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Measurable groupoids

and

associated von Neumann algebras

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1984

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Abstract

In this thesis, we study von Neumann algebras associated with measurable groupoids of type II_1. The aim of our study is to give approaches to the study of factors associated with actions of subgroups of $SL(n, \mathbb{Z})$ on the n-torus by linear automorphisms. This thesis consists of three parts. The first part is devoted to preliminaries. In the second part, we introduce the notion of an action of a semigroup on a Borel space and construct a homomorphism of a semigroup into the semigroup of injective endomorphisms of the associated von Neumann algebra. We study conjugacy problems for subalgebras associated with the above homomorphism. As an application, we construct countably infinite non-conjugate subalgebras of the full factor of type II_1 associated with the action of $SL(n, \mathbb{Z})$ on the n-torus. In the third part, we study full factors of type II_1 associated with the actions of some subgroups of $SL(2, \mathbb{Z})$ on the 2-torus. We construct minimal periodic automorphisms of these factors with various Connes' outer invariants. For every possible value of the invariant, we can find an automorphism with it among above automorphisms.

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Introduction.

The notion of measurable groupoids was first introduced by G. W. Mackey for the study of the ergodic theory [23]. Since then this notion has played important roles in various fields in mathematics, in particular, in the commutative ergodic theory and in the theory of operator algebras. In the theory of von Neumann algebras, the close relation to the ergodic theory has been known since Murray and von Neumann gave the group-measure space construction in [24]. This construction was generalized by W. Krieger [21], and then by J. Feldman and C. C. Moore [14]. In terms of the groupoid theory, they studied about principal measured groupoids with countable orbits. P. Hahn unified these constructions and showed the way to construct von Neumann algebras from more general measured groupoids [17]. By introducing the notion of transverse measures, A. Connes gave a new formulation for the groupoid theory [5].

In 1943, Murray and von Neumann showed the uniqueness of the hyperfinite factor of type II_1 in [25]. Corresponding to their result, H. A. Dye showed that measure preserving ergodic actions of singly generated groups on a non-atomic probability space are weakly equivalent [10, 11]. It should be noted that equivalence relations associated with weakly equivalent actions are isomorphic as measured groupoids. Recently, A. Connes, J. Feldman and B. Weiss showed that, for any amenable nonsingular countable equivalence relation $R \subset S \times S$, there exists a nonsingular transformation T of S such that, up to a null set, $R = \{(t, T^n t);\ t \in S,\ n \in \mathbb{Z}\}$ [6]. It follows that a principal, amenable and ergodic measured groupoid of type II_1 is unique up to isomorphism. As an opposite notion of amenability, K. Schmidt introduced the notion of strong ergodicity of measure-preserving actions of countable groups ([34], see also [7]). One of important examples of strongly ergodic actions is the action of $SL(n, \mathbb{Z})$ on the n-torus by linear automorphisms for n = 2, 3,.... Let G_n be the associated measured groupoid of this action. R. J. Zimmer showed that $\{G_n; n = 2, 3, \ldots\}$ are not isomorphic with each other [38]. Let $\mathcal{M}_n$ be the full factor of type II_1 associated with G_n. It seems to be very difficult to answer whether $\{\mathcal{M}_n; n = 3, 4, \ldots\}$ are isomorphic or not. (By using Property T, it can be shown that $\mathcal{M}_2$ is not isomorphic to $\mathcal{M}_n$ for $n \geq 3$.) Actions of non-amenable subgroups of $SL(2, \mathbb{Z})$ on the 2-torus are also strongly ergodic.[33]. It is not known whether groupoids associated with these actions are isomorphic or not.* This thesis is an attempt to understand well these groupoids and factors.

The organization of this thesis is as follows: Chapter I is a preliminary part. We review well-known facts about groupoids and von Neumann algebras. In Chapter II, we study cojugacy problems for subalgebras of von Neumann algebras associated with measurable groupoids. In Section 2.1, we introduce the notion of an action of a semigroup on a Borel space. This notion generalizes that of normalizer of a full group of a groupoid. For a measure-preserving action of a countable group C on a measure space (S, μ), we consider a semigroup B containing C such that elements of B correspond to Borel maps of S onto itself normalizing, in a sense, an action of C. The normalizing condition involves that an action of B is non-singular with respect to a transverse measure of a measurable groupoid associated with an action of C. Let $End_\Lambda(H)$ be a von Neumann algebra associated with an action of C. We construct a homomorphism Φ of a semigroup B into the semigroup of injective endomorphisms of $End_\Lambda(H)$. Thus we get a family $\{\Phi_b(End_\Lambda(H)); b \in B\}$ of subalgebras of $End_\Lambda(H)$. We shall study conjugacy problems for this family. We prove a necessary condition for $\Phi_{b_1}(End_\Lambda(H))$ and $\Phi_{b_2}(End_\Lambda(H))$ ($b_1, b_2 \in B$) to be inner conjugate. This condition is stated in terms of orbits of b_1 and b_2. In Section 2.2, we impose a one more condition on elements of B. Under this condition, we can calculate the coupling constants of associated subalgebras and show a sufficient condition for two subalgebras not to be conjugate. In Section 2.3, we apply the general argument in above sections to the action of $SL(n, \mathbb{Z})$ on $\mathbb{T}^n$. Let $\mathcal{M}_n$ be the factor of type II_1 obtained from this action by the group-measure space construction. Then we construct subalgebras $\{\mathcal{N}^n_k; k \in \mathbb{N}\}$ of $\mathcal{M}_n$ which are not conjugate with each other. For every $k \in \mathbb{N}$, $\mathcal{N}^n_k$ is isomorphic to $\mathcal{M}_n$ and its relative commutant in $\mathcal{M}_n$ is trivial. We remark that the index $[\mathcal{M}_n : \mathcal{N}^n_k]$ defined by V. Jones in [20] is k^n.

In Chapter III, we construct periodic automorphisms with various Connes' outer invariants for full factors of type II_1 associated with actions of certain subgroups of $SL(2, \mathbb{Z})$ on the 2-torus. Outer invariants of periodic automorphisms of factors of type II_1 were introduced by A. Connes and he showed that they are the complete invariants for the outer conjugacy classes of the periodic automorphisms of the hyperfinite factor of type II_1 [4]. V. F. R. Jones developed

Connes' idea and gave a complete classification up to conjugacy of the actions of a finite group on the hyperfinite II_1 factor [19]. Recently, A. Ocneanu gave the classification up to outer conjugacy of of the actions of discrete amenable groups on the hyperfinite factor of type II_1 [26]. In non-hyperfinite cases, however, the problem of classifying periodic automorphisms is still open. Let $F_2(n)$ be the subgroup of $SL(2, \mathbb{Z})$ generated by $\begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}$ for a natural number n. Then, $F_2(n)$ is isomorphic to the free group on two generators. We denote by $\mathcal{M}_n$ tne factor obtained by the crossed product from the action of $F_2(n)$ on the 2-torus by linear automorphisms. Then, $\mathcal{M}_n$ is a full factor of type II_1. (It is not known whether $\mathcal{M}_n$'s are isomorphic or not.) In Section 3.1, we construct a periodic automorphism $\sigma_{n,p}$ of $\mathcal{M}_n$ with period p if p is a divisor of 2n. In Section 3.2, by making use of $\sigma_{n,p}$, we construct a minimal periodic automorphism $\rho_{n,p,i}$ of $\mathcal{M}_n$ with Connes' outer invariant (p, γ^i), where $\gamma = \exp(2\pi\sqrt{-1}/p)$. Therefore, for each couple $(p, \gamma) \in \mathbb{N} \times \mathbb{C}$ with $\gamma^p = 1$, we can find a periodic automorphism with Connes' outer invariant (p, γ) in $\bigcup\{\mathrm{Aut}(\mathcal{M}_n);\ n = 1, 2, \ldots\}$. It also follows that, if $(p, i) \neq (p', i')$ for divisors p, p' of 2n and i, i' $= 0, \ldots, p - 1$, then $\rho_{n,p,i}$ and $\rho_{n,p',i'}$ are not outer conjugate [4, Proposition 1.4]. Since $\mathcal{M}_n$ is full, $\mathcal{M}_n \otimes R$ is not isomorphic to $\mathcal{M}_n$, where R is the hyperfinite factor of type II_1 [3, Corollary 2.3]. Therefore the construction of $\rho_{n,p,i}$ are essentially different from that of s_p^γ discussed in [4, Proposition 1.6]. Finally we remark a few facts. Let F_n be the free group on n generators ($n = 2, 3, \ldots, \infty$) and λ be its left regular representation. In [30], J. Phillips constructed, for $(p, \gamma) \in \mathbb{N} \times \mathbb{C}$ with $\gamma^p = 1$, an automorphism α_p^γ of $\lambda(F_{p+1})''$ with Connes' outer invariant (p, γ). He then defined an automorphism of $\lambda(F_\infty)''$ with Connes' outer invariant (p, γ) for every $(p, \gamma) \in \mathbb{N} \times \mathbb{C}$ with $\gamma^p = 1$. His construction, however, does not apply to $\lambda(F_2)''$. We also remark that, if $\lambda(F_2)''$ is considered canonically as a subalgebra of $\mathcal{M}_n$, then $\lambda(F_2)''$ is not globally invariant under $\rho_{n,p,i}$.

Chapter I. Preliminaries for measurable groupoids.

In this chapter, we shall establish definitions and notations about measurable groupoids and associated von Neumann algebras. The notion of measurable groupoids was first introduced by G. W. Mackey in [23], and then studied by many authors. There have been changes in terminology and notations during the development of the theory. We shall mainly use terms and notations given by A. Connes in [5]. The expositions in this chapter are taken from the following: A. Ramsay [31], P. Hahn [16], J. Feldman, P. Hahn and C. C. Moore [12], J. Renault [32] and A. Connes [5].

§ 1.1. Measurable groupoids.

In this section, we shall establish basic definitions and notations about measurable groupoids.

Definition 1.1. ([16, Definition 1.1.]) A groupoid is a set G, together with a distinguished subset $G^{(2)} \subset G \times G$, and maps $(\gamma, \gamma') \in G^{(2)} \mapsto \gamma\gamma' \in G$ (product) and $\gamma \in G \mapsto \gamma^{-1} \in G$ (inverse) such that

(i) $(\gamma^{-1})^{-1} = \gamma$,

(ii) $(\gamma, \gamma') \in G^{(2)}$ and $(\gamma', \gamma'') \in G^{(2)} \Longrightarrow (\gamma\gamma', \gamma'') \in G^{(2)}$, $(\gamma', \gamma'\gamma'') \in G^{(2)}$ and $(\gamma\gamma')\gamma'' = \gamma(\gamma'\gamma'')$,

(iii) $(\gamma^{-1}, \gamma) \in G^{(2)}$ and if $(\gamma, \gamma') \in G^{(2)}$, then $\gamma^{-1}(\gamma\gamma') = \gamma'$,

(iv) $(\gamma, \gamma^{-1}) \in G^{(2)}$ and if $(\gamma', \gamma) \in G^{(2)}$, then $(\gamma'\gamma)\gamma^{-1} = \gamma'$.

For $\gamma \in G$, $s(\gamma) = \gamma^{-1}\gamma$ is the <u>source</u> of γ and $r(\gamma) = \gamma\gamma^{-1}$ is the <u>range</u> of γ. The pair (γ, γ') belongs to $G^{(2)}$ if and only if $s(\gamma) = r(\gamma')$. As we have $\gamma s(\gamma) = \gamma$ and $r(\gamma)\gamma = \gamma$, the set $G^{(0)} = s(G) = r(G)$ is called the <u>unit space</u> of G. If A and B are subsets of G, one may form the following subsets of G;

$$A^{-1} = \{\gamma \in G;\ \gamma^{-1} \in A\},$$

$$AB = \{\gamma'' \in G;\ \gamma \in A,\ \gamma' \in B,\ \gamma'' = \gamma\gamma'\}.$$

A groupoid G is called <u>principal</u> if the map $(r, s): \gamma \in G \longmapsto (r(\gamma), s(\gamma)) \in G^{(0)} \times G^{(0)}$ is one-to-one, and it is called <u>transitive</u> if (r, s) is onto. Let G be a groupoid, E a subset of $G^{(0)}$. We set

$$G|E = \{\gamma \in G;\ r(\gamma) \in E \text{ and } s(\gamma) \in E\}.$$

Then $G|E$ becomes a groupoid with units E if we define $(G|E)^{(2)} = G^{(2)} \cap (G|E \times G|E)$. The groupoid $G|E$ is called the <u>reduction</u> of G by E.

For $u, v \in G^{(0)}$, we set $G^u = r^{-1}(u)$, $G_v = s^{-1}(u)$ and $G^u_v = G^u \cap G_v$. Two points $u, v \in G^{(0)}$ are called equivalent with respect to G, if there exists $\gamma \in G$ such that $r(\gamma) = u$ and $s(\gamma) = v$. If u and v are equivalent we write $u \sim v$. This equivalence relation is just the equivalence relation $(r, s)(G)$ on $G^{(0)}$. Its equivalence classes are called <u>orbits</u> (or G-orbits) and the orbit of u is denoted by $[u]$, that is, $[u] = \{v \in G^{(0)};\ u \sim v\}$. For a subset E of $G^{(0)}$, the saturation $[E]$ of E is the set $\{v \in G^{(0)};$ for some $u \in E,\ v \sim u\}$. If $E = [E]$, then E is said to be <u>saturated</u>.

<u>Example</u> 1.2. Let S be a set and $R \subset S \times S$ be an equivalence relation on S. Let $R^{(2)} = \{((t, u), (v, w)) \in R \times R;\ u = v\}$. With

product $(t, u)(u, w) = (t, w)$ and inverse $(t, u)^{-1} = (u, t)$, R is a principal groupoid. The unit space $R^{(0)} = \{(s, s); s \in S\}$ may be identified with S, and we have $r(t, u) = t$ and $s(t, u) = u$. Conversely if G is a principal groupoid, by identifying G with $(r, s)(G)$, G may be considered as an equivalence relation on $G^{(0)}$.

Definition 1.3. Let G_1 and G_2 be groupoids. A map $\rho\colon G_1 \to G_2$ is a homomorphism if $(\rho(\gamma), \rho(\gamma')) \in G_2^{(2)}$ and $\rho(\gamma)\rho(\gamma') = \rho(\gamma\gamma')$ whenever $(\gamma, \gamma') \in G_1^{(2)}$. We denote by $\rho^{(0)}$ the restriction of ρ to $G_1^{(0)}$. It is clear that $\rho^{(0)}$ is a map into $G_2^{(0)}$.

