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ON HYPERSINGULAR INTEGRALS

Dedicated to Professor Tatsuo Fuji'i'e on the occasion of his 60th birthday

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1. Introduction

The purposes of this paper are to generalize a result of E.M. Stein concerning hypersingular integrals and to give its application. The same problems for singular difference integrals are discussed in [2]. Let R^n be the n-dimensional Euclidean space and for each point $x=(x_1, \cdots, x_n)$ we write $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$. For a multi-index $\gamma=(\gamma_1, \cdots, \gamma_n)$ we denote $|\gamma|=\gamma_1+\cdots+\gamma_n$, $\gamma!=\gamma_1!\cdots\gamma_n!$, $x^\gamma=x_1^{\gamma_1}\cdots x_n^{\gamma_n}$ and $D^\gamma=D_1^{\gamma_1}\cdots D_n^{\gamma_n}$. The Riesz kernel κ_α of order $\alpha>0$ is defined as

$$\kappa_\alpha(x)=\begin{cases} |x|^{\alpha-n}, & \alpha<n \text{ or } \alpha\geqq n,\ \alpha-n \text{ not even} \\ (\delta_{\alpha,n}-\log|x|)|x|^{\alpha-n}, & \alpha\geqq n,\ \alpha-n \text{ even}\end{cases}$$

where $\delta_{\alpha,n}$ is the constant (see [1]).

For $\beta>0$, $\varepsilon>0$ and a positive integer l, we consider the integral (hypersingular integral)

$$H_\varepsilon^{l,\beta}u(x)=\int_{|t|\geqq\varepsilon}\frac{(R_t^l u)(x)}{|t|^{n+\beta}}dt$$

where $(R_t^l u)(x)$ is the remainder of order l, namely

$$(R_t^l u)(x)=u(x+t)-\Big(\sum_{|\gamma|\leqq l-1}D^\gamma u(x)/\gamma!\Big)t^\gamma .$$

We denote $\mu^{l,\alpha,\beta}(x)=H_1^{l,\beta}\kappa_\alpha(x)$. E.M. Stein showed in [3; p. 162] that $\mu^{1,\alpha,\alpha}\in L^1$ for $0<\alpha<2$. In § 2 we generalize this result (Theorems 2.3 and 2.14). R.L. Wheeden [4; Theorem 1] established an inequality of the following type for functions u in Bessel potential spaces:

$$||H_\varepsilon^{l,\beta}u||_p\leqq C||u||_{p,\beta} ,$$

where $||\ \ ||_p$ is the L^p-norm and $||\ \ ||_{p,\beta}$ is the norm in Bessel potential spaces.

In § 3 we give integral estimates of $H_\varepsilon^{l,\beta}u$ for Beppo Levi functions u (Theorem 3.7).

Throughout this paper, the letter C is used for a generic positive constant whose value may be different at each occurrence.

The author expresses his thanks to Prof. Fumi-Yuki Maeda who gave the proof of Lemma 2.12.

2. Hypersingular integrals of Riesz kernels

For a positive integer l, C^l denote the space of all l times continuously differentiable functions on R^n. For a point t in R^n we write $t'=t/|t|$ By Taylor's formula we have

Lemma 2.1. *If $\phi\in C^l$, then*

$$(R_t^l\phi)(x) = l\sum_{|\gamma|=l}\int_0^{|t|}\frac{(|t|-s)^{l-1}}{\gamma!}(t')^\gamma D^\gamma\phi(x+st')ds\,.$$

For $x, y\in R^n$, we put $L_{x,y}=\{sx+(1-s)y;\ 0\leqq s\leqq 1\}$ and $L_x=L_{x,0}$. We denote by $d(y, L_x)$ the distance between y and L_x. By Lemma 2.1 we obtain

Corollary 2.2. (i) *If $\phi\in C^l$, then*

$$|(R_t^l\phi)(y)|\leqq|t|^l\sum_{|\gamma|=l}(1/\gamma!)\max_{z\in L_{y,y+t}}|D^\gamma\phi(z)|.$$

(ii) *If $l>\alpha-n$, then for $d(y, L_{-t})\geqq|t|/2$ we have*

$$|(R_t^l\kappa_\alpha)(y)|\leqq C|t|^l|y|^{\alpha-l-n}\,.$$

Proof. (i) By Lemma 2.1 we have

$$\begin{aligned}|(R_t^l\phi)(y)|&\leqq l\sum_{|\gamma|=l}\int_0^{|t|}\frac{(|t|-s)^{l-1}}{\gamma!}|D^\gamma\phi(y+st')|ds\\&\leqq l\sum_{|\gamma|=l}(1/\gamma!)\max_{z\in L_{y,y+t}}|D^\gamma\phi(z)|\int_0^{|t|}(|t|-s)^{l-1}ds\\&=|t|^l\sum_{|\gamma|=l}(1/\gamma!)\max_{z\in L_{y,y+t}}|D^\gamma\phi(z)|.\end{aligned}$$

(ii) We note that if $d(y, L_{-t})\geqq|t|/2$, then

$$|y|/3\leqq|y+st'|\leqq 3|y|$$

for $0\leqq s\leqq|t|$. Further, if $|\gamma|=l$ and $l>\alpha-n$, then

$$|D^\gamma\kappa_\alpha(x)|\leqq C|x|^{\alpha-l-n}\,.$$

Hence by Lemma 2.1, for $d(y, L_{-t})\geqq |t|/2$ we have

$$\begin{aligned}|(R_t^l\kappa_\alpha)(y)&\leqq l\sum_{|\gamma|=l}\int_0^{|t|}\frac{(|t|-s)^{l-1}}{\gamma!}|D^\gamma\kappa_\alpha(st'+y)|ds\\&\leqq C\sum_{|\gamma|=l}\int_0^{|t|}\frac{(|t|-s)^{l-1}}{\gamma!}|st'+y|^{\alpha-l-n}ds\\&\leqq C|y|^{\alpha-l-n}\sum_{|\gamma|=l}\int_0^{|t|}\frac{(|t|-s)^{l-1}}{\gamma!}ds\\&=C|t|^l|y|^{\alpha-l-n}.\end{aligned}$$

For $\alpha>0$, $\beta>0$ and a positive integer l, we set

$$\mu^{l,\alpha,\beta}(y)=\int_{|t|\geqq 1}\frac{(R_t^l\kappa_\alpha)(y)}{|t|^{n+\beta}}dt$$

and

$$\mu_*^{l,\alpha,\beta}(y)=\int_{|t|\geqq 1}\frac{|(R_t^l\kappa_\alpha)(y)|}{|t|^{n+\beta}}dt\,.$$

We note that $\mu_*^{l,\alpha,\beta}(y)$ is finite for $\beta>\alpha-n$, $\beta>l-1$ and $y\neq 0$.

