

Title	The generic finiteness of the $m$ -canonical map for 3-folds of general type
Author(s)	Zhu, Lei
Citation	Osaka Journal of Mathematics. 42(4) P.873-P.884
Issue Date	2005-12
Text Version	publisher
URL	<a href="https://doi.org/10.18910/3508">https://doi.org/10.18910/3508</a>
DOI	10.18910/3508
rights	

*Osaka University Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/repo/ouka/all/>

## THE GENERIC FINITENESS OF THE $m$ -CANONICAL MAP FOR 3-FOLDS OF GENERAL TYPE

LEI ZHU

(Received June 4, 2004)

### Abstract

Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. We study the generic finiteness of the  $m$ -canonical map for such 3-folds. Suppose  $P_g(X) \geq 2$  and  $q(X) \geq 3$ . We prove that the  $m$ -canonical map is generically finite for  $m \geq 3$ , which is a supplement to Kollár's result. Suppose  $P_g(X) \geq 5$ . We prove that the 3-canonical map is generically finite, which improves Meng Chen's result.

### 0. Introduction

Throughout the ground field is always supposed to be algebraically closed of characteristic zero. Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. For all integer  $m > 0$ , one may define the so-called  $m$ -canonical map  $\phi_m$ , which is nothing but the rational map corresponding to the complete linear system  $|mK_X|$ . Many authors have studied the generic finiteness of  $\phi_m$  in quite different ways.

In 1986, J. Kollár presented the following theorem in his paper.

**Theorem 0.1** (Theorem (6.2) of [6]). *Let  $X$  be a smooth projective 3-fold of general type with  $q(X) \geq 4$ . Then  $\phi_k$  is generically finite for  $k \geq 3$ .*

Meanwhile, he pointed out that the bound is the best possible. During our study of generic finiteness of  $m$ -canonical map for threefolds, we find we get a better bound if we suppose  $P_g(X) \geq 2$ . We also improve a result of Meng Chen.

**Theorem 0.2** (Theorem 3.9 of [1]). *Let  $X$  be a projective minimal Gorenstein threefold of general type. Then  $\phi_3$  is generically finite whenever  $P_g(X) \geq 39$ .*

The following is our main theorem.

**Main Theorem.** *Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. If  $\phi_m$  is not generically finite whenever  $m \geq 3$ , then*

- (1)  $P_g(X) \leq 1$  whenever  $m \geq 6$ ;
- (2) either  $P_g(X) \leq 1$  or  $q(X) \leq 2$  if  $m = 3$  or  $4$ ;
- (3) either  $P_g(X) \leq 1$  or  $q(X) \leq 1$  if  $m = 5$ .

As a direct corollary, the following is a supplement to Kollár's result.

**Corollary 1.** Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. If  $P_g(X) \geq 2$  and  $q(X) \geq 3$ , then  $\phi_m$  is generically finite whenever  $m \geq 3$ .

Corollary 2 improves Theorem 0.2.

**Corollary 2.** Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities.  $\phi_3$  is generically finite whenever  $P_g(X) \geq 5$  or  $P_g(X) = 4$  and  $q(X) \geq 2$ .

As an application of our method, we will present more detailed results of the  $m$ -canonical map for 3-folds of general type.

## 1. Preliminaries

**(1.1) Kawamata-Ramanujam-Viehweg vanishing theorem.** We always use the vanishing theorem in the following form.

**Vanishing Theorem** ([7] or [9]). Let  $X$  be a smooth complete variety,  $D \in \text{Div}(X) \otimes \mathbb{Q}$ . Assume the following two conditions:

- (i)  $D$  is nef and big;
- (ii) the fractional part of  $D$  has supports with only normal crossings. Then  $H^i(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0$  for all  $i > 0$ .

Most of our notations are standard within algebraic geometry except the following which we are in favor of:  $\sim_{\text{lin}}$  means *linear equivalence* while  $\sim_{\text{num}}$  means *numerical equivalence* and  $=_{\mathbb{Q}}$  means  $\mathbb{Q}$ -numerical equivalence.

