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A Note on the Theory of Degree of Mapping in Euclidean Spaces

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1. Introduction

In a Japanese brochure we have established a theory of degree of mapping, at first for mappings in finite dimensional Euclidean spaces, which is based on the theory of infinitesimal analysis, and then extended it for a kind of mappings in Banach spaces[1].

But an essential difficulty lies in the proof of the continuity of the degree of mapping for differentiable mapping in Euclidean spaces. Recently M. Kneser has given an interesting theorem on the dependence of functions [2], to which the theorem of Sard on the critical values lies very close [5]. A special case of the theorem of Sard and Kneser concerning mapping of $(m+1)$-dimensional Euclidean open set into m-dimensional Euclidean space, combined with an idea of Birkhoff and Kellogg used for the theory of invariant points [1], affords us another way of the proof of the continuity of the degree of mapping, which we want to give in this note.

In § 2 we shall give a proof of the above mentioned special case of the theorem of Sard and Kneser. In § 3 we give the definition of the degree of mapping for differentiable mappings of class C^2 and prove its continuity. In § 4 will be given its extension to continuous mapping of bounded open sets in Euclidean spaces.

Let $x=(x_1, \cdots, x_m)$ be a point of an m-dimensional Euclidean space E^m. A function $f(x)$ defined on an open set in E^m is called *of class* C^p, if every p-th partial derivatives of f exists and is continuous. We use the notation $\partial_{x_i} f$ or $\partial_i f$ for the partial derivatives $\partial f/\partial x_i$ and $\partial^2_{ij} f$ for $\partial^2 f/\partial x_i \partial x_j$. Mapping means always a *continuous* mapping and a mapping f is called of class C^p, if every component of $f(x)$ is a function of class C^p.

2. Auxiliary theorem

Theorem 1. *Let D^{m+1} be an open set in E^{m+1}, and f be a mapping of D^{m+1} into E^m of class C^2. Let B^m_f be the point set in E^m defined by*

1) Essentially the same treatise is also published in Am. Jour. Math. 73 (1951), [3,] [4].

$$(1) \qquad B_r^m = \left\{ y : y = f(x),\ x \in D^{m+1},\ \text{Rank } (\partial_x f) \leqq r \right\}.$$

Then; $\qquad$ *measure of* $B_r^m{}_{(\text{in } E^m)} = 0$, *if* $r < m$.

To prove this we use the following

Lemma. *Let f be a mapping of D^{m+1} into E^m of class C^2. Then,*

$$(2) \qquad \textit{measure of } \ B_0^m{}_{(\text{in } E^m)} = 0 .$$

Proof: First we assume that $m \geqq 2$. Let A be the subset of D^{m+1} defined by $A = \{x : \partial_x f = 0\}$, and $Q = \{x : \alpha_i \leqq x_i \leqq \beta_i\ (i=1, \dots, m+1)\}$ $(\beta_i - \alpha_i = l)$ be any $(m+1)$-dimensional cube in D^{m+1}. Then $\| f(x') - f(x) \|_m \leqq M \| x' - x \|_{m+1}^2$ [2] for any $x \in A \frown Q$ and $x' \in Q$, where M is a definite constant. Divide Q into n^{m+1} equal small cubes $Q_\nu^{(n)} = \{x : \alpha_i^{(\nu)} \leqq x_i \leqq \beta_i^{(\nu)}\ (i=1, \dots, m+1)\} (\beta_i^{(\nu)} - \alpha_i^{(\nu)} = l/n)$. Then,

$$\text{measure of } \ f(Q_\nu^{(n)}) \leqq M^m (l/n)^{2m}, \quad \text{if} \quad Q_\nu^{(n)} \frown A \neq \{ \ \}^{3)} .$$

Therefore

$$\text{measure of } \ f(A \frown Q) \leqq n^{m+1} M^m (l/n)^{2m} \leqq (Ml^2)^m / n .$$

Thus for $n \to \infty$ we get: measure of $f(A \frown Q) = 0$. As $f(A) = B_0^m$ can be covered by an enumerable number of such $f(A \frown Q)$, we obtain: measure of $B_0^m = 0$.