2.1. Integrability of $\mu^{l,\alpha,\beta}$

For $1<q\leqq\infty$, we denote the conjugate exponent to q by q', namely, $(1/q)+(1/q')=1$. At first, we prove

Theorem 2.3. Let $1<q\leqq\infty$ and $\beta>\alpha-(n/q)$.
(i) *If $l-1<\alpha-(n-q)<l$, then $\mu_*^{l,\alpha,\beta}\in L^{q'}$.*
(ii) *If l is an odd number, then for $l-1<\alpha-(n/q)<l+1$ we have $\mu^{l,\alpha,\beta}\in L^{q'}$.*

Proof. (i) By Minkowski's inequality for integrals we obtain

$$\begin{aligned}I&=\left(\int|\mu_*^{l,\alpha,\beta}(y)|^{q'}dy\right)^{1/q'}\\&=\left(\int\left(\int_{|t|\geqq 1}\frac{|(R_t^l\kappa_\alpha)(y)|}{|t|^{n+\beta}}dt\right)^{q'}dy\right)^{1/q'}\\&\leqq\int_{|t|\geqq 1}\frac{dt}{|t|^{n+\beta}}\left(\int|(R_t^l\kappa_\alpha)(y)|^{q'}dy\right)^{1/q'}.\end{aligned}$$

Further, we have

$$\begin{aligned}\left(\int|(R_t^l\kappa_\alpha)(y)|^{q'}dy\right)^{1/q'}&\leqq\left(\int_{d(y,L_{-t})\geqq|t|/2}|(R_t^l\kappa_\alpha)(y)|^{q'}dy\right)^{1/q'}\\&\quad+\left(\int_{d(y,L_{-t})<|t|/2}|(R_t^l\kappa_\alpha)(y)|^{q'}dy\right)^{1/q'}\\&=I_1+I_2\,.\end{aligned}$$

Since $l>\alpha-(n/q)>\alpha-n$, it follows from Corollary 2.2 (ii) that

$$I_1\leqq C|t|^l(\int_{|y|\geqq|t|/2}|y|^{(\alpha-l-n)q'}dy)^{1/q'}=C|t|^{\alpha-(n/q)}.$$

Moreover it follows from $\alpha-(n/q)>l-1\geqq 0$ that

$$\begin{aligned}I_2&\leqq(\int_{d(y,L_{-t})<|t|/2}|\kappa_\alpha(y+t)|^{q'}dy)^{1/q'}\\&\quad+\sum_{|\gamma|\leqq l-1}(|t|^{|\gamma|}/\gamma!)(\int_{d(y,L_{-t})<|t|/2}|D^\gamma\kappa_\alpha(y)|^{q'}dy)^{1/q'}\\&\leqq C\begin{cases}|t|^{\alpha-(n/q)}, & \alpha-n \text{ is not a nonnegative even number}\\ |t|^{\alpha-(n/q)}(1+|\log|t||), & \alpha-n \text{ is a nonnegative even number}.\end{cases}\end{aligned}$$

Thus

$$I\leqq C\begin{cases}\displaystyle\int_{|t|\geqq 1}\frac{|t|^{\alpha-(n/q)}}{|t|^{n+\beta}}dt, & \alpha-n \text{ is not a nonnegative even number}\\ \displaystyle\int_{|t|\geqq 1}\frac{|t|^{\alpha-(n/q)}(1+|\log|t||}{|t|^{n+\beta}}dt, & \alpha-n \text{ is a nonnegative even number.}\end{cases}$$

By the condition $\beta>\alpha-(n/q)$, we obtain $I<\infty$.

(ii) Let l be a n odd number. We have

$$\begin{aligned}J&=(\int|\mu^{l,\alpha,\beta}(y)|^{q'}dy)^{1/q'}\\&\leqq\left(\int\left|\int_{|t|\geqq 1,|t|\leqq 2|y|/3}\frac{(R_t^l\kappa_\alpha)(y)}{|t|^{n+\beta}}dt\right|^{q'}dy\right)^{1/q'}\\&\quad+\left(\int\left|\int_{|t|\geqq 1,|t|>2|y|/3}\frac{(R_t^l\kappa_\alpha)(y)}{|t|^{n+\beta}}dt\right|^{q'}dy\right)^{1/q'}\\&=J_1+J_2.\end{aligned}$$

Since l is an odd number, we see that

$$\int_{|t|\geqq 1,|t|\leqq 2|y|/3}\sum_{|\gamma|=l}\frac{t^\gamma}{|t|^{n+\beta}}dt=0.$$

Hence we have

$$\begin{aligned}J_1&=\left(\int\left|\int_{|t|\geqq 1,|t|\leqq 2|y|/3}\frac{(R_t^{l+1}\kappa_\alpha)(y)}{|t|^{n+\beta}}dt\right|^{q'}dy\right)^{1/q'}\\&\leqq\int_{|t|\geqq 1}\frac{dt}{|t|^{n+\beta}}(\int_{|y|\geqq 3|t|/2}|R_t^{l+1}\kappa_\alpha(y)|^{q'}dy)^{1/q'}.\end{aligned}$$

Note that $|y|\geqq 3|t|/2$ implies $d(y, L_{-t})\geqq|t|/2$. Since $l+1>\alpha-(n/q)>\alpha-n$ and $\beta>\alpha-(n/q)$, we obtain from Corollary 2.2 (ii)

$$J_1 \leqq C \int_{|t|\geqq 1} \frac{dt}{|t|^{n+\beta}} \left(\int_{|y|\geqq 3|t|/2} (|t|^{l+1}|y|^{\alpha-l-1-n})^{q'} dy\right)^{1/q'}$$
$$= C \int_{|t|\geqq 1} \frac{|t|^{\alpha-(n/q)}}{|t|^{n+\beta}} dt < \infty .$$

In the same way as in (i) we obtain $J_2 < \infty$ since $\alpha-(n/q) > l-1 \geqq 0$ and $\beta > \alpha-(n/q)$. The proof of the theorem is complete.

2.2. Integrability of $\mu^{l,\alpha,\alpha}$

By Taylor's formula we have

Lemma 2.4. *If P is a polynomial of order m and $l>m$, then $(R_t^l P)(x)=0$.*

The symbol $\mathcal{D}$ denotes the LF-space of all C^∞-functions with compact support and $\mathcal{D}'$ stands for the topological dual of $\mathcal{D}$. For $f \in \mathcal{D}$, the Riesz potential of order α of f is defined as

$$U_\alpha^f(x) = \int \kappa_\alpha(x-y) f(y) dy .$$

We easily see

Lemma 2.5. *Let $f \in \mathcal{D}$. Then*
(i) *$U_\alpha^f \in C^\infty$.*
(ii) *For $|\gamma| < \alpha$*

$$D^\gamma U_\alpha^f(x) = \int D^\gamma \kappa_\alpha(x-y) f(y) dy .$$

(iii) *If $|\gamma| = l$ is odd, then*

$$\lim_{\varepsilon\to 0} \int_{|x-y|\geqq\varepsilon} D^\gamma \kappa_l(x-y) f(y) dy$$

exists for all x and

$$D^\gamma U_l^f(x) = \lim_{\varepsilon\to 0} \int_{|x-y|\geqq\varepsilon} D^\gamma \kappa_l(x-y) f(y) dy .$$

If $\alpha-n$ is not a nonnegative even number, then the Riesz kernel $\kappa_\alpha(x)$ is homogeneous of degree $\alpha-n$. Hence

$$D^\gamma \kappa_\alpha(\lambda x) = \lambda^{\alpha-|\gamma|-n} D^\gamma \kappa_\alpha(x) , \qquad \lambda > 0 . \tag{2.1}$$

When $\alpha-n$ is a nonnegative even number, we have

Lemma 2.6. *Let $\alpha-n$ be a nonnegative even number. Then*

$$D^\gamma \kappa_\alpha(\lambda x) = \lambda^{\alpha-|\gamma|-n}((-\log \lambda) D^\gamma \chi_\alpha(x) + D^\gamma \kappa_\alpha(x))$$

with $\chi_\alpha(x)=|x|^{\alpha-n}$.

