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PROPAGATION OF SINGULARITIES FOR A HYPERBOLIC EQUATION WITH NON-REGULAR CHARACTERISTIC ROOTS

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0. Introduction. Consider the Cauchy problem for a hyperbolic operator whose characteristic roots have variable multiplicities at most two. Then, we often meet the operator such that, even if the coefficients are infinitely differentiable, the characteristic roots are not infinitely differentiable. In this paper we prove in the above case, by constructing the fundamental solution, that the singularities of the solution propagate along the bicharacteristic curves.

Lax [13] constructed the asymptotic solution of the Cauchy problem for a strictly hyperbolic operator and investigated the propagation of singularities of the solution. For the operator with multiple characteristic roots many papers have been published. For examples, Chazarain [1], [2], Kumano-go [8], Ludwig [14] for operators with characteristics of constant multiplicity, and Flaschka-Strang [3], Ludwig-Granoff [15], Hata [5] for those with characteristics of variable multiplicity have constructed the fundamental solution or the asymptotic solution and investigated the propagation of singularities.

In these papers they assumed that characteristic roots are infinitely differentiable. In the present paper we treat hyperbolic operators such that the coefficients are infinitely differentiable, but the characteristic roots are not so with respect to the space variable. One of examples is

$$(0.1) \qquad L=\partial_t^2-(x_1^{2k}+x_2^{2k})(\partial_{x_1}^2+\partial_{x_2}^2)+ \text{ “a lower order term”} \quad \text{on } [0,T]\times R_x^2 \ (k\geqq 4)\,.$$

For such operators we study the Cauchy problem and investigate the propagation of singularities.

The outline of the present paper is as follows. Let L be a differential operator of second order in $\Omega=[0,T]\times R_x^n$:

$$(0.2) \qquad L=\partial_t^2+\sum_{\substack{|\nu|+j\leqq 2\\ j=0,1}} a_{\nu,j}(t,x)\partial_x^\nu\partial_t^j \qquad (a_{\nu,j}(t,x)\in\mathscr{B}^\infty(\Omega))\,,$$

whose characteristic roots $\lambda_i(t,x,\xi)$ $(i=1,2)$ are not infinitely differentiable with respect to x. Then, we modify the principal part of L by using the approximation theory in Section 2 in order that the characteristic roots $\tilde{\lambda}_i(t, x, \xi)$ $(i=1, 2)$ for the modified principal part are infinitely differentiable with respect to x. Then, applying the method of the case where the characteristic roots are infinitely differentiable, we can construct the fundamental solution. We note that the modified principal part is no longer differential operator.

In Section 1 we give the several classes of pseudo-differential operators and exhibit the results obtained by Kumano-go-Taniguchi-Tozaki [10] and Kumano-go-Taniguchi [11] on the theory of Fourier integral operators. In Section 2 we study the approximation theory for a non-regular symbol (Definition 1.4). We define an approximation of a non-regular sybmol by modifying that of Nagase [16], [17] and Kumano-go-Nagase [12]. In Section 3 we consider the approximation $\tilde{\lambda}_i(t, x, \xi)$ for a non-regular characteristic root $\lambda_i(t, x, \xi)$ and define the phase function $\phi(t, s; x, \xi)$ as the solution of the eiconal equation

$$\text{(0.3)} \qquad \partial_t\phi-\tilde{\lambda}(t, x, \nabla_x\phi) = 0\,, \qquad \phi|_{t=s} = x\cdot\xi\,.$$

Then, we investigate the wave front set of $P_\phi(X, D_x)u(x)$ for a Fourier integral operator $P_\phi(X, D_x)$ with symbol $p(x, \xi)$ and phase function ϕ. In Section 4 we prove the main theorem for hyperbolic operators of second order (Theorem 4.5). In Section 5 we shall extend the result in Section 4 to hyperbolic operators of higher order whose each characteristic root has multiplicity at most two.

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1. Definitions and Fourier integral operators

For multi-indexes $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\beta=(\beta_1, \cdots, \beta_n)$ of non-negative integers, and $x=(x_1, \cdots, x_n)\in R^n_x$, $\xi=(\xi_1, \cdots, \xi_n)\in R^n_\xi$, we use the usual notation:

$$|\alpha| = \alpha_1+\cdots+\alpha_n, \quad \alpha! = \alpha_1!\cdots\alpha_n!\,, \quad x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}\,,$$

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}\,, \quad \partial_{x_j} = \frac{\partial}{\partial x_j}\,, \quad D_x^\alpha = D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}\,, \quad D_{x_j} = -i\partial_{x_j}\,,$$

$$\langle x\rangle = (1+|x|^2)^{1/2}\,, \quad \nabla_x = (\partial_{x_1}, \cdots, \partial_{x_n})\,, \quad x\cdot\xi = x_1\xi_1+\cdots+x_n\xi_n\,.$$

$\alpha<\beta$ denotes that $\alpha_j\leqq\beta_j$ for all j and $\alpha\neq\beta$. Let $\mathcal{S}$ on R^n denote the Schwartz space of rapidly decreasing functions. $\mathcal{S}'$ is the dual spade of $\mathcal{S}$. For $u\in\mathcal{S}_x$ the Fourier transform $\hat{u}(\xi)=F[u](\xi)$ is defined by

$$F[u](\xi) = \int e^{-ix\cdot\xi}u(x)dx\,.$$

Then, for $\hat{u}(\xi)\in\mathcal{S}_\xi$ the inverse transform $\bar{F}[\hat{u}](\xi)$ is defined by

$$\bar{F}[\hat{u}](x) = \int e^{ix\cdot\xi}\hat{u}(\xi)đ\xi\,, \quad đ\xi = (2\pi)^{-n}d\xi\,.$$

For real s we define the Sobolev space H_s as the completion of $\mathcal{S}$ in the norm $\|u\|_s = \{\int\langle\xi\rangle^{2s}|\hat{u}(\xi)|^2 đ\xi\}^{1/2}$.

For $f(x)=(f_1(x), \cdots, f_n(x))$ $(f_j(x)\in C^1(R^n))$ we denote

$$\partial_x f = \nabla_x f = \left(\partial_{x_j} f_i\,; \begin{matrix} i \downarrow 1, \cdots, n \\ j \to 1, \cdots, n \end{matrix}\right).$$

We introduce the pseudo-differential operator. Definitions and notations are due to [6].

DEFINITION 1.1. We say that a C^∞-function $a(\eta, y)$ in $R^{2n}_{\eta,y}$ belongs to *the class* $\mathcal{A}^m_{\delta,\tau}$ for $-\infty<m<\infty$, $0\leqq\delta<1$ and $\tau\geqq 0$, when for any multi-indexes α, β we have

$$(1.1) \qquad |\partial^\alpha_\eta\partial^\beta_y a(\eta, y)| \leqq C_{\alpha,\beta}\langle\eta\rangle^{m+\delta|\beta|}\langle y\rangle^\tau\,,$$

for a constant $C_{\alpha,\beta}>0$, and set

$$\mathcal{A} = \bigcup_{0\leqq\delta<1}\ \bigcup_{0<\tau}\ \bigcup_{-\infty<m<\infty} \mathcal{A}^m_{\delta,\tau}\,.$$

It is clear that $\mathcal{A}^m_{\delta,\tau}$ is a Fréchet space with semi-norms

$$|a| = \max_{|\alpha+\beta|\leqq l}\{\inf\ C_{\alpha,\beta}\ \ \text{of (1. 1)}\}\,, \quad l = 0, 1, \cdots\,.$$

DEFINITION 1.2. For $a(\eta, y)\in\mathcal{A}^m_{\delta,\tau}$ we define *the oscillatory integral* $O_s[e^{-iy\cdot\eta}a(\eta, y)]$ by

$$\begin{aligned} O_s[e^{-iy\cdot\eta}a(\eta, y)] &= O_s\text{-}\iint e^{-iy\cdot\eta}a(\eta, y)dyđ\eta \\ &= \lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\chi(\varepsilon\eta, \varepsilon y)a(\eta, y)dyđ\eta\,, \end{aligned}$$

where $\chi(\eta, y)\in\mathcal{S}(R^{2n})$ such that $\chi(0, 0)=1$.

DEFINITION 1.3. i) We say that a C^∞-function $p(x, \xi)$ in R^{2n} belongs to *the* class $S^m_{\rho,\delta}$ $(-\infty<m<\infty,\ 0\leqq\delta\leqq\rho\leqq 1,\ \delta<1)$, when for any α, β we have

$$(1.2) \qquad |p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}\,,$$

for a constant $C_{\alpha,\beta}>0$, where $p^{(\alpha)}_{(\beta)}(x, \xi)=\partial^\alpha_\xi D^\beta_x p(x, \xi)$.

ii) We say that a C^∞-function $p(x, \xi, x', \xi')$ in R^{4n} belongs to *the class* $S^{m,m'}_{\rho,\delta}$ *of the double symbols* $(-\infty<m,\ m'<\infty,\ 0\leqq\delta\leqq\rho\leqq 1,\ \delta<1)$, when for any α, α', β, β' we have

$$(1.3)\qquad |p^{(\alpha,\alpha')}_{(\beta,\beta')}(x,\xi,x',\xi')| \leq C_{\alpha,\alpha',\beta,\beta'}\langle\xi\rangle^{m+\delta|\beta|-\rho|\alpha|}\langle\xi;\xi'\rangle^{\delta|\beta'|}\langle\xi'\rangle^{m'-\rho|\alpha'|},$$

for a constant $C_{\alpha,\alpha',\beta,\beta'}>0$, where $p^{(\alpha,\alpha')}_{(\beta,\beta')}(x,\xi,x',\xi')=\partial^{\alpha}_{\xi}\partial^{\alpha'}_{\xi'}D^{\beta}_{x}D^{\beta'}_{x'}p(x,\xi,x',\xi')$, $\langle\xi;\xi'\rangle=\sqrt{1+|\xi|^2+|\xi'|^2}$.

We often write

$$S^m_\rho = S^m_{\rho,1-\rho},\ S^{m,m'}_\rho = S^{m,m'}_{\rho,1-\rho} \qquad (\tfrac{1}{2}\leq\rho\leq 1).$$

Then, the pseudo-differential operator $P=p(X,D_x)$ with symbol $\sigma(P)(x,\xi)=p(x,\xi)$ is defined by

$$(1.4)\qquad Pu(x) = O_s\text{-}\iint e^{-iy\cdot\eta}p(x,\eta)u(x+y)dy\,đ\eta\,,\quad u\in\mathcal{S}\,.$$

In the same way for the double symbol $\sigma(P)=p(x,\xi,x',\xi')$ we define $p(X,D_x,X',D_{x'})$ by

$$(1.4)'\qquad Pu(x) = O_s\text{-}\iiiint e^{-i(y\cdot\eta+y'\cdot\eta')}p(x,\eta,x+y,\eta') u(x+y+y')dydy'đ\eta đ\eta'\,.$$

It is known that P of (1.4) and (1.4)$'$: $\mathcal{S}\to\mathcal{S}$ is continuous and if $p_i(x,\xi)\in S^{m_i}_{\rho,\delta}$ $(i=1,2)$, we have

$$\begin{cases} p_1(x,\xi)p_2(x',\xi')\in S^{m_1,m_2}_{\rho,\delta}, \\ P_1(X,D_x)P_2(X',D_{x'})u = P_1(X,D_x)\{P_2(X,D_x)u\}\,,\quad u\in\mathcal{S}\,, \end{cases}$$

(see [6]). Here we give the definition of class of non-regular symbols.

DEFINITION 1.4. Let $p(x,\xi)$ be a function such that for any α and β $(|\beta|\leq[l],\ l\geq 0)$ $p^{(\alpha)}_{(\beta)}(x,\xi)$ is continuous, where $[l]$ denotes the largest integer which is not bigger than l. Then, we say that $p(x,\xi)$ belongs to *the class* $S^m_{1,0;l}$, when the following i) and ii) are satisfied.

i) When $|\beta|\leq[l]$, for any α we have

$$(1.5)\qquad |p^{(\alpha)}_{(\beta)}(x,\xi)|\leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}\,.$$

ii) When $|\beta|=[l]$, we have for any α and $|x-y|\leq 1$

$$(1.6)\qquad |p^{(\alpha)}_{(\beta)}(x,\xi)-p^{(\alpha)}_{(\beta)}(y,\xi)|\leq C_{\alpha,\beta}|x-y|^{l-[l]}\langle\xi\rangle^{m-|\alpha|}\,,$$

where $C_{\alpha,\beta}>0$ is a constant.

DEFINITION 1.5. Let $p(x,\xi)\in S^m_\rho=S^m_{\rho,1-\rho}$ $(\tfrac{1}{2}\leq\rho\leq 1)$. Then, we say that $p(x,\xi)$ belongs to *the class* $S^m_\rho((l))$ for the positive number l, when the following i), ii) and iii) are satisfied.

i) $p_{(\beta)}^{(\alpha)}(x, \xi) \in S_\rho^{m-|\alpha|} \qquad (|\alpha+\beta| \leqq [l])$.

ii) We have for a constant $C_{\alpha,\beta}>0$

$$|p_{(\beta)}^{(\alpha)}(x, \xi)-p_{(\beta)}^{(\alpha)}(y, \xi)| \leqq C_{\alpha,\beta}|x-y|^{l-[l]}\langle\xi\rangle^{m-|\alpha|} \tag{1.7}$$
$$(|\alpha+\beta|=[l],\ |x-y| \leqq 1).$$

iii) $p_{(\beta)}^{(\alpha)}(x, \xi) \in S^{m-\rho|\alpha|+(1-\rho)(|\beta|-l)} \qquad ([l]+1 \leqq |\alpha+\beta|)$.

REMARK 1. For any $x \in R_x^n$ and $y \in R_y^n$ we get the estimates (1.6) and (1.7) by replacing $C_{\alpha,\beta}$ by another constant $C'_{\alpha,\beta}$, respectively.

REMARK 2. $S_{\rho,\delta}^m$, $S_{1,0\,;\,l}^m$ and $S_\rho^m((l))$ are Fréchet spaces provided with semi-norms

$$|p|_k^{(m)} = \max_{|\alpha+\beta| \leqq k} \inf \{ C_{\alpha,\beta} \ \text{ of } (1.2)\} ,$$

$$|p|_{k,l}^{(m)} = \max_{|\alpha| \leqq k, |\beta| \leqq [l]} \inf \{ C_{\alpha,\beta} \ \text{ of } (1.5) \text{ and } (1.6)\} ,$$

and

$$|p|_{k,((l))}^{(m)} \begin{cases} = \max\limits_{|\alpha+\beta| \leqq k} \inf |p_{(\beta)}^{(\alpha)}|_0^{(m-\rho|\alpha|)} & (k<[l]) , \\ = |p|_{[l]-1,((l))}^{(m)} + \max\limits_{|\alpha+\beta|=[l]} \inf \{C_{\alpha,\beta} \text{ of } (1.7)\} & (k=[l]) , \\ = |p|_{[l],((l))}^{(m)} + \max\limits_{[l]+1 \leqq |\alpha+\beta| \leqq k} |p|_0^{([l]-\rho|\alpha|+(1-\rho)(|\beta|-l))} & (k>[l]) , \end{cases}$$

respectively.

Now we summarize below the definitions and the fundamental theorems of of the theory of Fourier integral operators from Kumano-go [7], Kumano-go-Taniguchi-Tozaki [10] and Kumano-go-Taniguchi [11]. The proofs would be the given, only when the theorems are stated in extended forms.

DEFINITION 1.6. i) We say that a real valued C^2-function $\phi(x, \xi)$ in R^{2n} belongs to *the class* $\mathcal{P}(\tau)$ $(0 \leqq \tau < 1)$ of *phase functions,* if we have for $J(x, \xi) = \phi(x, \xi) - x \cdot \xi$

$$\|J\| = \sum_{|\alpha+\beta| \leqq 2} \sup \{|J_{(\beta)}^{(\alpha)}(x, \xi)|/\langle\xi\rangle^{1-|\alpha|}\} \leqq \tau . \tag{1.8}$$

ii) We say that a phase function $\phi(x, \xi)$ of class $\mathcal{P}(\tau)$ belongs to *the class* $\mathcal{P}_\rho(\tau)$ $(1/2<\rho \leqq 1)$, if $J(x, \xi)$ belongs to $S_\rho^1((2))$.

The Fourier integral operator P_ϕ with a symbol $\sigma(P)\,(x, \xi) = p(x, \xi) \in S_\rho^m$ and a phase function $\phi(x, \xi) \in \mathcal{P}_\rho(\tau)$ is defined by

$$(1.9)\qquad P_\phi u(x) = \int e^{i\phi(x,\xi)}p(x,\xi)\hat{u}(\xi)d\xi$$
$$= O_s\text{-}\int e^{i(\phi(x,\xi)-y\cdot\xi)}p(x,\xi)u(y)dyd\xi\,, \quad u\in\mathcal{S}\,.$$

Definition 1.7. Let ϕ_j belong to $\mathcal{P}(\tau_j)$, $j=1, 2, \cdots, \nu+1, \cdots$, with $\bar{\tau}_{\nu+1}\equiv \sum_{j=1}^{\nu+1}\tau_j\leqq\bar{\tau}^\infty\equiv\sum_{j=1}^\infty \tau_j\leqq\tau_0$ ($\leqq 1/8$). Then, we define $\Phi_{\nu+1}(x^0, \xi^{\nu+1})=\phi_1\#\phi_2\cdots\#\phi_{\nu+1}(x^0, \xi^{\nu+1})$ by

$$(1.10)\qquad \Phi_{\nu+1}(x^0, \xi^{\nu+1}) = \sum_{j=1}^{\nu}\{\phi_j(X_\nu^{j-1}, \Xi_\nu^j)-X_\nu^j\cdot\Xi_\nu^j\}+\phi_{\nu+1}(X_\nu^\nu, \xi^{\nu+1})$$
$$(X_\nu^0 = x^0)\,,$$

where $\{X_\nu^j, \Xi_\nu^j\}_{j=1}^\nu(x^0, \xi^{\nu+1})$ is defined as the solution of the equation

$$(1.11)\qquad \begin{cases} x^j = \nabla_\xi\phi_j(x^{j-1}, \xi^j)\,, \\ \xi^j = \nabla_x\phi_{j+1}(x^j, \xi^{j+1}) \qquad j = 1, 2, \cdots, \nu\,. \end{cases}$$

Theorem A (Theorem 1.8 and Theorem 1.9 in [10]). *a) Let* $\phi_j\in\mathcal{P}(\tau_j)$, $j=1, \cdots, \nu+1, \cdots$, $\bar{\tau}_\infty\leqq\tau_0$ ($\leqq 1/8$). *Then, we have*

$$\Phi_{\nu+1} = \phi_1\#\cdots\#\phi_{\nu+1}\in\mathcal{P}(c_0\bar{\tau}_{\nu+1}) \qquad (\bar{\tau}_{\nu+1} = \textstyle\sum_{j=1}^{\nu+1}\tau_j)\,,$$

for a constant $c_0>0$ *independent of* ν *and* τ_0, *and we have*

$$(1.12)\qquad \begin{cases} \nabla_{x^0}\Phi_{\nu+1}(x^0, \xi^{\nu+1}) = \nabla_x\phi_1(x^0, \Xi_\nu^1(x^0, \xi^{\nu+1}))\,, \\ \nabla_{\xi^{\nu+1}}\Phi_{\nu+1}(x^0, \xi^{\nu+1}) = \nabla_\xi\phi_{\nu+1}(X_\nu^\nu(x^0, \xi^{\nu+1}), \xi^{\nu+1})\,. \end{cases}$$

Furthermore, we obtain

$$(1.13)\qquad \begin{cases} \text{i)}\ \ \Phi_{\nu+1}\#\phi_{\nu+2} = \Phi_{\nu+2}\,, \\ \text{ii)}\ \ (\phi_1\#\phi_2)\#\phi_3 = \phi_1\#(\phi_2\#\phi_3) = \phi_1\#\phi_2\#\phi_3\,. \end{cases}$$

b) *We assume in a) furthermore that* $\{J_j/\tau_j\}_{j=1}^\infty$ *is bounded in* $S_\rho^1((2))$. *Then, for* $\boldsymbol{J}_{\nu+1}=\Phi_{\nu+1}-x^0\cdot\xi^{\nu+1}$ *we have*

$$\{\boldsymbol{J}_{\nu+1}/\bar{\tau}_{\nu+1}\}_{\nu=1}^\infty \quad \text{is bounded in } S_\rho^1((2))\,.$$

Definition 1.8. We say that a C^l-function $p(x, \xi)$ belongs to *the class* $S_{1,0\,;\,((l))}^m$ (l is an integer). When

$$|p_{(\beta)}^{(\alpha)}(x, \xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|} \qquad (|\alpha+\beta|\leqq l)\,.$$