Now let it be $m = 1$ and let A_1 be the subset of D^2 defined by

$$(3) \qquad A_1 = \left\{ x : \partial_i f = 0 \quad \text{for all } i,\ \ \partial_{ij}^2 f \neq 0 \quad \text{for some} \quad i, j \right\}.$$

Then

$$(4) \qquad \text{measure of } \ f(A_1)_{(\text{in } E^1)} = 0 .$$

Because: Let x^0 be any point of A_1 and i, j be such indices that $\partial_{ij}^2 f(x^0) \neq 0$. Putting $\partial_j f(x) = \varphi(x)$ we have $\partial_i \varphi(x^0) \neq 0$. Let us suppose $i = 2$. Then there exists a neighborhood U of x^0 such that for any point $x = (x_1, x_2) \in U$, which satisfies $\varphi(x) = 0$, holds the relation $x_2 = \psi(x_1)$, where $\psi(x_1)$ is a function of class C^1. Then for $x \in A_1 \frown U$ we have

$$f(x) = f(x_1, \psi(x_1)) = f^*(x_1) ,$$

where $f^*(x_1)$ is a function of class C^1. Hence

$$f(A_1 \frown U) \subset \left\{ y : \ y = f^*(x_1),\ (x_1, \psi(x_1)) \in U,\ \partial_1 f^* = 0 \right\}.$$

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2) $\| f \|_m = \max_{1 \leqq i \leqq m} |f_i|$, $\quad \| x \|_{m+1} = \max_{1 \leqq i \leqq m+1} |x_i|$.

3) { } means the empty set.

But we can easily prove

$$\text{measure of} \quad \left\{y: \ y = f^*(x_1), \ \partial_1 f^* = 0\right\}_{(\text{in } E^1)} = 0 .$$

Thus

$$(5) \qquad \text{measure of} \quad f(A_1 \cap U) = 0 .$$

As $f(A_1)$ can be covered by an enumerable number of such $f(A_1 \cap U)$, we obtain (4).

Now let A_0 be the subset of D^2 defined by

$$(6) \qquad A_0 = \left\{x: \ \partial_i f = 0, \ \partial_{ij}^2 f = 0 \quad \text{for all } i, j\right\} .$$

Then

$$(7) \qquad \text{measure of} \quad f(A_0)_{(\text{in } E^1)} = 0 .$$

Because: Let $Q = \{x: \ \alpha_i \leqq x_i \leqq \beta_{i\,(i=1,2)}\}\,(\beta_i - \alpha_i = l)$ be an arbitrary quadrate in D^2 and divide Q into n^2 small quadrates $Q_\nu^{(n)}$ as in the first part of the proof. Then for any given $\varepsilon > 0$ there exists a natural number n such that the inequality

$$|f(x') - f(x)| < \varepsilon \, \| x' - x \|_2^2$$

holds for any $x \in Q_\nu^{(n)} \cap A_0$, $x' \in Q_\nu^{(n)}$. Therefore

$$\text{measure of} \quad f(Q_\nu^{(n)}) < \varepsilon (l/n)^2 , \quad \text{if} \quad Q_\nu^{(n)} \cap A_0 \neq \{ \ \} .$$

Hence; measure of $f(Q \cap A_0) < n^2 \varepsilon (l/n)^2 = \varepsilon l^2$. But, as ε is arbitrarily small and $f(A_0)$ can be covered by an enumerable number of such $f(Q \cap A_0)$, we obtain (7). From (3), (4), (6) and (7) we get (2), since $B_0^1 = f(A_0) \cup f(A_1)$.

Proof of Theorem 1. Let A_r be the subset of D^{m+1} defined by

$$A_r = \left\{x: \ \text{Rank } (\partial_x f) = r\right\} .$$

Then by Lemma we have; measure of $f(A_0) = 0$. We shall prove by mathematical induction that

$$\text{measure of} \quad f(A_r) = 0 \quad \text{for} \quad r < m .$$

We assume that Theorem 1 holds for $0 \leqq r \leqq k-1$. Let x^0 be any point of A_k, then $\partial_i f_j(x^0) \neq 0$ for some i, j, and without loss of generality we can assume that $i = m+1$, $j = m$. Then by the theory of implicit functions, there exists a neighborhood U of x^0 such that through the relation

