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ACYCLIC FAKE SURFACES WHICH ARE SPINES OF 3-MANIFOLDS

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1. Introduction

In [1], we defined fake surfaces to study 3-manifolds with boundary from their spines. Let $\mathcal{F}(s, t)$ denote the set of all the acyclic closed fake surfaces P with $\# \mathcal{S}_2(P) = s$ and $\# \mathcal{S}_3(P) = t$ ($\#$ means the number of the connected components). In this paper, we consider about the subset $\mathcal{E}(s, t)$ of $\mathcal{F}(s, t)$ each of whose elements can be embedded in some 3-manifold.

A connected closed fake surface P is called a *normal spine*, if P can be embedded in a 3-manifold. That is, taking the regular neighborhood, we can regard P as a spine of a 3-manifold, when P is a normal spine. Of course, every element of $\mathcal{E}(s, t)$ is a normal spine.

We use the following notations. For a polyhedron P , \dot{P} means the boundary of P , that is, \dot{P} is the union of the free faces of P , and \bar{P} means the interior of P defined by $\bar{P} = P - \dot{P}$. \bar{P} means the closure of P , and I is the closed unit interval $[0, 1]$. For the other unexplained notations, see [1].

In §2, we prepare some lemmas for acyclic normal spines by defining the *connected sum* of closed fake surfaces and the *r-th complement*. In §3, we obtain the sufficient condition that $\mathcal{E}(s, t)$ is empty, that is, Theorem 1 states that $\mathcal{E}(s, t)$ is empty if $s \geq 2t$, (and, in the last section, we show that this is also the necessary condition). In §4, two types of *elementary deformation* of normal spines in the respective 3-manifolds are introduced and two invariants $\alpha(P)$ and $\beta(P)$ are defined for a closed fake surface P . And, in Theorem 2, we prove $\alpha(P) = r = \beta(P)$ when P is a *r-th complement*. In §5, all the elements of the set $\mathcal{E}(s, 2)$ are characterized geometrically using the concept of the *union* of closed fake surfaces, from which the Zeeman's conjecture is shown to be true for any element of $\mathcal{E}(s, 2)$, easily.

Zeeman's conjecture [2] : If P is a contractible 2-polyhedron, then $P \times I$ is collapsible where $I = [0, 1]$ is the closed unit interval.

In the last section, we obtain the geometrical characterizations of the elements of $\mathcal{E}(2t-1, t)$ and $\mathcal{E}(2t-2, t)$ for all integers $t \geq 1$ and $t \geq 2$, respectively. And, as the consequences, the Zeeman's conjecture for them follows.

Furthermore, in Theorem 6, we show that $\mathcal{E}(s, t)$ contains a spine of a 3-ball for any pair (s, t) with $1 \leq s \leq 2t-1$. Combining this with Theorem 1, we obtain the following.

Theorem. $\mathcal{E}(s, t)$ is empty if and only if $s \geq 2t$.

On the other hand, it is easily seen that $\mathcal{F}(s, t)$ is empty if and only if $t=0$. The sufficiency follows from Theorem 1 [1]. To show the necessity, replace a 2-ball B in $\dot{M}(P)$ of an element P of $\mathcal{E}(2t-1, t)$ by the element \mathcal{N}_{s-2t+1} so that $\dot{B} = \mathcal{N}_{s-2t+1}$ (for the definition of \mathcal{N}_{s-2t+1} , see Definition 6, §6, [1]).

Note that $\mathcal{E}(1, 1)$ consists of a unique element $F_{1,1}^1$ by Theorem 4 [1] which is named "Abalone" by H. Noguchi and the realization of an abalone in the Euclidean 3-space R^3 is written in Figure 0 which is shown by Y. Tsukui.

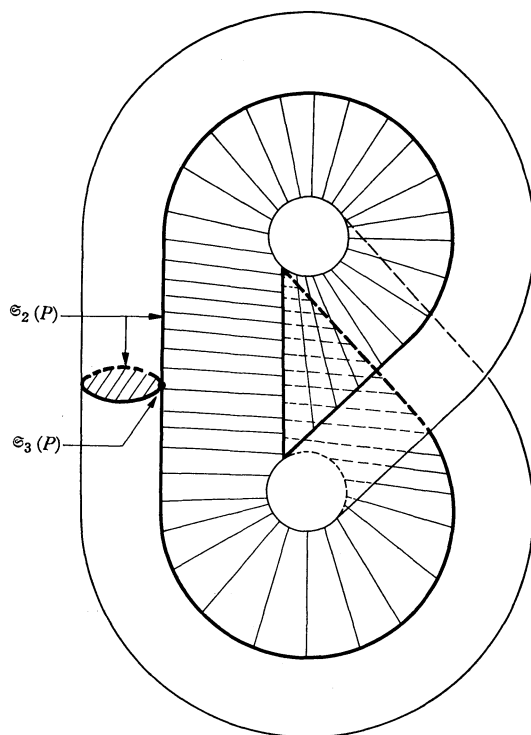


Fig. 0

The author thanks to Professors H. Noguchi, T. Homma, F. Hosokawa and all the members of their seminar All Japan Combinatorial Topology Study Group for their kind advices, suggestions and useful discussions.

2. Lemmas

DEFINITION 1. Let P_i be a closed fake surface with a 2-ball B_i in $\dot{M}(P)$, $i=1, 2$, and f a homeomorphism from \dot{B}_1 to \dot{B}_2 . We define the *connected sum* $P_1 \circ P_2$ of P_1 and P_2 with respect to B_1, B_2 and f by $P_1 \circ P_2 = ((P_1 - \dot{B}_1) \cup (P_2 - \dot{B}_2))/f$.

DEFINITION 2. First, define the 0-th *complement* to be an acyclic normal spine. A connected closed fake surface X is said to be a r -th *complement* if there exists an acyclic fake surface P such that $X \circ P$ is a $(r-1)$ -th complement.

DEFINITION 3. Let P be a fake surface. We say that a connected component U of $U(P)$ is *isolated* if $\mathcal{S}_3(U)$ is empty. And let $\nu(P)$ denote the number of the isolated components of $U(P)$.

Lemma 1. *Let P be a closed fake surface. If $U(P)$ is embeddable in an orientable 3-manifold, P is a normal spine.*

Proof. Let W be an orientable 3-manifold in which $U(P)$ is embedded, and let M be an element of $M(P)$ with boundary $\dot{M} = b_1 \cup \cdots \cup b_j$. Let us consider $M \times I$ and $A_i = b_i \times I$ where I denote the closed unit interval $[0, 1]$ and $\dot{M} = \dot{M} \times 1/2$, and the 2-nd derived neighborhood N_i of b_i in the boundary of the regular neighborhood N of $U(P)$ in $W \bmod \dot{U}(P)$, $i=1, \dots, j$. Since \dot{N} is a disjoint union of orientable closed 2-manifolds, there is a homeomorphism f_i from A_i onto N_i which is the identity on b_i . Then, we obtain a homeomorphism h_M from $\bigcup_i A_i = \dot{M} \times I$ onto $\bigcup_i N_i$ defined by f_i on each A_i . Define the 3-manifold

$$V = \bigcup_M ((N \cup (M \times I))/h_M),$$

that is, V is the 3-manifold obtained from N and $M(P) \times I$ by identifying A_i and N_i by f_i for all $i=1, \dots, j$ and for all elements M of $M(P)$. Obviously, P is embedded in the 3-manifold V , completing the proof.

Lemma 2. *Let P be a closed fake surface with $H_1(P)=0$. Then, P is a normal spine if and only if $U(P)$ can be embedded in R^3 , the Euclidean 3-space.*

Proof. Sufficiency follows immediately from Lemma 1. So, we prove Necessity. Let W be a 3-manifold in which P is embedded. Since W is orientable and $U(P)$ collapses to the 1-polyhedron $\mathcal{S}_2(P)$, the regular neighborhood N of $U(P)$ in W is a disjoint union of solid tori with certain genus. Then, N is embeddable in R^3 , and hence, so is the subpolyhedron $U(P)$.

