On the poset of pre-projective tilting modules over path algebras

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Contents

Chapter 1

Introduction

Tilting theory first appeared in an article by Brenner and Butler [3]. In that article the notion of a tilting module for finite dimensional algebras was introduced. Tilting theory now appears in many areas of mathematics, for example algebraic geometry, theory of algebraic groups and algebraic topology. Let T be a tilting module for a finite dimensional algebra A (see 3.1.1 below for the definition) and let $B = \mathrm{End}_A(T)$. Then Happel showed that the two bounded derived categories $\mathcal{D}^{\mathrm{b}}(A)$ and $\mathcal{D}^{\mathrm{b}}(B)$ are equivalent as triangulated category [8]. Therefore, classifying tilting modules is an important problem.

Theory of tilting-mutation introduced by Riedtmann and Schofield is an approach to this problem. They introduced a tilting quiver whose vertices are (isomorphism classes of) basic tilting modules and arrows correspond to mutations. Happel and Unger defined a partial order on the set of basic tilting modules and showed that the tilting quiver coincides with the Hasse quiver of this poset. This poset is now studied by many authors.

Notations

Let Q be a finite connected quiver without loops or oriented cycles. We denote by Q_0 (resp. Q_1) the set of vertices (resp. arrows) of Q. For any arrow $\alpha \in Q_1$ we denote by $s(\alpha)$ its starting point and denote by $t(\alpha)$ its target point (i.e. α is an arrow from $s(\alpha)$ to $t(\alpha)$). We call a vertex $x \in Q_0$ a source (resp. sink) if there is an arrow starting at x (resp. ending at x) and there is no arrow ending at x (resp. starting at x). Let kQ be the path algebra of Q over an algebraically closed field k. Denote by mod-kQ the category of finite dimensional right kQ-modules and by ind-kQ the full subcategory of indecomposable modules. For any module $M \in$ mod-kQ we denote by $|M|$ the number of pairwise non isomorphic indecomposable direct summands of M. Let $P(i)$ be the indecomposable projective module in mod-kQ associated with vertex $i \in Q_0$.

It is well-known that a path algebra kQ is representation-finite if and only if the underlying graph of Q is a Dynkin-graph. As a result, the poset of tilting modules over kQ shows a completely different behavior in the following two cases:

- Q is a Dynkin quiver,
- Q is a non Dynkin quiver.

We ask different questions in each of the cases and we have obtained two new results.

In chapter 3 we mainly consider Dynkin quivers. If Q is a Dynkin quiver then $\mathcal{T}(Q)$ is a finite poset. Moreover any tilting module is pre-projective (Definition 2.3.1). Therefore we have the following natural questions:

Question 1.0.1. (1) *How many vertices in* $\overrightarrow{\mathcal{T}}(Q)$ *are there?*
(2) *How many arrows in* $\overrightarrow{\mathcal{T}}(Q)$ *are there?*

Note that the underlying graph of $\overrightarrow{\mathcal{T}}(Q)$ may be embedded into the exchange graph, or the cluster complex, of the corresponding cluster algebra of finite type:the tilting modules of kQ correspond to positive clusters (cf.[4] and [17]). The number of positive clusters when the orientation is alternating is given by the following table [6, prop. 3.9]:

type	A_n	D_n	E_6	E_7	E_8
$\#\overrightarrow{\mathcal{T}}(Q)_0$	$\frac{1}{n+1}\binom{2n}{n}$	$\frac{3n-4}{2n}\binom{2(n-1)}{n-1}$	418	2431	17342

However the number of edges of this sub-diagram of positive clusters is not known in the cluster tilting theory. Note also that if we consider the similar problem for the exchange graph, it is not interesting, because the number of edges is $\frac{n}{2} \times \{$the number of clusters$\}$, and the number of vertices is given in [6, Proposition 3.8].

The first main result of this thesis is the following [K1].

Theorem 1.0.2. (1) $\#\overrightarrow{\mathcal{T}}(Q)_1$ *is independent of the orientation.*
(2) $\#\overrightarrow{\mathcal{T}}(Q)_1$ *is given by the following table.*

type	A_n	D_n	E_6	E_7	E_8
$\#\overrightarrow{\mathcal{T}}(Q)_1$	$\binom{2n-1}{n+1}$	$(3n-4)\binom{2(n-2)}{n-3}$	1140	8008	66976

Moreover, the above numbers may be expressed in a uniform way as the following formula:

$$\frac{n}{2}\left(1-\frac{1}{h-1}\right) \times \{\text{the number of positive clusters}\} \cdots (*),$$

where h is the Coxeter number. In this thesis we provide case by case proof for each type, however $(*)$ suggests that it should be possible to provide a uniform proof.

If Q is a non-Dynkin quiver, kQ is a representation-infinite algebra. In this case, to determine rigid modules is nearly impossible. However the pre-projective component of the Auslander-Reiten quiver of mod-kQ is completely determined. For example, there is a bijection between the set of (isomorphism classes of) indecomposable pre-projective modules over kQ and $\mathbb{Z}_{\geq 0} \times Q_0$.

In chapter 4 we consider the set $\mathcal{T}_{\mathrm{p}}(Q)$ of basic pre-projective tilting modules and study its combinatorial structure in the case when Q is a non-Dynkin quiver. For the purpose we have to answer to the following problem:

Question 1.0.3. *When does the* Ext^1_{kQ}*-group between two indecomposable pre-projective modules vanish?*

We introduce a function $l_Q : Q_0 \times Q_0 \to \mathbb{Z}_{\geq 0}$ and, by using this function, we give an answer to this question for any quiver satisfying the following condition (C):

$$\text{(C)} \qquad \delta(a) := \#\{\alpha \in Q_1 \mid s(\alpha) = a \text{ or } t(\alpha) = a\} \geq 2, \quad \forall a \in Q_0.$$

By applying this result we show the following [K2]:

Theorem 1.0.4. *If Q satisfies the condition* (C)*, then for any $T \in \mathcal{T}_{\mathrm{p}}$ there exists $(r_i)_{i \in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0}$ such that $T \simeq \bigoplus_{i \in Q_0} \tau_Q^{-r_i} P(i)$.*

Moreover, the map $\bigoplus_{i \in Q_0} \tau^{-r_i} P(i) \mapsto (r_i)_{i \in Q_0}$ induces a poset inclusion,

$$(\mathcal{T}_{\mathrm{p}}(Q), \leq) \to (\mathbb{Z}^{Q_0}, \leq^{\mathrm{op}}),$$

where $(r_i) \leq^{\mathrm{op}} (s_i) \overset{\mathrm{def}}{\Leftrightarrow} r_i \geq s_i$ for any $i \in Q_0$.

The above result says that if Q satisfies the condition (C), then study of the poset $\mathcal{T}_{\mathrm{p}}(Q)$ comes down to combinatorics on Z^{Q_0}. As an application of these results we determine the structure of Hasse-quiver $\overrightarrow{\mathcal{T}_{\mathrm{p}}}(Q)$ of $\mathcal{T}_p(Q)$, for any quiver Q which satisfies the following:
(i) Q has a unique source,
(ii) Q satisfies the condition (C),
(iii) $l(Q) := \max\{l_Q(x, y) \mid x, y \in Q_0\} \leq 1$.

References

[K1] R. Kase, The number of arrows in the quiver of tilting modules over a path algebra of Dynkin type, Tsukuba J. Math. 37 (2013), no. 1, 153-177

[K2] R. Kase, Pre-projective parts of tilting quivers over certain path algebras, Comm. Algebra, to appear.

Chapter 2

Preliminary

In this chapter we recall some fundamentals needed in this thesis. In section 1 we collect some important properties of finite dimensional path algebras. In section 2 we recall definitions of Auslander-Reiten translation, almost split sequences and Auslander-Reiten quivers. In section 3 we review Auslander-Reiten theory for path algebras.

2.1 Path algebras and quiver representations

Let Q be a finite quiver. We denote by Q_0 the set of vertices of Q and Q_1 the set of arrows of Q. For any path $w : x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_r} x_r$, we set $s(w) = x_0$ and $t(w) = x_r$. We regard $x \in Q_0$ as a path with length 0.

Definition 2.1.1. For any finite quiver Q, define a k-algebra kQ as follows:
• The set of paths in Q forms a basis of kQ.
• For any two paths $w : x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_r} x_r$ and $w' : y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\alpha_s} y_s$, the product is defined by

$$w \cdot w' := \begin{cases} x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_r} x_r = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_s} y_s & \text{if } x_r = y_0 \\ 0 & \text{if } x_r \neq y_0, \end{cases}$$

We call kQ the *path algebra* of Q over k.

Theorem 2.1.2. ([1] , [2]) *Let A be an indecomposable finite dimensional algebra. Then there exists a finite connected quiver Q and two-sided ideal I of kQ such that A is Morita equivalent to kQ/I.*

Remark 2.1.3. A path algebra kQ is finite dimensional if and only if Q is a finite acyclic quiver.

From now on we assume that Q is a finite connected acyclic quiver.

Theorem 2.1.4. ([1], [2]) *Let $A = kQ$ be a path algebra. Then global dimension of A is not more than 1. In other words,* $\mathrm{Ext}_A^i = 0$ *for any* $i > 1$.

Definition 2.1.5. We define a k-category rep Q as follows:
(Objects) A pair

$$(V, f) = ((V_x \mid x \in Q_0), (f_\alpha : V_{s(\alpha)} \to V_{t(\alpha)} \mid \alpha \in Q_1),$$

where V_x are finite dimensional k-vector spaces and f_α are k-linear maps.
(Morphisms) Let $(V,f) = ((V_x)_{x\in Q_0}, (f_\alpha)_{\alpha\in Q_1})$ and $(W,g) = ((W_x)_{x\in Q_0}, (g_\alpha)_{\alpha\in Q_1})$ be objects in rep Q. Then a morphism $\phi : (V,f) \to (W,g)$ is a collection $(\phi_x : V_x \to W_x)_{x\in Q_0}$ of k-linear maps such that, for any $\alpha \in Q_1$, the following diagram commutes:

$$\begin{tikzcd}
V_{s(\alpha)} \arrow[r, "f_\alpha"] \arrow[d, "\phi_{s(\alpha)}"'] & V_{t(\alpha)} \arrow[d, "\phi_{t(\alpha)}"] \\
W_{s(\alpha)} \arrow[r, "g_\alpha"] & W_{t(\alpha)}
\end{tikzcd} \qquad \forall\, \alpha \in Q_1$$

Then we call rep Q the category of finite dimensional representations of Q over k.

For a kQ-module M, we apply an idempotent $x \in Q_0$ to M and obtain Mx, which we denote by M_x.

Theorem 2.1.6. ([1], [2]) *There exists category equivalence*

$$F : \text{mod-}kQ \xrightarrow{\simeq} \text{rep } Q,$$

which sends M to $((M_x)_{x\in Q_0}, (M_\alpha : M_{s(\alpha)} \ni m \mapsto m\alpha \in M_{t(\alpha)})_{\alpha\in Q_1})$.

Proposition 2.1.7. ([1], [2]) *There are bijections between the following four sets.*
- *The set of vertices Q_0 of Q.*
- *The set of isomorphism classes of simple modules in* mod-kQ.
- *The set of isomorphism classes of projective modules in* ind-kQ.
- *The set of isomorphism classes of injective modules in* ind-kQ.

Remark 2.1.8. Let $A := kQ$ and $P(x) := xA$. Then $Q_0 \ni x \mapsto P(x) \in \text{mod-}kQ$ induces a bijection between Q_0 and the set of isomorphism classes of projective modules in ind-kQ.

Definition 2.1.9. Let $M \in \text{mod-}kQ$. We set

$$\underline{\dim}\, M = ((\underline{\dim}\, M)_x)_{x\in Q_0} := (\dim M_x)_{x\in Q_0},$$

and call it the *dimension vector* of M.

Then the following facts are well-known.

Theorem 2.1.10. (Gabriel) *Let Q be a finite connected acyclic quiver. Then kQ has finite representation type if and only if Q is a simply-laced Dynkin quiver. In this case* $\underline{\dim} : \text{mod-}kQ \to \mathbb{Z}_{\geq 0}^{Q_0}$ *induces a bijection between* ind-kQ *and the set of positive roots in the corresponding root system.*

Example 2.1.11. Let $Q = 1 \to 2 \to 3$. Denote by α_i the simple root of type A_3 associated with vertex $i \in Q_0$. Then the set of positive roots is as follows:

$$\Phi_{>0} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}.$$

On the other hand, we have

$$\text{ind-}kQ = \left\{ \begin{array}{l} k \to 0 \to 0,\ 0 \to k \to 0,\ 0 \to 0 \to k \\ k \xrightarrow{1} k \to 0,\ 0 \to k \xrightarrow{1} k,\ k \xrightarrow{1} k \xrightarrow{1} k \end{array} \right\},$$

and the correspondence via $\underline{\dim}$ is clear.

2.2 Almost split sequences and Auslander-Reiten quivers

Definition 2.2.1. (1) An epimorphism $h : P \to M$ is called a projective cover of $M \in \text{mod-}A$ if P is a projective module in mod-A and for any A-homomorphism $g : N \to P$ the surjectivity of $h \circ g$ implies the surjectivity of g.

(2) An exact sequence $P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0$ is called a minimal projective presentation of $M \in \text{mod-}A$ if $P_1 \xrightarrow{f} \operatorname{Ker} g$ and $P_0 \xrightarrow{g} M$ are projective covers.

Let $M \in \text{mod-}A$ and consider a minimal projective presentation

$$P_1 \xrightarrow{f} P_0 \to M \to 0.$$

Then we put $\tau M := \operatorname{Ker} D(f^*)$ where $D(-) := \operatorname{Hom}_k(-, k)$ and $(-)^* := \operatorname{Hom}_A(-, A)$. We call τM the *Auslander-Reiten translation* of M. Using minimal injective resolution we may define $\tau^{-1}N$, for $N \in \text{mod-}A$.

Proposition 2.2.2. ([1], [2]) *Let $X \in \text{ind-}A$. Then the following assertions hold.*
(1) $\tau X \neq 0$ *if and only if X is non-projective. In this case $X \simeq \tau^{-1}\tau X$.*
(2) $\tau^{-1}X \neq 0$ *if and only if X is non-injective. In this case $X \simeq \tau\tau^{-1}X$.*

Let $f \in \operatorname{Hom}_A(Y, Z)$. We call f a *right almost split morphism* if (a) it is not a split epimorphism and (b) any morphism $M \to Z$ which is not a split epimorphism factors through f. Dually we define a *left almost split morphism.*

Definition-Proposition 2.2.3. ([1], [2]) We call an exact sequence

$$0 \to X \xrightarrow{g} Y \xrightarrow{f} Z \to 0 \cdots (*)$$

an *almost split sequence* if $(*)$ satisfies following equivalent conditions.
(a) f is right almost split and g is left almost split.
(b) X is indecomposable and f is right almost split.
(c) Z is indecomposable and g is left almost split.
(d) $X \simeq \tau Z$ and f is right almost split.
(e) $Z \simeq \tau^{-1}X$ and g is left almost split.

Theorem 2.2.4. ([1], [2]) *If X is an indecomposable non-injective module, then there is a unique* (*up to isomorphism*) *almost split sequence*

$$0 \to X \to Y \to Z \to 0.$$

Dually if Z is an indecomposable non-projective module, then there is a unique (*up to isomorphism*) *almost split sequence*

$$0 \to X \to Y \to Z \to 0.$$

Definition-Theorem 2.2.5. ([1], [2]) We can define a quiver $\Gamma(A)$ as follows:
• The set of vertices $\Gamma(A)_0$ is the isomorphism classes of indecomposables.
• Consider an almost split sequence

$$0 \to \tau X \to E \to X \to 0.$$

Then we draw m arrows from τX to Y and m arrows from Y to X if E has m indecomposable direct summands which are isomorphic to Y.

Then we call $\Gamma(A)$ the *Auslander-Reiten quiver* of A.

Example 2.2.6. Let $Q = 1 \to 2 \to 3$. Then

$$\text{ind-}kQ = \left\{ \begin{array}{lcl} P(1) & = & k \to k \to k \\ P(2) & = & 0 \to k \to k \\ P(3) & = & 0 \to 0 \to k \\ \tau^{-1}P(2) & = & k \to k \to 0 \\ \tau^{-1}P(3) & = & 0 \to k \to 0 \\ \tau^{-2}P(3) & = & k \to 0 \to 0 \end{array} \right\}$$

and we have almost split sequences

$$0 \to P(3) \to P(2) \to \tau^{-1}P(3) \to 0,$$

$$0 \to P(2) \to P(1) \oplus \tau^{-1}P(3) \to \tau^{-1}P(2) \to 0,$$

$$0 \to \tau^{-1}P(3) \to \tau^{-1}P(2) \to \tau^{-2}P(3) \to 0.$$

Therefore the Auslander-Reiten quiver of mod-kQ is given by the following:

$$\begin{tikzcd}
 & & P(1) \arrow[dr] & & \\
 & P(2) \arrow[ur] \arrow[dr] & & \tau^{-1}P(2) \arrow[ll, dotted] \arrow[dr] & \\
P(3) \arrow[ur] & & \tau^{-1}P(3) \arrow[ll, dotted] \arrow[ur] & & \tau^{-2}P(3) \arrow[ll, dotted]
\end{tikzcd}$$

2.3 Auslander-Reiten theory for path algebras

Definition 2.3.1. We call a module $M \in \text{mod-}A$ a *pre-projective module* if $\tau^r M = 0$ for some $r \in \mathbb{Z}_{\geq 0}$.

