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Osaka University
Let $R$ be a Dedekind prime ring with the quotient ring $Q$. Let $F$ be any right additive topology (cf. [11]). Then $R$ is a topological ring with elements of $F$ as the neighborhoods of zero. Let $M$ be a topological right $R$-module with submodule neighborhoods of zero. $M$ is called $F$-linearly compact if

(a) it is Hausdorff,

(b) if every finite subset of the set of congruences $x \equiv m_a \pmod{N_a}$, where $N_a$ are closed submodules of $M$, has a solution in $M$, then the entire set of the congruences has a solution in $M$.

The purpose of this paper is to study the algebraic and topological properties of $F$-linearly compact modules.

After discussing some properties on $R$ which need in this paper, we show, in Section 2, that the Kaplansky’s duality theorem holds for $F$-linearly compact modules (Theorem 2.12). By using the duality theorem we determine, in Section 3, the algebraic and topological structures of $F$-linearly compact modules when $F$ is bounded. Moreover we define the concepts of $F^\omega$-pure injective and $F^\omega$-pure injective modules, and investigate the relations of between these concepts and $F$-linearly compact modules.

I wish to express my appreciation to the referee for his adequate advice.

1. Topologies on Dedekind prime rings

Throughout this paper, $R$ will denote a Dedekind prime ring which is not artinian, and $Q$ will denote the quotient ring of $R$. We will denote the $(R, R)$-bimodule $Q/R$ by $K$. A subring of $Q$ containing $R$ is called an overring of $R$. For any essential right ideal $I$, the left order of $I$ is defined by $0_I(I) = \{q \in Q | qI \subseteq I\}$. We define the inverse of $I$ to be $I^{-1} = \{q \in Q | IqI \subseteq I\}$. Then we obtain $II^{-1} = 0_I(I)$ and $I^{-1} I = R$. Let $I$ be a right ideal of $R$. By Theorem 1.3 of [1], $R/I$ is an artinian $R$-module if and only if $I$ is an essential right ideal of $R$. For any right ideal $I$ and any element $a$ of $R$, we define $a^{-1} I = \{r \in R | ar \in I\}$. Let $M$ be a (right $R$-) module. $M$ is said to be torsion if, for every $m \in M$, $mI = 0$ for some essential right ideal $I$. We say that $M$ is divisible if $M/J = M$ for every essential left ideal $J$ of $R$. Let $F$ be any (right additive) topology (cf. [11]). We
say that \( m \in M \) is an \( F \)-torsion element if \( O(m) = \{ r \in R \mid mr = 0 \} \subseteq F \), and denote the submodule of \( F \)-torsion elements by \( M_F \). If \( M_F = 0 \), then we say that \( M \) is \( F \)-torsion-free. A topology \( F \) is trivial if all modules are \( F \)-torsion or \( F \)-torsion-free. If \( F = \{ R \} \), then it is clear that all modules are \( F \)-torsion-free. Assume that \( F \) contains a non essential right ideal \( I \) of \( R \), then \( F \)-torsion module \( R/I \) is a direct sum of a torsion module and a non-zero projective module \( C \) by Theorem 2.1 of [1]. By Theorem 2.4 of [1], a finite copies of \( C \) contains \( R \) as right modules and so \( R \) is \( F \)-torsion. Hence all modules are \( F \)-torsion. So if \( F \) is a non-trivial topology, then \( F \) consists of essential right ideals. Conversely a topology \( F \) consists of essential right ideals, then it is non-trivial, because \( R \) is \( F \)-torsion-free and \( R/I \) is \( F \)-torsion (\( I \in F \)).

From now on, \( F \) will denote a non-trivial topology. We define \( Q_F = \lim_{\longrightarrow} I^{-1} \), where \( I \) ranges over all elements of \( F \). Clearly \( Q_F \) is an overring of \( R \).

**Proposition 1.1**

(i) The mapping \( F \to Q_F \) is one-to-one correspondence between all non-trivial topologies and all overrings of \( R \) properly containing \( R \).

(ii) A module \( M \) is \( F \)-torsion if and only if \( M \otimes Q_F = 0 \).

(iii) For any module \( M \), \( M_F = \text{Tor}(M, Q_F/R) \).

**Proof.** By Corollary 13.4 of [11], \( F \) is perfect. Hence (ii) and (iii) follow from Exercise 2 of [11, p. 81].

(i) Let \( Q_o \) be an overring of \( R \) properly containing \( R \). Then it is well known that \( Q_o \) is \( R \)-flat and that the inclusion map: \( R \to Q_o \) is an epimorphism (cf. [11, p. 75]). Hence, by Theorem 13.10 of [11], \( F_o = \{ I \mid Q_o \otimes Q_o = Q_o, I \text{ is a right ideal} \} \) is a topology. Since \( Q_o \otimes Q_o \simeq Q_o \) and \( Q_o \) is \( R \)-flat, we have \( Q_o \otimes R \otimes Q_o = 0 \). Hence \( 0 \neq Q_o/R \) is \( F_o \)-torsion. It is evident that \( R \) is not \( F_o \)-torsion. Hence \( F_o \) is non-trivial. Thus (i) follows from Theorem 13.10 of [11].

Let \( \{ S_\alpha \mid \alpha \in \Lambda \} \) be the representative class of simple modules which are non-isomorphic mutually. For any subset \( \Gamma \) of \( \Lambda \), we denote the set of essential right ideals \( I \) such that any composition factor of the module \( R/I \) is isomorphic to \( S_\gamma \) for some \( \gamma \in \Gamma \) by \( F(\Gamma) \).

