THE GENERIC FINITENESS OF THE $m$-CANONICAL MAP
FOR 3-FOLDS OF GENERAL TYPE

LEI ZHU

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Abstract

Let $X$ be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. We study the generic finiteness of the $m$-canonical map for such 3-folds. Suppose $P_k(X) \geq 2$ and $q(X) \geq 3$. We prove that the $m$-canonical map is generically finite for $m \geq 3$, which is a supplement to Kollár’s result. Suppose $P_3(X) \geq 5$. We prove that the 3-canonical map is generically finite, which improves Meng Chen’s result.

0. Introduction

Throughout the ground field is always supposed to be algebraically closed of characteristic zero. Let $X$ be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. For all integer $m > 0$, one may define the so-called $m$-canonical map $\phi_m$, which is nothing but the rational map corresponding to the complete linear system $|mK_X|$. Many authors have studied the generic finiteness of $\phi_m$ in quite different ways.

In 1986, J. Kollár presented the following theorem in his paper.

**Theorem 0.1** (Theorem (6.2) of [6]). Let $X$ be a smooth projective 3-fold of general type with $q(X) \geq 4$. Then $\phi_k$ is generically finite for $k \geq 3$.

Meanwhile, he pointed out that the bound is the best possible. During our study of generic finiteness of $m$-canonical map for threefolds, we find we get a better bound if we suppose $P_3(X) \geq 2$. We also improve a result of Meng Chen.

**Theorem 0.2** (Theorem 3.9 of [1]). Let $X$ be a projective minimal Gorenstein threefold of general type. Then $\phi_3$ is generically finite whenever $P_3(X) \geq 39$.

The following is our main theorem.

**Main Theorem.** Let $X$ be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $\phi_m$ is not generically finite whenever $m \geq 3$, then
(1) $P_g(X) \leq 1$ whenever $m \geq 6$;
(2) either $P_g(X) \leq 1$ or $q(X) \leq 2$ if $m = 3$ or $4$;
(3) either $P_g(X) \leq 1$ or $q(X) \leq 1$ if $m = 5$.

As a direct corollary, the following is a supplement to Kollár’s result.

**Corollary 1.** Let $X$ be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $P_g(X) \geq 2$ and $q(X) \geq 3$, then $\phi_m$ is generically finite whenever $m \geq 3$.

Corollary 2 improves Theorem 0.2.

**Corollary 2.** Let $X$ be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. $\phi_3$ is generically finite whenever $P_g(X) \geq 5$ or $P_g(X) = 4$ and $q(X) \geq 2$.

As an application of our method, we will present more detailed results of the $m$-canonical map for 3-folds of general type.

1. Preliminaries

**(1.1) Kawamata-Ramanujam-Viehweg vanishing theorem.** We always use the vanishing theorem in the following form.

**Vanishing Theorem** ([7] or [9]). Let $X$ be a smooth complete variety, $D \in \text{Div}(X) \otimes \mathbb{Q}$. Assume the following two conditions:
(i) $D$ is nef and big;
(ii) the fractional part of $D$ has supports with only normal crossings. Then $H^i(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0$ for all $i > 0$.

Most of our notations are standard within algebraic geometry except the following which we are in favor of: $\sim_{\text{lin}}$ means linear equivalence while $\sim_{\text{num}}$ means numerical equivalence and $\equiv_{\mathbb{Q}}$ means $\mathbb{Q}$-numerical equivalence.