$S_{1,0\,;\,((l))}^m$ is a Banach space provided with a norm

$$\|p\| = \max_{|\alpha+\beta|\leqq l}\sup\,\{|p_{(\beta)}^{(\alpha)}(x, \xi)|\langle\xi\rangle^{-(m-|\alpha|)}\}\,.$$

We study a hyperbolic operator of the form

(1.14) $$L = D_t - \lambda(t, X, D_x),$$

where $\lambda(t, x, \xi) \in B_t^0([0, T]; S^1_{1,0\,;\,((2))}) = B^0([0, T]; S^1_{1,0\,;\,((2))})$ and is real valued. Consider the eiconal equation

(1.15) $$\begin{cases} \partial_t \phi - \lambda(t, x, \nabla_x \phi) = 0, \\ \phi|_{t=s} = x \cdot \xi. \end{cases}$$

Then, we have

Proposition B (Theorem 3.2 in [7] and Proposition 2.2 in [10]). *Let $\phi \equiv \phi(t, s) \equiv \phi(t, s; x, \xi)$ be the solution of* (1.15) ($0 \leqq s \leqq t \leqq T$). *Then, we have for a constant $c_0 > 0$ and a small $T_0 > 0$*

(1.16) $$\begin{cases} \text{i)} \quad \phi(t, s) \in \mathcal{P}(c_0(t-s)) \qquad (0 \leqq s \leqq t \leqq T_0) \\ \text{ii)} \quad \{J(t, s)/(t-s)\}_{0 \leqq s \leqq t \leqq T_0} \textit{ is bounded in } S^1_{1,0\,;\,((2))}, \\ \text{iii)} \quad J(t, s) \in \bigcap_{j=0}^{1} B^j_{t,s}(\Delta_0; S^1_{1,0\,;\,((2-j))}) \\ \qquad \equiv \bigcap_{j=0}^{1} B^j(\Delta_0; S^1_{1,0\,;\,((2-j))}) \qquad (\Delta_0 = \{0 \leqq s \leqq t \leqq T_0\}), \end{cases}$$

and

(1.17) $$\partial_s \phi + \lambda(s, \nabla_\xi \phi(t, s), \xi) = 0,$$

where $J(t, s) = J(t, s; x, \xi) = \phi(t, s; x, \xi) - x \cdot \xi$. Particularly, if $\lambda(t, x, \xi) \in B^0([0, T]; S^1_\rho((2)))$, we have

(1.18) $$\begin{cases} \text{i)} \quad \phi(t, s) \in \mathcal{P}_\rho(c_0(t-s)) \qquad (0 \leqq s \leqq t \leqq T_0), \\ \text{ii)} \quad \{J(t, s)/(t-s)\}_{0 \leqq s \leqq t \leqq T_0} \textit{ is bounded in } S^1_\rho((2)), \\ \text{iii)} \quad J(t, s) \in \bigcap_{j=0}^{1} B^j(\Delta_0; S^1_\rho((2-j))). \end{cases}$$

Proof. Let $\{q(t, s), p(t, s)\} = \{(q_1, \cdots, q_n), (p_1, \cdots, p_n)\}$ be the solution of

(1.19) $$\begin{cases} \dfrac{dq}{dt} = -\nabla_\xi \lambda(t, q, p), \\ \dfrac{dp}{dt} = \nabla_x \lambda(t, q, p), \qquad \{q, p\}(s, s) = \{y, \xi\}. \end{cases}$$

Then, it is clear that

$$\begin{cases} q(t, s; y, \xi) - y = -\displaystyle\int_s^t \nabla_\xi \lambda(\tau, q(\tau, s), p(\tau, s)) d\tau, \\ p(t, s; y, \xi) - \xi = \displaystyle\int_s^t \nabla_x \lambda(\tau, q(\tau, s), p(\tau, s)) d\tau, \end{cases}$$

and for a small $T_0>0$ and a constant $C>0$ we have

(1.20) $$C^{-1}\langle\xi\rangle \leqq \langle p(t, s; y, \xi)\rangle \leqq C\langle\xi\rangle \qquad (0\leqq s\leqq t\leqq T_0).$$

Hence, we can see that

(1.21) $$\begin{cases} q(t, s)-y \in \bigcap_{j=0}^{1} B^j(\Delta_0; S^0_{1,0;((1-j))}), \\ p(t, s) \in \bigcap_{j=0}^{1} B^j(\Delta_0; S^1_{1,0;((1-j))}), \end{cases}$$

and

(1.22) $$\begin{cases} \{(q(t, s)-y)/(t-s)\}_{0\leqq s\leqq t\leqq T_0} \equiv \{(q(t, s)-y)/(t-s)\} \\ \qquad\qquad\qquad\qquad \text{is bounded in } S^0_{1,0;((1))}, \\ \{(p(t, s)-\xi)/(t-s)\} \quad \text{is bounded in } S^1_{1,0;((1))}. \end{cases}$$

So we have for a constant $T_0>0$ and $\varepsilon>0$

$$\|\partial_y q - I\| \leqq C(T_0-s) < (1-\varepsilon) \qquad (0\leqq s\leqq t\leqq T_0),$$

where for matrix $A=\left(a_{ij}; \begin{smallmatrix} i\downarrow 1, \cdots, n \\ j\rightarrow 1, \cdots, n\end{smallmatrix}\right)$ we define the norm $\|A\|$ by $\|A\|=\{\sum_{i,j}|a_{i,j}|^2\}^{1/2}$. Consequently, for the mapping: $R^n_y \ni y \rightarrow x=q(t, s; y, \xi)\in R^n_x$ with (t, s, ξ) as a parameter, there exists the inverse $y=y(t, s; x, \xi)$. Since $q(t, s; y(t, s; x, \xi), \xi)=x$, we have

(1.23) $$\begin{cases} y(t, s; x, \xi)-x \in \bigcap_{j=0}^{1} B^j(\Delta_0; S^0_{1,0;((1-j))}), \\ \{(y(t, s; x, \xi)-x)/(t-s)\} \quad \text{is bounded in } S^0_{1,0;((1))}. \end{cases}$$

Now setting

(1.24) $$u(t, s; y, \xi) = y\cdot\xi + \int_s^t \{\lambda - p\cdot\nabla_\xi\lambda\}(\tau, q(\tau, s), p(\tau, s))d\tau,$$

we have

$$\begin{aligned} &\partial_t\{\partial_{y_j}u - p(t, ,s)\cdot\partial_{y_j}q(t, s)\} \\ &= \partial_{y_j}\{\lambda(t, q, p) - p\cdot\nabla_\xi\lambda(t, q, p)\} - \{\nabla_x\lambda(t, q, p)\cdot\partial_{y_j}q - p\cdot\partial_{y_j}(\nabla_\xi\lambda(t, q, p))\} \\ &= 0 \end{aligned}$$

and

$$\partial_y u(s, s) - p(s, s)\cdot\partial_y q(s, s) = 0.$$

So we get

(1.25) $$\begin{cases} \partial_t u = \lambda(t, q, p) - p\cdot\nabla_\xi\lambda(t, q, p), \\ \partial_{y_j}u = p\cdot\partial_{y_j}q \qquad\qquad (j=1, \cdots, n). \end{cases}$$

Defining $\phi(t,s)=\phi(t,s;x,\xi)$ by

$$\phi(t,s)=u(t,s;y(t,s;x,\xi),\xi)\,, \tag{1.26}$$

we have

$$\begin{aligned}\partial_{x_j}\phi &= \nabla_y u(t,s;y(t,s),\xi)\cdot\partial_{x_j}y \\ &= p(t,s;y(t,s),\xi)\partial_y q\cdot\partial_{x_j}y \\ &= p_j(t,s;y(t,s),\xi)\,.\end{aligned} \tag{1.27}$$

So we get

$$\begin{aligned}\partial_t\phi &= \partial_t u(t,s;y(t,s),\xi)+\nabla_y u(t,s;y(t,s),\xi)\cdot\partial_t y \\ &= \lambda(t,q,p)-p\cdot\nabla_\xi\lambda(t,q,p)+p\partial_y q\cdot\partial_t y \\ &= \lambda(t,q,p)-p\cdot\nabla_\xi\lambda(t,q,p)+p\cdot\nabla_\xi\lambda(t,q,p) \\ &= \lambda(t,q(t,s;y(t,s),\xi),p(t,s;y(t,s),\xi)) \\ &= \lambda(t,x,\nabla_x\phi)\,.\end{aligned} \tag{1.28}$$

Consequently, $\phi(t,s)$ defined by (1.26) is the solution of (1.15). Since

$$\begin{aligned}&\partial_t{}^t(\nabla_\xi\phi(t,s;q(t,s;y,\xi),\xi)) \\ &\quad= \nabla_x\nabla_\xi\phi(t,s;q(t,s),\xi)^t\nabla_\xi\lambda+\nabla_x\nabla_\xi\phi(t,s;q(t,s),\xi)\frac{{}^t dq}{dt} \\ &\quad= 0\,,\end{aligned}$$

and $\nabla_\xi\phi(s,s;q(s,s;y,\xi),\xi)=y$, we have

$$y=\nabla_\xi\phi(t,s;q(t,s;y,\xi),\xi)\,.$$

Together with (1.27) we obtain

$$\begin{cases}\nabla_x\phi(t,s;x,\xi)=p(t,s,y(t,s;x,\xi),\xi)\,, \\ \nabla_\xi\phi(t,s;x,\xi)=y(t,s;x,\xi)\,, \\ y=\nabla_\xi\phi(t,s;q(t,s;y,\xi),\xi)\,.\end{cases} \tag{1.29}$$

Since from (1.26) we have

$$\begin{aligned}J(t,s) &= \phi(t,s)-x\cdot\xi \\ &= (y(t,s)-x)\cdot\xi+\int_s^t\{\lambda-p\cdot\nabla_\xi\lambda\}(\tau,q(\tau,s),p(\tau,s))d\tau\,|_{y=y(t,s)}\,,\end{aligned}$$

we can see by (1.22) and (1.23) that

$$\{J(t,s)/(t-s)\}\quad\text{is bounded in } S^1_{1,0\,;\,((1))}\,.$$

In the same way we obtain from (1.29)

$$\{J(t, s)/(t-s)\} \quad \text{is bounded in } S^1_{1,0,((2))}\,.$$

We note that

$$C^{-1}\langle\xi\rangle \leqq \langle\nabla_x\phi(t, s)\rangle \leqq C\langle\xi\rangle \qquad (0\leqq s\leqq t\leqq T_0)\,, \tag{1.30}$$

where $C>0$ is a constant. From (1.15) and (1.29) we can see that $\partial_t\partial_s\phi(t, s)$ and $\partial_s\nabla_x\phi(t, s)$ are continuous. By the same techniques of the proof of Proposition 2.2 in [10] we get

$$\partial_s\phi(t, s)+\lambda(s, \nabla_\xi\phi(t, s), \xi) = 0\,.$$

So we get

$$\left\{\begin{array}{l} \partial_t J = \partial_t\phi = \lambda(t, x, \nabla_x\phi)\,, \\ \partial_s J = \partial_s\phi = -\lambda(s, \nabla_\xi\phi, \xi)\,. \end{array}\right. \tag{1.31}$$

Hence, we obtain

$$J(t, s) \in \bigcap_{j=0}^{1} B^j(\Delta_0;\, S^1_{1,0\,;\,((2-j))})\,.$$

By the similar way to the proof of (1.16) we obtain (1.18). Q.E.D.

Now take $\lambda_j \in B^0([0, T];\, S^1_{1,0\,;\,((2))})$ $(j=1, 2, \cdots, \nu+1, \cdots)$ as λ of (1.14) and let $\phi_j(t, s)$ be the solution of (1.15) corresponding to λ_j. We define $\Phi=\Phi_{\nu+1}=\Phi_{1,\cdots,\nu+1}(t_0, \cdots, t_{\nu+1};\, x^0, \xi^{\nu+1})$ by

$$\Phi_{\nu+1}(t_0, \cdots, t_{\nu+1}) = \phi_1(t_0, t_1)\#\cdots\#\phi_{\nu+1}(t_\nu, t_{\nu+1})\,. \tag{1.32}$$

Theorem C (Theorem 2.3 in [10]). $\Phi(t_0, \cdots, t_{\nu+1})$ *of* (1.32) *satisfies*

1). $\partial_{t_j}\Phi = -\lambda_j(t_j, X^j_\nu, \Xi^j_\nu)+\lambda_{j+1}(t_j, X^j_\nu, \Xi^j_\nu) \qquad (0\leqq j\leqq \nu+1)$

$$(\lambda_0 = \lambda_{\nu+2} = 0,\ \Xi^0_\nu = \nabla_{x^0}\Phi,\ X^{\nu+1}_\nu = \nabla_{\xi^{\nu+1}}\Phi)\,.$$

2). *If $t_j=t_{j+1}=\tau$ for some j, we have*

$$\begin{aligned}&\Phi_{1,\cdots,\nu+1}(t_0, \cdots, t_{j-1}, \tau, \tau, t_{j+2}, \cdots, t_{\nu+1})\\ &= \Phi_{1,\cdots,j,j+2,\cdots,\nu+1}(t_0, \cdots, t_{j-1}, \tau, t_{j+2}, \cdots, t_{\nu+1})\end{aligned}$$

3). *If $\lambda_j(t, x, \xi)=\lambda_{j+1}(t, x, \xi)$ (therefore $\phi_j=\phi_{j+1}$) for some j, we have*

$$\Phi_{1,\cdots,\nu+1}(t, \cdots, t_{\nu+1}) = \Phi_{1,\cdots,j-1,j+1\,\cdots,\nu+1}(t_0, \cdots, t_{j-1}, t_{j+1}, \cdots, t_{\nu+1})\,.$$

REMARK 3. From Theorem C we have for Φ of (1.32)

$$\left\{\begin{array}{l} \partial_{t_0}\Phi-\lambda_1(t, x^0, \nabla_{x^0}\Phi) = 0\,, \\ \Phi\,|_{t_0=t_1} = \Phi_{2,3,\cdots,\nu+1}(t_1, t_2, \cdots, t_{\nu+1})\,. \end{array}\right. \tag{1.33}$$

Corollary D ([10]). *Let* $\lambda_i(t, x, \xi) \in \bigcap_{j=0}^{1} B^j([0, T]; S^1_\rho((2-j)))$ $(i=(1, 2))$. *Assume that*

$$\{\lambda-\lambda_1, \lambda-\lambda_2\} \in B^0([0, T]; S^0_\rho),$$

where $\{\lambda-\lambda_1, \lambda-\lambda_2\}$ *denotes the Poisson bracket for* $\lambda-\lambda_1$ *and* $\lambda-\lambda_2$, *which is defined by*

$$\partial_t\lambda_1-\partial_t\lambda_2-\nabla_x\lambda_1\cdot\nabla_\xi\lambda_2+\nabla_\xi\lambda_1\cdot\nabla_x\lambda_2 .$$

Then, for a small $T_0>0$ *we have*

$$\Phi_{2,1}(t_0, t_1, t_2; x^0, \xi^2)-\Phi_{1,2}(t_0, t_0-t_1+t_2, t_2; x^0, \xi^2) \in B^0(\Delta_1; S^0_\rho) .$$

Proof. We shall prove in the similar way to the proof of Corollary of Theorem 2.3 in [10]. Let $\{y(\sigma), \eta(\sigma)\} = \{y(\sigma, t_1, t_2; x^0, \xi^2), \eta(\sigma, t_1, t_2; x^0, \xi^2)\}$ be the solution of

$$(1.34)\qquad \begin{cases} \dfrac{dy}{d\sigma} = -\nabla_\xi\lambda_1(\sigma, y, \eta), \\ \dfrac{d\eta}{d\sigma} = \nabla_x\lambda_1(\sigma, y, \eta), \\ \{y, \eta\}\,|_{\sigma=t_1} = \{X^1_1(t_0, t_1, t_2), \Xi^1_1(t_0, t_1, t_2)\}, \end{cases}$$

where $\{X^1_1, \Xi^1_1\}$ is the solution of $X^1_1=\nabla_\xi\phi_1(t_0, t_1; x^0, \Xi^1_1)$, $\Xi^1_1=\nabla_x\phi_2(t_1, t_2; X^1_1, \xi^2)$. Then, we have $y(t_0)=x^0$, $\eta(t_0)=\nabla_{x^0}\Phi_{1,2}(t_0, t_1, t_2)$ and

$$(1.35)\qquad \begin{aligned} &\frac{d}{d\sigma}\{\lambda_2(\sigma, y(\sigma), \eta(\sigma))-\lambda_1(\sigma, y(\sigma), \eta(\sigma))\} \\ &= \partial_t\lambda_2-\partial_t\lambda_1+\nabla_x\lambda_2\cdot\frac{dy}{d\sigma}+\nabla_\xi\lambda_2\cdot\frac{d\eta}{d\sigma}-\nabla_x\lambda_1\cdot\frac{dy}{d\sigma}-\nabla_\xi\lambda_1\cdot\frac{d\eta}{d\sigma} \\ &= -\{\lambda-\lambda_1, \lambda-\lambda_2\}(\sigma, y(\sigma), \eta(\sigma)) . \end{aligned}$$

So we have

$$\begin{aligned} &\{\lambda_2(t_0, x^0, \nabla_{x^0}\Phi_{1,2})-\lambda_1(t_0, x^0, \nabla_{x^0}\Phi_{1,2})\}-\{\lambda_2(t_1, X^1_1, \Xi^1_1) \\ &\quad -\lambda_1(t_1, X^1_1, \Xi^1_1)\} = Q(t_0, t_1, t_2), \end{aligned}$$

where $Q(t_0, t_1, t_2)=Q(t_0, t_1, t_2; x^0, \xi^2)=-\int_{t_1}^{t_0}\{\lambda-\lambda_1, \lambda-\lambda_2\}(\sigma, y(\sigma), \eta(\sigma))d\sigma$. Consequently, we can see by Theorem C that

$$(1.36)\qquad \begin{cases} \partial_{t_0}\{\Phi_{1,2}(t_0, t_0-t_1+t_2, t_2)\}-\lambda_2(t_0, x^0, \nabla_{x^0}\Phi_{1,2}(t_0, t_0-t_1+t_2, t_2)) \\ = -Q(t_2, t_0-t_1+t_2, t_2) \\ \Phi_{1,2}(t_0, t_0-t_1+t_2, t_2)\,|_{t_0=t_1} = \phi_1(t_1, t_2) . \end{cases}$$

By Remark 3 we have

$$(1.37)\qquad \begin{cases} \partial_{t_0}\Phi_{2,1}(t_0, t_1, t_2)-\lambda_2(t_0, x^0, \nabla_{x^0}\Phi_{2,1}) = 0\,, \\ \Phi_{2,1}|_{t_0=t_1} = \phi_1(t_1, t_2)\,. \end{cases}$$

Hence, setting $u=\Phi_{1,2}(t_0, t_0-t_1+t_2, t_2)-\Phi_{2,1}(t_0, t_1, t_2)$, we see that u is the solution of

$$(1.38)\qquad \begin{cases} \partial_{t_0}u+H_1(t_0, t_1, t_2)\cdot\nabla_{x^0}u+H_2(t_0, t_1, t_2) = 0\,, \\ u|_{t_0=t_1} = 0\,, \end{cases}$$

where

$$(1.39)\qquad \begin{cases} H_1(t_0, t_1, t_2) = -\int_0^1 \nabla_\xi\lambda_2(t_0, x^0, \nabla_{x^0}\Phi_{2,1}(t_0, t_1, t_2) \\ \qquad +\theta(\nabla_{x^0}\Phi_{1,2}(t_0, t_1', t_2)-\nabla_{x^0}\Phi_{2,1}(t_0, t_1, t_2)))d\theta\ (t_1' = t_0-t_1+t_2)\,, \\ H_2(t_0, t_1, t_2) = Q(t_0, t_0-t_1+t_2, t_2)\,. \end{cases}$$

Since from Proposition B and its proof we have

$$\nabla_{x^0}\Phi_{1,2}\langle\xi^2\rangle^{-1},\ \eta(t)\langle\xi^2\rangle^{-1},\ y(t)-x^0\in B^0(\Delta_1;\ S^0_\rho((1)))\,,$$

and for a small $T_0>0$ and a constant $C>0$

$$C^{-1}\langle\xi^2\rangle\leqq\langle\nabla_{x^0}\Phi_{2,1}(t_0, t_1, t_2)+\theta(\nabla_{x^0}\Phi_{1,2}(t_0, t_1', t_2)-\nabla_{x^0}\Phi_{2,1}(t_0, t_1, t_2))\rangle\leq C\langle\xi^2\rangle\,,$$

we get

$$(1.40)\qquad \begin{cases} H_1(t_0, t_1, t_2)\in B^0(\Delta_1;\ S^0_\rho((1)))\,, \\ H_2(t_0, t_1, t_2)\in B^0(\Delta_1;\ S^0_\rho)\,. \end{cases}$$

We solve (1.38) along the characteristic curve. Let $q=q(t_0, t_1, t_2; y, \xi^2)=(q_1, \cdots, q_n)$ with (t_1, t_2, ξ^2) as a parameter be the solution of

$$(1.41)\qquad \frac{dq}{dt_0} = H_0(t_0, t_1, t_2;\ q,\ \xi^2),\ q|_{t_0=t_1} = y\,.$$

Then, we have for a small $T_0>0$ in the similar way to the proof of Proposition B

$$(1.42)\qquad \begin{cases} q(t_0, t_1, t_2; y, \xi^2)-y\in B^0(\Delta_1\ ;S^0_\rho((1)))\,, \\ y(t_0, t_1, t_2; x^0, \xi^2)-x^0\in B^0(\Delta_1;\ S^0_\rho((1)))\,, \end{cases}$$

where $y(t_0, t_1, t_2; x^0, \xi^2)$ is the inverse for the mapping $x^0=q(t_0, t_1, t_2; y, \xi^2)$: $R^n_y\ni y\to x^0\in R^n_{x^0}$ with (t_0, t_1, t_2, ξ^2) as a parameter. Then, the solution u of (1.38) has the form

$$(1.43)\qquad u(t_0, t_1, t_2) = -\int_{t_1}^{t_0} H_2(\tau, t_1, t_2;\ q(\tau, t_1, t_2; y, \xi^2), \xi^2)d\tau\,|_{y=y(t_0,t_1,t_2)}$$

Hence, we obtain Corollary D together with (1.40) and (1.42). Q.E.D.

