1. Introduction

Let \( k \) be a positive integer and \( R \) a proper subring of \( Q \) which contains \( k^{-1} \). We consider the complex \( K^* \)-cohomology theory, the Bott element \( \beta \) and the Adams operation \( \psi^k \). Since \( \psi^k(\beta) = k \cdot \beta \) it induces the stable Adams operation \( \psi^k_{2m} \) in \( K^{2m}(X; R) \), the \( R \)-coefficient \( K^* \)-cohomology theory. We consider the secondary cohomology theory \([9]\) associated to the stable operation \( 1 - \psi^k \) and denote the theory by \( Ad^*(X; R) \). That is, \( Ad^* \) is a cohomology theory satisfying the following exact sequence

\[
\cdots \to Ad^*(X; R) \to K^*(X; R) \xrightarrow{1-\psi^k} K^*(X; R) \to Ad^{*+1}(X; R) \to \cdots.
\]

When \( k \) is a prime power, the associated connective theory of \( Ad^*(X; R) \) coincides with the algebraic \( K \)-cohomology theory defined by Quillen \([6]\) and Seymour \([8]\). We consider a compact Lie group \( G \) and its classifying space \( BG \). As is well known, \( K^{2m}(BG; R) \) is a torsion free group and \( K^{2m+1}(BG; R) = 0 \). So it is easy to see that \( Ad^{2m}(BG; R) \) is a torsion free group. We denote the connected component of the unity by \( G_0 \), which is a closed normal subgroup of \( G \) and we consider the group of connected components \( G/G_0 \). Main result is the following.

**Theorem.** (i) For any integer \( m \neq 0 \), \( \tilde{Ad}^{2m}(BG; R) = 0 \).

(ii) When \( |G/G_0| \) is invertible in \( R \), then \( \tilde{Ad}^0(BG; R) = 0 \).

In section 2, we prove the Theorem for finite groups by considering their cyclic subgroups. In section 3, we prove the Theorem for compact connected Lie groups by considering their maximal tori. In section 4, we prove the Theorem for general compact Lie groups by considering their Cartan subgroups.

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2. Cyclic groups and finite groups

Let $R(G)$ be the complex representation ring of the compact Lie group $G$. The augmentation ideal $I = I(G)$ is the kernel of the induced ring homomorphism $R(G) \rightarrow R(\{1\})$. Let $p$ be a prime number. To compute $Ad^*(BG; R)$, we use

**Proposition 2.1** (Atiyah). *There holds the natural isomorphism*

$$\tilde{\alpha}: R(\hat{G}) \rightarrow K(BG).$$

*And when $G$ is a $p$-group, $I(G)$-adic completion coincides with the $p$-adic completion.*

Let $r$ be a positive integer. We consider the case $G = \mathbb{Z}/p^r$, the cyclic group of order $p^r$. Since $R(\mathbb{Z}/p^r) = \mathbb{Z}[\eta]/(\eta^{p^r} - 1)$, $K(B\mathbb{Z}/p^r)$ is a free $\mathbb{Z}_p$-module with basis $\{\eta_j - 1; 1 \leq j \leq p^r - 1\}$, where $\eta$ is the canonical line bundle over $B\mathbb{Z}/p^r$. In cases $p$ is not invertible in $R$,

$$K(B\mathbb{Z}/p^r; R) = \mathbb{Z}_p[\eta_j - 1; 1 \leq j \leq p^r - 1],$$

and in cases $p$ is invertible in $R$,

$$K(B\mathbb{Z}/p^r; R) = 0.$$

So we have

**Proposition 2.2.** *Let $p$ be a prime number which is invertible in $R$. Then, $\tilde{Ad}^m(B\mathbb{Z}/p^r; R) = 0$.*

As is well known, the $k$-th Adams operation $\psi^k$ acts on line bundles as $k$-times tensorial product. So when $k$ is prime to $p$, $\psi^k$ acts on the basis $\{\eta_j - 1; 1 \leq j \leq p^r - 1\}$ as a permutation. So we have

**Proposition 2.3.** *Let $p$ be a prime number which is not invertible in $R$. In this case, $p$ is prime to $k$. Then*

(i) $\tilde{Ad}^m(B\mathbb{Z}/p^r; R)$ is a free $\mathbb{Z}_p$-module of rank $r$.

(ii) For any integer $m \neq 0$, $\tilde{Ad}^m(B\mathbb{Z}/p^r; R) = 0$.