Proof. By Leibniz's formula we see that

$$D^\gamma\kappa_\alpha(x)=(\delta_{\alpha,n}-\log|x|)D^\lambda\chi_\alpha(x)+\sum_{0<\delta\leqq\gamma}{}_\gamma C_\delta D^\delta(\delta_{\alpha,n}-\log|x|)D^{\gamma-\delta}\chi_\alpha(x)$$

with ${}_\gamma C_\delta=\gamma!/\delta!(\gamma-\delta)!$. For $\delta>0$, we have

$$D^\delta(\delta_{\alpha,n}-\log|x|)=\frac{P_\delta(x)}{|x|^{2|\delta|}}$$

where $P_\delta(x)$ is a homogeneous polynomial of degree $|\delta|$. Hence we have

$$\begin{aligned}D^\gamma\kappa_\alpha(\lambda x)&=(\delta_{\alpha,n}-\log|\lambda x|)D^\gamma\chi_\alpha(\lambda x)+\sum_{0<\delta\leqq\gamma}{}_\gamma C_\delta\frac{P_\delta(\lambda x)}{|\lambda x|^{2|\delta|}}D^{\gamma-\delta}\chi_\alpha(\lambda x)\\&=(-\log\lambda)\lambda^{\alpha-|\gamma|-n}D^\gamma\chi_\alpha(x)+(\delta_{,\alpha n}-\log|x|)\lambda^{\alpha-|\gamma|-n}D^\gamma\chi_\alpha(x)\\&\quad+\sum_{0<\delta\leqq\gamma}{}_\gamma C_\delta\lambda^{\alpha-|\gamma|-n}\frac{P_\delta(x)}{|x|^{2|\delta|}}D^{\gamma-\delta}\chi_\alpha(x)\\&=\lambda^{\alpha-|\gamma|-n}(-\log\lambda)D^\gamma\chi_\alpha(x)+D^\gamma\kappa_\alpha(x)\,.\end{aligned}$$

Thus we obtain the lemma.

Lemma 2.7. *Let l be a positive integer, and moreover we assume that $l>\alpha-n$ in case $\alpha-n$ is a nonnegative even number. Then*

(i) $(R_t^l\kappa_\alpha)(y)=|t|^{\alpha-n}(R_{t'}^l\kappa_\alpha)(y/|t|)$,

(ii) $(R_t^l\kappa_\alpha)(y)=|y|^{\alpha-n}(R_{t/|y|}^l\kappa_\alpha)(y')$.

Proof. (i) If $\alpha-n$ is not a nonnegative even number, then by (2.1) we have

$$\begin{aligned}(R_t^l\kappa_\alpha)(y)&=|y+t|^{\alpha-n}-\sum_{|\gamma|\leqq l-1}(D^\gamma\kappa_\alpha(y)/\gamma!)t^\gamma\\&=|t|^{\alpha-n}\left(\left|\frac{y}{|t|}+t'\right|^{\alpha-n}-\sum_{|\gamma|\leqq l-1}\frac{D^\gamma\kappa_\alpha(y/|t|)}{\gamma!}(t')^\gamma\right)\\&=|t|^{\alpha-n}(R_{t'}^l\kappa_\alpha)(y/|t|)\,.\end{aligned}$$

If $\alpha-n$ is a nonnegative even number, then by Lemma 2.6 we see that

$$\begin{aligned}(R_t^l\kappa_\alpha)(y)&=(\delta_{\alpha,n}-\log|y+t|)|y+t|^{\alpha-n}-\sum_{|\gamma|\leqq l-1}(D^\gamma\kappa_\alpha(y)/\gamma!)t^\gamma\\&=|t|^{\alpha-n}\left\{\left(\delta_{\alpha,n}-\log\left|\frac{y}{|t|}+t'\right|-\log|t|\right)\left|\frac{y}{|t|}+t'\right|^{\alpha-n}\right.\\&\quad\left.-\sum_{|\gamma|\leqq l-1}\frac{D^\gamma\kappa_\alpha(y)}{\gamma!|t|^{\alpha-|\gamma|-n}}(t')^\gamma\right\}\end{aligned}$$

$$= |t|^{\alpha-n}\Big\{\Big(\delta_{\alpha,n}-\log\Big|\frac{y}{|t|}+t'\Big|-\log|t|\Big)\Big|\frac{y}{|t|}+t'\Big|^{\alpha-n}$$

$$-\sum_{|\gamma|\leqq l-1}(1/\gamma!)((-\log|t|)D^\gamma\chi_\alpha(y/|t|)+D^\gamma\kappa_\alpha(y/|t|))(t')^\gamma\Big\}$$

$$= |t|^{\alpha-n}(-\log|t|)\Big(\Big|\frac{y}{|t|}+t'\Big|^{\alpha-n}-\sum_{|\gamma|\leqq l-1}(D^\gamma\chi_\alpha(y/|t|)/\gamma!)(t'))^\gamma\Big)$$

$$+|t|^{\alpha-n}\Big\{\Big(\delta_{\gamma,n}-\log\Big|\frac{y}{|t|}+t'\Big|\Big)\Big|\frac{y}{|t|}+t'\Big|^{\gamma-n}-\sum_{|\gamma|\leqq l-1}\frac{D^\gamma\kappa_\alpha(y/|t|)}{\gamma!}(t')^\gamma\Big\}$$

$$= |t|^{\alpha-n}(-\log|t|)(R^l_{t'}\chi_\alpha)(y/|t|)+|t|^{\alpha-n}(R^l_{t'}\kappa_\alpha)(y/|t|)\,.$$

Since $l>\alpha-n$, by Lemma 2.4 we obtain the required equality.

Using (2.1) and Lemma 2.6 we can prove (ii) in the same way as in (i). We complete the proof of the lemma.

