

Title	Small submodules in a projective module and semi-T-nilpotent sets
Author(s)	Harada, Manabu
Citation	Osaka Journal of Mathematics. 14(2) P.355-P.364
Issue Date	1977
Text Version	publisher
URL	https://doi.org/10.18910/3535
DOI	10.18910/3535
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

SMALL SUBMODULES IN A PROJECTIVE MODULE AND SEMI-T-NILPOTENT SETS

MANABU HARADA

(Received September 10, 1975)

Let R be a ring with identity element and $(R)_I$ the ring of column-finite matrices over R with infinite degree I . N. Jacobson proposed to determine the Jacobson radical $J((R)_I)$ of $(R)_I$ in his book [5]. Many algebraists have been working on this problem. P.M. Patterson [8], N.E. Sexauer and J.E. Warnock [9] showed $J((R)_I) = (J(R))_I$ if and only if $J(R)$ is right T -nilpotent (cf. [4], Corollary 1 to Proposition 1). On the other hand, W. Liebert [6] gave an exact form of $J((R)_I)$ if R is domain and R. Slover [10,11] and R. Ware and J. Zelmanowitz [12] obtained an exact form of elements in $J((R)_I)$, which involved all results above.

In this note, we shall first give all types of small submodules in a free R -module $M = \sum_I \oplus u_\alpha R$. Since $J((R)_I)$ is determined by small submodules in M , we can obtain their results similarly to [12] and give another forms by means of locally, right semi- T -nilpotent sets of small submodules.

Finally, we shall give a characterization of right perfect module P by means of a structure of $(S)_I/J((S)_I)$, where $S = \text{End}_R(P)$.

1 Jacobson radicals

Throughout we shall assume that R is a ring with identity element and every module is a unitary right R -module. Let A be an R -module and B a submodule of A . B is called *small in A* if a fact: $A = B + T$ for a submodule T of A implies $A = T$. Let $\{M_\alpha\}_I$ be a set of R -modules and $M = \sum_I \oplus M_\alpha$. We put $S_M = \text{End}_R(M)$. We assume the elements in S_M operate on M from the left side. Furthermore, we can express them as the column-summable matrices with entries in $\text{Hom}_R(M_\sigma, M_\tau)$. If $M_\sigma = R$ for all σ , S_M is the ring $(R)_I$ of column-finite matrices over R .

Let $M = \sum_I \oplus u_\alpha R$ and S a small submodule in M . Then $S \subseteq \sum_I \oplus u_\alpha J(R)$.

In order to determine a type of S , we shall define a set of right semi- T -nilpotent, right ideals. Let $\{A_\alpha\}_K$ be a set of right ideals in R and K an infinite set. If $\{A_\alpha\}_K$ satisfies the following condition, we call $\{A_\alpha\}_K$ a *right*

semi- T -nilpotent set, (see a vanishing set of ideals in [12]).

For any countable subset $\{A_{\alpha_i}\}_{i=1}^{\infty}$ of $\{A_{\alpha}\}_K$ and $\{a_i | \in A_{\alpha_i}\}_{i=1}^{\infty}$, there exists n , depending on $\{a_i\}$, such that $a_n a_{n-1} \cdots a_1 = 0$.

Let $\{b_{\sigma}\}_K$ be a set of elements in R . If $\{b_{\sigma}R\}_K$ is a right semi- T -nilpotent set, we call $\{b_{\sigma}\}_K$ a right semi- T -nilpotent set. If we allow $\alpha_i = \alpha_j$ for $i \neq j$, we call $\{A_{\alpha}\}$ or $\{b_{\alpha}\}$ a right T -nilpotent set. If $A_{\alpha} = A$ for all α , then the above concept coincides with one of the usual T -nilpotency.

If K is a finite set, we understand $A_{\alpha} = 0$ for almost all α , then $\{A_{\alpha}\}_K$ is always a semi- T -nilpotent set, but not a T -nilpotent set. Now, we shall state the theorem which is substantially due to [12].

Theorem 1 ([12], Theorem 1). Let $M = \sum_I \oplus u_{\alpha}R$ be a free R -module with infinite basis u_{α} and S a submodule of M . Then the following statements are equivalent.

- 1) S is small in M .
- 2) Let $p_{\alpha}: M \rightarrow u_{\alpha}R$ be the projection of M onto $u_{\alpha}R$ and $A_{\alpha} = p_{\alpha}(S)$. Then $A_{\alpha} \subseteq J(R)$ and $\{A_{\alpha}\}_I$ is a right semi- T -nilpotent set.
- 3) There exists a right semi- T -nilpotent set $\{A_{\alpha}\}_I$ of right ideals in $J(R)$ such that $S \subseteq \sum \oplus u_{\alpha}A_{\alpha}$.