Lemma 3. (i) *Let X be a r -th complement. Then, we have $H_1(X)=0$ and $H_2(X)=Z + \cdots + Z$ of rank r .*

- (ii) *A r -th complement X is a normal spine.*
 (iii) *Let $X = X_1 \circ X_2$ be a r -th complement. Then, X_i is a r_i -th complement for $i = 1, 2$, and $r_1 + r_2 = r + 1$.*

Proof. The proof goes by induction on r . When $r = 0$, there is nothing to prove (i) and (ii). So, we prove (iii). By Lemma 14 [1], we may assume that X_1 is acyclic. Then, X_2 is a 1-st complement from the definition. Since X is a normal spine, X_1 is also a normal spine, by Lemma 2, because $U(X_1)$ is contained in $U(X)$ and is embeddable in R^3 . Thus, X_1 is a 0-th complement. Now, we consider the case $r \geq 1$. That is, there is an acyclic closed fake surface P such that $X \circ P$ is a $(r-1)$ -th complement, where the connected sum is taken with respect to the 2-balls B_X and B_P contained in $M(X)$ and $M(P)$ and a homeomorphism f from \dot{B}_X to \dot{B}_P . Define $Q = (X \circ P) \cup (\dot{B}_P^* v)$ where v is an ideal coing point over \dot{B}_P , that is, $(\dot{B}_P^* v)$ is the cone from v over \dot{B}_P and $(X \circ P) \cap (\dot{B}_P^* v) = \dot{B}_P$. Using the inductive hypothesis $H_1(X \circ P) = 0$ and $H_2(X \circ P) = Z + \cdots + Z$ of rank $r-1$, we obtain $H_1(Q) = 0$ and $H_2(Q) = Z + \cdots + Z$ of rank r by the Mayer-Vietoris exact sequence. Since $H_q(Q) = H_q(X) + H_q(P)$ and P is acyclic, we see $H_1(X) = 0$ and $H_2(X) = Z + \cdots + Z$ of rank r . This proves (i). By the inductive hypothesis, $U(X \circ P) = U(X) \cup U(P)$ can be embedded in R^3 . Then $U(X)$ is, of course, embeddable in R^3 , and hence, by Lemma 2, X is a normal spine. This proves (ii). Now, we may assume that the 2-ball B_X is contained in X_1 , because B_X can be moved away from X_2 when $B_X \cap (X_1 \cap X_2)$ is non-empty by an isotopy of X . Then, we can write $X \circ P = (X_1 \circ P) \circ X_2$. Then, by the inductive hypothesis, $(X_1 \circ P)$ is a r' -th complement and X_2 a r_2 -th one and $r' + r_2 = r$. Then, X_1 is a $(r' + 1)$ -th complement, because P is acyclic. Thus, we have $r_1 = r' + 1$, and hence $r_1 + r_2 = r + 1$. This completes the proof of Lemma 3.

Lemma 4. *Let P be a normal spine with $H_1(P) = 0$ and $H_2(P) = Z$. Then, $\mathfrak{S}_3(P)$ is empty if and only if P is a 2-sphere.*

Proof. Sufficiency is trivial. We prove Necessity. It is clear that a 2-sphere satisfies the required conditions and the other 2-manifolds do not. Hence Lemma 4 is true if P is a 2-manifold. So, we assume that $\mathfrak{S}_2(P)$ is non-empty and try to prove that such P does not exist. Let $U(P) = U_1 \cup \cdots \cup U_n$ where U_i means a connected component of $U(P)$ for $i = 1, \dots, n$. Then, each U_i must be isolated, because $\mathfrak{S}_3(P)$ is empty. And since P is a normal spine with $H_1(P) = 0$, U_i is neither $S \times \tau T$ nor $S \times \sigma T$, by Lemma 24 [1], Lemma 2 and Corollary to Theorem 1 [1]. That is, $U_i = S \times T$ for any $i = 1, \dots, n$. The proof goes by induction on n . When $n = 1$, $M(P)$ consists of three 2-balls by Lemma 12 [1] and Proposition 4 [1], and P is obtained from $M(P)$ by identifying their boundaries as indicated in Figure 1.

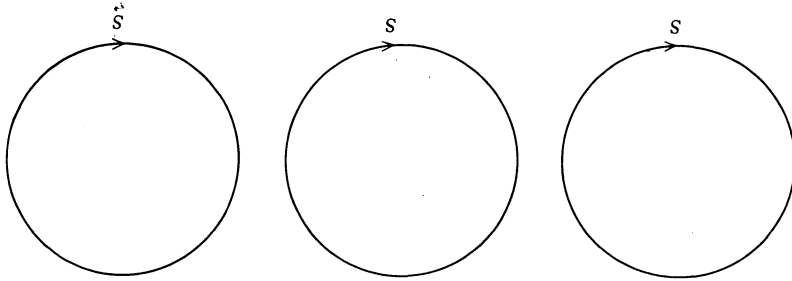


Fig. 1

Then, we have $H_2(P) = Z + Z$ which contradicts to our hypothesis $H_2(P) = Z$. Now, we deal with the case $n \geq 2$. Then, there is an element M with $\# \dot{M} \geq 2$ in $M(P)$ by Lemma 14 [1], and a boundary component b of M disconnects P into two fake surfaces P_1 and P_2 such that $\mathfrak{S}_2(P_i)$ is non-empty for both $i = 1, 2$, by Lemma 14 of [1]. Let $\tilde{P} = P \cup (b^*v)$ and $\tilde{P}_i = P_i \cup (b^*v)$, $i = 1, 2$, where v is an ideal coning point over b . Then, by the Mayer Vietoris exact sequence, we obtain $H_1(\tilde{P}) = 0$ and $H_2(\tilde{P}) = Z + Z$, and hence $H_1(\tilde{P}_i) = 0$ for both $i = 1, 2$, and $H_2(\tilde{P}_1) + H_2(\tilde{P}_2) = Z + Z$. Suppose $H_2(\tilde{P}_1) = 0$. Then, \tilde{P}_1 is an acyclic closed fake surface without 3-rd singularity, which is a contradiction to Theorem 1 [1]. Thus, we see $H_2(\tilde{P}_i) = Z$ for both $i = 1, 2$. Since P is a normal spine, \tilde{P}_i is also a normal spine by Lemma 2. And, clearly, $1 \leq \#U(\tilde{P}_i) \leq n - 1$ holds true, because $\mathfrak{S}_2(\tilde{P}_i)$ is non-empty. This contradicts to our inductive hypothesis, completing the proof.

REMARK. It is easy to see that a 2-sphere S^2 is a 1-st complement, because $S^2 \circ F_{1,1}^1$ is homeomorphic to $F_{1,1}^1$.

Lemma 5. Let $P = P_1 \circ P_2$ be an element of $\mathcal{E}(s, t)$. Suppose that P_1 is not acyclic. Then, P_1 is either a 2-sphere or a 1-st complement with $1 \leq \# \mathfrak{S}_2(P_1) \leq t - 1$.

Proof. By Lemma 14 [1], P_2 is acyclic, and hence $\# \mathfrak{S}_3(P_2) \geq 1$, by Theorem 1 [1]. Then, P_1 is a 1-st complement. Suppose $\# \mathfrak{S}_3(P_1) = 0$. Then, by Lemma 4, P_1 is a 2-sphere. And when $1 \leq \# \mathfrak{S}_3(P_1)$, we see $\# \mathfrak{S}_3(P_1) \leq t - 1$, because $\# \mathfrak{S}_3(P_2) \geq 1$ and $\# \mathfrak{S}_3(P_1) + \# \mathfrak{S}_3(P_2) = t$.

