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<th>Acyclic fake surfaces which are spines of 3-manifolds</th>
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1. Introduction

In [1], we defined fake surfaces to study 3–manifolds with boundary from their spines. Let \( \mathcal{F}(s, t) \) denote the set of all the acyclic closed fake surfaces \( P \) with \( \#\mathcal{E}_3(P) = s \) and \( \#\mathcal{F}_3(P) = t \) (\# means the number of the connected components). In this paper, we consider about the subset \( \mathcal{E}(s, t) \) of \( \mathcal{F}(s, t) \) each of whose elements can be embedded in some 3–manifold.

A connected closed fake surface \( P \) is called a normal spine, if \( P \) can be embedded in a 3–manifold. That is, taking the regular neighborhood, we can regard \( P \) as a spine of a 3–manifold, when \( P \) is a normal spine. Of course, every element of \( \mathcal{E}(s, t) \) is a normal spine.

We use the following notations. For a polyhedron \( P \), \( \bar{P} \) means the boundary of \( P \), that is, \( \bar{P} \) is the union of the free faces of \( P \), and \( P \) means the interior of \( P \) defined by \( P = P - \bar{P} \). \( \bar{P} \) means the closure of \( P \), and \( I \) is the closed unit interval \([0, 1]\). For the other unexplained notations, see [1].

In §2, we prepare some lemmas for acyclic normal spines by defining the connected sum of closed fake surfaces and the \( r \)-th complement. In §3, we obtain the sufficient condition that \( \mathcal{E}(s, t) \) is empty, that is, Theorem 1 states that \( \mathcal{E}(s, t) \) is empty if \( s \geq 2t \), (and, in the last section, we show that this is also the necessary condition). In §4, two types of elementary deformation of normal spines in the respective 3–manifolds are introduced and two invariants \( \alpha(P) \) and \( \beta(P) \) are defined for a closed fake surface \( P \). And, in Theorem 2, we prove \( \alpha(P) = r = \beta(P) \) when \( P \) is a \( r \)-th complement. In §5, all the elements of the set \( \mathcal{E}(s, 2) \) are characterized geometrically using the concept of the union of closed fake surfaces, from which the Zeeman’s conjecture is shown to be true for any element of \( \mathcal{E}(s, 2) \), easily.

Zeeman’s conjecture [2]: If \( P \) is a contractible 2–polyhedron, then \( P \times I \) is collapsible where \( I = [0, 1] \) is the closed unit interval.

In the last section, we obtain the geometrical characterizations of the elements of \( \mathcal{E}(2t-1, t) \) and \( \mathcal{E}(2t-2, t) \) for all integers \( t \geq 1 \) and \( t \geq 2 \), respectively. And, as the consequences, the Zeeman’s conjecture for them follows.
Furthermore, in Theorem 6, we show that $\mathcal{E}(s, t)$ contains a spine of a 3-ball for any pair $(s, t)$ with $1 \leq s \leq 2t - 1$. Combining this with Theorem 1, we obtain the following.

**Theorem.** $\mathcal{E}(s, t)$ is empty if and only if $s \geq 2t$.

On the other hand, it is easily seen that $\mathcal{F}(s, t)$ is empty if and only if $t = 0$. The sufficiency follows from Theorem 1 [1]. To show the necessity, replace a 2-ball $B$ in $\mathcal{M}(P)$ of an element $P$ of $\mathcal{E}(2t - 1, t)$ by the element $\mathcal{N}_{s - 2t + 1}$ so that $\hat{B} = \mathcal{N}_{s - 2t + 1}$ (for the definition of $\mathcal{N}_{s - 2t + 1}$, see Definition 6, §6, [1]).

Note that $\mathcal{E}(1, 1)$ consists of a unique element $F_{1,1}$ by Theorem 4 [1] which is named “Abalone” by H. Noguchi and the realization of an abalone in the Euclidean 3-space $\mathbb{R}^3$ is written in Figure 0 which is shown by Y. Tsukui.

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2. Lemmas

Definition 1. Let $P_i$ be a closed fake surface with a 2-ball $B_i$ in $\bar{M}(P)$, $i=1, 2$, and $f$ a homeomorphism from $\bar{B}_i$ to $\bar{B}_2$. We define the connected sum $P_1 \circ P_2$ of $P_1$ and $P_2$ with respect to $B_1, B_2$ and $f$ by $P_1 \circ P_2 = ((P_1 - \bar{B}_1) \cup (P_2 - \bar{B}_2))/f$.

Definition 2. First, define the 0-th complement to be an acyclic normal spine. A connected closed fake surface $X$ is said to be a $r$-th complement if there exists an acyclic fake surface $P$ such that $X \circ P$ is a $(r-1)$-th complement.

Definition 3. Let $P$ be a fake surface. We say that a connected component $U$ of $U(P)$ is isolated if $\emptyset_3(U)$ is empty. And let $\nu(P)$ denote the number of the isolated components of $U(P)$.

Lemma 1. Let $P$ be a closed fake surface. If $U(P)$ is embeddable in an orientable 3–manifold, $P$ is a normal spine.

Proof. Let $W$ be an orientable 3–manifold in which $U(P)$ is embedded, and let $M$ be an element of $M(P)$ with boundary $\bar{M} = \bar{b}_1 \cup \cdots \cup \bar{b}_j$. Let us consider $M \times I$ and $A_i = b_i \times I$ where $I$ denote the closed unit interval $[0, 1]$ and $M=M \times 1/2$, and the 2-nd derived neighborhood $N_i$ of $b_i$ in the boundary of the regular neighborhood $N$ of $U(P)$ in $W$ mod $\bar{U}(P)$, $i=1, \ldots, j$. Since $\bar{N}$ is a disjoint union of orientable closed 2–manifolds, there is a homeomorphism $f_i$ from $A_i$ onto $N_i$ which is the identity on $b_i$. Then, we obtain a homeomorphism $h_M$ from $\bigcup A_i = \bar{M} \times I$ onto $\bigcup N_i$ defined by $f_i$ on each $A_i$. Define the 3–manifold

$$V = \bigcup_{\mathcal{K}}((N \cup (M \times I))/h_M),$$

that is, $V$ is the 3–manifold obtained from $N$ and $M(P) \times I$ by identifying $A_i$ and $N_i$ by $f_i$ for all $i=1, \ldots, j$ and for all elements $M$ of $M(P)$. Obviously, $P$ is embedded in the 3–manifold $V$, completing the proof.

