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Series Representations of Certain Types of Arithmetical Functions

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1. Introduction. In this note we consider a special class of arithmetical functions which admit of seris expansion of the form (2.4). Two general theorems are proved in §2, the first being of representation-inversion type (Theorem 2.1), the second being an elementary asymptotic estimate for summatory functions (Theorem 2.2).

These two results are applied in §3 to functions arising from the generalized Euler function,

$$\phi_s(n) = \sum_{d\delta=n} \mu(d)\delta^s\,, \qquad \phi_1(n) = \phi(n)\,, \tag{1.1}$$

$\mu(n)$ denoting the Möbius symbol, and the generalized Dedekind function,

$$\psi_s(n) = \sum_{d\delta=n} \mu^2(d)\delta^s\,, \qquad \psi_1(n) = \psi(n)\,. \tag{1.2}$$

In particular, assuming $s>1$, we obtain in Theorem 3.2 estimates for the average order of $(\phi_s(n)/n^s)^k$, $(n^s/\phi_s(n))^k$, $(\psi_s(n)/n^s)^k$, $(n^s/\psi_s(n))^k$, where k is an arbitrary positive integer. These estimates are based on the series representations obtained in Theorem 3.1. Other combinations of $\phi_s(n)$ and $\psi_s(n)$ are also considered.

For a discussion of similar functions, corresponding to the case $s=1$ (excluded here), we mention Chowla [1] and Ward [5]. Their methods are quite different, however, from the one used in the present paper.

2. Two general theorems. The first theorem is based on the following well-known inversion formula for infinite series (cf. [4, Theorem 270]).

Lemma 2.1. *If $F(n)$ is an arithmetical function such that*

$$\sum_{r_1, r_2=1}^{\infty} |F(r_1 r_2)| \tag{2.1}$$

converges, then $F(n)$ may be represented in the form

$$F(n) = \sum_{r=1}^{\infty} g(nr)\,, \tag{2.2}$$

where

$$g(r) = \sum_{e=1}^{\infty} \mu(e)F(er)\,, \tag{2.3}$$

the series in (2.2) *being absolutely convergent.*

There is a converse to this result, but it will not be required in what follows.

A function $f(n)$ will be said to be *of type* A if

$$f(n) = \sum_{\substack{r=1 \\ (r,n)=1}}^{\infty} g(r)\,, \tag{2.4}$$

where the series is absolutely convergent. We now prove the following theorem relating to functions of this class. First define $\varepsilon(n)$ to be 1 or 0 according as $n=1$ or $n>1$.

Theorem 2.1. *If $f(n)$ is of the form*

$$f(n) = \sum_{d|n} \mu(d)F(d)\,, \tag{2.5}$$

where (2.1) *is convergent, then $f(n)$ is of type A and has the representation* (2.4), *with $g(r)$ determined by* (2.3). *Moreover, if $f(n)$ is of type A with a representation* (2.4), *then $f(n)$ can be represented in the form* (2.5), *with $F(n)$ determined by the absolutely convergent series in* (2.2).

Proof. Let $f(n)$ be defined by (2.5) such that (2.1) converges. Then by the preceding lemma, $F(n)$ has the absolutely convergent series representation (2.2), with $g(r)$ determined by (2.3). Thus by the fundamental property of $\mu(n)$,

$$f(n) = \sum_{d|n} \mu(d)F(d) = \sum_{d|n} \mu(d) \sum_{e=1}^{\infty} g(de) = \sum_{d|n} \mu(d) \sum_{\substack{r=1 \\ d|r}}^{\infty} g(r) = \sum_{r=1}^{\infty} g(r) \sum_{d|(n,r)} \mu(d)$$
$$= \sum_{r=1}^{\infty} g(r)\varepsilon((n,r))\,,$$

which is the same as (2.4). This completes the proof of the first half of the theorem, because the absolute convergence of (2.4) is a consequence of that of (2.2). The proof of the second half does not depend upon series inversion, and results on reversing the steps in the above computation.

REMARK 2.1. The actual form (2.3) of $g(r)$ in the Lemma is not required in the above proof. In fact, $g(r)$ in (2.4) is not necessarily unique (see Remark 3.2).

Before proving our next result we introduce the following notation.