**(1.2) Set up for canonical maps.** Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. Suppose  $P_g(X) \geq 2$ , we study the canonical map  $\phi_1$  which is usually a rational map. Take the birational modification  $\pi: X' \rightarrow X$ , according to Hironaka [5], such that

- (1)  $X'$  is smooth;
- (2)  $|K_{X'}|$  defines a morphism;
- (3) the fractional part of  $\pi^*(K_X)$  has supports with only normal crossings. Denote by  $g$  the composition of  $\phi_1 \circ \pi$ . So  $g: X' \rightarrow B \subseteq \mathbb{P}^{P_g(X)-1}$  is a morphism. Let  $g: X' \xrightarrow{f} B' \xrightarrow{s} B$  be the Stein factorization of  $g$ . We can write  $K_{X'} \sim_{\text{lin}} \pi^*(K_X) + E$

and  $K_{X'} \sim_{\text{lin}} M_1 + Z_1$ , where  $M_1$  is the movable part of  $|K_{X'}|$ .  $E$  is an effective  $\mathbb{Q}$ -divisor which is a  $\mathbb{Q}$ -linear combination of distinct exceptional divisors. We can also write  $\pi^*(K_X) \sim_{\text{lin}} M_1 + E'$ , where  $E' = Z_1 - E$  is actually an effective  $\mathbb{Q}$ -divisor.

If  $\dim \phi_1(X) = 2$ , we see that a general fiber of  $f$  is a smooth projective curve of genus  $g \geq 2$ . We say  $X$  is canonically fibered by curves of genus  $g$ .

If  $\dim \phi_1(X) = 1$ , we see that a general fiber  $S$  of  $f$  is a smooth projective surface of general type. We say that  $X$  is canonically fibered by surfaces with invariants  $(c_1^2, P_g) = (K_{S_0}^2, P_g(S))$ . Denote by  $\sigma : S \rightarrow S_0$  to be the contraction onto the minimal model.

**2. Proof of Main Theorem**

Let the notation be as in (1.2) throughout this section. We study  $\phi_m$  according to the value  $d := \dim \phi_1(X)$  and  $b := g(B)$ . Obviously  $1 \leq d \leq 3$ .

**Theorem 2.1.** *Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. If  $\phi_m$  is not generically finite whenever  $m \geq 6$ , then  $P_g(X) \leq 1$ .*

Proof. We suppose  $P_g(X) \geq 2$  and try to prove  $\phi_m$  is generically finite for all integers  $m \geq 6$ .

The case  $d = 2$ . Denote by  $S_1$  the general member of  $|M_1|$ . So  $S_1$  is a smooth projective surface of general type. We have

$$|K_{X'} + \lceil(m - 2)\pi^*(K_X)\rceil + S_1| \subseteq |mK_{X'}|.$$

Using (1.1), we have

$$|K_{X'} + \lceil(m - 2)\pi^*(K_X)\rceil + S_1| \Big|_{S_1} \supseteq |K_{S_1} + \lceil(m - 2)L\rceil|$$

where  $L := \pi^*(K_X)|_{S_1}$ . According to [10], we can reduce to the problem on surface  $S_1$  since

$$K_{X'} + \lceil(m - 2)\pi^*(K_X)\rceil$$

is effective. Since  $d = 2$ , we have  $h^0((m - 2)L) \geq 3$ . We know  $|K_{S_1} + \lceil(m - 2)L\rceil|$  gives a generically finite map by [2]. So does  $\phi_m$ .