$$y = f_m(x_1, \dots, x_{m+1}) , \quad x \in U ,$$

x_{m+1} can be expressed in the form

$$x_{m+1} = \psi(x_1, \dots, x_m, y),$$

where $\psi(x, y)$ is a function of class C^2. Thus for $x \in U$

$$f_i(x) = f_i(x_1, \dots, x_m, \psi(x_1, \dots, x_m, y)) = f_i^*(x_1, \dots, x_m, y),$$

where $f_i^*(x, y)$ is a function of class C^2. But

$$(8) \qquad \text{Rank}\left(\partial_{x_j} f_i \begin{smallmatrix} i=1,\dots,m \\ j=1,\dots,m+1 \end{smallmatrix}\right) = \text{Rank}\left(\partial_{x_j} f_i^*, \ \partial_y f_i^* \begin{smallmatrix} i=1,\dots,m \\ j=1,\dots,m \end{smallmatrix}\right)$$

$$= \text{Rank}\left(\partial_{x_j} f_i^*, \ \partial_y f_i^* \begin{smallmatrix} i=1,\dots,m-1 \\ j=1 \ \dots,m \end{smallmatrix}\right) + 1,$$

since $$\left(\partial_{x_j} f_i^*, \ \partial_y f_i^* \begin{smallmatrix} i=1,\dots,m \\ j=1,\dots,m \end{smallmatrix}\right) = \begin{pmatrix} \partial_{x_j} f_i^*, & \partial_y f_i^* & \begin{smallmatrix} i=1,\dots,m-1 \\ j=1,\dots,m \end{smallmatrix} \\ 0 & 1 & \end{pmatrix}.$$

Now let $B_r^{*m-1}(y) \subset E_{(x')}^{n-1}$ be defined (for each fixed y) by

$$B_r^{*m-1}(y) = \left\{x' : \ x' = f^*(x, y), \ (x, \psi) \in U, \quad \text{Rank}\,(\partial_x f^*) = r\right\}.$$

Then by our hypothesis

$$(9) \qquad \text{measure of} \quad B_r^{*m-1}(y) = 0 \qquad \text{for} \quad 0 \leqq r \leqq k-1.$$

But by (8)

$$\text{measure of } f(A_k \cap U) = \int \text{measure of } B_{k-1}^{*m-1}(y)dy.$$

Hence by (9); measure of $f(A_k \cap U) = 0$. As $f(A_k)$ can be covered by an enumerable number of such $f(A_k \cap U)$, we get

$$\text{measure of} \quad f(A_k) = 0.$$

This completes our proof.

Corollary 1. *If f is a mapping of an m-dimensional open set D^m into E^m of class C^2, then*

$$\textit{measure of } \left\{y : \ y = f(x), \quad \det(\partial_x f) = 0\right\} = 0 \,[4].$$

Proof: Let us define a mapping f^* of $D^{m+1} = D^m \times E^1$ into E^m by $f^*(x_1, \dots, x_m, x_{m+1}) = f(x_1, \dots, x_m)$. Then

$$\text{Rank}\left(\partial_{x_j} f_i \begin{smallmatrix} i=1,\dots,m \\ j=1 \ \dots,m \end{smallmatrix}\right) = \text{Rank}\left(\partial_{x_j} f_i^* \begin{smallmatrix} i=1,\dots,m \\ j=1,\dots,m+1 \end{smallmatrix}\right).$$

Thus Corollary 1 holds by Theorem 1.

4) Corollary 1 remains valid even when f is of class C^1. Cf. Nagumo [3].

3. Degree of mapping for mapping of class C^2

Let D be a *bounded open set* in E^m and f be a mapping of $\bar{D}$ (the closure of D) into E^m of class C^2. The set of all *critical values* of f on D, i.e., the set

$$\left\{y:\ y=f(x),\ x\in D,\quad \det(\partial_x f)=0\right\}$$

is called the *crease* of f. Let a be a point of E^m, which lies neither on $f(\bar{D}-D)$ nor on the crease of f. Then, by a theorem on implicit functions, the equation $f(x)=a$ has only isolated solutions. As $a\notin f(\bar{D}-D)$, the set $\{x: f(x)=a\}$ has no accumulation point on $\bar{D}-D$. Therefore there are only a *finite* number of points $x\in D$ such that $f(x)=a$. Let p be the number of points x for which $f(x)=a$ and $\det(\partial_x f)>0$ hold, and q be the number of those x for which $f(x)=a$ and $\det(\partial_x f)<0$ hold. Then we call the integer $p-q$ the *degree of mapping* of D at a by f, and denote it by