Lemma 6. Let P be an element of $\mathcal{F}(s, t)$ with an isolated component $U = S \times T$. Then, just one of the connected components of $P - \overline{U}$ is acyclic.

Proof. By Lemma 13 [1], $\overline{P - U}$ is the disjoint union of three connected fake surfaces P_1 , P_2 and P_3 . First, we show that at least one of P_1 , P_2 and P_3 is acyclic. Suppose that P_3 is not acyclic. Then, by Lemma 14 [1], we obtain an acyclic fake surface $P_0 = P_1 \cup U \cup P_2$. Since $U = S \times T$, we obtain an acyclic closed fake surface Q from P_0 by collapsing P_0 from its boundary \dot{P}_0 by the

natural way. And the 1-sphere $\mathfrak{S}_2(U)$ disconnects Q into two fake surfaces Q_1 and Q_2 so that P_i is contained in Q_i , for $i=1, 2$. Note that P_i is homeomorphic to Q_i , $i=1, 2$. Then, by the Mayer-Vietoris exact sequence, we obtain $H_2(Q_i)=0$ for both $i=1, 2$, and $H_1(Q_1)+H_1(Q_2)=Z$. Hence, either Q_1 or Q_2 is acyclic, that is, either P_1 or P_2 is acyclic. Suppose that there are two acyclic components P_1 and P_2 . Define $P_0=P_1 \cup U \cup P_2$. Then, we easily have $H_1(P_0)=0$ and $H_2(P_0)=Z$ which implies $H_2(P) \neq 0$. This proves Lemma 6.

Lemma 7. *Let P be an element of $\mathcal{E}(s, t)$ with $\nu(P) \geq 1$. Then, there is an isolated component U in $U(P)$ such that there exists a connected component Q of $\overline{P-U}$ with $\nu(Q)=0$ and $\#\mathfrak{S}_3(Q) \neq 0$.*

Proof. Let U_i be an isolated component of $U(P)$. Then, $U_i = S \times T$ by the same reason as in the proof of Lemma 4. And hence $\overline{P-U_i}$ has three connected components P_{i1}, P_{i2} and P_{i3} . By Lemma 6, we assume that P_{i3} is acyclic. Then, of course, P_{ij} is not acyclic, for $j=1, 2$. If we consider $\tilde{P}_{ij} = P_{ij} \cup (\dot{P}_{ij} * v_j)$, we see that \tilde{P}_{ij} is acyclic, for $j=1, 2$, by Lemma 14 [1]. And $\#\mathfrak{S}_3(P_{ij}) = \#\mathfrak{S}_3(\tilde{P}_{ij}) \neq 0$, by Theorem 1 [1], for $j=1, 2$. Now, it is sufficient to prove the following statement (*) by induction on $\nu = \nu(P_{i1})$.

(*) *Either (1) U_i is a required isolated component U in $U(P)$, or (2) we can find U in P_{i1} , holds true.*

Proof of ().* When $\nu=0$, there is nothing to prove by taking $U=U_i$ and $Q=P_{i1}$. So, we assume that (*) is true for $\nu(P_{i1}) \leq \nu-1$, and we deal with the case $\nu \geq 1$. Let U_k be an isolated component of $U(P)$ contained in P_{i1} . Then, either P_{k1} or P_{k2} is contained in P_{i1} , say P_{k1} . Then, (*) is true for P_{k1} , by the inductive hypothesis, because

$$\nu(P_{k1}) \leq \nu(P_{i1}) - 1 = \nu - 1.$$

Then, clearly, U is contained in P_{i1} , completing the proof.

3. The sufficient condition that $\mathcal{E}(s, t)$ be empty

Proposition 1. *Let P be an element of $\mathcal{E}(s, t)$. Then, we obtain $s \geq 2\nu(P) + 1$.*

Proof. The proof goes by induction on s . We see $s \geq 1$ by Theorem 1 [1], and when $s=1$, there is nothing to prove, because $\nu(P)=0$ by Theorem 1 [1] again. We deal with the case $s \geq 2$. If $U(P)$ contains no isolated component, that is, $\nu(P)=0$, Proposition 1 is trivially true for P . Thus, we may assume that there exist an isolated component U and a connected component Q of $\overline{P-U}$ with $\nu(Q)=0$ and $\#\mathfrak{S}_3(Q) \neq 0$ obtained in Lemma 7. Let us consider $X = \overline{P-Q}$, $Y = X \cup (\dot{X} * v)$ and $W = Q \cup (\dot{Q} * v)$ where v is an ideal coning point over the

1-sphere $\dot{X}=\dot{Q}$. Then, we can write $P=W\circ Y$, by identifying the 2-balls $(\dot{X}*v)$ and $(\dot{Q}*v)$. And, by Lemma 3, there are following two cases.

Case 1. W is a 0-th complement and Y is a 1-st one.

By Lemma 14 [1], X must be acyclic, and hence we can collapse X to an acyclic closed fake surface X' from \dot{X} by the natural way, because $U=S\times T$. Then, X' is also a normal spine by Lemma 2, and we easily have $1\leq\#\mathcal{S}_2(X')=s'\leq s-1$, because X' is acyclic and does not contain U . Hence, we have $s'\geq 2\nu(X')+1$, by the inductive hypothesis. Put $s''=\#\mathcal{S}_2(W)$. Then, we see $s=s'+s''+1$ and $\nu(P)=\nu(X')+1$. Hence,

$$\begin{aligned} s-2\nu(P) &= (s'-s''+1)-2(\nu(X')+1) \\ &= (s'-2\nu(X'))+(s''-1) \\ &\geq 1, \end{aligned}$$

because $\#\mathcal{S}_3(W)=\#\mathcal{S}_3(Q)\neq 0$ means $s''\neq 0$. Therefore, we obtains $\geq 2\nu(P)+1$.

Case 2. W is a 1-st complement and Y is a 0-th one.

In this case, we see $1\leq\#\mathcal{S}_2(Y)=s_1\leq s-1$, by the condition $s''\neq 0$. Then, by the inductive hypothesis, we obtain $s_1\geq 2\nu(Y)+1$, because Y is an acyclic normal spine by Lemma 2. And, in this case, we see $s=s_1+s''$ and $\nu(P)=\nu(Y)$ from which $s\geq 2\nu(P)+1$ follows by a similar calculation to Case 1. Thus, Proposition 1 is established.

Theorem 1. $\mathcal{E}(s, t)$ is empty if $s\geq 2t$.

Proof. Suppose that $\mathcal{E}(s, t)$ is non-empty. And let P be an element of $\mathcal{E}(s, t)$. Then, we have

$$s\geq 2\nu(P)+1\geq 2(s-t)+1$$

from Proposition 1. Hence $s\leq 2t-1$. This proves Theorem 1.

4. Elementary deformations of normal spines in the 3-manifolds

Let P be a normal spine in a 3-manifold V with nonempty 2-nd singularity, i. e. $\mathcal{S}_2(P)\neq\phi$. Suppose that there is a 1-ball A in P satisfying the following conditions (1) and (2).

- (1) $A\cap\mathcal{S}_2(P)=\dot{A}=a_1\cup a_2$.
- (2) a_1 and a_2 are vertices of $\mathcal{S}_2(P)-\mathcal{S}_3(P)$.

Taking the 2-nd derived neighborhood N of A in V , $\dot{N}-(\dot{N}\cap P)$ consists of four open 2-balls each of whose closures is a 2-ball B_i , $i=1, \dots, 4$. Let B_1 be the 2-ball contained in $\text{st}(a_1, V)$. Note that such a 2-ball is uniquely determined (see Figure 2). Then, we may regard the 3-ball $N=B_1\times I$ and hence we can collapse N to $\dot{N}-\dot{B}_1$ from the free face $B_1=B_1\times 0$.