We now consider a Borel structure on a groupoid. By a Borel space we mean a set S, together with a σ-algebra $\mathcal{B}(S)$ of subsets of S, called Borel sets. A map from one Borel space into another is itself called Borel if the inverse image of every Borel set is Borel. A one-to-one onto map which is Borel in both directions is called a Borel isomorphism. The Borel sets of a complete separable metric space are taken to be the σ-algebra generated by the open sets. A Borel space is called standard if it is Borel isomorphic to a Borel subset of a complete separable metric space. For a Borel space $(S, \mathcal{B}(S))$, if $\{u\} \in \mathcal{B}(S)$ for all $u \in S$, we say that S is separated. If there exists a countable subfamily $\{\sigma_i\}$ of $\mathcal{B}(S)$ which generates a separated sub σ-field of $\mathcal{B}(S)$, we say that S is countably separated, and if there exists $\{\sigma_i\}$ as above separating S which also generates $\mathcal{B}(S)$, we say that S is countably generated. One says that S is analytic if it is countably generated and there exists a standard space S' and a Borel map f of S' <u>onto</u> S. The Borel sets of any subset of a Borel space are taken to be the relative Borel sets. If f is a one-to-one Borel map of an analytic space S_1 into a countably generated space S_2, then f is a Borel isomorphism of S_1 onto $f(S_1)$. If S_1 is standard, then $f(S_1)$ is a Borel subset of S_2 [1, Chapter I, Proposition 2.5].

Definition 1.4. Let G be a groupoid such that the underlying space is also endowed with a Borel structure. If $G^{(2)}$ is a Borel set in the product structure on G × G, and $(\gamma, \gamma') \in G^{(2)} \longmapsto \gamma\gamma' \in G$ and $\gamma \in G \longmapsto \gamma^{-1} \in G$ are Borel maps, then G is called a measurable groupoid.

Note that the maps r and s are Borel. Two measurable groupoids G and G' are said to be isomorphic if there exists a Borel isomorphism ρ of G onto G' such that ρ and ρ^{-1} are algebraically homomorphisms.

§1.2. Measures on groupoids.

In this section, we define the notions of transverse functions and transverse measures which are introduced by A. Connes [5].

Definition 1.5. Let G be a measurable groupoid. A map ν of $G^{(0)}$ into the space of positive measures on G is called a transverse function on G if it satisfies the following properties;

(i) for all $u \in G^{(0)}$, ν^u is supported by G^u,

(ii) for every Borel set A of G, a map $u \longmapsto \nu^u(A) \in [0, +\infty]$ is Borel,

(iii) for all $\gamma \in G$, $\gamma\nu^{s(\gamma)} = \nu^{r(\gamma)}$,
where $\gamma\nu^{s(\gamma)}(A) = \nu^{s(\gamma)}(\gamma^{-1}A)$ for every Borel set A of G.

A transverse function ν is called <u>proper</u> if G is a union of an increasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of Borel subsets of G such that the function $u \in G^{(0)} \longmapsto \nu^u(A_n)$ is bounded for all $n \in \mathbb{N}$. Let $\mathcal{E}^+ = \mathcal{E}^+(G)$ denote the space of proper transverse functions on G. The order in $\mathcal{E}^+$ is defined by the following; for $\nu_1, \nu_2 \in \mathcal{E}^+$, $\nu_1 \leq \nu_2$ if $\nu_2^u - \nu_1^u$ is a positive measure for all $u \in G^{(0)}$. A map λ of $G^{(0)}$ into the space of positive measures on G is called a <u>kernel</u> on G if it satisfies the conditions (i), (ii) of Definition 1.5. For a Borel space S, $\mathcal{F}^+(S)$ denotes the space of non-negative Borel functions on S.

<u>Definition</u> 1.6. ([5, Chapter II, Definition 1]) A linear map Λ of $\mathcal{E}^+$ into $[0, +\infty]$ is called a <u>transverse</u> <u>measure</u> of module δ on G if it satisfies the following properties;

(a) Λ is normal, i.e. $\Lambda(\sup \nu_n) = \sup \Lambda(\nu_n)$ for every increasing sequence $\{\nu_n\}$ in $\mathcal{E}^+$ for which there exists $\nu \in \mathcal{E}^+$ such that $\nu_n \leq \nu$ for all $n \in \mathbb{N}$,

(b) Λ is of module δ, i.e. for every pair $\nu, \nu' \in \mathcal{E}^+$ and every kernel λ such that $\lambda^u(G) = 1$ for all $u \in G^{(0)}$

$$\nu*\delta\nu = \nu' \text{ implies } \Lambda(\nu) = \Lambda(\nu'),$$

where $(\nu*\delta\lambda)^u(f) = \iint f(\gamma\gamma')\delta(\gamma')\, d\lambda^{s(\gamma)}(\gamma')d\nu^u(\gamma)$ for every $u \in G^{(0)}$ and every $f \in \mathcal{F}^+(G)$.

A transverse measure Λ is called <u>semifinite</u> if, for all $\nu \in \mathcal{E}^+$, we have $\Lambda(\nu) = \sup \{\Lambda(\nu'); \nu' \leq \nu, \Lambda(\nu') < +\infty\}$, and it is called <u>σ-finite</u> if there exists a faithful transverse function ν of the form $\nu = \sup \nu_n$, $\Lambda(\nu_n) < +\infty$. A transverse measure Λ is called <u>unimodular</u> if $\delta = 1$.

Let Λ be a transverse measure on G. For $\nu \in \mathcal{E}^+$ and $f \in \mathfrak{F}^+(G^{(0)})$, we define an element $(f\circ s)\nu$ of $\mathcal{E}^+$ by $[(f\circ s)\nu]^u(g) = \int g(\gamma)f(s(\gamma))\, d\nu^u(\gamma)$ for $u \in G^{(0)}$ and $g \in \mathfrak{F}^+(G)$. The equality $\Lambda_\nu(f) = \Lambda((f\circ s)\nu)$ defines a positive measure Λ_ν on $G^{(0)}$. Then we have the following important result due to A. Connes.

<u>Theorem</u> 1.7. ([5, Chapter II, Theorem 3]) Let ν be a faithful proper transverse function on G. (A transverse function ν is called faithful if $\nu^u \neq 0$ for $u \in G^{(0)}$.) The map $\Lambda \longmapsto \Lambda_\nu$ is a bijection between the set of transverse measures of module δ on G and the set of positive measures μ on $G^{(0)}$ satisfying the following conditions;

$$\iint f(\gamma^{-1})\delta(\gamma^{-1})\, d\nu^u(\gamma)d\mu(\gamma) = \iint f(\gamma)\, d\nu^u(\gamma)d\mu(\gamma)$$

for all $f \in \mathfrak{F}^+(G)$.

Let μ be a measure on a Borel space S. The measure class of μ is denoted by $[\mu]$. For a Borel subset E of S, E is said to be <u>null</u> if $\mu(E) = 0$ and it is said to be <u>conull</u> if $\mu(S - E) = 0$. The characteristic function of E is denoted by χ_E.

Let ν be a proper transverse function on a measurable groupoid G and Λ be a transverse measure of module δ on G. We define a measure λ on G by

$$\lambda(f) = \iint f(\gamma)\, d\nu^u(\gamma)d\Lambda_\nu(u) \qquad \text{for } f \in \mathfrak{F}^+(G).$$

The pair $(G, [\lambda])$ is called a <u>measured groupoid</u>. For a Λ_ν-conull set E of $G^{(0)}$, $G|E$ is called an <u>inessential reduction</u> of $(G, [\lambda])$.

Let E be a saturated Borel set of $G^{(0)}$. If E is Λ_ν-null for all $\nu \in \mathcal{E}^+$, then E is called a <u>Λ-null</u> set. Let $\nu \in \mathcal{E}^+$ be faithful. For a saturated Borel set E of $G^{(0)}$, E is Λ_ν-null if and only if E is Λ-null [5, Chapter II, Proposition 8]. A saturated Borel set E of $G^{(0)}$ is called a <u>Λ-conull</u> set if $G^{(0)} - E$ is Λ-null.

§1.3. Von Neumann algebras associated with groupoids.

In this section, we define regular representations of measurable groupoids and then construct von Neumann algebras from them. All results in this section are due to A. Connes [5].

Let G be a measurable groupoid and H be a measurable field of Hilbert spaces on $G^{(0)}$.

<u>Definition</u> 1.8. A representation U of G in H is the object such that, for all $\gamma \in G$, $U(\gamma)$ is an isometry of $H_{s(\gamma)}$ onto $H_{r(\gamma)}$ satisfying;

(a) $U(\gamma_1{}^{-1}\gamma_2) = U(\gamma_1)^{-1}U(\gamma_2)$ for all $\gamma_1, \gamma_2 \in G$, $r(\gamma_1) = r(\gamma_2)$,

(b) for every pair ξ, η of Borel sections of H, the function (ξ, η) on G defined below is Borel;

$$(\xi, \eta)(\gamma) = \langle \xi_{r(\gamma)}, U(\gamma)\eta_{s(\gamma)} \rangle \quad \text{for all } \gamma \in G.$$

Let $\nu \in \mathcal{E}^+$ be a transverse function on G. For every $u \in G^{(0)}$, $L^2(G^u, \nu^u) = H_u$ is a Hilbert space and, for $\gamma \in G$, we define an isometry $L^\nu(\gamma)$ of $H_{s(\gamma)}$ onto $H_{r(\gamma)}$ by

$$(L^{\nu}(\gamma)f)(\gamma') = f(\gamma^{-1}\gamma') \qquad \text{for } f \in H_{s(\gamma)} \text{ and } \gamma' \in G^{r(\gamma)}.$$

The field $H^{\nu} = (H_u)_{u \in G^{(0)}}$ is endowed with the unique measurable structure for which the following sections are measurable; $u \longmapsto f|G^u$, where f is a Borel function on G such that $\int |f|^2 \, d\nu^u < \infty$ for all $u \in G^{(0)}$ and $f|G^u$ is the restriction of f to G^u. The representation (H^{ν}, L^{ν}) defined above is called the <u>left regular representation</u> associated with ν.

Let (H, U) and (H', U') be two representations of G. Let $T = (T_u)_{u \in G^{(0)}}$ be a measurable family of bounded operators T_u of H_u into H'_u ($u \in G^{(0)}$). A family T is called an <u>intertwining operator</u> of (H, U) to (H', U') if it satisfies the following conditions;

1) $\sup_{u \in G^{(0)}} \|T_u\| < \infty$,

2) for all $\gamma \in G$, $U'(\gamma)T_{s(\gamma)} = T_{r(\gamma)}U(\gamma)$.

We denote by $\mathrm{End}_G(H)$ the vector space of intertwining operators of (H, U) to itself.

Let (H, U) and (H', U') be representations of G. If H'_u is a closed subspace of H_u for all $u \in G^{(0)}$ and the restriction of $U(\gamma)$ to $H'_{s(\gamma)}$ is $U'(\gamma)$ for all $\gamma \in G$, then (H', U') is called a <u>sub-representation</u> of (H, U). The <u>direct sum</u> $(H \oplus H', U \oplus U')$ of representations (H, U) and (H', U') is defined by $(H \oplus H')_u = H_u \oplus H'_u$ for $u \in G^{(0)}$ and $(U \oplus U')(\gamma) = U(\gamma) \oplus U'(\gamma)$ for $\gamma \in G$. Two representations (H, U) and (H', U') are called <u>equivalent</u> if there exists an intertwining operator T of (H, U) to (H', U') such that T_u is an isometry of H_u onto H'_u for all $u \in G^{(0)}$. Let $\nu \in \mathcal{E}^+$ be a faithful transverse function on G. A representation (H, U) is said to be

<u>square-integrable</u> if it is equivalent to a subrepresentation of the direct sum of countably infinite copies of (H^{ν}, L^{ν}).

Let Λ be a transverse measure on G.

<u>Definition</u> 1.9. Let (H, U) be a square-integrable representation of G. Two elements $T, T' \in End_G(H)$ are said to be equivalent if there exists a Λ-conull set E of $G^{(0)}$ such that $T_u = T'_u$ for all $u \in E$. The vector space of equivalence classes of $End_G(H)$ is denoted by $End_{\Lambda}(H)$. Elements of $End_{\Lambda}(H)$ is called <u>random</u> <u>operators</u>.

For $T \in End_{\Lambda}(H)$, we set $\|T\|_{\infty} = \underset{\Lambda}{\text{ess sup}} \|T_u\|$. This definition has a sense because the function $u \longmapsto \|T_u\|$ is measurable and constant on every orbits of $G^{(0)}$.

<u>Theorem</u> 1.10. ([5, Chapter V, Theorem 2]) For every square-integrable representation (H, U) of G, the involutive normed algebra $End_{\Lambda}(H)$ is a von Neumann algebra.

§1.4. Principal groupoids with countable orbits.

In the following chapters we only consider principal groupoids with countable orbits, so we recall facts about them.

A groupoid G is said to have <u>countable</u> <u>orbits</u> if the orbit of u is a countable set for all $u \in G^{(0)}$. In the following, we suppose that G is a principal measurable groupoid with countable orbits and that G and $G^{(0)}$ are standard spaces. We set $S = G^{(0)}$. Since (r, s) is a one-to-one Borel map of G into S × S, it follows from §1 that (r, s) is a Borel isomorphism of G onto (r, s)(G). Therefore we may consider G as an equivalence relation on S, which is a groupoid as in Example 1.2. Let ν be a transverse function on G such that $\nu^t(\{(t, u)\}) = 1$ for all $t \in S$ and $(t, u) \in G^t$. We call such a transverse function the transverse function of counting measures. Let Λ be a transverse measure on G. We set $\mu = \Lambda_\nu$ and define a measure λ on G by $\lambda = \int \nu^s \, d\mu(s)$. We mean by a transformation of a Borel space S a Borel isomorphism of S onto itself. A transformation g of a measure space (S, μ) is said to be non-singular if, for every Borel set E of S, $\mu(E) = 0 \Longleftrightarrow \mu(gE) = 0$. The full group [G] of a measured groupoid $(G, [\lambda])$ is the group of non-singular transformations g of (S, μ) such that $[gt] = [t]$ for almost all $t \in S$. The normalizer N[G] of [G] is the group of non-singular transformations g of (S, μ) such that $[gt] = g[t]$ for almost all $t \in S$.