**(1.2) Set up for canonical maps.** Let $X$ be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Suppose $P_g(X) \geq 2$, we study the canonical map $\phi_3$ which is usually a rational map. Take the birational modification $\pi: X' \to X$, according to Hironaka [5], such that
(1) $X'$ is smooth;
(2) $|K_X|$ defines a morphism;
(3) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings. Denote by $g$ the composition of $\phi_3 \circ \pi$. So $g: X' \to B \subseteq \mathbb{P}^{P_g(X) - 1}$ is a morphism. Let $g: X' \xrightarrow{f} B' \xrightarrow{s} B$ be the Stein factorization of $g$. We can write $K_{X'} \sim_{\text{lin}} \pi^*(K_X) + E$
and $K_{X'} \sim_{\text{lin}} M_1 + Z_1$, where $M_1$ is the movable part of $|K_{X'}|$. $E$ is an effective $\mathbb{Q}$-divisor which is a $\mathbb{Q}$-linear combination of distinct exceptional divisors. We can also write $\pi^*(K_X) \sim_{\text{lin}} M_1 + E'$, where $E' = Z_1 - E$ is actually an effective $\mathbb{Q}$-divisor.

If $\dim \phi_1(X) = 2$, we see that a general fiber of $f$ is a smooth projective curve of genus $g \geq 2$. We say $X$ is canonically fibered by curves of genus $g$.

If $\dim \phi_1(X) = 1$, we see that a general fiber $S$ of $f$ is a smooth projective surface of general type. We say that $X$ is canonically fibered by surfaces with invariants $(c_1^2, P_g) = (K_{S_0}^2, P_g(S))$. Denote by $\sigma: S \to S_0$ to be the contraction onto the minimal model.

2. Proof of Main Theorem

Let the notation be as in (1.2) throughout this section. We study $\phi_m$ according to the value $d := \dim \phi_1(X)$ and $b := g(B)$. Obviously $1 \leq d \leq 3$.

**Theorem 2.1.** Let $X$ be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $\phi_m$ is not generically finite whenever $m \geq 6$, then $P_g(X) \leq 1$.

Proof. We suppose $P_g(X) \geq 2$ and try to prove $\phi_m$ is generically finite for all integers $m \geq 6$.

The case $d = 2$. Denote by $S_1$ the general member of $|M_1|$. So $S_1$ is a smooth projective surface of general type. We have

$$|K_{X'} + \gamma(m - 2)\pi^*(K_X)^\gamma + S_1| \subseteq |mK_{X'}|.$$

Using (1.1), we have

$$|K_{X'} + \gamma(m - 2)\pi^*(K_X)^\gamma + S_1|_{S_1} \supseteq |K_{S_1} + \gamma(m - 2)L|$$

where $L := \pi^*(K_X)|_{S_1}$. According to [10], we can reduce to the problem on surface $S_1$ since

$$K_{X'} + \gamma(m - 2)\pi^*(K_X)^\gamma$$

is effective. Since $d = 2$, we have $h^0((m - 2)L) \geq 3$. We know $|K_{S_1} + \gamma(m - 2)L|$ gives a generically finite map by [2]. So does $\phi_m$.

The case $d = 1$ and $b > 0$. Because $b > 0$, the movable part of $|K_X|$ is already base point free on $X$ and $M_1 \sim_{\text{num}} aS$ with $a \geq 2$. So one always has $\pi^*(K_X)|_S = \sigma^*(K_{S_0})$. Obviously we have