The case  $d = 1$  and  $b > 0$ . Because  $b > 0$ , the movable part of  $|K_X|$  is already base point free on  $X$  and  $M_1 \sim_{\text{num}} aS$  with  $a \geq 2$ . So one always have  $\pi^*(K_X)|_S = \sigma^*(K_{S_0})$ . Obviously we have

$$|K_{X'} + \lceil(m - 2)\pi^*(K_X)\rceil + M_1| \subseteq |mK_{X'}|$$

and

$$|K_{X'} + \lceil(m - 2)\pi^*(K_X)^\lrcorner + M_1| |_{S} = |K_S + \lceil(m - 2)L'^\lrcorner|_S|$$

where  $L' = \pi^*(K_X)$  by (1.1). According [8], we can reduce to the system  $|K_S + \lceil(m - 2)L'^\lrcorner|_S|$  on  $S$  since

$$K_{X'} + \lceil(m - 2)\pi^*(K_X)^\lrcorner$$

is effective and  $a \geq 2$ . While

$$|K_S + \lceil(m - 2)L'^\lrcorner|_S| \supseteq |K_S + \lceil(m - 2)L'|_S|$$

by Lemma 2.2 in [3], we see

$$|K_S + \lceil(m - 2)L'|_S| = |K_S + (m - 2)\sigma^*(K_{S_0})|.$$

The right system defines a generically finite map on  $S$  by [12]. So does  $\phi_m$ .

The case  $d = 1$  and  $b = 0$ . According to [6], we have

$$\mathcal{O}(1) \hookrightarrow f_*\omega_{X'}^2$$

and denote by

$$\varepsilon := f_*\omega_{X'/\mathbb{P}^1}^2 \hookrightarrow f_*\omega_{X'}^6.$$

The local sections of  $f_*\omega_{X'/\mathbb{P}^1}^2$  give the bicanonical map of the fiber  $S$  and they extend to global sections of  $\varepsilon$  because  $\varepsilon$  is generated by global sections. On the other hand,  $H^0(\mathbb{P}^1, \varepsilon)$  can distinguish different fibers of  $f$  because  $\deg(\varepsilon) > 0$ . So  $H^0(\mathbb{P}^1, \varepsilon)$  gives a generically finite map on  $X'$  and so does  $\phi_6$ , which means  $\phi_m$  is generically finite whenever  $m \geq 6$ . □

**Theorem 2.2.** *Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. If  $\phi_m$  is not generically finite where  $m = 3$  or  $4$ , then either  $P_g(X) \leq 1$  or  $q(X) \leq 2$ .*

*Proof.* We assume  $P_g(X) \geq 2$ . Since  $|3K_{X'}| \subseteq |4K_{X'}|$ , we study  $\phi_3$  according to  $d$  and  $b$ .

The case  $d = 2$ . Choose a 1-dimensional subsystem  $\Lambda \subseteq |K_{X'}|$  while taking a birational modification  $\pi_1: X' \rightarrow X$  such that the pencil  $\Lambda$  defines a morphism  $g_1: X' \rightarrow \mathbb{P}^1$ . We can even take further modification to  $\pi_1$  so that  $\pi_1^*(K_X)$  has supports with only normal crossings. Taking the stein factorization  $p: X' \rightarrow W'$ . We note that this fibration is different from the one which was defined in (1.2). Denote  $b_1 := g(W')$ . Let  $M$  be the movable part of the pencil. We obviously have  $M \leq K_{X'}$ . We can write  $M \sim_{\text{lin}} \sum_{i=1}^a F_i$  where  $a \geq 1$  and  $F_i$  is a fiber of  $p$  for all  $i$ .

Suppose  $b_1 > 0$ . We consider the system

$$|2K_{X'} + M| \subseteq |3K_{X'}|.$$

Now  $M$  contains at least 2 components  $F_1$  and  $F_2$ . By (1.1), we have a surjective map

$$H^0(X', K_{X'} + M) \rightarrow H^0(F_1, 2K_{F_1}) \oplus H^0(F_2, 2K_{F_2}).$$

This means  $\phi_{|2K_{X'}+M|}$  can distinguish  $F_1$  and  $F_2$  and the restriction to  $F_i$  is at least a bicanonical map. Then  $\phi_3$  is generically finite.