$$A[a, D, f]\quad (=p-q)\,.$$

As is easily seen, we have

$$A[a, D, f]=A[0, D, f-a]$$

Theorem 2. *For each $t(0\leqq t\leqq 1)$ let f_t be a mapping of $\bar{D}$ into E^m such that $f_t(x)$ is a function of class C^2 in (t, x) for $0\leqq t\leqq 1$, $x\in D$. Let a be a point of E^m such that*

$$(1)\qquad a\notin f_t(\bar{D}-D)\quad \textit{for}\quad 0\leqq t\leqq 1\,,$$

and a is neither on the crease of f_0 nor on that of f_1, then we have

$$(*)\qquad A[a, D, f_0]=A[a, D, f_1]\,.$$

Proof: Let K be the set in E^m defined by

$$(2)\quad K=\left\{y:\ y=f_t(x),\ 0\leqq t\leqq 1\,,\ x\in D\,,\quad \mathrm{Rank}\ (\partial_x f_t, \partial_t f_t)<m\right\}.$$

Then by Theorem 1,

$$(3)\qquad \text{measure of}\quad K=0\,.$$

At first let us suppose that $a\notin K$. Then by (1) and (2) there are at most a finite number of curves $\mathfrak{C}_\nu$: $x=\chi(s)$, $t=\tau(s)$ (for the parameter s is taken the arc length of $\mathfrak{C}_\nu$) in $E^{m+1}_{(x,t)}$ defined by the equation

$$(4)\qquad f_t(x)=a\,,$$

which are smooth and without multiple point because of

(5) $\qquad \text{Rank}\,(\partial_x f_t, \partial_t f_t) = m \quad \text{along } \mathfrak{C}_\nu .$

We determine the orientation of $\mathfrak{C}_\nu$ in such a way that

(6) $$\det \begin{pmatrix} \partial_{x_j} f_t^i & \partial_t f_t^i & {\scriptstyle i=1,\dots,m \atop j=1,\dots,m} \\ \chi_j' & \tau' & \end{pmatrix} > 0 \text{ [5]}.$$

The values of $\chi'(s)$ and $\tau'(s)$ at each point of $\mathfrak{C}_\nu$ will be determined by (6) and

(7) $$\begin{cases} \sum_{j=1}^m \partial_{x_j} f_t^i \cdot \chi_j' + \partial_t f_t^i \cdot \tau' = 0\,, & (i=1,\dots,m), \\ \sum_{j=1}^m \chi_j'^2 + \tau'^2 = 1 \text{ [6]}. \end{cases}$$

Then

(8) $\qquad \text{sign of } \tau'(s) = \text{sign of } \det(\partial_x f_t) \quad \text{along } \mathfrak{C}_\nu .$

Because:

$$\det \begin{pmatrix} \partial_{x_j} f_t^i & \partial_t f_t^i \\ \chi_j' & \tau' \end{pmatrix} \cdot \tau' = \det \begin{pmatrix} \partial_{x_j} f_t^i & \sum_{j=1}^m \partial_{x_j} f_t^i \cdot \chi_j' + \partial_t f_t^i \cdot \tau' \\ \chi_j' & \sum_{j=1}^m \chi_j'^2 + \tau'^2 \end{pmatrix}$$

$$= \det \begin{pmatrix} \partial_x f_t^i & 0 \\ \chi_j' & 1 \end{pmatrix} = \det \left(\partial_x f_t\right), \quad \text{(by (7))}.$$

Hence by (6) we get (8). Thus, for $t=0$ or $t=1$, $A\,[a, D, f_t]$ equals just the algebraic sum of the numbers of intersections of the hyperplane $t=0$ or $t=1$ by the curves $\mathfrak{C}_\nu$, taken positive or negative after the sense of intersection (the sign of τ'). Each $\mathfrak{C}$ is simply closed or runs from one of the two hyperplanes $t=0$ and $t=1$ to the same or another one of them without touching $\bar{D}-D$ (by (1)). Namely we have 4 possible cases for each $\mathfrak{C}_\nu$:

i) $\mathfrak{C}_\nu$ is a simply closed curve.

ii) $\mathfrak{C}_\nu$ runs from one of the two hyperplanes $t=0$ and $t=1$ to the same one.

iii) $\mathfrak{C}_\nu$ runs from the hyperplane $t=0$ to the hyperplane $t=1$.

iv) $\mathfrak{C}_\nu$ runs from the hyperplane $t=1$ to the hyperplane $t=0$.