We shall prove Theorem 1 in more general forms. First, we shall generalize the concept of right semi- T -nilpotent set of right ideals. Let $\{Q_{\alpha}\}_I$ be an infinite set of R -modules and $\{S_{\beta} | \subseteq Q_{\beta}\}_{\beta \in K \subseteq I}$ an infinite set of R -submodules. We take a countable subset $\{Q_{\alpha_i}\}$ of $\{Q_{\alpha}\}_K$ and a set of homomorphisms $f_i: Q_{\alpha_i} \rightarrow Q_{\alpha_{i+1}}$ such that $f_i(Q_{\alpha_i}) \subseteq S_{\alpha_{i+1}}$. If for any element t in Q_{α_1} there exists n , depending on t and $\{f_i\}$, such that $f_n f_{n-1} \cdots f_1(t) = 0$, then we call $\{f_i\}$ a locally (semi)- T -nilpotent set of homomorphisms. If for any countable subset $\{Q_{\alpha_i}\}$ and any set of homomorphisms f_i as above, $\{f_i\}$ is always locally (semi)- T -nilpotent, then we call $\{S_{\alpha}\}_K$ a locally (right) semi- T -nilpotent set of submodules.

The following lemma is obtained by [12].

Lemma 1. Let P be projective. Then $J(S_P) = \{f \in S_P | f(P) \text{ is small in } P\}$.

Lemma 2 ([4], Proposition 1). Let P be R -projective and S an R -submodule of P . If $\text{Hom}_R(P, S) \subseteq J(S_P)$, S is small in P , where $S_P = \text{End}_R(P)$.

Proof. We assume $P = S + T$ for some submodule T . Then we have a diagram:

$$\begin{array}{ccccccc}
 0 & \rightarrow & S \cap T & \rightarrow & S & \xrightarrow{\nu} & S/S \cap T \rightarrow 0 \\
 & & & & & \searrow h & \\
 & & & & & & \begin{array}{c} \cong f \\ P/T \\ \uparrow \nu' \\ P \end{array}
 \end{array}$$

Since P is projective, we have $h: P \rightarrow S$ such that $vh = fv'$. Hence, $S = h(P) + S \cap T$ and so $P = h(P) + T$. On the other hand, $h(P)$ is small in P from the assumption and Lemma 1. Therefore, $P = T$.

The following proposition implies 1) \rightarrow 2) \rightarrow 3) in Theorem 1.

Proposition 1. *Let $\{M_\alpha\}_I$ be a set of finitely generated R -modules and S a small submodule of $M = \sum_I \oplus M_\alpha$. Then $\{S_\alpha = p_\alpha(S)\}_I$ is a right semi- T -nilpotent set of submodules, where $p_\alpha: M \rightarrow M_\alpha$ is the projection.*

Proof. We may assume I is a well ordered, infinite set. Let $\{S_{\alpha_i}\}_1^\infty$ be any subset of $\{S_\alpha\}_I$ and $\{f_i: M_{\alpha_i} \rightarrow S_{\alpha_{i+1}}\}_1^\infty$ a given set. Let $\{m_\alpha^{(1)}, m_\alpha^{(2)}, \dots, m_\alpha^{n_\alpha}\}$ be a generator of M_α . Then for $m_{\alpha_i}^{(k)}$ there exists $s_i^{(k)}$ in S such that

$$s_i^{(k)} = s_{(\beta(i,k),1)} + s_{(\beta(i,k),2)} \cdots + f_i(m_{\alpha_i}^{(k)}) + \cdots + s_{(\beta(i,k),n_k)} \tag{*}$$

where $s_{(\beta(i,k),j)} \in S_{\beta(i,k),j}$.

Hence, we may assume

$$s_i^{(k)} \in \sum_{j=1}^{m_i} \oplus S_{\beta(i,j)} \quad \text{for all } k=1, 2, \dots, n \tag{**}$$