Lemma 2. Let $P$ be a closed fake surface with $H_1(P) = 0$. Then, $P$ is a normal spine if and only if $U(P)$ can be embedded in $R^3$, the Euclidean 3–space.

Proof. Sufficiency follows immediately from Lemma 1. So, we prove Necessity. Let $W$ be a 3–manifold in which $P$ is embedded. Since $W$ is orientable and $U(P)$ collapses to the 1–polyhedron $\emptyset_3(P)$, the regular neighborhood $N$ of $U(P)$ in $W$ is a disjoint union of solid tori with certain genus. Then, $N$ is embeddable in $R^3$, and hence, so is the subpolyhedron $U(P)$.

Lemma 3. (i) Let $X$ be a $r$-th complement. Then, we have $H_1(X) = 0$ and $H_2(X) = Z + \cdots + Z$ of rank $r$. 

(ii) A r-th complement $X$ is a normal spine.

(iii) Let $X = X_i \circ X_2$ be a r-th complement. Then, $X_i$ is a $r_i$-th complement for $i = 1, 2$, and $r_1 + r_2 = r + 1$.

Proof. The proof goes by induction on $r$. When $r = 0$, there is nothing to prove (i) and (ii). So, we prove (iii). By Lemma 14 [1], we may assume that $X_1$ is acyclic. Then, $X_2$ is a 1-st complement from the definition. Since $X$ is a normal spine, $X_1$ is also a normal spine, by Lemma 2, because $U(X_1)$ is contained in $U(X)$ and is embeddable in $R^3$. Thus, $X_1$ is a 0-th complement. Now, we consider the case $r \geq 1$. That is, there is an acyclic closed fake surface $P$ such that $X \circ P$ is a $(r - 1)$-th complement, where the connected sum is taken with respect to the 2-balls $B_X$ and $B_P$ contained in $M(X)$ and $M(P)$ and a homeomorphism $f$ from $B_X$ to $B_P$. Define $Q = (X \circ P) \cup (B_P \ast v)$ where $v$ is an ideal coing point over $B_P$, that is, $(B_P \ast v)$ is the cone from $v$ over $B_P$ and $(X \circ P) \cap (B_P \ast v) = B_P$. Using the inductive hypothesis $H_1(X \circ P) = 0$ and $H_2(X \circ P) = Z + \cdots + Z$ of rank $r - 1$, we obtain $H_1(Q) = 0$ and $H_2(Q) = Z + \cdots + Z$ of rank $r$ by the Mayer-Vietoris exact sequence. Since $H_3(Q) = H_3(X) + H_3(P)$ and $P$ is acyclic, we see $H_1(X) = 0$ and $H_2(X) = Z + \cdots + Z$ of rank $r$. This proves (i). By the inductive hypothesis, $U(X \circ P) = U(X) \cup U(P)$ can be embedded in $R^3$. Then $U(X)$ is, of course, embeddable in $R^3$, and hence, by Lemma 2, $X$ is a normal spine. This proves (ii). Now, we may assume that the 2-ball $B_X$ is contained in $X_1$, because $B_X$ can be moved away from $X_2$ when $B_X \cap (X_1 \cap X_2)$ is non-empty by an isotopy of $X$. Then, we can write $X \circ P = (X_1 \circ P) \circ X_2$. Then, by the inductive hypothesis, $(X_1 \circ P)$ is a $r'$-th complement and $X_2$ a $r_2$-th one and $r' + r_2 = r$. Then, $X_1$ is a $(r' + 1)$-th complement, because $P$ is acyclic. Thus, we have $r_1 = r' + 1$, and hence $r_1 + r_2 = r + 1$. This completes the proof of Lemma 3.

Lemma 4. Let $P$ be a normal spine with $H_1(P) = 0$ and $H_2(P) = Z$. Then, $\Theta_s(P)$ is empty if and only if $P$ is a 2-sphere.