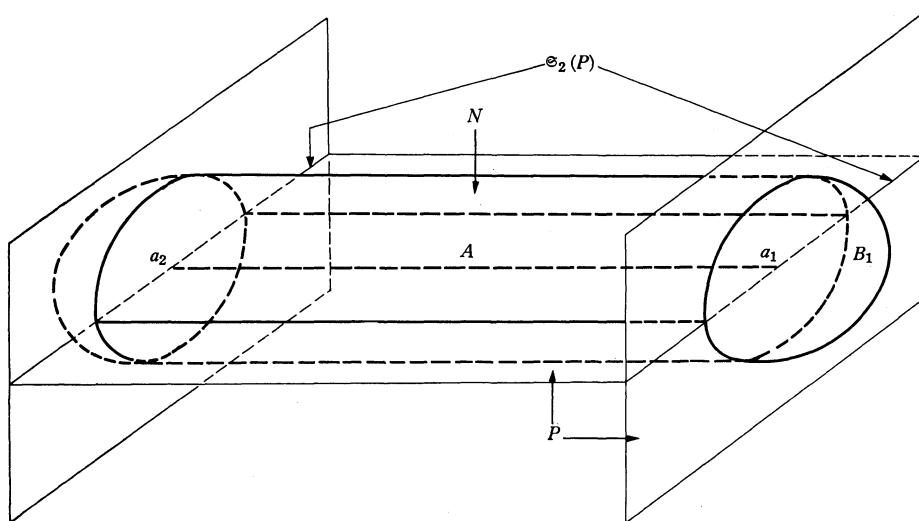


Fig. 2

DEFINITION 4. Define the normal spine $P(1)$ by

$$P(1) = (P - (P \cap N)) \cup (\dot{N} - \dot{B}_1),$$

and we say that $P(1)$ is obtained from P by an elementary deformation in V (with respect to A). Inductively, we can define $P(r)$ as a normal spine obtained from $P(r-1)$ by an elementary deformation in V , and we say that $P(r)$ is obtained from P by r times of elementary deformation in V .

DEFINITION 5. An elementary deformation is said to be of type *I*, if the boundary \dot{A} is contained in a connected component of $\mathfrak{S}_2(P)$, and of type *II* otherwise.

DEFINITION 6. Let P be a closed fake surface. We define the invariants $\alpha(P)$ and $\beta(P)$ by

$$\alpha(P) = \#M(P) - \#\mathfrak{S}_2(P) - \#\mathfrak{S}_3(P), \text{ and}$$

$$\beta(P) = \#\dot{M}(P) - 2\#\mathfrak{S}_2(P) - \#\mathfrak{S}_3(P) + 1.$$

Lemma 8. Let P be a normal spine of a 3-manifold V and $P(r)$ a normal spine obtained from P by r times of elementary deformation in V . Then, $P(r)$ is also a spine of V .

Proof. From the definition of $P(r)$, it is sufficient to prove that P and $P(1)$ are simple homotopy equivalent in V . Let N be the 2-nd derived neighborhood of A in V in the above definition. Then, P expands to $P \cup N$ and $P \cup N$ collapses to $P(1)$ in V , and hence P and $P(1)$ are simple homotopy equivalent in V .

The following two lemmas are immediate from Figure 2.

Lemma 9. *Let P be a normal spine in a 3-manifold V and $P(r)$ a normal spine obtained from P by r times of elementary deformation of type I in V . Then, we have;*

$$\begin{aligned} (1) \quad & \# \mathfrak{S}_2(P(r)) = \# \mathfrak{S}_2(P), \text{ and} \\ (2) \quad & \# \mathfrak{S}_3(P(r)) = \# \mathfrak{S}_3(P) + 2r. \end{aligned}$$

Lemma 10. *Let P be a normal spine in a 3-manifold V and $P(r)$ a normal spine obtained from P by r times of elementary deformation of type II in V . Then, we have;*

$$\begin{aligned} (1) \quad & \# \mathfrak{S}_2(P(r)) = \# \mathfrak{S}_2(P) - r, \\ (2) \quad & \# \mathfrak{S}_3(P(r)) = \# \mathfrak{S}_3(P) + 2r, \\ (3) \quad & \# M(P(r)) = \# M(P) + r, \text{ and} \\ (4) \quad & \# \dot{M}(P(r)) = \# \dot{M}(P). \end{aligned}$$

Proposition 2. *Let P be an element of $\mathcal{E}(s, t)$. Then, we obtain $\alpha(P) = 0 = \beta(P)$.*

Proof. The proof is done by induction on s . When $s=1$, Proposition 4 and Proposition 5 [1] give the answer. Suppose $s \geq 2$. Since P is connected, we can apply an elementary deformation of type II to P in some 3-manifold, and we obtain $P(1)$ which belongs to $\mathcal{E}(s-1, t+2)$ by Lemma 10. Then, by the inductive hypothesis and Lemma 10, we have

$$\begin{aligned} \alpha(P) &= \# M(P) - \# \mathfrak{S}_2(P) - \# \mathfrak{S}_3(P) \\ &= (\# M(P(1)) - 1) - s - t \\ &= ((s-1) + (t+2) - 1) - s - t \\ &= 0. \end{aligned}$$

And, by the same way, we can prove $\beta(P) = 0$.

Theorem 2. *Let X be an r -th complement. Then, we obtain $\alpha(X) = r = \beta(X)$.*

Proof. The proof is done by induction on r . When $r=0$, Proposition 2 gives the answer. We assume $r \geq 1$. Let P be an acyclic fake surface (closed) such that $X \circ P$ becomes an $(r-1)$ -th complement. Note that P is necessarily a 0-th complement. Clearly, the followings hold true.

$$\begin{aligned} \# \mathfrak{S}_2(X \circ P) &= \# \mathfrak{S}_2(X) + \# \mathfrak{S}_2(P), \\ \# \mathfrak{S}_3(X \circ P) &= \# \mathfrak{S}_3(X) + \# \mathfrak{S}_3(P), \\ \# M(X \circ P) &= \# M(X) + \# M(P) - 1, \\ \# \dot{M}(X \circ P) &= \# \dot{M}(X) + \# \dot{M}(P). \end{aligned}$$

Then, we have $\alpha(X \circ P) = \alpha(X) + \alpha(P) - 1$ and $\beta(X \circ P) = \beta(X) + \beta(P) - 1$. Thus, by the inductive hypothesis and Proposition 1 which means $\alpha(P) = 0 = \beta(P)$, we easily obtain $\alpha(X) = r = \beta(X)$.

5. $\mathcal{E}(s, 2)$.

DEFINITION 7. Let P_i be a closed fake surface with a 2-ball B_i in $\overset{\circ}{M}(P_i)$, $i=1, 2$, and let f be a homeomorphism from B_1 onto B_2 . We define the union $P_1 \oplus P_2$ of P_1 and P_2 with respect to B_1, B_2 and f by $P_1 \oplus P_2 = (P_1 \cup P_2)/f$.

Proposition 3. Let P be an element of $\mathcal{E}(3, 2)$. Then, we obtain $P = F_{1,1}^1 \oplus F_{1,1}^1$.