1 Special case. First, we assume that $\{\beta(i, j)\}_{i=1}^\infty \}_{j=1}^{m_i} \equiv \{1, 2, \dots, n, \dots\}$ in (**) and $\alpha_1 < \alpha_2 \leq m_1 < \alpha_3 \leq m_2 < \alpha_4 \leq \dots$. We put $M'_{\alpha_i} = \{m_{\alpha_i} + f_i(m_{\alpha_i}) \mid m_{\alpha_i} \in M_{\alpha_i}\} \subseteq M_{\alpha_i} \oplus M_{\alpha_{i+1}}$ and $M' = \sum_{\{\alpha \notin \alpha_i\}} M_\alpha + \sum_1^\infty M'_{\alpha_i} + S$. We shall show $M = M'$. For $m_{\alpha_1}^{(k)}$ we have

$$\begin{aligned} M' \supset M'_{\alpha_1} + S &\ni m_{\alpha_1}^{(k)} + f_1(m_{\alpha_1}^{(k)}) - s_1^{(k)} \\ &= -s_1^{(1,k)} - s_2^{(1,k)} - \dots + (m_{\alpha_1}^{(k)} - s_{\alpha_1}^{(1,k)}) - \dots - \underbrace{0}_{\alpha_2} - \dots - s_{m_1}^{(1,k)}. \end{aligned}$$

Hence, $m_{\alpha_1}^{(k)} \equiv s_{\alpha_1}^{(1,k)} \pmod{M'}$ and $s_{\alpha_1}^{(1,k)} \in S_{\alpha_1}$. Therefore, $(M_{\alpha_1} + M')/M' = (S_{\alpha_1} + M')/M'$. Since S_{α_1} is small in M_{α_1} , $(S_{\alpha_1} + M')/M'$ is small in $(M_{\alpha_1} + M')/M'$. Accordingly, $M_{\alpha_1} \subseteq M'$. Repeating those arguments we have $M = M'$. Since

S is small in M , $M = \sum_{\alpha \notin \{\alpha_i\}} \oplus M_\alpha \oplus \sum_1^\infty M'_{\alpha_i}$. Hence, there exists n such that $f_n f_{n-1} \cdots f_1(m_{\alpha_1}) = 0$ for $m_{\alpha_1} \in M_{\alpha_1}$ (see [1], Lemma 9).

2 General case. Since $\beta(i, j)$'s in (**) are countable, we may assume $\{\alpha, \beta(i, j)\} = \{1 = \alpha_1, 2, \dots, n, \dots\}$ after rearranging the order of indices. We shall denote the new index of α_i by $\sigma(\alpha_i)$, namely $\sigma(\alpha_1) = 1$. Then

$$s_1^{(k)} (= s_{\alpha_{i_2}}^{(k)}) = s_1^{(\alpha_{i_2}, k)} + s_2^{(\alpha_{i_2}, k)} + \dots + f_1(m_{\alpha_1}^{(k)}) + \dots + s_n^{(\alpha_{i_2}, k)}.$$

Since $\{f_i\}$ is infinite, there exists $i_3 > i_2$ such that $n(\alpha_{i_3}, k') \geq \sigma(\alpha_{i_3}) > \text{Max}_k(n(\alpha_{i_2}, k))$. Repeating those arguments, we obtain $\sigma(\alpha_1) = 1 < \sigma(\alpha_{i_2}) \leq \text{Max}_k(n(\alpha_{i_2}, k)) < \sigma(\alpha_{i_3}) \leq \text{Max}_k(n(\alpha_{i_3}, k')) < \sigma(\alpha_{i_4}) < \dots$ and $1 = i_1 < i_2 < i_3 < \dots$. Put $g_{j-1} = f_{i_{j-1}} \cdots f_{i_{j-1}}: M_{\alpha_{i_{j-1}}} \rightarrow S_{\alpha_{i_j}}$, ($g_1 = f_1$) and consider a countable subset $\{S_{\sigma(\alpha_{i_j})}\}_{j=1}^\infty$. Since $g_{j-1}(M_{\alpha_{i_{j-1}}}) \subseteq f_{i_{j-1}}(M_{\alpha_{i_{j-1}}})$, $g_{j-1}(m_{\alpha_{i_{j-1}}}^{(k)}) \in p_{\sigma(\alpha_{i_j})}(\bigoplus_{t=1}^{n(\alpha_{i_j}, k')} M_t) \cap S$. Hence, $\{S_{\sigma(\alpha_{i_j})}\}$ and $\{g_{j-1}\}$ satisfy the conditions of Special case 1. Accordingly, $0 = g_n g_{n-1} \cdots g_1(m_{\alpha_1}) = f_{i_{n+1}-1} \cdots f_1(m_{\alpha_1})$ for some n .