Let $\lambda_i(t, x, \xi)$ $(i=1, 2)$ satisfy the condition of Corollary D. Then, for any $p(t, \tau, s; x, \xi) \in B^0(\Delta_1; S^m_\rho)$ we have

$$(1.44) \qquad \int_x^t d\tau P_{\Phi_{1,2}}(t, \tau, s; X, D_x)u$$
$$= \int_s^t \exp(i\Phi_{1,2}(t, \tau, s))p(t, \tau, s; x, \xi)\hat{u}(\xi)d\xi d\tau$$
$$= \int_s^t \exp(i\Phi_{1,2}(t, t-\tau+s, s))p(t, t-\tau+s, s)\hat{u}(\xi)d\xi d\tau$$
$$= \int_s^t d\tau Q_{\Phi_{2,1}}(t, \tau, s; X, D_x)u,$$

where

$$\sigma(Q)(t, \tau, s; x, \xi)$$
$$= \exp(i(\Phi_{1,2}(t, t-\tau+s, s)-\Phi_{2,1}(t, \tau, s)))p(t, t-\tau+s, s; x, \xi)$$
$$(\in B^0(\Delta_1; S^m_\rho)).$$

Theorem E (Theorem 2.5 in [11]). *Let $p_j = p_j(x, \xi) \in S^{m_j}_\rho$ and $\phi_j = \phi_j(x, \xi) \in \mathcal{P}_\rho(\tau_j)$ $(j=1, \cdots, \nu+1, \cdots)$ $(1/2 < \rho \leqq 1)$. Assume that $\sum_{j=1}^\infty |m_j| < \infty$, $\bar{\tau}_\infty = \sum_{j=1}^\infty \tau_j \leqq \tau_0/8c_0$ with constants τ_0 and c_0 of Theorem A, and $\{J_j/\tau_j\}_{j=1}^\infty$ is bounded in $S^1_\rho((2))$. We also assume that for any l there exists a constant A_l such that*

$$(1.45) \qquad |p_j|_l^{(m_j)} \leqq A_l \qquad (j=1, 2, \cdots, \nu+1, \cdots).$$

Then, we have a symbol $q_\nu = q_\nu(x^0, \xi^{\nu+1}) \in S^{\bar{m}_{\nu+1}}$ $(\bar{m}_{\nu+1} = \sum_{j=1}^{\nu+1} m_j)$ which satisfies the followings. For a constant $C_l > 0$ independent of ν

$$(1.46) \qquad |q_\nu|_l^{(\bar{m}_{\nu+1})} \leqq C_l^\nu \qquad (\nu = 1, 2, \cdots),$$

holds, and if we set

$$(1.47) \qquad R_\nu = P_{1,\phi_1} \cdot P_{2,\phi_2} \cdots P_{\nu,\phi_\nu} - Q_{\Phi_{\nu+1}},$$

R_ν: $H_{-\infty} \to H_{-\infty}$ is a smoothing operator in the sense: For any σ and N we have for a constant $C_{\sigma,N} > 0$ independent of ν

$$(1.48) \qquad ||R_\nu||_{H_\sigma \to H_{\sigma+N}} \leqq C^\nu_{\sigma,N},$$

where $||\cdot||_{H_\sigma \to H_{\sigma+N}}$ denotes the operator norm of the mapping: $H_\sigma \to H_{\sigma+N}$.

2. Approximation theorems

Nagase in [16], [17] and Kumano-go-Nagase in [12] treat approximation theory for non-regular symbols. In this section we develop it.

DEFINITION 2.1. We say that $p(x, \xi)$ $(\in S^m_{\rho,\delta})$ belongs to *the class* $S^m_{\rho,\delta}(l)$

($l\geqq 0$), when the followings are satisfied.

i) $p_{(\beta)}(x,\xi)\in S^m_{\rho,\delta}\qquad (|\beta|\leqq[l])$,

ii)

$$|p^{(\alpha)}_{(\beta)}(x,\xi)-p^{(\alpha)}_{(\beta)}(y,\xi)|\leqq C_{\alpha,\beta}|x-y|^{l-[l]}\langle\xi\rangle^{m-\rho|\alpha|} \qquad (|\beta|=[l],\ |x-y|\leqq 1) \tag{2.1}$$

for any α,

iii) $p_{(\beta)}(x,\xi)\in S^{m+\delta(|\beta|-l)}_{\rho,\delta}\qquad ([l]+1\leqq|\beta|)$.

REMARK 1. It is clear that for another constant $C'_{\alpha,\beta}$ (2.1) is equivalent to

$$|p^{(\alpha)}_{(\beta)}(x,\xi)-p^{(\alpha)}_{(\beta)}(y,\xi)|\leqq C'_{\alpha,\beta}|x-y|^{l-[l]}\langle\xi\rangle^{m-\rho|\alpha|} \qquad (|\beta|=[l],\ x,y\in R^n). \tag{2.1$'$}$$

Lemma 2.2. *There exists a function $\psi(x)\in\mathcal{S}(R^n_x)$ such that*

$$\int\psi(x)dx=1\,,\quad \int\psi(x)x^\alpha dx=0\quad (|\alpha|\neq 0)\,. \tag{2.2}$$

Proof. Take a function $\phi(\xi)\in\mathcal{S}(R^n_\xi)$ such that

$$\phi(0)=1\,,\quad D^\alpha_\xi\phi(0)=0\quad (|\alpha|\neq 0)\,, \tag{2.3}$$

and set

$$\psi(x)=\int e^{ix\cdot\xi}\phi(\xi)d\xi\,.$$

Then, noting that

$$\int\psi(x)x^\alpha dx=(-D_\xi)^\alpha\phi(0)\,,$$

we see that $\psi(x)$ satisfies (2.2). Q.E.D.

Lemma 2.3. *Let $p(x,\xi)\in S^m_{1,0\,;\,l}$. By using $\psi(x)$ of Lemma 2.2 we set*

$$q(x,\xi)=\int\psi(\langle\xi\rangle^\delta y)p(x+y,\xi)dy\langle\xi\rangle^{\delta n}\,, \tag{2.4}$$

and we set for a multi-index $\omega\neq 0$

$$r(x,\xi)=\int\psi(\langle\xi\rangle^\delta y)(\langle\xi\rangle^\delta y)^\omega p(x+y,\xi)dy\langle\xi\rangle^{\delta n}\,. \tag{2.5}$$

Then, we get

$$q(x,\xi)\in S^m_{1,\delta}(l)\,,\quad r(x,\xi)\in S^{m-\delta l}_{1,\delta}\,.$$

Proof. First, we prove that $q(x,\xi)$ belongs to $S^m_{1,\delta}(l)$. We note that $q(x,\xi)$ has another expression

$$q(x, \xi) = \int \psi(\langle\xi\rangle^{\delta}(y-x))p(y, \xi)dy\langle\xi\rangle^{\delta n}. \tag{2.6}$$

We have

$$q_{(\beta)}(x, \xi) = \int \psi(\langle\xi\rangle^{\delta}y)p_{(\beta)}(x+y, \xi)dy\langle\xi\rangle^{\delta n}.$$
$$= \int \psi(\langle\xi\rangle^{\delta}(y-x))p_{(\beta)}(y, \xi)dy\langle\xi\rangle^{\delta n} \qquad (|\beta|\leqq[l]).$$

Hence, we get

$$q_{(\beta)}(x, \xi)\in S^m_{1,\delta} \qquad \text{for} \quad |\beta|\leqq[l].$$

When $|\beta|=[l]$, we write

$$\begin{aligned} &q_{(\beta)}(x_2, \xi)-q_{(\beta)}(x_1, \xi) \\ &= \int \psi(\langle\xi\rangle^{\delta}y)\{p_{(\beta)}(x_2+y, \xi)-p_{(\beta)}(x_1+y, \xi)\}dy\langle\xi\rangle^{\delta n}. \end{aligned} \tag{2.7}$$

Then, noting that $p(x, \xi)$ belongs to $S^m_{1,0\,;\,l}$ and

$$|\partial_\xi^\alpha\{\psi(\langle\xi\rangle^{\delta}y)\}|\leqq C_\alpha\langle\xi\rangle^{-|\alpha|}, \tag{2.8}$$

we get for any α

$$|q_{(\beta)}^{(\alpha)}(x_2, \xi)-q_{(\beta)}^{(\alpha)}(x_1, \xi)|\leqq C_{\alpha,\beta}|x_2-x_1|^{l-[l]}\langle\xi\rangle^{m-|\alpha|} \qquad (|\beta|=[l]). \tag{2.9}$$

Hence, we obtain ii) of Definition 2.1.

When $|\beta|\geqq[l]+1$, we have

$$q_{(\beta)}(x, \xi) = \int (-\langle\xi\rangle)^{\delta|\beta-\beta_0|}\psi^{(\beta-\beta_0)}(\langle\xi\rangle^{\delta}(y-x))p_{(\beta_0)}(y, \xi)dy\langle\xi\rangle^{\delta n},$$

where $\psi^{(\beta-\beta_0)}(y)=D_y^{(\beta-\beta_0)}\psi(y)$ and β_0 is a multi-index such that $|\beta_0|=[l]$ and $\beta_0<\beta$. Since

$$\int \psi^{(\beta-\beta_0)}(y)dy = 0 \qquad (|\beta-\beta_0|\neq 0),$$

we have

$$\begin{aligned} q_{(\beta)}(x, \xi) = (-\langle\xi\rangle)^{\delta|\beta-\beta_0|}\int &\psi^{(\beta-\beta_0)}(\langle\xi\rangle^{\delta}(y-x)) \\ &\times\{p_{(\beta_0)}(y, \xi)-p_{(\beta_0)}(x, \xi)\}dy\langle\xi\rangle^{\delta n}. \end{aligned} \tag{2.10}$$

So we have

$$\begin{aligned} |q_{(\beta)}^{(\alpha)}(x, \xi)| &\leqq C'_{\alpha,\beta}\int|\psi^{(\beta-\beta_0)}(\langle\xi\rangle^{\delta}(y-x))|(\langle\xi\rangle^{\delta}|y-x|)^{l-[l]}dy\langle\xi\rangle^{m-\delta(l-|\beta|)-|\alpha|+\delta n} \\ &\leqq C_{\alpha,\beta}\langle\xi\rangle^{m-\delta l+\delta|\beta|-|\alpha|}. \end{aligned}$$

Hence, we get

$$q_{(\beta)}(x,\xi)\in S_{1,\delta}^{m+\delta(|\beta|-l)}\qquad (|\beta|\geqq[l]+1).$$

Thus we can see that $q(x,\xi)$ belongs to $S_{1,\delta}^{m}(l)$.

Next we prove that $r(x,\xi)$ belongs to $S_{1,\delta}^{m-\delta l}$. Set

$$\psi_\omega(y)=\psi(y)y^\omega \qquad (\omega\neq 0). \tag{2.11}$$

Then, from (2.2) we get

$$\int \psi_\omega^{(\beta)}(y)y^\nu dy=0 \qquad \text{for any } \beta \text{ and } \nu. \tag{2.12}$$

Now we write by the Taylor expansion

$$\begin{aligned} p(y,\xi)=&\sum_{|\alpha|\leqq[l]} i^{|\alpha|}(y-x)^\alpha/(\alpha!)p_{(\alpha)}(x,\xi)\\ &+\sum_{|\gamma|\leqq[l]} i^{|\gamma|}(y-x)^\gamma/(\gamma!)q_\gamma(x,y;\xi), \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} &q_\gamma(x,y;\xi)\\ &=\begin{cases} [l]\int_0^1(1-\theta)^{[l]-1}\{p_{(\gamma)}(x+\theta(y-x),\xi)-p_{(\gamma)}(x,\xi)\}d\theta & ([l]\geqq 1),\\ p(y,\xi)-p(x,\xi) \qquad ([l]=0). \end{cases} \end{aligned} \tag{2.14}$$

Then, it is clear that

$$|q_\gamma(x,y;\xi)|\leqq C|y-x|^{l-[l]}\langle\xi\rangle^m. \tag{2.15}$$

From (2.12) and (2.13) we have

$$\begin{aligned} r(x,\xi)&=\int\psi_\omega(\langle\xi\rangle^\delta(y-x))p(y,\xi)dy\langle\xi\rangle^{\delta n}\\ &=\sum_{|\gamma|=[l]} i^{|\gamma|}/\gamma!\int\psi_\omega(\langle\xi\rangle^\delta(y-x))q_\gamma(x,y;\xi)(y-x)^\gamma dy\langle\xi\rangle^{\delta n}. \end{aligned}$$

Similarly, we have for any β

$$\begin{aligned} r_{(\beta)}(x,\xi)&=(-\langle\xi\rangle)^{\delta|\beta|}\int\psi_\omega^{(\beta)}(\langle\xi\rangle^\delta(y-x))p(y,\xi)dy\langle\xi\rangle^{\delta n}\\ &=(-\langle\xi\rangle)^{\delta|\beta|}\sum_{|\gamma|=[l]} i^{|\gamma|}/\gamma!\int\psi_\omega^{(\beta)}(\langle\xi\rangle^\delta(y-x))q_\gamma(x,y;\xi)\\ &\qquad\times(y-x)^\gamma dy\langle\xi\rangle^{\delta n}. \end{aligned} \tag{2.16}$$

Then, in the similar way to the proof for $q(x,\xi)$ we have

$$|r_{(\beta)}^{(\alpha)}(x,\xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{m-\delta l+\delta|\beta|-|\alpha|}. \tag{2.17}$$

Hence, we can see that $r(x,\xi)$ belongs to $S_{1,\delta}^{m-\delta l}$. Q.E.D.

Lemma 2.4. *For $p(x, \xi)\in S^m_{1,0\,;\,\sigma}$ $(0\leqq\sigma\leqq 2)$ set*

$$Q = Q(x, x', y; \xi) = p(x-y, \xi)-p(x, \xi)-p(x'-y, \xi)+p(x', \xi)\,. \tag{2.18}$$

Then, we get

i) *When $0\leqq\sigma<1$, we have for any τ $(0\leqq\tau\leqq\sigma)$*

$$|Q|\leqq 2|p|^{(m)}_{0\,;\,\sigma}|x'-x|^{\sigma-\tau}|y|^{\tau}\langle\xi\rangle^m\,, \tag{2.19}$$

ii) *When $1\leqq\sigma\leqq 2$, we have for any τ $(\sigma-1\leqq\tau\leqq 1)$*

$$|Q|\leqq n|p|^{(m)}_{0\,;\,\sigma}|x'-x|^{\sigma-\tau}|y|^{\tau}\langle\xi\rangle^m\,. \tag{2.20}$$

Proof. Set $K=|p|^{(m)}_{0\,;\,\sigma}$.

i) We have by the definition of $S^m_{1,0\,;\,\sigma}$

$$|p(x, \xi)-p(y, \xi)|\leqq K|x-y|^{\sigma}\langle\xi\rangle^m\,. \tag{2.21}$$

So it is clear that

$$|Q|\leqq 2K|x'-x|^{\sigma}\langle\xi\rangle^m\,,\quad |Q|\leqq 2K|y|^{\sigma}\langle\xi\rangle^m\,.$$

Hence, we have

$$|Q|=|Q|^{(\sigma-\tau)/\sigma}|Q|^{\tau/\sigma}\leqq 2K|x'-x|^{\sigma-\tau}|y|^{\tau}\langle\xi\rangle^m\,.$$

We obtain (2.19).

ii) We write

$$Q = (x-x')\cdot\int_0^1 \{\nabla_x p(x'-y+\theta(x-x'), \xi)-\nabla_x p(x'+\theta(x-x'), \xi)\}d\theta\,, \tag{2.22}$$

and

$$Q = -y\cdot\int_0^1 \{\nabla_x p(x-\theta y, \xi)-\nabla_x p(x'-\theta y, \xi)\}d\theta\,. \tag{2.23}$$

Then, we have

$$|Q|\leqq nK|x'-x|\,|y|^{\sigma-1}\langle\xi\rangle^m\,,\quad |Q|\leqq nK|x'-x|^{\sigma-1}|y|\langle\xi\rangle^m\,.$$

We get (2.20) in the case $\sigma=2$ $(\tau=1)$. When $1\leqq\sigma<2$, we also get for any τ $(\sigma-1\leqq\tau\leqq 1)$

$$\begin{aligned}|Q|&=|Q|^{(1-\tau)/(2-\sigma)}|Q|^{(\tau+1-\sigma)/(2-\sigma)}\\ &\leqq nK|x'-x|^{\sigma-\tau}|y|^{\tau}\langle\xi\rangle^m\,.\end{aligned}$$

Q.E.D.

Here, we shall define an approximation for $p(x, \xi)\in S^m_{1,0,l}$. As the approximation for p we define $\tilde{p}_\delta(x, \xi)$ by

$$(2.24)\qquad \tilde{p}_\delta(x, \xi) = \int \psi(\langle\xi\rangle^\delta y)p(x+y, \xi)dy\langle\xi\rangle^{\delta n},$$

where $0<\delta<1$. We shall often write $\tilde{p}_\delta=\tilde{p}$.

Theorem 2.5. *For $p(x, \xi)\in S^m_{1,0\,;\,l}$ we define $\tilde{p}_\delta(x, \xi)$ by (2.24). Then, we get*

$$\tilde{p}_\delta(x, \xi)\in S^m_{1,\delta}(l), \quad \tilde{p}_\delta(x, \xi)-p(x, \xi)\in \bigcap_{0\leqq\tau\leqq l} S^{m-\delta(l-\tau)}_{1,0\,;\,\tau}.$$

Proof. We can see from Lemma 2.3 that $\tilde{p}(x, \xi)$ belongs to $S^m_{1,\delta}(l)$. So we have only to prove that $r(x, \xi)=\tilde{p}(x, \xi)-p(x, \xi)$ belongs to $\bigcap_{0\leqq\tau\leqq l} S^{m-\delta(l-\tau)}_{1,0\,;\,\tau}$. Setting $s=[l-\tau]$, we write

$$(2.25)\qquad p(x+y, \xi) = \sum_{0\leqq|\beta|\leqq s} i^{|\beta|}y^\beta/(\beta!)p_{(\beta)}(x, \xi) + \sum_{|\gamma|=s} i^{|\gamma|}y^\gamma/(\gamma!)q_\gamma(x, y; \xi),$$

where

$$(2.26)\qquad q_\gamma(x, y; \xi)\begin{cases} = s\int_0^1(1-\theta)^{s-1}\{p_{(\gamma)}(x+\theta y, \xi)-p_{(\gamma)}(x, \xi)\}d\theta & (s\neq0), \\ = p(x+y, \xi)-p(x, \xi) & (s=0). \end{cases}$$

By (2.2) we have

$$(2.27)\qquad r(x, \xi) = \int \psi(\langle\xi\rangle^\delta y)\{p(x+y, \xi)-p(x, \xi)\}dy\langle\xi\rangle^{\delta n} = \sum_{|\gamma|=s} i^{|\gamma|}/(\gamma!)\int\psi(\langle\xi\rangle^\delta y)y^\gamma q_\gamma(x, y; \xi)dy\langle\xi\rangle^{\delta n}.$$

We devide the proof into two cases. First, we consider the case $[l-\tau]+[\tau]=[l]$. We see that the following I), II), III) are equivalent.

$$(2.28)\qquad \begin{cases} \text{I)} & [l-\tau]+[\tau]=[l], \\ \text{II)} & l-[l]\geqq\tau-[\tau], \\ \text{III)} & l-[l]\geqq l-\tau-[l-\tau]. \end{cases}$$

By (2.26) we have for any α and β ($|\beta|\leqq[\tau]$)

$$(2.29)\qquad |\partial^\alpha D^\beta_x q_\gamma(x, y; \xi)|\leqq C_{\alpha,\beta}(|y|^{l-[l]}+|y|)\langle\xi\rangle^{m-|\alpha|}.$$

Then, from (2.27) we have

$$(2.30)\qquad |r^{(\alpha)}_{(\beta)}(x, \xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{m-\delta(l-\tau)-|\alpha|} \qquad (|\beta|\leqq[\tau]).$$

Hence, r satisfies i) of Definition 1.4.

