [4]).This setting leads to a stochastic control problem with incomplete information and observations given by Cox process.

A classical approach to incomplete observation control problems is to first transform the problem into a so-called separated problem, where the unobservable part of the state is replaced by its conditional distribution. First, this requires solving the associated filtering problem, which already is non-standard and has been recently solved in [4] (also refer to [5]). Our major contribution here is to the control part of the separated problem that is approached in a non-classical manner. In particular, we shall restrict the rebalancing of investment strategies to only random times τ_k when prices change. Although slightly less general from a theoretical perspective, restricting trading to discrete, and particularly, random times is fairly realistic in finance, where in practice one cannot continuously rebalance a portfolio: think of the case with transaction costs or liquidity restrictions (for the latter context refer to [8], [9], [13], [16], [17], [18], where the authors consider illiquid markets, partly in addition to regime switching models as in this paper, but under complete information).

The thesis is organized as follows. In Chapter 1, we provide a precise definition of the market model, formulate the investor's strategy, and recall the filtering results of [4]. In Chapter 2, we consider the expected log-utility maximization

$$\sup E\{\log V_T\}, \tag{0.0.1}$$

where V_T is the total wealth of an investor at terminal time T. In Chapter 3, we consider expected power-utility maximization. Equivalently, we consider the risk-sensitive portfolio optimization,

$$\sup \frac{1}{\mu} \log E\{V_T^{\mu}\}, \tag{0.0.2}$$

where $\mu < 0$. In Chapter 4, we provide a numerical scheme for these problems to numerically compute the value functions.

Chapter 1

Preliminary

1.1 The market model and preliminary notations

Let θ_t be the hidden finite state Markovian factor process. With Q denoting its transition intensity matrix ($Q-$matrix) its dynamics are given by

$$d\theta_t = Q^* \theta_t dt + dM_t, \quad \theta_0 = \xi, \tag{1.1.1}$$

where M_t is a jump-martingale on a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. If N is the number of possible values of θ_t, we may without loss of generality take as its state space the set $E = \{e_1, \ldots, e_N\}$, where e_i is a unit vector for each $i = 1, \ldots, N$ (see [6]).

The evolution of θ_t may also be characterized by the process π_t given by the state probability vector that takes values in the set

$$\mathcal{S}_N := \{\pi \in \mathbb{R}^N \mid \sum_{i=1}^{N} \pi^i = 1, 0 \le \pi^i \le 1, i = 1, 2, \ldots, N\} \tag{1.1.2}$$

namely the set of all probability measures on E and we have $\pi_0^i = P(\xi = e_i)$. Denoting by $\mathcal{M}(E)$ the set of all finite nonnegative measures on E, it follows that $\mathcal{S}_N \subset \mathcal{M}(E)$. In our study it will be convenient to consider on $\mathcal{M}(E)$ the Hilbert metric $d_H(\pi, \bar{\pi})$ defined (see [1] [11] [12]) by

$$d_H(\pi, \bar{\pi}) := \log\left(\sup_{\bar{\pi}(A) > 0, A \subset E} \frac{\pi(A)}{\bar{\pi}(A)} \sup_{\pi(A) > 0, A \subset E} \frac{\bar{\pi}(A)}{\pi(A)} \right). \tag{1.1.3}$$

Notice that, while d_H is only a pseudo-metric on $\mathcal{M}(E)$, it is a metric on $\mathcal{S}_N$ ([1]).

In our market we consider m risky assets, for which the price processes $S^i = (S_t^i)_{t \ge 0}, i = 1, \ldots, m$ are supposed to satisfy

$$dS_t^i = S_t^i \{r^i(\theta_t) dt + \sum_j \sigma_j^i(\theta_t) dB_t^j\}, \tag{1.1.4}$$

for given coefficients $r^i(\theta)$ and $\sigma_j^i(\theta)$ and with B_t^j $(j = 1, \cdots, m)$ independent $(\mathcal{F}_t, P)-$Wiener processes. Letting $X_t^i = \log S_t^i$, by Itô's formula we have,in vector notation,

$$X_t = X_0 + \int_0^t r(\theta_s) - d(\sigma\sigma^*(\theta_s)) ds + \int_0^t \sigma(\theta_s) dB_s, \tag{1.1.5}$$

where by $d(\sigma\sigma^*(\theta))$ we denote the column vector $(\frac{1}{2}(\sigma\sigma^*)^{11}(\theta),\ldots,\frac{1}{2}(\sigma\sigma^*)^{mm}(\theta))$. As usual there is also a locally non-risky asset (bond) with price S_t^0 satisfying

$$dS_t^0 = r_0 S_t^0 dt \tag{1.1.6}$$

where r_0 stands for the short rate of interest. We shall also make use of discounted asset prices, namely

$$\tilde{S}_t^i := \frac{S_t^i}{S_t^0}, \quad \text{with} \quad \tilde{X}_t^i := \log \tilde{S}_t^i \tag{1.1.7}$$

for which, by Itô's formula

$$d\tilde{S}_t^i = \tilde{S}_t^i\{(r^i(\theta_t) - r_0)dt + \sum_j \sigma_j^i(\theta_t)dB_t^j\}, \tag{1.1.8}$$

$$d\tilde{X}_t^i = \{r^i(\theta_t) - r_0 - d(\sigma\sigma^*(\theta_t))^i\}dt + \sum_{j=1}^m \sigma_j^i(\theta_t)dB_t^j. \tag{1.1.9}$$

As already mentioned, the asset prices and thus also their logarithms are observed only at random times $\tau_0, \tau_1, \tau_2, \ldots$. The observations are thus given by the sequence $(\tau_k, \tilde{X}_{\tau_k})_{k\in\mathbb{N}}$ that forms a multivariate marked point process with counting measure

$$\mu(dt, dx) = \sum_k \mathbf{1}_{\{\tau_k<\infty\}}\delta_{\{\tau_k,\tilde{X}_{\tau_k}\}}(t,x)dtdx. \tag{1.1.10}$$

The corresponding counting process $\Lambda_t := \int_0^t \int_{\mathbb{R}^m} \mu(dt, dx)$ is supposed to be a Cox process with intensity $n(\theta_t)$, i.e. $\Lambda_t - \int_0^t n(\theta_s)ds$ is an $(\mathcal{F}_t, P)-$ martingale. We consider two sub-filtrations related to $(\tau_k, \tilde{X}_{\tau_k})_{k\in\mathbb{N}}$ namely

$$\begin{aligned} \mathcal{G}_t &:= \mathcal{F}_0 \vee \sigma\{\mu((0,s]\times B) : s \le t, B \in \mathcal{B}(\mathbb{R}^m)\}, \\ \mathcal{G}_k &:= \mathcal{F}_0 \vee \sigma\{\tau_0, \tilde{X}_{\tau_0}, \tau_1, \tilde{X}_{\tau_1}, \tau_2, \tilde{X}_{\tau_2}, \ldots, \tau_k, \tilde{X}_{\tau_k}\}. \end{aligned} \tag{1.1.11}$$

where, again for simplicity, $\mathcal{G}_k$ stands for$\mathcal{G}_{\tau_k}$. In our development below we shall often make use of the following notations. For the conditional (on $\mathcal{F}^\theta$) mean and variance of $\tilde{X}_t - \tilde{X}_{\tau_k}$ we set

$$\begin{aligned} m_k^\theta(t) &= \int_{\tau_k}^t [r(\theta_s) - r_0\mathbf{1} - d(\sigma\sigma^*(\theta_s))]ds, \\ \sigma_k^\theta(t) &= \int_{\tau_k}^t \sigma\sigma^*(\theta_s)ds \end{aligned} \tag{1.1.12}$$

and, for $z \in \mathbb{R}^m$, we set

$$\rho_{\tau_k,t}^\theta(z) \sim N(z; m_k^\theta(t), \sigma_k^\theta(t)) \tag{1.1.13}$$

namely the joint conditional (on $\mathcal{F}^\theta$) m-dimensional normal density function with mean vector $m_k^\theta(t)$ and covariance matrix $\sigma_k^\theta(t)$. In the symbol $\sim$ stands for "distributed according to".

1.2 Investment strategies, portfolios,objective

As mentioned in the Introduction, since observations take place at random time points τ_k, we shall consider investment strategies that are rebalanced only at those same time points τ_k.

Let N^i_t be the number of assets of type i held in the portfolio at time t, $\quad N^i_t = \sum_k \mathbf{1}_{[\tau_k,\tau_{k+1})}(t) N^i_k$. The wealth process is defined by

$$V_t := \sum_{i=0}^{m} N^i_t S^i_t.$$

Consider then the investment ratios

$$h^i_t := \frac{N^i_t S^i_t}{V_t},$$

and set, for simplicity of notation, $h^i_k := h^i_{\tau_k}$. The set of admissible investment ratios is given by

$$\bar{H}_m := \{(h^1,\ldots,h^m); h^1 + h^2 + \ldots + h^m \leq 1, 0 \leq h^i, i = 1,2,\ldots,m\}, \tag{1.2.1}$$

i.e. no shortselling is allowed and notice that $\bar{H}_m$ is bounded and closed. Put $h = (h^1,\cdots,h^m)$. Analogously to [14] define next a function $\gamma :\ \mathbb{R}^m \times \bar{H}_m \to \bar{H}_m$ by

$$\gamma^i(z,h) := \frac{h^i \exp(z^i)}{1 + \sum\limits_{i=1}^{m} h^i(\exp(z^i) - 1)}, \qquad i = 1,,\ldots,m. \tag{1.2.2}$$

Noticing that N_t is constant on $[\tau_k,\tau_{k+1})$, for $i = 1,\ldots,m$, and $t \in [\tau_k,\tau_{k+1})$ let

$$\begin{aligned}
h^i_t &= \frac{N^i_t S^i_t}{\sum_{i=0}^m N^i_t S^i_t} = \frac{N^i_k S^i_t}{\sum_{i=0}^m N^i_k S^i_t} \\
&= \frac{N^i_k S^i_{\tau_k} S^i_t/S^i_{\tau_k}}{\sum_{i=0}^m N^i_k S^i_{\tau_k} S^i_t/S^i_{\tau_k}} = \frac{h^i_k S^i_t/S^i_{\tau_k}}{\sum_{i=0}^m h^i_k S^i_t/S^i_{\tau_k}} = \frac{h^i_k S^0_{\tau_k}/S^0_t S^i_t/S^i_{\tau_k}}{\sum_{i=0}^m h^i_k S^0_{\tau_k}/S^0_t S^i_t/S^i_{\tau_k}} \\
&= \frac{h^i_k \exp(\tilde{X}^i_t - \tilde{X}^i_{\tau_k})}{h^0_k + \sum\limits_{i=1}^m h^i_k \exp(\tilde{X}^i_t - \tilde{X}^i_{\tau_k})} = \frac{h^i_k \exp(\tilde{X}^i_t - \tilde{X}^i_{\tau_k})}{1 + \sum\limits_{i=1}^m h^i_k (\exp(\tilde{X}^i_t - \tilde{X}^i_{\tau_k}) - 1)} \\
&= \gamma^i(\tilde{X}_t - \tilde{X}_{\tau_k}, h_k).
\end{aligned} \tag{1.2.3}$$

The set of admissible strategies $\mathcal{A}$ is defined by

$$\mathcal{A} := \{\{h_k\}_{k=0}^\infty | h_k \in \bar{H}_m,\ \mathcal{G}_k \text{ measurable for all } k \geq 0\}. \tag{1.2.4}$$

Furthermore, for $n > 0$, we let

$$\mathcal{A}^n := \{h \in \mathcal{A} | h_{n+i} = h_{\tau_{n+i}-} \text{ for all } i \geq 1\}. \tag{1.2.5}$$

Notice that, by the definition of $\mathcal{A}^n$, for all $\ k \geq 1,\ h \in \mathcal{A}^n$ we have

$$h^i_{n+k} = h^i_{\tau_{n+k}-}$$

$$\Leftrightarrow \frac{N^i_{n+k} S^i_{\tau_{n+k}}}{\sum_{i=0}^m N^i_{n+k} S^i_{\tau_{n+k}}} = \frac{N^i_{n+k-1} S^i_{\tau_{n+k}}}{\sum_{i=0}^m N^i_{n+k} S^i_{\tau_{n+k}}}$$

$$\Leftrightarrow N_{n+k} = N_{n+k-1}.$$

Therefore, for $k \geq 1$

$$N_{n+k} = N_n,$$

and

$$\mathcal{A}^0 \subset \mathcal{A}^1 \subset \cdots \mathcal{A}^n \subset \mathcal{A}^{n+1} \cdots \subset \mathcal{A}. \tag{1.2.6}$$

Remark 1.2.1. *Notice that, for a given finite sequence of investment ratios $h_0, h_1, \cdots, h_n$ such that h_k is an $\mathcal{G}_k-$measurable, $\bar{H}_m-$valued random variable for $k \leq n$, there exists $h^{(n)} \in \mathcal{A}^n$ such that $h_k^{(n)} = h_k$, $k = 0, \cdots, n$. Indeed, if N_t is constant on $[\tau_n, T)$, then for h_t we have $h_t = \gamma(\tilde{X}_t - \tilde{X}_{\tau_n}, h_n)$, $\forall t \geq \tau_n$. Therefore, by setting $h_\ell^{(n)} = h_\ell$, $\ell = 0, \cdots, n$, and $h_{n+k}^{(n)} = h_{\tau_{n+k}}$, $k = 1, 2, \cdots$, since the vector process S_t and the vector function $\gamma(\cdot, h_n)$ are continuous, we see that $h_{n+k}^{(n)} = h_{\tau_{n+k}-}$, $k = 1, 2, \cdots$.*

Finally, considering only self-financing portfolios, for their value process we have the dynamics

$$\frac{dV_t}{V_t} = [r_0 + h_t^*\{r(\theta_t) - r_0\mathbf{1}\}]dt + h_t^*\sigma(\theta_t)dB_t. \tag{1.2.7}$$

1.3 Filtering

As mentioned in the Introduction, the standard approach to stochastic control problems under incomplete information is to first transform them into a so-called separated problem, where the unobservable part of the state is replaced by its conditional (filter) distribution. This implies that we first have to study this conditional distribution and its (Markovian) dynamics, i.e. we have to study the associated filtering problem.

The filtering problem for our specific case, where the observations are given by a Cox process with intensity expressed as a function of the unobserved state, has been studied in [4] (see also [5]). In this section. we therefore summarize the main results from [4] in view of their use in our control problem in this paper. Recalling the definition of $\rho^\theta(z)$ in (1.1.13) and putting

$$\phi^\theta(\tau_k, t) = n(\theta_t)\exp(-\int_{\tau_k}^t n(\theta_s)ds), \tag{1.3.1}$$

for a given function $f(\theta)$ we let

$$\psi_k(f; t, x) := E[f(\theta_t)\rho^\theta_{\tau_k,t}(x - \tilde{X}_k)\phi^\theta(\tau_k, t)|\sigma\{\theta_{\tau_k}\} \vee \mathcal{G}_k] \tag{1.3.2}$$

$$\bar{\psi}_k(f; t) := \int \psi_k(f; t, x)dx = E[f(\theta_t)\phi^\theta(\tau_k, t)|\sigma\{\theta_{\tau_k}\} \vee \mathcal{G}_k] \tag{1.3.3}$$

$$\pi_t(f) = E[f(\theta_t)|\mathcal{G}_t] \tag{1.3.4}$$

with ensuing obvious meanings of $\pi_{\tau_k}(\psi_k(f; t, x))$ and $\pi_{\tau_k}(\bar{\psi}_k(f; t))$ where we consider $\psi_k(f; t, x)$ and $\bar{\psi}_k(f; t)$ as functions of θ_{τ_k}. The process $\pi_t(f)$ is called the *filter process* for $f(\theta_t)$.

We have the following lemma (see Lemma 4.1 in [4]), where by $\mathcal{P}(\mathcal{G})$ we denote the predictable σ -algebra on $\Omega \times [0, \infty)$ with respect to $\mathcal{G}$ and set $\tilde{\mathcal{P}}(\mathcal{G}) = \mathcal{P}(\mathcal{G}) \otimes \mathcal{B}(R^m)$.

Lemma 1.3.1. *The compensator of the random measure $\mu(dt,dx)$ in (1.1.10) with respect to $\tilde{\mathcal{P}}(\mathcal{G})$ is given by the following nonnegative random measure*

$$\nu(dt,dx) = \sum_k \mathbf{1}_{(\tau_k,\tau_{k+1}]}(t) \frac{\pi_{\tau_k}(\psi_k(1,t,x))}{\int_t^\infty \pi_{\tau_k}(\bar{\psi}_k(1,s))ds} dtdx. \tag{1.3.5}$$

The main filtering result is the following (see Theorem 4.1 in [4]).

Theorem 1.3.1. *For any bounded function $f(\theta)$, the differential of the filter $\pi_t(f)$ is given by*

$$\begin{aligned} d\pi_t(f) &= \pi_t(Lf)dt \\ &\quad + \int \sum_k \mathbf{1}_{(\tau_k,\tau_{k+1}]}(t)[\tfrac{\pi_{\tau_k}(\psi_k(f;t,x))}{\pi_{\tau_k}(\psi_k(1;t,x))} - \pi_{t-}(f)](\mu-\nu)(dt,dx), \end{aligned} \tag{1.3.6}$$

where L is the generator of the Markov process θ_t(namely $L = Q$).

Corollary 1.3.1. *We have*

$$\pi_{\tau_{k+1}}(f) = \left.\frac{\pi_{\tau_k}(\psi_k(f;t,x))}{\pi_{\tau_k}(\psi_k(1;t,x))}\right|_{t=\tau_{k+1}, x=\tilde{X}_{\tau_{k+1}}}. \tag{1.3.7}$$

Recall that in our setting θ_t is an N-state Markov chain with state space $E = \{e_1, \ldots, e_N\}$, where e_i is a unit vector for each $i = 1, \ldots, N$. One may then write $f(\theta_t) = \sum_{i=1}^N f(e_i)\mathbf{1}_{e_i}(\theta_t)$. For $i = 1, \ldots, N$ let $\pi_t^i = \pi_t(\mathbf{1}_{e_i}(\theta_t))$ and

$$r_{ji}(t,z) := E[\exp(\int_0^t -n(\theta_s)ds)\rho_{0,t}^\theta(z)|\theta_0 = e_j, \theta_t = e_i], \tag{1.3.8}$$

$$p_{ji}(t) := P(\theta_t = e_i|\theta_0 = e_j) \tag{1.3.9}$$

and, noticing that $\pi_t \in \mathcal{S}_N$, define the function $M : [0,\infty) \times \mathbb{R}^m \times \mathcal{S}_N \to \mathcal{S}_N$ by

$$M^i(t,x,\pi) := \frac{\sum_j n(e_i)r_{ji}(t,x)p_{ji}(t)\pi^j}{\sum_{ij} n(e_i)r_{ji}(t,x)p_{ji}(t)\pi^j}, \tag{1.3.10}$$

$$M(t,x,\pi) := (M^1(t,x,\pi), M^2(t,x,\pi), \ldots, M^N(t,x,\pi)). \tag{1.3.11}$$

For $A \subset E$

$$M(t,x,\pi)(A) := \sum_{i=1}^N M^i(t,x,\pi)1_{\{e_i \in A\}}. \tag{1.3.12}$$

The following corollary will be useful

Corollary 1.3.2. *For the generic i-th state one has*

$$\pi_{\tau_{k+1}}^i = M^i(\tau_{k+1} - \tau_k, \tilde{X}_{\tau_{k+1}} - \tilde{X}_{\tau_k}, \pi_{\tau_k}) \tag{1.3.13}$$

and the process $\{\tau_k, \pi_{\tau_k}, \tilde{X}_{\tau_k}\}_{k=1}^\infty$ is a Markov process with respect to $\mathcal{G}_k$.

Proof. The representation (1.3.13) and the fact that $\{\tau_k, \pi_{\tau_k}, \tilde{X}_{\tau_k}\}$ is a $\mathcal{G}_k-$adapted discrete stochastic processes on $[0,\infty) \times \mathcal{S}_N \times \mathbb{R}^m$ follow immediately from Corollary 1.3.1 and the preceding definitions. For the Markov property we calculate

$$\begin{aligned}
& P(\tau_{k+1} < t, \tilde{X}^1_{\tau_{k+1}} < x_1, \ldots, \tilde{X}^m_{\tau_{k+1}} < x_m | \mathcal{G}_k) \\
= \;& E[P(\tau_{k+1} < t, \tilde{X}^1_{\tau_{k+1}} < x_1, \ldots, \tilde{X}^m_{\tau_{k+1}} < x_m | \mathcal{G}_k \vee \mathcal{F}^\theta)|\mathcal{G}_k] \\
= \;& E[\textstyle\int_{\tau_k}^t P(\tilde{X}^1_{\tau_{k+1}} < x_1, \ldots, \tilde{X}^m_{\tau_{k+1}} < x_m | \mathcal{G}_k \vee \mathcal{F}^\theta) n(\theta_s) \exp(-\int_{\tau_k}^s n(\theta_u) du) ds | \mathcal{G}_k] \\
= \;& E[\textstyle\int_{\tau_k}^t \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_m} \rho_{\tau_k, s}(z - \tilde{X}_{\tau_k}) n(\theta_s) \exp(-\int_{\tau_k}^s n(\theta_u) du) ds dz | \mathcal{G}_k] \\
= \;& \textstyle\int_{\tau_k}^t \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_m} \displaystyle\sum_{ij} n(e_i) r_{ji}(s - \tau_k, z - \tilde{X}_{\tau_k}) p_{ji}(s - \tau_k) \pi^j_{\tau_k} ds dz,
\end{aligned}$$

and for any bounded measurable function g on $[0,\infty) \times \mathcal{S}_N \times \mathbb{R}^m$ it then follows that

$$\begin{aligned}
& E[g(\tau_{k+1}, \pi_{\tau_{k+1}}, \tilde{X}_{\tau_{k+1}})|\mathcal{G}_k] \\
= \;& E[g(\tau_{k+1}, M(\tau_{k+1} - \tau_k, \tilde{X}_{\tau_{k+1}} - \tilde{X}_{\tau_k}, \pi_{\tau_k}), \tilde{X}_{\tau_{k+1}})|\mathcal{G}_k] \\
= \;& E[E[g(\tau_{k+1}, M(\tau_{k+1} - \tau_k, \tilde{X}_{\tau_{k+1}} - \tilde{X}_{\tau_k}, \pi_{\tau_k}), \tilde{X}_{\tau_{k+1}})|\mathcal{G}(k) \vee \mathcal{F}^\theta]|\mathcal{G}_k] \\
= \;& E[\textstyle\int_{\tau_k}^\infty E[g(t, M(t - \tau_k, \tilde{X}_t - \tilde{X}_{\tau_k}, \pi_{\tau_k}), \tilde{X}_t) n(\theta_t) \exp(-\int_{\tau_k}^t n(\theta_s) ds)|\mathcal{G}_k \vee \mathcal{F}^\theta] dt | \mathcal{G}_k] \\
= \;& \textstyle\int_{\tau_k}^\infty \int_{\mathbb{R}^m} g(t, M(t - \tau_k, x - \tilde{X}_{\tau_k}, \pi_{\tau_k}), x) \sum_{ij} n(e_i) r_{ji}(t - \tau_k, x - \tilde{X}_{\tau_k}) p_{ji}(t - \tau_k) \pi^j_{\tau_k} dx dt,
\end{aligned}$$

where the last equation depends only on $\{\tau_k, \pi_{\tau_k}, \tilde{X}_{\tau_k}\}$ thus implying the Markov property. $\square$

Chapter 2

Log-utility maximization

In this chapter we consider expected log-utility maximization

$$\sup_{h\in\mathcal{A}} E[\log V_T|\tau_0 = 0, \pi_{\tau_0} = \pi]. \tag{2.0.1}$$

In Section 2.1 we introduce an operator that is important for the control results. The control part is then studied in section 2.2-2.5 with the main result stated in Theorem 2.5.1. Section 2.6 contains technical proofs.