For any non-negative integer $l$, there exist prime numbers $p_1, \ldots, p_i$, which are not invertible in $R$ and an integer $l_i$ which is invertible in $R$ satisfying $l = p_1^{l_1} \cdot p_2^{l_2} \cdots p_i^{l_i} \cdot l_i$. We consider a cyclic group $C = \mathbb{Z}/l$. Since $\mathbb{Z}/l = \mathbb{Z}/p_1^{l_1} \times \mathbb{Z}/p_2^{l_2} \times \cdots \times \mathbb{Z}/p_i^{l_i} \times \mathbb{Z}/l_i$, we use the results of Mahammed [5] and have

**Lemma 2.4.** $K(B\mathbb{Z}/l) = \bigoplus_{i=1}^{t} K(B\mathbb{Z}/p_i^{l_i}) \oplus K(B\mathbb{Z}/l_i)$

Since $K((B\mathbb{Z}/l)^{(n)})$ is a finite group, the lim$^1$-term is vanished. Thus
$K(BZ/l) = \lim \limits_{n \to \infty} K((BZ/l)^{(n)})$ and $K(BZ/l; R) = \lim \limits_{n \to \infty} K((BZ/l)^{(n)}; R)$. So we have

**Corollary 2.5.** $K(BZ/l; R) = \bigoplus_{i=1}^{t} K(BZ/p_i; R)$

Proposition 2.2, 2.3 and Corollary 2.5 induces

**Corollary 2.6.** (i) For any finite cyclic group $C$, $\widetilde{Ad}^{2m}(BC; R) = 0$ for any integer $m \neq 0$.

(ii) When $|C|$ is invertible in $R$, then $\widetilde{Ad}^{0}(BC; R) = 0$.

Thus we prove the Theorem for finite groups as follows.

**Proposition 2.7.** (i) For any finite group $G$, $\widetilde{Ad}^{2m}(BG; R) = 0$ for any integer $m \neq 0$.

(ii) When $|G|$ is invertible in $R$, then $\widetilde{Ad}^{0}(BG; R) = 0$.

Proof. Let $\{C_{\lambda}; \lambda \in \Lambda\}$ be the representative of the conjugacy class of cyclic subgroups of $G$. We consider the following commutative diagram where horizontal lines are exact:

$$
0 \longrightarrow \widetilde{Ad}^{2m}(BG; R) \longrightarrow \widetilde{K}^{2m}(BG; R) \quad \downarrow \quad \downarrow
0 \longrightarrow \bigoplus_{\lambda} \widetilde{Ad}^{2m}(BC_{\lambda}; R) \longrightarrow \bigoplus_{\lambda} \widetilde{K}^{2m}(BC_{\lambda}; R).
$$

Since the map

$$
\widetilde{K}^{2m}(BG; R) \longrightarrow \bigoplus_{\lambda} \widetilde{K}^{2m}(BC_{\lambda}; R)
$$

is an injection, [2], the left vertical homomorphism is also an injection. This completes the proof.

3. Tori and compact connected Lie groups

Let $S^{2n+1}$ be the unit sphere in $C^{n+1}$, the $(n+1)$-dimensional complex vector space. Then $S^1$ is an abelian group. The complex vector space structure on $C^{n+1}$ induces an $S^1$-action on $S^{2n+1}$ by

$$
z \cdot (z_0, \ldots, z_n) = (z \cdot z_0, \ldots, z \cdot z_n).
$$

We consider the $n$-dimensional complex projective space $CP^n = S^{2n+1}/S^1$. The infinite complex projective space $CP^\infty = \text{colim} CP^n$ is the classifying space of $S^1$.

Let $T^s$ be the $s$-dimensional torus. Then its classifying space $BT^s$ is the $s$-times cartesian product of $CP^\infty$. As is well known,

$$
K(BT^s; Q) = Q[[X_1, \ldots, X_s]],
$$

where $\xi_i$ is the canonical line bundle over $CP^n$ and $X_i = \xi_i - 1$ is the euler class.
of $\xi_i$. Let $J=(j_1, \cdots, j_s)$ be a sequence of non-negative integers. We denote $|J|=\sum_{i=1}^{s} j_i$ and $N(s, m)=\#\{J; |J|=m\}$.

**Lemma 3.1.** (i) For any non-negative integer $m$, the groups $Ad^{2m}((CP^n)^s; Q)$ and $Ad^{2m+1}((CP^n)^s; Q)$ are $N(s, m)$-times direct sum of $Q$ for any $n \geq m$. Moreover, the inclusion map $i: (CP^n)^s \rightarrow (CP^{n+1})^s$ induces the isomorphism of those groups.