From a special type of the above proof, we have

Corollary 1. *Let $\{N_\alpha\}_I$ be a set of R -modules and $\{T_\alpha | \subseteq N_\alpha\}_I$ a set of submodules. If $\sum_I \oplus T_\alpha$ is a small submodule of $\sum_I \oplus N_\alpha$, then $\{T_\alpha\}_I$ is a locally right semi- T -nilpotent set of submodules.*

Proposition 2. *Let $\{N_\alpha\}_I$ be a set of R -modules and $\{T_\alpha | \subseteq N_\alpha\}_I$. We put $N = \sum_I \oplus N_\alpha$, $T = \sum_I \oplus T_\alpha$ and $S_N = \text{End}_R(N)$. Then $J(S_N) \supseteq \text{Hom}_R(N, T)$ if and only if $\{T_\alpha\}_I$ is a locally right semi- T -nilpotent set of submodules and $\text{Hom}_R(N_\alpha, T_\alpha) \subseteq J(S_{N_\alpha})$.*

Proof. We assume $J(S_N) \supseteq \text{Hom}_R(N, T)$. Let $\{N_{\alpha_j}\}$ and $\{f_i: N_{\alpha_i} \rightarrow T_{\alpha_{i+1}}\}$ be given sets. We may assume $\alpha_j = j$ for all j . Then

$$\begin{pmatrix} 0 & & & & 0 \\ f_1 & 0 & & & \\ & f_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & f_n & 0 \\ 0 & & & & \ddots \end{pmatrix}$$

is in $J(S_N)$ from the assumption. Hence, $\{f_i\}$ is locally right T -nilpotent. It is clear that $J(S_{N_\alpha}) = e_\alpha J(S_N) e_\alpha \supseteq e_\alpha \text{Hom}_R(N, T) e_\alpha = \text{Hom}_R(N_\alpha, T_\alpha)$, where e_α is the projection of N onto N_α . Conversely, we shall show that $C_{\sigma\tau} = \text{Hom}_R(N_\tau, T_\sigma)$'s satisfy the conditions 1)~3) in [4], Lemma 5. 1) and 3) are clear from the assumptions and 2) is clear. We note that in the proof of [4], Lemma 5 we only used a fact that $C_{\sigma\tau} \text{Hom}_R(N_\sigma, N_\tau) \subseteq C_{\sigma\sigma}$. Hence, $J(S_N) \supseteq \text{Hom}_R(N, T)$ from [4], Lemma 5.

The following corollary implies 3) \rightarrow 1) in Theorem 1 and is the converse of Corollary 1 above in a restricted case.

Corollary 1 ([4], Theorem 3). *Let $\{P\}_I$ be a set of R -projectives and $\{S_\alpha | \subseteq P_\alpha\}$ a set of R -submodules. Then $\sum_I \oplus S_\alpha$ is small in $\sum_I \oplus P$ if and only if $\{S_\alpha\}_I$ is a locally, right semi- T -nilpotent set of small submodules S_α in P_α (see Remark 2 below).*

Corollary 2. *Let $\{P_\alpha\}_I$ and $\{S_\alpha\}_I$ be as above. Then for any set $\{Q_\beta\}_K$ such that $Q_\beta \overset{\varphi_\beta}{\approx} P_{\alpha(\beta)} \in \{P_\alpha\}_I \sum_K \oplus \varphi_\beta^{-1}(S_{\alpha(\beta)})$ is always small in $\sum_K \oplus Q_\beta$ if and only if $\{S_\alpha\}_I$ is a locally right T -nilpotent set of submodules.*

Proof. It is enough to show that S_α is small in P_α if $\{S_\alpha\}_I$ is locally right T -nilpotent. $A_\alpha = \text{Hom}_R(P_\alpha, S_\alpha)$ is a right ideal in S_{P_α} . Let $f \in A_\alpha$, then $f' = \sum_{n=0}^\infty f^n \in S_{P_\alpha}$ from the assumption. Hence, $(1-f)f' = 1$ and so $A_\alpha \subseteq J(S_{P_\alpha})$. Therefore, S_α is small in P_α from Lemma 2.

Corollary 3 ([4, 8, 9, 12, 13]). *Let $M = \sum_I \oplus u_\alpha R$. Then $J(M)$ is small in M if and only if $J(R)$ is right T -nilpotent.*

Corollary 4. *Let M be an R -module. If $\{A_m\}_{m \in M_0}$ is a right semi- T -nilpotent set of right ideals in $J(R)$, then $\sum_{M_0} mA_m$ is a small submodule in M . Conversely, if M is projective and S is a small submodule in M , then there exists a semi- T -nilpotent set $\{A_m\}_{M_0}$ of right ideals in $J(R)$ such that $S \subseteq \sum_{M_0} mA_m$, where M_0 is any set of generators of M (see Remark 3).*