2.1 A contraction operator

In this section we define a contraction operator (see Definition 2.1.1 below) that will be relevant for deriving the results on the value function. In view of its definition and in order to derive its properties, we need first to introduce some additional notions.

We start by defining an operator on $\mathcal{M}(E)$ as follows

$$K^i(t,x)\pi := \sum_j n(e_i) r_{ji}(t,x) p_{ji}(t)\pi^j, \tag{2.1.1}$$

$$K(t,x)\pi := (K^1(t,x)\pi, K^2(t,x)\pi, \ldots, K^N(t,x)\pi). \tag{2.1.2}$$

For $t \in [0,\infty),\ x \in \mathbb{R}^m,\ K^i(t,x)$ is a positive linear operator on $\mathcal{M}(E)$. For $A \subset E$ set

$$K(t,x)\pi(A) := \sum_{i=1}^N K^i(t,x)\pi 1_{\{e_i\in A\}}. \tag{2.1.3}$$

By the definition of $M^i(t,x,\pi)$ and $K^i(t,x)\pi$, setting $\kappa(t,x,\pi) := \sum_i K^i(t,x)\pi$, for $t \in [0,\infty),\ x \in \mathbb{R}^m,\ \pi \in \mathcal{M}(E)$ we have

$$M^i(t,x,\pi) = \frac{1}{\kappa(t,x,\pi)} K^i(t,x)\pi\ . \tag{2.1.4}$$

By the definition of the Hilbert metric $d_H(\cdot,\cdot)$, for $t\in[0,\infty),\ x\in\mathbb{R}^m,\ \pi,\bar\pi\in\mathcal{M}(E)$ we then have

$$\begin{aligned}
d_H(M(t,x,\pi),M(t,x,\bar\pi)) &= \log(\sup\tfrac{M(t,x,\pi)(A)}{M(t,x,\bar\pi)(A)}\sup\tfrac{M(t,x,\bar\pi)(A)}{M(t,x,\pi)(A)})\\
&= \log(\sup\tfrac{\frac{1}{\kappa(t,x,\pi)}K(t,x)\pi(A)}{\frac{1}{\kappa(t,x,\bar\pi)}K(t,x)\bar\pi(A)}\sup\tfrac{\frac{1}{\kappa(t,x,\bar\pi)}K(t,x)\bar\pi(A)}{\frac{1}{\kappa(t,x,\pi)}K(t,x)\pi(A)})\\
&= \log(\sup\tfrac{K(t,x)\pi(A)}{K(t,x)\bar\pi(A)}\sup\tfrac{K(t,x)\bar\pi(A)}{K(t,x)\pi(A)})\\
&= d_H(K(t,x)\pi,K(t,x)\bar\pi).
\end{aligned}\tag{2.1.5}$$

Applying [1], Lemma 3.4 in [11] and Theorem 1.1 in [12] , for the positive linear operator K on $\mathcal{M}(E)$ it then follows that

$$d_H(M(t,x,\pi),M(t,x,\bar\pi))=d_H(K(t,x)\pi,K(t,x)\bar\pi)\le d_H(\pi,\bar\pi)\tag{2.1.6}$$

for $t\in[0,\infty),\ x\in\mathbb{R}^m,\ \pi,\bar\pi\in\mathcal{S}_N$. By Lemma 3.4 in [11] , for $\forall\pi,\bar\pi\in\mathcal{S}_N$ we also have

$$\|\pi-\bar\pi\|_{TV}\le\frac{2}{\log 3}d_H(\pi,\bar\pi),\tag{2.1.7}$$

where $\|\cdot\|_{TV}$ is the total variation on $\mathcal{S}_N$.

We finally introduce a metric on $[0,\infty)\times\mathcal{S}_N\times\bar H_m$ by

$$|t-\bar t|+d_H(\pi,\bar\pi)+\sum_{i=1}^m|h^i-\bar h^i|\tag{2.1.8}$$

for $(t,\ \pi,\ h),(\bar t,\ \bar\pi,\ \bar h)\in[0,\infty)\times\mathcal{S}_N\times\bar H_m$ and considering the state space

$$\Sigma:=[0,\infty)\times\mathcal{S}_N,\tag{2.1.9}$$

let $C_b(\Sigma)$ be the set of bounded continuous functions $g:\Sigma\to\mathbb{R}$ with norm

$$\|g\|:=\max_{x\in\Sigma}|\ g(x)\ |\ .\tag{2.1.10}$$

Definition 2.1.1. *Let the operator $J:\ C_b(\Sigma)\to C_b(\Sigma)$ be given as follows*

$$\begin{aligned}
&Jg(\tau,\pi)\\
&:= \int_\tau^T\int_{\mathbb{R}^m}g(t,M(t-\tau,z,\pi))\sum_{ij}n(e_i)r_{ji}(t-\tau,z)p_{ji}(t-\tau)\pi^j dzdt\\
&= E[g(\tau_1,\pi_{\tau_1})1_{\{\tau_1<T\}}|\tau_0=\tau,\pi_{\tau_0}=\pi],
\end{aligned}\tag{2.1.11}$$

where M is defined in (1.3.11)-(1.3.12).

First we have

Lemma 2.1.1. *J is a contraction operator on $C_b(\Sigma)$ with contraction constant $c:=1-e^{-\bar nT}<1$, where $\bar n:=\max n(\theta)=\max_i n(e_i)$.*

Proof. For $\forall g \in C_b(\Sigma)$

$$\begin{aligned}
|\ Jg(t,\pi)\ | &= |\ E[g(\tau_1,\pi_1)1_{\{\tau_1<T\}}|\tau_0=t,\pi_{\tau_0}=\pi]\ | \\
&\leq \|g\|P(\tau_1<T|\tau_0=t) \\
&= \|g\|E[(1-\exp(-\textstyle\int_t^T n(\theta_t)dt))] \\
&\leq \|g\|(1-\exp(-\bar{n}(T-t)))
\end{aligned}$$

and so

$$\|Jg\| \leq c\|g\| \tag{2.1.12}$$

with c as specified in the statement. □

Let $C_{b,lip}(\Sigma)$ be the set of bounded and Lipschitz continuous functions $g:\ \Sigma \to \mathbb{R}$ and set for $g \in C_{b,lip}(\Sigma)$

$$N^\lambda(g) := \lambda\|g\| + [g]_{lip} \tag{2.1.13}$$

where,

$$[g]_{lip} := \sup_{\tau,\bar{\tau}\in[0,T]\ \pi,\bar{\pi}\in\mathcal{S}_N} \frac{|g(\tau,\pi)-g(\bar{\tau},\bar{\pi})|}{|\tau-\bar{\tau}|+d_H(\pi,\bar{\pi})}. \tag{2.1.14}$$

Note that $C_{b,lip}(\Sigma)$ is a Banach space with the norm $N^\lambda(g)$, for each $\lambda > 0$.
Take a sufficiently large constant λ such that

$$c' := (c + \max(\bar{n}, \frac{2}{\log 3})\frac{1}{\lambda}) < 1. \tag{2.1.15}$$

Proposition 2.1.1. *The operator J in Definition 2.1.1 is a contraction operator*

$$J: C_{b,lip}(\Sigma) \to C_{b,lip}(\Sigma)$$

with contraction constant c'.

Proof. Let us first prove that $Jg(t,\pi)$ is Lipschitz continuous with respect to t. By assumption, for all $g \in C_{b,lip}(\Sigma)$,

$$|g(\tau,\pi)-g(\bar{\tau},\pi)| \leq [g]_{lip}|\tau-\bar{\tau}|, \tag{2.1.16}$$

$$|g(\tau,\pi)-g(\tau,\bar{\pi})| \leq [g]_{lip}d_H(\pi,\bar{\pi}). \tag{2.1.17}$$

We change variables from t to $t+\tau$,

$$Jg(\tau,\pi) = \int_0^{T-\tau}\int_{\mathbb{R}^m} g(t+\tau, M(t,z,\pi))\sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt. \tag{2.1.18}$$

We then have

$$\begin{aligned}
&|Jg(\tau,\pi)-Jg(\bar{\tau},\pi)| \\
&= |\textstyle\int_{T-\bar{\tau}}^{T-\tau}\int_{\mathbb{R}^m} g(t+\tau,M(t,z,\pi))\sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \\
&\quad +|\textstyle\int_0^{T-\tau}\int_{\mathbb{R}^m}\{g(t+\tau,M(t,z,\pi))-g(t+\bar{\tau},M(t,z,\pi))\} \\
&\qquad\qquad \cdot\textstyle\sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \qquad (2.1.19) \\
&\leq \bar{n}\|g\||\tau-\bar{\tau}| + [g]_{lip}|\tau-\bar{\tau}|\ |\textstyle\int_0^{T-\tau}\int_{\mathbb{R}^m}\sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \\
&= \bar{n}\|g\||\tau-\bar{\tau}| + [g]_{lip}|\tau-\bar{\tau}|P(\tau_1<T|\tau_0=\tau,\pi_{\tau_0}=\pi) \\
&\leq (\bar{n}\|g\|+c[g]_{lip})|\tau-\bar{\tau}|.
\end{aligned}$$

Next, let us prove that $Jg(t,\pi)$ is Lipschitz continuous with respect to π.

$$\begin{aligned}
&|Jg(\tau,\pi)-Jg(\tau,\bar\pi)| \\
&\quad\le\ |\textstyle\int_0^{T-\tau}\int_{\mathbb{R}^m}\{g(t,M(t,z,\pi))-g(t,M(t,z,\bar\pi))\}\sum_{ij}n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \\
&\qquad+|\textstyle\int_0^{T-\tau}\int_{\mathbb{R}^m}g(t,M(t,z,\bar\pi))\sum_{ij}n(e_i)r_{ji}(t,z)p_{ji}(t)(\pi^j-\bar\pi^j)dzdt| \\
&\quad\le\ |\textstyle\int_0^{T-\tau}\int_{\mathbb{R}^m}[g]_{lip}d_H(M(t,z,\pi),M(t,z,\bar\pi))\sum_{ij}n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \\
&\qquad+\|g\|\tfrac{2}{\log 3}d_H(\pi,\bar\pi)P(\tau_1<T|\tau_0=\tau) \\
&\quad\le\ (\tfrac{2}{\log 3}\|g\|+c[g]_{lip})d_H(\pi,\bar\pi).
\end{aligned}\tag{2.1.20}$$

Therefore,

$$\begin{aligned}
[Jg]_{lip} &= \sup_{\tau,\bar\tau\in[0,T]\ \pi,\bar\pi\in\mathcal{S}_N}\frac{|Jg(\tau,\pi)-Jg(\bar\tau,\bar\pi)|}{|\tau-\bar\tau|+d_H(\pi,\bar\pi)} \\
&\le \sup_{\tau,\bar\tau\in[0,T]\ \pi,\bar\pi\in\mathcal{S}_N}\frac{|Jg(\tau,\pi)-Jg(\bar\tau,\pi)|+|Jg(\bar\tau,\pi)-Jg(\bar\tau,\bar\pi)|}{|\tau-\bar\tau|+d_H(\pi,\bar\pi)} \\
&\le \sup_{\tau,\bar\tau\in[0,T]\ \pi,\bar\pi\in\mathcal{S}_N}\frac{(\bar n\|g\|+c[g]_{lip})|\tau-\bar\tau|+(\frac{2}{\log 3}\|g\|+c[g]_{lip})d_H(\pi,\bar\pi)}{|\tau-\bar\tau|+d_H(\pi,\bar\pi)} \\
&\le \max(\bar n,\tfrac{2}{\log 3})\|g\|+c[g]_{lip}.
\end{aligned}\tag{2.1.21}$$

Finally, we obtain

$$\begin{aligned}
N^\lambda(Jg) &= \lambda\|Jg\|+[Jg]_{lip} \\
&\le c\lambda\|g\|+\max(\bar n,\tfrac{2}{\log 3})\|g\|+c[g]_{lip} \\
&\le c'\lambda\|g\|+c[g]_{lip} \\
&\le c'N^\lambda(g).
\end{aligned}\tag{2.1.22}$$

□

2.2 Preliminary results in view of the optimal strategy

Recall from (1.2.7) that the value process of a self financing portfolio satisfies

$$\frac{dV_t}{V_t}=[r_0+h_t^*\{r(\theta_t)-r_0\mathbf{1}\}]dt+h_t^*\sigma(\theta_t)dB_t.\tag{2.2.1}$$

We have by Itô's formula

$$\begin{aligned}
\log V_T=\log v_0 &+ \textstyle\int_0^T h_t^*\sigma(\theta_t)dB_t \\
&+ \textstyle\int_0^T[r_0+h_t^*\{r(\theta_t)-r_0\mathbf{1}\}-\frac{1}{2}h_t^*\sigma\sigma^*(\theta_t)h_t]dt.
\end{aligned}\tag{2.2.2}$$

Put

$$f(\theta,h):=r_0+h^*\{r(\theta)-r_0\mathbf{1}\}-\frac{1}{2}h^*\sigma\sigma^*(\theta)h\tag{2.2.3}$$

and notice that this function $f(\cdot)$ is bounded under our assumptions. The expected log-utility of terminal wealth then becomes

$$E[\log V_T|\tau_0 = 0, \pi_{\tau_0} = \pi] = \log v_0 + E[\int_0^T f(\theta_t, h_t)dt|\tau_0 = 0, \pi_{\tau_0} = \pi] \tag{2.2.4}$$

and, we want to consider the problem of maximization of expected terminal log-utility, namely

$$\sup_{h\in\mathcal{A}} E[\log V_T|\tau_0 = 0, \pi_{\tau_0} = \pi]. \tag{2.2.5}$$

Definition 2.2.1. *Let $\hat{C}(\tau, \pi, h)$ be defined by*

$$\begin{aligned} \hat{C}(\tau, \pi, h) &:= E[\textstyle\int_\tau^{T\wedge\tau_1} f(\theta_s, h_s)ds|\tau_0 = \tau, \pi_{\tau_0} = \pi] \\ &= \textstyle\int_\tau^T \int_{\mathbb{R}^m} \sum\limits_{i,j} f(e_i, \gamma(x, h))r_{ji}(t-\tau, x)p_{ji}(t-\tau)\pi^j dxdt, \end{aligned} \tag{2.2.6}$$

where $\gamma(x, h) = [\gamma^1(x^1, h), \cdots, \gamma^m(x^m, h)]$.

Lemma 2.2.1.

(i) For the function defined by (2.2.3), we have the following equation

$$E[\int_t^T f(\theta_s, h_s)ds|\tau_0 = t, \pi_{\tau_0} = \pi] = E[\sum_k \hat{C}(\tau_k, \pi_{\tau_k}, h_k)1_{\{\tau_k<T\}}|\tau_0 = t, \pi_{\tau_0} = \pi]. \tag{2.2.7}$$

(ii) $\hat{C}$ is a bounded and continuous function on $[0, T] \times \mathcal{S}_N \times \bar{H}_m$.

For the proof see Section 2.6

Corollary 2.2.1.

(i) There exists a Borel function $\hat{h}(\tau, \pi)$ such that $\sup_{h\in\bar{H}_m} \hat{C}(\tau, \pi, h) = \hat{C}(\tau, \pi, \hat{h}(\tau, \pi))$.

(ii) The function

$$C(t, \pi) := \sup_{h\in\bar{H}_m} \hat{C}(t, \pi, h). \tag{2.2.8}$$

is Lipschitz continuous with respect to t, π in the metric introduced in (2.1.8).

Proof. $\bar{H}_m$ is compact and $\hat{C}(\tau, \pi, h)$ is a bounded continuous function on $[0, T]\times\mathcal{S}_N\times\bar{H}_m$; there exists then a Borel function $\hat{h}(\tau, \pi)$ such that (2.2.8) holds. Furthermore, $\hat{C}(t, \pi, h)$ is uniformly Lipschitz continuous with respect to t, π.

□

2.3 Value function and first properties

We start with the following basic definition

Definition 2.3.1. *For given initial data ($\tau_0 = t, \pi_{\tau_0} = \pi$), where we now start at a generic time t, consider the following value function for $h \in \mathcal{A}$*

$$
\begin{aligned}
W(t,\pi,h.) &:= E[\textstyle\int_t^T f(\theta_s, h_s)ds|\tau_0 = t,\ \pi_{\tau_0} = \pi] \\
&= E[\sum_{k=0}^{\infty} \hat{C}(\tau_k, \pi_{\tau_k}, h_k)1_{\{\tau_k<T\}}|\tau_0 = t,\ \pi_{\tau_0} = \pi],
\end{aligned}
\tag{2.3.1}
$$

and define

$$
\begin{aligned}
W(t,\pi) &:= \sup_{h\in\mathcal{A}} W(t,\pi,h.) \\
&= \sup_{h\in\mathcal{A}} E[\int_t^T f(\theta_s, h_s)ds|\tau_0 = t,\ \pi_{\tau_0} = \pi] \\
&= \sup_{h\in\mathcal{A}} E[\sum_{k=0}^{\infty} \hat{C}(\tau_k, \pi_{\tau_k}, h_k)1_{\{\tau_k<T\}}|\tau_0 = t,\ \pi_{\tau_0} = \pi],
\end{aligned}
\tag{2.3.2}
$$

$$
\begin{aligned}
W^n(t,\pi) &:= \sup_{h\in\mathcal{A}^n} W(t,\pi,h.) \\
&= \sup_{h\in\mathcal{A}^n} E[\int_t^T f(\theta_s, h_s)ds|\tau_0 = t,\ \pi_{\tau_0} = \pi] \\
&= \sup_{h\in\mathcal{A}^n} E[\sum_{k=0}^{\infty} \hat{C}(\tau_k, \pi_{\tau_k}, h_k)1_{\{\tau_k<T\}}|\tau_0 = t,\ \pi_{\tau_0} = \pi],
\end{aligned}
\tag{2.3.3}
$$

where $\mathcal{A}^n$ was defined in (1.2.5).

Lemma 2.3.1. *For all $n \geq 0$ and $h \in \mathcal{A}^n$, we have the following equation*

$$
\begin{aligned}
W(t,\pi,h.) &= E[\sum_{k=0}^{n-1} \hat{C}(\tau_k, \pi_{\tau_k}, h_k)1_{\{\tau_k<T\}} \\
&\quad + \textstyle\int_{\tau_n}^T f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_n}, h_n))ds1_{\{\tau_n<T\}}|\tau_0 = t,\ \pi_{\tau_0} = \pi].
\end{aligned}
\tag{2.3.4}
$$

For the proof see Section 2.6.

Corollary 2.3.1. *For $n \geq 0$, $t \in [0,T]$, $\pi \in \mathcal{S}_N$ we have the following equation*

$$
\begin{aligned}
W^n(t,\pi) &= \sup_{h\in\mathcal{A}^n} E[\sum_{k=0}^{n-1} \hat{C}(\tau_k, \pi_{\tau_k}, h_k)1_{\{\tau_k<T\}} \\
&\quad + \textstyle\int_{\tau_n}^T f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_n}, h_n))ds1_{\{\tau_n<T\}}|\tau_0 = t,\ \pi_{\tau_0} = \pi].
\end{aligned}
\tag{2.3.5}
$$

2.4 An auxiliary value function

Recall the function $C(t,\pi)$ defined in Corollary 2.2.1 as well as the operator J from Definition 2.1.1. By Proposition 2.1.1 we have that J is a contraction operator on the Banach space $C_{b,lip}$ with its norm $N^{\lambda}(\cdot)$. Therefore, $\lim_{n\to\infty}\sum_{k=0}^{n} J^k C$ exists and so we introduce the

Definition 2.4.1. *Define the auxiliary value function* $\bar{W}(t,\pi)$ *as*

$$\bar{W} := \sum_{k=0}^{\infty} J^k C$$

The following lemma then holds

Lemma 2.4.1. *We have* $\bar{W} \in C_{b,lip}$ *and it satisfies*

$$\bar{W}(t,\pi) = C(t,\pi) + J\bar{W}(t,\pi). \tag{2.4.1}$$

Proof. Due always to the fact that (see Proposition 2.1.1) J is a contraction operator on the Banach space $C_{b,lip}$ with its norm $N^{\lambda}(\cdot)$, in addition to the existence of $\lim_{n\to\infty}\sum_{k=0}^{n} J^k C$ we also have

$$(I-J)^{-1}C = \sum_{k=0}^{\infty} J^k C,$$

from which the result follows. □

In view of deriving a recursion related to $\bar{W}(t,\pi)$ (value iteration), we start with the

Definition 2.4.2. *Define, for* $h \in \bar{H}_m$,

$$\bar{W}^0(t,\pi,h) := E[\int_t^T f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_t, h))ds | \tau_0 = t,\ \pi_{\tau_0} = \pi]. \tag{2.4.2}$$

Furthermore, let

$$\bar{W}^0(t,\pi) := \max_{h\in\bar{H}_m} \bar{W}^0(t,\pi,h), \tag{2.4.3}$$

and, for $n \geq 1$

$$\begin{aligned} \bar{W}^n(t,\pi) &:= C(t,\pi) + J\bar{W}^{n-1}(t,\pi) \\ &= \sum_{k=0}^{n-1} J^k C(t,\pi) + J^n \bar{W}^0(t,\pi). \end{aligned} \tag{2.4.4}$$

Remark 2.4.1. *The function* $\bar{W}^0(t,\pi,h)$ *in (2.4.2) is bounded and continuous with respect to* t,π,h. *This follows by an analogous proof as in Lemma 2.2.1(ii).*

We first state and prove the following lemma (later we need a relation from the proof)

Lemma 2.4.2.