(ii) For any negative integer $m$, the group $Ad^{2m}((CP^n)^s; Q)=0$.

Proof. $K^*((CP^n)^s; Q)=Q[\beta, \beta^{-1}, X_1, \cdots, X_n]/((X_i)^{s+2}; 1 \leq i \leq s)$. We consider the basis $\{\beta^{-n} \cdot X^j; 1 \leq j \leq n\}$ of the module $K^{2m}((CP^n)^s; Q)$, where $X^j$ is the multi index. We put the lexicographical order on the basis and represent the operation $1-\psi_{2m}$ by a matrix. This matrix is a triangular matrix with diagonal entries $1-k^{j^1-n}$. Clearly, the diagonal entry equal zero if and only if $|J|=m$.

**Corollary 3.2.** (i) For any non-negative integer $m$, the group $Ad^{2m}(BT^s; Q)$ is an $N(s, m)$-dimensional $Q$-module generated by $\{\beta^{-n} \cdot Y^j; |J|=m\}$, where $Y_j=\log(1+X_i)$.

(ii) For any negative integer $m$, the group $Ad^{2m}(BT^s; Q)=0$.

Proof. The space $BT^s$ is a colimit of the direct system $\{(CP^n)^s\}$. Since the limit-term is vanished, $Ad^{2m}(BT^s; Q)$ is an $N(s, m)$-dimensional $Q$-module. Easy computation shows that $\psi(Y_i)=k \cdot Y_i$. So when $|J|=m$, $(1-\psi_{2m}^k) (\beta^{-m} \cdot Y^j)=0$. It is easy to prove that $\{\beta^{-m} \cdot Y^j; |J|=m\}$ are linearly independent. This completes the proof.

**Proposition 3.3.** For any integer $m$, the group $\tilde{Ad}^{2m}(BT^s; R)=0$.

Proof. We consider the following commutative diagram where horizontal lines are exact:

$$
0 \rightarrow \tilde{Ad}^{2m}(BT^s; R) \rightarrow \tilde{K}^{2m}(BT^s; R) \\
\downarrow \hspace{1cm} \downarrow \\
0 \rightarrow \tilde{Ad}^{2m}(BT^s; Q) \rightarrow \tilde{K}^{2m}(BT^s; Q)
$$

Since $K(BT^s; R)=R[[X_1, \cdots, X_i]]$ is a subgroup of $K(BT^s; Q)=Q[[X_1, \cdots, X_i]]$, the left vertical homomorphism is an injection. Thus it is enough to show

**Lemma 3.4.** Let $m$ be a positive integer. If an element

$$
\sum_{|J|=m} a_J \cdot Y^j \in Q[[X_1, \cdots, X_i]]
$$

is a subgroup of $K(BT^s; Q)=Q[[X_1, \cdots, X_i]]$, then $a_J \neq 0$. This completes the proof.
is in
\[ R[[X_1, \ldots, X_i]], \]
then \(a_J = 0\) for all \(J\).

Proof. Since there exists at least one prime \(p\) satisfying \(p\) is not invertible in \(R\), it is enough to show the case \(R = \mathbb{Z}_p\). Using the fact that the denominator of the coefficients of the power series \(\log(1 + X)\) contains sufficient large \(p\)-power, the proof is only an induction.

Clearly, Proposition 3.3 implies

**Corollary 3.5.** If an element
\[ \alpha \in K^{2m}(BT^*; R) = R[[X_1, \ldots, X_i]], \]
satisfies \(k^{-m} \cdot \varphi^k \alpha = \alpha\) for some \(m\), then \(\alpha = 0\). Moreover this is also true when \(R\) is a ring of \(p\)-adic integers for some prime \(p\).

The following proposition contains a special case of the Theorem.

**Proposition 3.6.** For any compact connected Lie group \(G\), the group \(Ad^{2m}(BG; R) = 0\) for any integer \(m\).

Proof. Let \(T^*\) be the maximal torus of \(G\). We consider the following commutative diagram where horizontal lines are exact:
\[
0 \to \tilde{Ad}^{2m}(BG; R) \to \tilde{K}^{2m}(BG; R) \downarrow \downarrow \\
0 \to \tilde{Ad}^{2m}(BT^*; R) \to \tilde{K}^{2m}(BT^*; R).
\]
Since the group \(\tilde{K}^{2m}(BG; R)\) is a subgroup of \(\tilde{K}^{2m}(BT^*; R)\) \([2]\), the right vertical homomorphism is an injection. Thus the left vertical homomorphism is also an injection. This completes the proof.