Note that $M \in \text{mod-}A$ is pre-projective if and only if any indecomposable direct summand X of M is isomorphic to $\tau^{-r}P$ for some indecomposable projective module P and $r \in \mathbb{Z}_{\geq 0}$.

Proposition 2.3.2. ([*1*], [*2*]) *Let $A = kQ$. Then the following assertions hold.*
(1) *Q is a Dynkin quiver if and only if any indecomposable module X over A is pre-projective.*
(2) *If Q is a non-Dynkin quiver, then*

$$(x, r) \mapsto \tau^{-r}P(x)$$

induces a bijection between $Q_0 \times \mathbb{Z}_{\geq 0}$ and the set of (isomorphism classes of) indecomposable pre-projective modules.

Now we collect basic properties of the Auslander-Reiten translation for path algebras.

Proposition 2.3.3. ([1], [2], [7]) *Let $A = kQ$ be a path algebra and $M, N \in \text{ind-}A$. Then the following assertions hold.*
(1) *If M and N are non-injective modules, then*

$$\operatorname{Hom}_A(M, N) \simeq \operatorname{Hom}_A(\tau^{-1}M, \tau^{-1}N).$$

(2) (Auslander-Reiten duality) *There is a functorial isomorphism,*

$$D\operatorname{Hom}_A(M, N) \simeq \operatorname{Ext}^1_A(N, \tau M).$$

(3) *For any indecomposable non-projective module X and an almost split sequence*

$$0 \to \tau X \to E \to X \to 0,$$

we have

$$\dim \operatorname{Hom}(M, \tau X) - \dim \operatorname{Hom}(M, E) + \dim \operatorname{Hom}(M, X) = \begin{cases} 1 & X \simeq M \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.3.4. ([1], [2]) *Let Q be a finite connected non-Dynkin quiver, $a \in Q_0$ and $r \in \mathbb{Z}_{\geq 0}$. Then we have an almost split sequence*

$$0 \to \tau^{-r}P(a) \to \bigoplus_{\alpha: s(\alpha)=a} \tau^{-r-1}P(t(\alpha)) \oplus \bigoplus_{\beta: t(\beta)=a} \tau^{-r}P(s(\beta)) \to \tau^{-r-1}P(a) \to 0.$$

Example 2.3.5. We consider the following quiver

$$Q: 1 \rightrightarrows 2$$

Then the pre-projective component of the Auslander-Reiten quiver $\Gamma(kQ)$ of mod-kQ is given by the following:

$$\begin{tikzcd}
& P(1) \arrow[dr, shift left] \arrow[dr, shift right] & & \tau^{-1}P(1) \arrow[dr, shift left] \arrow[dr, shift right] & & \cdots \\
P(2) \arrow[ur, shift left] \arrow[ur, shift right] & & \tau^{-1}P(2) \arrow[ur, shift left] \arrow[ur, shift right] & & \tau^{-2}P(2) &
\end{tikzcd}$$

Let $d(X, Y) := \dim \operatorname{Ext}^1_{kQ}(X, Y)$. Then, from Proposition 2.3.3 and Theorem 2.3.4, we have the following.

Corollary 2.3.6. *Let Q be a finite connected non-Dynkin quiver and let $Y = \tau^{-s}P(y)$ be an indecomposable pre-projective module over kQ. Then we have*

$$d(\tau^{-r}P(x), Y) = \begin{cases} 0 & \text{if } (r, x) \not\succeq (s, y) \\ 1 & \text{if } (r, x) = (s, y) \end{cases}$$

and if $(r, x) \succ (s, y)$, then $d(\tau^{-r}P(x), Y)$ is equal to

$$-d(\tau^{-r+1}P(x), Y) + \sum_{\alpha: s(\alpha)=x} d(\tau^{-r}P(t(\alpha)), Y) + \sum_{\alpha: t(\alpha)=x} d(\tau^{-r+1}P(s(\alpha)), Y)$$

where $(r, x) \succeq (s, y)$ means either (1) $r > s$ or (2) $r = s$ and there is a path from x to y hold.

Chapter 3

Tilting modules

Throughout this thesis we consider tilting modules over path algebras and their poset-structure. Therefore, in section 1 we first recall the definition of tilting modules over path algebras. Second we define a tilted algebra of type Q. It is well known that its module category has connection with the module category of the path algebra kQ. In section 2 we recall the definition and basic properties of the poset of tilting modules.

3.1 Definition and properties

3.1.1 Definition and examples

In this subsection we will recall the definition of tilting modules and basic results for tilting modules.

Definition 3.1.1. Let $A = kQ$ be a path algebra.
(1) A module $M \in \text{mod-}A$ is a *partial tilting module* if $\mathrm{Ext}^1_A(M, M) = 0$.
(2) A partial tilting module $T \in \text{mod-}A$ is a *tilting module* if there is an exact sequence

$$0 \to A_A \to T_0 \to T_1 \to 0$$

with $T_i \in \mathrm{add}\, T$ $(i = 0, 1)$.

Lemma 3.1.2. (Bongartz). *For any partial tilting module $M \in \text{mod-}kQ$, there exists a module $C \in \text{mod-}kQ$ such that $M \oplus C$ is a tilting module.*

It is well known that if $T \in \text{mod-}kQ$ is a tilting module, then $|T| = \#Q_0$. In particular we have the following:

Corollary 3.1.3. *A module $T \in \text{mod-}kQ$ is a tilting module if and only if*
(1) $\mathrm{Ext}^1_{kQ}(T, T) = 0$,
(2) $|T| = \#Q_0$.

We denote by $\mathcal{T}(Q)$ the set of (isomorphism classes of) basic tilting modules in mod-kQ. Recall that a module is *basic* if any two distinct direct summands are non-isomorphic.

3.1.2 Tilted algebras

Definition 3.1.4. Let $A = kQ$. We call an algebra B a *tilted algebra* of type Q if there is a tilting module $T \in \text{mod-}A$ such that $B \cong \text{End}_A(T)$.

Let $T \in \text{mod-}A$ is a tilting module and $B = \text{End}_A(T)$. Then T is a (B, A)-bimodule, so that $\mathrm{D}(T)$ is a (A, B)-bimodule. We consider (full) subcategories

$$\begin{aligned} \mathcal{F} &= \mathcal{F}(T) := \{M \in \text{mod-}A \mid \text{Hom}_A(T, M) = 0\}, \\ \mathcal{T} &= \mathcal{T}(T) := \{M \in \text{mod-}A \mid \text{Ext}^1_A(T, M) = 0\} \end{aligned}$$

of mod-A. We also consider (full) subcategories

$$\begin{aligned} \mathcal{X} &= \mathcal{X}(T) := \{U \in \text{mod-}B \mid \text{Hom}_B(U, D(T)) = 0\}, \\ \mathcal{Y} &= \mathcal{Y}(T) := \{U \in \text{mod-}B \mid \text{Ext}_B(U, D(T)) = 0\} \end{aligned}$$

of mod-B. The following is the celebrated Brenner and Butler's tilting theorem.

Theorem 3.1.5. ([2] [3]) *The following assertions hold.*
(1) ${}_BT$ *is a tilting module, and a map* $a \mapsto (t \mapsto ta)$ *induces a* k*-algebra isomorphism*

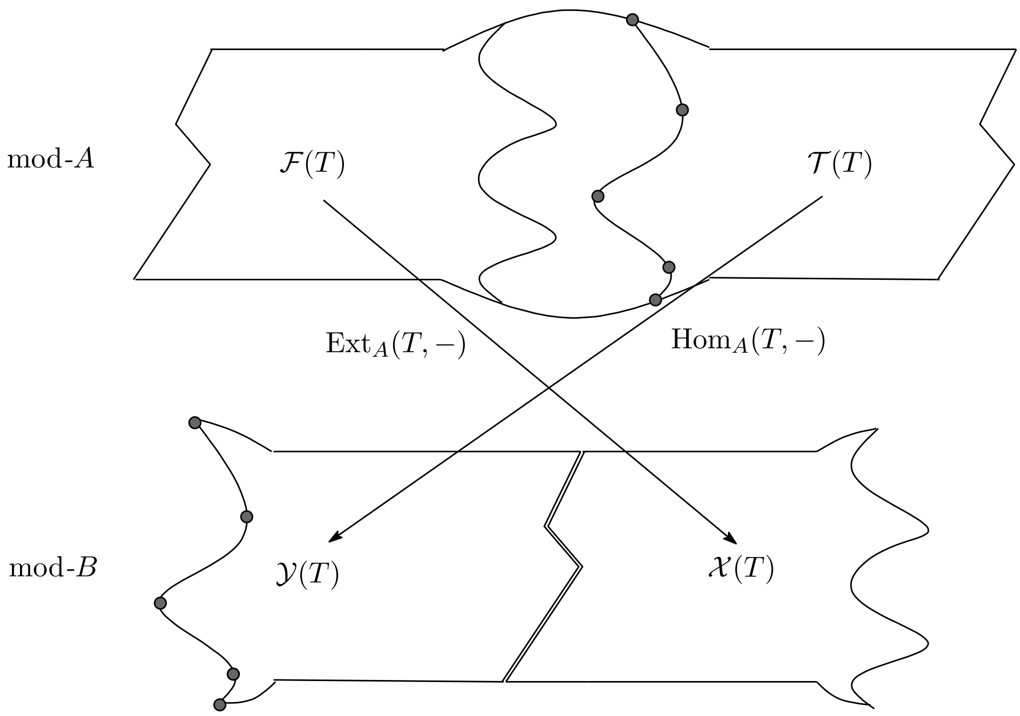
$$A \cong \text{End}_B(T)^{\text{op}}.$$

(2) $\text{Hom}_A(T, -)$ *induces category equivalence*

$$\text{Hom}_A(T, -) : \mathcal{T} \xrightarrow{\simeq} \mathcal{Y}.$$

(3) $\text{Ext}_A(T, -)$ *induces category equivalence*

$$\text{Ext}_A(T, -) : \mathcal{F} \xrightarrow{\simeq} \mathcal{X}.$$

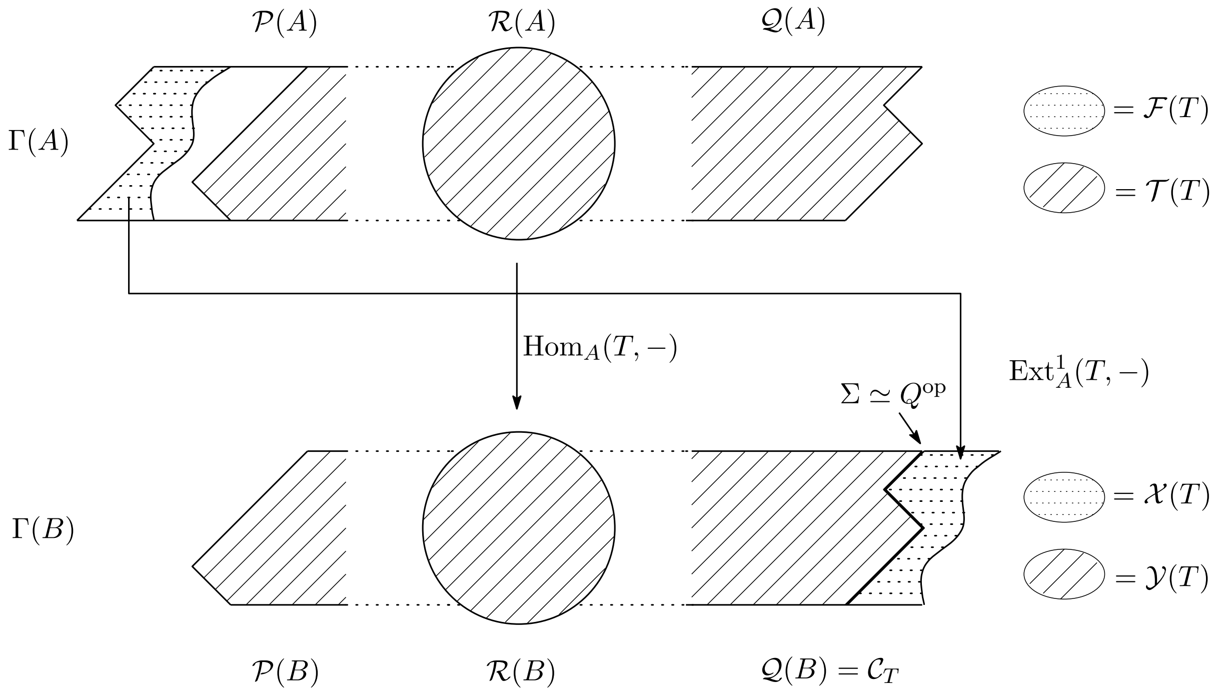


Remark 3.1.6. (i) (1), (2), (3) of above Theorem hold for an arbitrary finite dimensional algebra A.

(ii) $(\mathcal{F}, \mathcal{T})$ is a torsion pair of mod-A and $(\mathcal{Y}, \mathcal{X})$ is a split torsion-pair of mod-B [2].

Definition 3.1.7. Let $A = kQ$. We call an algebra B a *concealed algebra* of type Q if there is a pre-projective tilting module T such that $B \cong \mathrm{End}_A(T)$.

Theorem 3.1.8. [*2*, Chapter VIII, Theorem 4.5.] *Assume that Q is a non-Dynkin quiver and $A = kQ$. Let $T \in$ mod-A be a pre-projective tilting module and define $B := \mathrm{End}_A(T)$.*
(a) *$\mathcal{T}(T)$ contains all but finitely many non-isomorphic indecomposable A-modules, and any indecomposable A-module not in $\mathcal{T}(T)$ is pre-projective.*
(b) *The connecting component $\mathcal{C}_T$ determined by T (see [2] for the definition) is the unique pre-injective component $\mathcal{Q}(B)$ of $\Gamma(B)$. Moreover, $\mathcal{Q}(B)$ contains all indecomposable module in $\mathcal{X}(T)$ and no projective modules belong to $\mathcal{Q}(B)$.*
(c) *The images under the functor $\mathrm{Hom}_A(T, -)$ of the regular components from $\mathcal{R}(A)$ form a family $\mathcal{R}(B)$ of regular components of $\Gamma(B)$.*
(d) *The images under the functor $\mathrm{Hom}_A(T, -)$ of modules in $\mathcal{P}(A) \cap \mathcal{T}(T)$ form the unique pre-projective component $\mathcal{P}(B)$ of $\Gamma(B)$. Moreover $\mathcal{P}(B)$ contains no injective modules.*
(e) *We have $\Gamma(B) = \mathcal{P}(B) \sqcup \mathcal{R}(B) \sqcup \mathcal{Q}(B)$.*



3.2 Mutations and partial orders

In this section we recall the definition of an important partial order on $\mathcal{T}(Q)$ and study combinatorial properties of the poset $\mathcal{T}(Q)$ (cf.[10],[11],[12],[13]).

Definition-Proposition 3.2.1. [11, Lemma 2.1]. Let $T, T' \in \mathcal{T}(Q)$. Then the following relation $\leq$ defines a partial order on $\mathcal{T}(Q)$.

$$T \geq T' \overset{\text{def}}{\Leftrightarrow} \mathrm{Ext}^1_{kQ}(T, T') = 0.$$

Definition 3.2.2. The tilting quiver $\overrightarrow{\mathcal{T}}(Q)$ is defined as follows:
• $\overrightarrow{\mathcal{T}}(Q)_0 := \mathcal{T}(Q)$,
• $T \to T'$ in $\overrightarrow{\mathcal{T}}(Q)$ if $T \simeq M \oplus X$, $T' \simeq M \oplus Y$ for some $X, Y \in$ ind-kQ, $M \in$ mod-kQ and there is a non split exact sequence

$$0 \to X \to M' \to Y \to 0,$$

with $M' \in \operatorname{add} M$.

In this situation we call T' a *right mutation* of T at X and call T a *left mutation* of T' at Y.

Lemma 3.2.3. [*10*, Proof of Theorem 2.1]. *Let $T, T' \in \mathcal{T}(Q)$ with $T < T'$. Then there exists $T'' \in \mathcal{T}(Q)$ such that $T \leq T''$ and there is an arrow $T' \to T''$ in $\overrightarrow{\mathcal{T}}(Q)$.*

Theorem 3.2.4. [*10*, Theorem 2.1]. *The tilting quiver $\overrightarrow{\mathcal{T}}(Q)$ coincides with the Hasse-quiver of $(\mathcal{T}(Q), \leq)$.*

Remark 3.2.5. In this paper we define the Hasse-quiver $\overrightarrow{P}$ of a poset $(P, \leq)$ as follows:
(1) $\overrightarrow{P}_0 := P$,
(2) $x \to y$ in $\overrightarrow{P}$ if $x > y$ and there is no $z \in P$ such that $x > z > y$.