When $s\neq0$, for $|\beta|=[\tau]$ we have from (2.26)

(2.31) $$\begin{aligned}&\partial_\xi^\alpha D_x^\beta q_\gamma(x, y; \xi)-\partial_\xi^\alpha D_x^\beta q_\gamma(z, y; \xi)\\&= s\int_0^1 (1-\theta)^{s-1}\{p_{(\gamma+\beta)}^{(\alpha)}(x+\theta y, \xi)-p_{(\gamma+\beta)}^{(\alpha)}(x, \xi)\\&\qquad -p_{(\gamma+\beta)}^{(\alpha)}(z+\theta y, \xi)+p_{(\gamma+\beta)}^{(\alpha)}(z, \xi)\}\,d\theta\,.\end{aligned}$$

Since $|\gamma+\beta|=[l-\tau]+[\tau]=[l]$, we get

$$p_{(\gamma+\beta)}^{(\alpha)}(x, \xi)\in S_{1,0\,;\,l-[l]}^{m-|\alpha|}\,.$$

So we have by i) of Lemma 2.4

(2.32) $$\begin{aligned}&|\partial_\xi^\alpha D_x^\beta q_\gamma(x, y; \xi)-\partial_\xi^\alpha D_x^\beta q_\gamma(z, y; \xi)|\\&\leqq C|x-z|^{\tau-[\tau]}|y|^{l-\tau-[l-\tau]}\langle\xi\rangle^{m-|\alpha|}\,.\end{aligned}$$

When $s=0$, we also get (2.32). Since

(2.33) $$\begin{aligned}r_{(\beta)}(x, \xi)-r_{(\beta)}(z, \xi) = \sum_{|\gamma|=s} i^{|\gamma|}/(\gamma!)\int\psi(\langle\xi\rangle^\delta y)\times\\ y^\gamma\{D_x^\beta q_\gamma(x, y; \xi)-D_x^\beta q_\gamma(z, y; \xi)\}\,dy\langle\xi\rangle^{\delta n}\,,\end{aligned}$$

we obtain by the similar way to the proof of Lemma 2.3

(2.34) $$|r_{(\beta)}^{(\alpha)}(x, \xi)-r_{(\beta)}^{(\alpha)}(z, \xi)|\leqq C_{\alpha,\beta}|x-z|^{\tau-[\tau]}\langle\xi\rangle^{m-\delta(l-\tau)-|\alpha|}\,.$$

Hence, r satisfies ii) of Definition 1.4. Consequently, we see that r belongs to $S_{1,0\,;\,\tau}^{m-\delta(l-\tau)}$.

Next consider the case $[l-\tau]+[\tau]=[l]-1$. We can also prove in the similar way to the proof of the first case. We have for any α and β ($|\beta|\leqq[\tau]$)

(2.35) $$|\partial_\xi^\alpha D_x^\beta q_\gamma(x, y; \xi)|\leqq C_{\alpha,\beta}|y|\langle\xi\rangle^{m-|\alpha|}\,.$$

Then, using (2.28), we obtain (2.30). Now fix β ($|\beta|=[\tau]$). Then, since $|\gamma+\beta|=[l-\tau]+[\tau]=[l]-1$, we have

$$p_{(\gamma+\beta)}^{(\alpha)}(x, \xi)\in S_{1,0\,;\,1+l-[l]}^{m-|\alpha|}\,.$$

Hence, using $\tau-[\tau]>l-[l]$ from (2.28), we have (2.32) by ii) of Lemma 2.4. Thus we obtain (2.34). Q.E.D.

Theorem 2.6. *For $p_i(x, \xi)\in S_{1,0\,;\,l}^{m_i}$ $(i=1, 2)$ consider the approximations $\tilde{p}_i$ and $\widetilde{p_1p_2}$ defined by* (2.24). *Then, we get*

$$\tilde{p}_1(x, \xi)\tilde{p}_2(x, \xi)-\widetilde{p_1p_2}(x, \xi)\in S_{1,\delta}^{m_1+m_2-\delta l'}$$

where $l'=2l$ $([l]=0)$, $l'=l+1$ $([l]\neq 0)$.

Proof. We prove only the case $p_i(x, \xi)=a_i(x)\in\mathcal{B}^l$. Then, the general case would be proved in the similar way. We write

$$(2.36)\qquad 2(\tilde{a}_1\tilde{a}_2-\widetilde{a_1a_2})\,(x,\xi)$$
$$=-\iint \psi(\langle\xi\rangle^{\delta}(y-x))\psi(\langle\xi\rangle^{\delta}(y'-x))\{a_1(y)-a_1(y')\}\{a_2(y)-a_2(y')\}dydy'\langle\xi\rangle^{2\delta n},$$

and for $k=1, 2$

$$(2.37)\qquad a_k(y)=\sum_{|\alpha^k|\leqq[l]} i^{|\alpha^k|}(y-y')^{\alpha^k}/(\alpha^k!)a_{k(\alpha^k)}(y')+\sum_{|\gamma^k|=[l]} i^{|\gamma^k|}(y-y')^{\gamma^k}/(\gamma^k!)b_{k,\gamma^k}(y,y'),$$

where

$$(2.38)\qquad b_{k,\gamma^k}(y,y')$$
$$\begin{cases} =[l]\int_0^1(1-\theta)^{[l]-1}\{a_{k(\gamma^k)}(y'+\theta(y-y'))-a_{k(\gamma^k)}(y')\}d\theta & ([l]\neq 0),\\ =a_k(y)-a_k(y') & ([l]=0).\end{cases}$$

We have

$$(2.39)\qquad \{a_1(y)-a_1(y')\}\{a_2(y)-a_2(y')\}$$
$$=\sum_{1\leqq|\alpha^k|\leqq[l]} i^{|\alpha^1+\alpha^1|}(y-y')^{\alpha^1+\alpha^2}/(\alpha^1!\alpha^2!)a_{1(\alpha^1)}(y')a_{2(\alpha^2)}(y')$$
$$+\sum_{\substack{1\leqq|\alpha^1|\leqq[l]\\|\gamma^2|=[l]}} i^{|\alpha^1+\gamma^2|}(y-y')^{\alpha^1+\gamma^2}/(\alpha^1!\gamma^2!)a_{1(\alpha^1)}(y')b_{2,\gamma^2}(y,y')$$
$$+\sum_{\substack{1\leqq|\alpha^2|\leqq[l]\\|\gamma^1|=[l]}} i^{|\alpha^2+\gamma^1|}(y-y')^{\gamma^1+\alpha^2}/(\gamma^1!\alpha^2!)b_{1,\gamma^1}(y,y')a_{2(\alpha^2)}(y')$$
$$+\sum_{|\gamma^k|=[l]} i^{|\gamma^1+\gamma^2|}(y-y')^{\gamma^1+\gamma^2}/(\gamma^1!\gamma^2!)b_{1,\gamma^1}(y,y')b_{2,\gamma^2}(y,y')$$
$$\equiv\sum_{j=1}^{4} I_j(y,y').$$

and set

$$(2.40)\qquad \tilde{I}_j(x,\xi)=\iint \psi(\langle\xi\rangle^{\delta}(y-x))\psi(\langle\xi\rangle^{\delta}(y'-x))I_j(y,y')dydy'\langle\xi\rangle^{2\delta n}.$$

Then, we note that when $[l]=0$, $\tilde{I}_j$ $(j=1, 2, 3)$ disappears. Since

$$(2.41)\qquad |b_{k,\gamma^k}(y,y')|\leqq C|y-y'|^{l-[l]},$$

we have

$$(2.42)\qquad |I_j|\leqq C(|y-y'|^{1+[l]}+|y-y'|^{l+[l]})\qquad (j=2, 3),$$

and

$$(2.43)\qquad |I_4|\leqq C|y-y'|^{2l}.$$

So by the similar way to the proof of Theorem 2.5 we can see that

$$\tilde{I}_j \in S_{1,\delta}^{-\delta(l+1)} \quad (j=2,3), \quad \tilde{I}_4 \in S_{1,\delta}^{-2\delta l}.$$

Hence, we have only to show that $\tilde{I}_1$ belongs to $S_{1,\delta}^{-\delta(l+1)}$.

By (2.2) we have for each term of $\tilde{I}_1$

$$\begin{aligned}(2.44) \qquad & \iint \psi(\langle\xi\rangle^{\delta}(y-x))\psi(\langle\xi\rangle^{\delta}(y'-x))(y-y')^{\alpha^1+\alpha^2} \\ & \qquad \times a_{1(\alpha^1)}(y')a_{2(\alpha^2)}(y')dydy'\langle\xi\rangle^{2\delta n} \\ & = (-1)^{|\alpha^1+\alpha^2|}\int \psi(\langle\xi\rangle^{\delta}(y'-x))(\langle\xi\rangle^{\delta}(y'-x))^{\alpha^1+\alpha^2} \\ & \qquad \times a_{1(\alpha^1)}(y')a_{2(\alpha^2)}(y')dy'\langle\xi\rangle^{\delta n-\delta|\alpha^1+\alpha^2|}.\end{aligned}$$

Since $a_{1(\alpha^1)}(x)a_{2(\alpha^2)}(x) \in S_{1,0\,;\,l-k}^{0}$ ($k=\max(|\alpha^1|, |\alpha^2|)$), we can see by Lemma 2.3 that the right hand side of (2.44) belongs to $S_{1,\delta}^{-\delta(l-k)-\delta|\alpha^1+\alpha^2|}$. Then, noting that

$$l-k+|\alpha^1+\alpha^2| \geqq l+\min(|\alpha^1|, |\alpha^2|) \geqq l+1,$$

we get

$$\tilde{I}_1 \in S_{1,\delta}^{-\delta(l+1)}. \qquad \text{Q.E.D.}$$

3. Wave front set

DEFINITION 3.1. Let $u(x) \in H_{-\infty} = \bigcup_{S\in R} H_s$. Then, we say that $(x_0, \xi_0) \in R_x^n \times (R_\xi^n - \{0\})$ dose not belong to *the wave front set* $WF(u)$ *of* u, if there exists an $\varepsilon > 0$ *such that for any* $a(x) \in C_0^\infty(U_\varepsilon(x_0))$ and any $N>0$ we have

$$(3.1) \qquad \sup_{\Gamma_\varepsilon(\xi_0)} |F[au](\xi)| \leqq C_N(1+|\xi|)^{-N},$$

where $C_N > 0$ is a constant depending on $a(x)$ and N, and

$$(3.2) \qquad \begin{cases} U_\varepsilon(x_0) = \{x;\ |x-x_0| < \varepsilon\}, \\ \Gamma_\varepsilon(\xi_0) = \{\xi;\ |\xi/|\xi| - \xi_0/|\xi_0|| < \varepsilon,\ |\xi| > 1\}. \end{cases}$$

Theorem 3.2. *Let* $\phi(x,\xi) \in \mathcal{P}_\rho(\tau)$ $(1/2 < \rho \leqq 1)$ *and* $\psi(x,\xi) \in \mathcal{P}(\tau)$ *such that for a constant* $M>0$ *we have*

$$(3.3) \qquad \psi(x, \delta\xi) = \delta\psi(x,\xi) \qquad (|\xi| \geqq M,\ \delta \geqq 1).$$

We assume that there exists a constant a (<1) *such that for a constant* $C>0$

$$(3.4) \qquad |\partial_\xi^\alpha \partial_x^\beta(\phi-\psi)(x,\xi)| \leqq C\langle\xi\rangle^{a-|\alpha|} \qquad (|\alpha+\beta| \leqq 1).$$

Then, we have for any $p(x,\xi) \in S_\rho^m$

$$(3.5) \quad WF(P_\phi u) \subset \text{Conic}\,\{(x, \nabla_x\psi(x,\xi));\ (\nabla_\xi\psi(x,\xi), \xi) \in WF(u),\ |\xi| \geqq M\},$$

where for a set $A\subset R^n_x\times(R^n_\xi-\{0\})$ Conic A denotes the smallest conic set including A.

The proof will be given after Lemma 3.5. First, we state two corollaries.

Let real valued functions $\lambda_k(t, x, \xi)$ $(k=1, 2)$ belong to $B^0([0, T]; S^1_{1,0;2})$ and have the form for a constant $a<1$

$$(3.6)\qquad \lambda_k(t, x, \xi)=\lambda^1_k(t, x, \xi)+\lambda^0_k(t, x, \xi)\qquad (k=1, 2)$$

$$\left(\begin{array}{ll}\lambda^1_k(t, x, \xi)\in B^0([0, T]; S^1_\rho((2))) & (1/2<\rho\leqq 1)\,, \\ \lambda^0_k(t, x, \xi)\in B^0([0, T]; S^a_{1,0;1}), & \text{real valued.}\end{array}\right)$$

Assume that for a constant $M>0$ we have

$$(3.7)\qquad \lambda_k(t, x, \delta\xi)=\delta\lambda_k(t, x, \xi)\qquad (|\xi|\geqq M,\ \delta\geqq 1)\,.$$

Let $\phi_k(t, s; x, \xi)$ and $\psi_k(t, s; x, \xi)$ be the solutions of the eiconal equations of (1.15) corresponding to λ^1_k and λ_k, respectively. We define $\Phi_{i,j}(t, \tau, s; x, \xi)$ and $\Psi_{i,j}(t, \tau, s; x, \xi)$ $(1\leqq i, j\leqq 2)$ for a small $T_0>0$, respectively by

$$(3.8)\qquad \left\{\begin{array}{l}\Phi_{i,j}(t, \tau, s)=\phi_i(t, \tau)\#\phi_j(\tau, s)\,, \\ \Psi_{i,j}(t, \tau, s)=\psi_i(t, \tau)\#\psi_j(\tau, s)\,.\end{array}\right.$$

Then, for a constant $M_1>0$ we can easily get

$$(3.9)\qquad \Psi_{i,j}(t, \tau, s; x, \delta\xi)=\delta\Psi_{i,j}(t, \tau, s; x, \xi)\qquad (|\xi|\geqq M_1,\ \delta\geqq 1)\,.$$

We obtain

Corollary 3.3. *For any $p(x, \xi)\in S^m_\rho$ we have*

$$(3.10)\qquad WF(P_{\Phi_{i,j}}u)\subset \text{Conic}\ \{(x, \nabla_x\Psi_{i,j}(t, \tau, s; x, \xi)); (\nabla_\xi\Psi_{i,j}(t, \tau, s; x, \xi), \xi)\in WF(u),\ |\xi|\geqq M_1\}\qquad (0\leqq s\leqq\tau\leqq t\leqq T_0)\,.$$

Proof. If we can show (3.4) of Theorem 3.2 for $\Phi_{i,j}$ and $\Psi_{i,j}$, the proof is complete.

I) The case $i=j$. We have

$$(3.11)\qquad \Phi_{i,i}(t, \tau, s)=\phi_i(t, s)\,,\quad \Psi_{i,i}(t, \tau, s)=\psi_i(t, s)\,.$$

In what follows we omit the suffix i of λ_i, λ^1_i, λ^0_i, ϕ_i and ψ_i.

Then, we have

$$\begin{aligned}0&=\partial_t(\phi-\psi)(t, s)-\lambda^1(t, x, \nabla_x\phi)+\lambda(t, x, \nabla_x\psi)\\&=\partial_t(\phi-\psi)(t, s)-\lambda(t, x, \nabla_x\phi)+\lambda(t, x, \nabla_x\psi)+\lambda^0(t, x, \nabla_x\phi)\\&=\partial_t(\phi-\psi)(t, s)-\int_0^1\nabla_\xi\lambda(t, x, \nabla_x\psi+\theta(\nabla_x\phi-\nabla_x\psi))d\theta\cdot\nabla_x(\phi-\psi)(t, s)\\&\qquad+\lambda^0(t, x, \nabla_x\phi)\,.\end{aligned}$$

Set

$$\text{(3.12)}\qquad \begin{cases} H_1(t, s; x, \xi) = -\int_0^1 \nabla_\xi \lambda(t, x, \nabla_x\psi+\theta(\nabla_x\phi-\nabla_x\psi))d\theta\,, \\ H_2(t, s; x, \xi) = \lambda^0(t, x, \nabla_x\phi)\,. \end{cases}$$

Then, we can get $(\phi-\psi)(t, s; x, \xi)$ as the solution of

$$\text{(3.13)}\qquad \begin{cases} \partial_t u(t, s)+H_1(t, s; x, \xi)\cdot\nabla_x u(t, s)+H_2(t, s; x, \xi) = 0\,, \\ u(s, s) = 0\,. \end{cases}$$

Since for a small $T_0>0$ $\phi(t, s)$ and $\psi(t, s)$ belong to $\mathcal{P}(c_0(t-s))$ and we have

$$C^{-1}\langle\xi\rangle \leqq \langle\nabla_x\psi+\theta(\nabla_x\phi-\nabla_x\psi)\rangle \leqq C\langle\xi\rangle\,,$$

it follows that

$$\text{(3.14)}\qquad |H_{1(\beta)}^{(\alpha)}(t, s)| \leqq C\langle\xi\rangle^{-|\alpha|}, \quad |H_{2(\beta)}^{(\alpha)}(t, s)| \leqq C\langle\xi\rangle^{a-|\alpha|} \quad (|\alpha+\beta| \leqq 1,\ 0\leqq s\leqq t\leqq T_0)\,.$$

We can solve (3.13) in the explicit form along the characteristic curve by the same method of the proof of Proposition B and Corollary D in Section 1. Then, we can easily complete the proof.

II) The case $i\neq i$. By using Remark 3 in Section 1 we obtain (3.4) by the similar way to the first case. Q.E.D.

Let $\phi(x, \xi)\in\mathcal{P}(\tau)$ and set $F(\xi)=F(\xi; x, \eta) = -\nabla_x\phi(x, \xi)+\xi+\eta$. Then, since $\|\nabla_x\nabla_\xi\phi-I\|\leqq\tau<1$, we get

$$\begin{aligned} &|F(\xi_2)-F(\xi_1)| \\ &\leqq \Big(\int_0^1 \|\nabla_x\nabla_\xi\phi(x, \xi_1+\theta(\xi_2-\xi_1))-I\| d\theta\Big)|\xi_2-\xi_1| \\ &\leqq \tau|\xi_2-\xi_1|\,. \end{aligned}$$

So the mapping $\zeta=F(\xi)$: $R_\xi^n\ni\xi\to\zeta\in R_\zeta^n$ is a contraction. Defining $\xi=\xi(x, \eta)$ as the fixed point, $\xi(x, \eta)$ satisfies

$$\eta = \nabla_x\phi(x, \xi(x, \eta)) \equiv (\partial_{x_1}\phi, \cdots, \partial_{x_n}\phi)(x, \xi)|_{\xi=\xi(x,\eta)}$$

Hence, the mapping $\eta=\nabla_x\phi(x, \xi)$: $R_\xi^n\ni\xi\to\eta\in R_\eta^n$ has the inverse $\xi=\xi(x, \eta)\equiv \nabla_x\phi^{-1}(x, \eta)$. Similarly, the mapping $y=\nabla_\xi\phi(x, \xi)$: $R_x^n\ni x\to y\in R_y^n$ has the inverse $x=\nabla_\xi\phi^{-1}(y, \xi)$. Consequently, for any $(x, \xi)\in R^{2n}$ there exists a point $(y, \eta)\in R^{2n}$ such that

$$\text{(3.15)}\qquad x = \nabla_\xi\phi(y, \xi)\,, \quad \eta = \nabla_x\phi(y, \xi)\,.$$

Conversely, for any (y, η) there exists a point (x, ξ) which satisfies (3.15).

Now for $\lambda_k(t, x, \xi)$ $(k=1, 2)$ of Corollary 3.3 let $\{q^k, p^k\}(t, s; x, \xi)$ $(=\{(q_1^k, \cdots, q_n^k), (p_1^k, \cdots, p_n^k)\})$ be the characteristic strip, that is, $\{q^k, p^k\}$ is the solution of

$$(3.16)\qquad \begin{cases} \dfrac{dq^k}{dt} = -\nabla_\xi \lambda_k(t, q^k, p^k)\,, & \dfrac{dp^k}{dt} = \nabla_x \lambda_k(t, q^k, p^k))\,, \\ q^k(s, s) = x\,, & p^k(s, s) = \xi\,. \end{cases}$$

We set

$$(3.17)\qquad \{Q_{i,j}, P_{i,j}\}(t, \tau, s; x, \xi) = \{q^i, p^i\}(t, \tau; q^j(\tau, s; x, \xi), p^j(\tau, s; x, \xi)) \quad (1 \leqq i, j \leqq 2)\,.$$

Then, we obtain

Corollary 3.4. *Let $\Phi_{i,j}(t, \tau, s)$ be the phase function defined by* (3.8). *Then, we have for any $p(x, \xi) \in S_\rho^m$*

$$(3.18)\qquad WF(P_{\Phi_{i,j}}u) \subset \mathrm{Conic}\,[\{Q_{i,j}, P_{i,j}\}(t, \tau, s; x, \xi); (x, \xi) \in WF(u), |\xi| \geqq M]\,,$$

where M is a constant in (3.91).