Proof. First, we obtain $\nu(P) = 1$, because

$$\begin{aligned}\nu(P) &\geq \#\mathfrak{S}_2(P) - \#\mathfrak{S}_3(P) = 1, \\ \nu(P) &\leq (\#\mathfrak{S}_2(P) - 1)/2 = 1.\end{aligned}$$

The 2-nd inequality follows from Proposition 1. Let U denote the isolated component of $U(P)$ and P_i the connected component of $\overline{P-U}$, $i=1, 2, 3$. Since $\#\mathfrak{S}_3(P) = 2$, we may assume that P_2 contains no point of $\mathfrak{S}_3(P)$. We show that P_2 is acyclic. Suppose not. Then, $\tilde{P}_2 = P_2 \cup (\dot{P}_2 * v)$ is a acyclic closed fake surface without 3-rd singularity. This contradicts to Theorem 1 [1]. Putting $Q = \overline{P - P_2}$, we define $\tilde{Q} = Q \cup (\dot{Q} * v)$. Then, clearly, we can write $P = \tilde{P}_2 \circ \tilde{Q}$ using the 2-balls $(\dot{P}_2 * v)$ and $(\dot{Q} * v)$. Since P_2 is acyclic, \tilde{P}_2 is not acyclic, by Lemma 14 [1]. Then, by Lemma 5, \tilde{P}_2 is a 2-sphere, because $\#\mathfrak{S}_3(\tilde{P}_2) = \#\mathfrak{S}_3(P_2) = 0$. Hence P_2 is a 2-ball. Define $\tilde{P}_i = P_i \cup (\dot{P}_i * v_i)$, for $i=1, 3$. Then, \tilde{P}_i is an acyclic normal spine by Lemma 14 [1] and Lemma 2, because P_i is not acyclic by Lemma 6 for $i=1, 3$. Since \tilde{P}_i is acyclic, we see $\#\mathfrak{S}_3(\tilde{P}_i) \geq 1$, and hence $\#\mathfrak{S}_3(\tilde{P}_i) = 1$ by $\#\mathfrak{S}_3(\tilde{P}_1) + \#\mathfrak{S}_3(\tilde{P}_3) = \#\mathfrak{S}_3(P) = 2$. Similarly, we have $\#\mathfrak{S}_2(\tilde{P}_i) = 1$ for $i=1, 3$. Thus, \tilde{P}_i is an element of $\mathcal{E}(1, 1)$, that is, $\tilde{P}_i = F_{1,1}^1$, for $i=1, 3$. It is clear that P is obtained from \tilde{P}_1 and \tilde{P}_2 by identifying the 2-balls $(\dot{P}_1 * v_1)$ and $(\dot{P}_2 * v_2)$ to the 2-ball P_2 , that is, $P = \tilde{P}_1 \oplus \tilde{P}_2 = F_{1,1}^1 \oplus F_{1,1}^1$.

REMARK. The number of the elements of $\mathcal{E}(3, 2)$ is, clearly, at most 6.

Lemma 11. Let G be a 1-st complement. Suppose that $\#\mathfrak{S}_2(G) = 1 = \#\mathfrak{S}_3(G)$. Then, G is uniquely determined as described in Fig. 3.

Proof. We obtain the Homology groups $H_1(G) = 0$ and $H_2(G) = \mathbb{Z}$ by Lemma 3. By Theorem 2, we see $\alpha(G) = 1 = \beta(G)$ which implies $\#M(G) = 3$. Then, by Lemma 12 [1] and Proposition 4 [1], it is known that $M(G)$ consists of three 2-balls M_1, M_2 and M_3 . Then, we check all the possible cases as explained in the last half part of the proof of Theorem 2[1]. And we obtain the identification of M_1, M_2 and M_3 as shown in Fig. 3, uniquely.

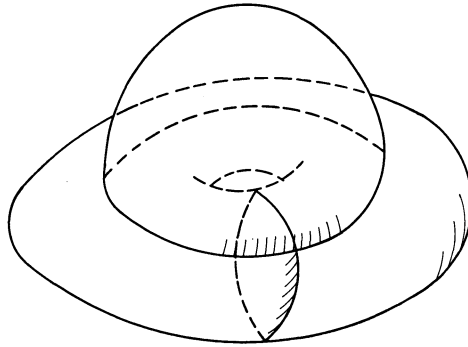
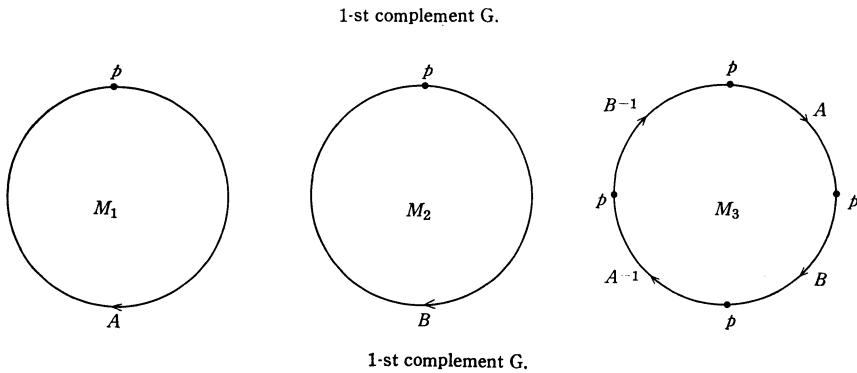


Fig. 3

REMARK. From now on, let G denote the unique 1-st complement obtained in Lemma 11.

REMARK. Let B_G be a 2-ball in $\dot{M}(G)$ and P an acyclic closed fake surface with a 2-ball B_P in $\dot{M}(P)$. Let $G \circ P$ be the connected sum with respect to B_G and B_P . Then, it is easy to see that $G \circ P$ is acyclic if and only if B_G is contained in M_3 (for M_3 , see Fig. 3). And, from now on, B_G denotes the 2-ball contained in M_3 .

Proposition 4. *Let P be an element of $\mathcal{E}(2, 2)$. Then, we obtain $P = G \circ F_{1,1}^1$.*

Proof. There exists an element M in $M(P)$ with $\# \dot{M} = 2$, because $\# M(P) = 4$ and $\# \dot{M}(P) = 5$ by Theorem 2. By cutting P along a boundary component of M and attaching a 2-ball to the boundary of each connected components, we can write $P = P_1 \circ P_2$ and we have $\# \mathcal{S}_3(P_i) \neq 0$ for $i = 1, 2$, because $\# \mathcal{S}_2(P_i) \neq 0$ is clear and $\nu(P) = 0$ implies $\nu(P_i) = 0$ for both $i = 1, 2$. Note that $\nu(P) = 0$ follows from Proposition 1. Then, by Lemma 3, We may assume that P_1 is a

1-st complement and P_2 is a 0-th one. Since $\# \mathfrak{S}_2(P_i) = 1 = \# \mathfrak{S}_3(P_i)$ for both $i = 1, 2$, we have $P_1 = G$ and $P_2 = F_{1,1}^1$, completing the proof.

REMARK. The number of the elements of $\mathcal{E}(2, 2)$ is at most 4.

Proposition 5. $\mathcal{E}(1, 2)$ consists of three elements $F_{1,2}^1$, $F_{1,2}^2$, and $F_{1,2}^3$ which are described in Fig. 4.

Proof. By the same way as explained in the last half part of the proof of Theorem 2 [1], we obtain the elements as shown in Fig. 4.