(i) We have the equation

$$\bar{W}^n(t,\pi) = E[\sum_{k=0}^{n-1} C(\tau_k, \pi_{\tau_k}) 1_{\{\tau_k<T\}} + \bar{W}^0(\tau_n, \pi_{\tau_n}) 1_{\{\tau_n<T\}} | \tau_0 = t,\ \pi_{\tau_0} = \pi]. \tag{2.4.5}$$

(ii) For any $\epsilon > 0$, we set $n_\epsilon := (\log(1-c') + \log\varepsilon - \log N^\lambda(\bar{W}^1 - \bar{W}^0))/\log c'$, where c' is the contraction constant defined in (2.1.15). For all $n > n_\epsilon$,

$$N^\lambda(\bar{W} - \bar{W}^n) < \epsilon. \tag{2.4.6}$$

Proof. We prove (i). For $n \geq 1$

$$\{\tau_{n-1} < T\} \supset \{\tau_n < T\}. \tag{2.4.7}$$

Therefore,

$$1_{\{\tau_{n-1}<T\}} 1_{\{\tau_n<T\}} = 1_{\{\tau_n<T\}}. \tag{2.4.8}$$

For all $g \in C_b([0,T] \times \mathcal{S}_N)$ and $n \geq 0$, we have

$$\begin{aligned} &E[g(\tau_n, \pi_{\tau_n}) 1_{\{\tau_n<T\}} | \tau_0 = t, \pi_{\tau_0} = \pi] \\ &= \quad E[E[g(\tau_n, \pi_{\tau_n}) 1_{\{\tau_n<T\}} | \mathcal{G}_{n-1}] 1_{\{\tau_{n-1}<T\}} | \tau_0 = t, \pi_{\tau_0} = \pi]. \end{aligned} \tag{2.4.9}$$

because $1_{\{\tau_{n-1}<T\}} E[1_{\{\tau_n<T\}} | \mathcal{G}_{n-1}] = E[1_{\{\tau_n<T\}} | \mathcal{G}_{n-1}]$. Then, since (see (2.1.11))

$$\begin{aligned} &E[g(\tau_n, \pi_{\tau_n}) 1_{\{\tau_n<T\}} | \mathcal{G}_{n-1}] \\ &= \quad \textstyle\int_{\tau_{n-1}}^T \int_{\mathbb{R}^m} g(t, M(t-\tau_{n-1}, z, \pi_{\tau_{n-1}})) \displaystyle\sum_{ij} n(e_i) r_{ji}(t-\tau_{n-1}, z) p_{ji}(t-\tau_{n-1}) \pi^j_{\tau_{n-1}} dz dt \\ &= \quad Jg(\tau_{n-1}, \pi_{\tau_{n-1}}), \end{aligned} \tag{2.4.10}$$

we have (see always (2.1.11))

$$\begin{aligned} E[g(\tau_n, \pi_{\tau_n}) 1_{\{\tau_n<T\}} | \tau_0 = t, \pi_{\tau_0} = \pi] &= E[Jg(\tau_{n-1}, \pi_{\tau_{n-1}}) 1_{\{\tau_{n-1}<T\}} | \tau_0 = t, \pi_{\tau_0} = \pi] \\ &= J^n g(t,\pi). \end{aligned} \tag{2.4.11}$$

We thus obtain

$$\begin{aligned} \bar{W}^n(t,\pi) &= \sum_{k=0}^{n-1} J^k C(t,\pi) + J^n \bar{W}^0(t,\pi) \\ &= E[\textstyle\sum_{k=0}^{n-1} C(\tau_k, \pi_{\tau_k}) 1_{\{\tau_k<T\}} + \bar{W}^0(\tau_n, \pi_{\tau_n}) 1_{\{\tau_n<T\}} | \tau_0 = t,\ \pi_{\tau_0} = \pi]. \end{aligned} \tag{2.4.12}$$

Next, we prove (ii). For any n,

$$\begin{aligned} N^\lambda(\bar{W} - \bar{W}^n) &= N^\lambda(\lim_{k\to\infty} \bar{W}^{n+k} - \bar{W}^n) = \lim_{k\to\infty} N^\lambda(\bar{W}^{n+k} - \bar{W}^n) \\ &\leq \lim_{k\to\infty} \sum_{i=0}^{k-1} N^\lambda(\bar{W}^{n+i+1} - \bar{W}^{n+i}) \leq N^\lambda(\bar{W}^{n+1} - \bar{W}^n) \sum_{i=0}^{\infty} (c')^i \\ &\leq N^\lambda(\bar{W}^1 - \bar{W}^0)(c')^n \sum_{i=0}^{\infty} (c')^i = \frac{(c')^n}{1-c'} N^\lambda(\bar{W}^1 - \bar{W}^0). \end{aligned} \tag{2.4.13}$$

□

Lemma 2.4.3. *For all* $n \geq 0$, *we have the equality*

$$W^n(t,\pi) = \bar{W}^n(t,\pi). \tag{2.4.14}$$

Proof. By Corollary 2.3.1, for all $n \geq 0$

$$\begin{aligned} W^n(t,\pi) &= \sup_{h\in\mathcal{A}^n} E[\sum_{k=0}^{n-1} \hat{C}(\tau_k,\pi_{\tau_k},h_k)1_{\{\tau_k<T\}} \\ &+ \int_{\tau_n}^T f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_n},h_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\pi_{\tau_0}=\pi]. \end{aligned} \tag{2.4.15}$$

Since $\bar{H}_m$ is compact and $\bar{W}^0(\tau,\pi,h)$ is a bounded continuous function on $[0,T]\times\mathcal{S}_N\times\bar{H}_m$, there exists a Borel function $w(\tau,\pi)$ such that $\sup_{h\in\bar{H}_m}\bar{W}^0(\tau,\pi,h)=\bar{W}^0(\tau,\pi,w(\tau,\pi))$. Furthermore, by Corollary 2.2.1(i) there exists a Borel function $\hat{h}(\tau,\pi)$ such that $\sup_{h\in\bar{H}_m}\hat{C}(\tau,\pi,h)=\hat{C}(\tau,\pi,\hat{h}(\tau,\pi))$ holds. For $n \geq 0$, we define the strategy

$$\begin{aligned} \tilde{h}_k &:= \hat{h}(\tau_k,\pi_{\tau_k}), && 0\leq k\leq n-1 \\ \tilde{h}_k &:= w(\tau_n,\pi_{\tau_n}), && k=n \\ \tilde{h}_k &:= \gamma(\tilde{X}_{\tau_k}-\tilde{X}_{\tau_n},\tilde{h}_n), && k>n. \end{aligned} \tag{2.4.16}$$

By definition of $\{\tilde{h}_k\}_{k\in\mathbb{N}}$, we have $\{\tilde{h}_k\}_{k\in\mathbb{N}}\in\mathcal{A}^n$. Using Lemma 2.4.2(i) and Lemma 2.3.1, for $n\geq 0, t\in[0,T],\pi\in\mathcal{S}_N$

$$\begin{aligned} \bar{W}^n(t,\pi) &= E[\sum_{k=0}^{n-1}\hat{C}(\tau_k,\pi_{\tau_k},\tilde{h}_k)1_{\{\tau_k<T\}} \\ &\quad + \int_{\tau_n}^T f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_n},\tilde{h}_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\pi_{\tau_0}=\pi] \\ &\leq W^n(t,\pi). \end{aligned} \tag{2.4.17}$$

Using again Lemma 2.3.1, (2.4.2) and Lemma 2.4.2(i), for all $n\geq 0, h\in\mathcal{A}^n, t\in[0,T],\pi\in\mathcal{S}_N$

$$\begin{aligned} W(t,\pi,h.) &= E[\sum_{k=0}^{n-1}\hat{C}(\tau_k,\pi_{\tau_k},h_k)1_{\{\tau_k<T\}} \\ &\qquad + \int_{\tau_n}^T f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_n},h_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\pi_{\tau_0}=\pi] \\ &= E[\sum_{k=0}^{n-1}\hat{C}(\tau_k,\pi_{\tau_k},h_k)1_{\{\tau_k<T\}}+\bar{W}^0(\tau_n,\pi_{\tau_n},h_n)1_{\{\tau_n<T\}}|\tau_0=t,\pi_{\tau_0}=\pi] \\ &\leq E[\sum_{k=0}^{n-1}C(\tau_k,\pi_{\tau_k})1_{\{\tau_k<T\}}+\bar{W}^0(\tau_n,\pi_{\tau_n})1_{\{\tau_n<T\}}|\tau_0=t,\pi_{\tau_0}=\pi] \\ &= \bar{W}^n(t,\pi). \end{aligned} \tag{2.4.18}$$

Therefore, we have

$$W^n(t,\pi)=\sup_{h\in\mathcal{A}^n}W(t,\pi,h.)\leq\bar{W}^n(t,\pi), \tag{2.4.19}$$

and so we obtain for all $n \geq 0$

$$W^n(t,\pi) = \bar{W}^n(t,\pi). \tag{2.4.20}$$

$\square$

Lemma 2.4.4. *For $n \geq 0$, we have the estimate*

$$\bar{W}^n(t,\pi) \leq \bar{W}^{n+1}(t,\pi) \leq \bar{W}(t,\pi) \leq W(t,\pi). \tag{2.4.21}$$

For the proof see Section 2.6.

Lemma 2.4.5. *The following estimate holds*

$$W(t,\pi) \leq \bar{W}(t,\pi) \tag{2.4.22}$$

for $t \in [0,T], \forall \pi \in \mathcal{S}_N$.

For the proof see Section 2.6.

2.5 Main result

Based on the previous sections we obtain now the main result of this section

Theorem 2.5.1.

(i) Approximation theorem :
For any $\epsilon > 0, n > n_\epsilon$,

$$N^\lambda(W - \bar{W}^n) < \epsilon, \tag{2.5.1}$$

where n_ϵ is the constant defined in Lemma 2.4.2(ii) and, modulo the additive term $\log v_0$, the function $W = W(t,\pi)$ is the optimal value function (see (2.2.4), (2.3.1), (2.3.2)), N^λ is the norm introduced in (2.1.13), and $\bar{W}^n$ are computed recursively according to (2.4.3) and (2.4.4).

(ii) Dynamic programming principle : for any $n > 0$

$$\begin{aligned} W(t,\pi) = \sup_{h\in\mathcal{A}^n} E[\sum_{k=0}^{n} \hat{C}(\tau_k, \pi_{\tau_k}, h_k) 1_{\{\tau_k < T\}} \\ + W(\tau_{n+1}, \pi_{\tau_{n+1}}) 1_{\{\tau_{n+1} < T\}} | \tau_0 = t, \pi_{\tau_0} = \pi]. \end{aligned} \tag{2.5.2}$$

.

(iii) Optimal value and optimal strategy for the Log Utility Maximization Problem : for the utility maximization under the initial conditions $V_0 = v_0, \tau_0 = 0, \pi_{\tau_0} = \pi$ we have

$$\begin{aligned} \sup_{h\in\mathcal{A}} E[\log V_T | \tau_0 = 0, \pi_{\tau_0} = \pi] &= \log v_0 + \sup_{h\in\mathcal{A}} E[\int_0^T f(\theta_t, h_t) dt | \tau_0 = 0, \pi_{\tau_0} = \pi] \\ &= \log v_0 + C(0,\pi) + \textstyle\sum_{k=1}^\infty E[\hat{C}(\tau_k, \pi_{\tau_k}, \hat{h}_k) 1_{\{\tau_k < T\}} | \tau_0 = 0, \pi_{\tau_0} = \pi], \end{aligned} \tag{2.5.3}$$

with $\hat{h}_k$ defined in Corollary 2.2.1, namely $\sup_{h\in\bar{H}_m} \hat{C}(\tau,\pi,h) = \hat{C}(\tau,\pi,\hat{h}(\tau,\pi))$ and $\hat{h}_k = \hat{h}(\tau_k, \pi_{\tau_k})$ and

$$\hat{h}_t^i = \gamma^i(\tilde{X}_t - \tilde{X}_{\tau_k}, \hat{h}_k), \quad \tau_k \leq t < \tau_{k+1} \tag{2.5.4}$$

Proof. Let us first prove (i). By Lemma 2.4.4 and Lemma 2.4.5,

$$W(t,\pi) = \bar{W}(t,\pi). \tag{2.5.5}$$

Therefore, applying Lemma 2.4.2(ii) one obtains

$$N^{\lambda}(W - \bar{W}^n) < \epsilon. \tag{2.5.6}$$

Next, let us prove (ii). By (2.5.5), Lemma 2.4.1, (2.4.11) and by Corollary 2.2.1

$$\begin{aligned} W(t,\pi) &= \bar{W}(t,\pi) = \sum_{k=0}^{n} J^k C + J^{n+1} W(t,\pi) \\ &= E[\sum_{k=0}^{n} C(\tau_k, \pi_{\tau_k}) 1_{\{\tau_k < T\}} + W(\tau_{n+1}, \pi_{\tau_{n+1}}) 1_{\{\tau_{n+1} < T\}} | \tau_0 = t, \pi_{\tau_0} = \pi] \\ &= \sup_{h \in \mathcal{A}^n} E[\sum_{k=0}^{n} \hat{C}(\tau_k, \pi_{\tau_k}, h_k) 1_{\{\tau_k < T\}} + W(\tau_{n+1}, \pi_{\tau_{n+1}}) 1_{\{\tau_{n+1} < T\}} | \tau_0 = t, \pi_{\tau_0} = \pi]. \end{aligned} \tag{2.5.7}$$

Finally, (iii) is an immediate consequence of (2.2.4), Lemma 2.2.1 and Lemma 2.4.3 and its proof. □

2.6 Proof of Lemma

Proof of Lemma 2.2.1.

Proof of statement (i). It follows from the two lemmas shown below.

Lemma 2.6.1. *We have the following representation,*

$$E[f(\theta_t, h_t)|\mathcal{G}_t] = \sum_{k\geq 0} 1_{]\tau_k, \tau_{k+1}]}(t) \frac{E[f(\theta_t, \gamma(\tilde{X}_t - \tilde{X}_{\tau_k}, h_k)) 1_{\{t \leq \tau_{k+1}\}} | \mathcal{G}_k]}{E[1_{\{t\leq \tau_{k+1}\}}|\mathcal{G}_k]}. \tag{2.6.1}$$

Proof. It suffices to prove that for any $\mathcal{G}_t-$adapted process Z_t

$$E[E[f(\theta_t, h_t)|\mathcal{G}_t] Z_t] = E[\sum_{k\geq 0} 1_{]\tau_k, \tau_{k+1}]}(t) \frac{E[f(\theta_t, \gamma(\tilde{X}_t - \tilde{X}_{\tau_k}, h_k)) 1_{\{t \leq \tau_{k+1}\}} | \mathcal{G}_k]}{E[1_{\{t\leq \tau_{k+1}\}}|\mathcal{G}_k]} Z_t]. \tag{2.6.2}$$

First notice that any $\mathcal{G}_t-$adapted process Z_t has the representation (see [3])

$$Z_t = \sum_{k\geq 0} 1_{]\tau_k, \tau_{k+1}]}(t) Z_k(t) + Z_\infty 1_{]\tau_\infty, \infty[}(t), \tag{2.6.3}$$

with the process $Z_k(t)$ being $\mathcal{G}_k \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. Furthermore, under our assumptions, for all $t > 0$, $\lim_{n\to\infty} 1_{\{\tau_n < t\}} = 0$ and thus

$$Z_t = \sum_{k\geq 0} 1_{]\tau_k, \tau_{k+1}]}(t) Z_k(t). \tag{2.6.4}$$

Note, finally, that $E[1_{\{\tau_k<t\leq\tau_{k+1}\}}|\mathcal{G}_k] = 1_{]\tau_k,\infty)}(t)E[1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]]$. We then have

$$\begin{aligned}
E[E[f(\theta_t,h_t)|\mathcal{G}_t]Z_t] &= E[f(\theta_t,h_t)\sum_{k\geq 0}1_{]\tau_k,\tau_{k+1}]}(t)Z_k(t)] \\
&= \sum_{k\geq 0}E[E[f(\theta_t,h_t)1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]1_{\{\tau_k<t\}}Z_k(t)] \\
&= \sum_{k\geq 0}E[\frac{E[f(\theta_t,h_t)1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}E[1_{]\tau_k,\tau_{k+1}]}(t)Z_k(t)|\mathcal{G}_k]] \\
&= E[\sum_{k\geq 0}1_{]\tau_k,\tau_{k+1}]}(t)\frac{E[f(\theta_t,h_t)1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}Z_t],
\end{aligned}$$

and thus we obtain (2.6.2) since

$$f(\theta_t,h_t) = \sum_{k=0}^{\infty}1_{[\tau_k,\tau_{k+1})}(t)f(\theta_t,\gamma(\tilde{X}_t-\tilde{X}_{\tau_k},h_k)),$$

which follows from (1.2.3). □

Lemma 2.6.2. *We have the following equation*

$$E[\int_t^T f(\theta_s,h_s)ds|\tau_0=t,\pi_{\tau_0}=\pi] = E[\sum_{k\geq 0}\hat{C}(\tau_k,\pi_{\tau_k},h_k)1_{\{\tau_k<T\}}|\tau_0=t,\pi_{\tau_0}=\pi] \tag{2.6.5}$$

with $\hat{C}(t,\pi,h)$ defined by (2.2.6) in Definition 2.2.1.

Proof. For simplicity, in the following formula we shall use the notation

$$E^{t,\pi}[\cdot] \equiv E[\cdot\,|\tau_0=t,\pi_{\tau_0}=\pi]$$

Using (2.6.1) we have similarly as above

$$\begin{aligned}
&E^{t,\pi}[\int_t^T E[f(\theta_s,h_s)|\mathcal{G}_s]ds] = E^{t,\pi}[\int_t^T\sum_{k\geq 0}1_{]\tau_k,\tau_{k+1}]}(s)\frac{E[f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_k},h_k))1_{\{s<\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{s\leq\tau_{k+1}\}}|\mathcal{G}_k]}ds] \\
&= E^{t,\pi}[\sum_{k\geq 0}\int_t^T 1_{]\tau_k,\infty)}(s)E[f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_k},h_k))1_{\{s<\tau_{k+1}\}}|\mathcal{G}_k]ds] \\
&= E^{t,\pi}[\sum_{k\geq 0}\int_t^T 1_{]\tau_k,\infty)}(s)E[e^{-\int_{\tau_k}^s n(\theta_u)du}f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_k},h_k))|\mathcal{G}_k]ds] \\
&= E^{t,\pi}[\sum_{k\geq 0}\int_t^T 1_{]\tau_k,\infty)}(s)E[E[e^{-\int_{\tau_k}^s n(\theta_u)du}f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_k},h_k))|\mathcal{G}_k\vee\sigma\{\theta_{\tau_k}\}]|\mathcal{G}_k]ds]
\end{aligned} \tag{2.6.6}$$

Since $(\theta_t,\tilde{X}_t)$ is a time homogeneous Markov process,

$$\begin{aligned}
&E[e^{-\int_{\tau_k}^s n(\theta_u)du}f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_k},h_k))|\mathcal{G}_k\vee\sigma\{\theta_{\tau_k}\}] \\
&= E[e^{-\int_0^t n(\theta_u)du}f(\theta_t,\gamma(\tilde{X}_t-x,h))|\theta_0=\theta,\tilde{X}_0=x]|_{t=s-\tau_k,\theta=\theta_k,x=\tilde{X}_{\tau_k},h=h_k}
\end{aligned} \tag{2.6.7}$$

We now have, recalling the definition of $r_{ji}(t,z)$ in (1.3.8),

$$\begin{aligned}
&E[e^{-\int_0^t n(\theta_s)ds} f(\theta_t,\gamma(\tilde{X}_t-x,h))|\theta_0=\theta,\tilde{X}_0=x]\\
&= E[e^{-\int_0^t n(\theta_s)ds} E[f(\theta_t,\gamma(\tilde{X}_t-x,h))|\mathcal{F}_t^\theta\vee\{\tilde{X}_0=x\}]|\theta_0=\theta,\tilde{X}_0=x]\\
&= E[e^{-\int_0^t n(\theta_s)ds}\int_{\mathbb{R}^m} f(\theta_t,\gamma(z,h))\rho_{0,t}^\theta(z)dz|\theta_0=\theta,\tilde{X}_0=x]\\
&= E[\int_{\mathbb{R}^m}\sum_{ij} 1_{\{\theta_t=e_i,\theta_0=e_j\}} f(e_i,\gamma(z,h))\\
&\qquad\qquad \times E[e^{-\int_0^t n(\theta_s)ds}\rho_{0,t}^\theta(z)|\theta_t=e_i,\theta_0=e_j]dz|\theta_0=\theta,\tilde{X}_0=x]\\
&= E[\int_{\mathbb{R}^m}\sum_{ij} 1_{\{\theta_t=e_i,\theta_0=e_j\}} f(e_i,\gamma(z,h))r_{ji}(t,z)dz|\theta_0=\theta,\tilde{X}_0=x]\\
&= \int_{\mathbb{R}^m}\sum_{ij} f(e_i,\gamma(z,h))r_{ji}(t,z)p_{ji}(t)1_{\{\theta=e_j\}}dz.
\end{aligned} \tag{2.6.8}$$

We finally have

$$\begin{aligned}
&E^{t,\pi}[\int_t^T f(\theta_s,h_s)ds] = E^{t,\pi}[\int_t^T E[f(\theta_s,h_s)|\mathcal{G}_s]ds]\\
&= E^{t,\pi}[\sum_{k\geq 0}\int_t^T 1_{]\tau_k,\infty)}(s)E[E[e^{-\int_{\tau_k}^s n(\theta_u)du} f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_k},h_k))|\mathcal{G}_k\vee\sigma\{\theta_{\tau_k}\}]|\mathcal{G}_k]ds]\\
&= E^{t,\pi}[\sum_{k\geq 0} 1_{\{\tau_k<T\}}\int_{\tau_k}^T\int_{\mathbb{R}^m}\sum_{ij} f(e_i,\gamma(z,h_k))r_{ji}(s-\tau_k,z)p_{ji}(s-\tau_k)\pi_{\tau_k}^j dzds]\\
&= E^{t,\pi}[\sum_{k\geq 0}\hat{C}(\tau_k,\pi_{\tau_k},h_k)1_{\{\tau_k<T\}}].
\end{aligned} \tag{2.6.9}$$

□

Proof of statement (ii) of Lemma 2.2.1.