$$|K_{X'} + \gamma(m - 2)\pi^*(K_X)^\gamma + M_1| \subseteq |mK_{X'}|$$
and
\[ |K_{X'} + \gamma(m - 2)\pi^*(K_X)\gamma + M_1|_S = |K_S + \gamma(m - 2)L\gamma|_S \]
where \( L' = \pi^*(K_X) \) by (1.1). According [8], we can reduce to the system \(|K_S + \gamma(m - 2)L\gamma|_S \) on \( S \) since
\[ K_{X'} + \gamma(m - 2)\pi^*(K_X)\gamma \]
is effective and \( a \geq 2 \). While
\[ |K_S + \gamma(m - 2)L\gamma|_S \supseteq |K_S + \gamma(m - 2)L'\gamma|_S \]
by Lemma 2.2 in [3], we see
\[ |K_S + \gamma(m - 2)L\gamma|_S = |K_S + (m - 2)\sigma^*(K_S)|_S \].
The right system defines a generically finite map on \( S \) by [12]. So does \( \phi_m \).
The case \( d = 1 \) and \( b = 0 \). According to [6], we have
\[ \mathcal{O}(1) \leftrightarrow f_*\omega_{X'}^2 \]
and denote by
\[ \varepsilon := f_*\omega_{X'/\mathbb{P}^1}^2 \leftrightarrow f_*\omega_{X'}^b. \]
The local sections of \( f_*\omega_{X'/\mathbb{P}^1}^2 \) give the bicanonical map of the fiber \( S \) and they extend to global sections of \( \varepsilon \) because \( \varepsilon \) is generated by global sections. On the other hand, \( H^0(\mathbb{P}^1, \varepsilon) \) can distinguish different fibers of \( f \) because \( \text{deg}(\varepsilon) > 0 \). So \( H^0(\mathbb{P}^1, \varepsilon) \) gives a generically finite map on \( X' \) and so does \( \phi_b \), which means \( \phi_m \) is generically finite whenever \( m \geq 6 \).

**Theorem 2.2.** Let \( X \) be a projective minimal threefold of general type with only \( \mathbb{Q} \)-factorial terminal singularities. If \( \phi_m \) is not generically finite where \( m = 3 \) or \( 4 \), then either \( P_g(X) \leq 1 \) or \( q(X) \leq 2 \).

**Proof.** We assume \( P_g(X) \geq 2 \). Since \( |3K_{X'}| \subseteq |4K_{X'}| \), we study \( \phi_3 \) according to \( d \) and \( b \).

The case \( d = 2 \). Choose a 1-dimensional subsystem \( \Lambda \subseteq |K_{X'}| \) while taking a birational modification \( \pi_1 : X' \to X \) such that the pencil \( \Lambda \) defines a morphism \( g_1 : X' \to \mathbb{P}^1 \). We can even take further modification to \( \pi_1 \) so that \( \pi_1^*(K_X) \) has supports with only normal crossings. Taking the stein factorization \( p : X' \to W' \). We note that this fibration is different from the one which was defined in (1.2). Denote \( b_1 := g(W') \). Let \( M \) be the movable part of the pencil. We obviously have \( M \leq K_{X'} \).

We can write \( M \sim_{\text{lin}} \sum_{i=1}^a F_i \) where \( a \geq 1 \) and \( F_i \) is a fiber of \( p \) for all \( i \).
Suppose $b_1 > 0$. We consider the system

$$[2K_X + M] \subseteq [3K_X].$$

Now $M$ contains at least 2 components $F_1$ and $F_2$. By (1.1), we have a surjective map

$$H^0(X', K_X + M) \to H^0(F_1, 2K_{F_1}) \oplus H^0(F_2, 2K_{F_2}).$$

This means $\phi_{[2K_X + M]}$ can distinguish $F_1$ and $F_2$ and the restriction to $F_i$ is at least a bicanonical map. Then $\phi_3$ is generically finite.

Suppose $b_1 = 0$. Now $M \sim_{\text{lin}} F$. Still by (1.1) and since $b_1 = 0$, we consider the following system

$$[K_F + \pi^*(K_X)]_F.$$

Assume $P_g(F) \geq 2$ and $[K_F]$ is composed of pencils otherwise $q(F) \leq 1$ or $\phi_3$ is generically finite. If $q(F) \leq 1$, then $q(X) \leq 1$ by virtue of Corollary 2.3 in [4]. If $[K_F]$ is composed of pencils, then $q(F) \leq 2$ according to [11]. So $q(X) \leq 2$.

The case $d = 1$ and $b > 0$. Now we have

$$[K_X + \pi^*(K_X) + M] \subseteq [3K_X].$$

One can replace $m$ with 3 in the corresponding proof of Theorem 2.1 and derive that $\phi_3$ is generically finite.