The Legendre totient, defined to be the number of positive integers $\leqq x$ relatively prime to n, will be denoted $\phi(x, n)$. In particular, $\phi(n, n)=\phi(n)$. The number of decompositions of n as a product of k positive integers, k a fixed integer $\geqq 1$, will be denoted by $\tau_k(n)$. In particular $\tau_1(n)\equiv 1$ and $\tau_2(n)=\tau(n)$ is the number of divisors of n. The number of square-free divisors of n will be denoted by $\theta(n)$. We shall need some known elementary estimates.

Lemma 2.2 (cf. [2, (3.9)]). *If* $x\geqq 1$, *then*

$$\phi(x, n) = \frac{x\phi(n)}{n}+h(n)\,, \tag{2.6}$$

where $h(n)=h(n, x)$ *is bounded as a function of* x; *more precisely, there exists a positive constant* c *such that* $|h(n)|\leqq c\theta(n)$ *for all* n *and all real* $x\geqq 1$.

Lemma 2.3. *If* k *is a given integer* $\geqq 1$, *then* $\tau_k(n)=O(n^{\varepsilon})$ *for all* $\varepsilon>0$.

REMARK 2.2. This result follows from the special case in which $k=2$ [4, theorem 314]; evidently we have also $\theta(n)=O(n^{\varepsilon})$ for all $\varepsilon>0$.

Theorem 2.2. *If* $f_s(n)$ *is defined by*

$$f_s(n) = \sum_{\substack{r=1\\(r,n)=1}}^{\infty} \frac{g(r)}{r^s}\,, \tag{2.7}$$

where $g(n)=O(n^{\varepsilon})$ *for all* $\varepsilon>0$ (*so that the series in* (2.7) *converges absolutely for* $s>1$), *then for* $x\geqq 1$

$$\sum_{n\leqq x} f_s(n) = \left(\sum_{n=1}^{\infty}\frac{g(n)\phi(n)}{n^{s+1}}\right)x+O(1)\,, \qquad s>1\,, \tag{2.8}$$

the series in (2.8) *being absolutely convergent.*

Proof. The absolute convergence of the series in (2.8) is a consequence of the absolute convergence of (2.7) and the fact that $\phi(n)\leqq n$. Moreover by (2.7), for $s>1$,

$$\sum_{n\leqq x} f_s(n) = \sum_{n\leqq x}\sum_{\substack{r=1\\(n,r)=1}}^{\infty}\frac{g(r)}{r^s} = \sum_{r=1}^{\infty}\frac{g(r)}{r^s}\sum_{\substack{n\leqq x\\(r,n)=1}} 1\,,$$

the rearrangement being justified by the absolute convergence of (2.7). Hence

$$\sum_{n \leq x} f_s(n) = \sum_{r=1}^{\infty} \frac{g(r)\phi(x, r)}{r^s}, \tag{2.9}$$

where the series is absolutely convergent for all $x \geqq 1$. Therefore, one may write, by (2.6),

$$\sum_{n \leq x} f_s(n) = x \sum_{r=1}^{\infty} \frac{g(r)\phi(r)}{r^{s+1}} + \sum_{r=1}^{\infty} \frac{g(r)h(r)}{r^s}, \qquad x \geqq 1, \tag{2.10}$$

the (absolute) convergence of the second series in (2.10) being assured by that of the first series and that of the series in (2.9). Moreover, by Lemma 2.2,

$$\left| \sum_{r=1}^{\infty} \frac{g(r)h(r)}{r^s} \right| \leqq \sum_{r=1}^{\infty} \frac{|g(r)h(r)|}{r^s} \leqq c \sum_{r=1}^{\infty} \frac{|g(r)|\theta(r)}{r^s},$$

and the latter series converges by Remark 2.2. The proof is complete.

3. Applications. In addition to the notation of the preceding sections, we shall require the following. Let $\lambda(n)$ denote Liouville's function, place $\theta'(n) = \lambda(n)\theta(n)$ and $q(n) = \mu^2(n)$, define $t(n)$ to be 1 or 0 according as n is or is not a square, and let $t'(n) = t(n)\mu(\sqrt{n})$. Also, let the k-tuple convolution of a function $f(n)$ be defined by

$$F_k(n) = \sum_{d_1 \cdots d_k = n} f(d_1) \cdots f(d_k), \qquad F_1(n) = f(n),$$

where the summation is over all $d_1, \cdots, d_k$ such that $d_1, \cdots, d_k = n$, k a positive integer. Clearly, $\tau_k(n)$ is the k-tuple convolution of $\tau_1(n) \equiv 1$. In addition, we shall use $\mu_k(n)$, $\lambda_k(n)$, $q_k(n)$, $\theta_k(n)$, $\theta'_k(n)$, $t_k(n)$, and $t'_k(n)$, to denote, respectively, the convolutions of $\mu(n)$, $\lambda(n)$, $q(n)$, $\theta(n)$, $\theta'(n)$, $t(n)$, and $t'(n)$.