The following result is important.

Theorem 1.11. ([13, Theorem 1]) Let G be a principal measurable groupoid with countable orbits. Suppose that G and $G^{(0)}$ are standard spaces. Then there exists a countable group C of transformations of $G^{(0)}$ such that G and the equivalence relation $\{(t, gt); t \in G^{(0)}, g \in C\}$ on $G^{(0)}$ are isomorphic as measurable groupoids.

Example 1.12. Let C be a countable group of transformations of a standard space S. We set $G = \{(t, gt); t \in S, g \in C\}$. Then G is a principal measurable groupoid with countable orbits such that $G^{(0)} = S$.

Since $\{(t, gt); t \in S\}$ is a Borel subset of $S \times S$ for each $g \in C$ [1, Chapter I, Proposition 2.2], G is a Borel subset of $S \times S$. It follows that G is a standard space. The groupoid G is called the orbit groupoid of an action of C on S. Let μ be a probability measure on S. Suppose that elements of C preserve the measure μ, i.e. $\mu(gE) = \mu(E)$ for every Borel subset E of S and $g \in C$. For the transverse function ν of counting measures on G, it follows from Theorem 1.7 that there exists a unique unimodular transverse measure Λ on G such that $\mu = \Lambda_\nu$.

Now, let G be an arbitrary principal measurable groupoid with countable orbits such that G and $G^{(0)}$ are standard. Let ν be the transverse function of counting measures on G and Λ be a unimodular transverse measure on G such that $\mu = \Lambda_\nu$ is a probability measure on $S = G^{(0)}$. We set $\lambda = \int \nu^t \, d\mu(t)$. In this case, every element of the full group $[G]$ of $(G, [\lambda])$ preserves μ [13, Corollary 1 of Proposition 2.2]. Let $L^2(G, \lambda)$ be the Hilbert space of square-integrable functions on G with respect to λ. We set $\mathcal{H} = L^2(G, \lambda)$. The von Neumann algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. We define a homomorphism U of $[G]$ into the unitary group of $\mathcal{L}(\mathcal{H})$ by

$$(U(g)\xi)(s, t) = \xi(s, g^{-1}t)$$

for $g \in [G]$, $\xi \in \mathcal{H}$ and λ-a.a. $(s, t) \in G$. We also define a *-homomorphism π of the von Neumann algebra $L^\infty(S, \mu)$ into $\mathcal{L}(\mathcal{H})$ by

$$(\pi(f)\xi)(s, t) = f(t)\xi(s, t)$$

for $f \in L^\infty(S, \mu)$, $\xi \in \mathcal{H}$ and $(s, t) \in G$. Let $\mathcal{M}(G)$ be a von Neumann algebra on $\mathcal{H}$ generated by $\{U(g);\ g \in [G]\}$ and $\{\pi(f);\ f \in L^\infty(S, \mu)\}$. We set $\mathcal{A}(G) = \{\pi(f);\ f \in L^\infty(S, \mu)\}$, which is a maximal abelian subalgebra of $\mathcal{M}(G)$. We shall call $\mathcal{M}(G)$ the von Neumann algebra associated with $(G, [\lambda])$. Note that $\mathcal{M}(G)$ is finite because Λ is unimodular and μ is finite. Let (H^ν, L^ν) be the left regular representation of G associated with ν. By a standard argument, one can show that $\mathcal{M}(G)$ is *-isomorphic to $\mathrm{End}_\Lambda(H^\nu)$.

If, for every saturated Borel set A of $G^{(0)}$, A is Λ-null or Λ-conull, then (G, Λ) is called <u>ergodic</u>. If this is the case, $\mathrm{End}_\Lambda(H)$ is a factor for every square-integrable representation (H, L) [5, §V, Corollary 8]. In the situation of the previous paragraph, $\mathcal{M}(G)$ is a factor of type II_1 if (G, Λ) is ergodic and μ is non-atomic.

We set $C^1 = \{z \in \mathbb{C};\ |z| = 1\}$. A Borel map $a: G \longrightarrow C^1$ is called a <u>cocycle</u> on G if there exists an inessential reduction G' of $(G, [\lambda])$ such that $a(\gamma_1\gamma_2) = a(\gamma_1)a(\gamma_2)$ for every $(\gamma_1, \gamma_2) \in G'^{(2)}$. A cocycle a is called a <u>coboundary</u> if there exists a Borel map $b: G^{(0)} \longrightarrow C^1$ such that $a(s, t) = b(s)\overline{b(t)}$ for λ-a.a. $(s, t) \in G$. We denote by $Z^1(G, C^1)$ the group of cocycles on G into C^1 and by $B^1(G, C^1)$ the group of coboundaries on G. Two functions on G are identified if they coincide λ-almost everywhere.

§1.5. Von Neumann algebras.

In this section, we recall some facts about Takesaki's duality theorem and Connes' invariants for periodic automorphisms. The expositions in this section are taken from [37] and [4].

1. Takesaki's duality theorem.

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and C be a locally compact abèlian group whose Haar measure is denoted by dg. The group of all *-automorphisms on $\mathcal{M}$ is denoted by Aut($\mathcal{M}$). A continuous action of C on $\mathcal{M}$ is a homomorphism α: $C \ni g \longmapsto \alpha_g \in \mathrm{Aut}(\mathcal{M})$ such that for each fixed $x \in \mathcal{M}$, the map; $g \in C \longmapsto \alpha_g(x) \in \mathcal{M}$ is σ-strongly* continuous. Let $L^2(C, \mathcal{H})$ denote the Hilbert space of square-integrable functions of C into $\mathcal{H}$ with respect to dg. For a continuous action α of C on $\mathcal{M}$, we define representations π^α of $\mathcal{M}$ and U^α of C on $L^2(C, \mathcal{H})$ as follows;

$$(\pi^\alpha(x)\xi)(h) = \alpha_h{}^{-1}(x)\xi(h),$$

$$(U^\alpha(g)\xi)(h) = \xi(g^{-1}h)$$

for $h, g \in C$ and $\xi \in L^2(C, \mathcal{H})$. The von Neumann algebra on $L^2(C, \mathcal{H})$ generated by $\pi^\alpha(\mathcal{M})$ and $U^\alpha(C)$ is called the <u>crossed product</u> of $\mathcal{M}$ by the action α of C, and denoted by $\mathcal{R}(\mathcal{M}, \alpha)$.

Let $\hat{C}$ be the dual group of C. We fix Haar measures dg in C and dp in C so that the Plancherel formula holds. Define a unitary representation $\hat{U}$ of $\hat{C}$ on $L^2(C, \mathcal{H})$ by

$$\hat{U}(p)\xi(g) = \overline{\langle g, p\rangle}\xi(g) \qquad \text{for } \xi \in L^2(C, \mathcal{H}),\ g \in C,\ p \in \hat{C},$$

where $\langle g, p\rangle$ denotes the value of $p \in \hat{C}$ at $g \in C$. We can define a continuous action $\hat{\alpha}$ of $\hat{C}$ on $\mathcal{R}(\mathcal{M}, \alpha)$ by

$$\hat{\alpha}_p(x) = \hat{U}(p)x\hat{U}(-p) \qquad \text{for } x \in \mathcal{R}(\mathcal{M}, \alpha),\ p \in \hat{C}.$$

We call $\hat{\alpha}$ the <u>dual action</u> of $\hat{C}$ on $\mathcal{R}(\mathcal{M}, \alpha)$. Then we have

$$\hat{\alpha}_p(U^\alpha(g)) = \overline{\langle g, p\rangle} U^\alpha(g) \qquad \text{for } g \in C,\ p \in \hat{C},$$
$$\hat{\alpha}_p(x) = x \qquad \text{for } x \in \mathcal{M},\ p \in \hat{C}.$$

Theorem 1.13. ([37, Theorem 4.5]) The crossed product $\mathcal{R}(\mathcal{R}(\mathcal{M}, \alpha), \alpha)$ is *-isomorphic to the tensor product $\mathcal{M} \otimes (L^2(C))$, where $\mathcal{L}(L^2(C))$ is the von Neumann algebra of all bounded linear operators on $L^2(C)$.

2. Connes' invariants for automorphisms.

Let $\mathcal{M}$ be a factor and α be a *-automorphism of $\mathcal{M}$. The outer period $p_o(\alpha)$ of α is a natural number such that α^i is outer for $i = 1, \ldots, p_o(\alpha) - 1$ and $\alpha^{p_o(\alpha)}$ is inner. The outer period of α is zero if all the non-zero powers of α are outer. Let U be a unitary operator in $\mathcal{M}$ such that $\alpha^{p_o(\alpha)} = \mathrm{Ad}\, U$, where $\mathrm{Ad}\, U(x) = UxU^*$ for all $x \in \mathcal{M}$. Then, the complex number $\gamma(\alpha)$ is defined by $\alpha(U) = \gamma(\alpha)U$. We see that $\gamma(\alpha)$ is a complex number of modulus 1, independent of the choice of U such that $\alpha^{p_o(\alpha)} = \mathrm{Ad}\, U$, and satisfying $\gamma(\alpha)^{p_o(\alpha)} = 1$. Two automorphisms α and β of $\mathcal{M}$ are called <u>outer conjugate</u> if there exists a $\sigma \in \mathrm{Aut}\,\mathcal{M}$ such that β and $\sigma\alpha\sigma^{-1}$ have the same image in $\mathrm{Out}\,\mathcal{M} = \mathrm{Aut}\,\mathcal{M}/\mathrm{Int}\,\mathcal{M}$, where $\mathrm{Int}\,\mathcal{M}$ denotes the group of all inner automorphisms of $\mathcal{M}$. If α and β are outer conjugate then $p_o(\alpha) = p_o(\beta)$, $\gamma(\alpha) = \gamma(\beta)$. The pair $(p_o(\alpha), \gamma(\alpha))$ is called <u>Connes' outer invariant</u> of α.

In the following, $\mathcal{M}$ is a factor of type II_1 with canonical trace τ ($\tau(1) = 1$). Let $\alpha = \mathrm{Ad}\, U$ be a periodic inner automorphism with period p. As U^p is a scalar λ_0, U is a finite linear combination of its spectral projections corresponding to the pth roots a_j of λ_0, say $U = \Sigma_{j=1}^{p} a_j e_j$, where e_j is the spectral projection of U corresponding to $\{a_j\}$. We define now the inner invariant $\varepsilon(\alpha)$ to be the probability measure $\Sigma\, \tau(e_j)\varepsilon_{a_j}$, determined up to rotation on $C^1 = \{z \in \mathbb{C}\,;\ |z| = 1\}$, where ε_a is the Dirac measure at a point a. For a periodic automorphism α of $\mathcal{M}$ with outer invariant (p_o, γ), we put $p_m = p_o \cdot \text{Order}\ \gamma$ and we put $\varepsilon(\alpha) = \varepsilon(\alpha^{p_m})$. This $\varepsilon(\alpha)$ is called <u>Connes' inner invariant</u> of α. The number $p_m(\alpha) = p_o(\alpha) \cdot \text{Order}\ \gamma(\alpha)$ is called the <u>minimal period</u> of α. For a periodic automorphism α of $\mathcal{M}$, if the period of α is equal to the minimal period of α, then α is called a <u>minimal periodic automorphism</u>.

Chapter II. Subalgebras of von Neumann algebras.

By a subalgebra of a von Neumann algebra $\mathcal{M}$, we mean a weakly closed *-subalgebra of $\mathcal{M}$ containing the identity of $\mathcal{M}$. For subalgebras $\mathcal{N}_1$, $\mathcal{N}_2$ of $\mathcal{M}$, $\mathcal{N}_1$ and $\mathcal{N}_2$ are called to be <u>conjugate</u> if there exists a *-automorphism α of $\mathcal{M}$ such that $\alpha(\mathcal{N}_1) = \mathcal{N}_2$, and they are called to be <u>inner</u> <u>conjugate</u> if there exists an inner automorphism α of $\mathcal{M}$ such that $\alpha(\mathcal{N}_1) = \mathcal{N}_2$. In this chapter, we study conjugacy problems for certain subalgebras of von Neumann algebras associated with measurable groupoids. In particular we show that, for some full factor $\mathcal{M}$ of type II_1, there exists a countable family of subalgebras of $\mathcal{M}$, which are *-isomorphic to $\mathcal{M}$, such that they are not conjugate with each other. All results in this chapter are taken from [27].

§ 2.1. Actions of semigroups on Borel spaces.

In this section, we introduce the notion of an action of a semigroup on a Borel space and study an inner conjugacy problem for certain subalgebras of von Neumann algebras associated with actions of semigroups.

Let A be a group, B be a sub-semigroup of A and C be a countable discrete normal subgroup of A which is contained in B. For a standard Borel space S, $\mathcal{B}(S)$ denotes the set of Borel maps of S <u>onto</u> itself which send Borel subsets of S to Borel subsets of S. For ϕ, $\psi \in \mathcal{B}(S)$, the product $\phi \circ \psi$ of ϕ and ψ is defined by $\phi \circ \psi(x) = \psi(\phi(x))$

$(x \in S)$. Then $\mathcal{B}(S)$ becomes a semigroup. Let α be a homomorphism of the semigroup B into the semigroup $\mathcal{B}(S)$ such that $\alpha(e)$ is the identity map on S, where e is the unit of A. We write xb instead of $\alpha(b)(x)$ $(x \in S,\ b \in B)$. Then S is a Borel C-space with respect to the restriction of the above action to C and $G = S \times C$ becomes a measurable groupoid, that is, $((x, c_1), (y, c_2)) \in G^{(2)}$ if and only if $y = xc_1$ and we have $(x, c_1)(xc_1, c_2) = (x, c_1c_2)$. Note that $G^{(0)}$ is identified with S. We assume that the action of C on S is free, that is, for every $x \in S$, $\{c \in C;\ xc = x\} = \{e\}$. Note that the orbit $[x]$ of x is $\{xc;\ c \in C\}$. The following lemma is clear.

<u>Lemma</u> 2.1. If b is an element of B, then $[xb] = [x]b$ for every $x \in S$.