The case $d = 1$ and $b = 0$. In this case we have

$$\pi^*(K_X) = aS + E'$$

where $a = P_g(X) - 1 \geq 1$ and

$$\left| K_X + \left( 2\pi^*(K_X) - \frac{E'}{a} \right)_S \right| = \left| K_S + \left( 2 - \frac{1}{a} \right)\pi^*(K_X) \right|_S.$$

If $\phi_3$ is not generically finite, nor is the map defined by the right system above. We suppose $P_g(S) \geq 2$. Then $[K_S]$ is composed of pencils and $q(S) \leq 2$ according to [11]. Thus $q(X) \leq 2$ by [4].

**Theorem 2.3.** Let $X$ be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $\phi_5$ is not generically finite, then either $P_g(X) \leq 1$ or $q(X) \leq 1$.

Proof. We suppose $P_g(X) \geq 2$.

The case $d = 2$. One can replace $m$ with 5 in the corresponding proof of Theorem 2.1 and derive that $\phi_5$ is generically finite.
The case $d = 1$ and $b > 0$. The proof of Theorem 2.2 implies that $\phi_S$ is generically finite since $|3K_X| \subseteq |5K_X|$. 

The case $d = 1$ and $b = 0$. We can write $\pi^*(K_X) = aS + E'$ where $a = P_g(X) - 1$ and

$$\left| K_{X'} + 4\pi^*(K_X) - \frac{E'}{a} \right| \subseteq |5K_{X'}|.$$

For the same reason, we consider the system

$$\left| K_S + \left(4 - \frac{1}{a}\right)\pi^*(K_X) \right|_S = \left| K_{X'} + 4\pi^*(K_X) - \frac{E'}{a} \right|_S$$

on surface $S$. If $P_g(X) \geq 3$, then we have

$$\mathcal{O}(2) \hookrightarrow f_*\omega_{X'}$$

and

$$f_*\omega_{X'/P^1}^2 \hookrightarrow f_*\omega_{X'}^4.$$ 

Thus $\phi_t$ is generically finite for the same reason given in the proof of Theorem 2.1. So is $\phi_S$.

Next we suppose $P_g(X) = 2$ and then $a = 1$. By [6],

$$\mathcal{O}(1) \hookrightarrow f_*\omega_{X'}$$

and

$$f_*\omega_{X'/P^1}^2 \hookrightarrow f_*\omega_{X'}^6.$$ 

We assume $P_g(S) \geq 2$ and then denote by $G$ the movable part of $|\sigma^*(K_S)|$, we have $6\pi^*(K_X)|_S \geq 2G$ since $|2\sigma^*(K_S)|$ is base-point-free for $P_g(S) > 0$. Furthermore, we suppose $|K_S|$ is composed of pencils otherwise $\phi_S$ is generically finite. Then we can write

$$\sigma^*(K_S) \sim_{\text{num}} bC + Z$$

where $C$ is a general fiber of the canonical map of $S$ and $b \geq P_g(S) - 1 \geq 1$. If $|K_S|$ is composed of irrational pencils, then $b \geq P_g(S) \geq 2$. Denote by $M'$ the movable part of

$$|7K_{X'} + S| \supseteq |K_{X'} + 6\pi^*(K_X) + S|.$$ 

Thus we have $M'|_S \geq 3G$ by Lemma 2.7 in [3]. Now we consider the subsystem

$$|K_{X'} + (7K_{X'} + S) + S| \subseteq |10K_X|.$$
From Theorem 2.1, we know $\phi_7$ is generically finite. Then $M'$ is nef and big. By (1.1), we have surjective map

$$H^0(X', K_{X'} + M' + S) \to H^0(S, K_S + M'|_S).$$

Then we see that $M_{10}|_S \geq 4G$ and thus $10\pi^*(K_X)|_S \geq 4G$. Pick up a general member $C$ of $|G|$. Then we can write