We start by proving that $\hat{C}(t,\pi,h)$ is Lipschitz continuouswith respect to t.

$$\begin{aligned}
\hat{C}(t,\pi,h) &= \int_t^T\int_{\mathbb{R}^m}\sum_{i,j} f(e_i,\gamma(x,h))r_{ji}(s-t,x)p_{ji}(s-t)\pi^j dxds\\
&= \int_0^{T-t}\int_{\mathbb{R}^m}\sum_{i,j} f(e_i,\gamma(x,h))r_{ji}(s,x)p_{ji}(s)\pi^j dxds.
\end{aligned} \tag{2.6.10}$$

Thus

$$\begin{aligned}
|\hat{C}(t,\pi,h)-\hat{C}(\bar{t},\pi,h)| &= |\int_{T-\bar{t}}^{T-t}\int_{\mathbb{R}^m}\sum_{i,j} f(e_i,\gamma(x,h))r_{ji}(s,x)p_{ji}(s)\pi^j dxds|\\
&\leq \|f\||t-\bar{t}|,
\end{aligned} \tag{2.6.11}$$

where $\|f\| := \sup_{e\in E,h\in\bar{H}_m}\|f(e,h)\|$. Next, let us prove that $C(t,\pi,h)$ is Lipschitz continuous with respect to π (in the metric introduced in (2.1.8)).

$$\begin{aligned} |\hat{C}(t,\pi,h)-\hat{C}(t,\bar{\pi},h)| &= |\int_0^{T-t}\int_{\mathbb{R}^m}\sum_{i,j} f(e_i,\gamma(x,h))r_{ji}(s,x)p_{ji}(s)(\pi^j-\bar{\pi}^j)dxds| \\ &\leq \|f\|T|\pi-\bar{\pi}| = \|f\|T\sum_{i=1}^N |\pi(e_i)-\bar{\pi}(e_i)| \\ &\leq \|f\|T\|\pi-\bar{\pi}\|_{TV} \leq \|f\|T\frac{2}{\log 3}d_H(\pi,\bar{\pi}), \end{aligned} \tag{2.6.12}$$

where we have used (2.1.7).

Next, let us prove that $C(t,\pi,h)$ is continuous with respect to h (always in the metric introduced in (2.1.8)). The function $f(e_i,h)$ is bounded and continuous with respect to h for all i. Furthermore, $\gamma(x,h)$ is continuous with respect to h for all $x\in\mathbb{R}^m$. Applying the dominated convergence theorem, for $h_n\subset\bar{H}_m$, s.t. $\lim_{n\to\infty} h_n = h\in\bar{H}_m$

$$\begin{aligned} \lim_{n\to\infty}\hat{C}(t,\pi,h_n) &= \int_0^{T-t}\int_{\mathbb{R}^m}\sum_{i,j}\lim_{n\to\infty} f(e_i,\gamma(x,h_n))r_{ji}(s,x)p_{ji}(s)\pi^j dxds \\ &= \int_0^{T-t}\int_{\mathbb{R}^m}\sum_{i,j} f(e_i,\gamma(x,h))r_{ji}(s,x)p_{ji}(s)\pi^j dxds \\ &= \hat{C}(t,\pi,h). \end{aligned} \tag{2.6.13}$$

$\hat{C}(t,\pi,h)$ is thus continuous with respect to each of the variables t,π,h. However, continuity in t,π is independent of the other variable. Hence, $\hat{C}(t,\pi,h)$ is a continuous function on $[0,T]\times\mathcal{S}_N\times\bar{H}_m$.

Proof of Lemma 2.3.1

Fix $n\geq 0$. Recall the definition of h_n^i given in section 2.2. Since S_t is continuous and V_t satisfies the self-financing condition, we obtain

$$h^i_{\tau_n-} = \frac{N^i_{n-1}S^i_{\tau_n-}}{V_{\tau_n-}} = \frac{N^i_{n-1}S^i_{\tau_n}}{V_{\tau_n}} = \frac{N^i_{n-1}S^i_{\tau_n}}{\sum_{i=0}^m N^i_n S^i_{\tau_n}}.$$

Using (1.2.3), (1.2.5), for *all* $k\geq 1, h\in\mathcal{A}^n, t\in[\tau_{n+k},T]$, one furthermore has

$$h^i_t = \gamma^i(\tilde{X}_t-\tilde{X}_{\tau_{n+k}},h_{n+k}) = \gamma^i(\tilde{X}_t-\tilde{X}_{\tau_n},h_n).$$

Therefore, using lemma 2.2.1 (i) for $h\in\mathcal{A}^n$

$$\begin{aligned} W(t,\pi,h.) = E[&\sum_{k=0}^{n-1}\int_{\tau_k}^{T\wedge\tau_{k+1}} f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_k},h_k))ds1_{\{\tau_k<T\}} \\ &+\sum_{k=n}^{\infty}\int_{\tau_k}^{T\wedge\tau_{k+1}} f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_k},h_k))ds1_{\{\tau_k<T\}}|\tau_0=t,\ \pi_{\tau_0}=\pi] \\ = E[&\sum_{k=0}^{n-1}\hat{C}(\tau_k,\pi_{\tau_k},h_k)1_{\{\tau_k<T\}} + \int_{\tau_n}^{T} f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_n},h_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\ \pi_{\tau_0}=\pi]. \end{aligned} \tag{2.6.14}$$

Proof of Lemma 2.4.4

By the definition of $\mathcal{A}^n$, for $n \geq 0, \mathcal{A}^n \subset \mathcal{A}^{n+1} \subset \mathcal{A}$, hence,

$$\sup_{h\in\mathcal{A}^n} W(t,\pi,h.) \leq \sup_{h\in\mathcal{A}^{n+1}} W(t,\pi,h.) \leq \sup_{h\in\mathcal{A}} W(t,\pi,h.). \tag{2.6.15}$$

By the definition of $W^n(t,\pi)$ and $W(t,\pi)$

$$W^n(t,\pi) \leq W^{n+1}(t,\pi) \leq W(t,\pi). \tag{2.6.16}$$

Using Lemma 2.4.3, for $n, m \geq 0$

$$\bar{W}^n(t,\pi) \leq \bar{W}^{n+m}(t,\pi) \leq W(t,\pi). \tag{2.6.17}$$

Letting $m \to \infty$

$$\bar{W}^n(t,\pi) \leq \bar{W}(t,\pi) \leq W(t,\pi). \tag{2.6.18}$$

Proof of Lemma 2.4.5

For $h \in \mathcal{A}$, $W(t,\pi,h)$ defined by (2.3.1) satisfies

$$\begin{aligned}
W(t,\pi,h.) &= E[\sum_{k=0}^{n-1} \hat{C}(\tau_k, \pi_{\tau_k}, h_k)1_{\{\tau_k<T\}}|\tau_0 = t, \pi_{\tau_0} = \pi] \\
&\qquad\qquad + E[\sum_{k=n}^{\infty} \hat{C}(\tau_k, \pi_{\tau_k}, h_k)1_{\{\tau_k<T\}}|\tau_0 = t, \pi_{\tau_0} = \pi] \\
&= E[\sum_{k=0}^{n-1} \hat{C}(\tau_k, \pi_{\tau_k}, h_k)1_{\{\tau_k<T\}} + \int_{\tau_n}^{T} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_n}, h_n))ds 1_{\{\tau_n<T\}} \\
&\qquad - \textstyle\int_{\tau_n}^{T} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_n}, h_n))ds 1_{\{\tau_n<T\}}|\tau_0 = t, \pi_{\tau_0} = \pi] \\
&\qquad + E[W(\tau_n, \pi_{\tau_n}, h.)1_{\{\tau_n<T\}}|\tau_0 = t, \pi_{\tau_0} = \pi]. \\
&\leq W^n(t,\pi) + |E[\textstyle\int_{\tau_n}^{T} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_n}, h_n))ds 1_{\{\tau_n<T\}}|\tau_0 = t, \pi_{\tau_0} = \pi]| \\
&\qquad + E[W(\tau_n, \pi_{\tau_n}, h.)1_{\{\tau_n<T\}}|\tau_0 = t, \pi_{\tau_0} = \pi] \\
&\leq \bar{W}^n(t,\pi) + 2\|f\|TP(\tau_n < T|\tau_0 = t).
\end{aligned} \tag{2.6.19}$$

because of the representation of $W^n(t,\pi)$ in Corollary 2.3.1 (equation (2.3.5)) and Lemma 2.4.3. Thus, by letting $n \to \infty$, we obtain

$$W(t,\pi,h.) \leq \bar{W}(t,\pi) \tag{2.6.20}$$

for all $h \in \mathcal{A}$.

Chapter 3

Power-utility maximization

Our problem of maximization of expected terminal power utility consists in determining

$$\begin{aligned}&\sup_{h\in\mathcal{A}}\frac{1}{\mu}\log E[V_T^\mu|\tau_0=0,\ \pi_0=\pi]\\&=\log v_0+\sup\nolimits_{h\in\mathcal{A}}\tfrac{1}{\mu}\log E[\tfrac{V_T^\mu}{V_0^\mu}|\tau_0=0,\ \pi_0=\pi],\end{aligned}\tag{3.0.1}$$

for $\mu<0$ as well as an optimal maximizing strategy $\hat{h}\in\mathcal{A}$.

Notice that, in order to study the optimization problem (3.0.1), it suffices to analyze the criterion function

$$W(t,\pi,h.):=\frac{1}{\mu}\log E[\frac{V_T^\mu}{V_t^\mu}|\tau_0=t,\ \pi_0=\pi].\tag{3.0.2}$$

The optimal value function will then be defined as

$$W(t,\pi):=\sup_{h\in\mathcal{A}}W(t,\pi,h.).\tag{3.0.3}$$

In section 3.1 consists mainly in estimation results and in establishing continuity properties, while section 3.2 and 3.3 contain results that will be used to obtain an approximation, of the type of “value iteration”, of the optimal value function and a Dynamic Programming principle that is specific to the given problem setting. These results serve also the purpose of obtaining a methodology to determine an optimal strategy. Section 3.5 contains auxiliary technical results.

In the sequel we shall for simplicity use mostly the shorthand notation

$$E^{t,\pi}[\cdot]\equiv E[\cdot\ |\tau_0=t,\pi_{\tau_0}=\pi]$$

and also use the following notations

$$
\begin{cases}
\bar{m} := \max_{0\le i\le m} \max_{1\le j\le N} m_i(e_j), \\
\underline{m} := \min_{0\le i\le m} \min_{1\le j\le N} m_i(e_j) \wedge 0 \quad \text{implying that } \underline{m} \le 0, \\
\bar{\sigma} := \max_{0\le i\le m} \max_{1\le j\le N} \sigma_i(e_j), \\
l(t) := E[|1-\exp(\mu|X_t - X_0|)|], \\
c := E[1_{\{\tau_1 \le T\}}], \\
\underline{n} := \min n(\theta) = \min_i n(e_i), \\
\bar{n} := \max n(\theta) = \max_i n(e_i).
\end{cases} \tag{3.0.4}
$$

3.1 Basic estimates

We start from the following representation of the criterion function.

Lemma 3.1.1. *For* $t \in [0,T], \pi \in \mathcal{S}_N$

$$
\begin{aligned}
& W(t,\pi,h_\cdot) \\
= \; & \tfrac{1}{\mu} \log E^{t,\pi}[\exp(\mu \sum_{k=1}^{\infty} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1}<T\}})],
\end{aligned} \tag{3.1.1}
$$

where,

$$
D(h,x) := \log(\sum_{i=0}^{m} h^i \exp(x^i)). \tag{3.1.2}
$$

Proof. Since $\sum_{i=0}^{m} h^i = 1$,

$$
D(h,0) = \log(\sum_{i=0}^{m} h^i) = 0 \tag{3.1.3}
$$

for $h \in \bar{H}_m$. For $k \ge 0,\ T \in [\tau_k, \tau_{k+1}]$

$$
\begin{aligned}
\frac{V_T^\mu}{V_{\tau_k}^\mu} &= (\sum_{i=0}^{m} \frac{N_T^i S_T^i}{V_{\tau_k}})^\mu = (\sum_{i=0}^{m} \frac{N_k^i S_{\tau_k}^i}{V_k} \frac{S_T^i}{S_{\tau_k}^i})^\mu = (\sum_{i=0}^{m} h_k^i \frac{S_T^i}{S_{\tau_k}^i})^\mu \\
&= (\sum_{i=0}^{m} h_k^i \exp(X_T^i - X_{\tau_k}^i))^\mu \\
&= \exp(\mu D(h_k, X_T - X_{\tau_k})).
\end{aligned} \tag{3.1.4}
$$

For $k \ge 1,\ T < \tau_k$

$$
\frac{V_{T\wedge\tau_{k+1}}^\mu}{V_{T\wedge\tau_k}^\mu} = \frac{V_T^\mu}{V_T^\mu} = 1, \tag{3.1.5}
$$

and

$$\exp(\mu D(h_k, X_{T\wedge\tau_{k+1}} - X_{T\wedge\tau_k})) = \exp(\mu D(h_k, X_T - X_T)) = 1. \tag{3.1.6}$$

Therefore, we obtain

$$\begin{aligned} E^{t,\pi}[(V_T/V_t)^\mu] &= E^{t,\pi}[\prod_{k=1}^{\infty} \frac{V^\mu_{T\wedge\tau_k}}{V^\mu_{T\wedge\tau_{k-1}}}] \\ &= E^{t,\pi}[\exp(\mu \sum_{k=1}^{\infty} D(h_{k-1}, X_{T\wedge\tau_k} - X_{T\wedge\tau_{k-1}}))] \qquad (3.1.7) \\ &= E^{t,\pi}[\exp(\mu \sum_{k=1}^{\infty} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1}<T\}})]. \end{aligned}$$

□

The representation of $W(t, \pi, h_.)$ in Lemma 3.1.1 leads us to define a function that will play a crucial role in the sequel, namely

Definition 3.1.1. *Let the function $\bar{W}^0(t, \pi, h)$ be defined as*

$$\bar{W}^0(t, \pi, h) := \frac{1}{\mu} \log E^{t,\pi}[\exp(\mu D(h, X_T - X_t))]. \tag{3.1.8}$$

For the function $\bar{W}^0(t, \pi, h)$ in the above Definition 3.1.1 we now obtain estimation and continuity results as stated in the following proposition.

Proposition 3.1.1. *For $t \in [0, T]$, $h \in \bar{H}_m$, we have the estimate:*

$$\exp((\mu\bar{m} + \frac{\mu\bar{\sigma}^2}{2})(T-t)) \le E^{t,\pi}[\exp(\mu D(h, X_T - X_t))] \le \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)), \tag{3.1.9}$$

from which it immediately follows that

$$(\underline{m} + \frac{\mu\bar{\sigma}^2}{2})(T-t) \le \bar{W}^0(t, \pi, h) \le (\bar{m} + \frac{\bar{\sigma}^2}{2})(T-t). \tag{3.1.10}$$

Furthermore, $\bar{W}^0(t, \pi, h)$ is a continuous function on $[0, T] \times \mathcal{S}_N \times \bar{H}_m$ and the following estimates hold:

$$|\ \exp(\mu\bar{W}^0(t, \pi, h)) - \exp(\mu\bar{W}^0(t, \bar{\pi}, h))| \le \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)) \frac{2}{\log 3} d_H(\pi, \bar{\pi}), \tag{3.1.11}$$

$$|\exp(\mu\bar{W}^0(t, \pi, h)) - \exp(\mu\bar{W}^0(\bar{t}, \pi, h))| \le \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)) l(t - \bar{t}) \ \ for\ \bar{t} < t, \tag{3.1.12}$$

where d_H was defined in (1.1.3).

Proof. See Section 3.5. □

We shall now introduce basic quantities that we shall use systematically throughout. The first one concerns useful function spaces, namely

Definition 3.1.2. *By $\mathcal{G}$ we denote the function space*

$$\mathcal{G} := \{g \in C([0,T] \times \mathcal{S}_N) \mid (\underline{m} + \frac{\mu\bar{\sigma}^2}{2})(T-t) \leq g(t,\pi) \leq (\bar{m} + \frac{\bar{\sigma}^2}{2})(T-t))\}. \tag{3.1.13}$$

Furthermore, we let $\mathcal{G}_1 \subset \mathcal{G}$ be the subspace

$$\mathcal{G}_1 := \left\{ g \in \mathcal{G} \mid |\exp(\mu g(t,\pi)) - \exp(\mu g(t,\bar{\pi}))| \leq \frac{2}{\log 3} d_H(\pi,\bar{\pi}) \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)) \right\} \tag{3.1.14}$$

and $\mathcal{G}_2 \subset \mathcal{G}$ be the subspace of functions $g \in \mathcal{G}$ satisfying, for $\bar{t} < t$,

$$|\exp(\mu g(t,\pi)) - \exp(\mu g(\bar{t},\pi))| \leq k(t-\bar{t}) \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)), \tag{3.1.15}$$

where $k(t)$ is some nonnegative function on $\mathbb{R}$ such that $k(0) = 0$ and continuous at 0.

The second one concerns a crucial auxiliary quantity that will play a major role in the proofs to follow, namely

Definition 3.1.3. *For each $g \in \mathcal{G}$ let $\hat{\xi} : [0,T] \times \mathcal{S}_N \times \bar{H}_m \to \mathbb{R}$ be the function defined by*

$$\hat{\xi}(t,\pi,h;g) := \frac{1}{\mu} \log E^{t,\pi}[\exp(\mu D(h, X_{T\wedge\tau_1} - X_t) + \mu 1_{\{\tau_1 < T\}} g(\tau_1, \pi_{\tau_1}))]. \tag{3.1.16}$$

We can now state and prove the following estimation result

Proposition 3.1.2. *For each $g \in \mathcal{G}$, we have the three estimates (c is as in (3.0.4))*

(i)

$$\begin{aligned} c\exp((\mu\bar{m} + \tfrac{\mu(\bar{\sigma})^2}{2})(T-t)) &\leq E^{t,\pi}[\exp\Big(\mu D(h, X_{\tau_1} - X_t) + \mu g(\tau_1, \pi_{\tau_1})\Big) 1_{\{\tau_1 \leq T\}}] \\ &\leq c\exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t)), \end{aligned} \tag{3.1.17}$$

(ii)

$$\begin{aligned} (1-c)\exp((\mu\bar{m} + \tfrac{\mu(\bar{\sigma})^2}{2})(T-t)) &\leq E^{t,\pi}[\exp\Big(\mu D(h, X_T - X_t)\Big) 1_{\{\tau_1 > T\}}] \\ &\leq (1-c)\exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t)) \end{aligned} \tag{3.1.18}$$

and

(iii)

$$(\underline{m} + \frac{\mu\bar{\sigma}^2}{2})(T-t) \leq \hat{\xi}(t,\pi,h;g) \leq (\bar{m} + \frac{\bar{\sigma}^2}{2})(T-t). \tag{3.1.19}$$

Furthermore, for all $g \in \mathcal{G}$, the function $\exp(\mu\hat{\xi}(t,\pi,h;g))$ is continuous with respect to h and for each $g \in \mathcal{G}_1$ we have

$$\begin{aligned} & |\exp(\mu\hat{\xi}(t,\pi,h;g) - \exp(\mu\hat{\xi}(t,\bar{\pi},h;g))| \\ \leq\ & \tfrac{2}{\log 3} d_H(\pi,\bar{\pi}) \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t))(1+c), \end{aligned} \tag{3.1.20}$$

while for each $g \in \mathcal{G}_2$ we have

$$\begin{aligned} & |\exp(\mu\hat{\xi}(t,\pi,h;g) - \exp(\mu\hat{\xi}(\bar{t},\pi,h;g))| \\ \leq\ & \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t))(2\tfrac{\bar{n}}{\underline{n}}(e^{\underline{n}(t-\bar{t})} - 1) + l(t-\bar{t}) + ck(t-\bar{t})). \end{aligned} \tag{3.1.21}$$

Proof. See Section 3.5. □

3.2 Basic approximation results

This section is intended to prepare for one of the two main results in Theorem 3.4.1 below, namely the approximation of the optimal value function, which involves a kind of "value iteration". We start by giving two definitions.

Definition 3.2.1. *Defining the operator J_μ on $C([0,T] \times \mathcal{S}_N)$ by*

$$J_\mu g(t,\pi) := \sup_{h \in \bar{H}_m} \hat{\xi}(t,\pi,h;g) \tag{3.2.1}$$

let, for $g \in \mathcal{G}$,

$$J_\mu^0 g(t,\pi) := g(t,\pi) \tag{3.2.2}$$

and, for $n \geq 1$,

$$J_\mu^n g(t,\pi) := J_\mu(J_\mu^{n-1} g(t,\pi)). \tag{3.2.3}$$

Definition 3.2.2. *Put*

$$\bar{W}^0(t,\pi) := \sup_{h \in \bar{H}_m} \bar{W}^0(t,\pi,h) = \sup_{h \in \bar{H}_m} \frac{1}{\mu} \log E[e^{\mu D(h, X_T - X_t)}], \tag{3.2.4}$$

and let ("value iteration")

$$\bar{W}^n(t,\pi) := J_\mu^n \bar{W}^0(t,\pi). \tag{3.2.5}$$

Then, we have the following proposition as a direct consequence of Proposition 3.1.1.