Proposition 3.2.6. [*10*, Corollary 2.2]. *If $\overrightarrow{\mathcal{T}}(Q)$ has a finite connected component $\mathcal{C}$, then $\overrightarrow{\mathcal{T}}(Q) = \mathcal{C}$. In particular, if Q is a* Dynkin-*quiver, then $\overrightarrow{\mathcal{T}}(Q)$ is connected.*

For $T \in \mathcal{T}(Q)$, we set

$$\begin{aligned} s(T) &:= \#\{T' \in \mathcal{T}(Q) \mid T \to T' \text{ in } \overrightarrow{\mathcal{T}}(Q)\} \\ e(T) &:= \#\{T' \in \mathcal{T}(A) \mid T' \to T \text{ in } \overrightarrow{\mathcal{T}}(Q)\} \end{aligned}$$

and define $\delta(T) := s(T) + e(T)$.

Proposition 3.2.7. [*12*, Proposition 3.2]. *We have*

$$\delta(T) = n - \#\{a \in Q_0 \mid (\underline{\dim} T)_a = 1\},$$

where $n = \#Q_0$.

Let M be a basic partial tilting module and $\operatorname{lk}(M) := \{T \in \mathcal{T}(Q) \mid M \in \operatorname{add} T\}$. Then we denote by $\overrightarrow{\operatorname{lk}}(M)$ the full sub-quiver of $\overrightarrow{\mathcal{T}}(Q)$ having $\operatorname{lk}(M)$ as the set of vertices (see [13]).

Proposition 3.2.8. *[13, Theorem 4.1] If M is faithful, then $\overrightarrow{\operatorname{lk}}(M)$ is connected.*

Example 3.2.9. (1) Let $Q = 1 \to 2 \to 3$. Then $\overrightarrow{\mathcal{T}}(Q)$ is given by the following:

$$
\begin{tikzcd}
& \boxed{\begin{array}{l} k \to k \to k \\ 0 \to k \to k \\ 0 \to 0 \to k \end{array}} \arrow[dl] \arrow[dr] & \\
\boxed{\begin{array}{l} k \to k \to k \\ 0 \to k \to k \\ 0 \to k \to 0 \end{array}} \arrow[d] & & \boxed{\begin{array}{l} k \to k \to k \\ k \to 0 \to 0 \\ 0 \to 0 \to k \end{array}} \arrow[ddl] \\
\boxed{\begin{array}{l} k \to k \to k \\ k \to k \to 0 \\ 0 \to k \to 0 \end{array}} \arrow[dr] & & \\
& \boxed{\begin{array}{l} k \to k \to k \\ k \to k \to 0 \\ k \to 0 \to 0 \end{array}} &
\end{tikzcd}
$$

(2) We consider the following quiver:

$$
Q : 1 \rightrightarrows 2
$$

Then $\overrightarrow{\mathcal{T}}(Q)$ has two connected components

$$
\begin{tikzcd}
\boxed{\begin{array}{l} k \to k^2 \\ 0 \to k \end{array}} \arrow[d] & & \vdots \\
\boxed{\begin{array}{r} k \to k^2 \\ k^2 \to k^3 \end{array}} \arrow[d] & & \boxed{\begin{array}{l} k^5 \to k^4 \\ k^4 \to k^3 \end{array}} \arrow[u] \\
\boxed{\begin{array}{l} k^3 \to k^4 \\ k^2 \to k^3 \end{array}} \arrow[d] & \text{and} & \boxed{\begin{array}{l} k^3 \to k^2 \\ k^4 \to k^3 \end{array}} \arrow[u] \\
\boxed{\begin{array}{l} k^3 \to k^4 \\ k^4 \to k^5 \end{array}} \arrow[d] & & \boxed{\begin{array}{l} k^3 \to k^2 \\ k^2 \to k^3 \end{array}} \arrow[u] \\
\vdots & & \boxed{\begin{array}{r} k \to 0 \\ k^2 \to k \end{array}} \arrow[u]
\end{tikzcd}
$$

Chapter 4

The number of arrows in tilting quivers

If Q is a Dynkin quiver, then kQ is representation-finite. In particular, $\overrightarrow{\mathcal{T}}(Q)$ is a finite quiver. Therefore there arise two natural questions:

- How many vertices in $\overrightarrow{\mathcal{T}}(Q)$ are there?
- How many arrows in $\overrightarrow{\mathcal{T}}(Q)$ are there?

The number of vertices $\#\overrightarrow{\mathcal{T}}(Q)_0$ is already known ([6]). So, in this chapter, we give the number of arrows $\#\overrightarrow{\mathcal{T}}(Q)_1$.

In the following we regard kQ-modules as objects of rep Q.

4.1 Theorem of Ladkani

In this section, we review [16]. Let x be a source of Q and $Q^{'} = \sigma_x Q$ be a quiver obtained by reversing all arrows starting at x. We define

$$\mathcal{T}(Q)^x := \{T \in \mathcal{T}(Q) \mid S(x) \in \operatorname{add} T\},$$

where $S(x) \in$ mod-kQ is the simple module associated with x. Similarly we define

$$\mathcal{T}(Q^{'})^x := \{T^{'} \in \mathcal{T}(Q^{'}) \mid S^{'}(x) \in \operatorname{add} T^{'}\},$$

where $S^{'}(x) \in$ mod-$kQ^{'}$ is the simple module associated with x.

Definition 4.1.1. Let $(X, \leq_X)$,$(Y, \leq_Y)$ be posets and $f : X \to Y$ an order-preserving function. Then we define the partial-orders $\leq^f_+$, $\leq^f_-$ of $X \sqcup Y$ as follows.

$$a \leq^f_+ b \Longleftrightarrow \begin{cases} a \leq_X b & \text{if } a, b \in X, \\ a \leq_Y b & \text{if } a, b \in Y, \\ f(a) \leq_Y b & \text{if } a \in X \text{ and } b \in Y. \end{cases}$$

$$a \leq^f_- b \Longleftrightarrow \begin{cases} a \leq_X b & \text{if } a, b \in X, \\ a \leq_Y b & \text{if } a, b \in Y, \\ a \leq_Y f(b) & \text{if } a \in Y \text{ and } b \in X. \end{cases}$$

For any $M \in$ mod-$k(Q \setminus \{x\})$, we define $F(M) \in$ mod-kQ as follows:

$$F(M)_a = \begin{cases} M_a & \text{if } a \neq x, \\ \oplus_{x\to y} M_y & \text{if } a = x \end{cases}$$

$$F(M)_{a\to b} = \begin{cases} M_{a\to b} & \text{if } a \neq x, \\ \oplus_{x\to y} M_y \overset{\text{projection}}{\longrightarrow} M_b & \text{if } a = x. \end{cases}$$

Similarly we define $F^{'}(M) \in$ mod-$kQ^{'}$ as follows:

$$F^{'}(M)_a = \begin{cases} M_a & \text{if } a \neq x, \\ \oplus_{y\to x} M_y & \text{if } a = x \end{cases}$$

$$F^{'}(M)_{a\to b} = \begin{cases} M_{a\to b} & \text{if } b \neq x, \\ M_a \overset{\text{injection}}{\longrightarrow} \oplus_{y\to x} M_y & \text{if } b = x. \end{cases}$$

Theorem 4.1.2. ([*16*]) (1) $T \mapsto F(T) \oplus S(x)$ *induces a poset isomorphism*

$$\iota_x : \mathcal{T}(Q \setminus \{x\}) \simeq \mathcal{T}(Q)^x.$$

Similarly $T^{'} \mapsto F^{'}(T) \oplus S^{'}(x)$ *induces a poset isomorphism*

$$\iota_x^{'} : \mathcal{T}(Q \setminus \{x\}) \simeq \mathcal{T}(Q^{'})^x.$$

(2) *There is an order-preserving map* $f : \mathcal{T}(Q) \setminus \mathcal{T}(Q)^x \to \mathcal{T}(Q)^x$ *such that*

$$\mathcal{T}(Q) \simeq ((\mathcal{T}(Q) \setminus \mathcal{T}(Q)^x) \sqcup \mathcal{T}(Q)^x, \leq^f_-).$$

Similarly there is an order-preserving map $f^{'} : \mathcal{T}(Q^{'}) \setminus \mathcal{T}(Q^{'})^x \to \mathcal{T}(Q^{'})^x$ *such that*

$$\mathcal{T}(Q^{'}) \simeq ((\mathcal{T}(Q^{'}) \setminus \mathcal{T}(Q^{'})^x) \sqcup \mathcal{T}(Q^{'})^x, \leq^{f^{'}}_+).$$

(3) *There exists an isomorphism of posets*

$$\rho_x : \mathcal{T}(Q) \setminus \mathcal{T}(Q)^x \to \mathcal{T}(Q^{'}) \setminus \mathcal{T}(Q^{'})^x$$

such that the following diagram commutes.

$$\begin{tikzcd}
 & \mathcal{T}(Q) \setminus \mathcal{T}(Q)^x \arrow[rr, "\rho_x", "\sim"'] \arrow[dl, "f"'] \arrow[dr] & & \mathcal{T}(Q^{'}) \setminus \mathcal{T}(Q^{'})^x \arrow[dl] \arrow[dr, "f^{'}"] & \\
\mathcal{T}(Q)^x & & \mathcal{T}(Q \setminus \{x\}) \arrow[ll, "\iota_x"', "\sim"] \arrow[rr, "\iota_x^{'}", "\sim"'] & & \mathcal{T}(Q^{'})^x
\end{tikzcd}$$

Corollary 4.1.3. ([*16*]) $\#\mathcal{T}(Q) = \#\mathcal{T}(Q^{'})$.

Remark 4.1.4. In [16] the partial order on $\mathcal{T}(Q)$ is defined by

$$T \geq T^{'} \iff \mathrm{Ext}^1_{kQ}(T^{'}, T) = 0 \ \ \text{(opposite to our definition)}.$$

4.2 The number of arrows

In this section we determine the number of arrows in $\overrightarrow{\mathcal{T}}(Q)$ for a Dynkin quiver Q. Let

$$\begin{aligned} \text{Gen}\,(M) &:= \{N \in \text{mod-}A \mid M' \overset{\text{surjection}}{\to} N \text{ for some } M' \in \text{add}\, M\}, \\ \text{Cogen}\,(M) &:= \{N \in \text{mod-}A \mid N \overset{\text{injection}}{\to} M' \text{ for some } M' \in \text{add}\, M\}. \end{aligned}$$

Lemma 4.2.1. ([5, Proposition 1.3]) *Let $A = kQ$ be a path algebra, $T = M \oplus Y \in \mathcal{T}(Q)$ with $Y \in \text{ind-}A$. If $Y \in \text{Gen}\,(M)$, then there exists a unique(up to isomorphism) indecomposable module X which is not isomorphic to Y such that $M \oplus X \in \mathcal{T}(Q)$ and there exists an exact sequence*

$$0 \longrightarrow X \longrightarrow E \longrightarrow Y \longrightarrow 0$$

with $E \in \text{add}\, M$.

Dually, if $Y \in \text{Cogen}\,(M)$ then there exists a unique (up to isomorphism) indecomposable module X which is not isomorphic to Y such that $M \oplus Y \in \mathcal{T}(Q)$ and there exists an exact sequence

$$0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0$$

with $E \in \text{add}\, M$.

Lemma 4.2.2. *Let $x \in Q_0$ and $T = M \oplus S(x) \in \mathcal{T}(Q)$. If x is a sink, then $S(x)$ is in* Cogen (M)*. Dually, If x is a source, $S(x)$ is in* Gen (M)*.*

Proof. For any $T \in \mathcal{T}(Q \setminus \{x\})$, we define $F(T) \in \text{mod-}kQ$ as follows,

$$F(T)_a = \begin{cases} T_a & \text{if } a \neq x, \\ \oplus_{y \to x} T_y & \text{if } a = x \text{ and } x \text{ is a sink}, \\ \oplus_{x \to y} T_y & \text{if } a = x \text{ and } x \text{ is a source}. \end{cases}$$

$$F(T)_{a \to b} = \begin{cases} T_{a \to b} & \text{if } a, b \neq x, \\ T_y \overset{\text{injection}}{\longrightarrow} \oplus_{y' \to x} T_{y'} & \text{if } a = y \text{ with } y \to x \text{ and if } b = x \text{ and } x \text{ is a sink}, \\ \oplus_{x \to y'} T_{y'} \overset{\text{projection}}{\longrightarrow} T_y & \text{if } b = y \text{ with } x \to y \text{ and if } a = x \text{ and } x \text{ is a source}. \end{cases}$$

Then, by Proposition 4.1.2, $T \longmapsto F(T) \oplus S(x)$ induces a bijection

$$\mathcal{T}(Q \setminus \{x\}) \overset{1:1}{\longleftrightarrow} \mathcal{T}(Q)^x.$$

Now if x is a sink then

$$S(x) \in \text{Cogen}\,(M) \Longleftrightarrow M_x \neq 0,$$

and if x is a source then

$$S(x) \in \text{Gen}\,(M) \Longleftrightarrow M_x \neq 0.$$

Therefore, the assertion follows from the fact that if $T \in \mathcal{T}(Q)$ then $(\underline{\dim} T)_a \geq 1$, for all $a \in Q_0$. $\square$

Lemma 4.2.3. *If x is a sink then*

$$\{\alpha \in \overrightarrow{\mathcal{T}}(Q)_1 \mid s(\alpha) \in \mathcal{T}(Q)^x, t(\alpha) \in \mathcal{T}(Q) \setminus \mathcal{T}(Q)^x\} \overset{1:1}{\longleftrightarrow} \mathcal{T}(Q)^x.$$

If x is a source then

$$\{\alpha \in \overrightarrow{\mathcal{T}}(Q)_1 \mid t(\alpha) \in \mathcal{T}(Q)^x, s(\alpha) \in \mathcal{T}(Q) \setminus \mathcal{T}(Q)^x\} \overset{1:1}{\longleftrightarrow} \mathcal{T}(Q)^x.$$

Here, for $T \overset{\alpha}{\to} T'$, $s(\alpha) = T$ and $t(\alpha) = T'$.

Proof. Suppose x is a sink, and let $T \in \mathcal{T}(Q)^x$. Then there exists a unique $T' \in \mathcal{T}(Q) \setminus \mathcal{T}(Q)^x$ such that $T \longrightarrow T'$ in $\overrightarrow{\mathcal{T}}(Q)$ (by Lemma 4.2.1, 4.2.2). □

Corollary 4.2.4.

$$\#\overrightarrow{\mathcal{T}}(Q)_1 = \#\overrightarrow{\mathcal{T}}(\sigma_x Q)_1.$$

In particular, if Q is a Dynkin quiver then $\#\overrightarrow{\mathcal{T}}(Q)_1$ depends only on the underlying graph of Q.

Proof. By Theorem 4.1.2 and Lemma 4.2.3 we get,

$$\begin{aligned} \#\overrightarrow{\mathcal{T}}(Q)_1 &= \#\overrightarrow{\mathcal{T}}(Q \setminus \{x\})_1 + \#\overrightarrow{\mathcal{T}}(\mathcal{T}(Q) \setminus \mathcal{T}(Q)^x)_1 + \#\mathcal{T}(Q)^x \\ &= \#\overrightarrow{\mathcal{T}}(\sigma_x Q)_1. \end{aligned}$$

□

4.2.1 case A

In this subsection we consider the quiver,

$$Q = \overset{1}{\circ} \to \overset{2}{\circ} \to \cdots \to \overset{n}{\circ}.$$

By Gabriel's Theorem, ind-$kQ = \{L(i,j) \mid 0 \le i < j \le n\}$ where

$$L(i,j)_a = \begin{cases} k & (i < a \le j) \\ 0 & \text{otherwise} \end{cases} \quad \text{and } L(i,j)_{a \to b} = \begin{cases} 1 & (i < a, b \le j) \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\tau L(i,j) = \begin{cases} L(i+1, j+1) & (j < n), \\ 0 & (j = n), \end{cases}$$

where τ is the Auslander-Reiten translation.

Definition 4.2.5. A pair of intervals $([i,j],[i',j'])$ is said to be *compatible* if

$$[i,j] \cap [i',j'] = \emptyset \text{ or } [i,j] \subset [i',j'] \text{ or } [i',j'] \subset [i,j].$$

Applying Auslander-Reiten duality

$$\mathrm{D}\,\mathrm{Ext}^1_{kQ}(M,N) \cong \mathrm{Hom}_{kQ}(N, \tau M),$$

we have the following:

Lemma 4.2.6. *We have*

$$\mathrm{Ext}^1_{kQ}(L(i,j), L(i^{'}, j^{'})) = 0 = \mathrm{Ext}^1_{kQ}(L(i^{'}, j^{'}), L(i,j))$$

if and only if $([i,j],[i^{'},j^{'}])$ *is compatible.*

Proof. It is obvious that $\mathrm{Hom}(L(i,j), L(i^{'}, j^{'})) \neq 0$ if and only if $i^{'} \leq i \leq j^{'} \leq j$. Therefore the assertion follows from this fact and the AR-duality. □

Lemma 4.2.7. *For any* $T \in \mathcal{T}(Q)$*, we have* $\delta(T) = n - 1$.