REMARK 3.1. By Lemma 2.3, $F_k(n) = O(n^\varepsilon)$ for every $\varepsilon > 0$ and each of the eight special functions $f(n)$ in the list above: Because $f(n) = O(n^\varepsilon)$ for each of these functions, a consequence of Remark 2.2 and the boundedness of $\mu(n)$, $\lambda(n)$, $t(n)$, and $q(n)$.

We also recall the fundamental properties of $\lambda(n)$,

$$\sum_{d \mid n} \lambda(d) = t(n), \qquad \sum_{d\delta = n} \lambda(d)q(\delta) = \varepsilon(n), \tag{3.1}$$

and two identities,

$$\sum_{d\delta = n} \mu(d)\lambda(\delta) = \theta'(n), \qquad \sum_{d\delta = n} \mu(d)q(\delta) = t'(n), \tag{3.2}$$

which are easily verified, using the multiplicative properties of the functions in question.

We now appeal Theorem 2.1 to expand a number of arithmetical functions in series of the form (2.4), and in fact, in the form of the series (2.7) appearing in Theorem 2.2. The Riemann zeta-function will be denoted $\zeta(s)$.

Theorem 3.1. *In the above notation, for all* s 1 *and positive integers* k,

$$\left(\frac{\phi_s(n)}{n^s}\right)^k = \frac{1}{\zeta^k(s)} \sum_{\substack{r=1 \\ (r,n)=1}}^{\infty} \frac{\tau_k(r)}{r^s}, \tag{3.3}$$

$$\left(\frac{n^s}{\phi_s(n)}\right)^k = \zeta^k(s) \sum_{\substack{r=1 \\ (r,n)=1}}^{\infty} \frac{\mu_k(r)}{r^s}, \tag{3.4}$$

$$\left(\frac{\psi_s(n)}{n^s}\right)^k = \frac{\zeta^k(s)}{\zeta^k(2s)} \sum_{\substack{r=1 \\ (r,n)=1}}^{\infty} \frac{\lambda_k(r)}{r^s}, \tag{3.5}$$

$$\left(\frac{n^s}{\psi_s(n)}\right)^k = \frac{\zeta^k(2s)}{\zeta^k(s)} \sum_{\substack{r=1 \\ (r,n)=1}}^{\infty} \frac{q_k(r)}{r^s}, \tag{3.6}$$

$$\left(\frac{\phi_s(n)}{\psi_s(n)}\right)^k = \frac{\zeta^k(2s)}{\zeta^{2k}(s)} \sum_{\substack{r=1 \\ (r,n)=1}}^{\infty} \frac{\theta_k(r)}{r^s}, \tag{3.7}$$

$$\left(\frac{\psi_s(n)}{\phi_s(n)}\right)^k = \frac{\zeta^{2k}(s)}{\zeta^k(2s)} \sum_{\substack{r=1 \\ (r,n)=1}}^{\infty} \frac{\theta'_k(r)}{r^s}, \tag{3.8}$$

$$\left(\frac{\phi_s(n)\psi_s(n)}{n^{2s}}\right)^k = \frac{1}{\zeta^k(s)} \sum_{\substack{r=1 \\ (r,n)=1}}^{\infty} \frac{t_k(r)}{r^s}, \tag{3.9}$$

$$\left(\frac{n^{2s}}{\phi_s(n)\psi_s(n)}\right)^k = \zeta^k(s) \sum_{\substack{r=1 \\ (r,n)=1}}^{\infty} \frac{t'_k(r)}{r^s}. \tag{3.10}$$

Proof. From (1.1) and (1.2), we have

$$\frac{\phi_s(n)}{n^s} = \sum_{d|n} \frac{\mu(d)}{d^s} = \prod_{p|n}\left(1-\frac{1}{p^s}\right), \tag{3.11}$$

$$\frac{\psi_s(n)}{n^s} = \sum_{d|n} \frac{\mu^2(d)}{d^s} = \prod_{p|n}\left(1+\frac{1}{p^s}\right), \tag{3.12}$$

where the products are over the prime divisors p of n. We apply Theorem 2.1 to these two functions.