**Proposition 1.2.** A non-empty family of right ideals of \( R \) is a non-trivial topology if and only if it is of the form \( F(\Gamma) \) for some subset \( \Gamma \) of \( \Lambda \).

**Proof.** First we shall prove that \( F(\Gamma) \) is a non-trivial topology. (i) If \( I \in F(\Gamma) \) and \( a \in R \), then \( a^{-1} I \in F(\Gamma) \), because \( R/a^{-1}I \simeq (aR+I)/I \). (ii) Let \( I \) be a right ideal of \( R \). Assume that there exists \( J \in F(\Gamma) \) such that \( a^{-1} I \subseteq F(\Gamma) \) for every \( a \in J \). Again, since \( R/a^{-1}I \simeq (aR+I)/I \) for every \( a \in J \), we obtain that \( (I+J)/I \) is a torsion module. Hence \( R/I \) is also torsion and so \( I \) is an essential right ideal. By Theorem 3.3 of [1], \( I+J = aR+I \) for some \( a \in J \), and thus \( R/a^{-1}I \simeq (I+J)/I \). Therefore \( I \in F(\Gamma) \). Thus \( F(\Gamma) \) is a topology. Since \( F(\Gamma) \)
consists of essential right ideals, it is non-trivial. Conversely let $F$ be any topology and let $\Gamma = \{ \gamma \in \Lambda | S_\gamma \simeq R/I \text{ for some } I \in F \}$. From Lemma 3.1 of [11], we have $\Gamma \neq \phi$. We shall prove that $F = F(\Gamma)$. For an essential right ideal $I$ of $R$, $I \in F$ if and only if $R/I \otimes Q_F = 0$ by Proposition 1.1 and so $F \supseteq F(\Gamma)$. Assume that $F \supseteq F(\Gamma)$. Then there is $I \in F$ such that some composition factor of $R/I$ is isomorphic to $S_\alpha$ for some $\alpha \in \Lambda - \Gamma$. So there are right ideals $J_1 \supseteq J_2 \supseteq I$ such that $J_1/J_2 \simeq S_\alpha$. Take $a \in J_1$ with $a \notin J_2$. Then we get: $R/a^{-1}J_2 \simeq J_1/J_2 \simeq S_\alpha$. Hence, since $a^{-1}J_2 \in F$, we have $\alpha \in \Gamma$, which is a contradiction.

**Corollary 1.3.** The lattice of all overrings of $R$ is a Boolean lattice.

The family $F_\Gamma$ of left ideals $J$ of $R$ such that $Q_\Gamma J = Q_\Gamma$ is a left additive topology. We call it the *left additive topology corresponding to $F$*. $F_\Gamma$ is also non-trivial by Proposition 1.1. Thus $F_\Gamma$ consists of essential left ideals of $R$. We put $Q_\Gamma = \varprojlim(J \in F_\Gamma)$. A module $M$ is said to be $F_\Gamma$-divisible if $MJ = M$ for every $J \subseteq F_\Gamma$. In a similar way, we define the concepts of $F_\Gamma$-torsion and $F$-divisible for any left module.

**Proposition 1.4.**

(i) $Q_\Gamma = Q_\Gamma$ and so $Q_\Gamma$ is $(F, F_\Gamma)$-divisible.

(ii) $K = K_\Gamma = Q_\Gamma/R$, where $K = Q/R$. Thus $K_\Gamma$ is also $(F, F_\Gamma)$-divisible.

(iii) Let $I$ be an essential right ideal of $R$. Then $I \in F$ if and only if $I^{-1}/R$ is $F_\Gamma$-torsion.

**Proof.** (i) follows from Proposition 1.1 of [10] and the definitions. (ii) is clear.

(iii) Since $Q_\Gamma$ is flat as $R$-modules, the sequence $0 \longrightarrow Q_\Gamma \longrightarrow Q_\Gamma \otimes I^{-1} \longrightarrow Q_\Gamma \otimes I^{-1}/R \longrightarrow 0$ is exact. Further, since $Q_\Gamma \otimes Q_\Gamma \approx Q_\Gamma$, we obtain that $I \in F$ if and only if $Q_\Gamma \otimes I^{-1}/R = 0$. So $I \in F$ if and only if $I^{-1}/R$ is $F_\Gamma$-torsion.

2. **Duality theorem for $F$-linearly compact modules**

Let $F$ be any non-trivial topology. We define $\hat{R}_F = \varprojlim R/I (I \in F)$ and $\hat{R}_\Gamma = \varprojlim R/J (J \in F_\Gamma)$. It is easy to see that both $\hat{R}_F$ and $\hat{R}_\Gamma$ are rings containing $R$ (cf. §4 of [10]). Let $M$ be an $F$-torsion module. Then $M$ is an $\hat{R}_F$-module as follows: For $m \in M$, $\hat{r} = (r + I) \in \hat{R}_F$, we define $mr = mr_\Gamma$, where $J \subseteq O(m)$. Similarly, an $F_\Gamma$-torsion left module is an $\hat{R}_\Gamma$-module.