$$3\pi^*(K_X)|_S - C - H \sim_{num} \frac{1}{2}\pi^*(K_X)|_S,$$

where $H$ is an effective divisor or zero. Since

$$|K_S + (3\pi^*(K_X))|_S| \supseteq |K_S + (3\pi^*(K_X))|_S - H|,$$

by (1.1) we have a surjective map

$$H^0(S, K_S + (3\pi^*(K_X))|_S - H) \to H^0(K_C + D)$$

where

$$D := (3\pi^*(K_X))|_S - H - C|_C.$$

Whether $|K_S|$ is composed of rational pencils or irrational pencils, we can reduce to the curve $C$. Since $C$ is nef on $S$, $\deg D > 0$. Thus $|K_C + D|$ gives a finite map and $\phi_S$ is generically finite. Then we can derive that if $\phi_S$ is not generically finite then $P_g(S) \leq 1$ and thus $q(S) \leq 1$. By virtue of Corollary 2.3 in [4], we have $q(X) \leq 1$. So we are done.

### 3. Generic finiteness of $\phi_m$

In this section we will keep the same notation as in (1.2) and denote $d := \dim_0(X)$ and $b := g(B)$.

**Corollary 3.1.** Let $X$ a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $P_g(X) \geq 2$, then $\phi_m$ is generically finite for all integers $m \geq 6$.

**Proof.** This is a direct result from Theorem 2.1.

**Corollary 3.2.** Let $X$ be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $P_g(X) \geq 3$. Then $\phi_m$ is generically finite when $m = 4$ or 5.
Proof. The proof of Theorem 2.3 implies the case \( m = 5 \). As for \( m = 4 \)
the case \( d = 2 \). One can still derive it from the proof of Theorem 2.1;
The case \( d = 1 \) and \( b > 0 \). The proof of Theorem 2.1 also implies \( \phi_4 \) is generically
finite as long as replacing \( m \) with 4 there.
The case \( d = 1 \) and \( b = 0 \). From proof of Theorem 2.3, we know this corollary is
true. \( \square \)

In the following, we will study \( \phi_3 \) and then present several probabilities.

**Corollary 3.3.** Let \( X \) be a projective minimal threefold of general type with only
\( \mathbb{Q} \)-factorial terminal singularities. Assume \( P_g(X) \geq 5 \). Then \( \phi_3 \) is generically finite.

Proof. The case \( d = 2 \). Denote by \( S_1 \) the general member of \( |M_1| \) where \( M_1 \) is
the movable part of \( |K_X| \). We have
\[
|K_X + \pi^*(K_X) + S_1| \subseteq |3K_X|
\]
and
\[
|K_X + \pi^*(K_X) + S_1|_{S_1} = |K_{S_1} + L|
\]
where \( L := \pi^*(K_X)|_{S_1} \). Since \( K_X + \pi^*(K_X) \) is effective, we can reduce to the
problem on the surface \( S_1 \) by [10]. Obviously \( h^0(L) \geq P_g(X) - 1 \geq 4 \). Then \( |K_{S_1} + L| \)
gives a generically finite map by [2], so does \( \phi_3 \).

The case \( d = 1 \) and \( b > 0 \). The proof of Theorem 2.2 implies this is true.
The case \( d = 1 \) and \( b = 0 \). Then
\[
\mathcal{O}(4) \hookrightarrow f_6\omega_{X'}
\]
and
\[
f_6\omega_{X'/\mathbb{P}^1}^{2} \hookrightarrow f_6\omega_{X'}^{3}.
\]
Thus \( \phi_3 \) is generically finite for the same reason given in the proof of Theorem 2.1. \( \square \)

**Corollary 3.4.** Let \( X \) be a projective minimal threefold of general type with only
\( \mathbb{Q} \)-factorial terminal singularities. Assume \( P_g(X) = 4 \) and \( d = 2 \). Then \( \phi_3 \) is generically
finite.