Suppose  $b_1 = 0$ . Now  $M \sim_{\text{lin}} F$ . Still by (1.1) and since  $b_1 = 0$ , we consider the following system

$$|K_F + \lceil \pi^*(K_X) \rceil|_F|.$$

Assume  $P_g(F) \geq 2$  and  $|K_F|$  is composed of pencils otherwise  $q(F) \leq 1$  or  $\phi_3$  is generically finite. If  $q(F) \leq 1$ , then  $q(X) \leq 1$  by virtue of Corollary 2.3 in [4]. If  $|K_F|$  is composed of pencils, then  $q(F) \leq 2$  according to [11]. So  $q(X) \leq 2$ .

The case  $d = 1$  and  $b > 0$ . Now we have

$$|K_{X'} + \lceil \pi^*(K_X) \rceil + M_1| \subseteq |3K_{X'}|.$$

One can replace  $m$  with 3 in the corresponding proof of Theorem 2.1 and derive that  $\phi_3$  is generically finite.

The case  $d = 1$  and  $b = 0$ . In this case we have

$$\pi^*(K_X) =_{\mathbb{Q}} aS + E'$$

where  $a = P_g(X) - 1 \geq 1$  and

$$\left| K_{X'} + \lceil 2\pi^*(K_X) - \frac{E'}{a} \rceil \right|_S = \left| K_S + \lceil \left(2 - \frac{1}{a}\right) \pi^*(K_X) \rceil \right|_S.$$

If  $\phi_3$  is not generically finite, nor is the map defined by the right system above. We suppose  $P_g(S) \geq 2$ . Then  $|K_S|$  is compose of pencils and  $q(S) \leq 2$  according to [11]. Thus  $q(X) \leq 2$  by [4]. □

**Theorem 2.3.** *Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. If  $\phi_5$  is not generically finite, then either  $P_g(X) \leq 1$  or  $q(X) \leq 1$ .*

*Proof.* We suppose  $P_g(X) \geq 2$ .

The case  $d = 2$ . One can replace  $m$  with 5 in the corresponding proof of Theorem 2.1 and derive that  $\phi_5$  is generically finite.

The case  $d = 1$  and  $b > 0$ . The proof of Theorem 2.2 implies that  $\phi_5$  is generically finite since  $|3K_{X'}| \subseteq |5K_{X'}|$ .

The case  $d = 1$  and  $b = 0$ . We can write  $\pi^*(K_X) =_Q aS + E'$  where  $a = P_g(X) - 1$  and

$$\left| K_{X'} + \lceil 4\pi^*(K_X) - \frac{E'}{a} \rceil \right| \subseteq |5K_{X'}|.$$

For the same reason, we consider the system

$$\left| K_S + \lceil \left(4 - \frac{1}{a}\right) \pi^*(K_X) \rceil \right|_S = \left| K_{X'} + \lceil 4\pi^*(K_X) - \frac{E'}{a} \rceil \right|_S$$

on surface  $S$ . If  $P_g(X) \geq 3$ , then we have

$$\mathcal{O}(2) \hookrightarrow f_*\omega_{X'}$$

and

$$f_*\omega_{X'/\mathbb{P}^1}^2 \hookrightarrow f_*\omega_{X'}^4.$$

Thus  $\phi_4$  is generically finite for the same reason given in the proof of Theorem 2.1. So is  $\phi_5$ .

Next we suppose  $P_g(X) = 2$  and then  $a = 1$ . By [6],

$$\mathcal{O}(1) \hookrightarrow f_*\omega_{X'}$$

and

$$f_*\omega_{X'/\mathbb{P}^1}^2 \hookrightarrow f_*\omega_{X'}^6.$$

We assume  $P_g(S) \geq 2$  and then denote by  $G$  the movable part of  $|\sigma^*(K_{S_0})|$ , we have  $6\pi^*(K_X)|_S \geq 2G$  since  $|2\sigma^*(K_{S_0})|$  is base-point-free for  $P_g(S) > 0$ . Furthermore, we suppose  $|K_S|$  is composed of pencils otherwise  $\phi_5$  is generically finite. Then we can write