Proposition 3.2.1. *For $\bar{W}^0(t,\pi)$ defined above we have that $\bar{W}^0(t,\pi) \in \mathcal{G}_1 \cap \mathcal{G}_2$ by setting $k(t) = l(t)$.*

Proof. By (3.1.10) in Proposition 3.1.1 we see that $\bar{W}^0(t,\pi) \in \mathcal{G}$ and, by (3.1.11) and (3.1.12) in Proposition 3.1.1 we also see that it belongs to $\mathcal{G}_1 \cap \mathcal{G}_2$ with $k(t) = l(t)$. □

We have now

Lemma 3.2.1. *For $g \in \mathcal{G}_1 \cap \mathcal{G}_2$ one has that $J_\mu g(t,\pi)$ is continuous with respect to t, π and $J_\mu g \in \mathcal{G}$.*

Proof. For $a, b \geq \gamma > 0, |a-b| < \varepsilon, \varepsilon > 0$, one has

$$\begin{array}{l} \log a - \log b = \log(a/b) = \log(1 + (a-b)/b) \leq \epsilon/\gamma \\ \log b - \log a = \log(b/a) = \log(1 + (b-a)/a) \leq \epsilon/\gamma \\ \Rightarrow |\log a - \log b| \leq \epsilon/\gamma, \end{array} \tag{3.2.6}$$

where we have used the inequality $\log(1+x) \leq x$ for $x \geq 0$. We set $a = \exp(\mu\hat{\xi}(t,\pi,h;g))$, $b = \exp(\mu\hat{\xi}(t,\bar{\pi},h;g))$ and use Proposition 3.1.2(iii), setting $\gamma = \exp(\{\mu\bar{m} + \frac{\mu\bar{\sigma}^2}{2}\}(T-t))$. From (3.1.20) in Proposition 3.1.2 it then follows that

$$\begin{array}{rl} & |\hat{\xi}(t,\pi,h;g) - \hat{\xi}(t,\bar{\pi},h;g)| \\ \leq & \frac{1}{|\mu|}\frac{2}{\log 3} d_H(\pi,\bar{\pi}) \exp((\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)(\bar{\sigma})^2}{2})(T-t))(1+c) \\ \leq & \frac{1}{|\mu|}\frac{2}{\log 3} d_H(\pi,\bar{\pi}) \exp((\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)(\bar{\sigma})^2}{2})T)(1+c). \end{array} \tag{3.2.7}$$

On the other hand, by analogous reasoning, from (3.1.21) in Proposition 3.1.2 it follows that

$$\begin{array}{rcl} |\hat{\xi}(t,\pi,h;g) - \hat{\xi}(\bar{t},\pi,h;g)| & \leq & \frac{1}{|\mu|}\exp((\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)(\bar{\sigma})^2}{2})(T-t)) \\ & & \times(2\left(\frac{\bar{n}}{\underline{n}}\right)(e^{\underline{n}(t-\bar{t})} - 1) + l(t-\bar{t}) + ck(t-\bar{t})) \\ & \leq & \frac{1}{|\mu|}\exp((\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)(\bar{\sigma})^2}{2})T) \\ & & \times(2\left(\frac{\bar{n}}{\underline{n}}\right)(e^{\underline{n}(t-\bar{t})} - 1) + l(t-\bar{t}) + ck(t-\bar{t})), \end{array} \tag{3.2.8}$$

namely $\hat{\xi}(t,\pi,h;g)$ is continuous with respect to (t,π), uniformly with respect to h, and hence $J_\mu g(t,\pi)$ is continuous. Further, because of Proposition 3.1.2 (iii) we see that $J_\mu g \in \mathcal{G}$.

□

Corollary 3.2.1. *Under the assumptions of Lemma 3.2.1, $J_\mu^n g \in \mathcal{G}$ for $n \geq 0$. Furthermore, there exists a Borel function $\hat{h}^{(n)}(t,\pi)$ such that*

$$\sup_{h\in\bar{H}_m} \hat{\xi}(t,\pi,h;J_\mu^n g) = \hat{\xi}(t,\pi,\hat{h}^{(n)}(t,\pi);J_\mu^n g), \quad n \geq 0. \tag{3.2.9}$$

Proof. Similar arguments to the proof of Lemma 3.2.1 apply to see that $J_\mu^n g \in \mathcal{G}$. Moreover, since $\bar{H}_m$ is compact and $\hat{\xi}(t,\pi,h;g)$ is a bounded continuous function on $[0,T] \times \mathcal{S}_N \times \bar{H}_m$, there exists a Borel function $\hat{h}^{(0)}(t,\pi)$ such that $\sup_{h\in\bar{H}_m} \hat{\xi}(t,\pi,h;g) = \hat{\xi}(t,\pi,\hat{h}^{(0)}(t,\pi);g)$. By the same reasoning we have (3.2.9) for general $n \geq 1$.

□

In what follows we denote by $\|g\|$ the norm of a function $g \in C([0,T] \times \mathcal{S}_N)$, namely

$$\|g\| := \sup_{(t,\pi)\in[0,T]\times\mathcal{S}_N} |g(t,\pi)|. \tag{3.2.10}$$

Lemma 3.2.2. *For each $g \in \mathcal{G}$ and $n \geq 1$, we have the following estimate*

$$\begin{array}{ll} & \|J_\mu^{n+1} g(t,\pi) - J_\mu^n g(t,\pi)\| \\ \leq & \frac{c^n}{|\mu|} \exp(\{\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)\bar{\sigma}^2}{2}\}T)|1 - \exp(\mu\|J_\mu g(t,\pi) - g(t,\pi)\|)| \end{array} \tag{3.2.11}$$

where, recall (3.0.4), $c \in (0,1)$.

Proof. Let us first prove that, for $n \geq 1$,

$$|\exp\{\mu J_\mu^{n+1} g(t,\pi)\} - \exp\{\mu J_\mu^n g(t,\pi)\}| \leq c^n e^{(\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)}|1 - e^{\mu\|J_\mu g - g\|}|. \tag{3.2.12}$$

To prove it for $n = 1$, using Proposition 3.1.2 (i), we see that

$$\begin{array}{ll} & |\exp(\mu\hat{\xi}(t,\pi,h;J_\mu g)) - \exp(\mu\hat{\xi}(t,\pi,h;g))| \\ \leq & E^{t,\pi}[e^{\mu D(h,X_{\tau_1}-X_t)}|e^{\mu J_\mu g(\tau_1,\pi_{\tau_1})} - e^{\mu g(\tau_1,\pi_{\tau_1})}|1_{\{\tau_1 \leq T\}}] \\ = & E^{t,\pi}[e^{\mu D(h,X_{\tau_1}-X_t)+\mu J_\mu g(\tau_1,\pi_{\tau_1})}|1 - e^{\mu(g(\tau_1,\pi_{\tau_1})-J_\mu g(\tau_1,\pi_{\tau_1}))}|1_{\{\tau_1 \leq T\}}] \\ \leq & |1 - e^{\mu\|J_\mu g - g\|}|E^{t,\pi}[e^{\mu D(h,X_{\tau_1}-X_t)+\mu J_\mu g(\tau_1,\pi_{\tau_1})}1_{\{\tau_1 \leq T\}}] \\ \leq & ce^{\mu\underline{m}(T-t)+\frac{(\mu\bar{\sigma})^2}{2}(T-t)}|1 - e^{\mu\|J_\mu g - g\|}|. \end{array} \tag{3.2.13}$$

Then, we have

$$\begin{array}{rcl} |e^{\mu J_\mu^2 g(t,\pi)} - e^{\mu J_\mu g(t,\pi)}| & = & |e^{\mu \sup_h \hat{\xi}(t,\pi,h;J_\mu g)} - e^{\mu \sup_h \hat{\xi}(t,\pi,h;g)}| \\ & = & |\inf_h e^{\mu\hat{\xi}(t,\pi,h;J_\mu g)} - \inf_h e^{\mu\hat{\xi}(t,\pi,h;g)}| \\ & \leq & \sup_h |e^{\mu\hat{\xi}(t,\pi,h;J_\mu g)} - e^{\mu\hat{\xi}(t,\pi,h;g)}| \\ & \leq & ce^{\mu\underline{m}(T-t)+\frac{(\mu\bar{\sigma}2)}{2}(T-t)}|1 - e^{\mu\|J_\mu g - g\|}|. \end{array}$$

Assuming that (3.2.12) holds for $n-1$, we will prove it for n. Using again Proposition 3.1.2 (i)

$$\begin{array}{l} |e^{\mu J_\mu^{n+1} g(t,\pi)} - e^{\mu J_\mu^n g(t,\pi)}| = |e^{\mu \sup_h \hat{\xi}(t,\pi,h;J_\mu^n g)} - e^{\mu \sup_h \hat{\xi}(t,\pi,h;J_\mu^{n-1} g)}| \\ = |\inf_h e^{\mu\hat{\xi}(t,\pi,h;J_\mu^n g)} - \inf_h e^{\mu\hat{\xi}(t,\pi,h;J_\mu^{n-1} g)}| \\ \leq \sup_h |e^{\mu\hat{\xi}(t,\pi,h;J_\mu^n g)} - e^{\mu\hat{\xi}(t,\pi,h;J_\mu^{n-1} g)}| \\ \leq \sup_h E^{t,\pi}[e^{\mu D(h,X_{\tau_1}-X_t)}|e^{\mu J_\mu^n g(\tau_1,\pi_{\tau_1})} - e^{\mu J_\mu^{n-1} g(\tau_1,\pi_{\tau_1})}|1_{\{\tau_1 \leq T\}}] \\ \leq \sup_h E^{t,\pi}[e^{\mu D(h,X_{\tau_1}-X_t)}c^{n-1}e^{(\mu\underline{m}+\frac{(\mu\bar{\sigma})^2}{2})(T-\tau_1)}1_{\{\tau_1 \leq T\}}|1 - e^{\mu\|J_\mu g - g\|}|] \\ \leq c^n e^{(\mu\underline{m}+\frac{(\mu\bar{\sigma})^2}{2})(T-t)}|1 - e^{\mu\|J_\mu g - g\|}|. \end{array} \tag{3.2.14}$$

Thus, (3.2.12) has been proved. Now we can complete the proof. Indeed, by using (3.2.6), we have

$$\begin{array}{l} |J_\mu^{n+1} g(t,\pi) - J_\mu^n g(t,\pi)| = \frac{1}{|\mu|}|\mu J_\mu^{n+1} g(t,\pi) - \mu J_\mu^n g(t,\pi)| \\ \leq \frac{1}{|\mu|} e^{-\mu\bar{m}(T-t)-\frac{\mu\bar{\sigma}^2}{2}(T-t)}|e^{\mu J_\mu^{n+1} g(t,\pi)} - e^{\mu J_\mu^n g(t,\pi)}| \\ \leq \frac{c^n}{|\mu|} e^{\mu(\underline{m}-\bar{m})(T-t)+\frac{(\mu^2-\mu)\bar{\sigma}^2}{2}(T-t)}|1 - e^{\mu\|J_\mu g - g\|}| \\ \leq \frac{c^n}{|\mu|} e^{\mu(\underline{m}-\bar{m})T+\frac{(\mu^2-\mu)\bar{\sigma}^2}{2}T}|1 - e^{\mu\|J_\mu g - g\|}| \end{array} \tag{3.2.15}$$

and hence obtain the present lemma by taking supremum with respect to (t, π).

$\square$

Corollary 3.2.2. *For $g \in \mathcal{G}_1 \cap \mathcal{G}_2$ we have that $\{J_\mu^n g(t,\pi)\}$ is a Cauchy sequence in $\mathcal{G}$ and, therefore, $\exists \lim_{n\to\infty} J_\mu^n g(t,\pi) \in \mathcal{G}$.*

Furthermore, for each $g_1,\ g_2 \in \mathcal{G}$, we have the estimates:

$$\begin{aligned} & \|J_\mu g_1 - J_\mu g_2\| \\ \leq\ & \tfrac{c}{|\mu|} \exp(\{\mu(\underline{m} - \bar{m}) + \tfrac{\mu(\mu-1)\bar{\sigma}^2}{2}\}T)|1 - \exp(\mu\|g_1 - g_2\|)| \\ \leq\ & c \exp(\{\mu(\underline{m} - \bar{m}) + \tfrac{\mu(\mu-1)\bar{\sigma}^2}{2}\}T)\|g_1 - g_2\|. \end{aligned} \tag{3.2.16}$$

Proof. By Corollary 3.2.1, $\{J_\mu^n g(t,\pi)\} \subset \mathcal{G}$. Further, by using Lemma 3.2.2, we can see that $\{J_\mu^n g(t,\pi)\}$ is a Cauchy sequence in $\mathcal{G}$. The proof of (3.2.16) is similar to that of Lemma 3.2.2. $\square$

Proposition 3.2.2. *We have that $\{\bar{W}^n(t,\pi)\}$ given by (3.2.5) is a Cauchy sequence in $\mathcal{G}$ and therefore $\exists \lim_{n\to\infty} \bar{W}^n(t,\pi) \in \mathcal{G}$.*

Proof. Thanks to Proposition 3.2.1, $\bar{W}^0(t,\pi)$ belongs to $\mathcal{G}_1 \cap \mathcal{G}_2$ with $k(t) = l(t)$ and we see that $\{\bar{W}^n(t,\pi)\}$ is a Cauchy sequence in $\mathcal{G}$ by Corollary 3.2.2. $\square$

3.3 Limiting results

In this section we perform some passages to the limit, which will complete our preliminary analysis in view of the main result in the next section. We start by defining some relevant quantities.

Definition 3.3.1. *We set*

$$\bar{W}(t,\pi) := \lim_{n\to\infty} \bar{W}^n(t,\pi), \tag{3.3.1}$$

which is justified by the previous Proposition 3.2.2,

$$W(t,\pi) := \sup_{h \in \mathcal{A}} W(t,\pi,h.), \tag{3.3.2}$$

and

$$W^n(t,\pi) := \sup_{h \in \mathcal{A}^n} W(t,\pi,h.), \tag{3.3.3}$$

where $W(t,\pi,h.)$ is the criterion function defined in (3.0.2).

The next lemma particularizes the representation result of Lemma 3.1.1.

Lemma 3.3.1. *For all $n \geq 0$ and $h \in \mathcal{A}^n$, we have the following equation*

$$\begin{aligned} W(t,\pi,h.) &= \tfrac{1}{\mu}\log E^{t,\pi}[\exp\Big(\mu\sum_{k=1}^{n} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}} \\ &\quad + \mu D(h_n, X_T - X_{\tau_n})1_{\{\tau_n<T\}}\Big)] \\ &= \tfrac{1}{\mu}\log E^{t,\pi}[\exp\Big(\mu\sum_{k=1}^{n} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}} \\ &\quad + \mu\bar{W}^0(\tau_n,\pi_{\tau_n},h_n)1_{\{\tau_n\leq T\}}\Big)]. \end{aligned} \tag{3.3.4}$$

Proof. By Lemma 3.1.1 and the definition of $\bar{W}^0(t,\pi,h)$ it suffices to prove the following equation for all $n > 0,\ k \geq n+1,\ h \in \mathcal{A}^n$. For $\tau_k < T \leq \tau_{k+1}$

$$\begin{aligned} &\exp(\mu\sum_{i=n}^{k-1} D(h_i, X_{\tau_{i+1}} - X_{\tau_i}) + \mu D(h_k, X_T - X_{\tau_k})) \\ =\ &\exp(\mu D(h_n, X_T - X_{\tau_n})). \end{aligned} \tag{3.3.5}$$

It can be seen as follows.

$$\begin{aligned} &\exp(\mu\sum_{i=n}^{k-1} D(h_i, X_{\tau_{i+1}} - X_{\tau_i}) + \mu D(h_k, X_T - X_{\tau_k})) \\ =\ &\prod_{i=n}^{k-1}(\sum_{j=0}^{m} h_i^j\frac{S^j_{\tau_{i+1}}}{S^j_{\tau_i}})^{\mu}(\sum_{j=0}^{m} h_k^j\frac{S^j_T}{S^j_{\tau_k}})^{\mu} = \prod_{i=n}^{k-1}(\sum_{j=0}^{m}\frac{N_i^j S^j_{\tau_{i+1}}}{V_{\tau_i}})^{\mu}(\sum_{j=0}^{m}\frac{N_k^j S^j_T}{V_{\tau_k}})^{\mu} \\ =\ &\prod_{i=n}^{k-1}(\sum_{j=0}^{m}\frac{N^j_{i+1}S^j_{\tau_{i+1}}}{V_{\tau_i}})^{\mu}(\sum_{j=0}^{m}\frac{N_k^j S_T^j}{V_{\tau_k}})^{\mu} = \prod_{i=n}^{k-1}(\frac{V_{\tau_{i+1}}}{V_{\tau_i}})^{\mu}(\sum_{j=0}^{m}\frac{N_T^j S_T^j}{V_{\tau_k}})^{\mu} \\ =\ &(\frac{N_T S_T}{V_{\tau_n}})^{\mu} = (\frac{V_T}{V_{\tau_n}})^{\mu} = \exp(\mu D(h_n, X_T - X_{\tau_n})), \end{aligned} \tag{3.3.6}$$

where we have used (3.1.4), the definition of $\mathcal{A}^n$ and the self financing property of the investment strategy. □

Corollary 3.3.1. *We have the following equation*

$$\begin{aligned} W^n(t,\pi) &= \sup_{h\in\mathcal{A}^n}\frac{1}{\mu}\log E^{t,\pi}[\exp\Big(\mu\sum_{k=1}^{n} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}} \\ &\quad + \mu\bar{W}^0(\tau_n,\pi_{\tau_n},h_n)1_{\{\tau_n\leq T\}}\Big)], \end{aligned} \tag{3.3.7}$$

for $n \geq 0,\ t \in [0,T],\ \pi \in \mathcal{S}_N$.

Proposition 3.3.1. *For each $n \geq 0$, we have*

$$\bar{W}^n(t,\pi) = W^n(t,\pi). \tag{3.3.8}$$

Furthermore,

$$\bar{W}(t,\pi) = W(t,\pi). \tag{3.3.9}$$

Proof. See Section 3.5. □

The next proposition is in the spirit of a Dynamic Programming principle, namely

Proposition 3.3.2. *We have $W = J_\mu W$. Namely, $W(t,\pi)$ satisfies the following equation*

$$\begin{aligned}&W(t,\pi)\\&= \sup_{h\in\bar{H}_m} \frac{1}{\mu} \log E^{t,\pi}[\exp(\mu D(h, X_{T\wedge\tau_1} - X_t) + \mu W(\tau_1, \pi_{\tau_1}) 1_{\{\tau_1 \le T\}})].\end{aligned} \tag{3.3.10}$$

Proof. We have, by using (3.2.16),

$$\begin{aligned}\|W - J_\mu W\| &\le \|W - J_\mu \bar{W}^n\| + \|J_\mu \bar{W}^n - J_\mu W\|\\&\le \|W - \bar{W}^{n+1}\| + C_1 \|\bar{W}^n - W\|,\end{aligned}$$

where $C_1 = c\exp(\{\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)\bar{\sigma}^2}{2}\}T)$. Hence, by sending n to ∞, we see that $\|W - J_\mu W\| = 0$. □

3.4 Main Theorem

From the preliminary analysis in previous Section we obtain now the main result in this chapter, namely an approximation result and a Dynamic Programming-type principle for the power-utility maximization problem.

Theorem 3.4.1.

(i) Approximation theorem
$\bar{W}^n$ computed according to (3.2.5) in Definition 3.2.2 are approximations to the solution of the original problem in the sense that, for any $\epsilon > 0, n > n_\epsilon$,

$$\|W - \bar{W}^n\| < \epsilon, \tag{3.4.1}$$

where,

$$n_\epsilon := \frac{\log(1-c)|\mu| + \log\epsilon - \log|1 - \exp(\mu\|J^1_\mu \bar{W}^0 - J^0_\mu \bar{W}^0\|)| - \{\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)\bar{\sigma}^2}{2}\}T}{\log c}.$$

(ii) Dynamic programming principle:
for $n \ge 0$

$$\begin{aligned}&W(t,\pi)\\&= \sup_{h\in\mathcal{A}^n} \frac{1}{\mu} \log E^{t,\pi}[\exp(\mu \sum_{k=1}^{n} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}}\\&+\mu W(\tau_n, \pi_{\tau_n}) 1_{\{\tau_n \le T\}})].\end{aligned} \tag{3.4.2}$$

(iii) Optimal value and optimal strategy for the Power Utility Maximization Problem For the utility maximization under initial condition $V_0 = v_0, \tau_0 = 0, \pi_0 = \pi$ we have

$$\begin{aligned}\sup_{h\in\mathcal{A}} \frac{1}{\mu}\log E^{0,\pi}[V_T^\mu] &= \log v_0 \\ &+\tfrac{1}{\mu}\log E^{0,\pi}[\exp(\mu\sum_{k=1}^{\infty} D(\hat{h}_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}})],\end{aligned} \tag{3.4.3}$$

where the optimal strategy in the k -th period, namely $\hat{h}_k$, is given by

$$\hat{h}_k = \hat{h}(\tau_k, \pi_{\tau_k}) \tag{3.4.4}$$

with $\hat{h}(\tau, \pi)$ defined by

$$\begin{aligned}&\sup_{h\in\bar{H}_m} \frac{1}{\mu}\log E^{t,\pi}[\exp(\mu D(h, X_{T\wedge\tau_1} - X_t) + \mu W(\tau_1, \pi_{\tau_1})1_{\{\tau_1\le T\}})] \\ =\ &\tfrac{1}{\mu}\log E^{t,\pi}[\exp(\mu D(\hat{h}(t,\pi), X_{T\wedge\tau_1} - X_t) + \mu W(\tau_1, \pi_{\tau_1})1_{\{\tau_1\le T\}})].\end{aligned} \tag{3.4.5}$$

Proof. First, we prove (i). For any n

$$\begin{aligned}\|\bar{W} - \bar{W}^n\| &= \|\lim_{k\to\infty}\bar{W}^{n+k} - \bar{W}^n\| = \lim_{k\to\infty}\|\bar{W}^{n+k} - \bar{W}^n\| \\ &\le \lim_{k\to\infty}\sum_{i=0}^{k-1}\|\bar{W}^{n+i+1} - \bar{W}^{n+i}\| = \lim_{k\to\infty}\sum_{i=0}^{k-1}\|J_\mu^{n+i+1}\bar{W}^0 - J_\mu^{n+i}\bar{W}^0\| \\ &\le \tfrac{1}{|\mu|}|1-\exp(\mu\|J_\mu^1\bar{W}^0 - J_\mu^0\bar{W}^0\|)|\exp(\{\mu(\underline{m}-\bar{m}) + \tfrac{\mu(\mu-1)\bar{\sigma}^2}{2}\}T)\sum_{i=0}^{\infty}c^{i+n} \\ &= \tfrac{1}{|\mu|}|1-\exp(\mu\|J_\mu^1\bar{W}^0 - J_\mu^0\bar{W}^0\|)|\exp(\{\mu(\underline{m}-\bar{m}) + \tfrac{\mu(\mu-1)\bar{\sigma}^2}{2}\}T)c^n\sum_{i=0}^{\infty}c^{i} \\ &= \tfrac{c^n}{(1-c)|\mu|}|1-\exp(\mu\|J_\mu^1\bar{W}^0 - J_\mu^0\bar{W}^0\|)|\exp(\{\mu(\underline{m}-\bar{m}) + \tfrac{\mu(\mu-1)\bar{\sigma}^2}{2}\}T),\end{aligned} \tag{3.4.6}$$

where we have used Lemma 3.2.2.