First take $f(n)=\phi_s(n)/n^s$, so that $F(n)=n^{-s}$ in Theorem 2.1. Since (2.1) converges, $f(n)$ must have a representation (2.4) with

$$g(r)= \sum_{e=1}^{\infty} \mu(e)F(er) = \frac{1}{r^s}\sum_{e=1}^{\infty}\frac{\mu(e)}{e^s} = \frac{1}{r^s\zeta(s)};$$

therefore,

$$\frac{\phi_s(n)}{n^s} = \frac{1}{\zeta(s)} \sum_{\substack{r=1 \\ (r,n)=1}}^{\infty} \frac{1}{r^s}. \tag{3.13}$$

Taking the k-th power of (3.13) yields (3.3), by Dirichlet multiplication, and (3.4) results on taking reciprocals.

Next we suppose $f(n)=\psi(n)/n^s$ in Theorem 2.1. Then by (3.12), $F(n)=\mu(n)n^{-s}$, and since (2.1) converges with this choice for $F(n)$, we have the representation (2.4) of $f(n)$, with

$$g(r) = \sum_{e=1}^{\infty} \mu(e)F(er) = \frac{1}{r^s} \sum_{r=1}^{\infty} \mu(e)\mu(er)e^{-s}.$$

Hence by (3.12), with primes denoted by p,

$$\begin{aligned} g(r) &= \frac{\mu(r)}{r^s} \sum_{\substack{e=1 \\ (e,r)=1}}^{\infty} \frac{\mu^2(e)}{e^s} = \left(\frac{\mu(r)}{r^s}\right) \frac{\prod_p \left(1+\frac{1}{p^s}\right)}{\prod_{p \mid r} \left(1+\frac{1}{p^s}\right)} \\ &= \left(\frac{\mu(r)}{\psi_s(r)}\right) \prod_p \left(\frac{1-\frac{1}{p^{2s}}}{1-\frac{1}{p^s}}\right) = \left(\frac{\mu(r)}{\psi_s(r)}\right) \frac{\zeta(s)}{\zeta(2s)}. \end{aligned}$$

With the latter value of $g(r)$ in (2.4), one obtains

$$\frac{\psi_s(n)}{n^s} = \frac{\zeta(s)}{\zeta(2s)} \sum_{\substack{r=1 \\ (r,n)=1}}^{\infty} \frac{\mu(r)}{\psi_s(n)}. \tag{3.14}$$

We can transform this expansion for $\Psi_s(n)/n^s$ by again using (3.12). In particular, we have

$$\begin{aligned} \frac{\zeta(2s)}{\zeta(s)}\left(\frac{\psi_s(n)}{n^s}\right) &= \sum_{\substack{r=1 \\ (r,n)=1}}^{\infty} \frac{\mu(r)}{\psi_s(r)} = \prod_{p \nmid n} \left(1-\frac{1}{p^s+1}\right) = \prod_{p \nmid n} \left(\frac{1}{1+p^{-s}}\right) \\ &= \prod_{p \nmid n} \left(\sum_{i=0}^{\infty} (-1)^i p^{-is}\right) = \prod_{p \nmid n} \left(\sum_{i=0}^{n} \frac{\lambda(p^i)}{p^{is}}\right), \end{aligned}$$

which may be restated as

$$\frac{\psi_s(n)}{n^s} = \frac{\zeta(s)}{\zeta(2s)} \sum_{\substack{r=1 \\ (r,n)=1}}^{\infty} \frac{\lambda(r)}{r^s}. \tag{3.15}$$

Taking k-th powers, (3.5) results from (3.15). On the basis of the second formula in (3.1), (3.6) is an immediate consequence of (3.5)

The other formulas follow similarly: (3.7) from (3.3) and (3.6);

(3. 8) from (3. 4) and (3. 5) on the basis of (3. 2a); (3. 9) from (3. 3) and (3. 5) by (3. 1a); and (3. 10) from (3. 4) and (3. 6) by (3. 2b). The proof is now complete.