We define an equivalence relation $\underset{b}{\sim}$ on S by the following; for x, $y \in S$, $x \underset{b}{\sim} y$ if and only if $xb \sim yb$, where $\sim$ denotes the equivalence relation with respect to G. Put $G \cdot b = \{(x, y) \in S \times S;\ x \underset{b}{\sim} y\}$. As in Example 1.2, $G \cdot b$ becomes a measurable groupoid. Note that G can be considered as a subgroupoid of $G \cdot b$ by the injection $(x, c) \longmapsto (x, xc)$ $((x, c) \in G)$. The saturation of E with respect to $G \cdot b$ is denoted by $[E]_b$.

<u>Lemma</u> 2.2. (i) If E is a saturated set with respect to G, then for every b, Eb and Eb^{-1} are saturated with respect to G, where $Eb^{-1} = \{x \in S;\ xb \in E\}$.

(ii) For $b \in B$ and $x \in S$, put $F = \{y \in S;\ yb = xb\}$. Then $[x]_b$ is a disjoint union of $\{[y]\}_{y \in F}$.

Proof. (i) By Lemma 2.1, it is clear that $[Eb] = Eb$. For $x \in [Eb^{-1}]$, there exists $y \in Eb^{-1}$ such that $x \sim y$. As we have $xb \sim yb$ and $yb \in E$, xb belongs to E. It follows that $[Eb^{-1}] = Eb^{-1}$.

(ii) It is clear that $\bigcup_{y \in F} [y] \subset [x]_b$. For $y \in [x]_b$, there exists $c \in C$ such that $xb = ybc$. Then, for some $c_1 \in C$, we have $xb = yc_1b$ and yc_1 belongs to F. Thus we have $[x]_b \subset \bigcup_{y \in F} [y]$. Now, suppose that, for y_1, $y_2 \in F$, y_1 belongs to $[y_2]$. There exist c_1, $c_2 \in C$ such that $y_1 = y_2c$ and $cb = bc_1$. We have

$$xb = y_1b = y_2bc_1 = xbc_1.$$

Since the action of C is free, this implies that $c = c_1 = e$. It follows that $y_1 = y_2$, and this means that $\{[y]\}_{y \in F}$ are disjoint. Q.E.D.

Definition 2.3. Let (A, B, G) be as above and Λ be a σ-finite transverse measure on G. A quartet (A, B, G, Λ) is called an action of B on S if it satisfies the following condition; for a G-saturated Borel set E of S, if E is Λ-null, then Eb and Eb^{-1} are Λ-null for every $b \in B$.

From now on we assume that (A, B, G, Λ) is an action of B on S. Let ν be the transverse function of counting measures on G and $(H, L) = (H^{\nu}, L^{\nu})$ be the left regular representation of G. We construct the von Neumann algebra $End_{\Lambda}(H)$ as in §1.3. We sometimes consider elements of $End_{\Lambda}(H)$ as elements of $End_G(H)$. Let b be an element of B. For every $x \in S$, define an isometry $\phi_{xb,x} = \phi^b_{xb,x}$ of H_x onto H_{xb} by

$$\phi_{xb,x}(f)(xb,\ c) = f(x,\ bcb^{-1})$$

for every $f \in H_x$ and $(xb,\ c) \in G^{xb}$.

<u>Lemma</u> 2.4. If c is an element of c, then

$$L(xb,\ b^{-1}cb) \circ \phi_{xcb,xc} = \phi_{xb,x} \circ L(x,\ c)$$

for every $x \in S$.

<u>Proof</u>. Note that, as C is a normal subgroup of A, bcb^{-1} and $b^{-1}cb$ belong to C for every $b \in B$. Since we have

$$(xb)(b^{-1}cb) = x(bb^{-1}cb) = xcb,$$

we have $s(xb,\ b^{-1}cb) = xcb$. For every $f \in H_{xc}$ and $c' \in C$, we have

$$L(xb,\ b^{-1}cb) \circ \phi_{xcb,xc}(f)(xb,\ c')$$

$$= f(xc,\ c^{-1}bc'b^{-1})$$

$$= \phi_{xb,x}\ L(x,\ c)(f)(xb,\ c'). \qquad \text{Q.E.D.}$$

<u>Proposition</u> 2.5. For an element $T = (T_x)_{x \in S}$ of $\mathrm{End}_\Lambda(H)$, put

$$\Phi_b(T)_x = \phi_{xb,x}{}^{-1} \circ T_{xb} \circ \phi_{xb,x} \qquad \text{for every } x \in S,$$

then $\Phi_b(T) = (\Phi_b(T)_x)_{x \in S}$ is an element of $\mathrm{End}_\Lambda(H)$.

<u>Proof</u>. Define a Borel structure for the field $H' = (H_{xb})_{x \in S}$ of Hilbert spaces by the following (c.f. [8, p.142, Definition 1]); if $\xi^i = (\xi^i{}_x)$ $(i = 1, 2, \ldots)$ is a fundamental sequence for the Borel structure of H, then $(\xi_{xb}{}^i)_{x \in S}$ $(i = 1, 2, \ldots)$ is a fundamental sequence for H'. Let f be a Borel function on G such that $\int |f|^2\, d\nu^x < +\infty$ for all $x \in S$. The restriction of f to G^x is denoted by f_x. Note that (f_x) is a Borel section of H and the Borel

structure of H is determined by the set of sections of this form (see §1.3). Then $(\phi_{xb,x}(f_x))_{x\in S}$ is a Borel section of H'. For Borel sections (f_x) and (g_x) of H of the above form, we have

$$\langle \Phi_b(T)_x f_x, g_x\rangle = \langle T_{xb}\phi_{xb,x}(f_x), \phi_{xb,x}(g_x)\rangle.$$

This implies that $\Phi_b(T)$ is a Borel field of operators for H.

Let $\gamma = (x, c_0)$ be an element of G. For $(x, c)\in G$ and $f\in H_{xc_0}$, we have

$$\begin{aligned}
&(L(\gamma)\Phi_b(T)_{xc_0}f)(x, c)\\
&= (T_{xc_0b}\circ\phi_{xc_0b,xc_0}(f))(xc_0b, b^{-1}c_0^{-1}cb)\\
&= (L(xb, b^{-1}c_0b)\circ T_{xc_0b}\circ\phi_{xc_0b,xc_0}(f))(xb, b^{-1}cb)\\
&= (T_{xb}\circ L(xb, b^{-1}c_0b)\circ\phi_{xc_0b,xc_0}(f))(xb, b^{-1}cb).
\end{aligned}$$

By Lemma 2.4, we have

$$\begin{aligned}
&(L(\gamma)\Phi_b(T)_{xc_0}f)(x, c)\\
&=T_{xb}\circ\phi_{xb,x}\circ L(x, c_0)(f)(xb, b^{-1}cb)\\
&= (\Phi_b(T)_x\circ L(\gamma)f)(x, c).
\end{aligned}$$

It follows that $\Phi_b(T)$ is an intertwining operator of (H, L) to itself. If T and T' are intertwining operators of (H, L) which coincide with each other Λ-almost everywhere, then $\Phi_b(T)$ and $\Phi_b(T')$ coincide with each other Λ-almost everywhere by the assumption of (A, B, G, Λ). Therefore $\Phi_b(T)$ is well-defined as an element of $\mathrm{End}_\Lambda(H)$. Q.E.D.

In the above proposition, we have constructed a map Φ_b of $\mathrm{End}_\Lambda(H)$ into itself. The following lemma shows that Φ_b is an injective endomorphism of $\mathrm{End}_\Lambda(H)$ and that the map $b\longmapsto\Phi_b$ is a homomorphism of the semigroup B into the semigroup of endomorphisms of $End_\Lambda(H)$.

Lemma 2.6. (i) If b_1 and b_2 are elements of B, then, for every $T \in End_\Lambda(H)$, $\Phi_{b_1} \circ \Phi_{b_2}(T) = \Phi_{b_1 b_2}(T)$.

(ii) If c is an element of C, Φ_c is an inner automorphism of $End_\Lambda(H)$.

(iii) For $b \in B$, the map Φ_b is an isometric *-homomorphism of $End_\Lambda(H)$ into $End_\Lambda(H)$.

Proof. (i) It is clear by an easy calculation.

(ii) Define a unitary operator U_x on H_x by

$$(U_x f)(x, c') = f(x, c'c^{-1}) \qquad \text{for } f \in H_x \text{ and } (x, c') \in G^x.$$

Then $U = (U_x)$ is a unitary element of $End_\Lambda(H)$ and we have $\Phi_c(T) = U^{-1}TU$ for every $T \in End_\Lambda(H)$.

(iii) It is clear that Φ_b is a *-homomorphism. Let T be an element of $End_\Lambda(H)$. For $\alpha \in \mathbb{R}_+$, we set $E_\alpha(T) = \{x \in S;\ \|T_x\| > \alpha\}$, which is a saturated Borel set. We have $E_\alpha(\Phi_b(T)) = E_\alpha(T)b^{-1}$ by the equation $\|\Phi_b(T)_x\| = \|T_{xb}\|$. Thus $E_\alpha(T)$ is Λ-null if and only if $E_\alpha(\Phi_b(T))$ is Λ-null. Recall that the norm of T is defined by

$$\|T\|_\infty = \inf\{\alpha \in \mathbb{R}_+;\ E_\alpha(T) \text{ is } \Lambda\text{-null}\}.$$

Hence we have $\|\Phi_b(T)\|_\infty = \|T\|_\infty$. Q.E.D.

If (x, y) is an element of $G \cdot b$, there exists a unique pair $(x_0, c) \in S \times C$ such that $x_0 b = yb$ and $x = x_0 c$ by Lemma 2.2. Define an isometry $L(x, y) = L^b(x, y)$ of H_y onto H_x by

$$L(x, y) = L(x_0, c)^{-1} \circ \phi^b_{x_0 b, x_0}{}^{-1} \circ \phi^b_{yb, y}.$$

Then we have the following:

Proposition 2.7. If b is an element of B, then (H, L^b) is a representation of $G\cdot b$.

Proof. For $(x, y) \in G\cdot b$ with $x_0 b = yb$ and $x = x_0 c$, we write $\psi(x, y)$ for c. We have, for $f \in H_y$ and $(x, c) \in G$,

$$(L^b(x, y)f)(x, c) = f(y, \psi(x, y)c).$$

Note that $\psi^{-1}(\{c\})$ is a Borel subset of $G\cdot b$ for every $c \in C$. Let f and g be a Borel functions on G as in the proof of Proposition 2.5. Then the function $(x, y, c) \longmapsto f(y, \psi(x, y)c)g(x, c)$ is Borel on $G\cdot b \times C$. Since we have

$$\langle L^b(x, y)f_y, g_x\rangle = \int f(y, \psi(x, y)c)g(x, c)\, d\nu^x(x, c),$$

the function $(x, y) \longmapsto \langle L^b(x, y)f_y, g_x\rangle$ is Borel on $G\cdot b$.

Let (x, y) and (y, z) be elements of $G\cdot b$ such that $x = x_0\psi(x, y)$, $y = y_0\psi(y, z)$, $x_0 b = yb$ and $y_0 b = zb$. We have

$$x_0\psi(y, z)^{-1}b = (x_0 b)(b^{-1}\psi(y, z)^{-1}b) = y\psi(y, z)^{-1}b = zb.$$

By putting $x_1 = x_0\psi(y, z)^{-1}$, we have $x_1 b = zb$ and $x_1\psi(y, z)\psi(x, y) = x$. It follows that $\psi(x, z) = \psi(y, z)\psi(x, y)$. Similarly we have $\psi(y, x) = \psi(x, y)^{-1}$. The equation $L^b((y, x)^{-1}(y, z)) = L^b(y, x)^{-1}L^b(y, z)$ follows immediately from the above equations. Q.E.D.

The following theorem characterizes the von Neumann algebra $\Phi_b(\mathrm{End}_\Lambda(H))$.

Theorem 2.8. Let b be an element of B. For $T \in End_\Lambda(H)$, T belongs to $\Phi_b(End_\Lambda(H))$ if and only if there exists a $G\cdot b$-saturated Λ-null set E of S such that, for $x, y \in S - E$, $x \underset{b}{\sim} y$ implies that

$$L^b(x, y)T_y = T_x L^b(x, y).$$

Proof. Let T be an element of $\Phi_b(End_\Lambda(H))$ with $T = \Phi_b(T')$ (T' $End_\Lambda(H)$). For $(x, y) \in G\cdot b$ with $x_0 b = yb$ and $x = x_0 c$, we have

$$\phi_{x_0 b, x_0}{}^{-1} \circ \phi_{yb,y} \circ T_y = T_{x_0} \circ \phi_{x_0 b, x_0}{}^{-1} \circ \phi_{yb,y}.$$

This implies that $L^b(x, y)T_y = T_x L^b(x, y)$ for every $(x, y) \in G\cdot b$.

