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ON Ad*-COHOMOLOGY GROUPS OF THE CLASSIFYING SPACES OF COMPACT LIE GROUPS

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Dedicated to Professor Nobuo Shimada on his sixtieth birthday

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1. Introduction

Let k be a positive integer and R a proper subring of Q which contains k^{-1}. We consider the complex K^*-cohomology theory, the Bott element β and the Adams operation ψ^k. Since $\psi^k(\beta)=k\cdot\beta$ it induces the stable Adams operation ψ^k_{2m} in $K^{2m}(X;R)$, the R-coefficient K^*-cohomology theory. We consider the secondary cohomology theory [9] associated to the stable operation $1-\psi^k_*$ and denote the theory by $Ad^*(X;R)$. That is, Ad^* is a cohomology theory satisfying the following exact sequence

$$\cdots\to Ad^*(X;R)\to K^*(X;R)\xrightarrow{1-\psi^k} K^*(X;R)\to Ad^{*+1}(X;R)\to\cdots.$$

When k is a prime power, the associated connective theory of $Ad^*(X;R)$ coincides with the algebraic K-cohomology theory defined by Quillen [6] and Seymour [8]. We consider a compact lie group G and its classifying space BG. As is well known, $K^{2m}(BG;R)$ is a torsion free group and $K^{2m+1}(BG;R)=0$. So it is easy to see that $Ad^{2m}(BG;R)$ is a torsion free group. We denote the connected component of the unity by G_0, which is a closed normal subgroup of G and we consider the group of connected components G/G_0. Main result is the following.

Theorem. (i) *For any integer* $m\neq 0$, $\widetilde{Ad}^{2m}(BG;R)=0$.
(ii) *When* $|G/G_0|$ *is invertible in* R, *then* $\widetilde{Ad}^0(BG;R)=0$.

In section 2, we prove the Theorem for finite groups by considering their cyclic subgroups. In section 3, we prove the Theorem for compact connected Lie groups by considering their maximal tori. In section 4, we prove the Theorem for general compact Lie groups by considering their Cartan subgroups.

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2. Cyclic groups and finite groups

Let $R(G)$ be the complex representation ring of the compact Lie group G. The augmentation ideal $I=I(G)$ is the kernel of the induced ring homomorphism $R(G)\to R(\{1\})$. Let p be a prime number. To compute $Ad^*(BG; R)$, we use

Proposition 2.1 (Atiyah). *There holds the natural isomorphism*

$$\hat{\alpha}\colon R(\hat{G})_1 \to K(BG)\,.$$

And when G is a p-group, $I(G)$-adic completion coincides with the p-adic completion.

Let r be a positive integer. We consider the case $G=Z/p^r$, the cyclic group of order p^r. Since $R(Z/p^r)=Z[\eta]/(\eta^{p^r}-1)$, $\tilde{K}(BZ/p^r)$ is a free $\hat{Z}_p$-module with basis $\{\eta^j-1;\ 1\leqq j\leqq p^r-1\}$, where η is the canonical line bundle over BZ/p^r. In cases p is not invertible in R,

$$\tilde{K}(BZ/p^r;\, R) = \hat{Z}_p\langle\eta^j-1;\ 1\leqq j\leqq p^r-1\rangle\,,$$

and in cases p is invertible in R,

$$\tilde{K}(BZ/p^r;\, R) = 0\,.$$

So we have

Proposition 2.2. *Let p be a prime number which is invertible in R. Then, $\widetilde{Ad}^{2m}(BZ/p^r;\, R)=0$.*

As is well known, the k-th Adams operation ψ^k acts on line bundles as k-times tensorial product. So when k is prime to p, ψ^k acts on the basis $\{\eta^j-1;\ 1\leqq j\leqq p^r-1\}$ as a permutation. So we have

Proposition 2.3. *Let p be a prime number which is not invertible in R. In this case, p is prime to k. Then*

(i) *$\widetilde{Ad}^0(BZ/p^r;\, R)$ is a free $\hat{Z}_p$-module of rank r.*

(ii) *For any integer $m\neq 0$, $\widetilde{Ad}^{2m}(BZ/p^r;\, R)=0$.*

For any non-negative integer l, there exist prime numbers $p_1, \cdots, p_t$ which are not invertible in R and an integer l_1 which is invertible in R satisfying $l=p_1^{r_1}\cdot p_2^{r_2}\cdots p_t^{r_t}\cdot l_1$. We consider a cyclic group $C=Z/l$. Since $Z/l=Z/p_1^{r_1}\times Z/p_2^{r_2}\times\cdots\times Z/p_t^{r_t}\times Z/l_1$, we use the results of Mahammed [5] and have