Proof. Sufficiency is trivial. We prove Necessity. It is clear that a 2-sphere satisfies the required conditions and the other 2-manifolds do not. Hence Lemma 4 is true if $P$ is a 2-manifold. So, we assume that $\Theta_s(P)$ is non-empty and try to prove that such $P$ does not exist. Let $U(P) = U_1 \cup \cdots \cup U_n$, where $U_i$ means a connected component of $U(P)$ for $i = 1, \ldots, n$. Then, each $U_i$ must be isolated, because $\Theta_s(P)$ is empty. And since $P$ is a normal spine with $H_1(P) = 0$, $U_i$ is neither $S \times \tau T$ nor $S \times \sigma T$, by Lemma 24 [1], Lemma 2 and Corollary to Theorem 1[1]. That is, $U_i = S \times T$ for any $i = 1, \ldots, n$. The proof goes by induction on $n$. When $n = 1$, $M(P)$ consists of three 2-balls by Lemma 12 [1] and Proposition 4 [1], and $P$ is obtained from $M(P)$ by identifying their boundaries as indicated in Figure 1.
Then, we have $H_2(P) = \mathbb{Z} + \mathbb{Z}$ which contradicts to our hypothesis $H_2(P) = \mathbb{Z}$. Now, we deal with the case $n \geq 2$. Then, there is an element $M$ with $\# M \geq 2$ in $M(P)$ by Lemma 14 [1], and a boundary component $b$ of $M$ disconnects $P$ into two fake surfaces $P_1$ and $P_2$ such that $\iota_2(P_i)$ is non-empty for both $i = 1, 2$, by Lemma 14 of [1]. Let $\bar{P} = P \cup (b* v)$ and $\bar{P}_i = P_i \cup (b* v)$, $i = 1, 2$, where $v$ is an ideal coning point over $b$. Then, by the Mayer Vietoris exact sequence, we obtain $H_1(\bar{P}) = 0$ and $H_2(\bar{P}) = \mathbb{Z} + \mathbb{Z}$, and hence $H_2(\bar{P}_i) = 0$ for both $i = 1, 2$, and $H_2(\bar{P}_1) + H_2(\bar{P}_2) = \mathbb{Z} + \mathbb{Z}$. Suppose $H_2(\bar{P}_i) = 0$. Then, $\bar{P}_i$ is an acyclic closed fake surface without 3-rd singularity, which is a contradiction to Theorem 1 [1]. Thus, we see $H_2(P_i) = \mathbb{Z}$ for both $i = 1, 2$. Since $P$ is a normal spine, $P_i$ is also a normal spine by Lemma 2. And, clearly, $1 \leq \#U(\bar{P}_i) \leq n - 1$ holds true, because $\iota_2(P_i)$ is non-empty. This contradicts to our inductive hypothesis, competing the proof.

**Remark.** It is easy to see that a 2-sphere $S^2$ is a 1-st complement, because $S^2 \cup F_1^1$ is homeomorphic to $F_1^1$.

**Lemma 5.** Let $P = P_1 \circ P_2$ be an element of $\mathcal{E}(s, t)$. Suppose that $P_1$ is not acyclic. Then, $P_1$ is either a 2-sphere or a 1-st complement with $1 \leq \# \iota_2(P_1) \leq t - 1$.

**Proof.** By Lemma 14 [1], $P_2$ is acyclic, and hence $\# \iota_2(P_2) \geq 1$, by Theorem 1 [1]. Then, $P_1$ is a 1-st complement. Suppose $\# \iota_2(P_1) = 0$. Then, by Lemma 4, $P_1$ is a 2-sphere. And when $1 \leq \# \iota_2(P_1)$, we see $\# \iota_2(P_1) \leq t - 1$, because $\# \iota_2(P_1) \geq 1$ and $\# \iota_1(P_1) + \# \iota_2(P_2) = t$.

**Lemma 6.** Let $P$ be an element of $\mathcal{E}(s, t)$ with an isolated component $U = S \times T$. Then, just one of the connected components of $P - U$ is acyclic.

**Proof.** By Lemma 13 [1], $P - U$ is the disjoint union of three connected fake surfaces $P_1$, $P_2$ and $P_3$. First, we show that at least one of $P_1$, $P_2$ and $P_3$ is acyclic. Suppose that $P_3$ is not acyclic. Then, by Lemma 14 [1], we obtain an acyclic fake surface $P_0 = P_1 \cup U \cup P_2$. Since $U = S \times T$, we obtain an acyclic closed fake surface $Q$ from $P_0$ by collapsing $P_0$ from its boundary $\bar{P}_0$ by the
natural way. And the 1-sphere $\mathbb{S}_1(U)$ disconnects $Q$ into two fake surfaces $Q_1$ and $Q_2$ so that $P_i$ is contained in $Q_i$, for $i=1,2$. Note that $P_i$ is homeomorphic to $Q_i$, $i=1,2$. Then, by the Mayer-Vietoris exact sequence, we obtain $H_2(Q_i)=0$ for both $i=1,2$, and $H_i(Q_1)+H_i(Q_2)=Z$. Hence, either $Q_1$ or $Q_2$ is acyclic, that is, either $P_1$ or $P_2$ is acyclic. Suppose that there are two acyclic components $P_1$ and $P_2$. Define $P_0=P_1 \cup U \cup P_2$. Then, we easily have $H_2(P_0)=0$ and $H_2(P_0)=Z$ which implies $H_2(P)=0$. This proves Lemma 6.

**Lemma 7.** Let $P$ be an element of $\mathcal{E}(s,t)$ with $\nu(P)\geq 1$. Then, there is an isolated component $U$ in $U(P)$ such that there exists a connected component $Q$ of $P-U$ with $\nu(Q)=0$ and $\# \mathbb{S}_3(Q)\neq 0$.

**Proof.** Let $U_i$ be an isolated component of $U(P)$. Then, $U_i=S \times T$ by the same reason as in the proof of Lemma 4. And hence $P-U_i$ has three connected components $P_{i1},P_{i2}$ and $P_{i3}$. By Lemma 6, we assume that $P_{i3}$ is acyclic. Then, of course, $P_{i3}$ is not acyclic, for $j=1,2$. If we consider $\tilde{P}_{ij}=P_{ij}\cup (\hat{P}_{ij} * v_{ij})$, we see that $\tilde{P}_{ij}$ is acyclic, for $j=1,2$, by Lemma 14 [1]. And $\# \mathbb{S}_3(\tilde{P}_{ij})=\# \mathbb{S}_3(\hat{P}_{ij})\neq 0$, by Theorem 1 [1], for $j=1,2$. Now, it is sufficient to prove the following statement (*) by induction on $\nu=\nu(P)$.

(*) Either (1) $U$ is a required isolated component $U$ in $U(P)$, or (2) we can find $U$ in $P_{i1}$, holds true.