4. General cases

First we consider the topological cyclic group \(T\). It is a cartesian product of a torus \(T^*\) and a finite cyclic group \(\mathbb{Z}/l\) and its classifying space is \(BT = (CP^*)^* \times B\mathbb{Z}/l\). As same as in section 2, we denote
\[ l = p_1^{e_1} \cdot p_2^{e_2} \cdots p_i^{e_i} \cdot l_1. \]
Buhštaber and Miščenko \([3]\) shows

**Lemma 4.1.** \(K(BT) = K(BT^*) \otimes K(B\mathbb{Z}/l)\)
\[ = Z \oplus \tilde{K}(BT^*) \oplus \tilde{K}(B\mathbb{Z}/l) \oplus \tilde{K}(BT^*) \otimes \tilde{K}(B\mathbb{Z}/l)\)
Considering the $R$-coefficient theory, we have

**Corollary 4.2.** $K(BT; R) = R \oplus \tilde{K}(BT^*; R) \oplus \hat{K}(BZ/l; R) \oplus \tilde{K}(BT^* ; R) \otimes \hat{K}(BZ/l; R)$.

When $l$ is invertible in $R$ the group $\hat{K}(BZ/l; R) = 0$. So we have

**Corollary 4.3.** When $|T/T_0|$ is invertible in $R$, then $\tilde{Ad}m(BT; R) = 0$ for any integer $m$.

This is the second part of the Theorem.

**Lemma 4.4.** For any integer $m \neq 0$, the group $\tilde{Ad}m(BT; R) = 0$.

Proof. We must prove the map $1 - \psi_{2m}^k$ is injective. We only need to prove that

$$1 - k^{-m} \cdot \psi^k : \tilde{K}(BT^*) \otimes \hat{K}(BZ/l; R) \to \tilde{K}(BT^*) \otimes \hat{K}(BZ/l; R)$$

is injective.

In section 2, we proved

$$\hat{K}(BZ/l; R) = \bigoplus_{i=1}^{l} \hat{Z}_{p^i} \langle g_{i,\mu} ; 1 \leq \mu \leq p^i - 1 \rangle$$

where $\eta_i$ is the image of the canonical line bundle of the map $K(BZ/p^i) \to K(BZ/l)$ and $g_{i,\mu} = (\eta_i)^{-1}$. So

$$K(BT^*) \otimes \hat{K}(BZ/l; R) = K(BT^*) \otimes \hat{Z}_{p^i} \langle g_{i,\mu} ; 1 \leq \mu \leq p^i - 1 \rangle.$$  

Thus any element of this group can be written as

$$\alpha = \sum_{1 \leq \mu \leq p^i - 1} a_{i,\mu} \cdot g_{i,\mu}$$

where $a_{i,\mu}$ are reduced elements of the ring $\hat{Z}_{p^i}[X_1, \ldots, X_s]$ and $M = p^i - 1$.

We assume that $(1 - k^{-m} \cdot \psi^k)\alpha = 0$. In section 2 we showed that the Adams operation $\psi^k$ act on $g$'s as permutation. By assumption, there exists an integer $e$ such that $(\psi^k)^e g_{i,\mu} = g_{i,\mu}$. So $a_{i,\mu}$ must satisfies $k^{-m} \cdot \psi^k a_{i,\mu} = a_{i,\mu}$. So using Corollary 3.5, we have $a_{i,\mu} = 0$.

Let $\{T_\lambda; \lambda \in \Lambda\}$ be the set of all Cartan subgroups of a compact Lie group $G$. Remember that $\hat{K}^{2n}(BG) \to \bigoplus \hat{K}^{2m}(BT_\lambda)$ is a monomorphism [7]. We consider the following commutative diagram with horizontal injective homomorphism:

$$0 \longrightarrow Ad^{2m}(BG; R) \longrightarrow K^{2m}(BG; R) $$

$$\downarrow$$

$$0 \longrightarrow \bigoplus \hat{Ad}^{2m}(BT_\lambda; R) \longrightarrow \bigoplus \hat{K}^{2m}(BT_\lambda; R).$$

Since the right vertical homomorphism is injective, the left vertical homomor-
bihomology groups

Proposition 4.5. For any integer $m \neq 0$, $\tilde{Ad}^{2m}(BG; R) = 0$.

This is the first part of the Theorem.

References


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