Lemma 2.4. $\tilde{K}(BZ/l)=\bigoplus_{i=1}^{t}\tilde{K}(BZ/p_i^{r_i})\oplus\tilde{K}(BZ/l_1)$

Since $K((BZ/l)^{(n)})$ is a finite group, the $\lim^1$-term is vanished. Thus

$K(BZ/l)=\lim_{n} K((BZ/l)^{(n)})$ and $K(BZ/l;\, R)=\lim_{n} K((BZ/l)^{(n)};\, R)$. So we have

Corollary 2.5. $\tilde{K}(BZ/l;\, R)=\bigoplus_{i=1}^{t}\tilde{K}(BZ/p_i^{r_i};\, R)$

Proposition 2.2, 2.3 and Corollary 2.5 induces

Corollary 2.6. (i) *For any finite cyclic group* C, $\widetilde{Ad}^{2m}(BC;\, R)=0$ *for any integer* $m\neq 0$.

(ii) *When* $|C|$ *is invertible in* R, *then* $\widetilde{Ad}^{0}(BC;\, R)=0$.

Thus we prove the Theorem for finite groups as follows.

Proposition 2.7. (i) *For any finite group* G, $\widetilde{Ad}^{2m}(BG;\, R)=0$ *for any integer* $m\neq 0$.

(ii) *When* $|G|$ *is invertible in* R, *then* $\widetilde{Ad}^{0}(BG;\, R)=0$.

Proof. Let $\{C_\lambda;\, \lambda\in\Lambda\}$ be the representative of the conjugacy class of cyclic subgroups of G. We consider the following commutative diagram where horizontal lines are exact:

$$\begin{array}{ccccc} 0 \longrightarrow & \widetilde{Ad}^{2m}(BG;\, R) & \longrightarrow & \tilde{K}^{2m}(BG;\, R) \\ & \downarrow & & \downarrow \\ 0 \rightarrow & \bigoplus_{\lambda}\widetilde{Ad}^{2m}(BC_\lambda;\, R) & \rightarrow & \bigoplus_{\lambda}\tilde{K}^{2m}(BC_\lambda;R)\,. \end{array}$$

Since the map

$$\tilde{K}^{2m}(BG;\, R)\rightarrow \bigoplus_{\lambda}\tilde{K}^{2m}(BC_\lambda;\, R)$$

is an injection, [2], the left vertical homomorphism is also an injection. This completes the proof.

3. Tori and compact connected Lie groups

Let S^{2n+1} be the unit sphere in C^{n+1}, the $(n+1)$-dimensional complex vector space. Then S^1 is an abelian group. The complex vector space structure on C^{n+1} induces an S^1-action on S^{2n+1} by

$$z\cdot(z_0,\, \cdots,\, z_n)=(z\cdot z_0,\, \cdots,\, z\cdot z_n)\,.$$

We consider the n-dimensional complex projective space $CP^n=S^{2n+1}/S^1$. The infinite complex projective space $CP^\infty=\operatorname{colim}_{n} CP^n$ is the classifying space of S^1. Let T^s be the s-dimensional torus. Then its classifying space BT^s is the s-times cartesian product of CP^∞. As is well known,

$$K(BT^s;\, Q)=Q[[X_1,\, \cdots,\, X_s]]\,,$$

where ξ_i is the canonical line bundle over CP^∞ and $X_i=\xi_i-1$ is the euler class of ξ_i. Let $J=(j_1, \cdots, j_s)$ be a sequence of non-negative integers. We denote $|J|=\sum_{i=1}^{s} j_i$ and $N(s, m)=\#\{J;\ |J|=m\}$.