Since S is a standard space, by [22, Theorem 6.3], there exist a Λ_ν-null Borel set N of S and a Borel set S_0 of S such that the restriction b_0 of b to S_0 is a one-to-one Borel map of S_0 onto S - N. If we put $N' = (\bigcup_{n=0}^{\infty} [N]b^n) \bigcup (\bigcup_{n=1}^{\infty} [N]b^{-n})$, then N' is a Λ-null saturated Borel set and b_0 is a one-to-one Borel map of $S_0 - N'$ onto $S - N'$. Therefore we may assume that b_0 is a one-to-one Borel map of S_0 onto S. Note that the inverse map $b_1 = b_0{}^{-1}$ of b_0 is Borel. For $x \in S$, the image of x under b_1 is denoted by xb_1. Now, suppose that an element $T = (T_x)$ of $End_\Lambda(H)$ and a Λ-null set E of S satisfy the condition of the theorem. We set

$$T'_x = \phi_{x,xb_1} \circ T_{xb_1} \circ \phi_{x,xb_1}{}^{-1} \qquad \text{for } x \in S.$$

As in the proof of Proposition 2.5, one can prove that $T' = (T'_x)_{x \in S}$ is a Borel field of operators with respect to H. Moreover if T_1 and T_2 are intertwining operators of (H, L) which coincide with each other Λ-almost everywhere, then T_1' and T_2' constructed as above coincide with each other Λ-almost everywhere. Let x be an element of $S - Eb$ and $\gamma = (x, c)$ be an element of G. If c' is an element of C such that $b^{-1}cb = c'$, then we have

$$xb_1cb = xb_1bc' = xc',$$

and there exists $y \in S_0$ such that $xb_1cb = yb$ i.e. $xc'b_1 = y$. By Lemma 2.4, we have

$$L(x, c') \circ \phi_{xc', xb_1c} = \phi_{x, xb_1} \circ L(xb_1, c).$$

This implies that

$$L(x, c') \circ \phi_{xc', xb_1c} \circ \phi_{yb, xb_1c}^{-1} \circ \phi_{yb, y}$$
$$= \phi_{x, xb_1} \circ L^b(xb_1, y).$$

The equation $\phi_{xc', xb_1c} \circ \phi_{yb, xb_1c}^{-1} \circ \phi_{yb, y} = \phi_{xc', y}$ implies that

$$L(x, c') \circ \phi_{xc', y} = \phi_{x, xb_1} \circ L^b(xb_1, y).$$

As $x \notin Eb$, we have

$$L(x, c') T'_{xc'}$$
$$= \phi_{x, xb_1} \circ L^b(xb_1, y) \circ T_y \circ \phi_{xc', y}^{-1}$$
$$= T'_x \circ L(x, c').$$

Therefore T' can be considered as an element of $End_\Lambda(H)$ as Eb is Λ-null. Note that $L^b(x, xbb_1) = \phi_{xb, x}^{-1} \circ \phi_{xb, xbb_1}$. Then, for $x \notin E$, we have

$$\Phi_b(T')_x = L^b(x, xbb_1) T_{xbb_1} L^b(x, xbb_1)^{-1}$$
$$= T_x$$

Thus T belongs to $\Phi_b(End_\Lambda(H))$. Q.E.D.

For the transverse function ν of counting measures on G, we set $\lambda = \int \nu^x \, d\Lambda_\nu(x)$, $\nu(H) = \int_S^\oplus H_x \, d\Lambda_\nu(x)$ and, for $T \in End_\Lambda(H)$, $\nu(T) =$

$\int_S^{\oplus} T_x \, d\Lambda_\nu(x)$. Then $\mathcal{M} = \nu(\mathrm{End}_\Lambda(H))$ is a von Neumann algebra on $\nu(H)$ which is isomorphic to $\mathrm{End}_\Lambda(H)$ [5, p.86, Theorem 4]. For $b \in B$, put $\mathcal{N}_b = \nu(\Phi_b(\mathrm{End}_\Lambda(H))$. Let $\mathcal{C}$ be a uniformly separable C*-subalgebra of $\mathcal{N}_b$ which is weakly dense in $\mathcal{N}_b$. The direct integral decomposition of the identity representation i of $\mathcal{C}$ is denoted by

$$i = \int_S^{\oplus} \hat{x} \, d\Lambda_\nu(x) \qquad [9, \text{Lemma } 8.3.1].$$

Note that, for $\nu(T) \in \mathcal{C}$, we have $\hat{x}(\nu(T)) = T_x$ for Λ_ν-a.a. x. As for the following proposition, compare with [18, 36].

<u>Proposition</u> 2.9. In the above situation, there exists a Λ-null set N of S such that, for $x, y \in S - N$, $\hat{x}$ and $\hat{y}$ are unitary equivalent if and only if $x \underset{b}{\sim} y$.

<u>Proof</u>. Let $\{T_i\}_{i=1}^{\infty}$ be a uniformly dense subset of $\mathcal{C}$. By Theorem 2.8, there exists a Λ-null set N_1 such that, if x, y $S - N_1$ and $x \underset{b}{\sim} y$, then $L^b(x, y)\hat{y}(T_i) = \hat{x}(T_i)L^b(x, y)$ for every i. This means that, for $x, y \in S - N_1$ with $x \underset{b}{\sim} y$, $\hat{x}$ and $\hat{y}$ are unitary equivalent.

Since S is a standard Borel space, we may assume that S has a compact metric topology which is compatible with the Borel structure of S. Let C(S) be the C*-algebra of all continuous functions on S. Let $\{g_i\}_{i=1}^{\infty}$ be a uniformly dense subset of C(S). For every bounded Borel function g on S, define an operator $\hat{g}_x$ on H_x by

$$(\hat{g}_x f)(\gamma) = g(s(\gamma))f(\gamma) \qquad \text{for } f \in H_x \text{ and } \gamma \in G^x.$$

Then $\hat{g} = (\hat{g}_x)_{x \in S}$ belongs to $\mathrm{End}_\Lambda(H)$ and we have $\Phi_b(\hat{g}) = (bg)^\wedge$, where bg is a Borel function on S defined by $(bg)(x) = g(xb)$ for every $x \in S$.

By [36, Theorem 1.1], we may suppose that $\{\nu((bg_i)^\wedge)\}_{i=1}^{\infty}$ is contained in $\mathcal{C}$. There exists a Λ-null set N_2 such that, for $x \in S - N_2$, $\hat{x}(\nu((bg_i)^\wedge)) = (bg_i)_x^\wedge$ for every i. Suppose that, for x, $y \in S - N_2$, $\hat{x}$ and $\hat{y}$ are unitary equivalent by means of an isometry V of H_x onto H_y. Then we have, for every i and $f \in H_x$,

$$\int g_i(xcb)|f(x, c)|^2 \, d\nu^x(x, c) = \int g_i(ycb)|(Vf)(y, c)|^2 \, d\nu^y(y, c).$$

Since $\{g_i\}$ is uniformly dense in C(S), this implies that, for a Borel set E of S, Eb^{-1} is $s_*(\nu^x)$-null if and only if it is $s_*(\nu^y)$-null, where $s_*(\nu^x)$ is a measure on S defined by $s_*(\nu^x)(E) = \nu^x(s^{-1}(E))$. As $[x]_b = [x]bb^{-1}$ and $s_*(\nu^x)([x]_b) > 0$, we have $s_*(\nu^y)([x]_b) > 0$. Since $s_*(\nu^y)$ is supported by [y], it follows that $[x]_b = [y]_b$. If we put $N = N_1 \cup N_2$, then the proposition follows. Q.E.D.

<u>Theorem</u> 2.10. Let b_1 and b_2 be elements of B. If there exists an inner automorphism α of $\mathrm{End}_\Lambda(H)$ such that

$$\alpha(\Phi_{b_1}(\mathrm{End}_\Lambda(H))) = \Phi_{b_2}(\mathrm{End}_\Lambda(H)),$$

then $[x]_{b_1} = [x]_{b_2}$ for Λ-a.e. $x \in S$.

<u>Proof</u>. Let $U = (U_x)$ be a unitary element of $\mathrm{End}_\Lambda(H)$ such that $\alpha = \mathrm{Ad}\, U$. Let $\mathcal{N}_i = \nu(\Phi_{b_i}(\mathrm{End}_\Lambda(H)))$ (i = 1, 2) and $\mathcal{C}_1$ be a uniformly separable C*-subalgebra of $\mathcal{N}_1$ which is weakly dense in $\mathcal{N}_1$. We set $\mathcal{C}_2 = \nu(U)\mathcal{C}_1\nu(U)^*$. For $x \in S$, $\hat{x}_i$ denotes the representation of $\mathcal{C}_i$ on H_x defined as above (i = 1, 2). The isomorphism of $\mathcal{C}_1$ onto $\mathcal{C}_2$

associated with α is also denoted by α. Then we have $U_x \hat{x}_1 U_x^* = \hat{x}_2 \circ \alpha$ for Λ-a.a. x. It follows that, for Λ-a.a. x, y, $\hat{x}_1$ and $\hat{y}_1$ are unitary equivalent by means of an isometry V of H_x onto H_y if and only if $(U_y V U_x^*)(\hat{x}_2 \circ \alpha)(U_y V U_x^*)^* = \hat{y}_2 \circ \alpha$. The last equation means that $\hat{x}_2$ and $\hat{y}_2$ are unitary equivalent. Therefore, by Proposition 2.9, there exists a Λ-null set N_1 such that, for $x, y \in S - N_1$, $x \underset{b_1}{\sim} y$ if and only if $x \underset{b_2}{\sim} y$. Note that $[N_1]_{b_1}$ is Λ-null by the condition of Definition 2.3. We set $N = [[N_1]_{b_1}]_{b_2}$, which is a Λ-null set. For $x \in S - N$, $[x]_{b_1}$ is contained in $S - N_1$ and $[x]_{b_2}$ is contained in $S - N$. From the above argument, we have $[x]_{b_1} = [x]_{b_2}$. Q.E.D.

Remark 2.11. Suppose that (G, Λ) is ergodic [see §1.4], then for every $b \in B$, the relative commutant of $\Phi_b(\mathrm{End}_\Lambda(H))$ in $\mathrm{End}_\Lambda(H)$ is the algebra of scalars.

§ 2.2. Conjugacy of subalgebras.

In this section, we give a sufficient condition for two subalgebras $\Phi_{b_1}(\mathrm{End}_\Lambda(H))$ and $\Phi_{b_2}(\mathrm{End}_\Lambda(H))$ $(b_1, b_2 \in B)$ not to be conjugate by any *-automorphism of $\mathrm{End}_\Lambda(H)$.

Let (A, B, G, Λ) be an action of B on S. Throughout this section, we assume that Λ is unimodular and that $\mu = \Lambda_\nu$ is a probability measure on S for the transverse function ν of counting measures. If this is the case, $\mathrm{End}_\Lambda(H)$ is finite.

Definition 2.12. An element b of B is said to be homogeneous of degree k if it satisfies the following conditions;

(i) there exists a Borel partition $\{S_i\}_{i=1}^{k}$ of S such that, for each i, the restriction b_i of b to S_i is a Borel isomorphism of S_i onto S,

(ii) if μ_i is the restriction of μ to S_i and $\mu_i\cdot(b_i b_j^{-1})$ is a measure on S_j defined by $\mu_i\cdot(b_i b_j^{-1})(E) = \mu(\{x \in S_i;\ x b_i b_j^{-1} \in E\})$ for every Borel set E of S_j, then $\mu_i\cdot(b_i b_j^{-1}) = \mu_j$ (i, j = 1,..., k).

Lemma 2.13. Let b be homogeneous of degree k.

(i) If $\mu\cdot b$ is a measure on S defined by $\mu\cdot b(E) = \mu(Eb^{-1})$ for every Borel set E of S, then $\mu\cdot b$ is equivalent to μ.

(ii) If $\mu\cdot b_i^{-1}$ is a measure on S_i defined by $\mu\cdot b_i^{-1}(E) = \mu(Eb_i)$ for every Borel set E of S_i, then $\mu\cdot b_i^{-1}$ is equivalent to μ_i (i = 1,...,k).

(iii) The equation $d((\mu\cdot b)\cdot b_i^{-1})/d\mu_i = k$ holds (i = 1,..., k).

Proof. (i) Note that, as C is a countable discrete group, for a Borel set E of S, E is μ-null if and only if [E] is Λ-null. It follows that $\mu\cdot b \prec \mu$. Suppose that E is $\mu\cdot b$-null, that is, Eb^{-1} is μ-null. Since $[Eb^{-1}]$ is Λ-null, $[Eb^{-1}]b$ is also Λ-null. Since E is contained in $[Eb^{-1}]b$, E is a μ-null set. Thus we have $\mu \prec \mu\cdot b$.

(ii) This can be proved by the same method as that of the proof of (i).

(iii) This follows from a straightforward calculation. Q.E.D.

Remark 2.14. If b is homogeneous, then the representation (H, L^b)

of G·b defined in Section 2.1 is a square integrable representation (see §1.3).

The measure $\int \nu^x \, d\mu(x)$ on G is denoted by λ. We set $\mathcal{M} = \nu(\mathrm{End}_\Lambda(H))$, $\mathcal{N}_b = \nu(\Phi_b(\mathrm{End}_\Lambda(H)))$ and $\mathcal{H} = \nu(H)$. The Hilbert space $\mathcal{H}$ can be identified with $L^2(G, \lambda)$. Define a partial isometry $U = U_b$ on $\mathcal{H}$ by

$$U(f)(x, c) = \left(\frac{d(\mu \cdot b)}{d\mu}\right)^{-1/2}(xb)f(xb, b^{-1}cb)$$

for $f \in \mathcal{H}$ and $(x, c) \in G$.

<u>Lemma</u> 2.15. Let b be homogeneous of degree k and e be the final projection of U_b.

(i) The space $e\mathcal{H}$ consists of all elements $f = (f_x)$ of $\mathcal{H}$ which satisfy the following; there exists a saturated null set $N = N_f$ of S such that $f_x = L^b(x, y)f_y$ if $xb = yb$ and $x, y \in S - N$.

(ii) The projection e belongs to the commutant $\mathcal{N}_b'$ of $\mathcal{N}_b$.

<u>Proof</u>. (i) Let $\mathcal{H}_0$ be the space consisting of all elements which satisfy the condition of (i). We write U for U_b. If $xb = yb$, then we have, for $f \in \mathcal{H}$,

$$\begin{aligned}(L^b(x, y)(Uf)_y)(x, c) &= (Uf)_y(y, c) \\ &= (Uf)_x(x, c).\end{aligned}$$

Hence $e\mathcal{H}$ is contained in $\mathcal{H}_0$. Conversely, let f be an element in $\mathcal{H}_0$. For $x, y \in S - N_f$ with $xb = yb$, we have $f(x, c) = f(y, c)$ for all $c \in C$. Fix an integer i with $1 \leq i \leq k$ and define an element g of $\mathcal{H}$ by

$$g(x, c) = k^{1/2}\left(\frac{d(\mu\cdot b_i^{-1})}{d\mu_i}\right)^{-1/2}(xb_i^{-1})f(xb_i^{-1}, bcb^{-1})$$

($(x, c) \in G$). Then we have, for $(x, c) \in G$,

$$U(g)(x, c)$$
$$= k^{1/2}\left(\frac{d(\mu\cdot b)}{d\mu}\right)^{-1/2}(xb)\left(\frac{d(\mu\cdot b_i^{-1})}{d\mu_i}\right)^{-1/2}(xbb_i^{-1})f(xbb_i^{-1}, c).$$

On the other hand, we have, for μ-a.a. $x \in S$,

$$\frac{d(\mu\cdot b)}{d\mu}(xb)\frac{d(\mu\cdot b_i^{-1})}{d\mu_i}(xbb_i^{-1})$$
$$= \frac{d((\mu\cdot b)\cdot b_i^{-1})}{d\mu_i}(xbb_i^{-1}) = k.$$

It follows that $U(g)(x, c) = f(xbb_i^{-1}, c)$ for μ-a.a. $x \in S$ and all $c \in C$. Since we have $f(xbb_i^{-1}, c) = f(x, c)$ for $x \notin N_f \cup N_f b_i b^{-1}$, we have $U(g) = f$. Hence $\mathcal{H}_0$ is contained in $e\mathcal{H}$.