Next, we prove (ii). Proceeding analogously to the proof of Lemma 3.5.5, for $n, m \in \mathbf{N}$, we take a sequence of functions $\tilde{h}^{n,k}(t,\pi),\ k = 0, 1, \ldots, n$ such that

$$\sup_{h\in H^m}\hat{\xi}(t,\pi,h;J_\mu^{n+m-k-1}\bar{W}^0) = \hat{\xi}(t,\pi,\tilde{h}^{n,k}(t,\pi);J_\mu^{n+m-k-1}\bar{W}^0)$$

and set

$$\tilde{h}_k^{n,k} := \tilde{h}^{n,k}(\tau_k, \pi_{\tau_k}),\quad k = 0, 1, \ldots, n$$

and

$$\tilde{h}_k^{n,k} = \gamma(\tilde{X}_{\tau_k} - \tilde{X}_{\tau_n}, \tilde{h}_n^{n,n}),\quad k \ge n+1.$$

Then, $\tilde{h}^{(n)} = \{\tilde{h}_k^{n,k}\}_k \in \mathcal{A}^n$. Therefore, similarly to the proof of Lemma 3.5.5, it follows that

$$\begin{aligned}\bar{W}^{n+m} &= \tfrac{1}{\mu}\log E^{t,\pi}[\exp\{\mu\textstyle\sum_{k=1}^n D(\tilde{h}^{n,k-1}(\tau_{k-1}, \pi_{\tau_{k-1}}), X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}\le T\}} \\ &\quad +\mu J_\mu^m\bar{W}^0(\tau_n, \pi_{\tau_n})1_{\{\tau_n\le T\}}\}].\end{aligned} \tag{3.4.7}$$

Since $J_\mu^m \bar{W}^0 \leq W$, we have

$$\begin{aligned}
\bar{W}^{n+m} &\leq \tfrac{1}{\mu}\log E^{t,\pi}[\exp\{\mu \textstyle\sum_{k=1}^n D(\tilde{h}^{n,k-1}(\tau_{k-1},\pi_{\tau_{k-1}}), X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}\leq T\}} \\
&\qquad +\mu W(\tau_n,\pi_{\tau_n})1_{\{\tau_n\leq T\}}\}] \\
&\leq \sup_{h\in\mathcal{A}^n} \tfrac{1}{\mu}\log E^{t,\pi}[\exp\{\mu \textstyle\sum_{k=1}^n D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}\leq T\}} \\
&\qquad +\mu W(\tau_n,\pi_{\tau_n})1_{\{\tau_n\leq T\}}\}].
\end{aligned} \tag{3.4.8}$$

Therefore, we obtain

$$\begin{aligned}
W(t,\pi) &\leq \sup_{h\in\mathcal{A}^n} \tfrac{1}{\mu}\log E^{t,\pi}[\exp\{\mu \textstyle\sum_{k=1}^n D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}\leq T\}} \\
&\qquad +\mu W(\tau_n,\pi_{\tau_n})1_{\{\tau_n\leq T\}}\}],
\end{aligned} \tag{3.4.9}$$

by letting $m \to \infty$. On the other hand, for each $h \in \mathcal{A}^n$, it follows that

$$\begin{aligned}
\bar{W}^{n+m}(t,\pi) &\geq \tfrac{1}{\mu}\log E^{t,\pi}[\exp\{\mu \textstyle\sum_{k=1}^n D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}\leq T\}} \\
&\qquad +\mu J_\mu^m \bar{W}^0(\tau_n,\pi_{\tau_n})1_{\{\tau_n\leq T\}}\}].
\end{aligned} \tag{3.4.10}$$

Then, by letting $m \to \infty$, we have

$$\begin{aligned}
W(t,\pi) &\geq \tfrac{1}{\mu}\log E^{t,\pi}[\exp\{\mu \textstyle\sum_{k=1}^n D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}\leq T\}} \\
&\qquad +\mu W(\tau_n,\pi_{\tau_n})1_{\{\tau_n\leq T\}}\}].
\end{aligned} \tag{3.4.11}$$

Hence,

$$\begin{aligned}
W(t,\pi) &\geq \sup_{h\in\mathcal{A}^n} \tfrac{1}{\mu}\log E^{t,\pi}[\exp\{\mu \textstyle\sum_{k=1}^n D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}\leq T\}} \\
&\qquad +\mu W(\tau_n,\pi_{\tau_n})1_{\{\tau_n\leq T\}}\}]
\end{aligned} \tag{3.4.12}$$

and thus we obtain (ii).

Part (iii) is an immediate consequence of the previous results in particular of the proof of point (ii) of this same theorem. □

3.5 Proof of Lemma

Proof of Proposition 3.1.1.

The proof is contained in the following two lemmas.

Lemma 3.5.1. *For $t \in [0,T]$, $h \in \bar{H}_m$, we have the estimate (3.1.9).*

Proof. Since x^μ is convex, Jensen's inequality applies and we obtain

$$E^{t,\pi}[\Big(\sum_{i=0}^m h^i \exp(X_T^i - X_t^i)\Big)^\mu] \leq E^{t,\pi}[\sum_{i=0}^m h^i \exp(\mu(X_T^i - X_t^i))]. \tag{3.5.1}$$

For each i and $t \in [0,T]$,

$$\begin{aligned} &\underline{m}(T-t) \leq \int_t^T m_i(\theta_s)ds \leq \bar{m}(T-t), \\ &\int_t^T \sigma_i^2(\theta_s)ds \leq \bar{\sigma}^2(T-t). \end{aligned} \tag{3.5.2}$$

Thus, we have

$$\begin{aligned} E^{t,\pi}[\exp(\mu(X_T^i - X_t^i))] &= E^{t,\pi}[\exp(\mu \int_t^T m_i(\theta_s)ds + \tfrac{\mu^2}{2} \int_t^T \sigma_i^2(\theta_s)ds)] \\ &\leq \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t)), \end{aligned} \tag{3.5.3}$$

and

$$E^{t,\pi}[\exp(\mu(X_T^0 - X_t^0))] = \exp(\mu r_0(T-t)) \leq \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)). \tag{3.5.4}$$

Therefore, from (3.5.1) it follows that

$$\begin{aligned} E^{t,\pi}[\Big(\sum_{i=0}^m h^i \exp(X_T^i - X_t^i)\Big)^\mu] &\leq \sum_{i=0}^m h^i \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)) \\ &= \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t)). \end{aligned} \tag{3.5.5}$$

To obtain the lower estimate, applying Jensen's inequality yields

$$(E^{t,\pi}[\sum_{i=0}^m h^i \exp(X_T^i - X_t^i)])^\mu \leq E^{t,\pi}[\Big(\sum_{i=0}^m h^i \exp(X_T^i - X_t^i)\Big)^\mu]. \tag{3.5.6}$$

Since x^μ is a decreasing function, we have

$$\begin{aligned} \Big(\sum_{i=0}^m h^i E^{t,\pi}[\exp(X_T^i - X_t^i)]\Big)^\mu &= \Big(\sum_{i=0}^m h^i E^{t,\pi}[\exp(\int_t^T m_i(\theta_s)ds + \frac{1}{2}\int_t^T \sigma_i^2(\theta_s)ds)]\Big)^\mu \\ &\geq \Big(\sum_{i=0}^m h^i \exp((\bar{m} + \frac{(\bar{\sigma})^2}{2})(T-t))\Big)^\mu \\ &= \exp((\mu\bar{m} + \tfrac{\mu(\bar{\sigma})^2}{2})(T-t)). \end{aligned} \tag{3.5.7}$$

□

Lemma 3.5.2. *$\bar{W}^0(t,\pi,h)$ in Definition 3.1.1 (see (3.1.8)) is a continuous function on $[0,T] \times \mathcal{S}_N \times \bar{H}_m$ and the estimates (3.1.11), (3.1.12) hold.*

Proof. Let us first prove the continuity of $\bar{W}^0(t,\pi,h)$ with respect to π. Owing to (2.1.7) and recalling $p_{ji}(t)$ that was defined in (1.3.9), we have

$$
\begin{aligned}
& |\exp(\mu\bar{W}^0(t,\pi,h)) - \exp(\mu\bar{W}^0(t,\bar{\pi},h))| \\
= \;& |\textstyle\int_{\mathbb{R}^m}(h^0\exp(r_0(T-t)) + \displaystyle\sum_{i=1}^m h^i\exp(y^i))^\mu \\
& \times \sum_{ij} E[\rho^\theta_{0,T-t}(y)|\theta_0 = e_j, \theta_{T-t} = e_i]p_{ji}(T-t)(\pi^j - \bar{\pi}^j)dyds| \\
= \;& |\sum_j \exp(\mu\bar{W}^0(t,e_j,h))(\pi^j - \bar{\pi}^j)| \qquad (3.5.8) \\
\leq \;& \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t)) \sum_j |\pi^j - \bar{\pi}^j| \\
\leq \;& \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t)) \|\pi^j - \bar{\pi}^j\|_{TV} \\
\leq \;& \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t)) \tfrac{2}{\log 3} d_H(\pi,\bar{\pi}).
\end{aligned}
$$

Next, we show the continuity of $\bar{W}^0(t,\pi,h)$ with respect to t. First notice that, due to the time homogeneity of the process (X_t,θ_t),

$$
\begin{aligned}
\exp(\mu\bar{W}^0(t,\pi,h)) &= E^{t,\pi}[\exp(\mu D(h, X_T - X_t))] \\
&= E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0))].
\end{aligned} \qquad (3.5.9)
$$

Notice furthermore that

$$|D(h,x) - D(h,y)| \leq |x-y| \qquad (3.5.10)$$

holds because

$$|\nabla_x D(h,x)| \leq 1. \qquad (3.5.11)$$

Therefore,

$$
\begin{aligned}
& |\exp(\mu\bar{W}^0(t,\pi,h)) - \exp(\mu\bar{W}^0(\bar{t},\pi,h))| \\
= \;& |E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0)) - \exp(\mu D(h, X_{T-\bar{t}} - X_0))]| \\
= \;& |E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0)) \Big(1 - \exp\big(\mu(D(h, X_{T-\bar{t}} - X_0) - D(h, X_{T-t} - X_0))\big)\Big)] \\
\leq \;& E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0))|1 - \exp(\mu|X_{T-\bar{t}} - X_{T-t}|)|] \\
= \;& E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0))E[|1 - \exp(\mu|X_{T-\bar{t}} - X_{T-t}|)||X_{T-t}]] \\
= \;& E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0))]l(t-\bar{t}) \\
\leq \;& \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t))l(t-\bar{t}).
\end{aligned} \qquad (3.5.12)
$$

Finally, we prove the continuity with respect to h. By the definition of $D(h,x)$ and Jensen's inequality,

$$\exp(\mu D(h,x)) = (\sum_{i=0}^m h^i\exp(x^i))^\mu \leq \sum_{i=0}^m h^i\exp(\mu x^i) \leq \sum_{i=0}^m \exp(\mu x^i). \qquad (3.5.13)$$

Therefore, for $m \geq 1$

$$\exp(\mu D(h, X_T - X_t)) \leq \sum_{i=0}^{m} \exp(\mu(X_T^i - X_t^i)). \tag{3.5.14}$$

Since $X_T - X_t = \{X_T^i - X_t^i\}_{i=1,\dots,m}$ is, conditionally on $\mathcal{F}^\theta$, Gaussian with mean $\{\int_t^T m_i(\theta_s)ds\}_{i=1,\dots,m}$ and covariance $\{\int_t^T (\sigma\sigma^*)^{ij}(\theta_s)ds\}_{i,j=1,\dots,m}$ we have

$$E^{t,\pi}[\exp(\mu(X_T^i - X_t^i))] < \infty. \tag{3.5.15}$$

Then, applying the dominated convergence theorem, for $h_j \subset \bar{H}_m$, s.t. $\lim_{j\to\infty} h_j = h \in \bar{H}_m$

$$\begin{aligned} \lim_{j\to\infty} \bar{W}^0(t,\pi,h_j) &= \tfrac{1}{\mu} \log E^{t,\pi}[\lim_{j\to\infty} \exp(\mu D(h_j, X_T - X_t))] \\ &= \tfrac{1}{\mu} \log E^{t,\pi}[\exp(\mu D(h, X_T - X_t))] \\ &= \bar{W}^0(t,\pi,h). \end{aligned} \tag{3.5.16}$$

□

Proof of Proposition 3.1.2.

Again, the proof is contained in the following two lemmas.

Lemma 3.5.3. *For each $g \in \mathcal{G}$, we have the three estimates (3.1.17),(3.1.18) and (3.1.19).*

Proof. Let us first set

$$I_1 = E^{t,\pi}[\exp\Big(\mu D(h, X_{\tau_1} - X_t) + \mu g(\tau_1, \pi_{\tau_1})\Big) 1_{\{\tau_1 \leq T\}}]$$

and

$$I_2 = E^{t,\pi}[\exp\Big(\mu D(h, X_T - X_t)\Big) 1_{\{\tau_1 > T\}}].$$

Recall also that $n(\theta_t)$ is the intensity of the Cox process describing the observations and that the dynamics of the filter process π_t was given in Corollary 1.3.1 in terms of the function $M(t,x,\pi)$.
(i) (estimate (3.1.17)). Since $g \in \mathcal{G}$, from the definition of $\rho_{t,T}^\theta(z)$ in (1.1.13) and from (3.5.2)

we obtain

$$
\begin{aligned}
I_1 &= E^{t,\pi}[\textstyle\int_t^T \int_{\mathbb{R}^m} (h^0 \exp(r_0(s-t)) + \displaystyle\sum_{i=1}^m h^i \exp(z^i))^\mu \exp\Big(\mu g(s, M(s-t,z,\pi))\Big) \\
&\quad \times \rho_{t,s}^\theta(z) n(\theta_s) \exp(-\textstyle\int_t^s n(\theta_u)du) dsdz] \\
&\geq E^{t,\pi}[\textstyle\int_t^T \int_{\mathbb{R}^m} \Big(h^0 \exp(r_0(s-t)) + \displaystyle\sum_{i=1}^m h^i \exp(z^i)\Big)^\mu \rho_{t,s}^\theta(z) dz \\
&\qquad \times\ \exp((\mu\bar{m} + \tfrac{\mu\bar{\sigma}^2}{2})(T-s)) n(\theta_s) \exp(-\textstyle\int_t^s n(\theta_u)du) ds] \\
&\geq E^{t,\pi}[\textstyle\int_t^T (h^0 \exp(r_0(s-t)) + \displaystyle\sum_{i=1}^m h^i \int_{\mathbb{R}^m} \exp(z^i) \rho_{t,s}^\theta(z) dz)^\mu \\
&\qquad \times\ \exp((\mu\bar{m} + \tfrac{\mu\bar{\sigma}^2}{2})(T-s)) n(\theta_s) \exp(-\textstyle\int_t^s n(\theta_u)du) ds] \\
&\geq E^{t,\pi}[\textstyle\int_t^T \exp((\mu\bar{m} + \tfrac{\mu\bar{\sigma}^2}{2})(s-t)) \\
&\qquad \times\ \exp((\mu\bar{m} + \tfrac{\mu\bar{\sigma}^2}{2})(T-s)) n(\theta_s) \exp(-\textstyle\int_t^s n(\theta_u)du) ds] \\
&= \exp((\mu\bar{m} + \tfrac{\mu(\bar{\sigma})^2}{2})(T-t)) E^{t,\pi}[1_{\{\tau_1 \leq T\}}],
\end{aligned}
\tag{3.5.17}
$$

by using Jensen's inequality. On the other hand, we obtain

$$
\begin{aligned}
I_1 &\leq E^{t,\pi}[\textstyle\int_t^T \int_{\mathbb{R}^m} (h^0 \exp(r_0(s-t)) + \displaystyle\sum_{i=1}^m h^i \exp(z^i))^\mu \rho_{t,s}^\theta(z) dz \\
&\qquad \times\ \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-s)) n(\theta_s) \exp(-\textstyle\int_t^s n(\theta_u)du) ds] \\
&\leq E^{t,\pi}[\textstyle\int_t^T \Big(h^0 \exp(\mu r_0(s-t)) + \displaystyle\sum_{i=1}^m h^i \int_{\mathbb{R}^m} \exp(\mu z^i) \rho_{t,s}^\theta(z) dz\Big) \\
&\qquad \times\ \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-s)) n(\theta_s) \exp(-\textstyle\int_t^s n(\theta_u)du) ds] \\
&\leq E^{t,\pi}[\textstyle\int_t^T \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(s-t)) \\
&\qquad \times\ \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-s)) n(\theta_s) \exp(-\textstyle\int_t^s n(\theta_u)du) ds] \\
&= \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t)) E^{t,\pi}[1_{\{\tau_1 \leq T\}}],
\end{aligned}
\tag{3.5.18}
$$

again by using Jensen's inequality and (3.5.2).

(ii) (estimate (3.1.18)). By using Jensen's inequality, we have

$$
\begin{aligned}
I_2 &= E^{t,\pi}[\textstyle\int_{\mathbb{R}_m}(h^0\exp(r_0(T-t)) \\
&\qquad + \displaystyle\sum_{i=1}^m h^i \exp(z^i))^\mu \rho^\theta_{t,T}(z)dz \int_T^\infty n(\theta_s)\exp(-\int_t^s n(\theta_u)du)ds] \\
&\geq E^{t,\pi}[(h^0\exp(r_0(T-t)) \\
&\qquad + \sum_{i=1}^m h^i \int_{\mathbb{R}^m} \exp(z^i)\rho^\theta_{t,T}(z)dz)^\mu \int_T^\infty n(\theta_s)\exp(-\int_t^s n(\theta_u)du)ds] \\
&\geq E^{t,\pi}[\{\sum_{i=0}^m h^i \exp((\bar{m}+\frac{(\bar{\sigma})^2}{2})(T-t)\}^\mu \int_T^\infty n(\theta_s)\exp(-\int_t^s n(\theta_u)du)ds] \\
&\geq E^{t,\pi}[\exp((\bar{m}+\tfrac{(\bar{\sigma})^2}{2})(T-t))^\mu \textstyle\int_T^\infty n(\theta_s)\exp(-\int_t^s n(\theta_u)du)ds] \\
&= \exp((\mu\bar{m}+\tfrac{\mu(\bar{\sigma})^2}{2})(T-t))E^{t,\pi}[1_{\{\tau_1>T\}}],
\end{aligned}
\tag{3.5.19}
$$

from (3.5.2), since the function x^μ is decreasing. On the other hand, by using Jensen's inequality, we have

$$
\begin{aligned}
I_2 &= E^{t,\pi}[\textstyle\int_{\mathbb{R}^m}(h^0\exp(r_0(T-t)) \\
&\qquad + \displaystyle\sum_{i=1}^m h^i \exp(z^i))^\mu \rho^\theta_{t,T}(z)dz \int_T^\infty n(\theta_s)\exp(-\int_t^s n(\theta_u)du)ds] \\
&\leq E^{t,\pi}[h^0\exp(\mu r_0(T-t)) \\
&\qquad + \sum_{i=1}^m h^i \int_{\mathbb{R}^m} \exp(\mu z^i)\rho^\theta_{t,T}(z)dz \int_T^\infty n(\theta_s)\exp(-\int_t^s n(\theta_u)du)ds] \\
&= E^{t,\pi}[\sum_{i=0}^m h^i \exp(\mu\int_t^T m_i(\theta_s)ds + \frac{\mu^2}{2}\int_t^T \sigma_i^2(\theta_s)ds) \\
&\qquad \times \textstyle\int_T^\infty n(\theta_s)\exp(-\int_t^s n(\theta_u)du)ds] \\
&\leq E^{t,\pi}[\exp((\mu\underline{m}+\tfrac{(\mu\bar{\sigma})^2}{2})(T-t))\textstyle\int_T^\infty n(\theta_s)\exp(-\int_t^s n(\theta_u)du)ds] \\
&= \exp((\mu\underline{m}+\tfrac{(\mu\bar{\sigma})^2}{2})(T-t))E^{t,\pi}[1_{\{\tau_1>T\}}],
\end{aligned}
\tag{3.5.20}
$$

because of (3.5.2).