**Proof of (*).** When $\nu=0$, there is nothing to prove by taking $U=U_i$ and $Q=P_{i1}$. So, we assume that (*) is true for $\nu(P_{i1})\leq \nu-1$, and we deal with the case $\nu\geq 1$. Let $v_k$ be an isolated component of $U(P)$ contained in $P_{i1}$. Then, either $P_{k1}$ or $P_{k2}$ is contained in $P_{i1}$, say $P_{k1}$. Then, (*) is true for $P_{k1}$, by the inductive hypothesis, because

$$\nu(P_{k1})\leq \nu(P_{i1})-1=\nu-1.$$ 

Then, clearly, $U$ is contained in $P_{i1}$, completing the proof.

3. The sufficient condition that $\mathcal{E}(s,t)$ be empty

**Proposition 1.** Let $P$ be an element of $\mathcal{E}(s,t)$. Then, we obtain $s\geq 2 \nu(P)+1$.

**Proof.** The proof goes by induction on $s$. We see $s\geq 1$ by Theorem 1 [1], and when $s=1$, there is nothing to prove, because $\nu(P)=0$ by Theorem 1 [1] again. We deal with the case $s\geq 2$. If $U(P)$ contains no isolated component, that is, $\nu(P)=0$, Proposition 1 is trivially true for $P$. Thus, we may assume that there exist an isolated component $U$ and a connected component $Q$ of $P-U$ with $\nu(Q)=0$ and $\# \mathbb{S}_3(Q)\neq 0$ obtained in Lemma 7. Let us consider $X=P-Q$, $Y=X \cup (X * v)$ and $W=Q \cup (Q * v)$ where $v$ is an ideal coning point over the
1-sphere \( \hat{X} = \hat{Q} \). Then, we can write \( P = W \circ Y \), by identifying the 2-balls \((\hat{X} \ast v)\) and \((Q \ast v)\). And, by Lemma 3, there are following two cases.

**Case 1.** \( W \) is a 0-th complement and \( Y \) is a 1-st one.

By Lemma 14 [1], \( X \) must be acyclic, and hence we can collapse \( X \) to an acyclic closed fake surface \( X' \) from \( \hat{X} \) by the natural way, because \( U = S \times T \). Then, \( X' \) is also a normal spine by Lemma 2, and we easily have \( 1 \leq \#S_3(X') = s' \leq s-1 \), because \( X' \) is acyclic and does not contain \( U \). Hence, we have \( s' \geq 2v(X') + 1 \), by the inductive hypothesis. Put \( s'' = \#S_3(W) \). Then, we see \( s = s' + s'' + 1 \) and \( v(P) = v(X') + 1 \). Hence,

\[
s - 2v(P) = (s' - s'' + 1) - 2(v(X') + 1) = (s' - 2v(X')) + (s'' - 1) \geq 1,
\]

because \( \#S_3(W) - \#S_3(Q) \neq 0 \) means \( s'' \neq 0 \). Therefore, we obtain \( s' \geq 2v(P) + 1 \).

**Case 2.** \( W \) is a 1-st complement and \( Y \) is a 0-th one.

In this case, we see \( 1 \leq \#S_3(Y) = s_1 \leq s-1 \), by the condition \( s' \neq 0 \). Then, by the inductive hypothesis, we obtain \( s_1 \geq 2v(Y) + 1 \), because \( Y \) is an acyclic normal spine by Lemma 2. And, in this case, we see \( s = s_1 + s'' \) and \( v(P) = v(Y) \) from which \( s \geq 2v(P) + 1 \) follows by a similar calculation to Case 1. Thus, Proposition 1 is established.

**Theorem 1.** \( \mathcal{E}(s, t) \) is empty if \( s \geq 2t \).

Proof. Suppose that \( \mathcal{E}(s, t) \) is non-empty. And let \( P \) be an element of \( \mathcal{E}(s, t) \). Then, we have

\[
s \geq 2v(P) + 1 \geq 2(s - t) + 1
\]

from Proposition 1. Hence \( s \leq 2t - 1 \). This proves Theorem 1.

**4. Elementary deformations of normal spines in the 3-manifolds**

Let \( P \) be a normal spine in a 3-manifold \( V \) with nonempty 2-nd singularity, i.e. \( \hat{S}_2(P) \neq \emptyset \). Suppose that there is a 1-ball \( A \) in \( P \) satisfying the following conditions (1) and (2).

(1) \( A \cap \hat{S}_2(P) = \hat{A} = a_1 \cup a_2 \).

(2) \( a_1 \) and \( a_2 \) are vertices of \( \hat{S}_2(P) - \hat{S}_3(P) \).

Taking the 2-nd derived neighborhood \( N \) of \( A \) in \( V \), \( \hat{N} - (\hat{N} \cap P) \) consists of four open 2-balls each of whose closures is a 2-ball \( B_i, \ i = 1, \ldots, 4 \). Let \( B_i \) be the 2-ball contained in \( st(a_i, V) \). Note that such a 2-ball is uniquely determined (see Figure 2). Then, we may regard the 3-ball \( N = B_1 \times I \) and hence we can collapse \( N \) to \( \hat{N} - B_1 \) from the free face \( B_1 = B_1 \times 0 \).
DEFINITION 4. Define the normal spine $P(1)$ by

$$P(1) = (P - (P \cap N)) \cup (\tilde{N} - \tilde{B}_1),$$

and we say that $P(1)$ is obtained from $P$ by an elementary deformation in $V$ (with respect to $A$). Inductively, we can define $P(r)$ as a normal spine obtained from $P(r-1)$ by an elementary deformation in $V$, and we say that $P(r)$ is obtained from $P$ by $r$ times of elementary deformation in $V$.

DEFINITION 5. An elementary deformation is said to be of type I, if the boundary $\tilde{A}$ is contained in a connected component of $\partial_2(P)$, and of type II otherwise.