(ii) Let T be an element of $\mathcal{N}_b$ and f be an element of $e\mathcal{H}$. For $x, y \in S - N_f$ with $xb = yb$, we have

$$L^b(x, y)T_y f_y = T_x L^b(x, y)f_y = T_x f_x.$$

Therefore Tf belongs to $e\mathcal{H}$ and e is an element of $\mathcal{N}_b'$. Q.E.D.

<u>Lemma</u> 2. 16. Let e be as in Lemma 2.15 and $\mathcal{H}_k$ be a k-dimensional Hilbert space whose complete othonormal system is $\{\delta_i\}_{i=1}^k$. For $f \otimes \delta_i \in e\mathcal{H} \otimes \mathcal{H}_k$, define an element $\psi(f \otimes \delta_i)$ of $\mathcal{H}$ by the following; $\psi(f \otimes \delta_i)_x$ is $k^{1/2}f_x$ if $x \in S_i$ and is 0 if $x \notin S_i$. Then ψ can be extended to an isometry of $e\mathcal{H} \otimes \mathcal{H}_k$ onto $\mathcal{H}$, which is denoted again by ψ. Moreover the von Neumann algebras $(\mathcal{N}_b)_e \otimes \mathbb{C}_k$ and $\mathcal{N}_b$ are spatially isomorphic by means of ψ, where $\mathbb{C}_k$ is the algebra of scalar

operators on $\mathcal{H}_k$.

Proof. For $f \in e\mathcal{H}$, we have

$$\begin{aligned}
&\|\psi(f \otimes \delta_i)\|^2 \\
&= k\int_{S_i} \|f_x\|^2 \, d\mu_i(x) \\
&= \Sigma_{j=1}^{k} \int_{S_i} \| L^b(xb_i b_j^{-1}, x) f_x \|^2 \, d\mu_i(x) \\
&= \Sigma_{j=1}^{k} \int_{S_j} \| f_x \|^2 \, d(\mu_i \cdot (b_i b_j^{-1}))(x) \qquad \text{(Lemma 2.15)} \\
&= \| f \|^2.
\end{aligned}$$

It follows that ψ can be extended to an isometry of $e\mathcal{H} \otimes \mathcal{H}_k$ into $\mathcal{H}$. Let f_0 be an element of $\mathcal{H}$ such that $\langle f_0, f\rangle = 0$ for all $f \in \psi(e\mathcal{H} \otimes \mathcal{H}_k)$. For $f \in \mathcal{H}$, define an element f^i of $e\mathcal{H}$ by $f^i(x, c) = f(xb_j b_i^{-1}, c)$ for $x \in S_j$ and $c \in C$ ($j = 1, \ldots, k$). Then we have

$$k^{1/2}\int_{S_i} \langle f_{0x}, f_x\rangle \, d\mu(x) = \langle f_0, \psi(f^i \otimes \delta_i)\rangle$$
$$= 0.$$

Since f and i are arbitrary, we have $f_0 = 0$. Hence ψ is onto.

Let T be an element of $(\mathcal{N}_b)_e$. There exists an element T' of $\mathcal{N}_b$ such that $T'_e = T$. For $f \otimes \delta_i \in e\mathcal{H} \otimes \mathcal{H}_k$, we have $\psi((T \otimes I)(f \otimes \delta_i))_x = (T'\psi(f \otimes \delta_i))_x$ for a.a. $x \in S$. It follows that $\psi(T \otimes I) = T'\psi$.. Therefore the map $T \otimes I \longmapsto \psi(T \otimes I)\psi^{-1}$ is an isomorphism of $(\mathcal{N}_b)_e \otimes \mathbb{C}_k$ onto $\mathcal{N}_b$. Q.E.D.

Lemma 2.17. Let U and e be as in Lemma 2.15. The von Neumann algebras $(\mathcal{N}_b)_e$ and $\mathcal{M}$ are spatially isomorphic by means of U.

Proof. For $T \in (\mathcal{N}_b)_e$, there exists a unique element T' of $\mathcal{M}$ such that $\Phi_b(T')_e = T$. As we have, for $f \in \mathcal{H}$,

$$U(f)_x = \left(\frac{d(\mu \cdot b)}{d\mu}\right)^{-1/2}(xb)\phi_{xb,x}{}^{-1}(f_{xb}),$$

we have

$$\begin{aligned}(TUf)_x &= \Phi_b(T')_x U(f)_x \\ &= \left(\frac{d(\mu \cdot b)}{d\mu}\right)^{-1/2}(xb)\phi_{xb,x}{}^{-1}(T'_{xb}f_{xb}) \\ &= U(T'f)_x.\end{aligned}$$

It follows that $T' = U^*TU$. Q.E.D.

From the above lemmas, we get the following theorem.

Theorem 2.18. If b is homogeneous of degree k, then $\mathcal{M} \otimes \mathbb{C}_k$ and $\mathcal{N}_b$ are spatially isomorphic.

Corollary 2.19. Suppose that (G, Λ) is ergodic. Let b_i be homogeneous of degree k_i $(i = 1, 2)$. If $k_1 \neq k_2$, then there are no automorphisms of $\mathrm{End}_\Lambda(H)$ which send $\Phi_{b_1}(\mathrm{End}_\Lambda(H))$ onto $\Phi_{b_2}(\mathrm{End}_\Lambda(H))$.

Proof. By the ergodicity of (G, Λ), $\mathcal{M}$ and $\mathcal{N}_{b_i}$ $(i = 1, 2)$ are factors. Since $\mathcal{M}$ is standard, the coupling constant of $\mathcal{N}_{b_i}$ is k_i $(i = 1, 2)$ [35, Corollary 7.22]. It follows that $\mathcal{N}_{b_1}$ and $\mathcal{N}_{b_2}$ cannot be spatially isomorphic if $k_1 \neq k_2$ [35, Theorem 8.3]. On the other hand, any automorphism of $\mathcal{M}$ is spatial [8, p.268, Corollary]. Therefore there are no automorphisms of $\mathcal{M}$ which send $\mathcal{N}_{b_1}$ onto $\mathcal{N}_{b_2}$.
Q.E.D.

§2.3. An example.

In this section, we apply the results of the previous sections to the action of the special linear group $SL(n, \mathbb{Z})$ of degree n on the n-dimensional torus $\mathbb{T}^n$ $(n \geq 2)$. Let A be the normalizer of $SL(n, \mathbb{Z})$ in the general linear group $GL(n, \mathbb{Q})$, and B be the semigroup consisting of all elements of A whose coefficients are integers. Note that B contains the elements of the form kI ($k \in \mathbb{Z} - \{0\}$), where I is the unit matrix. We set $C = SL(n, \mathbb{Z})$ and $S = \mathbb{T}^n$. The action of $b = (b_{ij}) \in B$ on $x = (x_1, \ldots, x_n) \in S$ is defined by

$$xb = (\Sigma_{j=1}^n b_{j1}x_j, \ldots, \Sigma_{j=1}^n b_{jn}x_j) \qquad (\text{mod } \mathbb{Z}^n).$$

Let G_n be the groupoid $S \times C$ associated with the above action and Λ be the transverse measure on G_n such that Λ_ν is the Lebesgue measure μ on S for the transverse function ν of counting measures. Note that Λ is unimodular and ergodic. Since the action of C on S is essentially free, (A, B, G, Λ) can be considered as an action of B on S. Let $\mathcal{M}_n$ be the von Neumann algebra $\nu(\text{End}_\Lambda(H))$ associated with the left regular representation (H^ν, L^ν) of G_n. Then $\mathcal{M}_n$ is a full factor of type II_1, which coincides with the factor obtained by the group-measure space construction. For $k \in \mathbb{Z} - \{0\}$, the element kI of B is homogeneous of degree $|k|^n$. We set $\mathcal{N}_k^n = \nu(\Phi'_{kI}(\text{End}_\Lambda(H)))$. The following theorem summarizes properties of $\mathcal{N}_k^n$ which are obtained from the results of the previous sections.

<u>Theorem</u> 2.20. (i) The subfactor $\mathcal{N}_k^n$ of $\mathcal{M}_n$ is spatially isomorphic to $\mathcal{M}_n \otimes \mathbb{C}_{|k|^n}$ ($k \in \mathbb{Z} - \{0\}$).

(ii) Elements of the family $\{\mathcal{N}_k^n\}_{k \in \mathbb{N}}$ are not conjugate with each other by any automorphism of $\mathcal{M}_n$.

Chapter III. Connes' invariants of periodic automorphisms.

In this chapter we calculate Connes' invariants for certain automorphisms of full factors of type II_1. In what follows, automorphisms of von Neumann algebras always mean *-preserving ones. Let $SL(2, \mathbb{Z})$ be the group of 2×2 matrices with integral entries and determinant 1, and $F_2(n)$ be the subgroup of $SL(2, \mathbb{Z})$ generated by $\begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}$, where n is a natural number. Then, for every n, $F_2(n)$ is isomorphic to the free group on two generators. Let S be the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ and μ be the normalized Lebesgue measure on S. The natural action of $SL(2, \mathbb{Z})$ on (S, μ) is defined as follows;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x, y) = (ax + by, cx + dy) \qquad (\mathrm{mod}\ \mathbb{Z}^2)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and $(x, y) \in S$. An action of $F_2(n)$ on (S, μ) is obtained by restricting the above action to $F_2(n)$. We denote by $\mathcal{M}_n$ the factor obtained by the crossed product from the above action of $F_2(n)$ on (S, μ). Using [33, Proposition 3.5], we can show that the action of $F_2(n)$ on (S, μ) is strongly ergodic. It follows from [2] that $\mathcal{M}_n$ is a full factor of type II_1. For every divisor p of 2n and $i = 0, \ldots, p - 1$, we shall construct a minimal periodic automorphism $\rho_{n,p,i}$ of $\mathcal{M}_n$ with Connes' outer invariant (p, γ^i), where $\gamma = \exp(2\pi\sqrt{-1}/p)$ (see §1.5, 2). All results in this chapter are taken from [28, 29].

§3.1. Construction of periodic automorphisms of $\mathcal{M}_n$.

Let n be a natural number and p be a divisor of 2n. In this section, we construct a periodic automorphism $\sigma_{n,p}$ of $\mathcal{M}_n$ with period p.

Let SL(2, $\mathbb{Z}$), $F_2(n)$ and (S, μ) be as above. We denote by $F_2(n, p)$ the subgroup of SL(2, $\mathbb{Z}$) generated by $\begin{pmatrix} 1 & 2n/p \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2pn & 1 \end{pmatrix}$, and by C(n, p) the subgroup of SL(2, $\mathbb{Z}$) generated by $F_2(n)$ and $F_2(n, p)$. Define an automorphism $\psi = \psi_{n,p}$ of C(n, p) by

$$\psi(c) = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} c \begin{pmatrix} 0 & 1/p \\ 1 & 0 \end{pmatrix} \qquad \text{for } c \in C(n, p).$$

Then we have ψ^2 = id. and $\psi(F_2(n)) = F_2(n, p)$. Actions of $F_2(n)$ and $F_2(n, p)$ on (S, μ) are defined as the restrictions of the natural action of SL(2, $\mathbb{Z}$) on (S, μ) to $F_2(n)$ and $F_2(n, p)$ respectively. Note that these actions of $F_2(n)$ and $F_2(n, p)$ are ergodic, measure-preserving and essentially free. Let $\phi = \phi_p$ be a Borel map of S onto itself defined by $\phi_p(x, y) = (y, px)$ for $(x, y) \in S$. The following property is important;

$$\phi_p(cs) = \psi_{n,p}(c)\phi_p(s)$$

for every $c \in C(n, p)$ and $s \in S$.

We define a measurable groupoids G_n and $H_{n,p}$ as follows;

$$G_n = \{(s, cs) \in S \times S;\ c \in F_2(n)\},$$

$$H_{n,p} = \{(s, cs) \in S \times S;\ c \in F_2(n, p)\}.$$

(see Example 1.2). Let ν (resp. ν') be the transverse function of counting measures of G_n (resp. $H_{n,p}$) and Λ (resp. Λ') be the unimodular transverse measure of G_n (resp. $H_{n,p}$) such that $\Lambda_\nu = \mu$ (resp. $\Lambda'_{\nu'} = \mu$). For left regular representations (H^ν, L^ν) and $(H^{\nu'}, L^{\nu'})$ of G_n and $H_{n,p}$ respectively, we set $\mathcal{M}_n = \mathrm{End}_\Lambda(H^\nu)$ and

$\mathcal{N}_{n,p} = \mathrm{End}_{\Lambda'}(H^{\nu'})$. We define a measure λ (resp. λ') on G_n (resp. $H_{n,p}$) by $\lambda = \int \nu^s \, d\mu(s)$ (resp. $\lambda' = \int \nu'^s \, d\mu(s)$). Let $\mathcal{H}_n$ denote the Hilbert space $L^2(G_n, \lambda)$ and $\mathcal{K}_{n,p}$ denote the Hilbert space $L^2(H_{n,p}, \lambda')$. We may consider $\mathcal{M}_n$ and $\mathcal{N}_{n,p}$ as algebras of operators on $\mathcal{H}_n$ and $\mathcal{K}_{n,p}$ respectively, as in Section 1.4. Let U (resp. U') be the homomorphism of $[G_n]$ (resp, $[H_{n,p}]$) into the unitary group of $\mathcal{M}_n$ (resp. $\mathcal{N}_{n,p}$) and let π (resp. π') be the *-homomorphism of $L^\infty(S, \mu)$ into $\mathcal{M}_n$ (resp. $\mathcal{N}_{n,p}$), which are defined as in Section 1.4. We set $\mathcal{A} = \pi(L^\infty(S, \mu))$ and $\mathcal{B} = \pi'(L^\infty(S, \mu))$.