**Lemma 2.1.** A module is $F$-linearly compact in the discrete topology if and only if it is $F$-torsion and artinian.

**Proof.** The sufficiency follows from Proposition 5 of [13]. Conversely assume that $M$ is $F$-linearly compact in the discrete topology. Take $m \in M$. Then, by the continuity of multiplication, there exists $I \in F$ such that $mI = 0$. 


Thus $M$ is $F$-torsion. By Lemma 2.3 of [9], $M$ is finite dimensional in the sense of Goldie. So the socle $S(M)$ of $M$ is finitely generated and $M$ is an essential extension of $S(M)$. Let $N$ be any submodule of $M$. Then, since $N$ is an open and closed submodule, $\bar{M}=M/N$ is also $F$-linearly compact in the discrete topology by Proposition 2 of [13]. Thus the socle $S(\bar{M})$ of $\bar{M}$ is also finitely generated and $\bar{M}$ is an essential extension of $S(\bar{M})$. This implies that $M$ is an artinian module by Proposition 2* of [12].

**Corollary 2.2.** Let $M$ be $F$-linearly compact and let $N$ be a submodule. Then $N$ is a neighborhood of zero if and only if $M/N$ is $F$-torsion and artinian.

**Proof.** If $N$ is a neighborhood of zero, then $M/N$ is $F$-linearly compact in the discrete topology. So the necessity follows from Lemma 2.1. Conversely, assume that $M/N$ is $F$-torsion and artinian. Let $\{M_\alpha\}$ be the set of submodule neighborhoods of zero. Since the topology is Hausdorff, $\cap M_\alpha=\emptyset$, and so $\cap \bar{M}_\alpha=\emptyset$ in $\bar{M}=M/N$. Therefore there are finite submodules $M_\alpha$, $\ldots$, $M_\beta$ such that $\cap \cap_{i=1}^\beta M_\alpha=\emptyset$, i.e., $\cap \cap_{i=1}^\beta M_\alpha\subseteq N$. Thus $N$ is open.

**Corollary 2.3.** If a module is $F$-linearly compact in two topologies, then these topologies coincide.

**Lemma 2.4.** A module is $F$-linearly compact if and only if it is an inverse limit of $F$-torsion and artinian modules.

**Proof.** The sufficiency follows from Proposition 4 of [13] and Lemma 2.1. To prove the necessity let $\{N_\alpha\}$ be the set of submodule neighborhoods of zero. Then the modules $M/N_\alpha$ with the natural maps: $[m+N_\alpha] \rightarrow [m+N_\beta]$, where $N_\alpha\subseteq N_\beta$, form an inverse system. Write $\hat{M}=\lim M/N_\alpha$. Then it is a topological module; each $M/N_\alpha$ has the discrete topology and the product topology on $\prod M/N_\alpha$ induces a subspace topology on $\hat{M}$. Since $\cap N_\alpha=0$, the canonical map $f: M \rightarrow \hat{M}$ is a monomorphism. It is easy to see that $f$ is a topological isomorphism from $M$ onto $f(M)$ and that $f(M)$ is dense in $\hat{M}$. On the other hand, $M$ is complete by Proposition 8 of [13] and so $f(M)=\hat{M}$. Further $M/N_\alpha$ is $F$-torsion and artinian by Corollary 2.2.

Following [11], a module $D$ is $F$-injective if $\text{Ext}(R/I, D)=0$ for every $I \in F$. By Proposition 6.2 of [11], $D$ is $F$-injective if and only if $\text{Ext}(T, D)=0$ for every $F$-torsion $T$. Further, since every $F$-torsion module $T$ can be embedded in an exact sequence $0 \rightarrow T \rightarrow \sum \oplus K_F$ with sufficiently many copies of $K_F$, $D$ is $F$-injective if and only if $\text{Ext}(K_F, D)=0$. For any module $M$, we denote the injective hull of $M$ by $E(M)$ and denote the $F$-injective hull of it by $E_F(M)$ (cf. [11]).

**Lemma 2.5.** (i) A module is $F$-injective if and only if it is $F_F$-divisible.
(ii) Let $M$ be a module with $M_F = 0$. Then $E_F(M) = M \otimes Q_F$.

Proof. (i) Assume that $D$ is $F$-injective. Let $J \subseteq F$. Then $J^{-1}/R$ is $F$-torsion by Proposition 1.4 and so the necessity follows from Proposition 3.2 of [10]. Conversely assume that $D$ is $F_I$-divisible. Let $I$ be any element of $F$. Then $I^{-1}/R = \sum_{i=1}^n \oplus R/J_i$ for $J_i \in F_I$. By Proposition 3.3 of [10], we have $R/I \cong \text{Hom}(I^{-1}/R, K_F) \cong \sum_{i=1}^n \oplus \text{Hom}(R/J_i, K_F) \cong \sum_{i=1}^n \oplus J_i^{-1}/R$, and so $\text{Ext}(R/I, D) \cong \sum_{i=1}^n \oplus \text{Ext}(J_i^{-1}/R, D) \cong \sum_{i=1}^n \oplus D/J_i = 0$. Therefore $D$ is $F$-injective.

(ii) By Proposition 1.1, $M_F = \text{Tor}(M, K_F)$. Hence from the exact sequence $0 \rightarrow R \rightarrow Q_F \rightarrow K_F \rightarrow 0$ we get an exact sequence $0 \rightarrow M \rightarrow M \otimes Q_F \rightarrow M \otimes K_F \rightarrow 0$.