Proof. Let $T \in \mathcal{T}(Q)$. Then the projective-injective module $L(0,n)$ is a direct summand of T. From this fact, we get $\delta(T) < n$.

Denote by X the set of indecomposable direct summands of T not isomorphic to $L(0,n)$ and define

$$a := \begin{cases} \max\{i \mid L(0,i) \in X\} & \text{if } L(0,i) \in X \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Lemma 4.2.6, we get

$$\mathrm{Ext}^1_{kQ}(T, L(a+1,n)) = 0 = \mathrm{Ext}^1_{kQ}(L(a+1,n), T).$$

By Ext $= 0$ condition, we can see that $L(a+1,n)$ is a direct summand of T. In particular,

$$(\underline{\dim} T)_i = 1 \Longleftrightarrow i = a+1.$$

The assertion follows from this fact and Proposition 3.2.7. □

Now it is easy to find the number of arrows in $\overrightarrow{\mathcal{T}}(Q)$, because it is equal to

$$\frac{1}{2} \sum_{T \in \mathcal{T}(\mathrm{Q})} \delta(T).$$

Namely, we obtain the result of Theorem 1.0.2 for type A_n.

Corollary 4.2.8. $\#\overrightarrow{\mathcal{T}}(Q)_1 = \frac{n-1}{2(n+1)}\binom{2n}{n} = \binom{2n-1}{n-2}$.

4.2.2 case D

Through this subsection, we consider the quiver

$$Q = Q_n = \begin{tikzcd}[row sep=small] & & & & n^+ \\ \overset{1}{\circ} \arrow[r] & \overset{2}{\circ} \arrow[r] & \cdots \arrow[r] & \overset{n-1}{\circ} \arrow[ur] \arrow[dr] & \\ & & & & n^- \end{tikzcd}$$

Then we have

$$\text{ind-}kQ = \{L(a,b) \mid 0 \leq a < b \leq n-1\} \cup \{L^{\pm}(a,n) \mid 0 \leq a \leq n-1\} \cup \{M(a,b) \mid 0 \leq a < b \leq n-1\}$$

where

$$
\begin{aligned}
L(a,b)_i &= \begin{cases} k & \text{if } a < i \le b, \\ 0 & \text{otherwise,} \end{cases} \\
L(a,b)_{i\to j} &= \begin{cases} 1 & \text{if } a < i < b, \\ 0 & \text{otherwise,} \end{cases} \\
L(a,n)^{\pm}_i &= \begin{cases} k & \text{if } a < i \le n-1 \textit{ or } i = n^{\pm}, \\ 0 & \text{otherwise,} \end{cases} \\
L(a,n)^{\pm}_{i\to j} &= \begin{cases} 1 & \text{if } a < i < n-1 \textit{ or } i = n-1, j = n^{\pm}, \\ 0 & \text{otherwise,} \end{cases} \\
M(a,b)_i &= \begin{cases} k & \text{if } a < i \le b \text{ or } i = n^{\pm}, \\ k^2 & \text{if } b < i \le n-1, \\ 0 & \text{otherwise,} \end{cases} \\
M(a,b)_{i\to j} &= \begin{cases} 1 & \text{if } a < i < b, \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } i = b, \\ (1,0) & \text{if } i = n-1, j = n^{+}, \\ (0,1) & \text{if } i = n-1, j = n^{-}, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } b < i < n-1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\tau L(a,b) &= \begin{cases} L(a+1,b+1) & \text{if } b < n-1, \\ M(0,a+1) & \text{if } b = n-1, \end{cases} \\
\tau L^{+}(a,n) &= \begin{cases} L^{-}(a+1,n) & \text{if } a < n-1, \\ 0 & \text{if } a = n-1, \end{cases} \\
\tau L^{-}(a,n) &= \begin{cases} L^{+}(a+1,n) & \text{if } a < n-1, \\ 0 & \text{if } a = n-1, \end{cases} \\
\tau M(a,b) &= \begin{cases} M(a+1,b+1) & \text{if } b < n-1, \\ 0 & \text{if } b = n-1. \end{cases}
\end{aligned}
$$

Lemma 4.2.9.

(1) $\mathrm{Ext}^1_{kQ}(L(a,b), L(a^{'},b^{'})) = 0 = \mathrm{Ext}^1_{kQ}(L(a^{'},b^{'}), L(a,b))$
$\Longleftrightarrow ([a,b],[a^{'},b^{'}])$: compatible.

(2) $\mathrm{Ext}^1_{kQ}(L(a,b), L^{\pm}(a^{'},n)) = 0 = \mathrm{Ext}^1_{kQ}(L^{\pm}(a^{'},n), L(a,b))$
$\Longleftrightarrow ([a,b],[a^{'},n])$: compatible.

(3) $\mathrm{Ext}^1_{kQ}(L(a,b), M(a^{'},b^{'})) = 0 = \mathrm{Ext}^1_{kQ}(M(a^{'},b^{'}), L(a,b))$
$\Longleftrightarrow ([a,b],[a^{'},n]), ([a,b],[b^{'},n])$: compatible.

(4) $\mathrm{Ext}^1_{kQ}(M(a,b), L^{\pm}(a^{'},n)) = 0 = \mathrm{Ext}^1_{kQ}(L^{\pm}(a^{'},n), M(a,b))$
$\Longleftrightarrow a \le a^{'} \le b$.

(5)$\mathrm{Ext}(L^{\pm}(a,n), L^{\pm}(a^{'},n)) = 0 = \mathrm{Ext}^1_{kQ}(L^{\pm}(a^{'},n), L^{\pm}(a,n))$ for all $a, a^{'}$.

(6) $\mathrm{Ext}^1_{kQ}(L^{+}(a,n), L^{-}(a^{'},n)) = 0 = \mathrm{Ext}^1_{kQ}(L^{-}(a^{'},n), L^{+}(a,n))$
$\Longleftrightarrow a = a^{'}$.

(7) $\mathrm{Ext}^1_{kQ}(M(a,b), M(a^{'},b^{'})) = 0 = \mathrm{Ext}^1_{kQ}(M(a^{'},b^{'}), M(a,b))$
$\Longleftrightarrow [a,b] \subset [a^{'},b^{'}]$ *or* $[a^{'},b^{'}] \subset [a,b]$.

Proof. (1) and (2) follow from the case A and (5),(6) are obvious. We prove (3).
(case $b < a^{'}$) It is obvious that

$$\mathrm{Ext}^1_{kQ}(L(a,b), M(a^{'},b^{'})) = 0 = \mathrm{Ext}^1_{kQ}(M(a^{'},b^{'}), L(a,b)).$$

(case $a < a^{'} \le b < b^{'}$) In this case we claim that

$$\mathrm{Hom}(M(a^{'},b^{'}), \tau L(a,b)) \neq 0.$$

In fact $0 \neq f = (f_i)_i \in \mathrm{Hom}(M(a^{'},b^{'}), \tau L(a,b))$ where

$$f_i = \begin{cases} 1 & \text{if } a^{'} < i \le b+1, \\ 0 & \text{otherwise.} \end{cases}$$

(case $a < a^{'} < b^{'} \le b < n-1$) In this case we claim that

$$\mathrm{Hom}(M(a^{'},b^{'}), \tau L(a,b)) \neq 0.$$

In fact $0 \neq f = (f_i)_i \in \mathrm{Hom}(M(a^{'},b^{'}), \tau L(a,b))$ where

$$f_i = \begin{cases} 1 & \text{if } a^{'} < i \le b^{'}, \\ (0,1) & \text{if } b^{'} < i \le b, \\ 0 & \text{otherwise.} \end{cases}$$

(case $a < a^{'} < b^{'} \le b = n-1$) In this case we also claim that

$$\mathrm{Hom}(M(a^{'},b^{'}), \tau L(a,b)) \neq 0.$$

In fact $0 \neq f = (f_i)_i \in \mathrm{Hom}(M(a^{'}, b^{'}), \tau L(a, n-1))$ where

$$f_i = \begin{cases} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } a^{'} < i \leq b^{'}, \\ 1 & \text{if } b^{'} < i \leq n-1 \text{ or } i = n^{\pm}, \\ 0 & \text{otherwise.} \end{cases}$$

(case $a^{'} \leq a < b < b^{'} < n-1$) In this case we claim that

$$\mathrm{Hom}(M(a^{'}, b^{'}), \tau L(a, b)) = 0 = \mathrm{Hom}(L(a, b), \tau M(a^{'}, b^{'})).$$

Let $f = (f_i)_i \in \mathrm{Hom}(M(a^{'}, b^{'}), \tau L(a, b))$. If $i \leq a+1$ or $b+1 < i \leq n-1$ or $i = n^{\pm}$ then $(\underline{\dim}\tau L(a, b))_i = 0$ and this implies $f_i = 0$. Note that

$$f_{a+2} = f_{a+3} = \cdots = f_{b+1}.$$

Now the commutative square for f_{a+1}, f_{a+2} shows $f_{a+2} = 0$. So

$$\mathrm{Hom}(M(a^{'}, b^{'}), \tau L(a, b)) = 0.$$

Similarly

$$\mathrm{Hom}(L(a, b), \tau M(a^{'}, b^{'})) = 0.$$

(case $a^{'} \leq a < b < b^{'} = n-1$) From the argument similar to the case ($a^{'} \leq a < b < b^{'} < n-1$) we can get

$$\mathrm{Hom}(M(a^{'}, n-1), \tau L(a, b)) = 0.$$

Since $M(a^{'}, n-1)$ is projective, we have

$$\mathrm{Hom}(L(a, b), \tau M(a^{'}, b^{'})) = 0.$$

(case $a^{'} \leq a < b^{'} \leq b < n-1$) In this case we claim that

$$\mathrm{Hom}(M(a^{'}, b^{'}), \tau L(a, b)) \neq 0.$$

In fact $0 \neq f = (f_i)_i \in \mathrm{Hom}(M(a^{'}, b^{'}), \tau L(a, b))$ where

$$f_i = \begin{cases} (1, -1) & \text{if } b^{'} < i \leq b+1, \\ 0 & \text{otherwise.} \end{cases}$$

(case $a^{'} \leq a < b^{'} \leq b = n-1$) In this case we also claim that

$$\mathrm{Hom}(M(a^{'}, b^{'}), \tau L(a, b)) \neq 0.$$

In fact $0 \neq f = (f_i)_i \in \mathrm{Hom}(M(a^{'}, b^{'}), \tau L(a, n-1))$ where

$$f_i = \begin{cases} 1 & \text{if } a^{'} < i \leq a+1 \text{ or } i = n^{\pm}, \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } a+1 < i \leq b^{'}, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } b^{'} < i \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

(case $b' \leq a$) From the argument similar to the case $a' \leq a < b < b'$, we get

$$\mathrm{Hom}(M(a',b'), \tau L(a,b)) = 0 = \mathrm{Hom}(L(a,b), \tau M(a',b')).$$

So we have proved (3). We can prove (4),(7) similarly. □

Lemma 4.2.10. *Let $T \in \mathcal{T}(Q)$.*

(1) *$L(0,n-1) \mid T$ implies $L^{\pm}(0,n) \mid T$.*

(2) *If $L^{+}(0,n) \mid T$ (resp. $L^{-}(0,n) \mid T$) and all indecomposable direct summands of T are insincere, then $L^{-}(0,n) \mid T$ (resp. $L^{+}(0,n) \mid T$).*

Proof. (1) : Suppose $L(0,n-1) \mid T$. Then

$$\mathrm{Ext}^1_{kQ}(T, L(0,n-1)) = 0 = \mathrm{Ext}^1_{kQ}(L(0,n-1), T)$$

and there exists an injection

$$\tau L^{\pm}(0,n) \longrightarrow \tau L(0,n-1).$$

Hence we have

$$\mathrm{Ext}^1_{kQ}(L^{\pm}(0,n), T) \simeq \mathrm{Hom}(T, \tau L^{\pm}(0,n)) = 0.$$

Since $L^{\pm}(0,n)$ is injective, we also get

$$\mathrm{Ext}^1_{kQ}(T, L^{\pm}(0,n)) = 0.$$

Thus we obtain $L^{\pm}(0,n) \mid T$.

(2): Suppose $L^{+}(0,n) \mid T$ and that all indecomposable direct summands of T are insincere. Since $(\underline{\dim} T)_{n^-} \neq 0$ from the fact that T is a tilting module, there exists some indecomposable direct summand N s.t.

$$(\underline{\dim} N)_{n^-} \neq 0.$$

If $N = M(a,b)$ then $\mathrm{Ext}^1_{kQ}(M(a,b), L^{+}(0,n)) = 0 = \mathrm{Ext}^1_{kQ}(L^{+}(0,n), M(a,b))$ so $a = 0$ and N is sincere. This is a contradiction. So $N = L^{-}(a,n)$ and $a = 0$ by $L^{+}(0,n) \mid T$. □

Lemma 4.2.11. *For all $T \in \mathcal{T}(Q)$ there exists some indecomposable direct summand N of T such that*

$$(\underline{\dim} N)_i \geq 1, \text{ for all } i \leq n-1.$$

Thus, $N = L(0,n-1)$, $L^{\pm}(0,n)$ or $M(0,b)$, for some b.

Proof. For an indecomposable direct summand N of T such that $(\underline{\dim} N)_1 = 1$, define

$$a(N) \stackrel{\mathrm{def}}{=} \sup\{i \mid 1 \leq i \leq n-1, (\underline{\dim} N)_i \geq 1\}.$$

Suppose that $\sup\ a(N) = a < n-1$, then $L(0,a) \mid T$. Therefore indecomposable direct summands of T are of the following form

$$\begin{array}{ll} L(a',b') & \text{for } b' \leq a \text{ or } a+1 \leq a', \\ L^{+}(a',n) & \text{for } a+1 \leq a', \\ M(a',b') & \text{for } a+1 \leq a'. \end{array}$$

Thus we have $(\underline{\dim} T)_{a+1} = 0$. This is a contradiction. □

Lemma 4.2.12. *We have*

$$\#\{i \mid 1 \leq i \leq n-1,\ (\underline{\dim} T)_i = 1\} \leq 1.$$

In particular, $\delta(T) \geq n-2$.

Proof. Let $i \neq n^{\pm}$ s.t. $(\underline{\dim} T)_i = 1$. Then we claim that

$$L(0, i-1) \mid T.$$

By Lemma 4.2.11 there exists a unique indecomposable direct summand N of T s.t.

$$(\underline{\dim} N)_j \geq 1 \text{ for all } j \leq n-1.$$

Hence, by Lemma 4.2.10, $N = M(0, b)$ for some $j \leq b \leq n-1$ and any indecomposable direct summand of T other than N is one of the following,

$$\begin{array}{ll} L(a,b) & \text{for } b \leq i-1 \text{ or } i \leq a, \\ L^{\pm}(a,n) & \text{for } i \leq a, \\ M(a,b) & \text{for } i \leq a. \end{array}$$

It implies

$$\operatorname{Ext}^1_{kQ}(T, L(0, i-1)) = 0 = \operatorname{Ext}^1_{kQ}(L(0, i-1), T),$$

so that

$$L(0, i-1) \mid T.$$

□

Corollary 4.2.13. *Let $T \in \mathcal{T}(Q)$. Then $\delta(T) \geq n-1$. The equality holds if and only if $L^{\pm}(0,n) \mid T$ and other indecomposable direct summands of T have the form $L(a,b)$ $(0 \leq a < b \leq n-1)$. In particular, we have*

$$\#\{T \in \mathcal{T}(Q) \mid \delta(T) = n-1\} = \frac{1}{n}\binom{2(n-1)}{n-1} = \frac{1}{n-1}\binom{2(n-1)}{n-2}.$$

Proof. Suppose that all indecomposable direct summands of T are insincere. Then, by Lemma 4.2.10 and Lemma 4.2.11, $L^{+}(0,n)$ and $L^{-}(0,n)$ are both direct summands of T. So $(\underline{\dim} T)_i = 1$ if and only if $i = n^{\pm}$. We have $\delta(T) \geq n-1$. If the equality holds then indecomposable direct summands of T not isomorphic to $L^{\pm}(0,n)$ are of the form $L(a,b)$.

Next we suppose there is a sincere indecomposable direct summand N of T. If $\delta(T) = n-2$ then, by Lemma 4.2.12, there is a unique $i \leq n-1$ s.t.

$$(\underline{\dim} T)_i = (\underline{\dim} T)_{n^{\pm}} = 1.$$

So all indecomposable direct summands of T not isomorphic to N are of the form $L(a,b)$ ($b < i$ or $i \leq a$). As their direct sum may be viewed as a rigid module in type $A_{i-1} \times A_{n-i-1}$, we get

$$\#\{L(a,b) \mid\ L(a,b) \mid T\} \leq (i-1) + (n-1-i) = n-2,$$

which is a contradiction. Next we consider the case $\delta(T) = n-1$.

(a) : $(\underline{\dim} T)_i = (\underline{\dim} T)_{n^+} = 1$, for a unique $i(\leq n-1)$. Then indecomposable direct summands of T other than N are of the following form:

$$\begin{array}{ll} L(a,b) & \text{for } b < i \text{ or } i \leq a, \\ L^-(a,n) & \text{for } i \leq a. \end{array}$$

We get by the same argument that

$$\#\{L \in \text{ind-}kQ \mid \ L \mid T,\ L \neq N\} \leq (i-1) + (n-i) = n-1,$$

which is a contradiction.