$$\sigma^*(K_{S_0}) \sim_{\text{num}} bC + Z$$

where  $C$  is a general fiber of the canonical map of  $S$  and  $b \geq P_g(S) - 1 \geq 1$ . If  $|K_S|$  is composed of irrational pencils, then  $b \geq P_g(S) \geq 2$ . Denote by  $M'$  the movable part of

$$|7K_{X'} + S| \supseteq |K_{X'} + \lceil 6\pi^*(K_X) \rceil + S|.$$

Thus we have  $M'|_S \geq 3G$  by Lemma 2.7 in [3]. Now we consider the subsystem

$$|K_{X'} + (7K_{X'} + S) + S| \subseteq |10K_X|.$$

From Theorem 2.1, we know  $\phi_7$  is generically finite. Then  $M'$  is nef and big. By (1.1), we have surjective map

$$H^0(X', K_{X'} + M' + S) \rightarrow H^0(S, K_S + M'|_S).$$

Then we see that  $M_{10}|_S \geq 4G$  and thus  $10\pi^*(K_X)|_S \geq 4G$ . Pick up a general member  $C$  of  $|G|$ . Then we can write

$$3\pi^*(K_X)|_S - C - H \sim_{\text{num}} \frac{1}{2}\pi^*(K_X)|_S$$

where  $H$  is an effective divisor or zero. Since

$$|K_S + \lceil 3\pi^*(K_X) \rceil|_S \supseteq |K_S + \lceil 3\pi^*(K_X) \rceil|_S - H|,$$

by (1.1) we have a surjective map

$$H^0(S, K_S + \lceil 3\pi^*(K_X) \rceil|_S - H) \rightarrow H^0(K_C + D)$$

where

$$D := (\lceil 3\pi^*(K_X) \rceil|_S - H - C)|_C.$$

Whether  $|K_S|$  is composed of rational pencils or irrational pencils, we can reduce to the curve  $C$ . Since  $C$  is nef on  $S$ ,  $\deg D > 0$ . Thus  $|K_C + D|$  gives a finite map and  $\phi_5$  is generically finite. Then we can derive that if  $\phi_5$  is not generically finite then  $P_g(S) \leq 1$  and thus  $q(S) \leq 1$ . By virtue of Corollary 2.3 in [4], we have  $q(X) \leq 1$ . So we are done.  $\square$

### 3. Generic finiteness of $\phi_m$

In this section we will keep the same notation as in (1.2) and denote  $d := \dim \phi_1(X)$  and  $b := g(B)$ .

**Corollary 3.1.** *Let  $X$  a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. If  $P_g(X) \geq 2$ , then  $\phi_m$  is generically finite for all integers  $m \geq 6$ .*

Proof. This is a direct result from Theorem 2.1.  $\square$

**Corollary 3.2.** *Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. Assume  $P_g(X) \geq 3$ . Then  $\phi_m$  is generically finite when  $m = 4$  or  $5$ .*



Proof. The proof of Theorem 2.3 implies the case  $m = 5$ . As for  $m = 4$ .

The case  $d = 2$ . One can still derive it from the proof of Theorem 2.1;

The case  $d = 1$  and  $b > 0$ . The proof of Theorem 2.1 also implies  $\phi_4$  is generically finite as long as replacing  $m$  with 4 there.

The case  $d = 1$  and  $b = 0$ . From proof of Theorem 2.3, we know this corollary is true. □

In the following, we will study  $\phi_3$  and then present several probabilities.