Lemma 3.1. (i) *For any non-negative integer m, the groups $Ad^{2m}((CP^n)^s; Q)$ and $Ad^{2m+1}((CP^n)^s; Q)$ are $N(s, m)$-times direct sum of Q for any $n\geqq m$. Moreover, the inclusion map $i: (CP^n)^s\to(CP^{n+1})^s$ induces the isomorphism of those groups.*

(ii) *For any negative integer m, the group $Ad^{2m}((CP^n)^s; Q)=0$.*

Proof. $K^*((CP^n)^s; Q)=Q[\beta, \beta^{-1}, X_1, \cdots, X_s]/((X_i)^{n+2};\ 1\leqq i\leqq s)$. We consider the basis $\{\beta^{-m}\cdot X^J;\ 1\leqq j_i\leqq n\}$ of the module $K^{2m}((CP^n)^s; Q)$, where X^J is the multi index. We put the lexicographical order on the basis and represent the operation $1-\psi_{2m}^k$ by a matrix. This matrix is a triangular matrix with diagonal entries $1-k^{|J|-m}$. Clearly, the diagonal entry equal zero if and only if $|J|=m$.

Corollary 3.2. (i) *For any non-negative integer m, the group $Ad^{2m}(BT^s; Q)$ is an $N(s, m)$-dimensional Q-module generated by $\{\beta^{-m}\cdot Y^J;\ |J|=m\}$, where $Y_i=\log(1+X_i)$.*

(ii) For any negative integer m, the group $Ad^{2m}(BT^s; Q)=0$.

Proof. The space BT^s is a colimit of the direct system $\{(CP^n)^s\}$. Since the $\lim^1$-term is vanished, $Ad^{2m}(BT^s; Q)$ is an $N(s, m)$-dimensional Q-module. Easy computation shows that $\psi^k(Y_i)=k\cdot Y_i$. So when $|J|=m$, $(1-\psi_{2m}^k)(\beta^{-m}\cdot Y^J)=0$. It is easy to prove that $\{\beta^{-m}\cdot Y^J;\ |J|=m\}$ are linearly independent. This completes the proof.

Proposition 3.3. *For any integer m, the group $\widetilde{Ad}^{2m}(BT^s; R)=0$.*

Proof. We consider the following commutative diagram where horizontal lines are exact:

$$\begin{array}{ccccc} 0 \to & \widetilde{Ad}^{2m}(BT^s; R) & \to & \tilde{K}^{2m}(BT^s; R) \\ & \downarrow & & \downarrow \\ 0 \to & \widetilde{Ad}^{2m}(BT^s; Q) & \to & \tilde{K}^{2m}(BT^s; Q) \end{array}$$

Since

$$K(BT^s; R) = R[[X_1, \cdots, X_s]]$$

is a subgroup of

$$K(BT^s; Q) = Q[[X_1, \cdots, X_s]],$$

the left vertical homomorphism is an injection. Thus it is enough to show

Lemma 3.4. *Let m be a positive integer. If an element*

$$\sum_{|J|=m} a_J\cdot Y^J\in Q[[X_1, \cdots, X_s]]$$

is in

$$R[[X_1, \cdots, X_s]],$$

then $a_J=0$ *for all* J.

Proof. Since there exists at least one prime p satisfying p is not invertible in R, it is enough to show the case $R=Z_{(p)}$. Using the fact that the denominator of the coefficients of the power series $\log(1+X)$ contains sufficient large p-power, the proof is only an induction.

Clearly, Proposition 3.3 implies

Corollary 3.5. *If an element*

$$\alpha \in K^{2m}(BT^s; R) = R[[X_1, \cdots, X_s]],$$

satisfies $k^{-m}\psi^k\alpha=\alpha$ *for some* m, *then* $\alpha=0$. *Moreover this is also true when* R *is a ring of* p*-adic integers for some prime* p.

The following proposition contains a special case of the Theorem.

Proposition 3.6. *For any compact connected Lie group* G, *the group* $Ad^{2m}(BG; R)=0$ *for any integer* m.

Proof. Let T^s be the maximal torus of G. We consider the following commutative diagram where horizontal lines are exact:

$$\begin{array}{ccccc} 0 \to & \widetilde{Ad}^{2m}(BG; R) & \to & \tilde{K}^{2m}(BG; R) \\ & \downarrow & & \downarrow \\ 0 \to & \widetilde{Ad}^{2m}(BT^s; R) & \to & \tilde{K}^{2m}(BT^s; R). \end{array}$$

Since the group $\tilde{K}^{2m}(BG; R)$ is a subgroup of $\tilde{K}^{2m}(BT^s; R)$ [2], the right vertical homomorphism is an injection. Thus the left vertical homomorphism is also an injection. This completes the proof.