Proof. One can easily derive it from above since \( h^0(L) \geq 3 \) in this case. \( \square \)

**Corollary 3.5.** Let \( X \) be a projective minimal threefold of general type with only
\( \mathbb{Q} \)-factorial terminal singularities. Assume \( P_g(X) \geq 2 \) and \( d = 1 \) and \( b > 0 \). Then \( \phi_3 \)
is generically finite.
Proof. This is just one part of the proof of Theorem 2.2. \qed

**Corollary 3.6.** Let $X$ be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $P_g(X) = 3$ and $d = 2$. Then $\phi_3$ is generically finite except $q(S_1) = 1$ or 2 and $[L]$ is composed of a rational pencil of genus $g = q(S_1) + 1$ where $S_1$ is the general member of $|M_1|$ and $L := [\pi^*(K_X)]_{S_1}$.

Proof. We only need to consider the system $[K_S + L]$. One can easily derive this result from Proposition 2.1 and 2.2 in [2]. \qed

**Proposition 3.7.** Let $X$ be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $P_g(X) = 4$ and $d = 1$ and $b = 0$. Then $\phi_3$ is generically finite if $P_g(S) \geq 2$.

Proof. One can easily see that we only need to study the system

$$
\left| K_S + \left[ \frac{5}{3} \pi^*(K_X) \right]_S \right| = \left| K_{S'} + \left[ \frac{5}{3} \pi^*(K_X) - \frac{E'}{3} \right]_S \right|
$$

since

$$\pi^*(K_X) = 3S + E'.
$$

Because

$$\mathcal{O}(S) \leftrightarrow f_S \omega_{S'},$$

we have

$$f_S \omega^{S'}_{X'/p_1} \leftrightarrow f_S \omega^{S'}_{S'}.
$$

Suppose $P_g(S) \geq 2$ and denote by $G$ the movable part of $|\sigma^*(K_S)|$. Then we have $5\pi^*(K_X)|_S \geq 3G$ since $\beta \sigma^*(K_S)|_S$ is base point free. Denote by $\overline{M}$ the movable part of

$$|6K_{S'} + S| \supseteq |K_{S'} + \gamma 5\pi^*(K_X) + S|.
$$

We know $\overline{M}|_S \geq 4G$ from Lemma 2.7 in [3]. Denote by $\overline{M}$ the movable part of $|2(6K_{S'} + S)|$. Now we consider the subsystem

$$|K_{S'} + 2(6K_{S'} + S) + S| \subseteq |14K_{S'}|.
$$

Since $\phi_{12}$ is generically finite, $\overline{M}$ is nef and big. By (1.1), we have a surjective map

$$H^0 \left( X', K_{S'} + \overline{M} + S \right) \rightarrow H^0 \left( S, K_S + \overline{M} \right)_{S'}.$$
Obviously we have $\overline{M}_S \geq 2\overline{M}_S$. So $M_{14}|_S \geq 9G$ by Lemma 2.7 in [3]. Thus $14\pi^*(K_X)|_S \geq 9G$. Then we can write

$$\frac{5}{3} \pi^*(K_X)|_S - G - H \sim_{\text{num}} \frac{1}{3} \pi^*(K_X)|_S$$

where $H$ is an effective divisor or zero. Pick up a general member $C$ of $[G]$. Then we have a surjective map

$$H^0 \left( S, K_S + \frac{\gamma}{3} \pi^*(K_X) \right) \to H^0(C, K_C + D)$$

by (1.1) where $D := (\gamma(5/3)\pi^*(K_X)|_S - C - H)|_C$. Since $C$ is nef on $S$, $|K_C + D|$ gives a finite map. Thus $\phi_3$ is generically finite. 

\begin{proposition}
Let $X$ be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $P_s(X) = 3$ and $d = 1$ and $b = 0$. Then $\phi_S$ is generically finite when $P_s(S) \geq 3$.
\end{proposition}

Proof. In this case, we have

$$\pi^*(K_X) = 2S + E'.$$

Then one can reduce to the system $[K_S + \gamma(3/2)\pi^*(K_X)|_S]$ since

$$\left| K_S + \frac{\gamma}{2} \pi^*(K_X) \right|_S = \left| K_{S'} + \frac{\gamma}{2} \pi^*(K_X) - \frac{E'}{2} \right|_S$$

by (1.1).