By Proposition 1.4 and (i), $M \otimes Q_F$ is $F$-injective and so $M \otimes Q_F = E_F(M)$.

**Corollary 2.6.** Let $M$ be a module. Then $M \otimes Q_F$ and $M \otimes K_F$ are both $F$-injective.

For a module $M$, we define $\hat{M}_{F_i} = \lim M/MJ (J \in F_I)$. $\hat{M}_{F_i}$ is an $\hat{R}_{F_i}$-module (cf. §4 of [10]). Similarly, for a left module $N$, we can define a left $\hat{R}_F$-module $\hat{N}_F$.

**Lemma 2.7.** Let $M$ be a module with $M_F = 0$. Then there are commutative diagrams:

$$
\begin{array}{ccc}
\hat{M}_{F_i} & \approx & \text{Hom}(K_F, M) \\
\uparrow & & \uparrow \alpha \\
M & = & M \\
\end{array}
$$

$$
\begin{array}{ccc}
M & = & M \\
\downarrow \beta \\
\hat{M}_{F_i} & \approx & \text{Ext}(K_F, M) \\
\end{array}
$$

where $\alpha(m)(q) = m \otimes q$ ($m \in M$, $q \in K_F$) and $\beta$ is the connecting homomorphism.

Proof. From the exact sequence $0 \rightarrow R \rightarrow Q_F \rightarrow K_F \rightarrow 0$, we get an exact sequence:

$$
0 = \text{Tor}(M, K_F) \rightarrow M \rightarrow M \otimes Q_F \rightarrow M \otimes K_F \rightarrow 0.
$$

Hence the assertion of the first diagram follows from the similar way as in Theorem 4.4 of [10]. Applying $\text{Hom}(K_F, \_)$ to the sequence (1), we obtain the exact sequence:

$$
\text{Hom}(K_F, M \otimes Q_F) \rightarrow \text{Hom}(K_F, M \otimes K_F) \rightarrow \text{Ext}(K_F, M) \rightarrow \text{Ext}(K_F, M \otimes Q_F).
$$

The first and last terms are zero, because $M \otimes Q_F$ is $F$-torsion-free and $F$-injective. Hence $\text{Hom}(K_F, M \otimes K_F) \approx \text{Ext}(K_F, M)$. We consider the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & M \rightarrow M \otimes Q_F \rightarrow M \otimes K_F \rightarrow 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & E(M) \rightarrow E(M)/M \rightarrow 0.
\end{array}
$$
If \([x+M]I=0\), where \(x\in E(M)\) and \(I\in F\), then \(xI\subseteq M\) and so \(x\in M\otimes Q_F\) by Proposition 6.3 of [11] and Lemma 2.5. Hence \([x+M]\in M\otimes K_F\). This implies that \((E(M)/M)_{\infty}=M\otimes K_F\). It is evident that \(E(M)_{\infty}=0\). Thus we have \(\text{Ext}(K_F, M)\cong \text{Hom}(K_F, E(M)/M)\cong \text{Hom}(K_F, M\otimes K_F)\). Now it is easy to see that \(\alpha=\beta\).

**Corollary 2.8.** (i) \(\hat{R}_F/R\) is \(F\)-divisible.

(ii) \(R/I\cong \hat{R}_F/I\hat{R}_F\) for every \(I\in F\).

Proof. (i) Applying Lemma 2.7 to the left module \(R\), we get an isomorphism: \(\hat{R}_F/R\cong \text{Ext}(Q_F, R)\). Since \(\text{Ext}(Q_F, R)\) is a left \(Q_F\)-module, it is \(F\)-divisible and so \(\hat{R}_F/R\) is also \(F\)-divisible.

(ii) It is evident that \(I\hat{R}_F\cap R=I\). Hence (ii) follows from (i).

By Lemma 2.4, \(\hat{R}_F\) is an \(F\)-linearly compact module in the topology which is defined by taking as a subbase of neighborhoods of zero the set \(\{\pi^{-1}(0)\cap \hat{R}_F\mid I\in F\}\), where \(\pi_I: \prod R/I\rightarrow R/I\) is the projection. Further we have

**Corollary 2.9.** (i) \(\pi^{-1}(0)\cap \hat{R}_F=I\hat{R}_F\) for every \(I\in F\).

(ii) \(\hat{R}_F\) is a complete topological ring in the topology which has the set \(\{I\hat{R}_F\mid I\in F\}\) as neighborhoods of zero.

Proof. (i) Clearly \(\pi^{-1}(0)\cap \hat{R}_F\supseteq I\hat{R}_F\). By Corollary 2.8, there exists a right ideal \(J\supseteq I\) such that \(J/I\cong [\pi^{-1}(0)\cap \hat{R}_F]/I\hat{R}_F\), i.e., \(\pi^{-1}(0)\cap \hat{R}_F=I\hat{R}_F+J=J\hat{R}_F\), because \(\pi^{-1}(0)\cap \hat{R}_F\) is an \(\hat{R}_F\)-module. From this fact we easily obtain that \(J=I\) and so \(\pi^{-1}(0)\cap \hat{R}_F=I\hat{R}_F\).

(ii) For any \(\hat{x}\in \hat{R}_F\), we define \(\hat{x}^{-1}(I\hat{R}_F)\equiv \{\hat{v}\in \hat{R}_F\mid \hat{x}\hat{v}\in I\hat{R}_F\}\), where \(I\in F\).