(iii) (estimate (3.1.19)). Since

$$\hat{\xi}(t,\pi,h;g) = \frac{1}{\mu}\log(I_1+I_2). \tag{3.5.21}$$

The estimate (3.1.19) follows from (i) and (ii). $\square$

Lemma 3.5.4. *For all $g \in \mathcal{G}$, the function $\exp(\mu\hat{\xi}(t,\pi,h;g))$is continuous with respect to h. Furthermore, for each $g \in \mathcal{G}_1$ the relation (3.1.20) holds and for each $g \in \mathcal{G}_2$ the relation (3.1.21) holds.*

Proof. Let us first prove the continuity of $\exp(\mu\hat{\xi}(t,\pi,h;g))$. From (3.5.13), we have for $m \geq 1$

$$\exp(\mu D(h, X_{T\wedge\tau_1} - X_t) + \mu g(\tau_1, \pi_{\tau_1})) \leq \sum_{i=0}^{m} \exp(\mu(X^i_{T\wedge\tau_1} - X^i_t) + \mu g(\tau_1, \pi_{\tau_{\tau_1}})). \qquad (3.5.22)$$

Similarly to (3.5.15), we have for each i

$$E^{t,\pi}[\exp(\mu(X^i_{T\wedge\tau_1} - X^i_t) + \mu g(\tau_1, \pi_{\tau_1}))] < \infty. \qquad (3.5.23)$$

Applying the dominated convergence theorem, for $h_n \subset \bar{H}_m$, s.t. $\lim_{n\to\infty} h_n = h \in \bar{H}_m$

$$\begin{aligned}
&\lim_{n\to\infty} \exp(\mu\hat{\xi}(t,\pi,h_n;g)) \\
= \; & E^{t,\pi}[\lim_{n\to\infty} \exp(\mu D(h_n, X_{T\wedge\tau_1} - X_t) + \mu g(\tau_1, \pi_{\tau_1}) 1_{\{\tau_1 \leq T\}})] \qquad (3.5.24)\\
= \; & \exp(\mu\hat{\xi}(t,\pi,h;g)).
\end{aligned}$$

We next prove that, for $g \in \mathcal{G}_1$, the relation (3.1.20) holds. For this purpose, recalling Corollary 1.3.1, we rewrite

$$\begin{aligned}
&\exp(\mu\hat{\xi}(t,\pi,h;g)) \\
&= E^{t,\pi}[\exp\Big(\mu D(h, X_{T\wedge\tau_1} - X_t) + \; \mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \pi)) 1_{\{\tau_1 \leq T\}}\Big)] \\
&= \sum_j E^{t,e_j}[\exp\Big(\mu D(h, X_{T\wedge\tau_1} - X_t) \qquad (3.5.25)\\
&+ \mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \pi)) 1_{\{\tau_1 \leq T\}}\Big)]\pi^j.
\end{aligned}$$

Then, recalling the Definition 3.1.16 of $\hat{\xi}(\cdot)$, from (3.1.19) in Proposition 3.1.2 and (2.1.7) it follows that

$$\begin{aligned}
& |\exp(\mu\hat{\xi}(t,\pi,h;g)) - \exp(\mu\hat{\xi}(t,\bar{\pi},h;g))| \\
= \; & |\sum_j E^{t,e_j}[\exp\Big(\mu D(h, X_{T\wedge\tau_1} - X_t) \\
& + \; \mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \pi)) 1_{\{\tau_1 \leq T\}}\Big)](\pi^j - \bar{\pi}^j) \\
& + \; \sum_j E^{t,e_j}[\exp(\mu D(h, X_{\tau_1} - X_t))\{\exp\Big(\mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \pi))\Big) \qquad (3.5.26)\\
& - \; \exp\Big(\mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \bar{\pi}))\Big)\} 1_{\{\tau_1 \leq T\}}]\bar{\pi}^j| \\
\leq \; & \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t)) \tfrac{2}{\log 3} d_H(\pi, \bar{\pi}) \\
& + \; E^{t,\bar{\pi}}[\exp(\mu D(h, X_1 - X_t)) |\exp\Big(\mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \pi))\Big) \\
& - \; \exp\Big(\mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \bar{\pi}))\Big)| 1_{\{\tau_1 \leq T\}})].
\end{aligned}$$

Furthermore, by the definition the definition of $\mathcal{G}_1$ (see (3.1.14) in Definition 3.1.2), using also(3.1.9)

$$\begin{aligned}
&|\exp\Big(\mu g(\tau_1, M(\tau_1-t, X_{\tau_1}-X_t,\pi))\Big)\\
&-\ \exp\Big(\mu g(\tau_1, M(\tau_1-t, X_{\tau_1}-X_t,\bar\pi))\Big)|\\
&\le \exp((\mu\underline{m}+\tfrac{(\mu\bar\sigma)^2}{2})(T-\tau_1))\\
&\times \tfrac{2}{\log 3} d_H(M(\tau_1-t, X_{\tau_1}-X_t,\pi), M(\tau_1-t, X_{\tau_1}-X_t,\bar\pi))\\
&\le \tfrac{2}{\log 3} d_H(\pi,\bar\pi)\exp((\mu\underline{m}+\tfrac{(\mu\bar\sigma)^2}{2})(T-\tau_1)).
\end{aligned} \tag{3.5.27}$$

Therefore, we obtain

$$\begin{aligned}
&|\exp(\mu\hat\xi(t,\pi,h;g))-\exp(\mu\hat\xi(t,\bar\pi,h;g))|\\
\le\ &\tfrac{2}{\log 3} d_H(\pi,\bar\pi)\exp((\mu\underline{m}+\tfrac{(\mu\bar\sigma)^2}{2})(T-t))(1+c).
\end{aligned} \tag{3.5.28}$$

Finally, to prove that for $g\in\mathcal{G}_2$ the relation (3.1.21) holds, we rewrite, using the time homogeneity of (X_t,θ_t),

$$\begin{aligned}
&\exp(\mu\hat\xi(t,\pi,h;g))\\
=\ &E^{t,\pi}[\exp(\mu D(h,X_T-X_t))1_{\{\tau_1>T\}}+\ \exp(\mu D(h,X_{\tau_1}-X_t)+\mu g(\tau_1,\pi_{\tau_1}))1_{\{\tau_1\le T\}}]\\
=\ &E^{t,\pi}[\exp(\mu D(h,X_{T-t}-X_0))1_{\{\tau_1>T-t\}}\\
&+\ \exp(\mu D(h,X_{\tau_1}-X_0)+\mu g(\tau_1+t,\pi_{\tau_1}))1_{\{\tau_1\le T-t\}}].
\end{aligned} \tag{3.5.29}$$

Therefore, recalling that $\bar t<t$,

$$\begin{aligned}
&|\exp(\mu\hat\xi(t,\pi,h;g))-\exp(\mu\hat\xi(\bar t,\pi,h;g))|\\
\le\ &|E^{0,\pi}[\exp(\mu D(h,X_{T-t}-X_0))\{1_{\{\tau_1>T-t\}}-1_{\{\tau_1>T-\bar t\}}\}]|\\
&+|E^{0,\pi}[\ \exp(\mu D(h,X_{\tau_1}-X_0)+\mu g(\tau_1+t,\pi_{\tau_1}))\ \{1_{\{\tau_1\le T-t\}}-1_{\{\tau_1\le T-\bar t\}}\}]|\\
&+|E^{0,\pi}[\{\exp(\mu D(h,X_{T-t}-X_0)-\exp(\mu D(h,X_{T-\bar t}-X_0))\}1_{\{\tau_1>T-\bar t\}}]|\\
&+|E^{0,\pi}[\ \exp(\mu D(h,X_{\tau_1}-X_0))\{\exp(\mu g(\tau_1+t,\pi_{\tau_1}))-\exp(\mu g(\tau_1+\bar t,\pi_{\tau_1})\}1_{\{\tau_1\le T-\bar t\}}]|\\
\equiv\ &J_1+J_2+J_3+J_4.
\end{aligned} \tag{3.5.30}$$

Now we have

$$\begin{aligned}
&J_1\le\exp((\mu\underline{m}+\tfrac{(\mu\bar\sigma)^2}{2})(T-t))P^{0,\pi}(T-t<\tau_1<T-\bar t)\\
=\ &\exp((\mu\underline{m}+\tfrac{(\mu\bar\sigma)^2}{2})(T-t))E^{t,\pi}[\textstyle\int_{T-t}^{T-\bar t} n(\theta_s)\exp(\int_0^s -n(\theta_u)du)ds]\\
\le\ &\exp((\mu\underline{m}+\tfrac{(\mu\bar\sigma)^2}{2})(T-t))\textstyle\int_{T-t}^{T-\bar t}\bar n\exp(-\underline{n}s)ds\\
\le\ &\exp((\mu\underline{m}+\tfrac{(\mu\bar\sigma)^2}{2})(T-t))\left(\tfrac{\bar n}{\underline{n}}\right)(e^{\underline{n}(t-\bar t)}-1).
\end{aligned} \tag{3.5.31}$$

We also have, using (3.1.19),

$$J_2\le\exp((\mu\underline{m}+\frac{(\mu\bar\sigma)^2}{2})(T-t))\left(\frac{\bar n}{\underline{n}}\right)(e^{\underline{n}(t-\bar t)}-1). \tag{3.5.32}$$

Further, since $|D(h,x) - D(h,y)| \leq |x-y|$ holds from (3.5.10), we obtain

$$\begin{aligned}
J_3 &\leq E^{0,\pi}[|\exp(\mu D(h, X_{T-t} - X_0)) - \exp(\mu D(h, X_{T-\bar{t}} - X_0))|] \\
&= E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0)) \\
&\quad \times |1 - \exp\big(\mu(D(h, X_{T-\bar{t}} - X_0) - D(h, X_{T-t} - X_0))\big)|]| \\
&\leq E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0))|1 - \exp(\mu|X_{T-\bar{t}} - X_{T-t}|)|] \\
&= E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0))E[|1 - \exp(\mu|X_{T-\bar{t}} - X_{T-t}|)||X_{T-t}]].
\end{aligned} \tag{3.5.33}$$

Since (X_t, θ_t) is a time homogeneous process, we have

$$\begin{aligned}
&E[|1 - \exp(\mu|X_{T-\bar{t}} - X_{T-t}|)||X_{T-t}] \\
&= E[|1 - \exp(\mu|\textstyle\int_{T-t}^{T-\bar{t}} r(\theta_s) - d(\sigma\sigma^*(\theta_s))ds + \int_{T-t}^{T-\bar{t}} \sigma(\theta_s)dB_s)||X_{T-t}] \\
&= E_{X_{T-t}}[|1 - \exp(\mu|\textstyle\int_{T-t}^{T-\bar{t}} r(\theta_s) - d(\sigma\sigma^*(\theta_s))ds + \int_{T-t}^{T-\bar{t}} \sigma(\theta_s)dB_s|)|] \\
&= E_{X_{T-t}}[|1 - \exp(\mu|\textstyle\int_0^{t-\bar{t}} r(\theta_s) - d(\sigma\sigma^*(\theta_s))ds + \int_0^{t-\bar{t}} \sigma(\theta_s)dB_s|)|] \\
&= E_{X_{T-t}}[|1 - \exp(\mu|X_{t-\bar{t}} - X_0|)|] \\
&= l(t-\bar{t}),
\end{aligned} \tag{3.5.34}$$

where l is the function defined in (3.0.4). Hence,we obtain

$$\begin{aligned}
J_3 &\leq E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0))]l(t-\bar{t}) \\
&\leq \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t))l(t-\bar{t}).
\end{aligned} \tag{3.5.35}$$

Since $g \in \mathcal{G}_2$, we have

$$\begin{aligned}
J_4 &\leq E^{0,\pi}[\exp(\mu D(h, X_{\tau_1} - X_0)) \\
&\quad \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t-\tau_1))k(t-\bar{t}))1_{\{\tau_1 \leq T-\bar{t}\}}] \\
&\leq c\exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t))k(t-\bar{t}),
\end{aligned} \tag{3.5.36}$$

by using (3.1.19) in Proposition 3.1.2. Putting all the estimates together, we finally obtain

$$\begin{aligned}
&|\exp(\mu\hat{\xi}(t,\pi,h;g)) - \exp(\mu\hat{\xi}(\bar{t},\pi,h;g))| \\
&\leq \exp((\mu\underline{m} + \tfrac{(\mu\bar{\sigma})^2}{2})(T-t))(2\left(\tfrac{\bar{n}}{\underline{n}}\right)(e^{n(t-\bar{t})} - 1) + l(t-\bar{t}) + ck(t-\bar{t})).
\end{aligned} \tag{3.5.37}$$

□

Proof of Proposition 3.3.1.

The equality(3.3.8) is shown in Lemma 3.5.5 below. This lemma is followed by Lemma 3.5.6 that is preliminary to Lemma 3.5.7, from which then (3.3.9) follows.

Lemma 3.5.5. *For each $n \geq 0$, the equality (3.3.8) holds.*

Proof. By definition we have

$$\bar{W}^0(t,\pi) \quad = \quad W^0(t,\pi). \tag{3.5.38}$$

Moreover, $\bar{W}^0(t,\pi) \in \mathcal{G}_1 \cap \mathcal{G}_2$ because of Proposition 3.2.1. Therefore, in Corollary 3.2.1, we set $g(t,\pi) = \bar{W}^0(t,\pi)$ and obtain a Borel function $\hat{h}^{(n)}(t,\pi)$ satisfying (3.2.9) for $n \geq 0$. Then,

$$\bar{W}^n(t,\pi) = J_\mu^n \bar{W}^0(t,\pi) = \sup_{h \in H_m} \hat{\xi}(t,\pi,h; J_\mu^{n-1}\bar{W}^0) = \hat{\xi}(t,\pi,\hat{h}^{(n-1)}(t,\pi); J_\mu^{n-1}\bar{W}^0).$$

We also have a Borel function $\bar{h}(t,\pi)$ such that

$$\bar{W}^0(t,\pi) = \sup_h \bar{W}^0(t,\pi,h) = \bar{W}^0(t,\pi,\bar{h}(t,\pi)).$$

We define a strategy $\bar{h}^{(n)} \in \mathcal{A}^n$ as follows.

$$\begin{aligned} \bar{h}_k^{(n)} &= \hat{h}^{(n-1-k)}(\tau_k, \pi_{\tau_k}),\ k = 0, \ldots, n-1, \\ \bar{h}_n^{(n)} &= \bar{h}(\tau_n, \pi_{\tau_n}), \\ \bar{h}_k^{(n)} &= \gamma(\tilde{X}_{\tau_k} - \tilde{X}_{\tau_n}, \bar{h}_n^{(n)}),\ k \geq n+1. \end{aligned} \tag{3.5.39}$$

First, to show that $\bar{W}_n(t,\pi) \leq W_n(t,\pi)$, we rewrite $\bar{W}^n$ as follows,

$$\begin{aligned} & \bar{W}^n(t,\pi) \\ = \;& \sup_{h \in \bar{H}_m} \hat{\xi}(t,\pi,h; J_\mu^{n-1}\bar{W}^0) \\ = \;& \hat{\xi}(t,\pi,\hat{h}^{(n-1)}(t,\pi); J_\mu^{n-1}\bar{W}^0) \\ = \;& \tfrac{1}{\mu} \log E^{t,\pi}[\exp(\mu D(\hat{h}^{(n-1)}(t,\pi), X_{T\wedge\tau_1} - X_t) + \ \mu \bar{W}^{n-1}(\tau_1, \pi_{\tau_1}) 1_{\{\tau_1 \leq T\}})] \\ = \;& \tfrac{1}{\mu} \log E^{t,\pi}[\exp(\mu D(\hat{h}^{(n-1)}(t,\pi), X_T - X_t)) 1_{\{\tau_1 > T\}} \\ & \qquad + \exp(\mu D(\hat{h}^{(n-1)}(t,\pi), X_{\tau_1} - X_t) + \ \mu \bar{W}^{n-1}(\tau_1, \pi_{\tau_1})) 1_{\{\tau_1 \leq T\}}] \\ = \;& \tfrac{1}{\mu} \log E^{t,\pi}[e^{\mu D(\hat{h}^{(n-1)}(t,\pi), X_T - X_t)} 1_{\{\tau_1 > T\}} + e^{\mu D(\hat{h}^{(n-1)}(t,\pi), X_{\tau_1} - X_t)} \\ & \qquad \cdot E^{\tau_1, \pi_{\tau_1}}[e^{\mu D(\hat{h}^{(n-2)}(\tau_1,\pi_1), X_{T\wedge\tau_2} - X_{\tau_1}) + \mu \bar{W}^{n-2}(\tau_2,\pi_{\tau_2}) 1_{\{\tau_2 \leq T\}}}] 1_{\{\tau_1 \leq T\}}]. \end{aligned} \tag{3.5.40}$$

Noting that

$$\begin{aligned} & E^{t,\pi}[e^{\mu D(\hat{h}^{(n-1)}(t,\pi), X_{\tau_1} - X_t)} \\ & \qquad \cdot E^{\tau_1, \pi_{\tau_1}}[e^{\mu D(\hat{h}^{(n-2)}(\tau_1,\pi_1), X_{T\wedge\tau_2} - X_{\tau_1}) + \mu \bar{W}^{n-2}(\tau_2,\pi_{\tau_2}) 1_{\{\tau_2 \leq T\}}}] 1_{\{\tau_1 \leq T\}}] \\ = \;& E^{t,\pi}[e^{\mu D(\hat{h}^{(n-1)}(t,\pi), X_{\tau_1} - X_t)} E^{\tau_1, \pi_{\tau_1}}[e^{\mu D(\hat{h}^{(n-2)}(\tau_1,\pi_{\tau_1}), X_T - X_{\tau_1})} 1_{\{\tau_2 > T\}}] 1_{\{\tau_1 \leq T\}}] \\ & \qquad + E^{t,\pi}[e^{\mu D(\hat{h}^{(n-1)}(t,\pi), X_{\tau_1} - X_t)} \\ & \qquad \cdot E^{\tau_1, \pi_{\tau_1}}[e^{\mu D(\hat{h}^{(n-2)}(\tau_1,\pi_{\tau_1}), X_{\tau_2} - X_{\tau_1}) + \mu \bar{W}^{n-2}(\tau_2,\pi_{\tau_2})} 1_{\{\tau_2 \leq T\}}] 1_{\{\tau_1 \leq T\}}], \end{aligned} \tag{3.5.41}$$

we have

$$
\begin{aligned}
&\bar{W}^n(t,\pi)\\
&= \tfrac{1}{\mu}\log E^{t,\pi}[e^{\mu D(\hat{h}^{(n-1)}(t,\pi),X_{T\wedge\tau_1}-X_t)+\mu D(\hat{h}^{(n-2)}(\tau_1,\pi_{\tau_1}),X_{T\wedge\tau_2}-X_{\tau_1})1_{\{\tau_1\le T\}}+\mu\bar{W}^{n-2}(\tau_2,\pi_{\tau_2})1_{\{\tau_2\le T\}}}]\\
&= \tfrac{1}{\mu}\log E^{t,\pi}[\exp(\mu \textstyle\sum_{k=1}^n D(\bar{h}^{(n)}(\tau_{k-1},\pi_{\tau_{k-1}}),X_{T\wedge\tau_k}-X_{\tau_{k-1}})1_{\{\tau_{k-1}\le T\}}\\
&\qquad\qquad\qquad\qquad +\mu\bar{W}^0(\tau_n,\pi_{\tau_n},\bar{h}^{(n)}(\tau_n,\pi_{\tau_n}))1_{\{\tau_n<T\}})],
\end{aligned}
\tag{3.5.42}
$$

inductively. By Corollary 3.3.1 we then have

$$
\begin{aligned}
&\bar{W}^n(t,\pi)\\
=\ & \tfrac{1}{\mu}\log E^{t,\pi}[\exp(\mu \textstyle\sum_{k=1}^n D(\bar{h}_n(\tau_{k-1},\pi_{\tau_{k-1}}),X_{T\wedge\tau_k}-X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}}\\
&\qquad +\mu\bar{W}^0(\tau_n,\pi_{\tau_n},\bar{h}_n(\tau_n,\pi_{\tau_n}))1_{\{\tau_n<T\}})]\\
\le\ & \sup_{h\in\mathcal{A}^n}\frac{1}{\mu}\log E^{t,\pi}[\exp\Big(\mu\sum_{k=1}^n D(h_{k-1},X_{T\wedge\tau_k}-X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}}\\
&\qquad +\mu\bar{W}^0(\tau_n,\pi_{\tau_n},h_n)1_{\{\tau_n\le T\}}\Big)]\\
=\ & W^n(t,\pi).
\end{aligned}
\tag{3.5.43}
$$

Next, we shall prove the converse inequality. By applying Lemma 3.3.1, we have for $h\in\mathcal{A}^n$

$$
\begin{aligned}
&W(t,\pi,h.)\\
=\ & \tfrac{1}{\mu}\log E^{t,\pi}[\exp(\mu\sum_{k=1}^n D(h_{k-1},X_{T\wedge\tau_k}-X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}}+\ \mu\bar{W}^0(\tau_n,\pi_{\tau_n},h_n)1_{\{\tau_n\le T\}})]\\
\le\ & \tfrac{1}{\mu}\log E^{t,\pi}[\exp(\mu\sum_{k=1}^n D(h_{k-1},X_{T\wedge\tau_k}-X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}}+\mu\bar{W}^0(\tau_n,\pi_{\tau_n})1_{\{\tau_n\le T\}})]\\
=\ & \tfrac{1}{\mu}\log E^{t,\pi}[\exp(\mu\sum_{k=1}^{n-1} D(h_{k-1},X_{\tau_k}-X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}})\\
&\times\exp(\mu D(h_{n-1},X_{T\wedge\tau_n}-X_{\tau_{n-1}})+\mu\bar{W}^0(\tau_n,\pi_{\tau_n})1_{\{\tau_n\le T\}})1_{\{\tau_{n-1}\le T\}}\\
&+\ \exp(\mu\sum_{k=1}^{n-1} D(h_{k-1},X_{T\wedge\tau_k}-X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}})1_{\{\tau_{n-1}>T\}}]\\
=\ & \tfrac{1}{\mu}\log E^{t,\pi}[\exp(\mu\sum_{k=1}^{n-1} D(h_{k-1},X_{\tau_k}-X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}})\\
&\times E^{\tau_{n-1},\pi_{\tau_{n-1}}}[\exp(\mu D(h_{n-1},X_{T\wedge\tau_n}-X_{\tau_{n-1}})+\mu\bar{W}^0(\tau_n,\pi_{\tau_n})1_{\{\tau_n\le T\}})]1_{\{\tau_{n-1}\le T\}}\\
&+\ \exp(\mu\sum_{k=1}^{n-1} D(h_{k-1},X_{T\wedge\tau_k}-X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}})1_{\{\tau_{n-1}>T\}}].
\end{aligned}
\tag{3.5.44}
$$

By the definition of $\hat{\xi}$ and $\bar{W}^1$we have

$$\begin{aligned}
& E^{\tau_{n-1},\pi_{\tau_{n-1}}}[\exp(\mu D(h_{n-1}, X_{T\wedge\tau_n} - X_{\tau_{n-1}}) + \mu\bar{W}^0(\tau_n, \pi_{\tau_n})1_{\{\tau_n\le T\}})] \\
= \ & \exp(\mu\hat{\xi}(\tau_{n-1}, \pi_{\tau_{n-1}}, h_{n-1}; \bar{W}^0)) \\
\ge \ & \exp(\mu \sup_{h\in\bar{H}_m} \hat{\xi}(\tau_{n-1}, \pi_{\tau_{n-1}}, h; \bar{W}^0)) = \exp(\mu\bar{W}^1(\tau_{n-1}, \pi_{\tau_{n-1}})).
\end{aligned} \tag{3.5.45}$$