Proof. Consider an epimorphism $\varphi: P = \sum_{M_0} \oplus u_m R \rightarrow M; \varphi(u_m) = m$. Since $\sum \oplus u_m A_m$ is small in P from Corollary 2, $\varphi(\sum \oplus u_m A_m) = \sum mA_m$ is small in M (see [4]). Conversely, we assume M is projective. M is a direct summand of P and so $\varphi i = 1_M$ for monomorphism i . $i(S)$ is also small in P and hence, there exists a right semi- T -nilpotent set $\{A_m\}_{M_0}$ of right ideals in $J(R)$ such that $i(S) \subseteq \sum \oplus u_m A_m$ from Theorem 1. Therefore, $S = \varphi i(S) \subseteq \sum_{M_0} mA_m$.

Corollary 5. *Let P be a projective R -module. Then the following statements are equivalent.*

- 1) $\{J(P)\}$ is itself a locally, right T -nilpotent set of submodules.
- 2) $J(S_P)$ is locally right T -nilpotent.
- 3) Any set of small submodules in P is a locally, right T -nilpotent set.

Proof. 1) \rightarrow 2). $\text{Hom}_R(P, J(P)) \subseteq J(S_P)$ from the proof of Corollary 2. Hence, $\text{Hom}_R(P, J(P)) = J(S_P)$ is locally right T -nilpotent.

2) \rightarrow 3). Let $\{S_i\}_I$ be a set of small submodules in P . Then $\text{Hom}_R(P, S_i) \subseteq J(S_P)$ from Lemma 1. Hence, $\{S_i\}$ is a locally, right- T -nilpotent set.

3) \rightarrow 1). First, we shall show that the union of small submodules $\{S_\alpha\}_I$ is also small in P . Consider the natural epimorphism: $\sum_I \oplus P_\alpha \rightarrow P \rightarrow 0, P_\alpha = P$. Then $\sum_I \oplus S_\alpha$ is small in $\sum_I \oplus P_\alpha$ from 3) and Corollary 1. Hence, $\cup_I S_\alpha$ is small in P . It is easily seen that pR is small in P , where $p \in J(P)$. Therefore, $J(P)$ is

small in P from the above.

The proof above shows that if $J(P)$ is not small in P , then there exists a locally, right non- T -nilpotent set of small submodules $\{S\}_I$.

Now, we shall give a general form of Theorem 1.

Theorem 1'. *Let $\{P_\alpha\}_I$ be a set of R -projectives. Let $M = \sum_I \oplus P_\alpha$ and S a submodule of M . Then the following statements are equivalent.*

- 1) S is small in M .
- 2) Let $p_\alpha: M \rightarrow P_\alpha$ be the projection of M onto P_α and $S_\alpha = p_\alpha(S)$. Then $\{S_\alpha\}_I$ is a locally, right semi- T -nilpotent set of small submodules.
- 3) There exists a locally, right semi- T -nilpotent set $\{S_\alpha\}_I$ of small submodules S_α in P_α such that $S \subseteq \sum_I \oplus S_\alpha$.

Proof. 2) \rightarrow 3) \rightarrow 1). It is clear from Corollary 1 to Proposition 2.
 1) \rightarrow 2). We shall prove it in a general form:

Lemma 3. *In Proposition 1, we assume every M_α is a summand of a direct sum Q_α of finitely generated R -modules $M_{\alpha\beta}$. Then the statement in Proposition 1 is valid.*

Proof. Put $Q_\alpha = \sum_{\beta \in I_\alpha} \oplus M_{\alpha\beta}$ and $M^* = \sum_I \oplus Q_\alpha$. Then M is a summand of M^* . Let i be the injection of M into M^* . Then $i(S)$ is small in M^* and $\sum_I \sum_{I_\alpha} \oplus p_{\alpha\beta}(i(S)) \supseteq \sum_I \oplus i(S_\alpha)$ and $i(S_\alpha) \subseteq \sum_{I_\alpha} \oplus p_{\alpha\beta}(i(S))$. Now, $\{p_{\alpha\beta}(i(S))\}_{I, I_\alpha}$ is a locally semi- T -nilpotent set from Proposition 1. Let $\{M_{\alpha_i}\}_1^\infty$ and $f_i: M_{\alpha_i} \rightarrow S_{\alpha_{i+1}}\}_1^\infty$ be given sets. Then we can extend f_i to $f'_i: Q_{\alpha_i} \rightarrow i(S_{\alpha_{i+1}})$ by sending a direct complement to zero. We shall denote f'_i by a column-finite matrix $(a_{\sigma\tau}^{(i)})$, where $a_{\sigma\tau}^{(i)} \in \text{Hom}_R(M_{\alpha_i\tau}, p_{\alpha_{i+1}\sigma}(i(S)))$. Let m be in M_{α_1} and $i(m) = \sum_{j=1}^t m_{\alpha_1\beta_j}$, $m_{\alpha_1\beta_j} \in M_{\alpha_1\beta_j}$. Then $f_1(m) = f'_1(i(m)) = \sum_{j=1}^t \sum_k a_{\sigma_k\beta_j}^{(1)}(m_{\alpha_1\beta_j})$, where $a_{\sigma_k\beta_j}^{(1)} = 0$ for almost all k .