Then we have the natural isomorphisms \(\hat{R}_F/\hat{x}^{-1}(I\hat{R}_F)\cong (\hat{x}\hat{R}_F+I\hat{R}_F)/I\hat{R}_F\cong J/I\) for some \(J\supseteq I\). Define \(\varphi_0[1+\hat{x}^{-1}(I\hat{R}_F)]=a+I\) \((a\in J)\). Then \(J=aR+I\) and so \(J/I\cong R/a^{-1}I\), where \(\gamma([a+I])=[1+a^{-1}I]\). Therefore we get the natural isomorphisms \(\hat{R}_F/\hat{x}^{-1}(I\hat{R}_F)\cong R/a^{-1}I\cong \hat{R}_F/(a^{-1}I)\hat{R}_F\). Thus we have \((a^{-1}I)\hat{R}_F=\hat{x}^{-1}(I\hat{R}_F)\). This implies that \(\hat{R}_F\) is a topological ring. The completeness of \(\hat{R}_F\) follows from Proposition 8 of [13].

Let \(\hat{F}=\{I\hat{R}_F\mid I\in F\}\). For any \(\hat{R}_F\)-module, we can define the concept of \(\hat{F}\)-linearly compact modules.

**Proposition 2.10.** A module is \(F\)-linearly compact if and only if it is an \(\hat{R}_F\)-module and is \(\hat{F}\)-linearly compact.

Proof. Assume that \(M\) is \(F\)-linearly compact. By Lemma 2.4, \(M\) is an \(\hat{R}_F\)-module. Let \(N\) be a closed submodule of \(M\). Then \(N\) is \(F\)-linearly compact by Proposition 3 of [13], and so it is an \(\hat{R}_F\)-submodule. Hence it is enough to
prove that $M$ is a topological $\hat{R}_F$-module. Take $m \in M, \hat{r} \in \hat{R}_F$. Then we define $m^{-1}N=\{s \in \hat{R}_F | ms \in N\}$ for any submodule neighborhood $N$ of zero. Since $M/N$ is $F$-torsion, we have $m^{-1}N \in \hat{F}$. Further we have $(m+N)(\hat{r}+m^{-1}N) \subseteq m\hat{r}$ + $N$ and so $M$ is a topological $\hat{R}_F$-module. Conversely assume that $M$ is $F$-linearly compact as an $\hat{R}_F$-module. Let $\{N_\alpha\}$ be the set of submodule neighborhoods of zero. Then, by a similar way as in Lemmas 2.1, 2.4 and Corollaries 2.2, 2.8, we have $M=\lim\cdots M/N_\alpha$ and $M/N_\alpha$ is $F$-torsion and artinian. Thus $M$ is $F$-linearly compact.

Let $M$ be $F$-linearly compact. $M^*$ will mean the module of all continuous homomorphisms from $M$ into $K_F$, where $K_F$ has been awarded the discrete topology. It is evident that an element $f \in \text{Hom}(M, K_F)$ is continuous if and only if $\text{Ker} f$ is open.

**Lemma 2.11.** Let $M$ be $F$-linearly compact. Then

(i) $M^*$ is an $\hat{R}_F$-module.

(ii) Let $N^*$ be a finitely generated left $\hat{R}_F$-submodule of $M^*$ and let $g \in \text{Hom}_{\hat{R}_F}(M^*, K_F)$. Then there exists an element $m \in M$ such that $(f)g=f(m)$ for every $f \in N^*$.

Proof. (i) For $f \in M^*$ and $\hat{r} \in \hat{R}_F$, we have $\text{Ker}(\hat{r}f) \supseteq \text{Ker} f$ and so $\hat{r}f \in M^*$.

We shall prove (ii) by Muller's method (cf. Lemma 1 of [8]). Write $N^*=\hat{R}_F f_1 + \cdots + \hat{R}_F f_n$, where $f_i \in N^*$, and let $W=\{(f_1(m), \ldots, f_n(m)) | m \in M\} \subseteq \sum^n \oplus K_F$. Assume that $x=((f_1)g, \ldots, (f_n)g) \in W$. Then $O(x)=\{r \in F | \bar{r}r=0\} \subseteq F$, where $x=[x+W]$ in $\sum \oplus K_F/W$. Hence there exists a map $\theta: xR \rightarrow K_F$ with $\theta(x) \neq 0$. Since $K_F$ is $F$-injective, the map $\theta$ is extended to the map $\theta': \sum \oplus K_F \rightarrow K_F$. Hence there exists a map $\varphi: \sum^n \oplus K_F \rightarrow K_F$ with $\varphi(x) \neq 0$. From Lemma 2.7 we have $\text{Hom}(\sum \oplus K_F, K_F)=\sum \oplus \hat{R}_F$ and so $\varphi=(\hat{r}_1, \ldots, \hat{r}_n)$ for some $\hat{r}_i \in \hat{R}_F$. Thus we get: $0 \neq \varphi(x)=\sum^n_{i=1} \hat{r}_i [(f_i)g]=\sum^n_{i=1} \hat{r}_i f_i g$ and so $\sum^n_{i=1} \hat{r}_i f_i g \neq 0$. On the other hand, $0=\varphi(w)=\sum^n_{i=1} \hat{r}_i f_i (m)$ for every $w=(f_1(m), \ldots, f_n(m))$, where $m \in M$. Hence $\sum^n_{i=1} \hat{r}_i f_i =0$, a contradiction.