DEFINITION 6. Let $P$ be a closed fake surface. We define the invariants $\alpha(P)$ and $\beta(P)$ by

$$\alpha(P) = \#M(P) - \#\mathcal{G}_2(P) - \#\mathcal{G}_3(P),$$

and

$$\beta(P) = \#\tilde{M}(P) - 2\#\mathcal{G}_2(P) - \#\mathcal{G}_3(P) + 1.$$ 

Lemma 8. Let $P$ be a normal spine of a 3-manifold $V$ and $P(r)$ a normal spine obtained from $P$ by $r$ times of elementary deformation in $V$. Then, $P(r)$ is also a spine of $V$.

Proof. From the definition of $P(r)$, it is sufficient to prove that $P$ and $P(1)$ are simple homotopy equivalent in $V$. Let $N$ be the 2-nd derived neighborhood of $A$ in $V$ in the above definition. Then, $P$ expands to $P \cup N$ and $P \cup N$ collapses to $P(1)$ in $V$, and hence $P$ and $P(1)$ are simple homotopy equivalent in $V$. 

Fig. 2
The following two lemmas are immediate from Figure 2.

**Lemma 9.** Let $P$ be a normal spine in a 3-manifold $V$ and $P(r)$ a normal spine obtained from $P$ by $r$ times of elementary deformation of type I in $V$. Then, we have:

1. $\#\mathfrak{S}_3(P(r)) = \#\mathfrak{S}_3(P)$, and
2. $\#\mathfrak{S}_3(P(r)) = \#\mathfrak{S}_3(P) + 2r$.

**Lemma 10.** Let $P$ be a normal spine in a 3-manifold $V$ and $P(r)$ a normal spine obtained from $P$ by $r$ times of elementary deformation of type II in $V$. Then, we have:

1. $\#\mathfrak{S}_3(P(r)) = \#\mathfrak{S}_3(P) - r$,
2. $\#\mathfrak{S}_3(P(r)) = \#\mathfrak{S}_3(P) + 2r$,
3. $\#M(P(r)) = \#M(P) + r$, and
4. $\#\hat{M}(P(r)) = \#\hat{M}(P)$.

**Proposition 2.** Let $P$ be an element of $\mathcal{E}(s, t)$. Then, we obtain $\alpha(P) = 0 = \beta(P)$.

Proof. The proof is done by induction on $s$. When $s=1$, Proposition 4 and Proposition 5 [1] give the answer. Suppose $s \geq 2$. Since $P$ is connected, we can apply an elementary deformation of type II to $P$ in some 3-manifold, and we obtain $P(1)$ which belongs to $\mathcal{E}(s-1, t+2)$ by Lemma 10. Then, by the inductive hypothesis and Lemma 10, we have

$$\alpha(P) = \#M(P) - \#\mathfrak{S}_3(P) - \#\mathfrak{S}_3(P)$$

$$= (\#M(P(1)) - 1) - s - t$$

$$= ((s - 1) + (t + 2) - 1) - s - t$$

$$= 0.$$ 

And, by the same way, we can prove $\beta(P) = 0$.

**Theorem 2.** Let $X$ be an $r$-th complement. Then, we obtain $\alpha(X) = r = \beta(X)$.

Proof. The proof is done by induction on $r$. When $r=0$, Proposition 2 gives the answer. We assume $r \geq 1$. Let $P$ be an acyclic fake surface (closed) such that $X \circ P$ becomes an $(r-1)$-th complement. Note that $P$ is necessarily a 0-th complement. Clearly, the followings hold true.

$$\#\mathfrak{S}_3(X \circ P) = \#\mathfrak{S}_3(X) + \#\mathfrak{S}_3(P),$$

$$\#\mathfrak{S}_3(X \circ P) = \#\mathfrak{S}_3(X) + \#\mathfrak{S}_3(P),$$

$$\#M(X \circ P) = \#M(X) + \#M(P) - 1,$$

$$\#\hat{M}(X \circ P) = \#\hat{M}(X) + \#\hat{M}(P).$$
Then, we have \( \alpha(X \circ P) = \alpha(X) + \alpha(P) - 1 \) and \( \beta(X \circ P) = \beta(X) + \beta(P) - 1 \). Thus, by the inductive hypothesis and Proposition 1 which means \( \alpha(P) = 0 = \beta(P) \), we easily obtain \( \alpha(X) = r = \beta(X) \).

5. \( \mathcal{E}(s, f) \).

Definition 7. Let \( P_i \) be a closed fake surface with a 2-ball \( B_i \) in \( \hat{M}(P_i) \), \( i = 1, 2 \), and let \( f \) be a homeomorphism from \( B_i \) onto \( B_i \). We define the union \( P_1 \oplus P_2 \) of \( P_1 \) and \( P_2 \) with respect to \( B_1 \) and \( B_2 \) and \( f \) by \( P_1 \oplus P_2 = (P_1 \cup P_2)/f \).

Proposition 3. Let \( P \) be an element of \( \mathcal{E}(3, 2) \). Then, we obtain \( P = F_{1,1}^1 \oplus F_{1,1}^1 \).