Lemma 2.8. *Let $f\in\mathcal{D}$.*
(i) *Let $l<\alpha+1$, and moreover we assume that $l>\alpha-n$ in case $\alpha-n$ is a nonnegative even number. Then*

$$(R^l_tU^f_\alpha)(x)=|t|^\alpha\int(R^l_{t'}\kappa_\alpha)(z)f(x-|t|z)dz\,.$$

(ii) *If l is an odd number, then*

$$(R^{l+1}_tU^f_l)(x)=|t|^l\lim_{\varepsilon\to0}\int_{|z|\geqq\varepsilon}(R^{l+1}_{t'}\kappa_\alpha)(z)f(x-|t|z)dz\,.$$

Proof. (i) Since $l-1<\alpha$, by Lemma 2.5 (ii) we have

$$\begin{aligned}(R^l_tU^f_\alpha)(x)&=U^f_\alpha(x+t)-\sum_{|\gamma|\leqq l-1}(D^\gamma U^f_\alpha(x)/\gamma!)t^\gamma\\&=\int(\kappa_\alpha(x+t-y)-\sum_{|\gamma|\leqq l-1}(D^\gamma\kappa_\alpha(x-y)/\gamma!)t^\xi)f(y)dy\\&=\int(R^l_t\kappa_\alpha)(x-y)f(y)dy\\&=\int(R^l_t\kappa_\alpha)(y)f(x-y)dy\,.\end{aligned}$$

Furthermore it follows from Lemma 2.7 (i) that

$$\begin{aligned}(R^l_tU^f_\alpha)(x)&=|t|^{\alpha-n}\int(R^l_{t'}\kappa_\alpha)(y/|t|)f(x-y)dy\\&=|t|^\alpha\int(R^l_{t'}\kappa_\alpha)(z)f(x-|t|z)dz\,.\end{aligned}$$

(ii) By Lemma 2.5 (ii) and (iii) we have

$$
\begin{aligned}
(R_t^{l+1}U_l^f)(x) &= U_l^f(x+t) - \sum_{|\gamma|\leqq l} (D^\gamma U_l^f(x)/\gamma!)t^\gamma \\
&= \int \kappa_l(x+t-y)f(y)dy - \sum_{|\gamma|\leqq l-1} (t^\gamma/\gamma!) \int D^\gamma \kappa_l(x-y)f(y)dy \\
&\quad - \sum_{|\gamma|=l} (t^\gamma/\gamma!) \lim_{\varepsilon\to 0} \int_{|x-y|\geqq\varepsilon|t|} D^\gamma\kappa_l(x-y)f(y)dy \\
&= \lim_{\varepsilon\to 0} \int_{|x-y|\geqq\varepsilon|t|} (R_t^{l+1}\kappa_l)(x-y)f(y)dy \\
&= \lim_{\varepsilon\to 0} \int_{|y|\geqq\varepsilon|t|} (R_t^{l+1}\kappa_l)(y)f(x-y)dy\,.
\end{aligned}
$$

By Lemma 2.7 (i) we see that

$$
\begin{aligned}
(R_t^{l+1}U_l^f)(x) &= \lim_{\varepsilon\to 0} |t|^{l-n} \int_{|y|\geqq\varepsilon|t|} (R_{t'}^{l+1}\kappa_l)(y/|t|)f(x-y)dy \\
&= \lim_{\varepsilon\to 0} |t|^{l} \int_{|z|\geqq\varepsilon} (R_{t'}^{l+1}\kappa_l)(z)f(x-|t|z)dz\,.
\end{aligned}
$$

Thus we obtain the lemma.

Corollary 2.9. (i) *If $\alpha<l<\alpha+1$, then $(R_t^l\kappa_\alpha)(y)$ is integrable as a function of y and for all $t\in R^n$*

$$\int_{R^n} (R_t^l\kappa_\alpha)(y)dy = 0\,.$$

(ii) *If l is an odd number, then $(R_t^{l+1}\kappa_l)(y)$ is integrable on $\{|y|\geqq\varepsilon\}$ $(\varepsilon>0)$ and for all $t\in R^n$*

$$\lim_{\varepsilon\to 0} \int_{|y|\geqq\varepsilon} (R_t^{l+1}\kappa_l)(y)\,dy = 0\,. \tag{2.2}$$

Proof. (i) We easily see that $(R_t^l\kappa_\alpha)(y)$ is locally integrable for $l<\alpha+1$. It follows from Corollary 2.2 (ii) that

$$|(R_t^l\kappa_\alpha)(y)| \leqq C\,|y|^{\alpha-l-n}$$

for $|y|\geqq|t|/2$. Hence for $\alpha<l<\alpha+1$, $(R_t^l\kappa_\alpha)(y)$ is integrable with respect to y. Next we shall show

$$\int (R_t^l\kappa_\alpha)(y)\,dy = 0\,.$$

Since

$$\int (R_t^l\kappa_\alpha)(y)dy = |t|^\alpha \int (R_{t'}^l\kappa_\alpha)(z)dz$$

by Lemma 2.7 (i), it suffices to show

$$\int (R_\xi^l \kappa_\alpha)(y)dy = 0$$

for all $|\xi|=1$. Let $|\xi|=1$. We take a function $f\in\mathcal{D}$ such that $f(0)=1$. By Lemma 2.8 (i) we have

$$(R_{r\xi}^l U_\alpha^f)(0) = r^\alpha \int (R_\xi^l \kappa_\alpha)(z)f(-rz)dz\,.$$

Since $l>\alpha$, it follows from Corollary 2.2 (i) that

$$\frac{(R_{r\xi}^l U_\alpha^f)(0)}{r^\alpha} \to 0 \qquad (r\to 0)\,.$$

On the other hand, since $(R_\xi^l \kappa_\alpha)(z)$ is integrable we see that

$$\int (R_\xi^l \kappa_\alpha)(z)f(-rz)dz \to \int (R_\xi^l \kappa_\alpha)(z)dz \qquad (r\to 0)\,.$$

Hence we obtain

$$\int (R_\xi^l \kappa_\alpha)(z)dz = 0\,.$$

(ii) We easily see that $(R_\xi^{l+1}\kappa_l)(y)$ is integrable on $\{|y|\geqq\varepsilon\}$. For (2.2) it suffices to show

$$\lim_{\varepsilon\to 0}\int_{|y|\geqq\varepsilon}(R_\xi^{l+1}\kappa_l)(y)\,dy = 0$$

for $|\xi|=1$. We take a function $f\in\mathcal{D}$ which satisfies the condition $f(x)=1$ on $\{|x|<1\}$. By Lemma 2.8 (ii) we have

$$(R_{r\xi}^{l+1}U_l^f)(0) = \lim_{\varepsilon\to 0} r^l \int_{|z|\geqq\varepsilon}(R_\xi^{l+1}\kappa_l)(z)f(-rz)dz\,.$$

It follows from Corollary 2.2 (i) that

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\int_{|z|\geqq\varepsilon}(R_\xi^{l+1}\kappa_l)(z)f(-rz)dz = 0\,.$$

We easily see that the left hand side is equal to

$$\lim_{\varepsilon\to 0}\int_{|z|\geqq\varepsilon}(R_\xi^{l+1}\kappa_l)(z)dz\,.$$

Hence we obtain the required property.