Let $G$ be a left $\hat{R}_F$-module. We denote the right module $\text{Hom}_{\hat{R}_F}(G, K_F)$ by $G^*$, and define its finite topology by taking the submodules $\text{Ann}_{\hat{R}_F}(N)=\{f \in G^* | (N)f=0\}$ as a fundamental system of neighborhoods of zero, where $N$ ranges over all finitely generated $\hat{R}_F$-submodules of $G$. The following theorem was proved by I. Kaplansky [4] for modules over commutative, complete discrete valuation rings.

**Theorem 2.12.** Let $M$ be an $F$-linearly compact module. Then $M$ is isomorphic to $M^{**}$ as topological modules.

Proof. Let $\alpha$ be the canonical homomorphism from $M$ into $M^{**}$ which is


defined by $\alpha(m)(f)=f(m)$, where $m \in M$ and $f \in M^*$.  

(i) First we shall prove that $\alpha$ is a monomorphism. To prove this, we assume that $\alpha(m)=0$ and $0 \neq m \in M$. Then there exists an open submodule $N$ with $N \ni m$. Let $\bar{m}=[m+N]$ in $M/N$. Then $O(\bar{m}) \in F$ by Lemma 2.1. So we can define a homomorphism $f: \bar{m}R \rightarrow K_F$ with $f(\bar{m})=0$. This map can be extended to a homomorphism $g$ from $M/N$ into $K_F$. Let $h: M \rightarrow M/N$ be the natural homomorphism. Then $g \cdot h \in M^*$ and $(g \cdot h)(m)=0$. This implies that $\alpha(m)=0$, a contradiction, and so $\alpha$ is a monomorphism.

(ii) Secondly, we shall prove that $\alpha$ is an epimorphism. Let $x$ be any element of $M^*$. Then, for every $f \in M^*$, there exists an element $m_f \in M$ such that $(f)x=f(m_f)$ by Lemma 2.11. We consider the congruences

$$x \equiv m_f (\ker f).$$

Again, by Lemma 2.11, any finite number of congruences (1) have a solution. Further $\ker f$ is open and so it is closed. By $F$-linearly compactness of $M$, there exists a solution $m \in M$. Hence $(f)x=f(m_f)=f(m)$ for every $f \in M^*$ and so $x=\alpha(m)$.

(iii) Finally we shall prove that $\alpha$ is a topological isomorphism. Let $S$ be any submodule neighborhood of zero in the finite topology. Then $S=\Ann_{M^*}(f)$ and $S \subseteq \ker f$, where $f \in M^*$. It is evident that $S=\ker f \cap \cdots \cap \ker f_n$ is open in $M$ and so it is open in the original topology. Conversely, let $N$ be any open submodule in the original topology. Then $M/N$ is $F$-torsion and artinian. So $M/N$ can be embedded in an exact sequence $0 \rightarrow M/N \rightarrow \sum_i \oplus K_F$ with finite copies of $K_F$. Let $\pi_i: \sum_i \oplus K_F \rightarrow K_F$ be the projection $(1 \leq i \leq n)$ and let $\eta: M \rightarrow M/N$ be the natural map. Then we have $N=\cap_i \ker g_i$, where $g_i=\pi_i \cdot \eta \in M^*$ and so $N$ is open in the finite topology.

3. In case $F$ is bounded.

A topology $F$ is said to be bounded if, for every $I \subseteq F$, there is a nonzero ideal $A$ such that $I \subseteq A$. When $F$ is bounded, we shall determine, in this section, the algebraic and topological structures of $F$-linearly compact modules. Let $P$ be a prime ideal of $R$ and let $F_P=\{I|I \supseteq P^n\}$ for some $n$, $I$ is a right ideal of $R$. Then $F_P$ is a bounded atom in the lattice of all topologies. $F$-linearly compact modules is called $P$-linearly compact. Write $\hat{R}_P=\lim R/P^n$. Then it is evident that $\hat{R}_P,F_P=\hat{R}_P=\hat{R}_{(F_P)}$, as rings. It is well-known that $\hat{R}_P$ is a prime, principal ideal ring and that $\hat{P}=P\hat{R}=\hat{R}_P P$, where $\hat{P}$ is the unique maximal ideal of $\hat{R}_P$. In this section, we shall use the following notations: $Q_P=Q_{F_P}$; $K_P=K_{F_P}$; $R(P^n)=e\hat{R}_P/e\hat{P}^n$; $R(P^n)=e\hat{R}_P/e\hat{P}^n$; $R(P^n)=\lim e\hat{R}_P/e\hat{P}^n$, where $e$ is a uniform idempotent in $\hat{R}_P$. First we shall study $P$-linearly compact modules.
Lemma 3.1. $Q \otimes \hat{R}_p$ is the quotient ring of $\hat{R}_p$.