Proof. I) The case $i=j$. We have

$$\begin{cases} \Phi_{i,i}(t, \tau, s) = \phi_i(t, s)\,, \quad \Psi_{i,i}(t, \tau, s) = \psi_i(t, s)\,, \\ \{Q_{i,i}, P_{i,i}\}(t, \tau, s) = \{q^i, p^i\}(t, s)\,. \end{cases}$$

We omit the suffix i. By (3.15) we can define $\{q', p'\}(t, s) \equiv \{q', p'\}(t, s; x, \xi)$ as the solution of

$$(3.19)\qquad x = \nabla_\xi \psi(t, s; q'(t, s), \xi)\,, \quad p'(t, s) = \nabla_x \psi(t, s; q'(t, s), \xi)\,.$$

Then, we have

$$\begin{aligned} 0 &= (\nabla_\xi \partial_t \psi)(t, s; q', \xi) + \nabla_x \nabla_\xi \psi(t, s; q', \xi)\, {}^t\frac{dq'}{dt} \\ &= \nabla_x \nabla_\xi \psi(t, s; q', \xi)^t \{\nabla_\xi \lambda(t, q', p') + \frac{dq'}{dt}\}\,. \end{aligned}$$

Since $\|\nabla_x \nabla_\xi \psi - I\| \leqq \tau < 1$, we have

$$\frac{dq'}{dt} = -\nabla_\xi \lambda(t\,, q', p')\,.$$

Similarly we get

$$\begin{cases} \dfrac{dq'}{dt} = -\nabla_\xi \lambda(t, q', p')\,, & \dfrac{dp'}{dt} = \nabla_x \lambda(t, q', p')\,, \\ q'(s, s) = x\,, & p'(s, s) = \xi\,. \end{cases}$$

Consequently, we have $q'(t, s)=q(t, s)$ and $p'(t, s)=p(t, s)$. Then, we get

$$x = \nabla_\xi \psi(t, s; q(t, s), \xi), \quad p(t, s) = \nabla_x \psi(t, s; q(t, s), \xi). \tag{3.20}$$

Hence, we obtain (3.18) by Corollary 3.3.

II) The case $i \neq j$. In the similar way to the proof of the case I) we get

$$\begin{cases} x = \nabla_\xi \Psi_{i,j}(t, \tau, s; Q_{i,j}(t, \tau, s), \xi), \\ P_{i,j}(t, \tau, s) = \nabla_x \Psi_{i,j}(t, \tau, s; Q_{i,j}(t, \tau, s), \xi), \end{cases} \tag{3.21}$$

which completes the proof. Q.E.D.

Lemma 3.5. *Let $\phi(x, \xi)$ and $\psi(x, \xi)$ satisfy the assumption of Theorem 3.2. Let $(x_0, \xi_0)\in R^n_x\times(R^n_\xi-\{0\})$ and define (y_0, η_0) as in (3.15) by.*

$$y_0 = \nabla_\xi \psi(x_0, \eta_0), \quad \xi_0 = \nabla_x \psi(x_0, \eta_0),$$

where we assume that $|\xi_0|$ is sufficient large. For any $\varepsilon>0$ and $R>0$ we set

$$\Gamma^1_{\varepsilon, R} = \{(x, \xi);\ |\nabla_x\phi(x, \xi)/|\nabla_x\phi(x, \xi)|-\xi_0/|\xi_0||<\varepsilon, \\ |x-x_0|<\varepsilon,\ |\xi|>R\}, \tag{3.22}$$

$$\Gamma^2_{\varepsilon, R} = \{(x, \xi);\ |\xi/|\xi|-\eta_0/|\eta_0||<\varepsilon, \\ |\nabla_\xi\phi(x, \xi)-y_0|<\varepsilon,\ |\xi|>R\}. \tag{3.23}$$

Then, for any $\varepsilon_2>0$ (resp. $\varepsilon_1>0$) there exist $\varepsilon_1>0$ (resp. $\varepsilon_2>0$) and $M'>0$ independent of (x_0, ξ_0) such that

$$\Gamma^1_{\varepsilon_1, R}\subset\Gamma^2_{\varepsilon_2, R} \quad (R\geqq M') \quad (\textit{resp. } \Gamma^1_{\varepsilon_1, R}\supset\Gamma^2_{\varepsilon_2, R}). \tag{3.24}$$

Proof. Generally, if $\varphi(x, \xi)\in\mathcal{P}(\tau)$, we have.

$$|\nabla_x\varphi-\xi|\leqq\tau\langle\xi\rangle.$$

So setting $\eta=\nabla_x\varphi(x, \xi)$, we get

$$\begin{cases} C^{-1}\langle\xi\rangle\leqq\langle\nabla_x\varphi(x, \xi)\rangle\leqq C\langle\xi\rangle, \\ C^{-1}\langle\eta\rangle\leqq\langle\nabla_x\varphi^{-1}(x, \eta)\rangle\leqq C\langle\eta\rangle, \end{cases} \tag{3.25}$$

for a constant $C>0$ depending only on τ. Consequently, for large R $\Gamma^i_{\varepsilon, R}$ $(i=1, 2)$ is well-defined.

We define $\tilde{\Gamma}^1_{\varepsilon, R}$ and $\tilde{\Gamma}^2_{\varepsilon, R}$ by

$$\tilde{\Gamma}^1_{\varepsilon, R} = \{(x, \xi);\ |\nabla_x\psi(x, \xi)/|\nabla_x\psi(x, \xi)|-\xi_0/|\xi_0||<\varepsilon, \\ |x-x_0|<\varepsilon,\ |\xi|>R\}, \tag{3.26}$$

$$\tilde{\Gamma}^2_{\varepsilon, R} = \{(x, \xi);\ |\xi/|\xi|-\eta_0/|\eta_0||<\varepsilon, \\ |\nabla_\xi\psi(x, \xi)-y_0|<\varepsilon,\ |\xi|>R\}. \tag{3.27}$$

Then, noting (3.3), we can easily see that for any $\varepsilon_2>0$ (resp. $\varepsilon_1>0$) there exist $\varepsilon_1>0$ (resp. $\varepsilon_2>0$) and $R_0>0$ such that

$$\tilde{\Gamma}^1_{\varepsilon_1,R}\subset\tilde{\Gamma}^2_{\varepsilon_2,R}\quad(R>R_0)\quad(\text{resp. }\tilde{\Gamma}^1_{\varepsilon_1,R}\supset\tilde{\Gamma}^2_{\varepsilon_2,R})\,. \tag{3.28}$$

We have from (3.4)

$$|\nabla_x\phi(x,\xi)/\langle\xi\rangle-\nabla_x\psi(x,\xi)/\langle\xi\rangle|\leqq C\langle\xi\rangle^{a-1}\,.$$

Consequently, using (3.25), we get

$$\begin{aligned}(3.29)\quad &|\nabla_x\phi(x,\xi)/|\nabla_x\phi(x,\xi)|-\nabla_x\psi(x,\xi)/|\nabla_x\psi(x,\xi)||\\ &\leqq\{|\nabla_x\phi-\nabla_x\psi|\times|\nabla_x\psi|+||\nabla_x\phi|-|\nabla_x\psi||\times|\nabla_x\psi|\}/(|\nabla_x\psi||\nabla_x\phi|)\\ &\leqq 2C\langle\xi\rangle^{a-1}\,.\end{aligned}$$

By (3.4) we have

$$|\nabla_\xi\phi(x,\xi)-\nabla_\xi\psi(x,\xi)|\leqq C\langle\xi\rangle^{a-1}\,. \tag{3.30}$$

Fix $\varepsilon=\varepsilon_2$ in (3.23). Then, by (3.30) we have for some R_0

$$\sup_{x,|\xi|\geqq R}|\nabla_\xi\phi(x,\xi)-\nabla_\xi\psi(x,\xi)|<\varepsilon_2/2\qquad(R\geqq R_0)\,.$$

So for $(x,\xi)\in\tilde{\Gamma}^2_{\varepsilon_2/2,R}$ $(R\geqq R_0)$ we have

$$|\nabla_\xi\phi(x,\xi)-y_0|\leqq|\nabla_\xi\phi(x,\xi)-\nabla_\xi\psi(x,\xi)|+|\nabla_\xi\psi(x,\xi)-y_0|<\varepsilon_2\,.$$

Consequently, we get

$$\tilde{\Gamma}^2_{\varepsilon_2/2,R}\subset\Gamma^2_{\varepsilon_2,R}\qquad(R\geqq R_0)\,. \tag{3.31}$$

By (3.28) we have for some ε_1' and R_0' $(\geqq R_0)$

$$\tilde{\Gamma}^1_{\varepsilon_1',R}\subset\tilde{\Gamma}^2_{\varepsilon_2/2,R}\qquad(R\geqq R_0')\,. \tag{3.32}$$

Next setting $\varepsilon_1=\varepsilon_1'/2$, from (3.29) we have for some R_0'' $(\geqq R_0')$

$$\sup_{x,|\xi|\geqq R}|\nabla_\xi\phi(x,\xi)/|\nabla_x\phi(x,\xi)|-\nabla_x\psi(x,\xi)/|\nabla_x\psi(x,\xi)||<\varepsilon_1\,.$$

So for $(x,\xi)\in\Gamma^1_{\varepsilon_1,R}$ $(R\geqq R_0')$ we have

$$|\nabla_x\psi(x,\xi)/|\nabla_x\psi(x,\xi)|-\xi_0/|\xi_0||<\varepsilon_1'\,.$$

Then, we get

$$\Gamma^1_{\varepsilon_1,R}\subset\tilde{\Gamma}^1_{\varepsilon_1',R}\qquad(R\geqq R_0'')\,.$$

Finally, for M' $(\geqq R_0''\geqq R_0'\geqq R_0)$ we obtain

$$\Gamma^1_{\varepsilon_1,R}\subset\tilde{\Gamma}^1_{\varepsilon_1',R}\subset\tilde{\Gamma}^2_{\varepsilon_2/2,R}\subset\Gamma^2_{\varepsilon_2,R}\qquad(R\geqq M')\,. \tag{3.33}$$

Convesely, for any ε_1 there exist ε_1', ε_2', ε_2 and $M'>0$ such that

$$\Gamma^1_{\varepsilon_1,R}\supset\tilde{\Gamma}^1_{\varepsilon_1',R}\supset\tilde{\Gamma}^2_{\varepsilon_2',R}\supset\Gamma^2_{\varepsilon_2,R} \qquad (R\geqq M'). \tag{3.34}$$

Thus the proof is complete. Q.E.D.

Proof of Theorem 3.2. For a constant M in (3.3) set

$$K=\text{Conic}\ \{(x, \nabla_x\psi(x, \xi));\ (\nabla_\xi\psi(x, \xi), \xi)\in WF(u),\ |\xi|\geqq M\}.$$

Then, we have only to prove that if $(x_0, \xi_0)\notin K$, (x_0, ξ_0) does not belong to $WF(P_\phi u)$. Assume that

$$(x_0, \xi_0)\notin K,$$

and define (y_0, η_0) as the solution of

$$y_0=\nabla_\xi\psi(x_0, \eta_0),\quad \xi_0=\nabla_x\psi(x_0, \eta_0) \qquad \text{(see (3.15))}.$$

Then, we may assume from (3.25) that $|\xi_0|$ is a sufficient large in order that $|\eta_0|\geqq M$. Then, we can see that

$$(y_0, \eta_0)\notin WF(u).$$

Corresponding $U_\varepsilon(y_0)$ and $\Gamma_\varepsilon(\eta_0)$ of (3.2), we choose $\tilde{a}(x)\in\mathcal{B}^\infty(R^n)$, $\tilde{b}(\xi)\in S^\infty$ such that

$$\begin{cases} \tilde{a}(x)=1 & \text{on } U_{\varepsilon_2}(y_0),\quad \text{supp}\ \tilde{a}\subset U_{2\varepsilon_2}(y_0),\\ \tilde{b}(\xi)=1 & \text{on } \Gamma_{\varepsilon_2}(\eta_0),\quad \text{supp}\ \tilde{b}\subset\Gamma_{2\varepsilon_2}(\eta_0). \end{cases} \tag{3.35}$$

Then, if we take an $\varepsilon_2>0$, we have

$$\tilde{b}(D_x)\tilde{a}(X')u\in H_\infty=\bigcap_s H_s. \tag{3.36}$$

We also consider $a(x)\in\mathcal{B}^\infty(R^n)$ and $b(\xi)\in S^\infty$ such that

$$\begin{cases} \text{supp}\ a(x)\subset U_{\varepsilon_1}(x_0).\\ \text{supp}\ b(\xi)\subset\Gamma_{\varepsilon_1}(\xi_0)\cap\{\xi;\ |\xi|\geqq\tilde{M}\}, \end{cases} \tag{3.37}$$

where ε_1 and $\tilde{M}$ are determined later. Then, if we get for any $a(x)$ and $b(\xi)$ of (3.37)

$$b(D_x)a(X')P_\phi u\in H_\infty, \tag{3.38}$$

we have

$$(x_0, \xi_0)\notin WF(P_\phi u).$$

Hence, we have only to prove (3.38).

We write for $\tilde{a}(x)$ and $\tilde{b}(\xi)$ of (3.35)

$$(3.39)\qquad \begin{aligned} & b(D_x)a(X')P_\phi u \\ &= b(D_x)a(X')P_\phi\tilde{b}(D_x)\tilde{a}(X')u+b(D_x)a(X')P_\phi\{I-\tilde{b}(D_x)\tilde{a}(X')\}u \\ &\equiv J_1u+J_2u \qquad (\text{I: identity map}). \end{aligned}$$

Then, since $\tilde{b}(D_x)\tilde{a}(X')u\in H_\infty$, we get

$$(3.40)\qquad J_1u\in H_\infty .$$

Consider the second term. We have by the expansion theorem in [7]

$$(3.41)\qquad \begin{aligned} \sigma(J_2)(x,\xi) &\sim b(\nabla_x\phi(x,\xi))a(x)p(x,\xi)(1-\tilde{b}(\xi)\tilde{a}(\nabla_\xi\phi(x,\xi)) \\ &+\sum_{\alpha^i,\beta^i} b^{(\alpha^1)}(\nabla_x\phi(x,\xi))a_{(\beta^1)}(x)p^{(\alpha^2)}_{(\beta^2)}(x,\xi)\{\partial_\xi^{\alpha^3}D_x^{\beta^3} \\ &\qquad\times(1-\tilde{b}(\xi)\tilde{a}(\nabla_\xi\phi(x,\xi))\}f_{\alpha^1,\beta^1,\alpha^2,\beta^2,\alpha^3,\beta^3}(x,\xi), \end{aligned}$$

where α^i *and* β^i $(1\leqq i\leqq 3)$ are multi-indexes and

$$f_{\alpha^1,\beta^1,\alpha^2,\beta^2,\alpha^3,\beta^3}(x,\xi)\in S^\infty .$$

Now we define ε_1 and $\tilde{M}$ by Lemma 3.4 such that

$$(3.42)\qquad \Gamma^1_{\varepsilon_1,R}\subset\Gamma^2_{\varepsilon_2,R}\qquad (R\geqq\tilde{M}).$$

If we have for some (x,ξ) $(|\xi|\geqq\tilde{M})$

$$b(\nabla_x\phi(x,\xi))a(x)\neq 0,$$

we have from (3.37)

$$|\nabla_x\phi(x,\xi)/|\nabla_x\phi(x,\xi)|-\xi_0/|\xi_0||<\varepsilon_1,\quad |x-x_0|<\varepsilon_1,\quad |\xi|\geqq\tilde{M}.$$

So we can see that $(x,\xi)\in\Gamma^1_{\varepsilon_1,\tilde{M}}$. Then, from (3.42) we get

$$|\xi/|\xi|-\eta_0/|\eta_0||<\varepsilon_2,\quad |\nabla_\xi\phi(x,\xi)-y_0|<\varepsilon_2,\quad |\xi|\geqq\tilde{M}.$$

Hence, from the definition of $\tilde{a}(x)$ and $\tilde{b}(\xi)$ we get

$$1-\tilde{b}(\xi)\tilde{a}(\nabla_\xi\phi(x,\xi))=0.$$

Consequently, the first term vanishes for (x,ξ) $(|\xi|\geqq\tilde{M})$. Similarly, we can see that the each term vanishes for (x,ξ) $(|\xi|\geqq\tilde{M})$. Therefore, we get

$$\sigma(J_2)(x,\xi)\in S^{-\infty}.$$

Finally, we get

$$J_2u\in H_\infty .$$

Hence, we obtain (3.38) together with (3.40). This completes the proof. Q.E.D.

4. Main theorems

Consider a hyperbolic operator

$$(4.1)\qquad \begin{aligned} L = {} & D_t^2+A_1(t, X, D_x)D_t+A_2(t, X, D_x) \\ & +B_0(t, X, D_x)D_t+B_1(t, X, D_x)+C_0(t, X, D_x) \\ & \text{on}\quad [0, T]\ (T>0) \qquad (A_j(t, x, \xi), B_j(t, x, \xi), \\ & C_j(t, x, \xi)\in B^1([0, T]; S_\rho^j),\ 1/2<\rho\leqq 1)\,. \end{aligned}$$

We assume that the characteristic roots $\lambda_i(t, x, \xi)$ $(i=1, 2)$ of the principal part of L (i.e. the roots of $\lambda^2+A_1(t, x, \xi)\lambda+A_2(t, x, \xi)=0$) are real valued.

In this section we shall consider the Cauchy problem

$$(P)\qquad Lu=0 \quad \text{on } [0, T]\,, \quad D_t^j u(0, x)=g_j(x) \qquad (j=0, 1)\,,$$

where $g_j(x)\in H_{-\infty}=\bigcup_s H_s$. For simplicity, we set

$$(4.2)\qquad \begin{cases} P_2 = P_2(t, X, D_t, D_x)=D_t^2+A_1(t, X, D_x)D_t+A_2(t, X, D_x)\,, \\ P_1=P_1(t, X, D_t, D_x)=B_0(t, X, D_x)D_t+B_1(t, X, D_x)\,, \\ P_0=P_0(t, X, D_x)=C_0(t, X, D_x)\,, \end{cases}$$

and

$$L=P(t, X, D_t, D_x)=P_2(t, X, D_t, D_x)+P_1+P_0\,.$$

DEFINITION 4.1. Let $p(x, \xi)\in S^m_{1,0\,;\,1}$. We say that $p_m(x, \xi)$ is *the principal symbol of* $p(x, \xi)$, if $p_m(x, \xi)$ and $(p-p_m)(x, \xi)$ belong to $S^m_{1,0\,;\,1}$ and $S^{m-1}_{1,0\,;\,1}$, respectively.

DEFINITION 4.2. Let $p(x, \xi)\in S^m_{1,0\,;\,1}$ and $p_m(x, \xi)$ be the principal symbol of $p(x, \xi)$. Then, we define *the subprincipal symbol with respect to* $p_m(x, \xi)$ $p_s(x, \xi)\in S^{m-1}_{1,0\,;\,0}$ by

$$(4.3)\qquad p_s(x, \xi)=(p-p_m)(x, \xi)+\frac{i}{2}\sum_{j=1}^{n}\partial_{\xi_j}\partial_{x_j}p_m(x, \xi)\,.$$

$p_2(t, x, \lambda, \xi)$ of (4.2) is the principal symbol of $p(t, x, \lambda, \xi)$. Throughout this section, when we define the subprincipal symbol of L of (4.1), we always take $p_2(t, x, \lambda, \xi)$ of (4.2) as the principal symbol.

We state the extension of the results obtained by M. Hata [5] (see also [4] and [15]).