Lemma 2.10. *Let $\beta>\alpha-n$, $\beta<l-1$, and moreover we assume that $l>\alpha-n$ in case $\alpha-n$ is a nonnegative even number. Then*

$$\mu^{l,\alpha,\beta}(y)=\frac{1}{|y|^{n+\beta-\alpha}}\int_{|z|\geq 1/|y|}\frac{(R_z^l\kappa_\alpha)(y')}{|z|^{n+\beta}}dz\,.$$

Proof. By Lemma 2.7 (ii) and the change of variables $t/|y|=z$, we have

$$\begin{aligned}\mu^{l,\alpha,\beta}(y)&=\int_{|t|\geq 1}\frac{(R_t^l\kappa_\alpha)(y)}{|t|^{n+\beta}}dt\\&=\int_{|t|\geq 1}\frac{|y|^{\alpha-n}(R_{t/|y|}^l\kappa_\alpha)(y')}{|t|^{n-\beta}}dt\\&=\frac{1}{|y|^{n+\beta-\alpha}}\int_{|z|\geq 1/|y|}\frac{(R_z^l\kappa_\alpha)(y')}{|z|^{n+\beta}}dz\,.\end{aligned}$$

REMARK 2.11. Let

$$\mu_\varepsilon^{l,\alpha,\beta}(y)=\int_{|t|\geq\varepsilon}\frac{(R_t^l\kappa_\alpha)(y)}{|t|^{n+\beta}}dt\,.$$

Then

$$\mu_\varepsilon^{l,\alpha,\beta}(y)=\frac{1}{|y|^{n+\beta-\alpha}}\int_{|z|\geq\varepsilon/|y|}\frac{(R_z^l\kappa_\alpha)(y')}{|z|^{n+\beta}}dz\,.$$

Lemma 2.12. *Let $K_m(x,y)=\sum_{|\gamma|=m}(D^\gamma\kappa_\alpha(x)/\gamma!)y^\gamma$. Then for $|x|=|y|$ we have $K_m(x,y)=K_m(y,x)$.*

Proof. Let $|x|=|y|$. We set $F_{x,y}(s)=\kappa_\alpha(x+sy)$ for $-1<s<1$. We easily see that $F_{x,y}(s)=F_{y,x}(s)$. Hence $(d^m/ds^m)F_{x,y}(0)=(d^m/ds^m)F_{y,x}(0)$. Since

$$(d^m/ds^m)F_{x,y}(0)=\sum_{|\gamma|=m}(m!/\gamma!)D^\gamma\kappa_\alpha(x)y^\gamma\,,$$

we obtain the required property.

Lemma 2.13. *Let $\beta>\alpha-n$, $\beta>l-1$, and moreover we assume that $l<\alpha-n$ in case $\alpha-n$ is a nonnegative even number. Then*

$$\mu^{l,\alpha,\beta}(y)=\frac{1}{|y|^{n+\beta-\alpha}}\int_{|v|\leq|y|}\frac{(R_{y'}^l\kappa_\alpha)(v)}{|v|^{\alpha-\beta}}dv\,.$$

Proof. By Lemma 2.10 we have

$$\mu^{l,\alpha,\beta}(y)=\frac{1}{|y|^{n+\beta-\alpha}}\int_{|u|\geq 1/|y|}\frac{(R_u^l\kappa_\alpha)(y')}{|u|^{n+\beta}}du\,.$$

Let $\alpha-n$ be not a nonnegative even number. By the inversion $u=v/|v|^2$, we see that

$$\mu^{l,\alpha,\beta}(y)=\frac{1}{|y|^{n+\beta-\alpha}}\int_{|v|\leq|y|}|v|^{\beta-\alpha}\left(\left||v|y'+\frac{v}{|v|}\right|^{\alpha-n}\right.$$
$$\left.-\sum_{|\gamma|\leqq l-1}(D^{\gamma}\kappa_{\alpha}(y')/\gamma!)|v|^{\alpha-2|\gamma|-n}v^{\gamma}\right)dv\,.$$

By Lemma 2.12 we have

$$\sum_{|\gamma|\leqq l-1}(D^{\gamma}\kappa_{\alpha}(y')/\gamma!)|v|^{\alpha-2|\gamma|-n}v^{\gamma}$$
$$=\sum_{m=0}^{l-1}|v|^{\alpha-m-n}\sum_{|\gamma|=m}(D^{\gamma}\kappa_{\alpha}(y')/\gamma!)(v/|v|)^{\gamma}$$
$$=\sum_{m=0}^{l-1}|v|^{\alpha-m-n}\sum_{|\gamma|=m}(D^{\gamma}\kappa_{\alpha}(v/|v|)/\gamma!)(y')^{\gamma}$$
$$=\sum_{m=0}^{l-1}\sum_{|\gamma|=m}(D^{\gamma}\kappa_{\alpha}(v)/\gamma!)/(y')^{\gamma}$$
$$=\sum_{|\gamma|\leqq l-1}(D^{\gamma}\kappa_{\alpha}(v)/\gamma!)(y')^{\gamma}\,.$$

Further, since $||v|y'+(v/|v|)|=|v+y'|$ by an easy argument, we have

$$\mu^{l,\alpha,\beta}(y)=\frac{1}{|y|^{n+\beta-\alpha}}\int_{|v|\leq|y|}|v|^{\beta-\alpha}\left(|v+y'|^{\alpha-n}-\sum_{|\gamma|\leqq l-1}\frac{D^{\gamma}\kappa_{\alpha}(v)}{\gamma!}(y')^{\gamma}\right)dv$$
$$=\frac{1}{|y|^{n+\beta-\alpha}}\int_{|v|\leq|y|}\frac{(R^{l}_{y'}\kappa_{\alpha})(v)}{|v|^{\alpha-\beta}}dv\,.$$

Next, let $\alpha-n$ be a nonnegative even number. By the inversion $u=v/|v|^2$, we see that

$$\mu^{l,\alpha,\beta}(y)=\frac{1}{|y|^{n+\beta-\alpha}}\int_{|v|\leq|y|}|v|^{\beta-\alpha}\left(\left(\delta_{\alpha,n}-\log\left||v|y'+\frac{v}{|v|}\right|+\log|v|\right)\right.$$
$$\left.\times\left||v|y'+\frac{v}{|v|}\right|^{\alpha-n}-\sum_{|\gamma|\leqq l-1}\frac{D^{\gamma}\kappa_{\alpha}(y')}{\gamma!}|v|^{\alpha-2|\gamma|-n}v^{\gamma}\right)dv\,.$$

By Lemmas 2.12 and 2.6 we have

$$\sum_{|\gamma|\leqq l-1}(D^{\gamma}\kappa_{\alpha}(y')/\gamma!)|v|^{\alpha-2|\gamma|-n}v^{\gamma}$$
$$=\sum_{|\gamma|\leqq l-1}(D^{\gamma}\kappa_{\alpha}(v)/\gamma!)(y')^{\gamma}+\log|v|)\sum_{|\gamma|\leqq l-1}(D^{\gamma}\chi_{\alpha}(v)/\gamma!)(y')^{\gamma}\,.$$

Therefore, by Lemma 2.4 we obtain

$$\mu^{l,\alpha,\beta}(y)=\frac{1}{|y|^{n+\beta-\alpha}}\int_{|v|\leq|y|}|v|^{\beta-\alpha}\Big((\delta_{\alpha,n}-\log|v+y'|)|v+y'|^{\alpha-n}$$
$$-\sum_{|\gamma|\leqq l-1}\frac{D^{\gamma}\kappa_{\alpha}(v)}{\gamma!}(y')^{\gamma}+(\log|v|)\Big(|v+y'|^{\alpha-n}-\sum_{|\gamma|\leqq l-1}\frac{D^{\gamma}\chi_{\alpha}(v)}{\gamma!}(y')^{\gamma}\Big)\Big)dv$$
$$=\frac{1}{|y|^{n+\beta-\alpha}}\int_{|v|\leq|y|}|v|^{\beta-\alpha}((R^{l}_{y'}\kappa_{\alpha})(v)+(\log|v|)(R^{l}_{y'}\chi_{\alpha})(v))dv$$

$$= \frac{1}{|y|^{n+\beta-\alpha}} \int_{|v|\leq|y|} \frac{(R_{y'}^{l}\kappa_\alpha)(v)}{|v|^{\alpha-\beta}} dv .$$

The proof of the lemma is completed.