Proof. From the exact sequence $0 \to R \to \hat{R}_p \to \hat{R}_p/R \to 0$, we get the exact sequence: $0 = \text{Tor}(K_p, \hat{R}_p/R) \to K_p \to K_p \otimes \hat{R}_p \to K_p \otimes \hat{R}_p/R \to 0$, since $\hat{R}_p/R$ is $P$-divisible and has no $P$-primary submodules, and so $K_p \simeq K_p \otimes \hat{R}_p$. Hence we have the exact sequence $0 \to \hat{R}_p \to Q \otimes \hat{R}_p \to K_p \to 0$. Thus $Q \otimes \hat{R}_p$ is an essential extension of $\hat{R}_p$ as a right $\hat{R}_p$-module. Since $\hat{R}_p^n = \hat{R}_p^n \hat{R}_p = R^n$ and $\hat{R}_p$ is bounded, local, we obtain that $Q \otimes \hat{R}_p$ is divisible as an $\hat{R}_p$-module. Hence $Q \otimes \hat{R}_p$ is an $\hat{R}_p$-injective hull of $\hat{R}_p$. By Theorem of [2, p 69], it is the maximal quotient ring of $\hat{R}_p$ in the sense of [2] and so it is the quotient ring of $\hat{R}_p$.

For an $\hat{R}_p$-module $M$, we let $M^t = \text{Hom}_{\hat{R}_p}(M, K_p)$.

Lemma 3.2. (i) $R(P^n)^t \simeq R(P^n)^l$.
(ii) $R(P^n)^t \simeq R(P^n)^e$.
(iii) $(e\hat{R}_p)^t \simeq R(P^n)^l$.
(iv) $[e(Q \otimes \hat{R}_p)]^t \simeq (Q \otimes \hat{R}_p)^e$.

These modules are all $P$-linearly compact.

Proof. (i) is evident. (ii) $R(P^n)^t = \lim R(P^n)^t \simeq \lim R(P^n)^l \simeq \hat{R}_p e$.

(iii) $R(P^n)_l$ is $F_p$-torsion and artinian. Hence it is $P$-linearly compact and so $R(P^n)_l \simeq [R(P^n)_l]^t \simeq \lim R(P^n)_l^t \simeq \lim R(P^n)^t \simeq (e\hat{R}_p)^t$.

(iv) From the exact sequence $0 \to e\hat{R}_p \to e(Q \otimes \hat{R}_p) \to R(P^n) \to 0$, we get the exact sequence $0 \to \hat{R}_p e \to [e(Q \otimes \hat{R}_p)]^t \to R(P^n)^l \to 0$ as left $\hat{R}_p$-modules. Let $f$ be any element of $[e(Q \otimes \hat{R}_p)]^t$. Assume that $P^nf = 0$ for some $n$. Then $P^nf(e(Q \otimes \hat{R}_p)) = 0$ implies that $0 \simeq f(e(Q \otimes \hat{R}_p)) = f(e(Q \otimes \hat{R}_p)) = 0$. Hence $[e(Q \otimes \hat{R}_p)]^t$ is torsion-free as a left $\hat{R}_p$-module. Thus $[e(Q \otimes \hat{R}_p)]^t$ is an essential extension of $\hat{R}_p e$. Hence we may assume that $\hat{R}_p e \subseteq [e(Q \otimes \hat{R}_p)]^t \subseteq (Q \otimes \hat{R}_p)^e$. From Lemma 3.2 of [6], we easily obtain that $[e(Q \otimes \hat{R}_p)]^t = (Q \otimes \hat{R}_p)^e$.

By Lemma 2.1, $R(P^n)$ and $R(P^n)$ are $P$-linearly compact in the discrete topology. By Lemma 2.4 and Corollary 2.9, $e\hat{R}_p$ is $P$-linearly compact in the $P$-adic topology. $e(Q \otimes \hat{R}_p)$ is a topological module by taking as neighborhoods of zero the submodules $\{e\hat{P}^n | n = 0, \pm 1, \pm 2, \cdots \}$. Further the exact sequence $0 \to e\hat{R}_p \to e(Q \otimes \hat{R}_p) \to R(P^n) \to 0$ satisfies the assumption of Proposition 9 of [13] and so $e(Q \otimes \hat{R}_p)$ is $P$-linearly compact in the above topology.

Lemma 3.3. Let $0 \to L \to M \to N \to 0$ be an exact sequence of $\hat{R}_p$-modules. If the sequence is $P^u$-pure in the sense of [7], then the exact sequence $0 \to N^t \to M^t \to L^t \to 0$ is also $P^u$-pure.

Proof. Since $\hat{R}_p$ is a principal ideal ring, the proof of the lemma is similar to the one of Proposition 4.1 of [3] (see, also Lemma 1.1 of [7]).
**Theorem 3.4.** (i) A module is $P$-linearly compact if and only if it is isomorphic, as topological modules to a direct product of modules of the following types: $R(P^n)$, $R(P^\omega)$, $e\hat{R}_P$, $e(Q\otimes\hat{R}_P)$, where $e$ is a uniform idempotents in $\hat{R}_P$ and the topologies of these modules are defined in the proof of Lemma 3.2.

(ii) A module $M$ is $P$-linearly compact, then $M^*$ is isomorphic to a direct sum of modules of the following types: $R(P^n)$, $R(P^\omega)$, $\hat{R}_Pe$, $(Q\otimes\hat{R}_Pe)$. where $e$ is a uniform idempotent in $\hat{R}_P$.