$$f_2 f_1(m) = f_2 f'_1(i(m)) = \sum_{j=1}^t \sum_{k'} \sum_k a_{k'\sigma_k}^{(2)} a_{\sigma_k\beta_j}^{(1)}(m_{\alpha_1\beta_j})$$

Since $p_{\alpha\beta}(i(S))$ is locally semi- T -nilpotent, we obtain n such that $f_n f_{n-1} \cdots f_1(m) = 0$ from Konig Graph Theorem.

Let M be an R -module. We can correspond (not necessarily unique) any element in $S_M = \text{End}_R(M)$ to a column-finite matrix $(a_{\sigma\tau})$ over R by making use of generators.

Theorem 2. *Let P be R -projective. Then $f \in J(S_P)$ if and only if f corresponds to a matrix above such that $\{\sum_\tau a_{\sigma\tau} R\}_\sigma$ is a right semi- T -nilpotent set of right ideals in $J(R)$ (cf. [12]).*

It is clear from Corollary 4 to Proposition 2.

Theorem 2' *Let $\{P_\alpha\}_I$ be a set of R -projective modules, $P = \sum_I \oplus P_\alpha$ and $S_P = \text{End}_R(P)$. Then*

$$J(S_P) = \bigcup_{(S_\sigma)} \begin{pmatrix} [P_1, S_1][P_2, S_1] \cdots [P_\sigma, S_1] \cdots \\ [P_1, S_2][P_2, S_2] \cdots [P_\sigma, S_2] \cdots \\ \dots\dots\dots \\ [P_1, S_\sigma][P_2, S_\sigma] \cdots [P_\sigma, S_\sigma] \cdots \\ \dots\dots\dots \end{pmatrix}$$

where matrices are column-summable, $\{S_\alpha | \subseteq P_\alpha\}$ runs through all locally, right semi- T -nilpotent sets of small submodules S_α in P_α and $[P_\sigma, S_\tau] = \text{Hom}_R(P_\sigma, S_\tau)$.

Proof. It is clear $J(S_P) \supseteq ([P_\sigma, S_\tau])$ from Lemma 1 and Corollary 1 to Proposition 2. Let $f \in J(S_P)$ and $f = (f_{\sigma\tau}), f_{\sigma\tau} \in [P_\tau, P_\sigma]$. Then $p_\sigma(f(P)) = \sum_\tau f_{\sigma\tau}(P_\tau) (= S_\sigma)$. Since $f(P)$ is small in P , $\{S_\sigma\}_I$ is a right semi- T -nilpotent set of submodules from Theroem 1'.

Corollary 1.

$$J((R)_I) = \bigcup_{(A_\sigma)} \begin{pmatrix} A_1, A_1, \dots \\ A_2, A_2, \dots \\ \dots\dots\dots \\ A_\sigma, A_\sigma, \dots \end{pmatrix}$$

where $\{A_\sigma\}_I$ runs through all the right semi- T -nilpotent sets of right ideals in $J(R)$ and all permutations $\{A_{\pi(\sigma)}\}$ of $\{A_\sigma\}$.

Corollary 2 ([10, 11, 12]). *Let $(a_{\sigma\tau})$ be in $(R)_I$. Then the following statements are equivalent.*

- 1) $(a_{\sigma\tau}) \in J((R)_I)$.
- 2) $\{\sum_\tau a_{\sigma\tau}R\}_\sigma$ is a right semi- T -nilpotent set.
- 3) Any set $\{a_{\sigma\tau}\}$ is a right semi- T -nilpotent set, where almost all σ 's are distinct.

Proof. 3) \rightarrow 2) We can prove it from Konig Graph Theorem. Other implications are clear from Corollary 1.