In the first two cases, i) and ii), $\mathfrak{C}_\nu$ has no effect on $A\,[a, D, f_t]$ for $t=0$ and $t=1$. Let p be the number of $\mathfrak{C}_\nu$ of the case iii), and q be the number of $\mathfrak{C}_\nu$ of the case iv). Then we have

$$A\,[a, D, f_0] = A\,[a, D, f_1] = p - q\,.$$

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5) $f_t = (f_t^1, \dots f_t^m,)$

6) The values of x' and τ' are as follows: $x_k' = \Delta_k / |\Delta|$, $\tau' = \Delta_t / |\Delta|$, where $\Delta_k = (-1)^{m-k+1} \det\left(\partial_{x_j} \partial_t^{\delta_{it}} f_t^i \; {\scriptstyle i=1,\dots,m \atop j \neq k}\right)$, $\Delta_t = \det\left(\partial_{x_j} f_t^i \; {\scriptstyle i=1,\dots,m \atop j=1,\dots,m}\right)$ and $|\Delta| = \left(\sum_{k=1}^m \Delta_k^2 + \Delta_t^2\right)^{\frac{1}{2}}$. The determinant on the left side of (6) is equal to $|\Delta|\,(>0)$.

Thus the theorem is proved for the case $a \notin K$.

Now we consider the case $a \in K$. Since $a \notin f_t(\bar{D}-D)$, a is not a critical value of f_t for $t=0, 1$ and the critical sets of f_0 and f_1 are closed in D, there exists a neighborhood U of a such that U is free from the critical values of f_0 and f_1, and

$$(9) \qquad U \frown f_\nu(\bar{D}-D) = \{\ \} \quad (\nu=0,1) .$$

As the measure of K is 0, there exists a point a' in $U-K$. Put

$$(10) \qquad a(s) = (1-s)a + sa' \quad \text{for} \quad 0 \leqq s \leqq 1 .$$

Then

$$(11) \qquad A[a(s), D, f_\nu] = A[0, D, f_\nu - a(s)] \quad (\nu=0,1) .$$

For such x that $f_\nu(x) = a(s)$ $(0 \leqq s \leqq 1)$ $(\nu=0,1)$, we have

$$\text{Rank}\left(\partial_x(f_\nu - a(s)),\ \partial_s(f_\nu - a(s))\right) = \text{Rank}\left(\partial_x f_\nu,\ a'(s)\right) = m \quad (\nu=0,1) .$$

Therefore, by the first part of the proof, taking $f_\nu - a(s)$. instead of f_t, we have by (10)

$$A[0, D, f_\nu - a] = A[0, D, f_\nu - a'] \quad (\nu=0,1) ,$$

hence by (11)

$$(12) \qquad A[a, D, f\ ] = A[a', D, f_\nu] \quad (\nu=0,1) .$$

But as $a' \notin K$, by the first part of the proof, we have

$$A[a', D, f_0] = A[a', D, f_1] .$$

Then by (12) we obtain (*). The proof is thus complete.

Corollary 2. *Let f be a mapping of $\bar{D}$ into E^m of class C^2 and a be a point of E^m such that $a \notin f(\bar{D}-D)$. Let a_1, a_2 be any two points of E^m, which are not critical values of f and such that*

$$|a'_\nu - a| < \text{dist}(a, f(\bar{D}-D)) \quad (\nu=1,2)\ ^{7)}.$$

Then

$$A[a_1, D, f] = A[a_2, D, f] .$$

Proof: Put $a(t) = (1-t)a_1 + ta_2 (0 \leqq t \leqq 1)$, then

$$A[a(t), D, f] = A[0, D, f - a(t)] ,$$

and $0 \notin f(\bar{D}-D) - a(t)$ for $0 \leqq t \leqq 1$. Thus Corollary 2 follows from Theorem 1.