Lemma 4.3 (*c.f.* [5]). *We assume that L of* (4.1) *satisfies the following conditions* i), ii) *and* iii).

i) $B_0(t, x, \xi)\in \bigcap_{j=0}^{1} B^j([0, T]; S_\rho^0((1-j)))$.

ii) $\lambda_{i(\alpha)}^{(\alpha)}(t, x, \xi) \in \bigcap_{j=0}^{1} B^j([0, T]; S_\rho^{1-|\alpha|}((2-j-|\alpha|)))$ $\quad (|\alpha| \leqq 1)$.

iii) *There exists a symbol* $\mu(t, x, \xi) \in \bigcap_{j=0}^{1} B^j([0, T]; S_\rho^0((1-j)))$ *such that*

$$p_s(t, x, \lambda_1(t, x, \xi), \xi) - \frac{i}{2}\{\lambda-\lambda_1, \lambda-\lambda_2\} - \mu(\lambda_1-\lambda_2) \in B^0([0, T]; S_\rho^0). \tag{4.4}$$

Here, $p_s(t, x, \lambda, \xi)$ *denotes the subprincipal symbol of* $p(t, x, \lambda, \xi)$ *in* $R^{2(n+1)}$ *and* $\{\lambda-\lambda_1, \lambda-\lambda_2\}$ *denotes the Poisson bracket (see Corollary D in Section 1). Then, there exist* $R_i(t, x, \xi) \in \bigcap_{j=0}^{1} B^j([0, T]; S_\rho^0((1-j)))$ $(i=1, 2)$ *and* $R_0(t, x, \xi) \in B^0([0, T]; S_\rho^0)$ *such that*

$$\begin{aligned} L = \{&D_t - \lambda_2(t, X, D_x) + R_2(t, X, D_x)\}\{D_t - \lambda_1(t, X, D_x) \\ &+ R_1(t, X, D_x)\} + R_0(t, X, D_x). \end{aligned} \tag{4.5}$$

Proof. If $R_i(t) \in \bigcap_{j=0}^{1} B^j([0, T]; S_\rho^0((1-j)))$ and $R_0(t) \in B^0([0, T]; S_\rho^0)$, we can write

$$\begin{aligned} &\{D_t - \lambda_2(t, X, D_x) + R_2(t, X, D_x)\}\{D_t - \lambda_1(t) + R_1(t)\} + R_0(t) \\ &\equiv D_t^2 - (\lambda_1+\lambda_2)(t, X, D_x)D_t + (\lambda_1\lambda_2)(t, X, D_x) + (R_1+R_2) \\ &\quad \times (t, X, D_x)D_t + (i\partial_t\lambda_1 - i\nabla_\xi\lambda_2\cdot\nabla_x\lambda_1 - \lambda_2 R_1 - R_2\lambda_1) \\ &\qquad\qquad (\text{mod. } B^0([0, T]: S_\rho^0)), \end{aligned}$$

where $\sigma(\lambda_1+\lambda_2)(t, x, \xi) = \lambda_1(t, x, \xi) + \lambda_2(t, x, \xi)$, $\sigma(\lambda_1\lambda_2)(t, x, \xi) = \lambda_1(t, x, \xi)\lambda_2(t, x, \xi)$. Comparing with (4.1), we have

$$\begin{cases} B_0(t, x, \xi) = R_1(t, x, \xi) + R_2(t, x, \xi), \\ B_1(t, x, \xi) \equiv \{i(\partial_t\lambda_1 - \nabla_\xi\lambda_2\cdot\nabla_x\lambda_1) - \lambda_2 R_1 - R_2\lambda_1\}(t, x, \xi) \\ \qquad\qquad (\text{mod. } B^0([0, T]; S_\rho^0)). \end{cases}$$

So we get

$$\begin{aligned} &(\lambda_1-\lambda_2)R_1(t, x, \xi) \\ &\equiv B_0\lambda_1 + B_1 - i(\partial_t\lambda_1 - \nabla_\xi\lambda_2\cdot\nabla_x\lambda_1) \\ &\equiv p_s(t, x, \lambda_1(t, x, \xi), \xi) - \frac{i}{2}\{\lambda-\lambda_1, \lambda-\lambda_2\} \\ &\qquad + \frac{i}{2}\sum_{j=1}^{n}(\partial_{\xi_j}\partial_{x_j}\lambda_1)(\lambda_1-\lambda_2) \\ &\equiv (\mu + \frac{i}{2}\sum_{j=1}^{n}\partial_{\xi_j}\partial_{x_j}\lambda_1)(\lambda_1-\lambda_2) \qquad (\text{mod. } B^0([0, T]; S_\rho^0)). \end{aligned} \tag{4.6}$$

Define $R_i(t, x, \xi)$ $(i=1, 2)$ by

$$(4.7)\qquad \begin{cases} R_1 = \mu + \dfrac{i}{2}\sum_{j=1}^{n} \partial_{\xi_j}\partial_{x_j}\lambda_1 , \\ R_2 = B_0 - \Big(\mu + \dfrac{i}{2}\sum_{j=1}^{n} \partial_{\xi_j}\partial_{x_j}\lambda_1\Big) . \end{cases}$$

Then, we can see that

$$\begin{cases} R_i(t, x, \xi) \in \bigcap_{j=0}^{1} B^j([0, T]; S^0_\rho((1-j))) , \\ L \equiv \{D_t - \lambda_2(t, x, D_x) + R_2(t)\}\{D_t - \lambda_1(t) + R_1(t)\} \\ \qquad\qquad (\mathrm{mod.}\ B^0([0, T]; S^0_\rho)) . \end{cases}$$

We obtain Lemma 4.3. Q.E.D.

Let $\phi_i(t, s; x, \xi)$ $(i=1, 2)$ be the solution of the eiconal equation (1.15) corresponding to $\lambda_i(t, x, \xi)$ in Lemma 4.3. Then, we have

Theorem 4.4 (*c.f.* [5]). *Let L be the operator which has the properties of Lemma* 4.3. *We also assume that*

iv) $\{\lambda-\lambda_1, \lambda-\lambda_2\} \in B^0([0, T]; S^0_\rho)$.

Then, the fundamental solution $E_k(t, s)$ $(k=0, 1)$ of the Cauchy problem (P) (i.e. $LE_k=0$ on $[s, T]$, $\partial_t^j E_k(s, s)=\delta_{j,k}$ $(j, k=0, 1)$) can be constructed for a small $T_0>0$ in the form

$$(4.8)\qquad E_k(t, s) = \sum_{i=1}^{2} H^k_{i,\phi_i}(t, s; X, D_x) + \int_s^t H^k_{3,\phi_1\#\phi_2}(t, t_1, s; X, D_x)dt_1 \qquad (0 \leqq s \leqq t \leqq T_0) ,$$

where

$$\begin{cases} \sigma(H^k_i)(t, s; x, \xi) \in B^2_t \cap B^0_{t,s}(\Delta_0; S^{1-k}_\rho) \qquad (k=0, 1,\ i=1, 2) , \\ \sigma(H^k_3)(t, t_1, s; x, \xi) \in B^0_{t,t_1,s}(\Delta_1; S^{1-k}_\rho) \\ \qquad (\Delta_\nu = \{0 \leqq s = t_{\nu+1} \leqq \cdots \leqq t_1 \leqq t_0 = t\}) . \end{cases}$$

Proof. The equality (4.5) is valid from Lemma 4.3. So setting for $t \in [0, T]$

$$(4.9)\qquad v_0(t, x) = u(t, s) ,\quad v_1(t, x) = (D_t - \lambda_1(t, X, D_x) + R_1(t))v_0 ,$$

we have the system for $V(t, x) = {}^t(v_0(t, x), v_1(t, x))$

$$(4.10)\qquad \boldsymbol{L} = D_t - \begin{pmatrix} \lambda_1, & 0 \\ 0, & \lambda_2 \end{pmatrix}(t, X, D_x) + \begin{pmatrix} R_1, & -1 \\ R_0, & R_2 \end{pmatrix}(t, X, D_x) .$$

Then, the initial data V_0 for (4.10) becomes

$$(4.11)\qquad V_0 = G(0, X, D_x)\begin{pmatrix} g_0 \\ g_1 \end{pmatrix},\quad G(s, x, \xi) = \begin{pmatrix} 1 & , & 0 \\ -(\lambda_1 - R_1), & & 1 \end{pmatrix}(s, x, \xi) .$$

Now we shall construct the fundamental solution $\boldsymbol{E}(t, s)$ for $\boldsymbol{L}$ by the same method of the proof of Theorem 3.1 in Kumano-go-Taniguchi [11], where

$$\boldsymbol{L}\boldsymbol{E}(t, s) = 0 \qquad \text{on } [s, T], \quad \boldsymbol{E}(s, s) = I \quad \text{(unit matrix)}.$$

Set

$$(4.12) \qquad \boldsymbol{I}_\phi(t, s) = \begin{pmatrix} I_{\phi_1}(t, s), & 0 \\ 0, & I_{\phi_2}(t, s) \end{pmatrix},$$

where $I_{\phi_i}(t, s)$ is Fourier integral operator with symbol 1 and phase function $\phi_i(t, s; x, \xi)$. Then, we have

$$(4.13) \qquad \boldsymbol{L}\boldsymbol{I}_\phi(t, s) = \boldsymbol{F}_\phi(t, s),$$

where $\boldsymbol{F}_\phi(t, s) = \sum_{j=1}^{2} F_{j,\phi_j}(t, s)$ is a matrix of Fourier integral operators with $\phi_j(t, s; x, \xi)$ and symbols $F_j(t, s; x, \xi) \in B^0_{t,s}(\Delta_0; S^0_\rho)$. From (4.13) the fundamental solution for $\boldsymbol{L}$, as the continuous operator from the Sobolev space H_σ into itself for any fixed σ, is constructed in the form:

$$(4.14) \qquad \boldsymbol{E}(t, s) = \boldsymbol{I}_\phi(t, s) + \int_s^t \boldsymbol{I}_\phi(t, \theta) \sum_{\nu=1}^{\infty} W_\nu(\theta, s) d\theta .$$

Here, $\{W_\nu(t, s)\}_{\nu=1}^{\infty}$ are defined by

$$(4.15) \qquad \begin{cases} W_1(t, s) = -i\boldsymbol{F}_\phi(t, s), \\ W_{\nu+1}(t, s) = \int_s^t W_1(t, \theta) W_\nu(\theta, s) d\theta \qquad (\nu = 1, 2, \cdots, \text{ c.f. [6] and [11]}). \end{cases}$$

We note that $W_{\nu+1}(t, s)$ can be written in the form

$$(4.16) \qquad W_{\nu+1}(t, s) = \int_s^t \int_s^{t_1} \cdots \int_s^{t_{\nu-1}} W^{(\nu+1)}(t, t_1, \cdots, t_\nu, s) dt_\nu \cdots dt_1$$
$$(W^{(\nu+1)} = W_1(t, t_1) W_1(t_1, t_2) \cdots W_1(t_\nu, s),\ t_0 = t),$$

and $W^{(\nu+1)}$ has the form

$$(4.17) \qquad W^{(\nu+1)}(t, t_1, \cdots, t_\nu, s) = \sum_{j_1, \cdots, j_{\nu+1}=1}^{2} (-i)^{\nu+1} F_{j_1, \phi_{j_1}}(t, t_1) \cdots F_{j_{\nu+1} \phi_{j_{\nu+1}}}(t_\nu, s).$$

By Theorem A we have for a small $T_0 > 0$

$$(4.18) \qquad \begin{cases} \phi_j(t_{k-1}, t_k) \in \mathcal{P}_\rho(c_0(t_{k-1} - t_k)) \qquad (t_0 = t,\ t_{\nu+1} = s), \\ \phi_{j_1}(t, t_1) \# \cdots \# \phi_{j_{\nu+1}}(t_\nu, s) \in \mathcal{P}_\rho(c_1(t - s)) \qquad (0 \leqq s \leqq t \leqq T_0), \end{cases}$$

for constants $c_0 > 0$, $c_1 > 0$ (see also Theorem A). Then, by Theorem E we can find the Fourier integral operators

$$\tilde{W}_{j_1,\cdots,j_{\nu+1},\Phi j_1,\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s),$$

with phase functions

$$\Phi_{j_1\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s) = \phi_{j_1}(t, t_1)\#\phi_{j_2}(t_1, t_2)\#\cdots\#\phi_{j_{\nu+1}}(t_\nu, s),$$

and symbols $\tilde{W}_{j_1,\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s)$ of class $B^0(\Delta_\nu; S^0_\rho)$ $(=B^0_{t_0,t_1,\cdots,t_{\nu+1}}(\Delta_\nu; S^0_\rho))$, such that for any l we have semi-norm estimates with respect to the component $\tilde{W}^{ij}_{i_1,\cdots,j_{\nu+1}}$

$$|\tilde{W}^{ij}_{j_1,\cdots,j_{\nu+1}}|^{(0)}_l \leqq C^\nu_l, \tag{4.19}$$

for a constant $C_l>0$ independent of ν, and for any l, real σ and integer $N>0$ we have the estimates of the operator norm

$$||\tilde{W}_{j_1,\cdots,j_{\nu+1},\Phi j_1,\cdots,j_{\nu+1}} - F_{j_1,\phi j_1}\cdots F_{j_{\nu+1}\phi j_{\nu+1}}||_{H_\sigma\to H_{\sigma+N}} \leqq C^\nu_{\sigma,N} \tag{4.20}$$

for a constant $C_{\sigma,N}>0$ independent of ν.

Set

$$\tilde{W}_{\nu+1,\Phi_{\nu+1}}(t, t_1, \cdots, s) = \sum_{j_1,\cdots,j_{\nu+1}=1}^{2} (-i)^{\nu+1}\tilde{W}_{j_1,\cdots,j_{\nu+1},\Phi j_1,\cdots,j_{\nu+1}}(t, t_1, \cdots, t_\nu, s), \tag{4.21}$$

$$W_{-\infty}(t, s) = \int_s^t \boldsymbol{I}_\phi(t, \theta)\{\sum_{\nu=1}^{\infty}\int_s^\theta\int_s^{t_1}\cdots\int_s^{t_{\nu-1}}(W^{(\nu+1)} - \tilde{W}_{\nu+1,\Phi_{\nu+1}})(\theta, t_1, \cdots, t_\nu, s)dt_\nu\cdots dt_1\}d\theta \qquad (t_0=\theta). \tag{4.22}$$

Then, we can see by (4.20) and Proposition 3.2 in [11] that

$$W_{-\infty}(t, s) \in B^1_t \cap B^0_{t,s}(\Delta_0; S^{-\infty}).$$

Therefore, we obtain the fundamental solution $\boldsymbol{E}(t, s)$ for $\boldsymbol{L}$ in the form

$$\begin{aligned}\boldsymbol{E}(t, s) = \boldsymbol{I}_\phi(t, s) &+ \int_s^t \boldsymbol{I}_\phi(t, \theta)W_1(\theta, s)d\theta \\ &+ \sum_{\nu=1}^{\infty}\int_s^t\int_s^\theta\int_s^{t_1}\cdots\int_s^{t_{\nu-1}} \boldsymbol{I}_\phi(t, \theta)\tilde{W}_{\nu+1,\Phi_{\nu+1}}(\theta, t_1, \cdots, t_\nu, s) \\ &\times dt_\nu\cdots dt_1 d\theta + W_{-\infty}(t, s).\end{aligned} \tag{4.23}$$

We note that we did not use the assumption iv) until now. If we use iv), then, we can apply (1.44) to the right hand side of (4.23). For example,

$$\begin{aligned}&\int_s^t\int_s^\theta I_{\phi_1}(t, \theta)W_{\Phi_{1,2}}(\theta, t_1, s)dt_1 d\theta \\ &= \int_s^t\int_s^\theta I_{\phi_1}(t, \theta)W'_{\Phi_{1,2}}(\theta, t_1, s)dt_1 d\theta\end{aligned}$$

$$= \int_s^t \int_s^\theta W''(t, \theta, t_1, s)_{\Phi_{1,2}(t,t,s)} dt_1 d\theta$$
$$= \int_s^t W'''_{\Phi_{1,2}}(t, t_1, s) dt_1 ,$$

where

$$\begin{cases} W'(\theta, t_1, s) \\ = W(\theta, \theta - t_1 + s, s) \exp i\{\Phi_{2,1}(\theta, \theta - t_1 + s, s) - \Phi_{1,2}(\theta, t_1, s)\} , \\ W'''(t, t_1, s) = \int_{t_1}^t W''(t, \theta, t_1, s) d\theta . \end{cases}$$

By repeating this process, we obtain the expression

$$\begin{aligned} \boldsymbol{E}(t, s) = \hat{W}_{1,\phi_1}(t, s; X, D_x) + \hat{W}_{2,\phi_2}(t, s) \\ + \int_s^t \hat{W}_{3,\phi_1 \# \phi_2}(t, t_1, s) dt_1 + W_{-\infty}(t, s) , \end{aligned} \tag{4.24}$$

where

$$\begin{cases} \hat{W}_i(t, s) \in B_t^1 \cap B_{t,s}^0(\Delta_0; S_\rho^0) \qquad (i = 1, 2) , \\ \hat{W}_3(t, t_1, s) \in B^0(\Delta_1; S_\rho^0) . \end{cases}$$

Then, we can see from (4.9) and (4.11) that the fundamental solution $E_k(t, s)$ ($k=0, 1$) for L can be represented by

$$E_k(t, s) = \text{“the first component of } E(t, s) G(s, X, D_x) \begin{pmatrix} \delta_{0,k} \\ \delta_{1,k} \end{pmatrix} \text{”} .$$

We get (4.8). Q.E.D.

Remark 1. The formal adjoint L^* of L in Theorem 4.4 also satisfies the assumption of Theorem 4.4. Then, we get the fundamental solution $E_k^*(t, \tau)$ ($k=0, 1$) ($0 \leqq t \leqq \tau \leqq T_0$) of the backward initial value problem

$$L^* E_k^*(t, \tau) = 0 \quad \text{on } [t, \tau] , \quad \partial_t^j E_k^*(\tau, \tau) = \delta_{j,k} \qquad (j, k = 0, 1) ,$$

in the similar way to the proof of Theorem 4.4. Consequently, we can prove the uniqueness of the Cauchy problem (P) (c.f. [7]).

Remark 2. We get the fundamental solution of the form (4.23) without the condition iv) in Theorem 4.4. But, then, the statement for the propagation of singularities will not be simple (c.f. see [11]).

We consider L of (4.1) whose characteristic roots are non-regular. Hereafter, we always assume the following I) and II) for L in (4.1).

Condition I): $A_j(t, x, \xi)$, $B_j(t, x, \xi)$ and $C_j(t, x, \xi)$ in (4.1) belong to $B^\infty([0, T]; S_{1,0}^j)$, respectively.

Condition II): We have

$$\begin{cases} A_j(t, x, \delta\xi) = \delta^j A_j(t, x, \xi) & (j = 1, 2)\,, \\ B_j(t, x, \delta\xi) = \delta^j B_j(t, x, \xi) & (j = 0, 1)\,, \\ C_0(t, x, \delta\xi) = C_0(t, x, \xi) & (|\xi| \geqq M > 0,\ \delta \geqq 1)\,. \end{cases}$$

Let $\phi_i(t, s; x, \xi)$ $(i=1, 2)$ be the solution of (1.15) corresponding to $\tilde{\lambda}_i(t, x, \xi)$, where $\tilde{\lambda}_i(t, x, \xi)$ is the approximation of $\lambda_i(t, x, \xi)$ defined by (2.24) in Section 2, that is,

$$\tilde{\lambda}_i(t, x, \xi) = \int \psi(\langle\xi\rangle^{(1-\rho)}y)\lambda_i(t, x+y, \xi)dy\langle\xi\rangle^{\delta n}\,. \tag{4.25}$$

For $\lambda_i(t, x, \xi)$ $(i=1, 2)$ we define $\{q^i, p^i\}(t, s)$ by (3.16) corresponding to λ_i and define $\{Q_{1,2}, P_{1,2}\}(t, \tau, s)$ by (3.17) corresponding to λ_1 and λ_2. Then, we obtain

Theorem 4.5. *Assume the conditions* I) *and* II) *for L of* (4.1). *We also assume the following* i), ii) *and* iii).

i) *Characteristic roots $\lambda_i(t, x, \xi)$ $(i=1, 2)$ belong to $B^1([0, T]; S^1_{1,0\,;\,3+\sigma})$* $(\sigma>0)$.

ii) *The Poisson bracket $\{\lambda-\lambda_1, \lambda-\lambda_2\}=0$.*

iii) *There exists a symbol $\mu(t, x, \xi)\in B^0([0, T]; S^0_{1,0\,;\,1+\sigma})\cap B^1([0, T]; S^0_{1,0\,;\,0})$ such that*

$$p_s(t, x, \lambda_1(t, x, \xi), \xi)-\mu(\lambda_1-\lambda_2)\in B^0(([0, T]; S^0_{1,0\,;\,0})\,,$$

where $p_s(t, x, \lambda, \xi)$ denotes the subprincipal symbol in $R^{2(n+1)}_{t,x,\lambda,\xi}$ of L. Then, the fundamental solution $E_k(t, s)$ $(k=0, 1)$ exists for some $T_0>0$ and is constructed in the form (4.8) *by using $\rho=(2+\sigma)/(4+\sigma)$. Moreover, the solution $u(t, x)$ of the Cauchy problem (P) is unique and then, we obtain*

$$\begin{aligned} WF(u(t))\subset \bigcup_{0\leqq\tau\leqq t} \mathrm{Conic}\,(\{Q_{1,2}, P_{1,2})(t, \tau, 0; x, \xi); (x, \xi) \\ \in WF(G),\ |\xi|\geqq M\}\,, \end{aligned} \tag{4.26}$$

where $WF(G)=\bigcup_{j=0}^{1} WF(g_j)$ and M is a large constant.