An action of the cyclic group $W_p = \mathbb{Z}/p\mathbb{Z}$ on (S, μ) is defined by $is = w_p^{\,i}s$ for $i \in W_p$ and $s \in S$, where w_p is an automorphism of S such that $w_p(x, y) = (x + 1/p, y)$ for $(x, y) \in S$. Since w_p commutes with elements of $C(n, p)$, we can define an action of $D_{n,p} = F_2(n, p) \times W_p$ on (S, μ) by

$$(c, i)s = cis \qquad \text{for } (c, i) \in D_{n,p} \text{ and } s \in S.$$

The action of $D_{n,p}$ is ergodic, measure-preserving and essentially free. For $i \in W_p$, define a unitary operator V_i' on $\mathcal{K}_{n,p}$ by

$$(V_i'f)(s, t) = f((-i)s, (-i)t)$$

for $f \in \mathcal{K}_{n,p}$, $(s, t) \in H_{n,p}$. Then, we can define an action β of W_p on $\mathcal{N}_{n,p}$ by $\beta_i = \mathrm{Ad}\, V_i'$ $(i \in W_p)$. Let $\mathcal{R}(\mathcal{N}_{n,p}, \beta)$ be the crossed product of $\mathcal{N}_{n,p}$ by β (see §1.5, 1). It is isomorphic to the factor $\mathcal{R}(L^\infty(S, \mu), D_{n,p})$ obtained by the crossed product from the action of $D_{n,p}$ on (S, μ).

Throughout this section we fix n and p, and we omit indices n and p if there is no confusion.

Let H_0 denote the measurable groupoid $\{(s, ds) \in S \times S;\ d \in D_{n,p}\}$ and ν_0 be the transverse function of counting measures on H_0. We define a measure λ_0 on H_0 by $\lambda_0 = \int \nu_0{}^s \, d\mu(s)$. Let I_p be the transitive groupoid $\{0, 1, \ldots, p-1\} \times \{0, 1, \ldots, p-1\}$ (see §1.1) and λ_1 denote the counting measure on I_p. Define a Borel partition $\{S_i\}_{i=0}^{p-1}$ of S by

$$S_i = [ip^{-1}, (i+1)p^{-1}) \times \mathbb{T},$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a 1-torus. For $i = 0, \ldots, p-1$, let ϕ_i be the inverse of the restriction of ϕ to S_i. It is clear that ϕ_i is a Borel isomorphism of S onto S_i. Then there exists a Borel isomorphism of the groupoid H_0 onto the groupoid $G_n \times I_p$ such that

$$u(s, t) = ((\phi(s), \phi(t)), (i, j))$$

for $(s, t) \in H_0$, $s \in S_i$, $t \in S_j$. The inverse of u is given by

$$u^{-1}((s, t), (i, j)) = (\phi_i(s), \phi_j(t))$$

for $((s, t), (i, j)) \in G_n \times I_p$. As the measure class of λ_0 is sent to that of $\lambda \times \lambda_1$ by u, the factors associated with $(H_0, [\lambda_0])$ and $(G_n \times I_p, [\lambda \times \lambda_1])$ are isomorphic. The factors associated with $(H_0, [\lambda_0])$ and $(G_n \times I_p, [\lambda \times \lambda_1])$ are $\mathcal{R}(L^\infty(S, \mu), D_{n,p})$ and $\mathcal{M}_n \otimes M_p$ respectively , where M_p is the algebra of all $p \times p$ complex matrices. Therefore factors $\mathcal{M}_n \otimes M_p$ and $\mathcal{R}(\mathcal{N}_{n,p}, \beta)$ are isomorphic.

For $c \in F_2(n)$, let $\{E_i(c) = E_i(c; n, p)\}_{i=0}^{p-1}$ be a Borel partition of S such that $E_i(c) = \phi(\psi(c)S_i \cap S_0)$. Define a projection $e_i(c) = e_i(c; n, p)$ in $\mathcal{A}$ by $e_i(c) = \pi(\chi_{E_i(c)})$ for $i = 0, \ldots, p-1$ and $c \in F_2(n)$, where $\chi_{E_i(c)}$ is a characteristic function of $E_i(c)$. The family $\{e_i(c)\}_{i=0}^{p-1}$ is a partition of the identity in $\mathcal{A}$. Let

$\{e_{ij};\ i, j = 0,\ldots, p-1\}$ be a system of matrix units in M_p . For $i = 0,\ldots, p-1$, we define a unitary operator u_i in M_p by $u_i = \Sigma_{j=0}^{p-1} e_{i+j,j}$, where $i + j$ means $i + j$ modulo p. Let π^β be the canonical isomorphism of $\mathcal{N}_{n,p}$ into $\mathcal{R}(\mathcal{N}_{n,p}, \beta)$ and U^β be the canonical representation of W_p into the group of unitary operators in $\mathcal{R}(\mathcal{N}_{n,p}, \beta)$ (see §1.5, 1). By straightforward calculations, we can prove the following.

<u>Lemma</u> 3.1. There exists a unique isomorphism Ψ of $\mathcal{R}(\mathcal{N}_{m,p}, \beta)$ onto $\mathcal{M}_n \otimes M_p$ with the following properties;

(i) $\Psi(\pi^\beta(\pi'(h))) = \Sigma_{i=0}^{p-1} \pi(h\circ\phi_i) \otimes e_{ii}$ for $h \in L^\infty(S, \mu)$,

(ii) $\Psi(\pi^\beta(U_c')) = \Sigma_{i=0}^{p-1} (e_i(\psi(c))U_{\psi(c)}) \otimes u_i^*$ for $c \in F_2(n, p)$,

(iii) $\Psi(U_i^\beta) = 1 \otimes u_i$ for $i \in W_p$.

Let $\hat{W}_p$ be the dual group of W_p. We consider $\hat{W}_p$ as the group $\{\gamma \in \mathbb{C} : \gamma^p = 1\}$. Then we have $\langle i, \gamma\rangle = \gamma^i$. Let $\hat{\beta}$ be the dual action of β (§1.5, 1). For $\gamma = \exp(2\pi\sqrt{-1}/p)$, by restricting the automorphism $\Psi\circ\hat{\beta}_\gamma\circ\Psi^{-1}$ of $\mathcal{M}_n \otimes M_p$ to $\mathcal{M}_n$, we have the following proposition.

<u>Proposition</u> 3.2. Let n be a natural number and p be a divisor of 2n. If $\gamma = \exp(2\pi\sqrt{-1}/p)$, then there is a unique automorphism $\sigma_{n,p}$ of $\mathcal{M}_n$ such that

(i) $\sigma_{n,p}(X) = X$ for $X \in \mathcal{A}$,

(ii) $\sigma_{n,p}(e_i(c;n, p)U_c) = \gamma^{-i}e_i(c;n, p)U_c$ for $c \in F_2(n)$ and $i = 0,\ldots, p-1$.

§3.2. Connes' invariants of certain automorphisms.

In this section we construct a minimal periodic automorphism $\rho_{n,p,i}$ of $\mathcal{M}_n$ with Connes' outer invariant $(p, \exp(2\pi\sqrt{-1}i/p))$ for every divisor p of 2n and $i = 0,\ldots, p-1$.

Let Γ_p be the Borel automorphism of S such that $\Gamma_p(x, y) = (x, y + 1/p)$ for $(x, y) \in S$. Then, Γ_p commutes with every element of $F_2(n)$. We define a unitary operator V_p on $\mathcal{H}_n$ by

$$(V_p f)(s, t) = f(\Gamma_p^{-1}(s), \Gamma_p^{-1}(t))$$

for $f \in \mathcal{H}_n$, $(s, t) \in G_n$. We write τ_p for the automorphism Ad V_p of $\mathcal{M}_n$. For $s \in S$, let $[s]$ be the orbit of s with respect to $F_2(n)$, that is, $[s] = \{cs;\ c \in F_2(n)\}$. Since we have $[\Gamma_p(s)] \neq [s]$ for μ-a.a. $s \in S$, it follows from [15, §8] that $\tau_p^{\,i}$ is an outer automorphism for $i = 1,\ldots, p-1$.

Now, we can state the main theorem.

<u>Theorem</u> 3.3. Let n be a natural number and p be a divisor of 2n. For $i = 0,\ldots, p-1$, define an automorphism $\rho_{n,p,i}$ of $\mathcal{M}_n$ by

$$\rho_{n,p,i} = \sigma_{n,p}\tau_p^{-i}.$$

Then, the Connes' outer invariant of $\rho_{n,p,i}$ is (p, γ^i), where $\gamma = \exp(2\pi\sqrt{-1}/p)$, and its Connes' inner invariant is ε_1, where ε_1 is the Dirac measure on C^1 supported by $\{1\}$. In particular, $\rho_{n,p,i}$ is minimal periodic.

To prove the theorem, we will need two lemmas. In what follows, we write ρ, σ and τ for $\rho_{n,p,i}$, $\sigma_{n,p}$ and τ_p respectively.

<u>Lemma</u> 3.4. Let q be the order of τ^{-i}. For $\ell = 0, \ldots, q-1$, define a Borel set E_ℓ of S by

$$E_\ell = \mathbb{T} \times [\ell q^{-1}, (\ell + 1)q^{-1}).$$

If h is an element of $L^\infty(S, \mu)$ such that

$$h = \Sigma_{\ell=0}^{q-1} \gamma^{r\ell} \chi_{E_\ell},$$

where $r = pq^{-1}$, then $\rho^p = \mathrm{Ad}\, \pi(h)$.

<u>Proof</u>. For a while, let c be the fixed element $\begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}$ of $F_2(n)$. Then, the Borel set $E_0(c)$ defined in Section 3.1 is the union of the following Borel sets;

$$\{(x, y) \in S;\ 0 \le x \le (2n)^{-1}y,\ 0 \le y < 1\},$$

$$\{(x, y) \in S;\ (2n)^{-1}(y+pk-1) < x \le (2n)^{-1}(y+pk),\ 0 \le y < 1\}$$

$(k = 1, \ldots, (2n/p) - 1)$,

$$\{(x, y) \in S;\ (2n)^{-1}(y+2n-1) < x < 1,\ 0 \le y < 1\},$$

and, for $j = 1, \ldots, p - 1$, $E_j(c)$ is the union of Borel sets

$$\{(x, y) \in S;\ (2n)^{-1}(y+pk-(j+1)) < x \le (2n)^{-1}(y+pk-j),\ 0 \le y < 1\}$$

$(k = 1, \ldots, 2n/p)$. We define a Borel partition $\{A(k);\ k = 0, \ldots, 2n - 1\}$ of S by the following; A(0) is the union of the following two sets

$$\{(x, y) \in S;\ 0 \le x \le (2n)^{-1}y,\ 0 \le y < 1\},$$

$$\{(x, y) \in S;\ (2n)^{-1}(y+2n-1) < x < 1,\ 0 \le y < 1\},$$

and, for $k = 1, \ldots, 2n - 1$, A(k) is the set

$$\{(x, y) \in S;\ (2n)^{-1}(y+k-1) < x \le (2n)^{-1}(y+k),\ 0 \le y < 1\}.$$

For $k = 0, \dots, 2n-1$, let $m(k)$ be an element of $\{1, \dots, p\}$ such that $m(k) \equiv k \pmod p$. Then, $E_j(c)$ is the union of sets $A(k)$ with $m(k) = p - j$. Therefore we have

$$\sigma(U_c) = \Sigma_{k=0}^{2n-1} \gamma^k \pi(\chi_{A(k)}) U_c.$$

As we have $\tau(U_c) = U_c$ and $\sigma(X) = X$ for $X \in \mathcal{A}$, we have

$$\rho^p(U_c) = \Pi_{j=0}^{p-1} (\Sigma_{k=0}^{2n-1} \gamma^k \tau^{-ij}(\pi(\chi_{A(k)}))U_c.$$

As the number of $\{j \in \{0, \dots, p-1\};\ j \equiv j_0 \pmod q)\}$ is r for $j_0 = 0, \dots, q-1$, we have

$$\rho^p(U_c) = \Pi_{j=0}^{q-1} (\Sigma_{k=0}^{2n-1} \gamma^k \tau^{-ij}(\pi(\chi_{A(k)}))^r U_c$$
$$= \Pi_{j=0}^{q-1} (\Sigma_{k=0}^{2n-1} \gamma^{kr} \pi(\chi_{\Gamma_p^{-ij} A(k)}) U_c.$$

Define an element h' in $L^\infty(S, \mu)$ by

$$h' = \Pi_{j=0}^{q-1} (\Sigma_{k=0}^{2n-1} \gamma^{kr} \chi_{\Gamma_p^{-ij} A(k)}.$$

It is clear that $\rho^p(U_c) = \pi(h')U_c$.

Since the order of $\Gamma_p{}^i$ is q, there exists a natural number i_0 relatively prime with q such that $i/p = i_0/q$. For $j = 0, \dots, q-1$, let $\theta(j)$ be an element of $\{0, \dots, q-1\}$ such that $\theta(j) \equiv i_0 j \pmod q$. If we define a Borel isomorphism ω of S onto itself by $\omega(x, y) = (x + (2nq)^{-1}, y)$, then, by a straightforward calculation, we know that

$$\Gamma_p^{-ij} A(k) \cap E_0 = \omega^{\theta(j)} A(k) \cap E_0$$

for $j = 0, \dots, q-1$, $k = 0, \dots 2n-1$. Notice that the set $\{\omega^{\theta(j)};\ j = 0, \dots, q-1\}$ coincides with $\{\omega^j;\ j = 0, \dots, q-1\}$.