Proof. (i) Since each of these modules does admit a $P$-linearly compact topology, the sufficiency is evident from Proposition 1 of [13]. Conversely, let $M$ be $P$-linearly compact. Then $M^*$ is a left $\hat{R}_P$-module and $\hat{R}_P$ is a complete $g$-discrete valuation ring in the sense of [6] (cf. p. 432 of [6]). So $M^*$ possesses a basic submodule $B$ by Theorem 3.6 of [6]. Further any finitely generated module and any injective module over a Dedekind prime ring are both a direct sum of indecomposable modules. Hence, from the definition of basic submodules, Corollary 4.4 of [6] and Lemma 3.1 we have $B=\sum_i\bigoplus R(P^n)i\bigoplus \hat{R}_Pe$ and $M^*B=\sum_iR(P^n)i\bigoplus (Q\otimes\hat{R}_Pe)$. By Theorem 1.5 of [7] and Lemmas 3.2, 3.3, the exact sequence $0\rightarrow (M^*/B)^*\rightarrow M^*/B^*\rightarrow 0$ splits and so, from Theorem 2.12 and Lemma 3.2, we get:

$$M \cong \prod_i R(P^n)\bigoplus \prod_i R(P^\omega) \bigoplus \prod_i \hat{R}_P \bigoplus \prod_i (Q\otimes\hat{R}_P).$$

The right sided module is $P$-linearly compact and so, by Corollary 2.3, $\varphi$ is an isomorphism as topological modules.

Since the topology of the left sided of (1) is the product topology, (ii) follows easily from Lemma 3.2.

From Theorem 1.5 of [7], Theorem 3.4 and definitions, we have the following chain of implications;

$(P^\omega$-pure injective) $\Rightarrow$ $(P^\omega$-pure injective) $\Rightarrow (P$-linearly compact).

Let $F$ be a bounded topology and let $M$ be $F$-linearly compact. Then we know from Lemma 2.4 that $M=\lim M_i$, where $M_i$ is $F$-torsion and artinian. By the same way as in Theorem 3.2 of [5], we have $M_i=\sum_i\bigoplus M_{ip}$, where $M_{ip}=\{x\in M_i | xP^n=0 \text{ for some } n\}$ and $P$ ranges over all prime ideals contained in $F$. Write $M_p=\lim M_{ip}$. Then $M_p$ is $P$-linearly compact and $M$ is isomorphic naturally to $\prod M_p$ as topological modules, where $\prod M_p$ will carry the product topology. It is evident that $K_F=\sum_i K_{ip}$, where $P$ ranges over all prime ideals in $F$. Further we can easily prove that $M^*=\sum_i M^*_p$ and that $M^{**}=\prod M^{**}_p$, where $M^*_p$ consists of all continuous maps of $M_p$ into $K_p$. Thus, from Theorem 3.4, we have
Theorem 3.5. Let $F$ be a bounded topology. Then

(i) A module is $F$-linearly compact if and only if it is isomorphic as topological modules to a direct product of modules of the following types: $R(P^n)$, $R(P^\infty)$, $e_P\hat{R}_P$, $e_P(Q\otimes\hat{R}_P)$, where $P$ ranges over all prime ideals in $F$ and $e_P$ is a uniform idempotent in $\hat{R}_P$.

(ii) If $M$ is $F$-linearly compact, then $M^*$ is isomorphic to a direct sum of modules of the following types: $R(P^n)$, $R(P^\infty)$, $\hat{R}_Pe_P$, $(Q\otimes\hat{R}_P)e_P$.

Let $F$ be any topology. A short exact sequence

$$0 \to L \to M \to N \to 0$$

is said to be $F^\omega$-pure if $MJ \cap L = LJ$ for every $J \in F$, and $(E)$ is said to be $F^\omega$-pure if the induced sequence $0 \to L_F \to M_F \to N_F \to 0$ is splitting exact. A module is called $F^\omega(F^\omega)$-pure injective if it has the injective property relative to the class of $F^\omega(F^\omega)$-pure exact sequences. The structure of $F^\omega$-pure injective modules is investigated in the forthcoming paper.

Lemma 3.6. Let $F$ be a bounded topology. Then $(E)$ is $F^\omega$-pure if and only if $(E)$ is $P^\omega$-pure for every prime ideal $P \in F$.

Proof. For any prime ideal $P$, it is clear that $P \in F$ if and only if $P \in F_1$. So the necessity is evident. Conversely assume that $(E)$ is $P^\omega$-pure for $P \in F$. Let $J$ be any element of $F_1$. Then there is a nonzero ideal $A$ with $J \supseteq A$. Write $A = P_1^P \cdots P_n^P$, where $P_i$ are prime ideals. Then $P_i \in F$ and $X/XA \cong X/XP_1^P \oplus \cdots \oplus X/XP_n^P$ for every module $X$. Hence by Lemma 1.1 of [7] the sequence $0 \to L/LA \to M/MA \to N/NA \to 0$ is splitting exact. Hence $MJ \cap L = LJ$ and so $(E)$ is $F^\omega$-pure.

From the same ways as (1.2), (1.4), (1.5) of [7] and Lemma 3.6 we have

Proposition 3.7. Let $F$ be a bounded topology. Then a module $G$ is $F^\omega$-pure injective if and only if it is isomorphic to the module $E(GF^\omega) \oplus \prod P \hat{G}_P$, where $P$ ranges over all prime ideals in $F$, $GF^\omega = \bigcap G(J \in F_1)$ and $\hat{G}_P = \lim \to G/PG^\omega$.

Let $F$ be a bounded topology. Then from Theorem 3.5, Proposition 3.7 and definitions, we get the following chain of implications:

$(F$-linearly compact) $\Rightarrow$ $(F^\omega$-pure injective) $\Rightarrow$ $(F^\omega$-pure injective).

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References