By $J_f((R)_I)$ we shall denote the set of matrices in $(J(R))_I$ almost all of whose rows are zero. On the other hand, we denote a small submodule $\sum_I^n \oplus u_{\alpha_i} J(R)$ in $M = \sum_I \oplus u_\alpha R$ by $J(\alpha_1, \alpha_i, \dots, \alpha_n)(M)$. Then we have

Corollary 3. *The following statements are equivalent.*

- 1) $J((R)_I) = J_f((R)_I)$

- 2) Every small submodule in M is contained in some $J(\alpha_1, \alpha_2, \dots, \alpha_n)(M)$.
- 3) There are no non-trivial, infinite right semi- T -nilpotent sets of elements in $J(R)$ (cf. [6], Theorem 1).

Proof. 1) \rightarrow 2). We assume 2) is not satisfied. Then there exists a small submodule S in M which is not contained in any $J(\alpha'_1, \alpha'_2, \dots, \alpha'_n)(M)$. Hence, for a suitable sequence $\{\alpha_i, \alpha_i \neq \alpha_j \text{ for } i \neq j\}$, there exist elements s_i^* in S such that

$$s_i^* = \dots + u_{\alpha_i} s_{\alpha_i} + \dots, s_{\alpha_i} \neq 0 \in J(R).$$

We define f in S_M by setting

$$f(u_i) = s_i^* \text{ and } f(u_\alpha) = 0 \text{ for } u_\alpha \notin \{u_i\}_I^\infty.$$

Since $f(M) \subseteq S$, $f \in J((R)_I) = J_f((R)_I)$ from lemma 1. Therefore, $f(M) \subseteq J(\beta_1, \beta_2, \dots, \beta_m)(M)$, which is a contradiction. Other implications are clear.

REMARKS 1. If $\{T_\alpha\}_I$ is locally, right T -nilpotent in Proposition 2, $J(S_N) \supseteq \text{Hom}_R(N, T)$ (see the proof of Corollary 2 to Proposition 2).

2. Let Z be the ring of integers and p prime. Put $N_\alpha = Z_{p^\infty}$ for all α in Proposition 2. Then $\text{End}_Z(Z_{p^\infty}) = \hat{Z}_p$; the ring of p -adic completions and S_N is the ring of column-summable matrices $(a_{\sigma\tau})$ over \hat{Z}_p . Furthermore, $J(S_N) = \{(a_{\sigma\tau}) \mid a_{\sigma\tau} \in p\hat{Z}_p\}$ from [1], Theorem 9 and Proposition 10. Let $A_n = \{a \in Z_{p^\infty} \mid ap^n = 0\}$. Then $\{T_\alpha\}$ is a locally semi- T -nilpotent if and only if $T_\alpha = A_{n(\alpha)}$ for almost all α . On the other hand, $\text{Hom}_Z(Z_{p^\infty}, A_n) = 0$. Hence, $\text{Hom}_Z(N, T) = 0$ if $T_\alpha = A_{n(\alpha)}$ for all α and so $J(S_N) \neq \bigcup_T \text{Hom}_Z(N, T)$ (cf. Theorem 2'). Furthermore, let $A = \sum_1^\infty \oplus A_i$ and $M = \sum_1^\infty \oplus Z_{p^\infty} \xrightarrow{\varphi} Z_{p^\infty}$ the natural epimorphism. Then $\varphi(A) = Z_{p^\infty}$ and so A is not small in M (cf. Corollary 1 to Proposition 2). Hence, every small submodule in M is of a type $A^{(n)} = \{m \in M \mid mp^n = 0\}$ (use the similar argument above and the proof of Proposition 1).

3. Let Q be the rationals. Then Q is an injective and flat Z -module. It is clear that Z is a small submodule in Q . Put $A = \sum_1^\infty \oplus Q_i$; $Q_i = Q$ and $\varphi: A \rightarrow Q$ by setting $\varphi(q_i) = (1/i)q_i$; $q_i \in Q_i$. Since $\text{Hom}_Z(Q, Z) = 0$, $\{Z\}$ is a locally T -nilpotent set of small submodules. However, $\varphi(\sum \oplus Z) = Q$ and so $\sum \oplus Z$ is not small in A (see Corollary 1 to Proposition 2). Furthermore, $J(Z) = 0$ and so Z is not of a form in Corollary 4 to Proposition 2.

4. If R is a right perfect ring, $MJ(R) = J(M)$ is a unique maximal one among small submodules in an R -module M . Hence, every set of small submodules is a locally, right semi- T -nilpotent set and so almost results above are trivially valid without any assumptions: finitely generated and projective.

5. It is clear that

$$J((R)_I) = \begin{pmatrix} A_1, A_1 \cdots \\ A_2, A_2 \cdots \\ \dots\dots\dots \\ A_\sigma, A_\sigma \cdots \\ \dots\dots\dots \end{pmatrix}$$

for a right semi- T -nilpotent set of right ideals A_σ if and only if $A_\sigma = J(R)$ for all σ and $J(R)$ is right T -nilpotent.