Proof. First, we obtain \( v(P) = 1 \), because
\[
\nu(P) \geq \#\mathcal{E}_2(P) - \#\mathcal{E}_3(P) = 1,
\nu(P) \leq \left( \#\mathcal{E}_3(P) - 1 \right)/2 = 1.
\]
The 2-nd inequality follows from Proposition 1. Let \( U \) denote the isolated component of \( U(P) \) and \( P_i \) the connected component of \( P - U \), \( i = 1, 2, 3 \). Since \( \#\mathcal{E}_3(P) = 2 \), we may assume that \( P_2 \) contains no point of \( \mathcal{E}_3(P) \). We show that \( P_2 \) is acyclic. Suppose not. Then, \( \bar{P}_2 = P_2 \cup (\bar{P}_2 \ast v) \) is an acyclic closed fake surface without 3-rd singularity. This contradicts to Theorem 1 [1]. Putting \( Q = P - P_2 \), we define \( \bar{Q} = Q \cup (\bar{Q} \ast v) \). Then, clearly, we can write \( P = \bar{P}_2 \ast \bar{Q} \) using the 2-balls \( (\bar{P}_2 \ast v) \) and \( (\bar{Q} \ast v) \). Since \( P_2 \) is acyclic, \( \bar{P}_2 \) is not acyclic, by Lemma 14 [1]. Then, by Lemma 5, \( \bar{P}_2 \) is a 2-sphere, because \( \#\mathcal{E}_3(\bar{P}_2) = \#\mathcal{E}_3(\bar{P}_2) = 0 \). Hence \( P_2 \) is a 2-ball. Define \( \bar{P}_i = P_i \cup (\bar{P}_i \ast v_i) \), for \( i = 1, 3 \). Then, \( \bar{P}_i \) is an acyclic normal spine by Lemma 14 [1] and Lemma 2, because \( P_i \) is not acyclic by Lemma 6 for \( i = 1, 3 \). Then, \( P_i \) is an element of \( \mathcal{E}(3, 2) \), that is, \( \bar{P}_i = F_{1,1}^1 \), for \( i = 1, 3 \). It is clear that \( P \) is obtained from \( \bar{P}_1 \) and \( \bar{P}_2 \) by identifying the 2-balls \( (\bar{P}_1 \ast v_1) \) and \( \bar{P}_2 \ast v_2 \) to the 2-ball \( P_2 \), that is, \( P = \bar{P}_1 \oplus \bar{P}_2 = F_{1,1}^1 \oplus F_{1,1}^1 \).

Remark. The number of the elements of \( \mathcal{E}(3, 2) \) is, clearly, at most 6.

Lemma 11. Let \( G \) be a 1-st complement. Suppose that \( \#\mathcal{E}_3(G) = 1 = \#\mathcal{E}_4(G) \). Then, \( G \) is uniquely determined as described in Fig. 3.

Proof. We obtain the Homology groups \( H_1(G) = 0 \) and \( H_2(G) = \mathbb{Z} \) by Lemma 3. By Theorem 2, we see \( \alpha(G) = 1 = \beta(G) \) which implies \( \#M(G) = 3 \). Then, by Lemma 12 [1] and Proposition 4 [1], it is known that \( M(G) \) consists of three 2-balls \( M_1, M_2 \) and \( M_3 \). Then, we check all the possible cases as explained in the last half part of the proof of Theorem 2 [1]. And we obtain the identification of \( M_1, M_2 \) and \( M_3 \) as shown in Fig. 3, uniquely.
Remark. From now on, let $G$ denote the unique 1-st complement obtained in Lemma 11.

Remark. Let $B_G$ be a 2-ball in $\hat{M}(G)$ and $P$ an acyclic closed fake surface with a 2-ball $B_P$ in $\hat{M}(P)$. Let $G \circ P$ be the connected sum with respect to $B_G$ and $B_P$. Then, it is easy to see that $G \circ P$ is acyclic if and only if $B_G$ is contained in $M_3$ (for $M_3$, see Fig. 3). And, from now on, $B_G$ denotes the 2-ball contained in $M_3$.

Proposition 4. Let $P$ be an element of $E(2, 2)$. Then, we obtain $P = G \circ P_{1,1}$.  

Proof. There exists an element $M$ in $M(P)$ with $\#M = 2$, because $\#M(P) = 4$ and $\#\hat{M}(P) = 5$ by Theorem 2. By cutting $P$ along a boundary component of $M$ and attaching a 2-ball to the boundary of each connected component, we can write $P = P_1 \circ P_2$ and we have $\#S_i(P_i) \neq 0$ for $i = 1, 2$, because $\#S_i(P_i) \neq 0$ is clear and $\nu(P) = 0$ implies $\nu(P_i) = 0$ for both $i = 1, 2$. Note that $\nu(P) = 0$ follows from Proposition 1. Then, by Lemma 3, We may assume that $P_1$ is a
1–st complement and \( P_2 \) is a 0-th one. Since \( \# \mathcal{E}(P_i) = 1 = \# \mathcal{E}(P_i) \) for both \( i = 1, 2 \), we have \( P_1 = G \) and \( P_2 = F_{1,1} \), completing the proof.

REMARK. The number of the elements of \( \mathcal{E}(2, 2) \) is at most 4.

**Proposition 5.** \( \mathcal{E}(1, 2) \) consists of three elements \( F_{1,2}^1, F_{1,2}^2, \) and \( F_{1,2}^3 \) which are described in Fig. 4.

Proof. By the same way as explained in the last half part of the proof of Theorem 2 [1], we obtain the elements as shown in Fig. 4.

![Diagram](attachment:fig4.png)
Remark. The element $F^1_{1,1}$ of $\mathcal{C}(1,2)$ is well-known as "Bing's House with two rooms".

**Theorem 3.** Zeeman's conjecture holds true for any element $P$ of $\mathcal{E}(s,2)$, that is, $P \times I$ is collapsible.

**Proof.** Case 1. When $s=3$, we see $P=F^1_{1,1} \oplus F^1_{1,1}$ by Proposition 3, and hence, $P \times I$ is collapsible by Proposition 8 of [1].

Case 2. When $s=2$, we obtain $P=G \circ P$, from Proposition 4. Then, by the same way as Case 2 in the proof of Theorem 3 [1], $P \times I$ is collapsible, because $G-B_0$ is collapsible.

Case 3. When $s=1$, $P \times I$ is collapsible by the same way as Case 1 in the proof of Theorem 3 [1], by attaching a 3-ball to $M$ (for $M$, see Fig. 4).

6. $\mathcal{E}(s,t)$ with $1 \leq s \leq 2t-1$.

In this section, we characterize, geometrically, the elements of the sets $\mathcal{E}(2t-1, t)$ and $(2t-2, t)$ and prove the converse of Theorem 1.