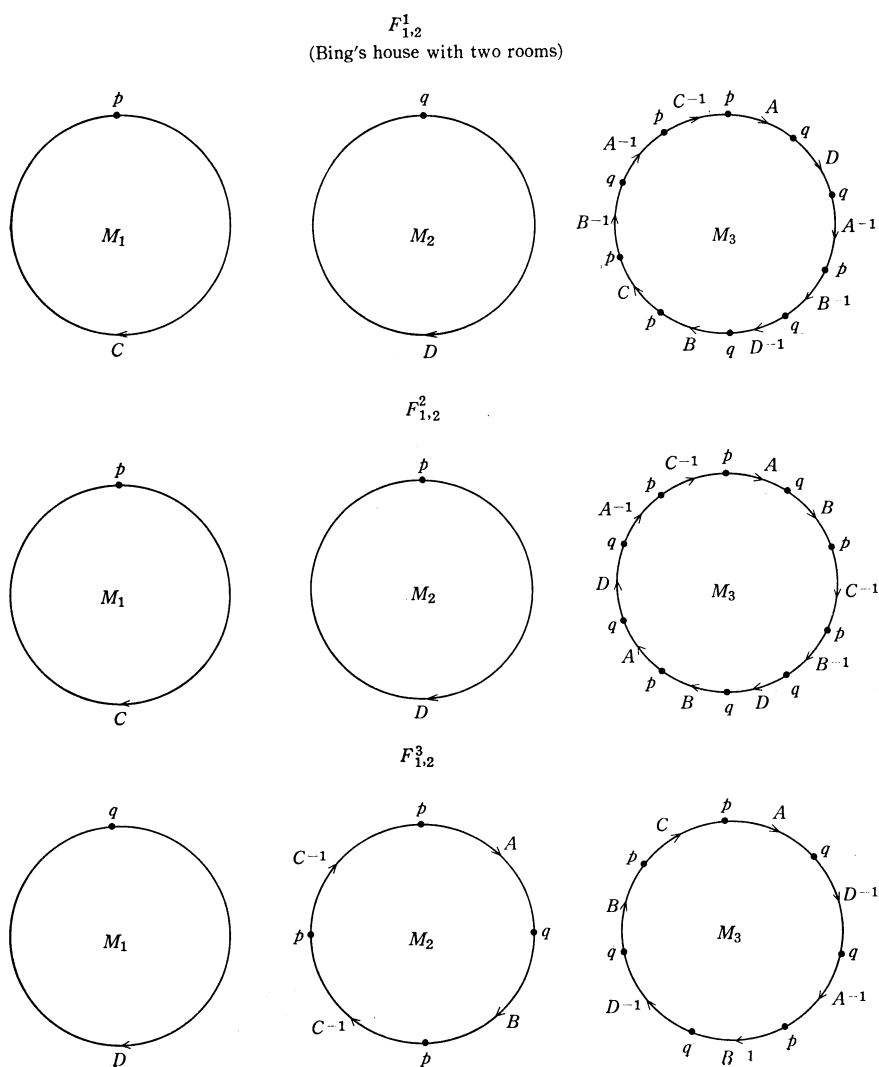


Fig. 4

REMARK. The element $F_{1,2}^1$ of $\mathcal{E}(1, 2)$ is well-known as “Bing’s House with two rooms”.

Theorem 3. *Zeeman’s conjecture holds true for any element P of $\mathcal{E}(s, 2)$, that is, $P \times I$ is collapsible.*

Proof. Case 1. When $s=3$, we see $P = F_{1,1}^1 \oplus F_{1,1}^1$ by Proposition 3, and hence, $P \times I$ is collapsible by Proposition 8 of [1].

Case 2. When $s=2$, we obtain $P = G \circ P$, from Proposition 4. Then, by the same way as Case 2 in the proof of Theorem 3 [1], $P \times I$ is collapsible, because $G - \hat{B}_G$ is collapsible.

Case 3. When $s=1$, $P \times I$ is collapsible by the same way as Case 1 in the proof of Theorem 3 [1], by attaching a 3-ball to M_1 (for M_1 , see Fig. 4).

6. $\mathcal{E}(s, t)$ with $1 \leq s \leq 2t-1$.

In this section, we characterize, geometrically, the elements of the sets $\mathcal{E}(2t-1, t)$ and $(2t-2, t)$ and prove the converse of Theorem 1.

Theorem 4. *Let P be an element of $\mathcal{E}(s, t)$ with $s=2t-1$ and $t \geq 2$. Then, we can write $P = P_1 \oplus P_2$ where P_i belongs to $\mathcal{E}(s_i, t_i)$ with $s_i = 2t_i - 1$, $t_1 + t_2 = t$ and $t_i \geq 1$, $i=1, 2$.*

Proof. The proof goes by induction on t . When $t=2$, Proposition 3 gives the answer. So, we assume $t \geq 3$. Since $s=2t-1$, we obtain $\nu(P)=t-1$, because

$$t-1 = s-t \leq \nu(P) \leq (s-1)/2 = t-1.$$

by Proposition 1. Hence $\nu(P) \geq 1$. Let U and Q be the isolated component of $U(P)$ and the connected component of $\overline{P-U}$ obtained in Lemma 7. Now, we show that Q is not acyclic. Suppose not. Then, $\hat{A} = A \cup (\hat{A}^*v)$ must be acyclic by Lemma 14 [1], where $A = \overline{P-Q}$. And we have $\nu(\hat{A}) = \nu(P)$ and $\#\mathcal{S}_2(\hat{A}) \leq s-1$, because, by Lemma 7, $\nu(Q)=0$ and $\#\mathcal{S}_3(Q) \neq 0$ implies $\#\mathcal{S}_2(Q) \neq 0$. Then, we obtain

$$\#\mathcal{S}_2(\hat{A}) \leq s-1 = 2t-2 = 2\nu(A)$$

which contradicts to Proposition 1, because \hat{A} is a normal spine by Lemma 2. Thus, Q is not acyclic and hence A is acyclic. Then, A collapses naturally to an acyclic normal spine A_1 from \hat{A} . Note that $U = S \times T$. And $\nu(A_1) = \nu(P) - 1$ is trivial. Then, we have $\#\mathcal{S}_2(A_1) = s-2$, because

$$\begin{aligned} s-2 &\geq \#\mathcal{S}_2(A_1) \geq 2\nu(A_1) + 1 \\ &= 2\nu(P) - 1 \\ &= 2t-3 \\ &= s-2. \end{aligned}$$

And we see $\#\mathcal{S}_3(A_1) \geq t-1$, because

$$t-2 = \nu(P)-1 = \nu(A_1) \geq s-2 - \#\mathcal{S}_3(A_1).$$

Since $\#\mathcal{S}_3(Q) \neq 0$ by Lemma 7, we obtain $\#\mathcal{S}_3(A_1) = t-1$. Therefore, A_1 is an element of $\mathcal{E}(s'_1, t'_1)$ with

$$s'_1 = s-2 = 2t-3 = 2(t-1)-1 = 2t'_1-1.$$

And consequently, we see $\#\mathcal{S}_2(Q) = 1 = \#\mathcal{S}_3(Q)$. Let S denote the base space of the T -bundle $U = S \times T$.

Case 1. Suppose that S bounds a 2-ball in $M(A_1)$. Let $\tilde{Q} = Q \cup (\dot{Q} * v)$. Then, \tilde{Q} belongs to $\mathcal{E}(1, 1)$. And it is easy to write $P = A_1 \oplus Q$ by identifying the 2-balls B and $(\dot{Q} * v)$. Putting $P_1 = A_1$ and $P_2 = Q$, the required conditions in Theorem 4 are satisfied.

Case 2. Suppose that S does not bound a 2-ball in $M(A_1)$. By the inductive hypothesis, we can write $A_1 = A_2 \oplus A_3$ with respect to the 2-balls B_2 and B_3 contained in $M(A_2)$ and $M(A_3)$, respectively, where A_i belongs to $\mathcal{E}(s'_i, t'_i)$ with $s'_i = 2t'_i - 1$, $t'_2 + t'_3 = t'_1$ and $t'_i \geq 1$, $i = 1, 2$. Since S does not bound a 2-ball in $M(A_1)$, S is contained in either $A_2 - B_2$ or $A_3 - B_3$, say $A_2 - B_2$. Let us define $P_1 = A_2 \cup U \cup Q$ and $P_2 = A_3$. Then, using the 2-balls B_2 and B_3 , we can write $P = P_1 \oplus P_2$. And it is clear that P_1 belongs to $\mathcal{E}(s'_2 + 2, t'_2 + 1)$. And hence, $s'_2 + 2 = (2t'_2 - 1) + 2 = 2(t'_2 + 1) - 1$. Thus, the required conditions in Theorem 4 are satisfied. And Theorem 4 is now established.