It follows that, for $s \in E_0$,

$$h'(s) = \Pi_{j=0}^{q-1} (\Sigma_{k=0}^{2n-1} \gamma^{kr} \chi_{\omega^{\theta(j)} A(k)}$$

$$= \Pi_{j=0}^{q-1} \left(\Sigma_{k=0}^{2n-1} \gamma^{kr} \chi_{\omega^j A(k)}\right). \qquad (1)$$

We define a partition $\{B(k, \ell);\ k = 0,\ldots, 2nq - 1,\ \ell = 0,\ldots, q - 1\}$ of S by the following; for each ℓ, $B(0, \ell)$ is the union of the following two sets

$$\{(x,y) \in S;\ 0 \le x \le (2n)^{-1}(y - \ell q^{-1}),\ \ell q^{-1} \le y < (\ell+1)q^{-1}\},$$

$$\{(x,y) \in S;\ (2n)^{-1}(y + (2nq - \ell - 1)q^{-1}) < x < 1,\ \ell q^{-1} \le y < (\ell+1)q^{-1}\},$$

and, for $k = 1,\ldots, 2nq - 1$, $B(k, \ell)$ is the set

$$\{(x,y) \in S;\ (2n)^{-1}(y + (k - \ell - 1)q^{-1}) < x \le (2n)^{-1}(y + (k - \ell)q^{-1}),$$
$$\ell q^{-1} \le y < (\ell+1)q^{-1}\}.$$

It is clear that $\omega(B(k, \ell)) = B(k + 1, \ell)$, where $k + 1$ means $k + 1$ modulo $2nq$. For $k = 0,\ldots, 2nq - 1$, if k satisfies that $(m - 1)q \le k \le mq - 1$ $(m = 1,\ldots, n)$, we have

$$B(k, 0) \subset \omega^j A(m) \quad \text{for } j = 0,\ldots, k - ((m - 1)q + 1),$$

$$B(k, 0) \subset \omega^j A(m - 1) \quad \text{for } j = k - (m - 1)q,\ldots, q - 1.$$

Therefore, for $s \in B(k, 0)$ with $(m - 1)q \le k \le mq - 1$, we have

$$h'(s) = \left(\Pi_{j=0}^{k-((m-1)q+1)} \gamma^{mr} \chi_{\omega^j A(m)}(s)\right) \times$$
$$\left(\Pi_{j=k-(m-1)q}^{q-1} \gamma^{(m-1)r} \chi_{\omega^j A(m-1)}(s)\right)$$
$$= \gamma^{kr}.$$

Since $\{B(k, 0);\ k = 0,\ldots, 2nq - 1\}$ is a partition of E_0, we have

$$h'|E_0 = \Sigma_{k=0}^{2nq-1} \gamma^{kr} \chi_{B(k,0)}.$$

It is clear that the following equations hold;

$$h'(\Gamma_p{}^i s) = h'(s) \qquad \text{for all } s \in S,$$

$$\Gamma_p{}^{ij} B(k, 0) = B(k, \theta(j))$$

for $j = 0,\ldots, q - 1$, $k = 0,\ldots, 2nq - 1$. Fore each $\ell = 0,\ldots, q - 1$, there exists an element $j(\ell)$ in $\{0,\ldots, q-1\}$ such that $\theta(j(\ell)) = \ell$. For every $s \in E_\ell$, as $\Gamma_p^{-ij(\ell)}s$ is in E_0, we have

$$\begin{aligned} h'(s) &= h'(\Gamma_p^{-ij(\ell)}s) \\ &= \Sigma_k \gamma^{kr} \chi_{\Gamma_p^{ij(\ell)}B(k,0)}(s) \\ &= \Sigma_k \gamma^{kr} \chi_{B(k,\ell)}(s). \end{aligned}$$

Thus we have

$$h' = \Sigma_{\ell=0}^{q-1} \Sigma_{k=0}^{2nq-1} \gamma^{kr} \chi_{B(k,\ell)}.$$

On the other hand, as the set cE_ℓ is

$$\bigcup_{k=0}^{2n} \{(x,y) \in S;\ (2n)^{-1}(y+k-(\ell+1)q^{-1}) < x \leq (2n)^{-1}(y+k-\ell q^{-1}),$$
$$0 \leq x < 1,\ 0 \leq y < 1\},$$

the set $E_\ell \cap cE_{\ell'}$ is $\bigcup_{k=0}^{2n-1} B(qk - \ell' + \ell, \ell)$ if $\ell \geq \ell'$, and is $\bigcup_{k=1}^{2n} B(qk - \ell' + \ell, \ell)$ if $\ell < \ell'$. For $s \in E_\ell \cap cE_{\ell'}$, we have

$$h(s)\overline{h(c^{-1}s)} = \gamma^{(\ell-\ell')r},$$

and for $s \in B(qk - \ell' + \ell, \ell)$, we have

$$h'(s) = \gamma^{(\ell-\ell')r}.$$

This implies that

$$h'(s) = h(s)\overline{h(c^{-1}s)} \qquad \text{for all } s \in S.$$

Therefore we have, for $f \in \mathcal{H}_n$ and $(s, t) \in G_n$,

$$\begin{aligned} (\mathrm{Ad}\,\pi(h)(U_c)f)(s, t) &= h(t)\overline{h(c^{-1}t)}f(s, c^{-1}t) \\ &= (\pi(h')U_c f)(s, t). \end{aligned}$$

It follows that $\mathrm{Ad}\,\pi(h)(U_c) = \rho^p(U_c)$, where $c = \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}$.

Next, let c be the element $\begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$ of $F_2(n)$. As $e_0(c) = 1$ and $e_i(c) = 0$ for $i = 1,\ldots, p-1$, we have $\sigma(U_c) = U_c$. Furthermore we have $\tau(U_c) = U_c$. It follows that $\rho(U_c) = U_c$. On the other hand, as $cE_\ell = E_\ell$ for $\ell = 0,\ldots, q-1$, we have $h(c^{-1}s) = h(s)$ for all $s \in S$.

This implies that $\mathrm{Ad}\,\pi(h)(U_c) = U_c$. Hence we have $\mathrm{Ad}\,\pi(h)(U_c) = \rho^p(U_c)$, where $c = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$.
As $\tau(\mathcal{A}) = \mathcal{A}$ and $\sigma(X) = X$ for $X \in \mathcal{A}$, we have $\rho^p(X) = \tau^{-ip}(X) = X$ for all $X \in \mathcal{A}$. It follows that $\mathrm{Ad}\,\pi(h)(X) = \rho^p(X)$ for all $X \in \mathcal{A}$. Since $\mathcal{M}_n$ is generated by $\mathcal{A}$ and $\{U_c;\ c = \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}\}$, the assertion of the lemma follows. Q.E.D.

Lemma 3.5. For $j = 1, \ldots, p-1$, ρ^j is an outer automorphism.

Proof. Let q be the order of τ^i. Notice that we have $\rho^j(X) = \tau^{-ij}(X)$ for all $X \in \mathcal{A}$. If j is not a multiple of q, then ρ^j is an outer automorphism because τ^{-ij} is outer (c.f.[15, §8]).

Now, we assume that $j = mq$ with $m < r$, where $r = pq^{-1}$. Suppose that ρ^j is inner and we will show that this leads us to a contradiction. Then, since $\rho^j(X) = X$ for all $X \in \mathcal{A}$, there exists an element h in $L^\infty(S, \mu)$ with $|h| = 1$ such that $\rho^j = \mathrm{Ad}\,\pi(h)$. Notice that we have, for $c \in F_2(n)$, $f \in \mathcal{H}_n$ and $(s, t) \in G_n$,

$$(\mathrm{Ad}\,\pi(h)(U_c)f)(s, t) = h(t)\overline{h(c^{-1}t)}f(s, c^{-1}t).$$

First, let c be the element $\begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$ of $F_2(n)$. In this case, as we have $\rho^j(U_c) = U_c$, we have

$$h(t) = h(c^{-1}t) \qquad \text{for } \mu\text{-a.a. } t \in S.$$

Let $\tilde{\mu}$ be the normalized Lebesgue measure on the 1-torus $\mathbb{T}$. Since the map $x \longmapsto x + 2ny$ is an ergodic transformation on $(\mathbb{T}, \tilde{\mu})$ for every irrational number y, we know that, for $\tilde{\mu}$-a.a. $y \in \mathbb{T}$, h_y is constant $\tilde{\mu}$-almost everywhere on $\mathbb{T}$, where h_y is the Borel function on $\mathbb{T}$ such that $h_y(x) = h(x, y)$ for $(x, y) \in S$.

Next, let c be the element $\begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}$ of $F_2(n)$. We define an element h' of $L^\infty(S, \mu)$ by $h'(s) = h(s)\overline{h(c^{-1}s)}$ for $s \in S$. It is clear that $\rho^j(U_c) = \pi(h')U_c$. As in the proof of Lemma 3.4, we can show that

$$h' = \Sigma_{\ell=0}^{q-1} \Sigma_{k=0}^{2nq-1} \gamma^{km} \chi_{B(k,\ell)} \qquad \mu\text{-a.e.} \qquad (2)$$

Thus, we have $h(s) = h(c^{-1}s)$ for μ-a.a. $s \in B(0, \ell)$ $(\ell = 0, \ldots, q - 1)$. Therefore there exist a μ-conull Borel set S' of S, a $\tilde{\mu}$-conull Borel set A of $\mathbb{T}$ and a Borel map α of $\mathbb{T}$ into C^1 such that

(i) S'_y is empty if $y \notin A$, where $S'_y = \{x \in \mathbb{T};\ (x, y) \in S'\}$,

(ii) $\tilde{\mu}(S'_y) = 1$ for all $y \in A$,

(iii) $h(x, y) = \alpha(y)$ for all $(x, y) \in S'$,

(iv) $h(s) = h(c^{-1}s)$ for all $s \in B(0, \ell) \cap S'$ $(\ell = 0, \ldots, q - 1)$.

For every $y_0 \in \mathbb{T}$ with $\ell q^{-1} < y_0 < (\ell + 1)q^{-1}$, there exists a positive number ε such that, if we put $D = \{(x, y) \in S,\ 0 \leq x < \varepsilon, |y - y_0| < \varepsilon\}$, then D is contained in $B(0, \ell)$. Define Borel subsets D^+, D^- of S and D_0^+, D_0^- of $\mathbb{T}$ by the following;

$$D^+ = [0, \varepsilon) \times [y_0, y_0 + \varepsilon),$$

$$D^- = [0, \varepsilon) \times (y_0 - \varepsilon, y_0],$$

$$D_0^+ = [y_0, y_0 + \varepsilon) \cap A,$$

$$D_0^- = (y_0 - \varepsilon, y_0] \cap A.$$

We set $F = c^{-1}(D^+ \cap S') \cap (D^- \cap S')$ and $\overline{F} = c^{-1}D^+ \cap D^-$. It is clear that $\mu(\overline{F} - F) = 0$. Then, there exists a Borel subset $\tilde{D}_0^-$ of D_0^- with $\tilde{\mu}(D_0^- - \tilde{D}_0^-) = 0$ such that

$$\tilde{\mu}(F_y) = \tilde{\mu}(\overline{F}_y) = \varepsilon/2n \qquad \text{for all } y \in \tilde{D}_0^-.$$

For $y \in \tilde{D}_0^-$, we denote by R_y the set of $y' \in D_0^+$ for which there exists an element (x', y') of $D^+ \cap S'$ such that $c^{-1}(x', y') \in F \cap L_y$, where $L_y = \{(x, y) \in S';\ 0 \leq x < \varepsilon\}$. Let β be the Borel isomorphism of $[y_0, y_0 + \varepsilon)$ onto $\overline{F}_y = [(2n)^{-1}(-y + y_0), (2n)^{-1}(-y + y_0 + \varepsilon))$ such that $\beta(y') = (2n)^{-1}(y' - y)$. As we have $\beta(R_y) = F_y$, we have

$$\tilde{\mu}(R_y) = 2n\tilde{\mu}(F_y) = \varepsilon.$$

We fix an element y in $\tilde{D}_0^-$. For every $y' \in R_y$ and $(x', y') \in D^+ \cap S'$ with $c^{-1}(x', y') \in F \cap L_y$, we have

$$\begin{aligned} \alpha(y') &= h(x', y') \\ &= h(c^{-1}(x', y')) = \alpha(y). \end{aligned}$$

Moreover, for every $y' \in \tilde{D}_0^-$, there exists an element y'' in $R_y \cap R_{y'}$ because $\tilde{\mu}(R_y) = \tilde{\mu}(R_{y'}) = \varepsilon$, and we have

$$\alpha(y') = \alpha(y'') = \alpha(y).$$

Since $R_y \cup \tilde{D}_0^-$ is a subset of $(y_0 - \varepsilon, y_0 + \varepsilon)$ such that $\tilde{\mu}(R_y \cup \tilde{D}_0^-) = 2\varepsilon$, α is constant $\tilde{\mu}$-almost everywhere on $(y_0 - \varepsilon, y_0 + \varepsilon)$. It follows that α is constant $\tilde{\mu}$-almost everywhere on $(\ell q^{-1}, (\ell + 1)q^{-1})$. This implies that h is constant μ-almost everywhere on E_ℓ $(\ell = 0, \ldots, q - 1)$. Therefore there exist complex numbers α_ℓ $(\ell = 0, \ldots, q - 1)$ such that

$$h = \Sigma_{\ell=0}^{q-1} \alpha_\ell \chi_{E_\ell} \qquad \mu\text{-a.e.}$$

Thus, for a.a. $s \in E_0$, the value $h'(s)$ is in $\{\alpha_0\overline{\alpha}_0, \alpha_0\overline{\alpha}_1, \ldots, \alpha_0\overline{\alpha}_{q-1}\}$. On the other hand, by (2), for every $k = 0, \ldots, \text{Order}(\gamma^m) - 1$, h' takes the value γ^{mk} on a subset of E_0 with a positive measure. Since the order of γ^m is strictly greater than q, we get a contradiction.

Q.E.D.

<u>Proof of Theorem 3.3.</u> It follows from Lemmas 3.4 and 3.5 that the outer period of ρ is p. Let i' be a non-negative integer such that i' and q are relatively prime and such that i/p = i'/q. Then we have ${\Gamma_p}^i E_\ell = E_{\ell+i'}$, where $\ell + i'$ means $\ell + i'$ modulo q. As we have $\rho(\pi(h)) = \tau^{-i}(\pi(h))$, we have, for $f \in \mathcal{H}_n$ and a.a. $(s, t) \in G_n$,

$$(\rho(\pi(h))f)(s, t) = h({\Gamma_p}^i t)f(s, t).$$

On the other hand, we have

$$\begin{aligned} h({\Gamma_p}^i t) &= \Sigma_{\ell=0}^{q-1} \gamma^{r\ell} \chi_{{\Gamma_p}^{-i} E_\ell}(t) \\ &= \gamma^{ri'}(\Sigma_{\ell=0}^{q-1} \gamma^{r(\ell-i')} \chi_{E_{\ell-i'}}(t)) \\ &= \gamma^i h(t). \end{aligned}$$

It follows that $\rho(\pi(h)) = \gamma^i \pi(h)$. Therefore the Connes' outer invariant of ρ is (p, γ^i). The remainder of the assertion of the theorem is clear. Q.E.D.

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