Now we prove

Theorem 2.14. *Let* $1<q\leqq\infty$. *If* $l-1<\alpha-(n/q)<i$, *then* $\mu^{l,\alpha,\alpha}\in L^{q'}$. *Moreover, when l is an odd number,* $\mu^{l,\alpha,\alpha}\in L^{q'}$ *still holds for* $l-1<\alpha-(n/q)<l+1$.

Proof. In case $1<q<\infty$, the conclusion follows from Theorem 2.3 since $\alpha>\alpha-(n/q)$. Hence it is sufficient to show that if $l-1<\alpha<l$, then $\mu^{l,\alpha,\alpha}\in L^1$, and that if l is odd and $l-1<\alpha<l+1$, then $\mu^{l,\alpha,\alpha}\in L^1$.

First let $l-1<\alpha<l$. By Lemma 2.13 and $\alpha>l-1$, we see

$$\int_{|y|\leq 1/2} |\mu^{l,\alpha,\alpha}(y)|\,dy \leqq \int_{|y|\leq 1/2} \frac{1}{|y|^n} \int_{|v|\leq|y|} |(R_{y'}^{l}\kappa_\alpha)(v)|\,dv\,dy<\infty .$$

Since $\alpha<l<\alpha+1$, by Lemma 2.13 and Corollary 2.9 (i) we have

$$\begin{aligned}\mu^{l,\alpha,\alpha}(y) &= \frac{1}{|y|^n} \int_{|v|\leq|y|} (R_{y'}^{l}\kappa_\alpha)(v)dv \\ &= \frac{1}{|y|^n} \int_{|v|>|y|} (R_{y'}^{l}\kappa_\alpha)(v)dv .\end{aligned}$$

Hence by Corollary 2.2 (ii) we obtain

$$\begin{aligned}\int_{|y|>1/2} |\mu^{l,\alpha,\alpha}(y)|\,dy &= \int \frac{1}{|y|^n} \left|\int_{|v|>|y|} (R_{y'}^{l}\kappa_\alpha)(v)dv\right| dy \\ &\leqq C \int_{|y|>1/2} \frac{1}{|y|^n} \int_{|v|>|y|} |v|^{\alpha-l-n}\,dv\,dy \\ &= C\int_{|y|>1/2} |y|^{\alpha-l-n}\,dy<\infty\end{aligned}$$

since $l<\alpha$. Tnus $\mu^{l,\alpha,\alpha}\in L^1$ for $l-1<\alpha<l$.

Next let l be an odd number and $l<\alpha<l+1$. Since $\alpha>l>l-1$, we see

$$\int_{|y|\geqq 1/2} |\mu^{l,\alpha,\alpha}(y)|\,dy<\infty$$

by the same reason as above. Since l is an odd number and $l<\alpha$, for $|\gamma|=l$ we have

$$\int_{|v|\leq|y|} D^\gamma\kappa_\alpha(v)dv = 0 .$$

Hence by Lemma 2.13 we have

$$\mu^{l,\alpha,\alpha}(y) = \frac{1}{|y|^n}\int_{|v|\leq|y|}(R^l_{y'}\kappa_\alpha)(v)dv = \frac{1}{|y|^n}\int_{|v|\leq|y|}(R^{l+1}_{y'}\kappa_\alpha)(v)dv\,.$$

Furthermore, because of $\alpha<l+1<\alpha+1$ by Corollary 2.9 (i) we obtain

$$\mu^{l,\alpha,\alpha}(y) = \frac{1}{|y|^n}\int_{|v|>|y|}(R^{l+1}_{y'}\kappa_\alpha)(v)\,dv\,.$$

Therefore by Corollary 2.2 (ii) we have

$$\begin{aligned}\int_{|y|>1/2}|\mu^{l,\alpha,\alpha}(y)|\,dy &\leqq C\int_{|y|>1/2}\frac{1}{|y|^n}\int_{|v|>|y|}|v|^{\alpha-l-1-n}\,dv\,dy\\ &= C\int_{|y|>1/2}|y|^{\alpha-l-1-n}dy<\infty\end{aligned}$$

since $\alpha<l+1$.

Finally let l be an odd number and $\alpha=l$. We easily see

$$\int_{|y|\leq 1/2}|\mu^{l,l,l}(y)|\,dy<\infty\,.$$

Since l is an odd number, for $|\gamma|=l$ we obtain

$$\int_{\varepsilon<|v|\leq|y|}D^\gamma\kappa_l(v)dv = 0\,,\qquad \varepsilon>0\,.$$

Hence by Lemma 2.13 and Corollary 2.9 (ii) we have

$$\begin{aligned}\mu^{l,l,l}(y) &= \frac{1}{|y|^n}\int_{|v|\leq|y|}(R^l_{y'}\kappa_l)(v)dv\\ &= \frac{1}{|y|^n}\lim_{\varepsilon\to 0}\int_{\varepsilon<|v|\leq|y|}(R^{l+1}_{y'}\kappa_l)(v)dv\\ &= \frac{1}{|y|^n}\int_{|v|>|y|}(R^{l+1}_{y'}\kappa_l)(v)dv\,.\end{aligned}$$

Therefore by Corollary 2.2 (ii) we obtain

$$\begin{aligned}\int_{|y|>1/2}|\mu^{l,l,l}(y)|\,dy &\leqq C\int_{|y|>1/2}\frac{1}{|y|^n}\int_{|v|>|y|}|v|^{-1-n}\,dv\,dy\\ &= C\int_{|y|>1/2}|y|^{-1-n}dy<\infty\,.\end{aligned}$$

The proof of the theorem is completed.

2.3. $\mu_*^{l,\alpha,\beta}$-potentials

For a locally integrable function f we set

$$V^f_{l,\alpha,\beta}(x)=\int \mu_*^{l,\alpha,\beta}(y)f(x-y)dy\,.$$

Theorem 2.15. *Let f be a nonnegative L^p-function, $\beta>\alpha-(n/p)$ and $\beta>l-1$. If $\alpha-(n/p)<l<\alpha+1$, then $V^f_{l,\alpha,\beta}(x)<\infty$ for almost every x. In particular, if $\alpha-(n/p)<l<\alpha-(n/p)+1$, then $V^f_{l,\alpha,\beta}(x)<\infty$ for all x.*

Proof. We have

$$\begin{aligned}V^f_{l,\alpha,\beta}(x)&=\int_{|t|\geqq 1}\frac{dt}{|t|^{n+\beta}}\int_{d(y,L_{-t})\geqq|t|/2}|(R^l_t\kappa_\alpha)(y)|f(x-y)dy\\&\quad+\int f(x-y)dy\int_{|t|\geqq 1,d(y,L_{-t})<|t|/2}\frac{|(R^l_t\kappa_\alpha)(y)|}{|t|^{n+\beta}}dt\\&=V_1(x)+V_2(x)\,.\end{aligned}$$