Therefore, for $h \in \mathcal{A}^n$, we have inductively

$$\begin{aligned}
& W(t, \pi, h.) \\
\le \ & \tfrac{1}{\mu} \log E^{t,\pi}[\exp(\mu \sum_{k=1}^{n-1} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}} + \mu\bar{W}^1(\tau_n, \pi_{\tau_n})1_{\{\tau_n\le T\}})] \\
\le \ & \tfrac{1}{\mu} \log E^{t,\pi}[\exp(\mu D(h, X_{T\wedge\tau_1} - X_0) + \mu\bar{W}^{n-1}(\tau_1, \pi_{\tau_1})1_{\{\tau_1\le T\}})] \\
\le \ & \bar{W}^n(t, \pi).
\end{aligned} \tag{3.5.46}$$

□

Lemma 3.5.6. *For all $h \in \mathcal{A}$, we have*

$$\begin{aligned}
& \lim_{n\to\infty} \frac{1}{\mu} \log E^{t,\pi}[\exp\Big(\mu \sum_{k=1}^{n} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}} \\
& \qquad + \mu W^0(\tau_n, \pi_{\tau_n}, h_n)1_{\{\tau_n\le T\}}\Big)] \\
= \ & W(t, \pi, h.).
\end{aligned} \tag{3.5.47}$$

Proof.

$$\begin{aligned}
& E^{t,\pi}[\exp\Big(\mu \sum_{k=1}^{n} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}\le T\}} \\
& + \mu W^0(\tau_n, \pi_{\tau_n}, h_n)1_{\{\tau_n\le T\}}\Big)] \\
= \ & E^{t,\pi}[\exp\Big(\mu \sum_{k=1}^{n} D(h_{k-1}, X_{\tau_k} - X_{\tau_{k-1}}) + \mu W^0(\tau_n, \pi_{\tau_n}, h_n)\Big)1_{\{\tau_n\le T\}}] \\
& + E^{t,\pi}[\exp\Big(\mu \sum_{k=1}^{n} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}\le T\}}\Big)1_{\{\tau_n>T\}}] \\
=: \ & I_1(n) + I_2(n).
\end{aligned} \tag{3.5.48}$$

We shall first give an estimate for $I_1(n)$. From (3.1.9) in Proposition 3.1.1 it follows that for $h \in \bar{H}_m$

$$\exp(\mu W^0(t, \pi, h)) \le \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T - t)). \tag{3.5.49}$$

Therefore, we have

$$
\begin{aligned}
I_1(n) &\le E^{t,\pi}[\exp\Big(\mu\sum_{k=1}^{n} D(h_{k-1}, X_{\tau_k}-X_{\tau_{k-1}}) + \mu(\underline{m}+\frac{(\mu\bar{\sigma})^2}{2})(T-\tau_n)\Big)1_{\{\tau_n\le T\}}] \\
&\le E^{t,\pi}[\exp\Big(\mu\sum_{k=1}^{n-1} D(h_{k-1}, X_{\tau_k}-X_{\tau_{k-1}}\Big)E^{\tau_{n-1},\pi_{\tau_{n-1}}}[\exp\Big(\mu D(h_{n-1}, X_{\tau_n}-X_{\tau_{n-1}}) \\
&\quad + \mu(\underline{m}+\tfrac{(\mu\bar{\sigma})^2}{2})(T-\tau_n)\Big)1_{\{\tau_n<T\}}]1_{\{\tau_{n-1}\le T\}}] \\
&\le cE^{t,\pi}[\exp\Big(\mu\sum_{k=1}^{n-1} D(h_{k-1}, X_{\tau_k}-X_{\tau_{k-1}}\Big)\exp((\mu\underline{m}+\frac{(\mu\bar{\sigma})^2}{2})(T-\tau_{n-1}))1_{\{\tau_{n-1}\le T\}}],
\end{aligned}
\tag{3.5.50}
$$

by using Proposition 3.1.2(i) because clearly $\exp((\mu\underline{m}+\frac{(\mu\bar{\sigma})^2}{2})(T-t))\in\mathcal{G}$. Thus, we obtain inductively

$$
\begin{aligned}
I_1(n) &\le cE^{t,\pi}[\exp\Big(\mu\sum_{k=1}^{n-2} D(h_{k-1}, X_{\tau_k}-X_{\tau_{k-1}})\Big) \\
&\quad \times\ E^{\tau_{n-2},\pi_{\tau_{n-2}}}[\exp\Big(\mu D(h_{n-2}, X_{\tau_{n-1}}-X_{\tau_{n-2}}) \\
&\quad +(\mu\underline{m}+\tfrac{(\mu\bar{\sigma})^2}{2})(T-\tau_{n-1})\Big)1_{\{\tau_{n-1}<T\}}]1_{\{\tau_{n-2}\le T\}}] \\
&\le c^2E^{t,\pi}[\exp\Big(\mu\sum_{k=1}^{n-2} D(h_{k-1}, X_{\tau_k}-X_{\tau_{k-1}})\Big) \\
&\quad \times\ \exp\Big((\mu\underline{m}+\tfrac{(\mu\bar{\sigma})^2}{2})(T-\tau_{n-2})\Big)1_{\{\tau_{n-2}\le T\}}]] \\
&\le c^n\exp((\mu\underline{m}+\tfrac{(\mu\bar{\sigma})^2}{2})(T-t)),
\end{aligned}
\tag{3.5.51}
$$

and therefore we see that

$$\lim_{n\to\infty} I_1(n)=0.$$

On the other hand, since $1_{\{\tau_n>T\}}=\sum_{j=0}^{n-1}1_{\{\tau_j<T\le\tau_{j+1}\}}$, we have

$$
\begin{aligned}
I_2(n) &= E^{t,\pi}[\exp\Big(\mu\sum_{k=1}^{n} D(h_{k-1}, X_{T\wedge\tau_k}-X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}}\Big)1_{\{\tau_n\ge T\}}] \\
&= E^{t,\pi}[\textstyle\sum_{j=0}^{n-1}\exp\Big(\mu\displaystyle\sum_{k=1}^{n} D(h_{k-1}, X_{T\wedge\tau_k}-X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}}\Big)1_{\{\tau_j<T\le\tau_{j+1}\}}] \\
&= E^{t,\pi}[\sum_{j=0}^{n-1}\exp\Big(\mu\sum_{k=1}^{j+1} D(h_{k-1}, X_{T\wedge\tau_k}-X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}} \\
&\quad +\mu\textstyle\sum_{k=j+2}^{n} D(h_{k-1}, X_{T\wedge\tau_k}-X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}}\Big)1_{\{\tau_j<T\le\tau_{j+1}\}}].
\end{aligned}
\tag{3.5.52}
$$

Noting that $\{\tau_k < T\} \cap \{T \le \tau_{j+1}\} = \emptyset$ for all $k \ge j+1$, we have

$$\begin{aligned}
&\exp\Big(\mu \sum_{k=j+2}^{n} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1}<T\}}\Big) 1_{\{\tau_j < T \le \tau_{j+1}\}} \\
= \;& \exp\Big(\mu \sum_{k=j+2}^{\infty} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1}<T\}}\Big) 1_{\{\tau_j < T \le \tau_{j+1}\}} \\
= \;& 1_{\{\tau_j < T \le \tau_{j+1}\}}
\end{aligned} \tag{3.5.53}$$

and that

$$\begin{aligned}
\lim_{n\to\infty} I_2(n) \;=\;& \lim_{n\to\infty} E^{t,\pi}[\exp\Big(\mu \sum_{k=1}^{n} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1}<T\}}\Big) 1_{\{\tau_n \ge T\}}] \\
=\;& \lim_{n\to\infty} E^{t,\pi}[\sum_{j=0}^{n-1} \exp\Big(\mu \sum_{k=1}^{\infty} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1}<T\}}\Big) 1_{\{\tau_j < T \le \tau_{j+1}\}}] \\
=\;& E^{t,\pi}[\sum_{j=0}^{\infty} \exp\Big(\mu \sum_{k=1}^{\infty} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1}<T\}}\Big) 1_{\{\tau_j < T \le \tau_{j+1}\}}] \\
=\;& E^{t,\pi}[\exp\Big(\mu \sum_{k=1}^{\infty} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1}<T\}}\Big)].
\end{aligned} \tag{3.5.54}$$

Therefore, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} E^{t,\pi}[\exp\Big(\mu \sum_{k=1}^{n} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1}<T\}} + \ \mu W^0(\tau_n, \pi_{\tau_n}, h_n) 1_{\{\tau_n \le T\}}\Big)] \\
&= \lim_{n\to\infty} I_1(n) + I_2(n) = \exp(\mu W(t,\pi,h.)),
\end{aligned} \tag{3.5.55}$$

having used Lemma 3.1.1. This completes the proof. □

Lemma 3.5.7. *The equality (3.3.9) holds.*

Proof. By the definition of $\mathcal{A}^n$, the inclusions $\mathcal{A}^n \subset \mathcal{A}^{n+1} \subset \mathcal{A}$ hold for $n \ge 0$ and we have

$$\sup_{h\in\mathcal{A}^n} W(t,\pi,h.) \le \sup_{h\in\mathcal{A}^{n+1}} W(t,\pi,h.) \le \sup_{h\in\mathcal{A}} W(t,\pi,h.). \tag{3.5.56}$$

From the definition of $W^n(t,\pi)$ and $W(t,\pi)$ it follows that

$$W^n(t,\pi) \le W^{n+1}(t,\pi) \le W(t,\pi). \tag{3.5.57}$$

Therefore, from Lemma 3.5.5 we have

$$\bar{W}^n(t,\pi) \le \bar{W}^{n+1}(t,\pi) \le W(t,\pi). \tag{3.5.58}$$

Thus, from Proposition 3.2.2 and (3.3.1), we obtain

$$\bar{W}(t,\pi) \le W(t,\pi). \tag{3.5.59}$$

On the other hand, for $h \in \mathcal{A}$

$$\begin{aligned}\bar{W}^n(t,\pi) &= W^n(t,\pi)\\ &\geq \frac{1}{\mu}\log E^{t,\pi}[\exp\Big(\mu\sum_{k=1}^{n} D(h_{k-1}, X_{T\wedge\tau_k} - X_{\tau_{k-1}})1_{\{\tau_{k-1}<T\}} \\ &\qquad + \mu W^0(\tau_n, \pi_{\tau_n}, h_n)1_{\{\tau_n \leq T\}}\Big)].\end{aligned} \tag{3.5.60}$$

Letting $n \to \infty$ and applying Lemma 3.5.5,

$$\bar{W}(t,\pi) \geq W(t,\pi,h.) \tag{3.5.61}$$

and hence, we obtain

$$\bar{W}(t,\pi) = W(t,\pi). \tag{3.5.62}$$

□

Chapter 4

Numerical analysis

In this chapter, we construct a numerical scheme for the log-utility value function. In Section 4.1, we construct an approximation filter and a value function. In Section 4.2, we compute the value function and confirm convergence of the value iteration numerically.

4.1 Numerical scheme

4.1.1 Numerical approximation for filter

We construct a uniform discretization grid $t_0 = 0 < t_1 < t_2 <, \dots, < t_n = T, .\delta := t_l - t_{l-1}$ on $[0, T]$ and approximate the transition probability

$$P(\theta_{t_l} = e_i | \theta_{t_{l-1}} = e_j) = \begin{cases} q_{ij}\delta & i \neq j \\ 1 - \sum_{i \neq k} q_{ik}\delta & i = j. \end{cases} \tag{4.1.1}$$

We define (discrete)occupation time as

$$I_i(l) = \sum_{k=1}^{l} 1_{\{\theta_{t_k} = e_i\}} \tag{4.1.2}$$

$$I(l) := (I_1(l), I_2(l), \dots, I_N(l)) \tag{4.1.3}$$

and approximate the following integral

$$\int_0^{t_l} n(\theta_s) ds \approx \sum_{i=1}^{N} n(e_i) I_i(l) \delta. \tag{4.1.4}$$

$$O(l) := \{(o_1, o_2, \dots, o_N) \in \mathbb{Z}^N, \sum_{i=1}^{N} o_i = l, 0 \leq o_i \leq l \ for \ all \ i\}. \tag{4.1.5}$$

We approximate the filter

$$
\begin{aligned}
&\sum_j n(e_i) r_{ji}(t_l,x) p_{ji}(t_l)\pi^j \\
&= \textstyle\sum_j E[n(e_i)1_{\{\theta_{t_l}=e_i\}}\exp(\int_0^{t_l} -n(\theta_s)ds)\rho^\theta_{0,t_l}(z)|\theta_0=e_j]\pi_j \\
&\approx \sum_{1\le i,j\le N, o\in O(l)} n(e_i)\exp(\sum_{1\le k\le N} n(e_k)o_k\delta)\rho^\theta_{0,t_l}(z;\sum_{1\le k\le N} m(e_k)o_k\delta, \sum_{1\le k\le N}\sigma(e_k)o_k\delta) \\
&\times\ P(\theta_{t_l}=e_i, I(l)=o|\theta_{t_0}=e_j)\pi^j.
\end{aligned}
\tag{4.1.6}
$$

4.1.2 Numerical approximation for value function

We approximate the control space

$$\bar{H}_m \approx \bar{H}^p_m := \{h_1, h_2, \ldots, h_p, h_i \in \bar{H}_m \ for\ all\ i\} \tag{4.1.7}$$

$$\bar{W}_0(t_i,\pi) = \sup_{h\in\bar{H}_m} \bar{W}_0(t_i,\pi,h) \approx \max_{j\in H^p_m} \bar{W}_0(t_i,\pi,h_j) \tag{4.1.8}$$

$$C(t_i,\pi) = \sup_{h\in\bar{H}_m} \hat{C}(t_i,\pi,h) \approx \max_{j\in H^p_m} \hat{C}(t_i,\pi,h_j). \tag{4.1.9}$$

We approximate the value function and operator J by a Monte Carlo simulation. $X^i_k(j,l), \tau_k(j,l)$ are the j-th sample paths starting at $\theta_0 = e_l$

$$
\begin{aligned}
\bar{W}^0(t,\pi,h) &= E[\textstyle\int_t^T f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_t,h))ds|\tau_0=t,\ \pi_{\tau_0}=\pi] \\
&= E[\log(\textstyle\sum_{i=0}^m N^i_0 S^i_T)|\tau_0=t,\ \pi_{\tau_0}=\pi] \\
&= \textstyle\sum_{l=1}^N E[\log(\sum_{i=0}^m N^i_0 S^i_T)|\tau_0=t,\ \theta_{\tau_0}=e_l]\pi^l \\
&\approx \tfrac{1}{M}\textstyle\sum_{j=1}^M\sum_{l=1}^N \log(\sum_{i=0}^m N^i_0\exp(X^i_T(j,l)-X^i_0)\pi^l
\end{aligned}
\tag{4.1.10}
$$

$$
\begin{aligned}
\hat{C}(t,\pi,h) &= E[\textstyle\int_\tau^{T\wedge\tau_1} f(\theta_s,h_s)ds|\tau_0=t,\pi_{\tau_0}=\pi] \\
&= E[\log(\textstyle\sum_{i=0}^m N^i_0 S^i_{T\wedge\tau_1})|\tau_0=t,\pi_{\tau_0}=\pi] \\
&= \textstyle\sum_{l=1}^N E[\log(\sum_{i=0}^m N^i_0 S^i_{T\wedge\tau_1})|\tau_0=t,\theta_{\tau_0}=e_l]\pi_l \\
&\approx \tfrac{1}{M}\textstyle\sum_{j=1}^M\sum_{l=1}^N \log(\sum_{i=0}^m N^i_0\exp(X^i_{T\wedge\tau_1(j,l)}(j,l)-X^i_0)\pi^l
\end{aligned}
\tag{4.1.11}
$$

$$
\begin{aligned}
J\bar{W}^n(\tau,\pi) &= E[\bar{W}^n(\tau_1,\pi_{\tau_1})1_{\{\tau_1<T\}}|\tau_0=\tau,\pi_{\tau_0}=\pi] \\
&= \textstyle\sum_{l=1}^N E[\bar{W}^n(\tau_1,\pi_{\tau_1})1_{\{\tau_1<T\}}|\tau_0=\tau,\theta_{\tau_0}=e_l]\pi_l \\
&\approx \tfrac{1}{M}\textstyle\sum_{j=1}^M\sum_{l=1}^N \bar{W}^n(\tau_1(j,l),M(\tau_1(j,l)-\tau_0,X_{\tau_1(j,l)}(j,l)-X_0),\pi)1_{\{\tau_1(j,l)<T\}}\pi^l
\end{aligned}
\tag{4.1.12}
$$

$$\bar{W}^{n+1}(\tau,\pi) = \hat{C}(t,\pi) + J\bar{W}^n(\tau,\pi). \tag{4.1.13}$$

4.2 Numerical results

4.2.1 List of parameter settings

$N = 2$ (two-state Markov chain)
$m = 1$ (one risky asset and bank account)
$n(e_1) = 1, n(e_2) = 2$(intensity for observation time)
$\sigma(e_1) = 0.5, \sigma(e_2) = 0.3$(volatility of risky asset)
$r(e_1) = 0.08, r(e_2) = 0.06$(return of risky asset)
$r_0 = 0.02$(interest rate)
$q_{11} = q_{22} = 1, q_{12} = q_{21} = -1$(Q-matrix)

4.2.2 Numerical results

In figures 4.1-4.2, we compute $W_0(t, \pi, h)$ and $C(t, \pi, h)$, respectively. These functions are concave with respect to control variable h. Therefore,we compute a unique optimal strategy.
In figure 4.3,we compute the value function and confirm that it is a monotonically increasing function with respect to iteration number and converges.These numerical results are consistent with Lemma 2.4.4 and Theorem 2.5.1.

Figure 4.1: $W_0(t, \pi, h)$

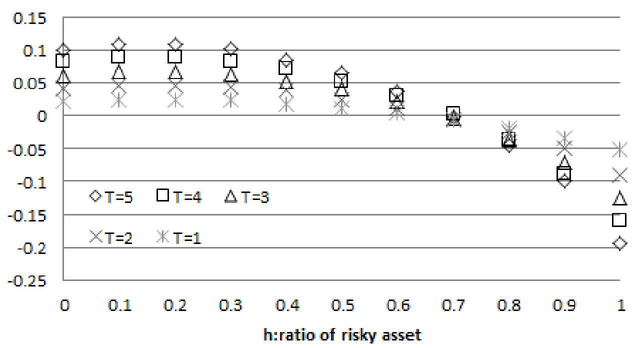


Figure 4.2: $C(t,\pi,h)$

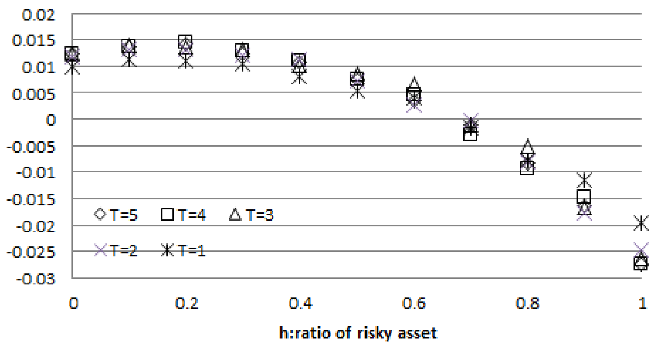
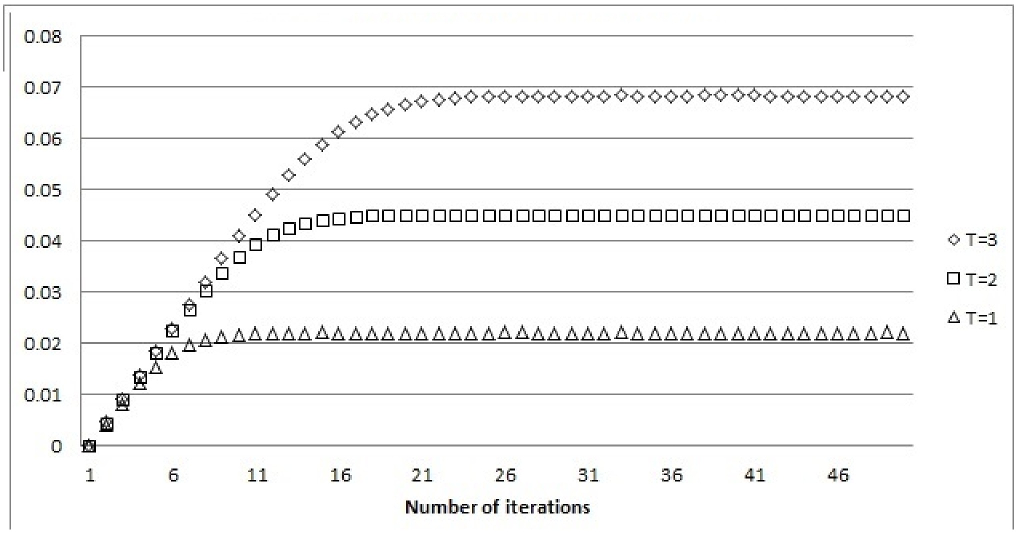


Figure 4.3: Convergence $\sum_{k=0}^{n} J^k C$



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List of Publications

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List of Talks

(1) *Pricing Exotic Derivatives with Variance Curve model*, 藤本　一文, 金融工学 数理計量ファイナンスの諸問題 2009, 大阪大学 中之島センター, 2009.12.4

(2) *Pricing Exotic Derivatives with Variance Curve model*, 藤本　一文, 日本金融 証券計量・工学学会 夏季大会, 成城大学, 2010.7.30

(3) CVA と信用リスクモデリング, 藤本　一文, 日本金融・証券計量・工学学会 信用リスク理論研究部会, 学術総合センター , 2011.8.26

(4) フィルターの安定性について, 藤本　一文, 日本応用数理学会 数理ファイナンス研究部会　研究会, 立命館大学東京キャンパス, 2012.9.15

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(6) *Expected utility maximization under incomplete information and with Cox-processes observations*, 藤本　一文, 金融工学・数理計量ファイナンスの諸問題 2012, 大阪大学中之島センター, 2012.12.1

(7) *Expected utility maximization under incomplete information and with Cox-processes observations*, K. Fujimoto, Workshop on Finance, Stochastics and Asymptotic Analysis, Osaka University, 2013.2.12

(8) *Expected utility maximization under incomplete information and with Cox-processes observations*, 藤本　一文, 日本数学会 2013 年年会, 京都大学, 2013.3.20