(b) : $(\underline{\dim} T)_i = (\underline{\dim} T)_{n^-} = 1$, for a unique $i(\leq n-1)$. Then, similar to (a), we reach a contradiction.

(c) : $(\underline{\dim} T)_{n^\pm} = 1$. Then indecomposable direct summands of T not isomorphic to N are of the form $L(a,b)$. Thus

$$\#\{L(a,b) \mid \ L(a,b) \mid T\} \leq n-1.$$

It is a contradiction. Therefore we have $\delta(T) \geq n$ and $\delta(T) = n-1$ does not occur in this case.

Thus we have proved that if $\delta(T) = n-1$ then $L^\pm(0,n) \mid T$ and the other indecomposable direct summands of T have the form $L(a,b)$. The converse implication is clear. $\square$

Now we define subsets $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2$ of $\mathcal{T}(Q)$ by

$$\begin{array}{lll} \mathcal{T}_0 & := & \{T \in \mathcal{T}(Q) \mid \delta(T) = n+1\}, \\ \mathcal{T}_1 & := & \{T \in \mathcal{T}(Q) \mid \delta(T) = n\}, \\ \mathcal{T}_2 & := & \{T \in \mathcal{T}(Q) \mid \delta(T) = n-1\}. \end{array}$$

Lemma 4.2.14. *Fix $1 \leq i \leq n-1$, then there is a bijection*

$$\{T \in \mathcal{T}_1 \mid (\underline{\dim} T)_i = 1\} \overset{1:1}{\longleftrightarrow}$$
$$\mathcal{T}(\circ \to \circ \to \cdots \to \overset{i-1}{\circ}) \times \{T \in \mathcal{T}(Q_{n-i+1}) \mid (\underline{\dim} T)_1 = 1,\ \delta(T) = n-i+1\}.$$

Proof. Let $T \in \mathcal{T}_1$ such that $(\underline{\dim} T)_i = 1$, for a unique $i(\leq n-1)$. By Lemma 4.2.10 and Lemma 4.2.11 there exists a unique $j = j(T)(\geq i)$ s.t. $M(0,j) \mid T$. Now let

$$X(T) = \{L(a,b) \mid \ L(a,b) \mid T,\ b < i\}$$

and

$$Y(T) = \{N \in \text{ind-}kQ \mid \ N \mid T\} \setminus \{X(T) \cup \{M(0,j)\}\}.$$

We define the maps

$$\varphi_T :\ X(T) \longrightarrow \text{ind-}k(\circ \to \circ \to \cdots \to \overset{i-1}{\circ})$$

and

$$\psi_T :\ Y(T) \longrightarrow \text{ind-}kQ_{n-i+1},$$

by

$$(\varphi_T(N))_a = (N)_a \ (1 \leq a < i),$$

$$(\psi_T(N))_a = (N)_{a+i-1} \ (\text{let } (n-i+1)^\pm + i - 1 = n^\pm).$$

Then

$$T \longmapsto \left(\bigoplus_{x \in X(T)} \varphi_T(x), \bigoplus_{y \in Y(T)} \psi_T(y) \bigoplus M(0, j(T) - i + 1) \right)$$

induces a bijection between

$$\{T \in \mathcal{T}_1 \mid (\underline{\dim} T)_i = 1\}$$

and

$$\mathcal{T}(\circ \to \circ \to \cdots \to \overset{i-1}{\circ}) \times \{T \in \mathcal{T}(Q) \mid (\underline{\dim} T)_1 = 1,\ \delta(T) = n - i + 1\}.$$

□

Let us define the following subsets of $\mathcal{T}_1$:

$$\begin{array}{lcl}
\mathcal{A}_\pm & := & \left\{ T \in \mathcal{T}_1 \mid \begin{array}{l} \text{all indecomposable direct summands of } T \text{ are insincere} \\ \text{and } (\underline{\dim} T)_{n^\pm} = 1 \end{array} \right\}, \\
\mathcal{B}_\pm & := & \{T \in \mathcal{T}_1 \mid (\underline{\dim} T)_{n^\pm} = 1, \text{ there exists some } j \text{ s.t. } M(0,j) \mid T\}, \\
\mathcal{B}_\pm(j) & := & \{T \in \mathcal{B}_\pm \mid \ M(0,j) \mid T\}, \\
\mathcal{C} & := & \{T \in \mathcal{T}_1 \mid \delta(T) = n,\ (\underline{\dim} T)_1 = 1\}, \\
\mathcal{C}(j) & := & \{T \in \mathcal{C} \mid \ M(0,j) \mid T\}.
\end{array}$$

Theorem 4.2.15. (1) : $\mathcal{A}_\pm = \emptyset$.
(2) : $\mathcal{B}_\pm(j) \overset{1:1}{\longleftrightarrow} \{T' \in \mathcal{T}(\circ \to \cdots \to \overset{n}{\circ}) \mid \min\{j' \mid L(j', n-1) \mid T'\} = j\}$. *In particular,*

$$\mathcal{B}_\pm \overset{1:1}{\longleftrightarrow} \mathcal{T}(\circ \to \cdots \to \overset{n}{\circ}) \setminus \{T' \in \mathcal{T}(\circ \to \cdots \to \overset{n}{\circ}) \mid L(0, n-1) \mid T'\},$$

and we have

$$\#\mathcal{B}_\pm = \frac{1}{n+1}\binom{2n}{n} - \frac{1}{n}\binom{2(n-1)}{n-1}.$$

(3) : $\mathcal{C}(j) \overset{1:1}{\longleftrightarrow} \{T' \in \mathcal{T}(Q_{n-1}) \mid j = j'(T') + 1\}$
where

$$j'(T') = \sup\{b \mid L^+(b, n-1) \text{ or } L^-(b, n-1) \text{ or } M(a,b) \mid T \text{ for some } a\}.$$

In particular,

$$\mathcal{C} \overset{1:1}{\longleftrightarrow} \mathcal{T}(Q_{n-1}),$$

and we have

$$\#\mathcal{C} = \frac{3n-4}{2n}\binom{2(n-1)}{n-1}.$$

Proof. (1) Suppose that there exists some $T \in \mathcal{A}_+$. Then, by Lemma 4.2.11, we have $L^\pm(0,n) \mid T$. Now there exists some indecomposable direct summand N of T not isomorphic to $L^-(0,n)$ s.t. $(\underline{\dim} N)_{n^-} = 1$.

If $N = M(a,b)$ or $L^-(a,n)$ then $a = 0$. This is a contradiction because $L^\pm(0,n) \mid T$. Therefore $\mathcal{A}_+ = \emptyset$ and similarly we have $A_- = \emptyset$.

(2) Define

$$\varphi : \{L(a,b) \mid 0 \le a < b \le n-1\} \cup \{L^-(a,n) \mid 0 \le a \le n-1\} \longrightarrow \text{ind-}k(\circ \to \circ \to \cdots \to \overset{n}{\circ})$$

and

$$\psi : \text{ind-}k(\circ \to \circ \to \cdots \to \overset{n}{\circ}) \longrightarrow \{L(a,b) \mid 0 \le a < b \le n-1\} \cup \{L^-(a,n) \mid 0 \le a \le n-1\}$$

by

$$(\varphi(L))_a = \begin{cases} L_a & \text{if } 0 \le a \le n-1, \\ L_{n^-} & \text{if } a = n, \end{cases}$$
$$(\psi(L'))_a = \begin{cases} L'_a & \text{if } 0 \le a \le n-1, \\ L'_n & \text{if } a = n^-, \\ 0 & \text{if } a = n^+, \end{cases}$$

respectively. Then $\varphi \circ \psi = 1 = \psi \circ \varphi$. Define

$$Z(T) := \{N \in \text{ind-}kQ \mid N \mid T, N \not\cong M(0,j)\}$$

and

$$Y(T') := \{N \in \text{ind-}kQ \mid N \mid T'\}.$$

Then it is easy to see that the maps induce a bijection

$$\mathcal{B}_+(j) \overset{1:1}{\longleftrightarrow} \{T' \in \mathcal{T}(\circ \to \cdots \to \overset{n}{\circ}) \mid \min\{j' \mid L(j', n-1) \mid T'\} = j\}$$

by

$$T \mapsto \bigoplus_{L \in Z(T)} \varphi(L)$$

and its inverse

$$T' \mapsto \left(\bigoplus_{L' \in Y(T')} \psi(L') \right) \oplus M(0,j).$$

In fact, if $T \in \mathcal{B}_+(j)$ then all indecomposable direct summands of T not isomorphic to $M(0,j)$ are either

$$L(a,b) \ (a \ge j \text{ or } b < j) \text{ or } L^-(a,n) \ (a \le j),$$

which implies $L(j, n-1), L^-(j,n) \mid T$. It follows

$$\min\{j' \mid L(j', n-1) \mid \bigoplus_{L \in Z(T)} \varphi(L)\} = j.$$

Conversely, if

$$T' \in \{T' \in \mathcal{T}(\circ \to \cdots \to \overset{n}{\circ}) \mid \min\{j' \mid L(j', n-1) \mid T'\} = j\}$$

then $\left(\bigoplus_{L' \in Y(T')} \psi(L') \right) \oplus M(0,j) \in \mathcal{B}_+(j)$.

(3) Define

$$\varphi : \{N \in \text{ind-}kQ_n \mid (\underline{\dim} N)_1 = 0\} \longrightarrow \text{ind-}kQ_{n-1}$$

and

$$\psi : \text{ind-}kQ_{n-1} \longrightarrow \{N \in \text{ind-}kQ_n \mid (\underline{\dim} N)_1 = 0\}$$

by the obvious way. Then $\varphi \circ \psi = 1 = \psi \circ \varphi$. Define

$$Z(T) := \{N \in \text{ind-}kQ \mid N \mid T, N \not\cong M(0,j)\}$$

and

$$Y(T^{'}) := \{N \in \text{ind-}kQ \mid N \mid T^{'}\}.$$

Then they induce a bijection

$$\mathcal{C}(j) \xleftrightarrow{1:1} \{T^{'} \in \mathcal{T}(Q_{n-1}) \mid j = j^{'}(T^{'}) + 1\}$$

by

$$T \longmapsto \bigoplus_{N \in Z(T)} \varphi(N).$$

The inverse map is

$$T^{'} \longmapsto \left(\bigoplus_{N^{'} \in Y(T^{'})} \psi(N^{'}) \right) \oplus M(0, j+1).$$

In fact, if $T \in \mathcal{C}(j)$ then

$$Z(T) \subset \{L(a,b) \mid 1 \leq a < b < j\} \cup \{L^{\pm}(b,n) \mid 1 \leq b \leq j\} \cup \{M(a,b) \mid 1 \leq a < b \leq j\}.$$

It implies $M(1,j) \mid T$ and $j^{'}\left(\bigoplus_{N \in Z(T)} \varphi(N)\right) = j - 1$. Conversely, if $j = j^{'}(T^{'}) + 1$ then

$$\left(\underline{\dim} \oplus_{N^{'} \in Y(T^{'})} \psi(N^{'})\right)_a \begin{cases} \geq 1 & (a \geq 2) \\ = 0 & (a = 1). \end{cases}$$

It implies

$$\left(\bigoplus_{N^{'} \in Y(T^{'})} \psi(N^{'}) \right) \oplus M(0, j) \in \mathcal{C}(j).$$

□

Corollary 4.2.16.

$$\#\mathcal{T}_1 = 3\binom{2(n-1)}{n-2}.$$

Proof. First we claim that

$$\sum_{i=1}^{n} \frac{1}{i(n+1-i)} \binom{2(i-1)}{i-1} \binom{2(n-i)}{n-i} = \frac{1}{n+1} \binom{2n}{n}.$$

This follows from the fact that

$$\mathcal{T}(\circ \to \cdots \to \overset{n}{\circ}) = \bigsqcup \{T \in \mathcal{T}(\circ \to \cdots \to \overset{n}{\circ}) \mid \min\{i^{'} \mid L(i^{'}, n) \mid T,\ i^{'} > 0\} = i\}.$$

Thus, by Lemma 4.2.14 and Theorem 4.2.15, $\#\mathcal{T}_1$ is equal to

$$2\left(\frac{1}{n+1}\binom{2n}{n}-\frac{1}{n}\binom{2(n-1)}{n-1}\right)+\sum_{i=1}^{n-1}\frac{3(n-i)-1}{2i(n-i+1)}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}$$

$$=2\left\{\left(\frac{1}{n+1}\binom{2n}{n}-\frac{1}{n}\binom{2(n-1)}{n-1}\right)-\sum_{i=1}^{n-1}\frac{1}{i(n-i+1)}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}\right\}$$

$$+\sum_{i=1}^{n-1}\frac{3}{2i}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}$$

$$=\frac{3}{2}\sum_{i=1}^{n-1}\frac{1}{i}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}.$$

Now let

$$a_n=\sum_{i=1}^{n}\frac{1}{i}\binom{2(i-1)}{i-1}\binom{2(n+1-i)}{n+1-i}$$

and

$$f(X)=\left(\sum_{i=1}^{n}\frac{1}{i}\binom{2(i-1)}{i-1}X^i\right)^2.$$

Then the coefficient of X^{n+1} in $f'(X)$ is equal to

$$2a_n-2\binom{2n}{n}.$$

On the other hand, using the claim above, the coefficient of X^{n+2} in $f(X)$ is equal to

$$\sum_{i=1}^{n+1}\frac{1}{i(n-i+2)}\binom{2(i-1)}{i-1}\binom{2(n-i+1)}{n-i+1}-\frac{2}{n+1}\binom{2n}{n}$$

$$=\frac{1}{n+2}\binom{2(n+1)}{n+1}-\frac{2}{n+1}\binom{2n}{n}.$$

Therefore we have

$$2a_n=\binom{2(n+1)}{n+1}-\frac{2}{n+1}\binom{2n}{n}=4\binom{2n}{n-1}.$$

We conclude that

$$\#\mathcal{T}_1=\frac{3}{2}a_{n-1}=3\binom{2(n-1)}{n-2}.$$

□

Corollary 4.2.17. *We have*

$$\#\mathcal{T}_0 = \frac{3(n-1)}{n+1}\binom{2(n-1)}{n-2}.$$

Proof. Recall that the number of tilting modules is given in the table from the introduction, $\#\mathcal{T}_1$ is given in Corollary 4.2.16 and $\#\mathcal{T}_2$ is given in Corollary 4.2.13. Hence

$$\begin{aligned}\#\mathcal{T}_0 &= \frac{3n-1}{2(n+1)}\binom{2n}{n} - 3\binom{2(n-1)}{n-2} - \frac{1}{n}\binom{2(n-1)}{n-1} \\ &= \frac{3(n-1)}{n+1}\binom{2(n-1)}{n-2}.\end{aligned}$$

□

We have reached the first main result of this thesis for type D_{n+1}.

Theorem 4.2.18.

$$\#\overrightarrow{\mathcal{T}}(Q)_1 = (3n-1)\binom{2(n-1)}{n-2}.$$

Proof. In fact, $\#\overrightarrow{\mathcal{T}}(Q)_1$ is equal to

$$\begin{aligned}&\frac{1}{2}\left\{\frac{n-1}{n-1}\binom{2(n-1)}{n-2} + 3n\binom{2(n-1)}{n-2} + 3(n-1)\binom{2(n-1)}{n-2}\right\} \\ =&(3n-1)\binom{2(n-1)}{n-2}.\end{aligned}$$

□

4.2.3 case E_6, E_7, E_8

By using AR-sequences, it is possible to calculate the dimension vector of any indecomposable modules and the dimension of the space of the morphism between any two indecomposable modules. Thus, by using AR-duality and Proposition 3.2.7, we can calculate the number of arrows in the tilting quiver. I have written a computer program in C++ and results for E_6, E_7, E_8 are as in the list in the first main theorem.

Chapter 5

Poset-structure of $\mathcal{T}_{\mathrm{p}}(Q)$

This chapter contains the second main result of this thesis. In section 1 we give basic properties of the poset of pre-projective tilting modules. In section 2 we give an criterion for Ext^1_{kQ}-vanishing for pre-projective modules over kQ under the condition (C) from the introduction. This criterion plays an important role in this chapter. In section 3 we determine the structure of $\overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q)$ in the case of $l(Q) := \max\{l_Q(x, y) \mid x, y \in Q_0\} \leq 1$.

From now on we use the following notations:

- $\mathcal{T}_{\mathrm{p}}(Q) := \{T \in \mathcal{T}(Q) \mid T \text{ is pre-projective}\}$.
- $\overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q)$:the full sub-quiver of $\overrightarrow{\mathcal{T}}(Q)$ with $\overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q)_0 := \mathcal{T}_{\mathrm{p}}(Q)$.
- $\mathrm{lk}_{\mathrm{p}}(M) := \mathrm{lk}(M) \cap \mathcal{T}_{\mathrm{p}}(Q)$.
- $\overrightarrow{\mathrm{lk}}_{\mathrm{p}}(M)$:the full sub-quiver of $\overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q)$ with $\overrightarrow{\mathrm{lk}}_{\mathrm{p}}(M)_0 := \mathrm{lk}_{\mathrm{p}}(M)$.