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7) $|a' - a|$ is the distance between a and a'.

Remark: By virtue of Corollary 2 we can define $A[a, D, f]$ *even when a is a critical valve of f, provided that $a \notin f(\bar{D}-D)$.* Indeed we have only to define

$$A[a, D, f] = A[a', D, f],$$

where a' is any point of E^m not on the crease of f and such that

$$|a'-a| < \text{dist}(a, f(\bar{D}-D)).$$

The existence of such a' is assured by the fact that the measure of the crease of f is 0 (by Corollary 1).
$A[a, D, f]$ being thus defined we have:

Theorem 3. *Theorem 2 holds even if a is a critical value of f* $(\nu=0, 1)$.

4. Degree of mapping for general case

Now let F be a continuous mapping of $\bar{D}$ (D is a bounded open set in E^m) into E^m and a be a point of E^m such that $a \notin F(\bar{D}-D)$.

Definition. *Let f be any mapping of $\bar{D}$ into E^m of class C^2 such that*

$$|f(x)-F(x)| < \text{dist}(a, F(\bar{D}-D)) \quad \textit{for} \quad x \in \bar{D}.$$

Then we define the degree of mapping of D at a by F by

$$A[a, D, F] = A[a, D, f].$$

To legitimize this definition we use the following:

Theorem 4. *Let f_ν $(\nu=0, 1)$ be mappings of $\bar{D}$ into E^m of class C^2 such that*

$$|f(x)-F(x)| < \text{dist}(a, F(\bar{D}-D)) \quad (\nu=0, 1).$$

Then

$$(*) \qquad A[a, D, f_0] = A[a, D, f_1].$$

Proof: Put $f_t(x) = (1-t)f_0(x) + tf_1(x)$ $(0 \leq t \leq 1)$. Then for each $t \in \langle 0, 1 \rangle$, f_t is a mapping of $\bar{D}$ into E^m such that $f_t(x)$ is a function of class C^2 in (x, t) for $0 \leq t \leq 1$, $x \in D$ and

$$|f_t(x)-F(x)| < \text{dist}(a, F(\bar{D}-D)) \quad \text{for} \quad 0 \leq t \leq 1.$$

Hence $a \notin f_t(\bar{D}-D)$ for $0 \leq t \leq 1$. Thus by Theorem 2 we get $(*)$.

Theorem 5. *For each $t (0 \leq t \leq 1)$ let f_t be a mapping of $\bar{D}$ into E^m such that $f_t(x)$ is continuous for $0 \leq t \leq 1$, $x \in \bar{D}$. Let $a(t)$ be a point of E^m such that $a(t)$ is a continuous function of t for $0 \leq t \leq 1$ and $a \notin f_t(\bar{D}-D)$. Then $A[a(t), D, f_t]$ is constant for $0 \leq t \leq 1$.*

Proof: Let τ be any fixed value of t from $0 \leqq t \leqq 1$. Then there exists a $\delta = \delta(\tau) > 0$ and a mapping F of $\bar{D}$ into E^m of class C^2 such that

$$(1) \qquad |F(x) - f_t(x)| < \operatorname{dist}(a(t), f_t(\bar{D} - D)) \quad \text{for} \quad |t - \tau| < \delta\,,$$

and

$$(2) \qquad |a(t) - a(\tau)| < \operatorname{dist}(a(\tau), F(\bar{D} - D)) \quad \text{for} \quad |t - \tau| < \delta\,.$$

Then by the definition of $A[a, D, f_t]$ and (1)

$$(3) \qquad A[a(t), D, f] = A[a(t), D, F] = A[0, D, F - a(t)]\,.$$

But by (2) $0 \notin F(\bar{D} - D) - a(t)$ for $|t - \tau| < \delta$. Then by Theorem 3 and (3) $A[0, D, F - a(t)] = A[a(t), D, f_t]$ is constant for $|t - \tau| < \delta$. Applying the Borel's covering theorem to the closed interval $0 \leqq t \leqq 1$ we obtain the constancy of $A[a(t), D, f_t]$ for $0 \leqq t \leqq 1$.

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