**Corollary 3.3.** *Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. Assume  $P_g(X) \geq 5$ . Then  $\phi_3$  is generically finite.*

Proof. The case  $d = 2$ . Denote by  $S_1$  the general member of  $|M_1|$  where  $M_1$  is the movable part of  $|K_{X'}|$ . We have

$$|K_{X'} + \lceil \pi^*(K_X) \rceil + S_1| \subseteq |3K_{X'}|$$

and

$$|K_{X'} + \lceil \pi^*(K_X) \rceil + S_1|_{S_1} = |K_{S_1} + L|$$

where  $L := \lceil \pi^*(K_X) \rceil_{S_1}$ . Since  $K_{X'} + \lceil \pi^*(K_X) \rceil$  is effective, we can reduce to the problem on the surface  $S_1$  by [10]. Obviously  $h^0(L) \geq P_g(X) - 1 \geq 4$ . Then  $|K_S + L|$  gives a generically finite map by [2], so does  $\phi_3$ .

The case  $d = 1$  and  $b > 0$ . The proof of Theorem 2.2 implies this is true.

The case  $d = 1$  and  $b = 0$ . Then

$$\mathcal{O}(4) \hookrightarrow f_*\omega_{X'}$$

and

$$f_*\omega_{X'/\mathbb{P}^1}^2 \hookrightarrow f_*\omega_{X'}^3.$$

Thus  $\phi_3$  is generically finite for the same reason given in the proof of Theorem 2.1. □

**Corollary 3.4.** *Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. Assume  $P_g(X) = 4$  and  $d = 2$ . Then  $\phi_3$  is generically finite.*

Proof. One can easily derive it from above since  $h^0(L) \geq 3$  in this case. □

**Corollary 3.5.** *Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. Assume  $P_g(X) \geq 2$  and  $d = 1$  and  $b > 0$ . Then  $\phi_3$  is generically finite.*

Proof. This is just one part of the proof of Theorem 2.2. □

**Corollary 3.6.** *Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. Assume  $P_g(X) = 3$  and  $d = 2$ . Then  $\phi_3$  is generically finite except  $q(S_1) = 1$  or  $2$  and  $|L|$  is composed of a rational pencil of genus  $g = q(S_1) + 1$  where  $S_1$  is the general member of  $|M_1|$  and  $L := \lceil \pi^*(K_X) \rceil|_{S_1}$ .*

Proof. We only need to consider the system  $|K_{S_1} + L|$ . One can easily derive this result from Proposition 2.1 and 2.2 in [2]. □

**Proposition 3.7.** *Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. Assume  $P_g(X) = 4$  and  $d = 1$  and  $b = 0$ . Then  $\phi_3$  is generically finite if  $P_g(S) \geq 2$ .*

Proof. One can easily see that we only need to study the system

$$\left| K_S + \lceil \frac{5}{3} \pi^*(K_X) \rceil \right|_S = \left| K_{X'} + \lceil 2\pi^*(K_X) - \frac{E'}{3} \rceil \right|_S$$

since

$$\pi^*(K_X) =_{\mathbb{Q}} 3S + E'.$$

Because

$$\mathcal{O}(3) \hookrightarrow f_*\omega_{X'},$$

we have

$$f_*\omega_{X'/\mathbb{P}^1}^3 \hookrightarrow f_*\omega_{X'}^5.$$

Suppose  $P_g(S) \geq 2$  and denote by  $G$  the movable part of  $|\sigma^*(K_{S_0})|$ . Then we have  $5\pi^*(K_X)|_S \geq 3G$  since  $|3\sigma^*(K_{S_0})|$  is base point free. Denote by  $\overline{M}$  the movable part of

$$|6K_{X'} + S| \supseteq |K_{X'} + \lceil 5\pi^*(K_X) \rceil + S|.$$

We know  $\overline{M}|_S \geq 4G$  from Lemma 2.7 in [3]. Denote by  $\overline{\overline{M}}$  the movable part of  $|2(6K_{X'} + S)|$ . Now we consider the subsystem

$$|K_{X'} + 2(6K_{X'} + S) + S| \subseteq |14K_{X'}|.$$

Since  $\phi_{12}$  is generically finite,  $\overline{\overline{M}}$  is nef and big. By (1.1), we have a surjective map

$$H^0(X', K_{X'} + \overline{\overline{M}} + S) \rightarrow H^0(S, K_S + \overline{\overline{M}}|_S).$$