It follows from Corollary 2.2 (ii), Hölder's inequality, $\alpha-(n/p)<l$ and $\beta>\alpha-(n/p)$ that

$$\begin{aligned}V_1(x)&\leqq C\int_{|t|\geqq 1}\frac{dt}{|t|^{n+\beta}}\int_{|y|\geqq|t|/2}|t|^l|y|^{\alpha-l-n}f(x-y)dy\\&\leqq C\int_{|t|\geqq 1}\frac{|t|^l}{|t|^{n+\beta}}dt\Big(\int_{|y|\geqq|t|/2}|y|^{(\alpha-l-n)p'}dy\Big)^{1/p'}\|f\|_p\\&=C\|f\|_p\int_{|t|\geqq 1}\frac{|t|^{\alpha-(n/p)}}{|t|^{n+\beta}}dt<\infty\,.\end{aligned}$$

For $V_2(x)$ we see that

$$\begin{aligned}V_2(x)&\leqq\int f(x-y)dy\int_{|t|\geqq 1,d(y,L_{-t})<|t|/2}\frac{|\kappa_\alpha(y+t)|}{|t|^{n+\beta}}dt\\&\quad+C\sum_{|\gamma|\leqq l-1}\int|D^\gamma\kappa_\alpha(y)|f(x-y)dy\int_{|t|\geqq 1,d(y,L_{-t})<|t|/2}\frac{|t|^{|\gamma|}}{|t|^{n+\beta}}dt\\&=V_{21}(x)+V_{22}(x)\,.\end{aligned}$$

Let $\alpha-n$ be not a nonnegative even number. Since

$$\int_{|t|\geqq 1,d(y,L_{-t})<|t|/2}\frac{|y+t|^{\alpha-n}}{|t|^{n+\beta}}dt\leqq C(1+|y|)^{\alpha-\beta-n}\,,$$

we obtain

$$V_{21}(x)\leqq C\int(1+|y|)^{\alpha-\beta-n}f(x-y)dy<\infty$$

because of $\beta>\alpha-(n/p)$. By $\beta>l-1$ we have

$$\begin{aligned}V_{22}(x) &\leqq C \sum_{|\gamma|\leqq l-1} \int |y|^{\alpha-|\gamma|-n} f(x-y)dy \int_{|t|\geqq 1, d(y,L_{-t})<|t|/2} \frac{|t|^{|\gamma|}}{|t|^{n+\beta}} dt \\ &\leqq C \sum_{|\gamma|\leqq l-1} \int |y|^{\alpha-|\gamma|-n}(1+|y|)^{-\beta+|\gamma|} f(x-y)dy \\ &\leqq C \sum_{|\gamma|\leqq l-1} \int_{|y|<1} |y|^{\alpha-|\gamma|-n} f(x-y)dy + C \sum_{|\gamma|\leqq l-1} \int_{|y|\geqq 1} |y|^{\alpha-\beta-n} f(x-y)dy \\ &= V_{221}(x) + V_{222}(x)\,.\end{aligned}$$

It follows from $\beta>\alpha-(n/p)$ that $V_{222}(x)<\infty$ for all x. Moreover, by $l-1<\alpha$ and Young's inequality we obtain $V_{221}(x)<\infty$ for almost every x. In particular, if $l-1<\alpha-(n/p)$, then by Hölder's inequality we have $V_{221}(x)<\infty$ for all x. In case $\alpha-n$ is a nonnegative even number, the proof is similar. Therefore we obtain the theorem.

3. Hypersingular integrals of Beppo Levi functions

For an integer $k<\alpha$, we set

$$\kappa_{\alpha,k}(x, y) = \begin{cases} (R_x^{k+1}\kappa_\alpha)(-y)\,, & 0\leqq k<\alpha\,, \\ \kappa_\alpha(x-y)\,, & k\leqq -1\,. \end{cases}$$

Further, for a locally integrable function f we set

$$U_{\alpha,k}^f(x) = \int \kappa_{\alpha,k}(x, y)f(y)dy\,.$$

We denote by $[r]$ the integral part of a real number r and by $f|_E$ the restriction of a function f to a set E.

Lemma 3.1. ([1]) *Let $k=[\alpha-(n/p)]$ and $f\in L^p$. If $\alpha-(n/p)$ is not a nonnegative integer, then $U_{\alpha,k}^f$ exists, and if $\alpha-(n/p)$ is a nonnegative integer, then $U_{\alpha,k-1}^{f_1}$ and $U_{\alpha,k}^{f_2}$ exist where $f_1=f|_{B_1}$, $f_2=f-f_1$ and $B_1=\{|x|<1\}$.*

Lemma 3.2. *Let $k=[\alpha-(n/p)]$ and $f\in L^p$.*
(i) *If $\alpha-(n/p)$ is not a nonnegative integer, then for $\alpha-(n/p)<l<\alpha+1$ we have*

$$(R_t^l U_{\alpha,k}^f)(x) = \int (R_t^l \kappa_\alpha)(y)f(x-y)dy\,.$$

(ii) *If $\alpha-(n/p)$ is a nonnegative integer, then for $\alpha-(n/p)<l<\alpha+1$ we have*

$$(R_t^l(U_{\alpha,k-1}^{f_1}+U_{\alpha,k}^{f_2}))(x) = \int (R_t^l\kappa_\alpha)(y)f(x-y)dy$$

where f_1 and f_2 are as in Lemma 3.1.

Proof. By $l-1<\alpha$, $l>\alpha-(n/p)$ and Lemma 2.4 we have

$$\begin{aligned}
(R_t^l U_{\alpha,k}^f)(x) &= U_{\alpha,k}^f(x+t) - \sum_{|\gamma|\leq l-1} (D^\gamma U_{\alpha,k}^f(x)/\gamma!)t^\gamma \\
&= \int (\kappa_\alpha(x+t-y) - \sum_{|\delta|\leq k} ((x+t)^\delta/\delta!) D^\delta \kappa_\alpha(-y)) f(y) dy \\
&\quad - \sum_{|\gamma|\leq l-1} (t^\gamma/\gamma!) D_x^\gamma \int (\kappa_\alpha(x-y) - \sum_{|\delta|\leq k} (x^\delta/\delta!) D^\delta \kappa_\alpha(-y)) f(y) dy \\
&= \int (\kappa_\alpha(x+t-y) - \sum_{|\gamma|\leq l-1} (t^\gamma/\gamma!) D_x^\gamma \kappa_\alpha(x-y)) f(y)\, dy \\
&\quad - \sum_{|\delta|\leq k} \int (((x+t)^\delta/\delta!) - \sum_{|\gamma|\leq l-1} (t^\gamma/\gamma!) D^\gamma(x^\delta/\delta!)) D^\delta \kappa_\alpha(-y) f(y)\, dy \\
&= \int (R_t^l \kappa_\alpha)(x-y) f(y) dy - \sum_{|\delta|\leq k} \int (R_t^l(x^\delta/\delta!)) D^\gamma \kappa_\alpha(-y) f(y)\, dy \\
&= \int (R_t^l \kappa_\alpha)(y) f(x-y) dy\,.
\end{aligned}$$

Since the proof of (ii) is similar, we obtain the lemma.