Proof. In this proof, we always use the approximation (4.25) for fixed $\rho=(2+\sigma)/(4+\sigma)$.

We define the symbol $A'_j(t, x, \xi)$ $(j=1, 2)$, $B'_j(t, x, \xi)$ $(j=0, 1)$ and $C'_0(t, x, \xi)$ by

$$\begin{cases} A'_1 = -(\tilde{\lambda}_1+\tilde{\lambda}_2)(t, x, \xi)\,, \quad A'_2 = \tilde{\lambda}_1(t, x, \xi)\tilde{\lambda}_2(t, x, \xi)\,, \\ B'_0 = B_0+A_1-A'_1\,, \quad B'_1 = B_1\,, \quad C'_0 = C_0+A_2-A'_2\,. \end{cases} \tag{4.27}$$

Then, it is clear that we can write

$$\begin{aligned} L = D_t^2+A'_1(t, X, D_x)D_t+A'_2(t, X, D_x) \\ +B'_0(t, X, D_x)D_t+B'_1(t, X, D_x)+C'_0(t, X, D_x)\,. \end{aligned} \tag{4.28}$$

We can see that

$$(4.29)\qquad \begin{cases} A'_j(t, x, \xi)\in B^1([0, T]; S^j_{1,1-\rho}(3+\sigma)) & (j=1, 2), \\ B'_j(t, x, \xi)\in B^1([0, T]; S^j_\rho((1))) & (j=0, 1), \\ C'_0(t, x, \xi)\in B^1([0, T]; S^0_\rho). \end{cases}$$

In fact the relation (4.29) is clear for A'_j $(j=1, 2)$ and B'_1 by Theorem 2.5. Since we have by Theorem 2.5

$$A_1-A'_1 = A_1-\tilde{A}_1\in B^1([0, T]; S^{-\infty}),$$

we get $B'_0\in B^1([0, T]; S^0_\rho((1)))$. By Theorem 2.6 we have

$$\begin{aligned} A_2-A'_2 &= A_2-\tilde{A}_2-(\tilde{\lambda}_1\tilde{\lambda}_2-\widetilde{\lambda_1\lambda_2}) \\ &\in B^1([0, T]; S^{2-(1-\rho)(4+\sigma)}_\rho) \ (=B^1([0, T]; S^0_\rho)). \end{aligned}$$

So we get $C'_0\in B^1([0, T]; S^0_\rho)$.

Next set

$$(4.30)\qquad \begin{cases} p'_2(t, x, \lambda, \xi) = \lambda^2+A'_1(t, x, \xi)\lambda+A'_2(t, x, \xi), \\ p'_1(t, x, \lambda, \xi) = B'_0(t, x, \xi)\lambda+B'_1(t, x, \xi), \\ p'_0(t, x, \xi) = C'_0(t, x, \xi). \end{cases}$$

We can rewrite L in the form

$$(4.31)\qquad \begin{aligned} L &= D_t^2+A'_1(t, X, D_x)D_t+A'_2(t, X, D_x)+B'_0D_t+B'_1+C'_0 \\ &= P'_2(t, X, D_t, D_x)+P'_1(t, X, D_t, D_x)+P'_0(t, X, D_x). \end{aligned}$$

We prove that L in the form (4.31) satisfies the assumption of Theorem 4.4. We get $B'_0(t, x, \xi)\in B^1([0, T]; S^1_\rho((1)))$ from (4.29). It is clear from (4.27) that the characteristic roots of L in the form (4.31) are $\tilde{\lambda}_i(t, x, \xi)$ $(i=1, 2)\in B^1([0, T]; S^1_{1,1-\rho}(3+\sigma))$. So the condition i) and ii) in Lemma 4.3 are satisfied. Consequently, we have only to prove the condition iii) and iv) in Theorem 4.4.

Since $\lambda_i(t, x, \xi)$ $(i=1, 2)$ belong to $B^1([0, T]; S^1_{1,0\,;\,3+\sigma})$, we get

$$(4.32)\qquad \begin{cases} \nabla_x\tilde{\lambda}_i(t, x, \xi) = \widetilde{\nabla_x\lambda_i}(t, x, \xi), \\ \nabla_\xi\tilde{\lambda}_i(t, x, \xi) = \widetilde{\nabla_\xi\lambda_i}(t, x, \xi)+H_i(t, x, \xi) \qquad (i=1, 2), \end{cases}$$

where

$$(4.33)\qquad \begin{aligned} &H_i(t, x, \xi) \\ &= \int \psi(y)y\cdot(\nabla_x\lambda_i(t, x+\langle\xi\rangle^{-(1-\rho)}y, \xi))dy\nabla_\xi(\langle\xi\rangle^{-(1-\rho)}). \end{aligned}$$

From the assumption ii) we have

$$\partial_t\lambda_1-\partial_t\lambda_2+\nabla_\xi\lambda_1\cdot\nabla_x\lambda_2-\nabla_x\lambda_1\cdot\nabla_\xi\lambda_2 = 0.$$

Taking the approximation for the both sides, we get

$$
\begin{aligned}
(4.34)\qquad & \{\lambda-\tilde{\lambda}_1, \lambda-\tilde{\lambda}_2\}(t, x, \xi) \\
&= \partial_t\tilde{\lambda}_1-\partial_t\tilde{\lambda}_2+\nabla_\xi\tilde{\lambda}_1\cdot\nabla_x\tilde{\lambda}_2-\nabla_x\tilde{\lambda}_1\cdot\nabla_\xi\tilde{\lambda}_2 \\
&= (\nabla_\xi\tilde{\lambda}_1\cdot\nabla_x\tilde{\lambda}_2-\widetilde{\nabla_\xi\lambda_1\cdot\nabla_x\lambda_2})-(\nabla_x\tilde{\lambda}_1\cdot\nabla_\xi\tilde{\lambda}_2-(\widetilde{\nabla_x\lambda_1\cdot\nabla_\xi\lambda_2}) \\
&= (\widetilde{\nabla_\xi\lambda_1}\cdot\widetilde{\nabla_x\lambda_2}-\widetilde{\nabla_\xi\lambda_1\cdot\nabla_x\lambda_2})-(\widetilde{\nabla_x\lambda_1}\cdot\widetilde{\nabla_\xi\lambda_2}-\widetilde{\nabla_x\lambda_1\cdot\nabla_\xi\lambda_2}) \\
&\qquad\qquad +H_1\cdot\widetilde{\nabla_x\lambda_2}-\widetilde{\nabla_x\lambda_1}\cdot H_2 .
\end{aligned}
$$

Since $\nabla_\xi\lambda_i(t, x, \xi)$ and $\nabla_x\lambda_i(t, x, \xi)$ belong to $B^1([0, T]; S^0_{1,0\,;\,3+\sigma})$ and $B^1([0, T]; S^1_{1,0\,;\,2+\sigma})$, respectively, by Theorem 2.6 we can see that

$$
\begin{aligned}
&\widetilde{\nabla_\xi\lambda_1}\cdot\widetilde{\nabla_x\lambda_2}-\widetilde{\nabla_\xi\lambda_1\cdot\nabla_x\lambda_2},\ \widetilde{\nabla_x\lambda_1}\cdot\widetilde{\nabla_\xi\lambda_2}-\widetilde{\nabla_x\lambda_1\cdot\nabla_x\lambda_2} \\
&\qquad \in B^1([0, T]; S_\rho^{1-(1-\rho)(3+\sigma)})\subset B^1([0, T]; S^0_\rho) .
\end{aligned}
$$

By Lemma 2.3 we also have

$$
H_i(t, x, \xi)\in B^0([0, T]; S_\rho^{-(3+\sigma)(1-\rho)}) .
$$

Consequently, it follows that $H_1\cdot\widetilde{\nabla_x\lambda_2}(t, x, \xi)$ and $H_2\cdot\widetilde{\nabla_x\lambda_1}(t, x, \xi)$ belong to $B^0([0, T]; S^0_\rho)$. Hence, from (4.34) we can see that $\{\lambda-\tilde{\lambda}_1, \lambda-\tilde{\lambda}_2\}$ belongs to $B^0([0, T]; S^0_\rho)$.

Next we can easily have

$$
\begin{aligned}
(4.35)\qquad & p_s(t, x, \lambda_1(t, x, \xi), \xi)-\frac{i}{2}\{\lambda-\lambda_1, \lambda-\lambda_2\} \\
&= B_0\lambda_1+B_1+C_0+\frac{i}{2}[\partial_t\partial_\lambda(\lambda-\lambda_1)(\lambda-\lambda_2) \\
&\qquad +\sum_{j=1}^{n}\partial_{\xi_j}\partial_{x_j}(\lambda-\lambda_1)(\lambda-\lambda_2)-\{\lambda-\lambda_1, \lambda-\lambda_2\}]|_{\lambda=\lambda} \\
&= B_0\lambda_1+B_1+C_0-i\partial_t\lambda_1+i\nabla_x\lambda_1\cdot\nabla_\xi\lambda_2 \\
&\qquad -\frac{i}{2}\sum_{j=1}^{n}(\partial_{\xi_j}\partial_{x_j}\lambda_1)(\lambda_1-\lambda_2) .
\end{aligned}
$$

Then, by the similar way to the proof of the condition iv), we get

$$
\begin{aligned}
(4.36)\qquad & p'_s(t, x, \tilde{\lambda}_1(t, x, \xi), \xi)-\frac{i}{2}\{\lambda-\tilde{\lambda}_1, \lambda-\tilde{\lambda}_2\} \\
&\qquad -\tilde{\mu}(\tilde{\lambda}_1-\tilde{\lambda}_2)\in B^0[0, T]; S^0_\rho) ,
\end{aligned}
$$

where $p'_s(t, x, \lambda, \xi)$ is the subprincipal symbol of L in the form (4.31). Hence, we obtain

$$
(4.37)\qquad p'_s(t, x, \tilde{\lambda}_1(t, x, \xi), \xi)-\tilde{\mu}(\tilde{\lambda}_1-\tilde{\lambda}_2)\in B^0([0, T]; S^0_\rho) .
$$

Using $\tilde{\mu}(t, x, \xi)\in \bigcap_{j=0}^{1} B^j([0, T]; S_\rho^0((1-j)))$, we can see that L in the form (4.31) satisfies iii) in Lemma 4.3. Finally, we obtain (4.8).

It is clear by Remark 1 that the solution $u(t, x)$ of the Cauchy problem (P) is unique. We shall show (4.26). We can easily see by Theorem 2.5 that $(\tilde{\lambda}_i-\lambda_i)(t, x, \xi)$ belongs to $B^0([0, T]; S_{1,0;2}^{1-(1-\rho)(1+\sigma)})$. Hence, considering together with the Condition I), we obtain (4.26) by Corollary 3.4. Thus the proof is complete. Q.E.D.

REMARK 3. We can see from (4.36) and Remark 2 that we get the fundamental solution in the form (4.23) without the condition ii) in Theorem 4.5.

REMARK 4. K. Taniguchi in [18] recently proved Theorem E in Section 1 when the case $\rho=1/2$. By using this we can also prove Theorem 4.5 when σ in assumption i) and iii) of Theorem 4.5 equals to zero.

EXAMPLE 1. For the differential operator in $[0, T]\times R_x^2$

$$L = D_t^2-(x_1^8+x_2^8)(D_{x_1}^2+D_{x_2}^2)+x_1^6D_{x_1}+x_2^6D_{x_2},$$

we can see that L satisfies the condition of Theorem 4.5.

EXAMPLE 2. For the differential operator in $[0, T]\times R_x^2$

$$L = D_t^2-(x_1^8D_{x_1}^2+x_2^8D_{x_2}^2)+x_1^6D_{x_1}+x_2^6D_{x_2},$$

we obtain

$$L = D_t^2-(x_1^8D_{x_1}^2+x_2^8D_{x_2}^2)\alpha(D_x)+x_1^8D_{x_1}+x_2^6D_{x_2}-(x_1^8D_{x_1}^2+x_2^8D_{x_2}^2)(I-\alpha(D_x)),$$

where $\alpha(\xi)$ belongs to $C^\infty(R^2)$ such that

$$\alpha(\xi) = 0 \quad (|\xi|\leqq 1), \qquad \alpha(\xi) = 1 \quad (|\xi|\geqq 2).$$

Then, we can see that L satisfies the condition of Theorem 4.5.

EXAMPLE 3. Consider a hyperbolic operator in $[0, T]\times R_x^2$

$$L = D_t^2-2(D_{x_1}+D_{x_2})D_t-\{(x_1+t)^{10}+(x_2+t)^{10}-1\}(D_{x_1}+D_{x_2})^2 \\ +a(t, x)D_t-a(t, x)(D_{x_1}+D_{x_2})+b(t, x),$$

where $a(t, x)$, $b(t, x)\in \mathcal{B}^\infty(R_{t,x}^3)$. Then, we obtain as the characteristic roots

$$\lambda_\pm = (1\pm\sqrt{(x_1+t)^{10}+(x_2+t)^{10}})(\xi_1+\xi_2)\in B^1([0, T]; S_{1,0;4}^1).$$

We can see that L also satisfies the condition ii) and iii) of Theorem 4.5.

5. Hyperbolic operators of higher order

In this section we shall treat hyperbolic operators of higher order with non-regular characteristic roots, whose each multiplicity is at most two.

Consider a hyperbolic operator

$$(5.1) \qquad L = D_t^m + \sum_{j=1}^m A_j(t, X, D_x) D_t^{m-j} + \sum_{j=1}^m B_j(t, X, D_x) D_t^{m-j} \quad \text{on} \quad [0, T]$$

$$(A_j(t, x, \xi) \in B^\infty([0, T]; S^j),\ B_j(t, x, \xi) \in B^\infty([0, T]; S^{j-1})) .$$

We assume that for a constant $M>0$ we have

$$(5.2) \qquad A_j(t, x, \delta\xi) = \delta^j A_j(t, x, \xi) \qquad (|\xi| \geqq M,\ \delta \geqq 1) .$$

Throughout this section let $\lambda^m + \sum_{j=1}^m A_j(t, x, \xi)\lambda^{m-j}$ be the principal part of L. We assume that the characteristic roots of the principal part of L are real valued and that multiplicity of each characteristic root is at most two. That is, the characteristic roots $\lambda_1^{(l)}$, $\lambda_2^{(l)}$ and $\lambda_1^{(j)}(1 \leqq l \leqq k,\ k+1 \leqq j \leqq m-k)$ satisfy for a constant $C_0>0$

$$(5.3) \qquad \inf_{[0, T] \times R_x^n} |\lambda_i^{(j)}(t, x, \xi) - \lambda_{i'}^{(j')}(t, x, \xi)| \geqq C_0 |\xi|$$

$$(i, i' = 1, 2,\ 1 \leqq j \neq j' \leqq m-k) ,$$

where $\lambda_i^{(j)} = \lambda_1^{(j)}$ $(k+1 \leqq j \leqq m-k)$.

We study the Cauchy problem

$$(\mathrm{P}.2) \qquad \begin{cases} Lu = 0 & \text{on} \quad [0, T] , \\ \partial_t^j u|_{t=0} = g_j(x) \quad (\in H_{-\infty}) & (j = 0, \cdots, m-1) . \end{cases}$$

First we state a proposition on the subprincipal symbol.

Proposition 5.1. *Let $p(x, \xi) \in S^{m_1}$, $q(x, \xi) \in S^{m_2}$ and $p_0(x, \xi)$, $q_0(x, \xi)$ be the principal symbols, respectively. Set $h(x, \xi) = \sigma(P \cdot Q)(x, \xi)$ where $\sigma(P \cdot Q)(x, \xi)$ denotes the single symbol of $P(X, D_x) \cdot Q(X, D_x)$ (i.e. $H(X, D_x) = P \cdot Qu$, $u \in \mathcal{S}$, c.f. [6]). Then, we get*

$$(5.4) \qquad h_s(x, \xi) \equiv p_s(x, \xi) q(x, \xi) + p(x, \xi) q_s(x, \xi) - \frac{i}{2}\{p, q\}(x, \xi) \quad (\mathrm{mod.}\ S^{m_1+m_2-2}) ,$$

where $h_s(x, \xi)$, $p_s(x, \xi)$, $q_s(x, \xi)$ denote the subprincipal symbols with respect to $p_0(x, \xi) q_0(x, \xi)$, $p_0(x, \xi)$, $q_0(x, \xi)$, respectively and $\{p, q\}(x, \xi)$ denotes the Poisson bracket, that is,

$$\{p, q\}(x, \xi) = (\nabla_\xi p \cdot \nabla_x q - \nabla_x p \cdot \nabla_\xi \xi q)(x, \xi) .$$

Proof. Set

$$p_1(x,\xi)=p(x,\xi)-p_0(x,\xi)\,,\quad q_1(x,\xi)=q(x,\xi)-q_0(x,\xi)\,. \tag{5.5}$$

By the expansion theorem of the double symbol (c.f. [6]), we have

$$h(x,\xi)\equiv p_0(x,\xi)q_0(x,\xi)+(p_0q_1+p_1q_0-i\nabla_\xi p_0\cdot\nabla_x q_0)(x,\xi) \quad (\text{mod. } S^{m_1+m_2-2})\,. \tag{5.6}$$

Hence, noting that the principal symbol of h is p_0q_0, we get

$$\begin{aligned}
h_s(x,\xi) &\equiv (p_0q_1+p_1q_0-i\nabla_\xi p_0\cdot\nabla_x q_0)+\frac{i}{2}\sum_{j=1}^n \partial_{\xi_j}\partial_{x_j}(p_0q_0)\\
&\equiv p_0q_1+p_1q_0-i\nabla_\xi p_0\cdot\nabla_x q_0+\frac{i}{2}(\nabla_\xi p_0\cdot\nabla_x q_0+\nabla_x p_0\cdot\nabla_\xi q_0)\\
&\quad+\frac{i}{2}\sum_{j=1}^n((\partial_{\xi_j}\partial_{x_j}q_0)p_0+q_0\partial_{\xi_j}\partial_{x_j}p_0)\\
&\equiv p_sq_0+p_0q_s-\frac{i}{2}(\nabla_\xi p_0\cdot\nabla_x q_0-\nabla_x p_0\cdot\nabla_\xi q_0)\\
&\equiv p_sq+pq_s-\frac{i}{2}\{p,q\} \qquad (\text{mod. } S^{m_1+m_2-2})\,.
\end{aligned}$$

We obtain (5.4). Q.E.D.

Corollary 5.2. *Let $p(x,\xi)\in S^{m_1}$, $q(x,\xi)\in S^{m_2}$, $r(x,\xi)\in S^{m_3}$ and let $p_0(x,\xi)$, $q_0(x,\xi)$, $r_0(x,\xi)$ be the principal symbols, respectively. Set $h(x,\xi)=\sigma(P\cdot Q\cdot R)(x,\xi)$, where $\sigma(P\cdot Q\cdot R)(x,\xi)$ denotes the single symbol of $P\cdot Q\cdot R$. Then, we get*

$$h_s(x,\xi)\equiv p_sqr+pq_sr+pqr_s-\frac{i}{2}(\{p,q\}r+\{q,r\}p-\{r,p\}q) \quad (\text{mod. } S^{m_1+m_2+m_3-2})\,, \tag{5.7}$$

where h_s, p_s, q_s and r_s denote the subprincipal symbols with respect to $p_0q_0r_0$, p_0, q_0 and r_0, respectively.

Proof. Since $p_0(x,\xi)q_0(x,\xi)r_0(x,\xi)$ is the principal symbol of $h(x,\xi)$, h_s is well defined.