Obviously we have  $\overline{M}|_S \geq 2\overline{M}|_S$ . So  $M_{14}|_S \geq 9G$  by Lemma 2.7 in [3]. Thus  $14\pi^*(K_X)|_S \geq 9G$ . Then we can write

$$\frac{5}{3}\pi^*(K_X)\Big|_S - G - H \sim_{\text{num}} \frac{1}{9}\pi^*(K_X)\Big|_S$$

where  $H$  is an effective divisor or zero. Pick up a general member  $C$  of  $|G|$ . Then we have a surjective map

$$H^0\left(S, K_S + \left\lceil \frac{5}{3}\pi^*(K_X) \right\rceil - H\right) \rightarrow H^0(C, K_C + D)$$

by (1.1) where  $D := (\lceil (5/3)\pi^*(K_X) \rceil|_S - C - H)|_C$ . Since  $C$  is nef on  $S$ ,  $|K_C + D|$  gives a finite map. Thus  $\phi_3$  is generically finite.  $\square$

**Proposition 3.8.** *Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. Assume  $P_g(X) = 3$  and  $d = 1$  and  $b = 0$ . Then  $\phi_3$  is generically finite when  $P_g(S) \geq 3$ .*

*Proof.* In this case, we have

$$\pi^*(K_X) =_Q 2S + E'.$$

Then one can reduce to the system  $|K_S + \lceil (3/2)\pi^*(K_X) \rceil|_S|$  since

$$\left| K_S + \left\lceil \frac{3}{2}\pi^*(K_X) \right\rceil \right|_S = \left| K_{X'} + \left\lceil 2\pi^*(K_X) - \frac{E'}{2} \right\rceil \right|_S$$

by (1.1).

If  $|K_S|$  is not composed of pencils, then  $\phi_3$  is generically finite.

If  $|K_S|$  is composed of pencils, then we can write  $K_S \sim_{\text{num}} bC + Z''$  where  $b \geq P_g(S) - 1 \geq 2$ . Since

$$\mathcal{O}(2) \hookrightarrow f_*\omega_{X'}$$

and

$$f_*\omega_{X'/\mathbb{P}^1}^3 \hookrightarrow f_*\omega_{X'}^6,$$

we have  $6\pi^*(K_X)|_S \geq 3G$  where  $G$  is the movable part of  $|\sigma^*(K_{S_0})|$ . By Lemma 2.7 in [3] and (1.1) considering the system  $|K_{X'} + \lceil 6\pi^*(K_X) \rceil + S|$ , we have  $M'|_S \geq 4G$  where  $M'$  is the movable part of  $|7K_{X'} + S|$ . Then considering the subsystem

$$|K_{X'} + (7K_{X'} + S) + S| \subseteq |9K_{X'}|,$$

by (1.1), we have a surjective map

$$H^0(X', K_{X'} + M' + S) \rightarrow H^0(S, K_S + M'|_S).$$

Denote by  $M''$  the movable part of  $|K_{X'} + (7K_{X'} + S) + S|$ . Then  $M''|_S \geq 5G$ . So  $9\pi^*(K_X)|_S \geq 5G$ . Then we can write

$$\frac{3}{2}\pi^*(K_X)\Big|_S - C - H \sim_{\text{num}} \frac{3}{5}\pi^*(K_X)\Big|_S$$

where  $H$  is an effective divisor or zero. Thus we can reduce to the problem on the smooth curve  $C$  of  $g \geq 2$ . Then we are done.  $\square$

**Proposition 3.9.** *Let  $X$  be a projective minimal threefold of general type with only  $\mathbb{Q}$ -factorial terminal singularities. Assume  $P_g(X) = 2$  and  $d = 1$  and  $b = 0$ . Then  $\phi_3$  is generically finite when  $P_g(S) \geq 4$ .*

*Proof.* We can write  $\pi^*(K_X) = \mathbb{Q}S + E'$  and reduce to the problem on the system  $|K_S + \lceil \pi^*(K_X) \rceil|_S|$  on surface  $S$ .