**Theorem 4.** Let $P$ be an element of $\mathcal{E}(s,t)$ with $s=2t-1$ and $t \geq 2$. Then, we can write $P=P_1 \oplus P_2$ where $P_i$ belongs to $\mathcal{E}(s_i, t_i)$ with $s_i=2t_i-1$, $t_i+t_2=t$ and $t_i \geq 1$, $i=1,2$.

**Proof.** The proof goes by induction on $t$. When $t=2$, Proposition 3 gives the answer. So, we assume $t \geq 3$. Since $s=2t-1$, we obtain $\nu(P)=t-1$, because

$$t-1=s-t \leq \nu(P) \leq (s-1)/2=t-1.$$  

by Proposition 1. Hence $\nu(P) \geq 1$. Let $U$ and $Q$ be the isolated component of $\overline{U(P)}$ and the connected component of $\overline{P-U}$ obtained in Lemma 7. Now, we show that $Q$ is not acyclic. Suppose not. Then, $\tilde{A}=A \cup (\tilde{A}*v)$ must be acyclic by Lemma 14 [1], where $\tilde{A}=A-P-Q$. And we have $\nu(\tilde{A})=\nu(P)$ and $\#\mathbb{S}_g(A) \leq s-1$, because, by Lemma 7, $\nu(Q)=0$ and $\#\mathbb{S}_g(Q) \neq 0$ implies $\#\mathbb{S}_g(Q) \neq 0$. Then, we obtain

$$\#\mathbb{S}_g(A) \leq s-1 = 2t-2 = 2\nu(A)$$

which contradicts to Proposition 1, because $\tilde{A}$ is a normal spine by Lemma 2. Thus, $Q$ is not acyclic and hence $A$ is acyclic. Then, $A$ collapses naturally to an acyclic normal spine $A_1$ from $\tilde{A}$. Note that $U=S \times T$. And $\nu(A_1)=\nu(P)-1$ is trivial. Then, we have $\#\mathbb{S}_g(A_1)=s-2$, because

$$s-2 \geq \#\mathbb{S}_g(A_1) \geq 2\nu(A_1)+1$$

$$= 2\nu(P)-1$$

$$= 2t-3$$

$$= s-2.$$
And we see \( \#S_i(A_i) \geq t - 1 \), because
\[
t - 2 = v(P) - 1 = v(A_i) \geq s - 2 - \#S_i(A_i).
\]
Since \( \#S_i(O) \neq 0 \) by Lemma 7, we obtain \( \#S_i(A_i) = t - 1 \). Therefore, \( A_i \) is an element of \( \mathcal{E}(s_i', t_i') \) with
\[
s_i' = s - 2 = 2t - 3 = 2(t - 1) - 1 = 2t_i' - 1.
\]
And consequently, we see \( \#S_i(O) = \#S_i(O) \). Let \( S \) denote the base space of the \( T \)-bundle \( U = S \times T \).

Case 1. Suppose that \( S \) bounds a 2-ball in \( M(A_i) \). Let \( \bar{Q} = Q \cup (Q^* v) \). Then, \( \bar{Q} \) belongs to \( \mathcal{E}(1, 1) \). And it is easy to write \( P = A_1 \oplus Q \) by identifying the 2-balls \( B \) and \( (Q^* v) \). Putting \( P_1 = A_1 \) and \( P_2 = Q \), the required conditions in Theorem 4 are satisfied.

Case 2. Suppose that \( S \) does not bound a 2-ball in \( M(A_i) \). By the inductive hypothesis, we can write \( A_1 = A_2 \oplus A_3 \) with respect to the 2-balls \( B_2 \) and \( B_3 \) contained in \( M(A_2) \) and \( M(A_3) \), respectively, where \( A_i \) belongs to \( \mathcal{E}(s_i', t_i') \) with \( s_i' = 2t_i' - 1 \), \( t_i' + t_i' = t_i' \) and \( t_i' \geq 1 \). Since \( S \) does not bound a 2-ball in \( M(A_i) \), \( S \) is contained in either \( A_2 - B_2 \) or \( A_3 - B_3 \), say \( A_2 - B_2 \). Let us define \( P_1 = A_2 \cup U \cup Q \) and \( P_2 = A_3 \). Then, using the 2-balls \( B_2 \) and \( B_3 \), we can write \( P = P_1 \oplus P_2 \). And it is clear that \( P_1 \) belongs to \( \mathcal{E}(s_2' + 2, t_2' + 1) \). And hence, \( s_2' + 2 = (2t_2' - 1) + 2 = 2(t_2' + 1) - 1 \). Thus, the required conditions in Theorem 4 are satisfied. And Theorem 4 is now established.

**Corollary to Theorem 4.** For any element \( P \) of \( \mathcal{E}(2t - 1, t) \) with \( t \geq 1 \), the Zeeman's conjecture holds true, that is, \( P \times I \) is collapsible.

Proof. By Theorem 4, \( \mathcal{E}(2t - 1, t) \) is contained in \( C \) defined in §9 [1], for any integer \( t \geq 1 \). Then, \( P \times I \) is collapsible by Proposition 8 [1].

In order to characterize the elements of \( \mathcal{E}(s, t) \) in the case \( s = 2t - 2 \), we extend the definition of the union of closed fake surfaces as follows.

**Definition 8.** Let \( P_i \) be a closed fake surface with an acyclic fake surface \( A_i \) such that the boundary \( \overline{A}_i \) is a 1-sphere contained in \( \overline{M(P_i)} \) and \( A_i \) is a connected component of \( P \) disconnected by \( \overline{A}_i \), \( i = 1, 2 \). Suppose that there is a homeomorphism \( f \) from \( A_i \) onto \( A_i \). Define the union \( P_1 \oplus P_2 \) of \( P_1 \) and \( P_2 \) with respect to \( A = A_1 = A_2 \) and \( f \) by \( P_1 \oplus P_2 = (P_1 \cup P_2)/f \).