4. General cases

First we consider the topological cyclic group T. It is a cartesian product of a torus T^σ and a finite cyclic group Z/l and its classifying space is $BT=(CP^\infty)^\sigma \times BZ/l$. As same as in section 2, we denote

$$l = p_1^{r_1}\cdot p_2^{r_2} \cdots p_t^{r_t}\cdot l_1.$$

Buhštaber and Miščenko [3] shows

Lemma 4.1.
$$\begin{aligned} K(BT) &= K(BT^\sigma)\hat{\otimes} K(BZ/l) \\ &= Z \oplus \tilde{K}(BT^\sigma) \oplus \tilde{K}(BZ/l) \oplus \tilde{K}(BT^\sigma)\hat{\otimes}\tilde{K}(BZ/l) \end{aligned}$$

Considering the R-coefficient theory, we have

Corollary 4.2. $K(BT;R)=R\oplus\tilde{K}(BT^{\sigma};R)\oplus\tilde{K}(BZ/l;R)\oplus\tilde{K}(BT^{\sigma})\hat{\otimes}\tilde{K}(BZ/l;R)$.

When l is invertible in R the group $\tilde{K}(BZ/l;R)=0$. So we have

Corollary 4.3. *When* $|T/T_0|$ *is invertible in* R, *then* $\widetilde{Ad}^{2m}(BT;R)=0$ *for any integer* m.

This is the second part of the Theorem.

Lemma 4.4. *For any integer* $m\neq 0$, *the group* $\widetilde{Ad}^{2m}(BT;R)=0$.

Proof. We must prove the map $1-\psi^k_{2m}$ is injective. We only need to prove that

$$1-k^{-m}\cdot\psi^k\colon \tilde{K}(BT^{\sigma})\hat{\otimes}\tilde{K}(BZ/l;R)\to\tilde{K}(BT^{\sigma})\hat{\otimes}\tilde{K}(BZ/l\colon R)$$

is injective.

In section 2, we proved

$$\tilde{K}(BZ/l;R)=\bigoplus_{i=1}^{t}\hat{Z}_{p_i}\langle g_{i,\mu};\ 1\leqq\mu\leqq p_i^{r_i}-1\rangle$$

where η_i is the image of the canonical line bundle of the map $K(BZ/p_i^{r_i})\to K(BZ/l)$ and $g_{i,\mu}=(\eta_i)^{\mu}-1$. So

$$K(BT^{\sigma})\hat{\otimes}K(BZ/l;R)=K(BT^{\sigma})\hat{\otimes}\hat{Z}_{p_i}\langle g_{i,\mu};\ 1\leqq\mu\leqq p_i^{r_i}-1\rangle\,.$$

Thus any element of this group can be written as

$$\alpha=\sum_{\substack{1\leqq t\leqq i\\ 1\leqq\mu\leqq M}}a_{i,\mu}\cdot g_{i,\mu}$$

where $a_{i,\mu}$ are reduced elements of the ring $\hat{Z}_{p_i}[[X_1,\cdots,X_s]]$ and $M=p_i^{r_i}-1$.

We assume that $(1-k^{-m}\cdot\psi^k)\alpha=0$. In section 2 we showed that the Adams operation ψ^k act on g's as permutation. By assumption, there exists an integer e such that $(\psi^k)^e g_{i,\mu}=g_{i,\mu}$. So $a_{i,\mu}$ must satisfies $k^{-em}\psi^{k^e}a_{i,\mu}=a_{i,\mu}$. So using Corollary 3.5, we have $a_{i,\mu}=0$.

Let $\{T_\lambda;\ \lambda\in\Lambda\}$ be the set of all Cartan subgroups of a compact Lie group G. Remember that $\tilde{K}^{2m}(BG)\to\oplus\tilde{K}^{2m}(BT_\lambda)$ is a monomorphism [7]. We consider the following commutative diagram with horizontal injective homomorphism:

$$\begin{array}{ccccc}
0 \longrightarrow & Ad^{2m}(BG;R) & \longrightarrow & K^{2m}(BG;R) \\
& \downarrow & & \downarrow \\
0 \to & \bigoplus_{\lambda} Ad^{2m}(BT_\lambda;R) & \to & \bigoplus_{\lambda} K^{2m}(BT_\lambda;R)\,.
\end{array}$$

Since the right vertical homomorphism is injective, the left vertical homomorphism is also injective. Together with the Lemma 4.4, we have

Proposition 4.5. *For any integer* $m \neq 0$, $\widetilde{Ad}^{2m}(BG; R)=0$.

This is the first part of the Theorem.

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