If $[K_S]|_S$ is not composed of pencils, then $\phi_3$ is generically finite.

If $[K_S]|_S$ is composed of pencils, then we can write $K_S \sim_{\text{num}} bC + Z''$ where $b \geq P_s(S) - 1 \geq 2$. Since

$$\mathcal{O}(2) \hookrightarrow f_*\omega_{S'},$$

and

$$f_*\omega_{S'/\mathbb{P}^1}^3 \hookrightarrow f_*\omega_{S'}^6,$$

we have $6\pi^*(K_X)|_S \geq 3G$ where $G$ is the movable part of $|\pi^*(K_S)|$. By Lemma 2.7 in [3] and (1.1) considering the system $|K_{S'} + \gamma 6\pi^*(K_X) + S|$, we have $M'|_S \geq 4G$ where $M'$ is the movable part of $|7K_{S'} + S|$. Then considering the subsystem

$$|K_{S'} + (7K_{S'} + S) + S| \subseteq |9K_{S'}|,$$
by (1.1), we have a surjective map

$$H^0(X', K_{X'} + M' + S) \rightarrow H^0(S, K_S + M'|_S).$$

Denote by $M''$ the movable part of $[K_{X'} + (7K_{X'} + S)]$. Then $M''|_S \geq 5G$. So

$$9\pi^*(K_X)|_S \geq 5G.$$ Then we can write

$$\frac{3}{2}\pi^*(K_X)|_S - C - H \sim_{\text{num}} \frac{3}{5}\pi^*(K_X)|_S$$

where $H$ is an effective divisor or zero. Thus we can reduce to the problem on the smooth curve $C$ of $g \geq 2$. Then we are done.

**Proposition 3.9.** Let $X$ be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $P_g(X) = 2$ and $d = 1$ and $b = 0$. Then $\phi_3$ is generically finite when $P_g(S) \geq 4$.

Proof. We can write $\pi^*(K_X) = Q S + E'$ and reduce to the problem on the system $[K_S + \pi^*(K_X)]|_S$ on surface $S$.

If $|K_S|$ is not composed of pencils, then we are done.

If $|K_S|$ is composed of pencils, we can write $\sigma^*(K_S) \sim_{\text{num}} bC + Z''$ where $b \geq P_g(S) - 1 \geq 3$. Now

$$O(1) \leftrightarrow f_s\omega_{X'}$$

and

$$f_s\omega_{X'/P}^2 \leftrightarrow f_s\omega_{X'}^5.$$ Then we see that $M'|_S \geq 3G$ where $M'$ is the movable part of $[7K_{X'} + S]$ and $G$ the movable part of $\sigma^*(K_S)$. Then consider the subsystem

$$[K_{X'} + (7K_{X'} + S) + S] \subseteq [10K_{X'}].$$

Denote by $M''$ the movable part of the left system above. By (1.1) we have a surjective map

$$H^0(X', K_{X'} + M' + S) \rightarrow H^0(S, K_S + M'|_S)$$

and then $M''|_S \geq 4G$. Thus $10\pi^*(K_X)|_S \geq 4G$. We can write

$$\pi^*(K_X)|_S - C - H \sim_{\text{num}} \frac{1}{6}\pi^*(K_X)|_S$$

where $H$ is an effective divisor or zero. Then we can consider the system $[K_C + D]$ on curve $C$ where $D \sim_{\text{num}} (\frac{1}{6}\pi^*(K_X)|_S)|_C$. So we are done.
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References


Department of Applied Mathematics
Tongji University
Shanghai 200092, P.R. China
e-mail: lzhued@hotmail.com

Current address:
Doctor 05 Grade One
Institute of Mathematics
Fudan University
Shanghai 200433, P.R. China
e-mail: 051018003@fudan.edu.cn