Let $h^{(1)}(x,\xi)$ be the single symbol of $P(X,D_x)\cdot Q(X,D_x)$ and $h_s^{(1)}(x,\xi)$ be the subprincipal symbol of $h^{(1)}(x,\xi)$ with respect to $p_0(x,\xi)q_0(x,\xi)$. Then, by Proposition 5.1 we have

$$h_s^{(1)}(x,\xi)\equiv p_sq+pq_s-\frac{i}{2}\{p,q\} \qquad (\text{mod. } S^{m_1+m_2-2})\,,$$

and

$$h_s(x,\xi)\equiv h_s^{(1)}r+h^{(1)}r_s-\frac{i}{2}\{h^{(1)},r\} \qquad (\text{mod. } S^{m_1+m_2+m_3-2})\,.$$

Hence, we get

$$h_s(x, \xi) \equiv p_s qr + pq_s r + h^{(1)} r_s - \frac{i}{2}(\{p, q\} r + \{h^{(1)}, r\})$$
$$\equiv p_s qr + pq_s r + pqr_s - \frac{i}{2}(\{p, q\} r + \{q, r\} p - \{r, p\} q)$$
$$(\text{mod. } S^{m_1+m_2+m_3-2}). \qquad \text{Q.E.D.}$$

Now consider L of (5.1). Noting the assumption (5.3), by the factorization theorem in Kumano-go [8] we can write

$$(5.8) \qquad L = P_1^{(m-k1)} \cdots P_1^{(k+1)} \cdot P_2^{(k)} \cdots P_2^{(1)} + \sum_{j=1}^{m} R_j(t, X, D_x) D_t^{m-j}$$
$$(R_j(t, x, \xi) \in B^\infty([0, T]; S^{-\infty}),$$

where

$$(5.9) \qquad P_2^{(l)} = P_2^{(l)}(t, X, D_t, D_x)$$
$$= D_t^2 + A_1^{(l)}(t, X, D_x) D_t + A_2^{(l)}(t, X, D_x)$$
$$+ B_0^{(l)}(t, X, D_x) D_t + B_1^{(l)}(t, X, D_x)$$
$$(A_i^{(l)}(t, x, \xi),\ B_i^{(l)}(t, x, \xi) \in B^\infty([0, T]; S^i),\ 1 \leqq l \leqq k),$$

$$(5.10) \qquad \lambda^2 + A_1^{(l)}(t, x, \xi)\lambda + A_2^{(l)}(t, x, \xi)$$
$$= (\lambda - \lambda_1^{(l)}(t, x, \xi))(\lambda - \lambda_2^{(l)}(t, x, \xi)),$$

and

$$(5.11) \qquad P_1^{(j)} = P_1^{(j)}(t, X, D_t, D_x) = D_t - \lambda_1^{(j)}(t, X, D_x) + R_0^{(j)}(t, X, D_x)$$
$$(\lambda_1^{(j)}(t, x, \xi) \in B^\infty([0, T]; S^1),\ R_0^{(j)}(t, x, \xi) \in B^\infty([0, T]; S^0),$$
$$k+1 \leqq j \leqq m-k).$$

Let $L_s(t, x, \lambda, \xi)$ and $p_{2s}^{(l)}(t, x, \lambda, \xi)$ $(1 \leqq l \leqq k)$ be the subprincipal symbols of $\sigma(L)$ (t, x, λ, ξ) and $p_2^{(k)}(t, x, \lambda, \xi)$ with respect to the principal symbols

$$\lambda^m + \sum_{j=1}^{m} A_j(t, x, \xi)\lambda^{m-j},$$

and

$$\lambda^2 + A_1^{(l)}(t, x, \xi)\lambda + A_2^{(l)}(t, x, \xi),$$

respectively. Then, we obtain in the same way as in Hata [5]

Lemma 5.3. *We also assume for L of* (5.1) *that*

$$(5.12) \qquad \lambda_i^{(k)}(t, x, \xi) \in B^\infty([0, T]; S^1_{1,0;3+\sigma}) \qquad (\sigma > 0)$$
$$(i = 1, 2, 1 \leqq l \leqq k).$$

Then, the following a) *and* b) *are equivalent.*

a) *There exist symbols $\mu_l(t, x, \xi) \in B^\infty([0, T]; S^{m-2}_{1,0;1+\sigma})$ $(1 \leqq l \leqq k)$ such that*

$$(5.13)\qquad L_s(t, x, \lambda_1^{(l)}(t, x, \xi), \xi) - \frac{i}{2}\{\lambda - \lambda_1^{(l)}, \lambda - \lambda_2^{(l)}\}$$
$$\times Q_l(t, x, \lambda_1^{(l)}(t, x, \xi), \xi) - \mu_l(\lambda_1^{(l)} - \lambda_2^{(l)}) \in B^\infty([0, T]; S^{m-2}_{1,0\,;\,0}),$$

where

$$Q_l(t, x, \lambda, \xi) = \prod_{\substack{j=1\\ j\neq l}}^{k} (\lambda - \lambda_1^{(j)}(t, x, \xi))(\lambda - \lambda_2^{(j)}(t, x, \xi)) \prod_{j=k+1}^{m-k} (\lambda - \lambda_1^{(j)}(t, x, \xi)).$$

b) *There exist symbols* $\nu_l(t, x, \xi) \in B^\infty([0, T]; S^0_{1,0\,;\,1+\sigma})$ $(1 \leqq l \leqq k)$ *such that*

$$(5.14)\qquad p_{2s}^{(l)}(t, x, \lambda_1^{(l)}(t, x, \xi), \xi) - \frac{i}{2}\{\lambda - \lambda_1^{(l)}, \lambda - \lambda_2^{(l)}\}$$
$$-\nu_l(\lambda_1^{(l)} - \lambda_2^{(l)}) \in B^\infty([0, T]; S^0_{1,0\,;\,0}).$$

Proof. Let $S_1(t, x, \lambda, \xi)$ be the single symbol of

$$P_1^{(m-k)} \cdots P_1^{(k+1)} \cdot P_2^{(k)} \cdots P_2^{(l+1)},$$

and let $S_2(t, x, \lambda, \xi)$ be the single symbol of

$$P_2^{(l-1)} \cdots P_2^{(1)}.$$

Then, we consider the subprincipal symbols S_{1s} and S_{2s} of S_1 and S_2 with respect to the principal symbols

$$\prod_{j=l+1}^{k} (\lambda - \lambda_1^{(j)}(t, x, \xi))(\lambda - \lambda_2^{(j)}(t, x, \xi)) \prod_{j=k+1}^{m-k} (\lambda - \lambda_1^{(j)}(t, x, \xi))$$

and

$$\prod_{j=1}^{l-1} (\lambda - \lambda_1^{(j)}(t, x, \xi))(\lambda - \lambda_2^{(j)}(t, x, \xi)),$$

respectively. So we can apply Corollary 5.2 as $p = S_1$, $q = p_2^{(l)}$ and $r = S_2$. We have

$$(5.15)\qquad L_s(t, x, \lambda, \xi) \equiv S_{1s} p_2^{(l)} S_2 + S_1 p_{2s}^{(l)} S_2 + S_1 p_2^{(l)} S_{2s}$$
$$-\frac{i}{2}(\{S_1, p_2^{(l)}\} S_2 + \{p_2^{(l)}, S_2\} S_1 - \{S_2, S_1\} p_2^{(l)})$$
$$(\mathrm{mod.}\ S^{m-2}(R^{2(n+1)}_{t,x,\lambda,\xi})).$$

Hence, we get from (5.12)

$$(5.16)\qquad L_s(t, x, \lambda_1^{(l)}(t, x, \xi), \xi)$$
$$\equiv S_1 S_2 p_{2s}^{(l)}(t, x, \lambda_1^{(l)}(t, x, \xi), \xi) + \frac{i}{2} S_3(t, x, \xi)(\lambda_1^{(l)} - \lambda_2^{(l)})$$
$$(\mathrm{mod.}\ B^\infty([0, T]; S^{m-2}_{1,0\,;\,2+\sigma}(R^{2n}_{x,\xi}))),$$

where $S_3(t, x, \xi) \in B^\infty([0, T]; S^{m-2}_{1,0;1+\sigma})$.

First, we derive b) from a). Comparing (5.13) with (5.16), we have

$$
\begin{aligned}
(5.17) \qquad 0 &\equiv S_1 S_2(t, x, \lambda_1^{(l)}, \xi) p_{2s}^{(l)}(t, x, \lambda_1^{(l)}, \xi) + \left(\frac{i}{2} S_3 - \mu_l\right)(\lambda_1^{(l)} - \lambda_2^{(l)}) \\
&\quad - \frac{i}{2}\{\lambda - \lambda_1^{(l)}, \lambda - \lambda_2^{(l)}\} Q_l(t, x, \lambda_1^{(l)}, \xi) \\
&\equiv p_{2s}^{(l)}(t, x, \lambda_1^{(l)}, \xi) Q_l(t, x, \lambda_1^{(l)}, \xi) + \left(\frac{i}{2} S_3 - \mu_l\right)(\lambda_1^{(l)} - \lambda_2^{(l)}) \\
&\quad - \frac{i}{2}\{\lambda - \lambda_1^{(l)}, \lambda - \lambda_2^{(l)}\} Q_l(t, x, \lambda_1^{(l)}, \xi) \\
&\qquad\qquad (\mathrm{mod.}\ B^\infty([0, T]; S^{m-2}_{1,0;0})).
\end{aligned}
$$

It is easy to see that $Q_l(t, x, \lambda_1^{(l)}(t, x, \xi), \xi)^{-1}$ belongs to $B^\infty([0, T]; S^{-(m-2)}_{1,0;3+\sigma})$. Hence, defining $\nu_l(t, x, \xi)$ by

$$
(5.18) \qquad \nu_l(t, x, \xi) = -Q_l(t, x, \lambda_1^{(l)}, \xi)^{-1}\left(\frac{i}{2} S_3 - \mu_l\right),
$$

we can see that $\nu_l(t, x, \xi)$ satisfies b). We get b).

Convesely, assume b). Setting

$$
\mu_l(t, x, \xi) = \left(\frac{i}{2} S_3 + \nu_l(t, x, \xi) Q_l(t, x, \lambda_1^{(l)}, \xi)\right),
$$

we get a) in the same way. Q.E.D.

Let $\phi_i^l(t, s; x, \xi)$ $(i=1, 2,\ 1 \leqq l \leqq k)$ and $\phi_1^j(t, s; x, \xi)$ $(k+1 \leqq j \leqq m-k)$ be the solution of (1.15) corresponding to $\tilde{\lambda}_{i\delta}^{(l)}$ and $\tilde{\lambda}_{1\delta}^{(j)}$ which are the approximations defined by (2.24) for $\lambda_i^{(l)}$ and $\lambda_1^{(j)}$, respectively. The index δ will be determined later. Let $\{q^j, p^j\}(t, s; x, \xi)$ $(k+1 \leqq j \leqq m-k)$ be the solutions of (3.16) corresponding to $\lambda_1^{(j)}$ and define $\{Q_{1,2}^{(l)}, P_{1,2}^{(l)}\}(t, \tau, s; x, \xi)$ $(1 \leqq l \leqq k)$ by (3.17) corresponding to $\lambda_1^{(l)}$ and $\lambda_2^{(l)}$. Then, we obtain

Theorem 5.4. *We assume for L of* (5.1) *the following conditions* i), ii) *and* iii).

i) $\lambda_i^{(l)}(t, x, \xi)$ $(i=1, 2,\ 1 \leqq l \leqq k)$ *belongs to* $B^\infty([0, T]; S^1_{1,0;3+\sigma})$ $(\sigma > 0)$.

ii) $\{\lambda - \lambda_1^{(l)}, \lambda - \lambda_2^{(l)}\} = 0 \qquad (1 \leqq l \leqq k)$.

iii) *There exist symbols* $\mu_l(t, x, \xi) \in B^\infty([0, T]; S^{m-2}_{1,0;1+\sigma})$ $(1 \leqq l \leqq k)$ *such that*

$$
(5.19) \qquad L_s(t, x, \lambda_1^{(l)}, \xi) - \mu_l(\lambda_1^{(l)} - \lambda_2^{(l)}) \in B^\infty([0, T]; S^{m-2}_{1,0;0}).
$$

Then, we have the unique solution $u(t, x)$ of the Cauchy problem $(P.2)$ in $[0, T_0]$ for some $T_0 > 0$. Setting $WF(G) = \bigcup_{j=0}^{m-1} WF(g_j)$, we have

$$(5.20)\qquad WF(u(t,x))\subset \bigcup_{l=1}^{k}\bigcup_{0\leqq\tau\leqq t}\text{Conic}\,\{(Q_{1,2}^{(l)}, P_{1,2}^{(l)})(t,\tau,0;x^0,\xi^0);$$
$$(x^0,\xi^0)\in WF(G),\ |\xi^0|\geqq M\}\cup\bigcup_{j=k+1}^{m-k}\text{Conic}\,\{(q^j,p^j),$$
$$(t,0;x^0,\xi^0);(x^0,\xi^0)\in WF(G),\ |\xi^0|\geqq M\},$$

for a large constant $M>0$.

Proof. From (5.8) we may assume that L has the form

$$(5.21)\qquad L=P_1^{(m-k)}\cdots P_1^{(k+1)}\cdot P_2^{(k)}\cdots P_2^{(1)}.$$

we can see from the assumptions i), ii), iii) and Lemma 5.3 that $p_2^{(l)}$ $(1\leqq l\leqq k)$ satisfies the conditions of Theorem 4.5. Hence, in the same way to the proof of Theorem 4.5, we get

$$(5.22)\qquad P_2^{(l)}=\{D_t-\tilde{\lambda}_2^{(l)}(t,X,D_x)+R_2^{(l)}(t,X,D_x)\}\{D_t-\tilde{\lambda}_1^{(l)}(t,X,D_x)+R_1^{(l)}(t,X,D_x)\}+R_0^{(l)}(t,X,D_x),$$

where

$$\tilde{\lambda}_i^{(l)}(t,x,\xi)=\int\psi(y)\lambda_i^{(l)}(t,x+\langle\xi\rangle^{-(1-\rho)}y,\xi)dy\qquad\left(\rho=\frac{2+\sigma}{4+\sigma}\right),$$

and

$$\begin{cases} R_i^{(l)}(t,x,\xi)\in B^\infty([0,T];S_\rho^0((1)))\qquad(i=1,2),\\ R_0^{(k)}(t,x,\xi)\in B^\infty([0,T];S_\rho^0).\end{cases}$$

Setting

$$(5.23)\qquad \partial_i^{(l)}=D_t-\tilde{\lambda}_i^{(l)}(t,X,D_x)+R_i^{(l)}(t,X,D_x)\qquad(i=1,2,\ 1\leqq l\leqq k),$$

we have

$$P_2^{(l)}=\partial_2^{(l)}\cdot\partial_1^{(l)}+R_0^{(l)}.$$

Then, set

$$(5.24)\qquad\begin{cases} v_0=u,\quad v_{2l-1}=\partial_1^{(l)}\cdot P_2^{(l-1)}\cdots P_2^{(1)}u & (1\leqq l\leqq k),\\ v_{2l}=P_2^{(l)}\cdot P_2^{(l-1)}\cdots P_2^{(1)}u & (1\leqq l\leqq k),\\ v_j=P_1^{(j-k)}\cdots P_1^{(k+1)}\cdot P_2^{(1)}\cdots P_2^{(1)}u & (2k+1\leqq j\leqq m-1),\end{cases}$$

and set

$$V={}^t(v_0,v_1,\cdots,v_{m-1}).$$

Then, from (5.24) we have for (P.2)

$$(5.25)\qquad\begin{cases} \boldsymbol{L}V=(\boldsymbol{D}_t-\boldsymbol{A}(t,X,D_x)+\boldsymbol{B}(t,X,D_x))V=0,\\ V|_{t=s}=\boldsymbol{M}(X,D_x)^t(g_0,g_1,\cdots,g_{m-1}),\end{cases}$$

where

$$(5.26)\qquad \begin{cases} \boldsymbol{D}_t = \begin{pmatrix} D_t & & & 0 \\ & D_t & & \\ & & \ddots & \\ 0 & & & D_t \end{pmatrix}, \quad \boldsymbol{A}(t, x, \xi) = \begin{pmatrix} \tilde{\lambda}_1^{(1)} & & & 0 \\ & \tilde{\lambda}_2^{(1)} & & \\ & & \ddots & \\ 0 & & & \lambda_1^{(m-k)} \end{pmatrix}, \\ \boldsymbol{B}(t, x, \xi) = \begin{pmatrix} R_1^{(1)}, & -1 & & & 0 \\ R_0^{(1)}, & R_2^{(1)} & \ddots & & \\ & \ddots & \ddots & & \\ & & \ddots & -1 & \\ 0 & & & \ddots & R_1^{(m-k)} \end{pmatrix} \end{cases}$$

and

$$(5.27)\qquad \boldsymbol{M}(x, \xi) = \left(m_{jk}(x, \xi)\,;\ \begin{matrix} j \downarrow 1, \cdots, m \\ k \to 1, \cdots, m \end{matrix}\right)$$

$$(m_{jj}=1\,, \quad m_{jk}=0\ (j<k)\,, \quad m_{jk} \in S_\rho^{j-k}\ (j>k))\,.$$

Now it is clear from (5.3) and Theorem 2.5 that

$$(5.28)\qquad \begin{cases} \inf\limits_{[0,T]\times R_x^n} |\tilde{\lambda}_i^{(l)}(t, x, \xi) - \tilde{\lambda}_{i'}^{(l')}(t, x, \xi)| \geqq C_0|\xi|\,, \\ \inf\limits_{[0,T]\times R_x^n} |\tilde{\lambda}_i^{(l)}(t, x, \xi) - \lambda_1^{(j)}(t, x, \xi)| \geqq C_0|\xi| \end{cases}$$

$$(i, i'=1, 2\,, \quad 1 \leqq l \neq l' \leqq k\,, \quad j=k+1, \cdots, m-k)\,,$$

where $C_0>0$ is a constant. Hence, we can apply the perfect diagonaliser of Kumano-go[9] to the block

$$\begin{pmatrix} \tilde{\lambda}_1^{(l)} & 0 \\ 0 & \tilde{\lambda}_2^{(l)} \end{pmatrix} \quad (1 \leqq l \leqq k)\,, \qquad (\lambda_1^j) \quad (k+1 \leqq j \leqq m-k)\,.$$

Then, we have by using some matrix $\boldsymbol{N}(t) = \boldsymbol{N}(t, X, D_x) = \left(n_{ij}(t, X, D_x);\ \begin{matrix} i \downarrow 1, \cdots, m \\ j \to 1, \cdots, m \end{matrix}\right)$ $(n_{ij}(t, x, \xi) \in B^\infty([0, T]; S_\rho^0))$

$$(5.29)\qquad \boldsymbol{Q}(t)\boldsymbol{L}\boldsymbol{N}(t) \equiv \boldsymbol{D}_t - \boldsymbol{A}(t, X, D_x) + \boldsymbol{C}(t, X, D_x) \quad (\text{mod. } B^\infty([0, T]; S^{-\infty})\,.$$

Here $\boldsymbol{Q}(t) = \boldsymbol{Q}(t, X, D_x)$ is the parametrix of $\boldsymbol{N}(t, X, D_x)$ and

$$\boldsymbol{C}(t, x, \xi) = \begin{pmatrix} \boldsymbol{C}_1 & & & & 0 \\ & \ddots & & & \\ & & \boldsymbol{C}_k & & \\ 0 & & & c_{k+1} & \\ & & & & \ddots \\ & & & & & c_{m-k} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{C}_1 = \left(c_{ij}^{(l)}(t, x, \xi); \begin{smallmatrix} i \downarrow 1, 2 \\ j \to 1, 2 \end{smallmatrix}\right) (c_{ij}^{(l)}(t, x, \xi) \in B^\infty([0, T]; S_\rho^0) \\ (1 \leqq l \leqq k)), \quad c_j(t, x, \xi) \, (\in B^\infty([0, T]; S_\rho^0)); \text{ scalar} \\ (k+1 \leqq j \leqq m-k). \end{pmatrix}$$

Then, we get the fundamental solution $\boldsymbol{E}_0(t, s)$ for $\boldsymbol{Q}(t)\boldsymbol{L}\boldsymbol{N}(t)$ in the similar way to the proof of Theorem 4.5 in the form

$$\boldsymbol{E}_0(t, s) \equiv \begin{pmatrix} \boldsymbol{E}_1(t, s) & & & 0 \\ & \ddots & & \\ & & \boldsymbol{E}_k(t, s) & \\ & & & \ddots \\ 0 & & & E_{m-k}(t, s) \end{pmatrix} \quad (\text{mod. } B^\infty(\Delta_0; S^{-\infty})), \tag{5.30}$$

where $\boldsymbol{E}_l(t, s)$ $(1 \leqq l \leqq k)$ has the form (4.24) whose symbols are infinitely differentiable with respect to (t, τ, s), and we have

$$E_j(t, s) = W_{j, \phi_1^j}(t, s; X, D_x) \qquad (W_j(t, s; x, \xi) \in B^\infty(\Delta_0; S_\rho^0), \ k+1 \leqq j \leqq m-k). \tag{5.31}$$

Then, we get the fundamental solution $\boldsymbol{E}(t, s)$ for $\boldsymbol{L}$

$$\boldsymbol{E}(t, s) = \boldsymbol{N}(t)\boldsymbol{E}_0(t, s)\boldsymbol{Q}(s). \tag{5.32}$$

Therefore, there exists the solution $u(t,x)$ which has the form

$$u(t, x) = \text{"the first row of } \boldsymbol{E}(t, 0)\boldsymbol{M}(X, D_x)^t(g_0, \cdots, g_{m-1})\text{"}. \tag{5.33}$$

Hence, in the similar way to the proof of Theorem 4.5 we obtain (5.20). Thus the proof is complete. Q.E.D.

REMARK 1. Assume a) of Lemma 5.3 in stead of ii) and iii) in Theorem 5.4. Then, the fundamental solution can be represented by the Fourier integral operators (see Remark 3 in Section 4).

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