5.1 Elementary properties of $\mathcal{T}_{\mathrm{p}}(Q)$

Lemma 5.1.1. *Let $T \in \mathcal{T}_{\mathrm{p}}(Q)$ and $T \leq T' \in \mathcal{T}(Q)$. Then $T' \in \mathcal{T}_{\mathrm{p}}(Q)$. In particular $\overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q)$* (resp. $\overrightarrow{\mathrm{lk}}_{\mathrm{p}}(M)$) *is the Hasse-quiver of* $(\mathcal{T}_{\mathrm{p}}, \leq)$ (resp. $(\mathrm{lk}_{\mathrm{p}}(M), \leq)$).

Proof. Let X be an indecomposable direct summand of T'. If X is not pre-projective, then $\mathrm{Ext}^1_{kQ}(\tau^{-r}P, X) \simeq \mathrm{Ext}^1_{kQ}(P, \tau^r X) = 0$ for any projective module P. Therefore $\mathrm{Ext}^1_{kQ}(T, X) = 0$. Since $\mathrm{Ext}^1_{kQ}(X, T) = 0$, we obtain $X \in \mathrm{add}\, T$. Therefore we get a contradiction. □

We set $\mathcal{T}_{\mathrm{p}}(\geq T) := \{T' \in \mathcal{T}_{\mathrm{p}}(Q) \mid T' \geq T\}$.

Lemma 5.1.2. *We have $\#\mathcal{T}_{\mathrm{p}}(\geq T) < \infty$ for any $T \in \mathcal{T}_{\mathrm{p}}(Q)$.*

Proof. Let $T \in \mathcal{T}_{\mathrm{p}}(Q)$. By Auslander-Reiten duality, we have

$$\mathrm{Ext}^1_{kQ}(X, T) = 0 \Leftrightarrow \tau X \in \mathcal{F}(T).$$

Therefore Theorem 3.1.8 implies that $\mathcal{T}_{\mathrm{p}}(\geq T) < \infty$. □

Proposition 5.1.3. *Let $T, T' \in \mathcal{T}_{\mathrm{p}}(Q)$. If $T > T'$, then there is a path from T to T' in $\overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q)$.*

Proof. This follows from Lemma 3.2.3 and Lemma 5.1.2. □

5.2 Criterion for Ext-vanishing

For any vertex $x \in Q_0$, we set $s(x) := \{\alpha \in Q_1 \mid s(\alpha) = x\}$, $t(x) := \{\beta \in Q_1 \mid t(\beta) = x\}$ and $\delta(x) := \#s(x) + \#t(x)$. Now we consider the following condition:

$$\text{(C)} \qquad \delta(a) := \#\{\alpha \in Q_1 \mid s(\alpha) = a \text{ or } t(\alpha) = a\} \geq 2, \quad \forall a \in Q_0.$$

Lemma 5.2.1. *Assume Q satisfies the condition* (C). *If there is an arrow $\gamma : x \to y$ in Q. Then*

$$\dim\ \mathrm{Ext}^1_{kQ}(\tau^{-r}P(y), X) \leq \dim\ \mathrm{Ext}^1_{kQ}(\tau^{-r}P(x), X) \leq \dim\ \mathrm{Ext}^1_{kQ}(\tau^{-r-1}P(y), X)$$

for any $r \geq 0$ and $X \in \text{ind-}kQ$.

Proof. If X is not a pre-projective module, then the assertion is obvious. Therefore we assume X is a pre-projective module. In this case, by Proposition 2.3.3, we can assume that $X := P(a)$ for some vertex $a \in Q$.

Without loss of generality, we can assume $Q_0 = \{0, 1, \cdots, n-1\}$ and for any $i < j$, there is no arrows $i \to j$. For any vertex $i \in Q_0$, we set $P(i + rn) := \tau^{-r}P(i)$ and

$$d_a(i + rn) := \dim\ \mathrm{Ext}^1_{kQ}(P(i + rn), P(a)).$$

With this notation the inequality to be proved is

$$d_a(y + rn) \leq d_a(x + rn) \leq d_a(y + (r+1)n).$$

We prove the assertion by induction on $y + rn$.
($r = 0$) In this case $d_a(y) = d_a(x) = 0$.
($n \leq y + rn < a + n$) In this case $d_a(y + rn) = 0$ and

$$d_a(y + (r+1)n) = \sum_{\alpha \in s(y)} d_a(t(\alpha) + (r+1)n) + \sum_{\beta \in t(y)} d_a(s(\beta) + rn) \geq d_a(x + rn).$$

($y + rn \geq a + n$) In this case we obtain

$$\begin{aligned} d_a(x + rn) &= -d_a(x + (r-1)n) + \textstyle\sum_{\alpha \in s(x)} d_a(t(\alpha) + rn) \\ &\quad + \textstyle\sum_{\beta \in t(x)} d_a(s(\beta) + (r-1)n) \\ &= d_a(y + rn) - d_a(x + (r-1)n) + \textstyle\sum_{\alpha \in s(x) \setminus \{\gamma\}} d_a(t(\alpha) + rn) \\ &\quad + \textstyle\sum_{\beta \in t(x)} d_a(s(\beta) + (r-1)n) \\ &\geq d_a(y + rn) \end{aligned}$$

The last inequality follows from the induction hypothesis. Similarly we obtain

$$d_a(y + (r+1)n) \geq d_a(x + rn).$$

$\square$

Let $\widetilde{Q}$ be a quiver obtained from Q by adding new edges $-\alpha : y \to x$ for any $\alpha : x \to y$. For a path $w : x_0 \overset{\alpha_1}{\to} x_1 \overset{\alpha_2}{\to} \cdots \overset{\alpha_r}{\to} x_r$ in $\widetilde{Q}$, we put $c^+(w) := \#\{t \mid \alpha_t \in Q_1 \subset \tilde{Q}_1\}$. Then we define

$$l_Q(i,j) := \begin{cases} \min\{c^+(w) \mid w : \text{path from } i \text{ to } j \text{ in } \tilde{Q}\} & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

Proposition 5.2.2. *If Q satisfies the condition* (C)*, then*

$$\mathrm{Ext}^1_{kQ}(\tau^{-r}P(i), \tau^{-s}P(j)) = 0 \Leftrightarrow r \leq s + l_Q(j,i)$$

Proof. ($\Rightarrow$) Let $w : j = x_0 \overset{\alpha_1}{\to} x_1 \to \cdots \overset{\alpha_t}{\to} x_t = i$ be a path which attains $l_{j,i} := l_Q(j,i)$. $l(j,i) := l_Q(j,i) = c^+(w)$ and let $\{k_1 < k_2 < \cdots < k_{l(j,i)}\} = \{k \mid \alpha_k \in Q_1\}$. If there exists $r > l(j,i)$ such that $\mathrm{Ext}^1_{kQ}(\tau^{-r}P(i), P(j)) = 0$, then Lemma 5.2.1 implies

$$\begin{aligned}
&0 = d_j(x_t + rn) \geq d_j(x_{k_{l(j,i)}} + rn) \geq d_j(x_{k_{l(j,i)}-1} + (r-1)n) \\
&\geq \cdots \geq d_j(x_{k_1} + (r - l(j,i) + 1)n) \geq d_j(x_{k_1 - 1} + (r - l(j,i))n) \\
&\geq d_j(j + (r - l(j,i))n)) \geq d_j(j + (r - l(j,i) - 1)n)) \geq \cdots \geq d_j(j+n) > 0.
\end{aligned}$$

Therefore we get a contradiction.

We now suppose that $\mathrm{Ext}^1_{kQ}(\tau^{-r}P(i), \tau^{-s}P(j)) = 0$ with $r > s + l(j,i)$. Then, by Proposition 2.3.3, we reduce this situation to the above case and reach a contradiction.

($\Leftarrow$) We put

$$\mathcal{A}(j) := \{(i,r) \mid r \leq l(j,i),\ \mathrm{Ext}^1_{kQ}(\tau^{-r}P(i), P(j)) \neq 0\}.$$

If $\mathcal{A}(j) \neq \emptyset$, then we take $r := \min\{r \mid (i,r) \in \mathcal{A}(j) \text{ for some } i\}$. Let $i \in Q_0$ such that $(i,r) \in \mathcal{A}(j)$ and $(i', r) \notin \mathcal{A}(j)$ for any $i' \leftarrow i$ in Q. Since

$$0 < d_j(i + rn) \leq \sum_{\alpha \in s(i)} d_j(t(\alpha) + rn) + \sum_{\beta \in t(i)} d_j(s(\beta) + (r-1)n),$$

one of the following holds.
(1) $d_j(t(\alpha) + rn) \neq 0$ for some $\alpha \in s(i)$,
(2) $d_j(s(\beta) + (r-1)n) \neq 0$ for some $\beta \in t(i)$.
Note that $r \leq l(j,i) \leq l(j, t(\alpha))$ for any $\alpha \in s(i)$ and $r - 1 \leq l(j,i) - 1 \leq l(j, s(\beta))$ for any $\beta \in t(i)$. Therefore we obtain $d_j(t(\alpha) + rn) = 0 = d_j(s(\beta) + (r-1)n)$ for any $\alpha \in s(i)$ and $\beta \in t(i)$. We get a contradiction. In particular $\mathcal{A}(j) = \emptyset$.

Suppose that there exists $i \in Q_0$ and $r, s \in \mathbb{Z}_{\geq 0}$ such that $r \leq s + l(j,i)$ and

$$\mathrm{Ext}^1_{kQ}(\tau^{-r}P(i), \tau^{-s}P(j)) \neq 0.$$

If $r < s$, then

$$\mathrm{Ext}^1_{kQ}(\tau^{-r}P(i), \tau^{-s}P(j)) \simeq \mathrm{Ext}^1_{kQ}(P(i), \tau^{-(s-r)}P(j)) = 0.$$

If $r \geq s$, then Proposition 2.3.3 implies $(i, r-s) \in \mathcal{A}(j)$. Therefore we get a contradiction. $\square$

Lemma 5.2.3. *Let $T = \bigoplus_{i \in Q_0} \tau^{-r_i}P(i)$ and $T' = \bigoplus_{i \in Q_0} \tau^{-r'_i}P(i)$ be basic pre-projective tilting modules. If $T \to T'$ in $\overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q)$, then there exists $i \in Q_0$ such that $r'_i = r_i + 1$ and $r'_j = r_j$, for any $j \neq i$.*

Proof. By the definition of the tilting quiver there exists $i \in Q_0$ such that $r_i < r'_i$ and $r_j = r'_j$ for any $j \neq i$. Assume that $r'_i = r_i + t$. Then Proposition 5.2.2 shows

$$r_j - l(i,j) \leq r_i < r_i + t \leq r_j + l(j,i),\ \forall j \neq i.$$

Thus we obtain $T'' := \tau^{-r_i - 1}P(i) \oplus (\bigoplus_{j \neq i} \tau^{-r_j}P(j)) \in \mathcal{T}_{\mathrm{p}}(Q)$. Since $T > T'' \geq T'$, we get $T' = T''$. $\square$

For any quiver Q satisfying the condition (C), put

$$L(Q) := \{(r_i)_{i\in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0} \mid r_j \leq r_i + l_Q(i,j)\} \subset \mathbb{Z}^{Q_0}.$$

Then as an immediate corollary of Proposition 5.2.2, and Lemma 5.2.3, we have the following.

Corollary 5.2.4. *Assume Q satisfies the conditions* (C)*. Then*

$$(r_i)_{i\in Q_0} \mapsto \bigoplus_{i=0}^{n-1} \tau^{-r_i} P(i)$$

induces an isomorphism of posets

$$(L(Q), \leq^{\mathrm{op}}) \simeq (\mathcal{T}_{\mathrm{p}}(Q), \leq)$$

where $(r_i)_{i\in Q_0} \geq^{\mathrm{op}} (r_i^{'})_{i\in Q_0} \overset{\mathrm{def}}{\Leftrightarrow} r_i \leq r_i^{'}$ *for any* $i \in Q_0$.

In particular $\overrightarrow{\mathcal{T}_{\mathrm{p}}}(Q)$ *is a full sub-quiver of Hasse-quiver of* $(\mathbb{Z}^{Q_0}, \leq^{\mathrm{op}})$.

Proposition 5.2.5. *Let* $i \in Q_0$ *and* $T(i) := \bigoplus_{j\in Q_0} \tau^{-l(i,j)} P(j)$*. Then* $T(i)$ *is the unique minimal element of* $\mathrm{lk}_{\mathrm{p}}(P(i))$.

Proof. Let $j, j^{'} \in Q_0$. By the definition of l_Q, we get

$$l_Q(i,j) \leq l_Q(i,j^{'}) + l_Q(j^{'},j).$$

Therefore we obtain $T(i) \in \mathrm{lk}_{\mathrm{p}}(P(i))$.

Let $T := \bigoplus_{j\in Q_0} \tau^{-r_j} P(j) \in \mathrm{lk}_{\mathrm{p}}(P(i))$. Then $r_j \leq r_i + l(i,j) = l(i,j)$. Therefore Corollary 5.2.4 shows $T \geq T(i)$. □

Theorem 5.2.6. *Assume that Q has the unique source $s \in Q_0$ and satisfies the condition* (C)*. Then the following assertions hold.*
(1) $\mathcal{T}_{\mathrm{p}}(Q)$ *is the disjoint union of* $\mathrm{lk}_{\mathrm{p}}(\tau^{-r}P(s))$ *for all* $r \geq 0$.
(2) τ^{-r} *gives a quiver isomorphism*

$$\tau^{-r} : \overrightarrow{\mathrm{lk}_{\mathrm{p}}}(P(s)) \simeq \overrightarrow{\mathrm{lk}_{\mathrm{p}}}(\tau^{-r}P(s)).$$

(3) $\#\mathrm{lk}_{\mathrm{p}}(P(s)) \leq 2^{n-1}$.
(4) *Let* $T \in \mathrm{lk}_{\mathrm{p}}(\tau^{-r}P(s))$ *and* $T^{'} \in \mathrm{lk}_{\mathrm{p}}(\tau^{-r^{'}}P(s))$*. If there is an arrow* $T \to T^{'}$*, then* $0 \leq r^{'} - r \leq 1$

Proof. Without loss of generality, we can assume $Q_0 = \{0, 1, \cdots, n-1\}$ and for any $i < j$, there is no arrows $i \to j$. We note that $s = n-1$.

(1) This follows from Proposition 5.2.2.

(2) It is obvious that τ^{-r} induces an injection

$$\overrightarrow{\mathrm{lk}}_{\mathrm{p}}(P(s)) \to \overrightarrow{\mathrm{lk}}_{\mathrm{p}}(\tau^{-r}P(s))$$

as a quiver. Therefore it is sufficient to show

$$\tau^{-r} : \mathrm{lk_p}(P(s)) \to \mathrm{lk_p}(\tau^{-r}P(s))$$

is surjective.

Let $T \in \mathrm{lk_p}(\tau^{-r}P(s))$. By Corollary 5.2.4, there exists $(r_i)_{\in Q_0\setminus\{s\}}$ such that

$$T \simeq (\bigoplus_{i\in Q_0\setminus\{s\}} \tau^{-r_i}P(i)) \oplus \tau^{-r}P(s).$$

Since $l(i,s) = 0$ for any $i \in Q_0$, Proposition 5.2.2 shows $r_i \geq r$ for any $i \in Q_0$. Therefore $\tau^r T \in \mathrm{lk_p}(P(s))$.

(3) For any $i \in Q_0$ put $t(i) := \max\{j \in Q_0 \mid j \to i\}$. Let $T = \bigoplus_{i\in Q_0} \tau^{-r_i}P(i)$. Then $T \in \mathrm{lk_p}(P(s))$ only if

$$r_s = 0, r_{n-2} \in \{0,1\}, \cdots, r_i \in \{r_{t(i)}, r_{t(i)}+1\}, \cdots, r_0 \in \{r_{t(0)}, r_{t(0)}+1\}.$$

Therefore we obtain $\#Q_0 \leq 2^{n-1}$.

(4) Let $T = \bigoplus_{i\in Q_0} \tau^{-r_i}P(i) \in \mathrm{lk_p}(\tau^{-r}P(s))$ and $T' = \bigoplus_{i\in Q_0} \tau^{-r'_i}P(i) \in \mathrm{lk_p}(\tau^{-(r+t)}P(s))$. Namely $\tau^{-r}P(s) \in \mathrm{add}\, T$ and $\tau^{-(r+t)}P(s) \in \mathrm{add}\, T'$. Thus the claim follows from Lemma 5.2.3. $\square$

Example 5.2.7. We give three examples. Recall that for any two elements (r_i), (r'_i) of $\mathbb{Z}^n$ ($n \geq 1$), $(r_i) \geq^{\mathrm{op}} (r'_i)$ means $r_i \leq r'_i$ for any i.