Corollary to Theorem 4. *For any element P of $\mathcal{E}(2t-1, t)$ with $t \geq 1$, the Zeeman's conjecture holds true, that is, $P \times I$ is collapsible.*

Proof. By Theorem 4, $\mathcal{E}(2t-1, t)$ is contained in \mathcal{C}_t defined in §9 [1], for any integer $t \geq 1$. Then, $P \times I$ is collapsible by Proposition 8 [1].

In order to characterize the elements of $\mathcal{E}(s, t)$ in the case $s = 2t - 2$, we extend the definition of the union of closed fake surfaces as follows.

DEFINITION 8. Let P_i be a closed fake surface with an acyclic fake surface A_i such that the boundary \dot{A}_i is a 1-sphere contained in $\mathring{M}(P_i)$ and A_i is a connected component of P disconnected by \dot{A}_i , $i = 1, 2$. Suppose that there is a homeomorphism f from A_1 onto A_2 . Define the union $P_1 \oplus_A P_2$ of P_1 and P_2 with respect to $A = A_1 = A_2$, and f by $P_1 \oplus_A P_2 = (P_1 \cup P_2)/f$.

Then, in general, we obtain the following.

Proposition 6. (1) *Let P be an element of $\mathcal{E}(s, t)$ with $\nu(P) \geq 1$. Then, there exists an acyclic fake surface A in P such that we can write $P = P_1 \oplus_A P_2$.*

(2) If we can write $P = P_1 \oplus_A P_2$ for an element P of $\mathcal{E}(s, t)$, we obtain the following conditions.

- (i) P_i belongs to $\mathcal{E}(s_i, t_i)$, $i = 1, 2$.
- (ii) $s_i \geq \#\mathfrak{S}_2(A) + 1$, $i = 1, 2$.
- (iii) $t_i \geq \#\mathfrak{S}_3(A) + 1$, $i = 1, 2$.
- (iv) $s_1 + s_2 - \#\mathfrak{S}_2(A) = s - 1$.
- (v) $t_1 + t_2 - \#\mathfrak{S}_3(A) = t$.

Proof. Since $\nu(P) \geq 1$, there exists an isolated component U in $U(P)$. And we see $U = S \times T$, because P belongs to $\mathcal{E}(s, t)$. Then, by Lemma 6, there exists an acyclic component A in $\overline{P - U}$, uniquely, and the other components than A of $\overline{P - U}$ are denoted by Q_1 and Q_2 . Note that $\#\mathfrak{S}_3(Q_i) \neq 0$ for $i = 1, 2$, because $\tilde{Q}_i = Q_i \cup (\tilde{Q}_i * v_i)$ is an acyclic normal spine and hence $\#\mathfrak{S}_3(Q_i) = \#\mathfrak{S}_3(\tilde{Q}_i) \neq 0$, by Theorem 1 [1]. Now, unpasting P at A , we obtain two closed fake surfaces P_1 and P_2 , and it is clear that P can be written $P = P_1 \oplus_A P_2$. This proves (1). And it is also clear that P_i is an acyclic normal spine for $i = 1, 2$, that is, P_i belongs to $\mathcal{E}(s_i, t_i)$, because both P and A are acyclic. We may assume $P_i \supset Q_i$, for $i = 1, 2$. Then, the conditions (ii) and (iii) are proved by $\mathfrak{S}_j(Q_i) \cup \mathfrak{S}_j(A) = \mathfrak{S}_j(P_i)$, for $i = 1, 2$, and $j = 2, 3$. The condition (iv) follows from the facts $\mathfrak{S}_2(P_1) \cup U \cup \mathfrak{S}_2(P_2) = \mathfrak{S}_2(P)$ and $\mathfrak{S}_2(A) \subset \mathfrak{S}_2(P_i)$, for both $i = 1, 2$. The last condition (v) is also satisfied by $\mathfrak{S}_3(P_1) \cup \mathfrak{S}_3(P_2) = \mathfrak{S}_3(Q_1) \cup \mathfrak{S}_3(Q_2) \cup \mathfrak{S}_3(A) = \mathfrak{S}_3(P)$ and $\mathfrak{S}_3(A) \subset \mathfrak{S}_3(P_i)$ for both $i = 1, 2$.

REMARK. Let G be the 1-st complement obtained in Lemma 11 and B_G the 2-ball in M_3 of G (see Remark to Lemma 11). From now on, $G - \mathring{B}_G$ is denoted by G_0 .

Theorem 5. Let P be an element of $\mathcal{E}(s, t)$ with $s = 2t - 2$ and $t \geq 3$. Then, we can write $P = P_1 \oplus_A P_2$ so that A is either a 2-ball or G_0 and P_i belongs to $\mathcal{E}(s_i, t_i)$, $i = 1, 2$. And if A is a 2-ball, we obtain $s_1 = 2t_1 - 1$ and $s_2 = 2t_2 - 2$. If A is G_0 , we obtain $s_i = 2t_i - 2$, for both $i = 1, 2$.

Proof. This theorem is also proved by induction on t by the similar argument to the proof of Theorem 4. However, the preparation is more complicated. In this case, we obtain $\nu(P) = t - 2$, because

$$t - 2 = s - t \leq \nu(P) \leq (s - 1)/2 < t - 1.$$

We can find an isolated component U in $U(P)$ and a connected component Q in $\overline{P - U}$ with $\nu(Q) = 0$ and $\#\mathfrak{S}_3(Q) \neq 0$, by Lemma 7.

Step 1. In this step, we study about Q .

Case 1. Suppose that Q is acyclic.

In this case, we show $\#\mathcal{S}_2(Q)=1=\#\mathcal{S}_3(Q)$ which implies $Q=G_0$, because $\tilde{Q}=\dot{Q}\cup(Q*v)$ is a 1-st complement.

Since Q is acyclic and $\nu(Q)=0$, we obtain $\tilde{F}=F\cup(\dot{F}*v)$ is a acyclic normal spine with $\nu(\tilde{F})=\nu(P)$, where $F=\overline{P-Q}$. Then, we see $\#\mathcal{S}_2(\tilde{F})=s-1$ and $\#\mathcal{S}_3(\tilde{F})=t-1$. because

$$\begin{aligned} s-1 &\geq \#\mathcal{S}_2(\tilde{F}) \geq 2\nu(\tilde{F})+1 \\ &= 2t-3 \\ &= s-1, \end{aligned}$$

and

$$\begin{aligned} t-1 &\geq \#\mathcal{S}_3(\tilde{F}) \geq (\#\mathcal{S}_2(\tilde{F})+1)/2 \\ &= t-1, \end{aligned}$$

by Proposition 1 and Theorem 1. Hence, we obtain the required condition $\#\mathcal{S}_2(Q)=1=\#\mathcal{S}_3(Q)$, because

$$\#\mathcal{S}_j(\tilde{F})+\#\mathcal{S}_j(Q)=\#\mathcal{S}_j(P)$$

is true for $j=2, 3$.

Case 2. Suppose that Q is not acyclic.

In this case, F is acyclic and hence we obtain an acyclic normal spine F_1 from F by a natural collapsing. And we have $\nu(F_1)=\nu(P)-1=t-3$. Then, by the similar arargument to the proof of Theorem 4 and Case 1 in this step, we can prove that the pair $(\#\mathcal{S}_2(F_1), \#\mathcal{S}_3(F_1))$ is either $(s-2, t-1)$ or $(s-3, t-2)$. Thus, we obtain the following statement (*).