REMARK 3.3. Let $f \in L^p$. Then for $\alpha-(n/p)<l<\alpha+1$ we have

$$\int |(R_t^l \kappa_\alpha)(y) f(x-y)|\, dy < \infty$$

for almost evrey x. In particular, if $\alpha-(n/p)<l<\alpha-(n/p)+1$, then

$$\int |(R_t^l \kappa_\alpha)(y) f(x-y)|\, dy < \infty$$

for all x.

For a positive integer l and $\beta>0$, we set

$$H_\varepsilon^{l,\beta} u(x) = \int_{|t|\geq \varepsilon} \frac{(R_t^l u)(x)}{|t|^{n+\beta}} dt\,, \qquad \varepsilon>0\,.$$

Recall that in Remark 2.11 we defined $\mu^{l,\alpha,\beta}$ as

$$\mu_\varepsilon^{l,\alpha,\beta}(y) = \int_{|t|\geq\varepsilon} \frac{(R_t^l \kappa_\alpha)(y)}{|t|^{n+\beta}} dt\,.$$

Proposition 3.4. *Let* $\alpha-(n/p)<l<\alpha+1$, $\beta>\alpha-(n/p)$, $\beta>l-1$, $k=[\alpha-(n/p)]$ *and* $f \in L^p$.

(i) *If* $\alpha-(n/p)$ *is not a nonnegative integer, then*

$$H_\varepsilon^{l,\beta} U_{\alpha,k}^f(x) = \int \mu_\varepsilon^{l,\alpha,\beta}(y) f(x-y) dy\,.$$

(ii) *If* $\alpha-(n/p)$ *is a nonnegative integer, then*

$$H_{\alpha}^{l,\beta}(U_{\alpha,k-1}^{f_1}+U_{\alpha,k}^{f_2})(x)=\int \mu_{\varepsilon}^{l,\alpha,\beta}(y)f(x-y)dy\,.$$

Proof. (i) By Lemma 3.2, Theorem 2.15 and Fibini's theorem we have

$$\begin{aligned}H_{\varepsilon}^{l,\beta}U_{\alpha,k}^{f}(x)&=\int_{|t|\geq\varepsilon}\frac{(R_t^l U_{\alpha,k}^f)(x)}{|t|^{n+\beta}}dt\\&=\int_{|t|\geq\varepsilon}\frac{dt}{|t|^{n+\beta}}\int (R_t^l\kappa_\alpha)(y)f(x-y)\,dy\\&=\int f(x-y)dy\int_{|t|\geq\varepsilon}\frac{(R_t^l\kappa_\alpha)(y)}{|t|^{n+\beta}}dt\\&=\int \mu_{\varepsilon}^{l,\alpha,\beta}(y)f(x-y)dy\,.\end{aligned}$$

The proof of (ii) is similar. Hence the proposition is proved.

Lemma 3.5. *Let $\beta>\alpha-n$, $\beta>l-1$, and moreover we assume that $l>\alpha-n$ in case $\alpha-n$ is a nonnegative even number. Then*

$$\frac{1}{\varepsilon^{n+\beta-\alpha}}\mu^{l,\alpha,\beta}\Big(\frac{y}{\varepsilon}\Big)=\mu_{\varepsilon}^{l,\alpha,\beta}(y)\,.$$

Proof. This lemma follows from Remark 2.11.

Let R_j $(j=1,\cdots,n)$ be the Riesz transforms, namely

$$R_jf(x)=\lim_{\varepsilon\to 0}c_n\int_{|x-y|\geq\varepsilon}\frac{x_j-y_j}{|x-y|^{n+1}}f(y)dy$$

with $c_n=\Gamma((n+1)/2)\pi^{-(n+1)/2}$ for $f\in L^p$. For a multi-index $\gamma=(\gamma_1,\cdots,\gamma_n)$ we set

$$R^\gamma=R_1^{\gamma_1}\cdots R_n^{\gamma_n}\,.$$

For a positive integer m and $p>1$, the Beppo Levi space $\mathcal{L}_m^p$ is defined as follows:

$$\mathcal{L}_m^p=\{u\in\mathcal{D}';\ |u|_{m,p}=\sum_{|\gamma|=m}||D^\gamma u||_p<\infty\}.$$

Lemma 3.6. ([1]) *Let $k=[m-(n/p)]$ and $u\in\mathcal{L}_m^p$.*

(i) *If $m-(n/p)$ is not a nonnegative integer, then u can be represented as*

$$u(x)=\sum_{|\delta|\leq m-1}a_\delta x^\delta+U_{m,k}^f(x)$$

where $f=c_{m,n}\sum_{|\gamma|=m}(m!/\gamma!)R^\gamma D^\gamma u$ and $c_{m,n}$ is the constant.

(ii) *If $m-(n/p)$ is a nonnegative integer, then u can be represented as*

$$u(x)=\sum_{|\delta|\leq m-1}a_\delta x^\delta+U_{m,k-1}^{f_1}(x)+U_{m,k}^{f_2}(x)$$

where $f_1 = f|_{B_1}$, $f_2 = f - f_1$ *and* f *is as in* (i).

Theorem 3.7. *Let* $u \in \mathcal{L}_m^p$.

(i) *If* $n < q < \infty$, $p \leqq q$, $\beta > m - (n/q)$ *and* $(1/r) = (1/p) - (1/q)$, *then*

$$||H_\varepsilon^{m,\beta} u||_r \leqq C\varepsilon^{m-\beta-(n/q)} |u|_{m,p}\,.$$

(ii) *If* $\beta \geqq m$, *then*

$$||H_\varepsilon^{m,\beta} u||_p \leqq C\varepsilon^{m-\beta} |u|_{m,p}\,.$$

Proof. Let $m-(n/p)$ be not a nonnegative integer. By Lemma 3.6 (i) we have

$$u(x) = \sum_{|\delta| \leqq m-1} a_\delta x^\gamma + U_{m,k}^f(x)\,.$$

By Lemma 2.4 we see that $H_\varepsilon^{m,\beta} u = H_\varepsilon^{m,\beta} U_{m,k}^f$. Therefore by Proposition 3.4 we obtain

$$H_\varepsilon^{m,\beta} u(x) = \int \mu_\varepsilon^{m,m,\beta}(y) f(x-y) dy\,.$$

Hence by Young's inequality we have

$$||H_\varepsilon^{m,\beta} u||_r \leqq ||\mu_\varepsilon^{m,m,\beta}||_{q'} ||f||_p\,.$$

Consequently, (i) and (ii) follows from Theorems 2.3, 2.14, Lemma 3.5 and $||f||_p \leqq C|u|_{m,p}$. In case $m-(n/p)$ is a nonnegative integer, the proof is similar. Thus we obtain the theorem.

References

[1] T. Kurokawa: *Riesz potentials, higher Riesz transforms and Beppo Levi spaces,* Hiroshima Math. J. **18** (1988), 541–597

[2] T. Kurokawa: *On singular difference integrals,* Preprint.

[3] E.M. Stein: Singular integrals and differentiability properties of functions, Princeton Univ. Press. Princeton, 1970.

[4] R.L. Wheeden: *On hypersingular integrals and Lebesgue spaces of differentiable functions II,* Trans. Amer. Math. Soc. **139** (1969), 37–53.

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