(1) Consider the following quiver

$$Q = \begin{tikzcd}[column sep=small] & & \overset{n-1}{\circ} \arrow[drr] & & \\ \overset{s}{\circ} \arrow[urr] \arrow[dr] & & & & \overset{1}{\circ} \\ & \underset{n-2}{\circ} \arrow[r] & \cdots \arrow[r] & \underset{2}{\circ} \arrow[ur] & \end{tikzcd}$$

Let

$L_0 = \{(0,0,0\cdots 0,0,0), (1,0,0\cdots 0,0,0), (1,1,0\cdots 0,0,0), \cdots, (1,1,1\cdots 1,0,0)\}$
$L_1 = \{(1,0,0\cdots 0,1,0), (1,1,0\cdots 0,1,0), (1,1,1\cdots 0,1,0), \cdots, (1,1,1\cdots 1,1,0)\}$
$L_2 = \{(2,1,0\cdots 0,1,0), (2,1,1\cdots 0,1,0), \cdots, (2,1,1\cdots 1,1,0)\}$
$L_3 = \{(2,2,1,0\cdots 0,1,0), \cdots, (2,2,1\cdots 1,1,0)\}$
$\vdots$
$L_{n-3} = \{(2,2,2\cdots 2,1,1,0)\}$

and

$$T(a,b) = \begin{cases} b\text{-th elment of } L_a & 0 \leq a \leq n-3,\ 1 \leq b \leq n-1-a \\ T(b-n+a, n-b) + (1,\cdots 1) & 0 \leq a \leq n-3,\ n-1-a < b \leq n-1 \\ T(x,b) + (2r, 2r, \cdots, 2r) & a = x + (n-2)r\ (0 \leq x < n-2), 1 \leq b \leq n-1. \end{cases}$$

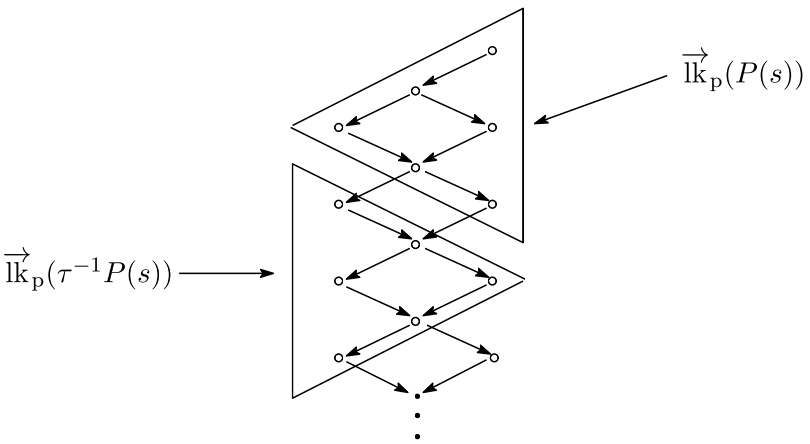
Then we get

$$(\amalg L_a, \leq^{\mathrm{op}}) \simeq (\mathrm{lk_p}(P(n-1)), \leq)$$

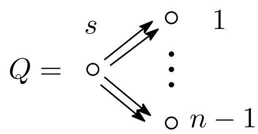
and

$$(L(Q), \leq^{\mathrm{op}}) = (\{T(a,b) \mid a \in \mathbb{Z}_{\geq 0}, 1 \leq b \leq n-1\}, \leq^{\mathrm{op}}) \simeq (\mathbb{Z}_{\geq 0} \times \{1, \cdots, n-1\}, \leq^{\mathrm{op}}).$$

In particular $\overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q) = \mathbb{Z}_{\geq 0}\overrightarrow{A}_{n-1}$. If $n = 4$, then $\overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q)$ is given as follows:



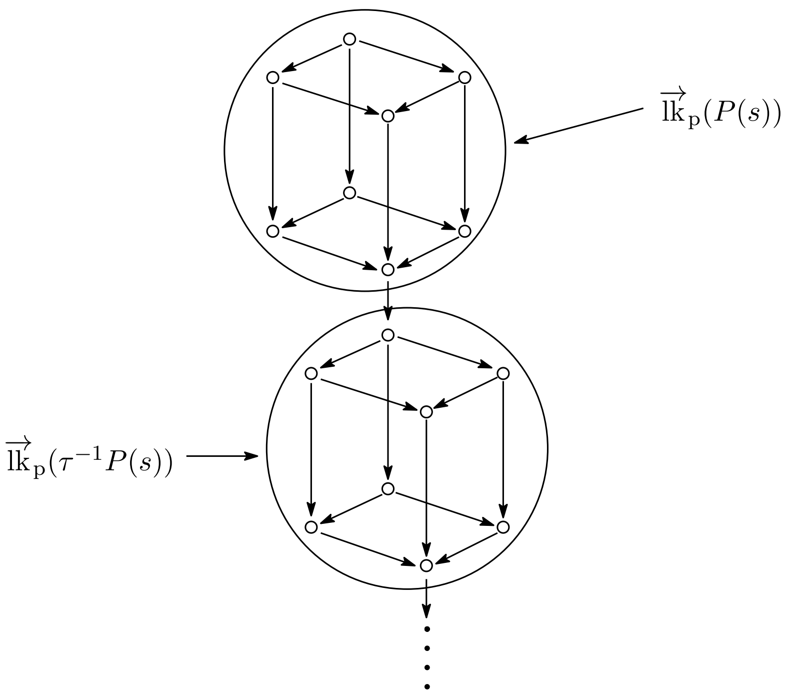
(2) Consider the following quiver



It is easy to check that

$$(\{0,1\}^{n-1}, \leq^{\mathrm{op}}) \simeq (\mathrm{lk}_{\mathrm{p}}(P(s)), \leq)$$

Therefore the underlying graph of $\overrightarrow{\mathrm{lk}}_{\mathrm{p}}(P(s))$ is isomorphic to $(n-1)$-dimensional cube. In the case $n = 4$, $\overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q)$ is given as follows:



(3) Let Q be the following quiver:

$$0 \longrightarrow 1 \rightrightarrows 2$$

Note that Q does not satisfy the condition (C). Let

$$\begin{cases} a_n^i & := & \dim \operatorname{Ext}^1_{kQ}(\tau^{-1-n}P(0), P(i)) \\ b_n^i & := & \dim \operatorname{Ext}^1_{kQ}(\tau^{-1-n}P(1), P(i)) \quad \forall n \geq 0. \\ c_n^i & := & \dim \operatorname{Ext}^1_{kQ}(\tau^{-1-n}P(2), P(i)) \end{cases}$$

Then we obtain the following equations:

$$\begin{cases} a_{n+1}^i & = & b_{n+1}^i - a_n^i \\ b_{n+1}^i & = & a_n^i + 2c_{n+1}^i - b_n^i \\ c_{n+1}^i & = & 2b_n^i - c_n^i. \end{cases}$$

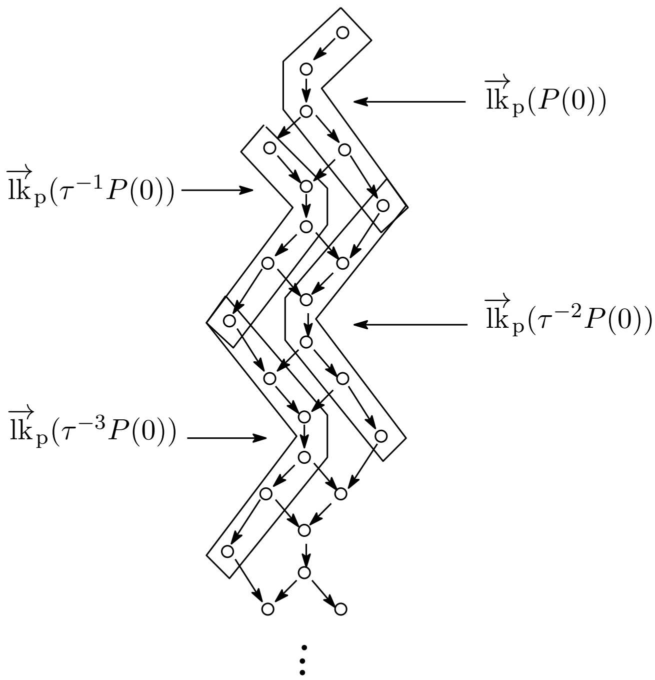
Therefore we get

$$\begin{aligned} b_{n+2}^i &= 2c_{n+2}^i + a_{n+1}^i - b_{n+1}^i \\ &= 2c_{n+2}^i - a_n^i \\ &= 4b_{n+1}^i - 2c_{n+1}^i - a_n^i \\ &= 4b_{n+1}^i - b_{n+1}^i - b_n^i \\ &= 3b_{n+1}^i - b_n^i. \end{aligned}$$

Note that the above equations show $b_{n+1}^i - b_n^i \geq b_n^i - b_{n-1}^i \geq \cdots \geq b_1^i - b_0^i > 0$ for any $i = 0, 1, 2$. Since $a_{n+2}^i = b_{n+2}^i - a_{n+1}^i = b_{n+2}^i - b_{n+1}^i + a_n^i$ and $c_{n+2}^i = b_{n+1}^i - c_{n+1}^i = 2(b_{n+1}^i - b_n^i) + c_n^i$, we obtain $a_{n+2}^i > a_n^i$ and $c_{n+2}^i > c_n^i$. Now it is easy to check that,

$$\operatorname{Ext}^1_{kQ}(\tau^{-r}P(i), \tau^{-s}P(j)) = 0 \Leftrightarrow r \leq s + l_Q(j,i) \text{ or } (i = j = 0,\ r = s+2).$$

Therefore $\overrightarrow{\mathcal{T}_{\mathrm{p}}}(Q)$ is given by the following:



5.3 The case of $l(Q) \leq 1$

In this section we consider the quiver $\overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q)$ for those Q that satisfy the following conditions:
(i) Q has a unique source,
(ii) Q satisfies the condition (C),
(iii) $l(Q) := \max\{l_Q(x,y) \mid x, y \in Q_0\} \leq 1$.
In this case we can completely determine poset-structure of $\mathcal{T}_p(Q)$.

Denote by $\mathcal{Q}$ the set of finite connected acyclic quivers. For any quiver $Q \in \mathcal{Q}$ define a new quiver Q° by adding arrows which directly connects sources and sinks. Namely for each pair (x, y) of a source x and a sink y we add an arrow $x \to y$. Then we define subsets $\mathcal{A}, \mathcal{B}, \mathcal{A}^\circ$ of $\mathcal{Q}$ as follows:

$$\begin{array}{lcl} \mathcal{A} & := & \{Q \in \mathcal{Q} \mid Q \text{ has the unique source}\}, \\ \mathcal{B} & := & \{Q \in \mathcal{A} \mid Q \text{ has the unique sink}\}, \\ \mathcal{A}^\circ & := & \{Q^\circ \mid Q \in \mathcal{A}\}. \end{array}$$

Note that

$$\mathcal{A}^\circ = \left\{ Q \in \mathcal{Q} \;\middle|\; \begin{array}{l} \bullet \;\; Q \text{ has a unique source} \\ \bullet \;\; Q \text{ satisfies the condition (C)} \\ \bullet \;\; l(Q) \leq 1 \end{array} \right\}.$$

Definition 5.3.1. We define maps $\overrightarrow{\amalg} : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$, $\phi : \mathcal{A}^\circ \times \mathcal{A}^\circ \to \mathcal{A}^\circ$, $\psi : \mathcal{A}^\circ \to \mathcal{B}$ and $\Psi : \mathcal{A}^\circ \times \mathcal{A}^\circ \to \mathcal{B} \times \mathcal{B}$ as follows:

$$(1) \left\{ \begin{array}{lcl} (Q \overrightarrow{\amalg} Q')_0 & := & Q_0 \amalg Q'_0, \\ (Q \overrightarrow{\amalg} Q')_1 & := & Q_1 \amalg Q'_1 \amalg \left\{ y \to x' \;\middle|\; \begin{array}{l} y \in Q_0 : \text{there is no arrow starting at } y \\ x' \in Q'_0 : \text{there is no arrow ending at } x' \end{array} \right\}. \end{array} \right.$$

(2) $\phi(Q, Q') := (Q \overrightarrow{\amalg} Q')^\circ$.
(3) $\psi(Q) := \overrightarrow{\mathrm{lk}}_{\mathrm{p}}(P_Q)$, where P_Q is the indecomposable projective module associated with the unique source.
(4) $\Psi(Q, Q') := (\psi(Q'), \psi(Q))$.

Proposition 5.3.2. *The following diagram is commutative.*

$$\begin{tikzcd} \mathcal{A}^\circ \times \mathcal{A}^\circ \arrow[r, "\Psi"] \arrow[d, "\phi"'] & \mathcal{B} \times \mathcal{B} \arrow[d, "\overrightarrow{\amalg}"] \\ \mathcal{A}^\circ \arrow[r, "\psi"'] & \mathcal{B} \end{tikzcd}$$

Proof. Let $Q(1), Q(2) \in \mathcal{A}^\circ$, $s(k)$ a unique source of $Q(k)$ $(k = 1, 2)$ and

$$T = \bigoplus_{x \in \phi(Q(1), Q(2))_0} \tau^{-r_x} P(x).$$

Then Corollary 5.2.4 shows $T \in \psi(\phi(Q(1), Q(2)))_0$ if and only if one of the following holds:
(i) $(r_x)_{x \in Q(2)_0} \in L(Q(2))$ with $r_{s(2)} = 0$ and $r_y = 0$ for any $y \in Q(1)_0$,
(ii) $(r_y)_{y \in Q(1)_0} \in L(Q(1))$ with $r_{s(1)} = 0$ and $r_x = 1$ for any $x \in Q(2)_0$.

Now it is easy to check that

$$\psi(\phi(Q(1), Q(2))) = \psi(Q(2)) \overrightarrow{\amalg} \psi(Q(1)).$$

□

Let $\alpha \in Q_0$. Then we write

$$Q \rightsquigarrow Q \setminus \{\alpha\}$$

if $\alpha \in Q_1$ satisfies either (1) or (2) below:
(1) $s(\alpha)$ is not a source or $t(\alpha)$ is not a sink, and there exists a path $w \neq \alpha$ from $s(\alpha)$ to $t(\alpha)$.
(2) $s(\alpha)$ is a source, $t(\alpha)$ is a sink, and there are at least two arrows from $s(\alpha)$ to $t(\alpha)$ and another path from $s(\alpha)$ to $t(\alpha)$.

Let $\mathcal{S} := \{Q \in \mathcal{A}^\circ \mid$ there is no quiver $Q^{'}$ such that $Q \rightsquigarrow Q^{'}\}$. It is easy to check that if

$$Q \rightsquigarrow \cdots \rightsquigarrow Q^{'} \in \mathcal{S}, \; Q \rightsquigarrow \cdots \rightsquigarrow Q^{''} \in \mathcal{S},$$

then $Q^{'} = Q^{''}$ and in this case put $\pi(Q) := Q^{'} = Q^{''}$. Now we define an equivalence relation $\sim$ on $\mathcal{A}^\circ$ as follows:

$$Q \sim Q^{'} \overset{\text{def}}{\Leftrightarrow} \pi(Q) = \pi(Q^{'}).$$

Remark 5.3.3. It is easy to check that if $Q \sim Q^{'}$, then

$$l_Q(i,j) = l_{Q^{'}}(i,j), \quad \forall i,j \in Q_0 = Q^{'}_0.$$

Lemma 5.3.4. *Let $Q, Q^{'} \in \mathcal{A}^\circ$. If $Q \sim Q^{'}$, then $\psi(Q) = \psi(Q)$. In particular we obtain a map*

$$\psi/_\sim : \mathcal{A}^\circ/_\sim \to \mathcal{B}.$$

Proof. This follows from Corollary 5.2.4 and Remark 5.3.3. □

Lemma 5.3.5. *Let $Q(1), Q(2), Q^{'}(1), Q^{'}(2) \in \mathcal{A}^\circ$. If $Q(i) \sim Q^{'}(i)$ $(i = 1, 2)$, then $\phi(Q(1), Q(2)) \sim \phi(Q^{'}(1), Q^{'}(2))$. In particular we get a map*

$$\phi/_\sim : \mathcal{A}^\circ/_\sim \times \mathcal{A}^\circ/_\sim \to \mathcal{A}^\circ/_\sim.$$

Proof. Let $Q, Q^{'}, Q^{''} \in \mathcal{A}^\circ$ with $Q \rightsquigarrow Q^{'} = Q \setminus \{\alpha\}$. By the definition we get

$$\phi(Q^{'}, Q^{''}) = \phi(Q, Q^{''}) \setminus \{\alpha\}.$$

Since one can check that α satisfies (1) in $\phi(Q, Q^{''})$,we have

$$\phi(Q, Q^{''}) \rightsquigarrow \phi(Q^{'}, Q^{''}).$$

We repeat this procedure and obtain

$$\phi(Q, Q^{''}) \rightsquigarrow \phi(\pi(Q), Q^{''}).$$

Similarly we have that

$$\phi(Q, Q^{''}) \rightsquigarrow \phi(Q, \pi(Q^{''})).$$

Therefore we obtain $\phi(Q, Q^{''}) \rightsquigarrow \phi(\pi(Q), \pi(Q^{''}))$. □

Let C^n be the Hasse-quiver of $(\{0,1\}^n, \leq^{\mathrm{op}})$ and $Q \in \mathcal{A}^\circ$ with $Q_0 = \{s, 1, 2, \cdots n-1\}$ where s is the unique source of Q. Then a map

$$\rho : P(s) \oplus (\bigoplus_{i=1}^{n-1} \tau^{-r_i} P(i)) \mapsto (r(i))_i$$

induces a full embedding $\psi(Q) \to C^{n-1}$ of quivers. Therefore we can identify $\psi(Q)$ as a full sub-quiver of C^{n-1}. For any $T \in C_0^{n-1}$ denote by T_i the i-th entry of T. We note that $(0, \cdots, 0), (1, \cdots 1) \in \psi(Q)$.