2 Perfect modules

We shall add here a characterization for a finitely generated projective module to be perfect.

Theorem 3. *Let P be a finitely generated projective module and $M = \sum_1^\infty \oplus P$. Then P is perfect if and only if $S_M/J(S_M)$ is a regular ring in the sense of Von Neumann and every idempotent in $S_M/J(S_M)$ is lifted to S_M (cf. [3], Theorem 1).*

Proof. If P is perfect, the statements are obtained by [7]. Conversely, Let $S_P = \text{End}_R(P)$. Then $S_M = (S_P)_I$. Let \bar{e} be an idempotent in $(J(S_P))_I/J(S_M)$. We may assume e is idempotent in $(J(S_P))_I$ from the assumption. Since $J(S_P) = \text{Hom}_R(P, J(P))$ from Lemma 1, $e(M) \subseteq \sum \oplus J(P) = \sum \oplus PJ(R) = MJ(R)$. Hence, $e(M) = e(M)J(R)$. Therefore, $e = 0$. On the other hand, $S_M/J(S_M)$ is regular and so $J(S_M) = (J(S_P))_I$. Accordingly, $J(S_P)$ is right T -nilpotent and $S_P/J(S_P)$ is semi-simple artinian from [4], Corollary to Lemma 2. Thus, $P = \sum_1^n \oplus P_i$ and $\text{End}_R(P_i)$ is a local ring, which implies P is perfect from [2], Theorem 6.

Corollary 1. *Let R be a semi-simple artinian ring if and only if $J(R)$ contains no non-trivial right semi- T -nilpotent sets and $S_M/J(S_M)$ is a regular ring, where $M = \sum \oplus u_i R$.*

Proof. If $J(R)$ contains no right semi- T -nilpotent sets, then $J(S_M) = J_f(S_M)$. For any elements $(a_{\sigma\tau}), (b_{\sigma\tau})$ in S_M , $(a_{\sigma\tau}) \equiv (b_{\sigma\tau}) \pmod{J(S_M)}$ implies $a_{\sigma\tau} = b_{\sigma\tau}$ for almost all σ . Let aE be in S_M and $a \in R$, where E is the identity matrix in S_M . Then there exists $(b_{\sigma\tau})$ in S_M such that $aE(b_{\sigma\tau})aE \equiv aE \pmod{J(S_M)}$. Hence, there exists σ such that $ab_{\sigma\sigma}a = a$ from the above. Therefore, R is regular and $J(R) = 0$. Since $(R)_I = (R)_I/(J(R))_I$ is regular, R is artinian from [4], Corollary to Lemma 2.

References

- [1] M. Harada and Y. Sai: *On categories of indecomposable modules I*, Osaka J. Math. **7** (1970), 323–344.
- [2] M. Harada: *Perfect categories I*, *ibid.*, **10** (1973), 329–341.
- [3] M. Harada and T. Ishii: *Perfect rings and the exchange property*, Osaka J. Math. **12** (1975), 483–491.
- [4] M. Harada and H. Kanbara: *On categories of projective modules*, Osaka J. Math. **8** (1971), 471–483.
- [5] N. Jacobson: *Structure of rings*, Amer. Math. Soc. Colloq. XXXVII, 1956.
- [6] W. Liebert: *Radical of some endomorphism rings*, J. Reine Angew. Math. **261/263** (1973), 166–170.
- [7] E. Mares: *Semi-perfect modules*, Math. Z. **83** (1963), 347–360.
- [8] E.M. Patterson: *On radicals of rings of row-finite matrices*, Proc. Roy. Soc. Edinburgh Set A. **66** (1962), 24–46.
- [9] N.E. Sexauer and J.E. Warnock: *The radical of the row-finite matrices over an arbitrary ring*, Trans. Amer. Math. Soc. **139** (1965), 286–295.
- [10] R. Solter: *The Jacobson radical of row-finite matrices*, J. Algebra **12** (1969), 345–359.
- [11] ———: *A note on the radical of row-finite matrices*, Glasgow Math. J. **23** (1972), 80–81.
- [12] R. Ware and J. Zelmanowitz: *The radical of the endomorphism ring of a projective module*, Proc. Amer. Math. Soc. **26** (1970), 15–20.
- [13] K. Motose: *On Sexauer-Warnock theorem*, J. Fac. Sci. Shinshu Univ. **5** (1970), 33–34.