REMARK 3. 2. We note the two distinct representations (3. 14) and (3. 15) of type (2. 4) for the function $\psi_s(n)/n^s$ $(s>1)$.

The series identities of the preceding theorem will now be used to obtain the following estimates.

Theorem 3. 2. *Suppose that* $s>1$ *and that* k *is a positive integer. Then for* $x\geqq 1$,

$$\sum_{n\leqq x}\left(\frac{\phi_s(n)}{n^s}\right)^k=\frac{1}{\zeta^k(s)}\left(\sum_{n=1}^{\infty}\frac{\tau_k(n)\phi(n)}{n^{s+1}}\right)x+O(1)\,, \tag{3.16}$$

$$\sum_{n\leqq x}\left(\frac{n^s}{\phi_s(n)}\right)^k=\zeta^k(s)\left(\sum_{n=1}^{\infty}\frac{\mu_k(n)\phi(n)}{n^{s+1}}\right)x+O(1)\,, \tag{3.17}$$

$$\sum_{n\leqq x}\left(\frac{\psi_s(n)}{n^s}\right)^k=\frac{\zeta^k(s)}{\zeta^k(2s)}\left(\sum_{n=1}^{\infty}\frac{\lambda_k(n)\phi(n)}{n^{s+1}}\right)x+O(1)\,, \tag{3.18}$$

$$\sum_{n\leqq x}\left(\frac{n^s}{\psi_s(n)}\right)^k=\frac{\zeta^k(2s)}{\zeta^k(s)}\left(\sum_{n=1}^{\infty}\frac{q_k(n)\phi(n)}{n^{s+1}}\right)x+O(1)\,, \tag{3.19}$$

$$\sum_{n\leqq x}\left(\frac{\phi_s(n)}{\psi_s(n)}\right)^k=\frac{\zeta^k(2s)}{\zeta^{2k}(s)}\left(\sum_{n=1}^{\infty}\frac{\theta_k(n)\phi(n)}{n^{s+1}}\right)x+O(1)\,, \tag{3.20}$$

$$\sum_{n\leqq x}\left(\frac{\psi_s(n)}{\phi_s(n)}\right)^k=\frac{\zeta^{2k}(s)}{\zeta^k(2s)}\left(\sum_{n=1}^{\infty}\frac{\theta_k'(n)\phi(n)}{n^{s+1}}\right)x+O(1)\,, \tag{3.21}$$

$$\sum_{n\leqq x}\left(\frac{\phi_s(n)\psi_s(n)}{n^{2s}}\right)^k=\frac{1}{\zeta^k(s)}\left(\sum_{n=1}^{\infty}\frac{t_k(n)\phi(n)}{n^{s+1}}\right)x+O(1)\,, \tag{3.22}$$

$$\sum_{n\leqq x}\left(\frac{n^{2s}}{\phi_s(n)\psi_s(n)}\right)^k=\zeta^k(s)\left(\sum_{n=1}^{\infty}\frac{t_k'(n)\phi(n)}{n^{s+1}}\right)x+O(1)\,. \tag{3.23}$$

All occurring series are absolutely convergent.

Proof. Theorem 2. 2 is applicable to the functions in question by reason of Remark 3. 1 and the series representations of Theorem 3. 1. The theorem results on applying (2. 8).

We specialize to the case $k=2$ in (3. 16) and (3. 17) to obtain the following estimates in product form.

Corollary 3. 2. 1. *If* $s>1$, *then for* $x\geqq 1$,

$$\sum_{n\leqq x}\left(\frac{\phi_s(n)}{n^s}\right)^2=x\prod_p\left(1-\frac{2}{p^{s+1}}+\frac{1}{p^{2s+1}}\right)+O(1)\,, \tag{3.24}$$

$$\sum_{n\leqq x}\left(\frac{n^s}{\phi_s(n)}\right)^2=x\prod_p\left(1+\frac{2}{p(p^s-1)}+\frac{1}{p(p^s-1)^2}\right)+O(1)\,, \tag{3.25}$$

the products ranging over the primes p.

The details of expanding the series in (3. 16) and (3. 17) with $k=2$ are omitted. These product expansions, as infinite products for each of the series occuring in Theorem 3. 2, can be obtained by using a result of Erdös concerning mean values of arithmetical functions [3, (2)]. Erdös's method, however, yields only remainder terms of order $o(x)$ for the problems under discussion.

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