Proposition 5.3.6. *Let $Q \in \mathcal{A}^\circ$. Then $\psi(Q) = K(1) \overrightarrow{\amalg} K(2)$ for some quivers $K(1), K(2) \in \mathcal{B}$ if and only if there exists $(Q(1), Q(2)) \in \mathcal{A}^\circ \times \mathcal{A}^\circ$ such that $\psi(Q(i)) = K(i)$ $(i = 1, 2)$ and $\phi(Q(2), Q(1)) \sim Q$.*

Proof. First assume that there is $(Q(1), Q(2)) \in \mathcal{A}^\circ \times \mathcal{A}^\circ$ such that $\phi(Q(2), Q(1)) \sim Q$. Then define $\psi(Q(i)) =: K(i)$ $(i = 1, 2)$. Then Proposition 5.3.2, Lemma 5.3.4 and Lemma 5.3.5 imply

$$\psi(Q) = K(1) \overrightarrow{\amalg} K(2).$$

Next assume that $\psi(Q) = K(1) \overrightarrow{\amalg} K(2)$ for some quivers $K(1), K(2) \in \mathcal{B}$. By the above Lemma 5.3.5, we can assume that $Q \in \mathcal{S}$. Let T be a minimum element of K_1 and T' a maximum element of K_2. Lemma 5.2.3 implies that there is a unique vertex $i \in Q_0$ such that $T_i = 0$ and $T'_i = 1$. Note that $T'' \leq T$ or $T'' > T$ for any $T'' \in \psi(Q)$. We note also that $T'' > T'$ or $T'' \leq T'$ for any $T'' \in \psi(Q)$. Since $T' < T(i) \leq T$ we have $T = T(i)$. If $T(j) \leq T = T(i)$ then $l(i, j) \leq l(j, j) = 0$. If $T(j) > T(i)$ then $l(j, i) \leq l(i, i) = 0$. Therefore for any $j \leq n-1$ there is either a path from i to j or a path from j to i. Thus we obtain that $Q = (Q'(2) \overrightarrow{\amalg} Q'(1))^\circ$, where $Q'(1) \in \mathcal{A}$ (resp. $Q'(2) \in \mathcal{A}$) is the full sub-quiver of Q with $Q'(1)_0 = \{j \mid j \neq i, l(j, i) = 0\}$ (resp. $Q'(2)_0 = \{j \mid l(i, j) = 0\}$) (here we use the fact $Q \in \mathcal{S}$). Let $Q(i) := Q'(i)^\circ$, then $(Q'(2) \overrightarrow{\amalg} Q'(1))^\circ \sim \phi(Q(2), Q(1))$.

Now it is sufficient to show that $\psi(Q(i)) = K(i)$. Consider an injective map

$$\iota : \psi(Q(1)) \to K(1) \text{ given by } \iota(T'')_j := \begin{cases} T''_j & j \in Q(1)_0 \\ 0 & \text{otherwise.} \end{cases}$$

We show that ι is surjective.

Let $T'' \in K(1)$. Then $T''_j \leq T''_{j'} + l_Q(j', j) = T''_{j'} + l_{Q(1)}(j', j)$ for any $j, j' \in Q(1)_0$. Since $T'' \geq T(i)$ we have $T''_j \leq T(i)_j = l_Q(i, j) = 0$ for any $j \in Q(2)_0$. Therefore we obtain $T'' \in \iota(\psi(Q(1)))$. In particular we get $\psi(Q(1)) = K(1)$. Similarly we obtain $\psi(Q(2)) = K(2)$. □

Definition 5.3.7. Let T, T' be vertices of C^n.
(1) We set $T \vee T' := (\min\{T_i, T'_i\})_i$.
(2) We set $T \wedge T' := (\max\{T_i, T'_i\})_i$.

We consider the following properties for a full sub-quiver K of C^n.
$(\mathrm{i})_n$ $(0, \cdots 0), (1, \cdots, 1) \in K_0$.
$(\mathrm{ii})_n$ for any $T >^{\mathrm{op}} T'$ in K, there is a path from T to T' in K.
$(\mathrm{iii})_n$ $T \vee T', T \wedge T' \in K_0$ for any $T, T' \in K_0$
We put

$$\mathcal{L}_n := \{K \in \mathcal{B} \mid K \text{ satisfies } (\mathrm{i})_n, (\mathrm{ii})_n, (\mathrm{iii})_n\} \text{ and } \mathcal{L} := \amalg \mathcal{L}_n.$$

Remark 5.3.8. Note that $(C^n, \vee, \wedge)$ is a finite distributive lattice. Let L be a finite distributive lattice with a maximum element x and a minimum element y. Then there is a unique positive integer n such that there exists a lattice-embedding

$$\iota : L \to C^n$$

which satisfies $\iota(x) = (0, 0, \cdots 0)$ and $\iota(y) = (1, 1, \cdots 1)$. Thus the following are equivalent:
(1) L is a finite distributive lattice.
(2) L is isomorphic to some element of $\mathcal{L}$.

Lemma 5.3.9. *For any $K \in \mathcal{L}_n$, we define a relation $\leq_K$ on $\{1, 2, \cdots, n-1\}$ as follows:*

$$i \leq_K j \overset{\text{def}}{\Leftrightarrow} T_i \geq T_j \text{ for any } T \in K.$$

Proof. It is sufficient to show that

$$i \leq_K j \leq_K i \Rightarrow i = j.$$

Assume $T_i = T_j$ for any $T \in K$. Now there is a path $(0, \cdots, 0) = T^0 \to T^1 \to \cdots \to T^r = (1, \cdots, 1)$ in K. Therefore there exists a positive integer a such that $T_i^{a-1} = T_j^{a-1} = 0$ and $T_i^a = T_j^a = 1$. In particular we obtain $i = j$. $\square$

Definition 5.3.10. We define a map $\psi^- : \mathcal{L} \to \mathcal{S}$ as follows:

$$\psi^-(K) := (\{\overset{s}{\circ}\}\overrightarrow{\amalg} Q'(K))^\circ,$$

where $Q'(K)$ is the Hasse-quiver of $(\{1, 2, \cdots, n-1\}, \leq_K)$.

Lemma 5.3.11. *If $Q \in \mathcal{A}^\circ$, then $\psi(Q) \in \mathcal{L}$.*

Proof. Let $K = \psi(Q)$ with $Q \in \mathcal{A}^\circ$ and $n := \#Q_0$. Then we have already seen that K is a full sub-quiver of C^{n-1} and $(0, \cdots, 0), (1, \cdots, 1) \in K_0$. Therefore K satisfies the condition $(\mathrm{i})_{n-1}$. Note that K also satisfies the condition $(\mathrm{ii})_{n-1}$ (Lemma 5.1.3).

Therefore we prove that K satisfies the condition $(\mathrm{iii})_{n-1}$. By Corollary 5.2.4, it is sufficient to show that

$$(T \vee T')_i \leq (T \vee T')_j + l_Q(j, i) \text{ and } (T \wedge T')_i \leq (T \wedge T')_j + l_Q(j, i) \quad (\forall i, j),$$

for any vertices T, T' of K. Let $T, T' \in K_0$ and $i, j \in Q_0$. If $\min\{T_i, T_i'\} = 0$ or $l_Q(j, i) = 1$, then $\min\{T_i, T_i'\} \leq \min\{T_j, T_j'\} + l_Q(j, i)$. Assume that $\min\{T_i, T_i'\} = 1$ and $l_Q(j, i) = 0$. In this case $1 = T_i \leq T_j$ and $1 = T_i' \leq T_j'$. Therefore we obtain $T_j = T_j' = 1$. In particular $\min\{T_i, T_i'\} \leq \min\{T_j, T_j'\} + l_Q(j, i)$ for any $i, j \in Q_0$. Thus we get $T \vee T' \in K_0$. Similarly we obtain $T \wedge T' \in K_0$. $\square$

Lemma 5.3.12. *We obtain $Q \sim \psi^-(\psi(Q))$ for any $Q \in \mathcal{A}^\circ$. In particular the map*

$$\psi/_\sim : \mathcal{A}^\circ/_\sim \to \mathcal{B}$$

is injective.

Proof. It is sufficient to show that if $Q \in \mathcal{S}$ then $Q = \psi^-(\psi(Q))$. Let $s \neq i \to j$ in Q. Then

$$T_i \leq T_j + l_Q(j,i) = T_j$$

for any $T \in \psi(Q)$. If $j <_{\psi(Q)} j^{'} <_{\psi(Q)} i$, then we have

$$l(j^{'},i) = T(j^{'})_i \leq T(j^{'})_{j^{'}} = 0 \text{ and } l(j,j^{'}) = T(j)_{j^{'}} \leq T(j)_j = 0.$$

Therefore we obtain that there exists a path

$$i \to \cdots \to j^{'} \to \cdots \to j$$

in Q. Since $Q \in \mathcal{S}$, we get a contradiction. In particular $i \to j$ in $Q^{'}(\psi(Q))$.

On the other hand if $i \to j$ in $Q^{'}(\psi(Q))$ then

$$l_Q(j,i) = T(j)_i \leq T(j)_j = 0.$$

Therefore there is a path

$$i \to j^{'} \to \cdots \to j$$

in Q. Since

$$T_i \leq T_{j^{'}} + l(j^{'},i) = T_{j^{'}} \leq T_j + l(j,j^{'}) = T_j$$

for any $T \in \psi(Q)$, we have

$$j \leq_{\psi(Q)} j^{'} <_{\psi(Q)} i.$$

In particular we obtain $j = j^{'}$.

Hence we obtain $Q \setminus \{s\} = Q^{'}(\psi(Q))$. We conclude that

$$Q = (\{\overset{s}{\circ}\} \overrightarrow{\amalg} Q^{'}(\psi(Q)))^{\circ} = \psi^-(\psi(Q)).$$

□

Lemma 5.3.13. *Let $K \in \mathcal{B}$. Then the following are equivalent.*
(1) $K = \psi(Q)$ for some $Q \in \mathcal{A}^{\circ}$.
(2) $K \in \mathcal{L}$.

Proof. ((1) $\Rightarrow$ (2)) It follows from Lemma 5.3.11.

((2) $\Rightarrow$ (1)) Let $K \in \mathcal{L}_n$. It is sufficient to show $K_0 = \psi(\psi^-(K))_0$.

First let $T \in K_0$ and $i,j \in \psi^-(K)_0 \setminus \{s\}$. If $l(j,i) := l_{\psi^-(K)}(j,i) = 0$ then $j \leq_K i$. Thus we obtain $T_i \leq T_j$. If $l(j,i) = 1$ then $T_i \leq T_j + l(j,i)$. Hence we obtain

$$T_i \leq T_j + l_{\psi^-(K)}(j,i) \quad (\forall i,j).$$

This implies that

$$T \in \psi(\psi^-(K))_0.$$

Next suppose that $\psi(\psi^-(K))_0 \setminus K_0 \neq \emptyset$. Note that $\psi(\psi^-(K)) \in \mathcal{L}_n$. Then the conditions (i)$_n$ and (ii)$_n$ give that

$$\{T \in \psi(\psi^-(K))_0 \setminus K_0 \mid T^{'} \to T \text{ for some } T^{'} \in K_0\} \neq \emptyset.$$

Let T be a minimal element of $\{T \in \psi(\psi^-(K))_0 \setminus K_0 \mid T^{'} \to T \text{ for some } T^{'} \in K_0\}$ and $T^{'} \in K_0$ with $T^{'} \to T$. Then the conditions (i)$_n$ and (ii)$_n$ imply that there exists $T^{''} \in K_0$ such that

$T' \to T''$. Now there exists i, j such that $0 = T_j < T_i = 1$, $T_i' = T_j' = 0$ and $0 = T_i'' < T_j'' = 1$. Note that Lemma 5.3.12 implies $\leq_K = \leq_{\psi(\psi^-(K))}$. In fact if we let $Q := \psi^-(K)$, then

$$\begin{aligned} i \leq_{\psi(Q)} j &\Leftrightarrow l_{\psi^-(\psi(Q))}(i,j) = 0 \\ &\Leftrightarrow l_Q(i,j) = 0 \\ &\Leftrightarrow i \leq_K j. \end{aligned}$$

Since $T_i > T_j$, we have

$$i \not\geq_{\psi(\psi^-(K))} j \Leftrightarrow i \not\geq_K j.$$

In particular there exists $S \in K_0$ such that $S_i > S_j$. Let $T''' := T \wedge T''$. Then the minimality of T gives $T''' \in K_0$. Since

$$((T' \wedge S) \vee T''')_a = \begin{cases} \min\{\max\{T_a, S_a\}, T_a\} & a \neq i, j, \\ 1 & a = i, \\ 0 & a = j, \end{cases}$$

we obtain that $T = (T' \wedge S) \vee T''' \in K_0$. Therefore we get a contradiction.

□

Corollary 5.3.14. *ψ induces a bijection between $\mathcal{S}$ and $\mathcal{L}$.*

Now Theorem 5.2.6 gives the following result.

Theorem 5.3.15. (1) *For any $Q \in \mathcal{A}^\circ$, there exists $K \in \mathcal{L}$ such that $\overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q) = \overrightarrow{\amalg} K$.*
(2) *For any $K \in \mathcal{L}$, there exists $Q \in \mathcal{A}^\circ$ such that $\overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q) = \overrightarrow{\amalg} K$.*

Theorem 5.3.15 says that a poset which is obtained from infinitely many copies of a finite distributive lattice may be realized as the pre-projective tilting Hasse quiver of $Q \in \mathcal{A}^\circ$ and vice verse.

Finally we give an equivalent condition for two quivers in $\mathcal{A}^\circ$ to have the same pre-projective tilting Hasse quiver.

Corollary 5.3.16. *Let $Q(1), Q(2) \in \mathcal{A}^\circ$. Then the following are equivalent.*
(1) $\overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q(1)) = \overrightarrow{\mathcal{T}}_{\mathrm{p}}(Q(2))$.
(2) *There exists $t_1, t_2 \in \mathbb{Z}_{>0}$ and $Q \in \mathcal{A}^\circ$ such that,*

$$Q(1) \sim (\underbrace{Q\overrightarrow{\amalg}Q\overrightarrow{\amalg}\cdots\overrightarrow{\amalg}Q}_{t_1})^\circ \text{ and } Q(2) \sim (\underbrace{Q\overrightarrow{\amalg}Q\overrightarrow{\amalg}\cdots\overrightarrow{\amalg}Q}_{t_2})^\circ.$$

Proof. ((2) $\Rightarrow$ (1)) This follows from Proposition 5.3.6, Lemma 5.3.4 and Lemma 5.3.5.

((1) $\Rightarrow$ (2)) Let

$$\psi(Q(i)) = S^1(i)\overrightarrow{\amalg}S^2(i)\overrightarrow{\amalg}\cdots\overrightarrow{\amalg}S^{r_i}(i) \ (S^t(i) \in \mathcal{B})$$

be a decomposition with r_i being maximal ($i = 1, 2$) and $r := \gcd(r_1, r_2)$.

Consider a homomorphism $f : \mathbb{Z} \to \mathbb{Z}/r_1\mathbb{Z} \oplus \mathbb{Z}/r_2\mathbb{Z}$ where $f(t) = (t \mod r_1, t \mod r_2)$. Let $1 \leq a \leq r_1$ and $1 \leq b \leq r_2$. Then the condition (1) implies

$$a \equiv b \bmod r \Rightarrow (a \bmod r_1, b \bmod r_2) \in \operatorname{Im} f \Rightarrow S^a(1) = S^b(2).$$

Therefore we get $S^{x+tr}(1) = S^x(2) = S^x(1)$ and $S^{x+tr}(2) = S^x(1) = S^x(2) (x \leq r)$. In particular we get,

$$\psi(Q(1)) = \underbrace{S \overrightarrow{\amalg} S \cdots \overrightarrow{\amalg} S}_{t_1},\ \psi(Q(2)) = \underbrace{S \overrightarrow{\amalg} S \cdots \overrightarrow{\amalg} S}_{t_2},$$

where $S = S^1(1) \overrightarrow{\amalg} S^2(1) \overrightarrow{\amalg} \cdots \overrightarrow{\amalg} S^r(1)$ and $t_i = \frac{r_i}{r}$ $(i = 1, 2)$. By Proposition 5.3.6, there exists a quiver Q satisfying $\psi(Q) = S$. Then Lemma 5.3.12 shows Q satisfies the condition (2). $\square$

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