If  $|K_S|$  is not composed of pencils, then we are done.

If  $|K_S|$  is composed of pencils, we can write  $\sigma^*(K_{S_0}) \sim_{\text{num}} bC + Z''$  where  $b \geq P_g(S) - 1 \geq 3$ . Now

$$\mathcal{O}(1) \hookrightarrow f_*\omega_{X'}$$

and

$$f_*\omega_{X'/\mathbb{P}^1}^2 \hookrightarrow f_*\omega_{X'}^6.$$

Then we see that  $M'|_S \geq 3G$  where  $M'$  is the movable part of  $|7K_{X'} + S|$  and  $G$  the movable part of  $\sigma^*(K_{S_0})$ . Then consider the subsystem

$$|K_{X'} + (7K_{X'} + S) + S| \subseteq |10K_{X'}|.$$

Denote by  $M''$  the movable part of the left system above. By (1.1) we have a surjective map

$$H^0(X', K_{X'} + M' + S) \rightarrow H^0(S, K_S + M'|_S)$$

and then  $M''|_S \geq 4G$ . Thus  $10\pi^*(K_X)|_S \geq 4G$ . We can write

$$\pi^*(K_X)|_S - C - H \sim_{\text{num}} \frac{1}{6}\pi^*(K_X)\Big|_S$$

where  $H$  is an effective divisor or zero. Then we can consider the system  $|K_C + D|$  on curve  $C$  where  $D \sim_{\text{num}} (\lceil (1/6)\pi^*(K_X) \rceil|_S)|_C$ . So we are done.  $\square$

ACKNOWLEDGMENT. I am in debted to Meng Chen who gives me a lot of encouragement and guidance.

---

### References

- [1] M. Chen: *On Pluricanonical Maps for Threefolds of General Type II*, Osaka J. Math. **38** (2001), 451–468.
- [2] M. Chen and E. Viehweg: *Bicanonical and adjoint linear system on surfaces of general type*, Pacific J. Math. (to appear).
- [3] M. Chen: *Canonically Stability in terms of singularities index for algebraic 3-folds*, Math. Proc. Cambridge. Phil. Soc. **131** (2001), 241–264.
- [4] M. Chen: *On Canonical derived families of surfaces of general type over curves*, Comm. in Algebras. **29** (2001), 4597–4618.
- [5] H. Hironaka: *Resolution of singularities of an algebraic variety over a field of characteristic zero I, II*, Ann. of Math. **79** (1964), 109–203, 205–326.
- [6] J. Kollár: *Higher direct images of dualizing sheaves I*, Ann. of Math. **123** (1986), 11–42.
- [7] Y. Kawamata: *A generalization of Kodaira-Ramanujam's vanishing theorem*, Math. Ann. **261** (1982), 43–46.
- [8] K. Matsuki: *On pluricanonical maps for threefolds of general type*, J. Math. Soc. Japan, **38** (1986), 339–359.
- [9] E. Viehweg: *Vanishing theorems*, J. Reine Angew. Math. **335** (1982), 1–8.
- [10] S.G. Tankeev: *On  $n$ -dimensional canonically polarized varieties and varieties of fundamental type*, Izv. Akad. Nauk SSSR. Ser. Math. **35** (1971), 31–44.
- [11] G. Xiao: *L'irrégularité des surfaces de type général dont le système canonique est composé d'un pinceau*, Compositio Math. **56** (1985), 251–257.
- [12] G. Xiao: *Finitude de l'application bicanonique des surfaces de type général*, Bull. Soc. Math. France, **113** (1985), 23–51.

Department of Applied Mathematics  
Tongji University  
Shanghai 200092, P.R. China  
e-mail: lzhued@hotmail.com

Current address:  
Doctor 05 Grade One  
Institute of Mathematics  
Fudan University  
Shanghai 200433, P.R. China  
e-mail: 051018003@fudan.edu.cn