(*) $(\#\mathcal{S}_2(Q), \#\mathcal{S}_3(Q))=(k, k)$ if and only if $(\#\mathcal{S}_2(F_1), \#\mathcal{S}_3(F_1))=(s-1-k, t-k)$, for $k=1, 2$.

Step 2. Suppose $t=3$ (the 1-st step of induction).

Then, $\nu(P)=t-2=1$. Then, by Proposition 6, we can write $P=P_1\underset{A}{\oplus}P_2$.

Since $\nu(P)=1$ implies $\nu(P_i)=0$ for both $i=1, 2$, we obtain the two possibility. That is, if $\#\mathcal{S}_3(A)=0$, then A is a 2-ball by Lemma 4 or Lemma 5. And hence P_i belongs to $\mathcal{E}(i, i)$ for $i=1, 2$. And if $\#\mathcal{S}_3(A)\neq 0$, we see $A=G_0$ by Step 1 (Case 1), because $\nu(A)=0$. Hence, we can write $P=P_1\underset{G_0}{\oplus}P_2$, and P_i belongs to $\mathcal{E}(2, 2)$ for both $i=1, 2$, by Proposition 6.

Step 3. We deal with the case $t\geq 4$.

Case 1. Suppose that Q is acyclic.

In this case, take $A=Q$. Then, $Q=G_0$ by Case 1 of Step 1, and hence, $P=P_1\underset{G_0}{\oplus}P_2$ and P_i belongs to $\mathcal{E}(s_i, t_i)$, $i=1, 2$. By Proposition 6, we obtain $s_1+s_2=s$ and $s_i\geq 2$ and $t_1+t_2=t+1$ and $t_i\geq 2$. Put $s_i=2t_i-u_i$, $i=1, 2$. Then, we obtain $u_1+u_2=4$, because

$$\begin{aligned}
2t - (u_1 + u_2 - 2) &= (2t_1 - u_1) + (2t_2 - u_2) \\
&= s_1 + s_2 \\
&= s \\
&= 2t - 2.
\end{aligned}$$

Since $u_i \geq 1$ by Theorem 1, for both $i=1, 2$, we see that the pair (u_1, u_2) is either $(1, 3)$ or $(2, 2)$. Suppose $u_1=1$. Then, P_1 must be an element of $\mathcal{E}(2t_1-1, t_1)$. But, for any integer $t_1 \geq 1$, it is clear, from Theorem 4, that no element of $\mathcal{E}(2t_1-1, t_1)$ contains G_0 as a subpolyhedron. Thus, (u_1, u_2) must be $(2, 2)$, and hence $s_i = 2t_i - 2$ for both $i=1, 2$. This completes the proof of this case.

Case 2. Suppose that Q is not acyclic.

In this case, the construction of P_1 and P_2 highly resembles to the last Case 2 in the proof of Theorem 4. We use the statement (*) in Case 2 in Step 1. When $k=1$, we can write $F_1 = F_2 \oplus_{\mathcal{A}} F_3$ by the inductive hypothesis. And if $k=2$, we can write $F_1 = F_2 \oplus F_3$ by Theorem 4. And we obtain P_1 and P_2 as required in Theorem 5.

Thus, Theorem 5 is established.

When we define the set \mathcal{C} of acyclic normal spines obtained from $\mathcal{E}(1, 1)$ and $\mathcal{E}(2, 2)$ using $P_1 \oplus_{\sigma_0} P_2$ and $P_1 \oplus P_2$ as the set \mathcal{C}_t defined in §9 in [1], we have the following proposition by the similar reason to that of Proposition 8 [1].

Proposition 7. *Let P be an element of \mathcal{C} . Then, $P \times I$ is collapsible.*

And we have the following as a corollary to Theorem 5, because $\mathcal{E}(2t-2, t)$ is contained in \mathcal{C} by Theorem 5.

Corollary to Theorem 5. *For any element P of $\mathcal{E}(2t-2, t)$ with $t \geq 2$, the Zeeman's conjecture is true, that is, $P \times I$ is collapsible.*

We prepare the following lemmas to prove Theorem 6.

Lemma 12. *$\mathcal{E}(1, t)$ contains a spine of a 3-ball, for any integer $t \geq 1$.*

Proof. Suppose that t is odd, that is, $t=2r+1$. When $r=0$, there is nothing to prove, because the unique element $F_{1,1}^1$ (abalone) of $\mathcal{E}(1, 1)$ is a spine of a 3-ball by Theorems 3 and 4 [1]. We construct a normal spine of a 3-ball in $\mathcal{E}(1, t)$ inductively. Let P be an element of $\mathcal{E}(1, 2(r-1)+1)$ which is a spine of a 3-ball V . Then, we can apply an elementary deformation of type I to P in V , and we obtain a normal spine $P(1)$ of V , by Lemma 8. Then, by Lemma 9, it is clear that $P(1)$ belongs to $\mathcal{E}(1, t)$. When t is even, we obtain a spine of a 3-ball in $\mathcal{E}(1, t)$ by the same way as above from an element of $\mathcal{E}(1, 2)$ which is non-empty by Proposition 5 and it is known, by Theorem 3, that any

element of $\mathcal{E}(1, 2)$ is a spine of a 3-ball.

Lemma 13. *Suppose that G_0 is embedded in a 3-ball V properly, that is, $G_0 \cap \dot{V} = \dot{G}_0$. Then, V collapses to G_0 .*

Proof. Let N be the regular neighborhood of G_0 in V meeting the boundary regularly, that is, $N \cap \dot{V}$ is a regular neighborhood of \dot{G}_0 in \dot{V} . Since G_0 is collapsible and \dot{G}_0 is a 1-sphere, N is a 3-ball and $N \cap \dot{V}$ is an annulus. Then, $\overline{V-N}$ is the disjoint union of two 3-balls V_1 and V_2 . And, clearly, $N \cap V_i = \dot{N} \cap \dot{V}_i = F_i$ is a 2-ball for $i=1, 2$. Then, V collapses to N by collapsing each V_i to F_i and N collapses to G_0 . Thus, V collapses to G_0 .

Lemma 14. *Let P be a normal spine of a 3-manifold W , that is, W collapses to P . Then, $G \circ P$ is also a spine of W , where the connected sum is taken with respect to B_G .*

Proof. Let B_P be the 2-ball of P used in the connected sum $G \circ P$, and let N be the 2-nd derived neighborhood of B_P in $W \bmod \dot{B}_P$. Note that we can expand P to $P \cup N$ in W . It is possible to replace B_P by G_0 in N to satisfy $G_0 \cap \dot{N} = \dot{G}_0 = \dot{B}_P$, because N is a 3-ball and \dot{G}_0 and \dot{B}_P are 1-spheres. Then, by Lemma 13, N collapses to G_0 , and hence $P \cup N$ collapses to $(P - B_P) \cup G_0$ which is clearly $G \circ P$. Thus, $G \circ P$ is a spine of W .

Theorem 6. *$\mathcal{E}(s, t)$ contains a spine of a 3-ball for any pair (s, t) with $1 \leq s \leq 2t-1$.*

Proof. By Lemma 12 and Corollary to Theorem 4, each of $\mathcal{E}(1, t)$ and $\mathcal{E}(2t-1, t)$ contains a spine of a 3-ball for any integer $t \geq 1$. So, assuming $2 \leq s \leq 2t-2$, we construct a spine Q of a 3-ball in $\mathcal{E}(s, t)$ inductively. Suppose that P is a spine of a 3-ball in $\mathcal{E}(s-1, t-1)$. Define $Q = G \circ P$. Then, by Lemma 14, Q is also a spine of a 3-ball and clearly Q belongs to $\mathcal{E}(s, t)$.

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