Then, in general, we obtain the following.

**Proposition 6.** (1) Let \( P \) be an element of \( \mathcal{E}(s, t) \) with \( v(P) \geq 1 \). Then, there exists an acyclic fake surface \( A \) in \( P \) such that we can write \( P = P_1 \oplus P_2 \).
(2) If we can write $P = P_1 \# P_2$ for an element $P$ of $\mathcal{E}(s, t)$, we obtain the following conditions.

(i) $P_i$ belongs to $\mathcal{E}(s_i, t_i)$, $i = 1, 2$.
(ii) $s_i \leq \# \mathbb{S}_3(A) + 1$, $i = 1, 2$.
(iii) $t_i \leq \# \mathbb{S}_3(A) + 1$, $i = 1, 2$.
(iv) $s_1 + s_2 - \# \mathbb{S}_3(A) = s - 1$.
(v) $t_1 + t_2 - \# \mathbb{S}_3(A) = t$.

Proof. Since $\nu(P) \geq 1$, there exists an isolated component $U$ in $U(P)$. And we see $U = S \times T$, because $P$ belongs to $\mathcal{E}(s, t)$. Then, by Lemma 6, there exists an acyclic component $A$ in $P - U$, uniquely, and the other components than $A$ of $P - U$ are denoted by $Q_1$ and $Q_2$. Note that $\# \mathbb{S}_3(Q_i) \neq 0$ for $i = 1, 2$, because $Q_i = Q_i \cup (Q_i * v_i)$ is an acyclic normal spine and hence $\# \mathbb{S}_3(Q_i) = \# \mathbb{S}_3(Q_i) \neq 0$, by Theorem 1 [1]. Now, unpasting $P$ at $A$, we obtain two closed fake surfaces $P_1$ and $P_2$, and it is clear that $P$ can be written $P = P_1 \# P_2$. This proves (1). And it is also clear that $P_i$ is an acyclic normal spine for $i = 1, 2$, that is, $P_i$ belongs to $\mathcal{E}(s_i, t_i)$, because both $P$ and $A$ are acyclic. We may assume $P_i \supset Q_i$, for $i = 1, 2$. Then, the conditions (ii) and (iii) are proved by $\mathbb{S}_3(Q_i) \cup \mathbb{S}_3(A) = \mathbb{S}_3(P_i)$ for $i = 1, 2$, and $j = 2, 3$. The condition (iv) follows from the facts $\mathbb{S}_3(P_i) \cup U \cup \mathbb{S}_3(P_2) = \mathbb{S}_3(P)$ and $\mathbb{S}_3(A) \subset \mathbb{S}_3(P_i)$, for both $i = 1, 2$. The last condition (v) is also satisfied by $\mathbb{S}_3(P_i) \cup \mathbb{S}_3(P_2) = \mathbb{S}_3(Q_i) \cup \mathbb{S}_3(Q_2) \cup \mathbb{S}_3(A) = \mathbb{S}_3(P)$ and $\mathbb{S}_3(A) \subset \mathbb{S}_3(P_i)$ for both $i = 1, 2$.

Remark. Let $G$ be the 1-st complement obtained in Lemma 11 and $B_G$ the 2-ball in $M_s$ of $G$ (see Remark to Lemma 11). From now on, $G - B_G$ is denoted by $G_o$.

Theorem 5. Let $P$ be an element of $\mathcal{E}(s, t)$ with $s = 2t - 2$ and $t \geq 3$. Then, we can write $P = P_1 \# P_2$ so that $A$ is either a 2-ball or $G_o$ and $P_i$ belongs to $\mathcal{E}(s_i, t_i)$, $i = 1, 2$. And if $A$ is a 2-ball, we obtain $s_1 = 2t_1 - 1$ and $s_2 = 2t_2 - 2$. If $A$ is $G_o$, we obtain $s_i = 2t_i - 2$, for both $i = 1, 2$.

Proof. This theorem is also proved by induction on $t$ by the similar argument to the proof of Theorem 4. However, the preparation is more complicated. In this case, we obtain $\nu(P) = t - 2$, because

$$t - 2 = s - t \leq \nu(P) \leq (s - 1)/2 < t - 1.$$

We can find an isolated component $U$ in $U(P)$ and a connected component $Q$ in $P - U$ with $\nu(Q) = 0$ and $\# \mathbb{S}_3(Q) = 0$, by Lemma 7.
Step 1. In this step, we study about $Q$.

Case 1. Suppose that $Q$ is acyclic.

In this case, we show $\#_2(Q) = 1 = \#_3(Q)$ which implies $Q = G_0$, because $Q = \hat{Q} \cup (Q^*)$ is a 1-st complement.

Since $Q$ is acyclic and $\nu(Q) = 0$, we obtain $\hat{F} = F \cup (F^*)$ is an acyclic normal spine with $\nu(\hat{F}) = \nu(P)$, where $F = P - Q$. Then, we see $\#_2(\hat{F}) = s - 1$ and $\#_3(\hat{F}) = t - 1$. Because

$$s - 1 \geq \#_2(\hat{F}) \geq 2\nu(\hat{F}) + 1 = 2t - 3 = s - 1,$$

and

$$t - 1 \geq \#_3(\hat{F}) \geq (\#_2(\hat{F}) + 1)/2 = t - 1,$$

by Proposition 1 and Theorem 1. Hence, we obtain the required condition $\#_2(Q) = 1 = \#_3(Q)$, because

$$\#_2(\hat{F}) + \#_3(Q) = \#_2(P)$$

is true for $j = 2, 3$.

Case 2. Suppose that $Q$ is not acyclic.

In this case, $F$ is acyclic and hence we obtain an acyclic normal spine $F_1$ from $F$ by a natural collapsing. And we have $\nu(F_1) = \nu(P) - 1 = t - 3$. Then, by the similar argument to the proof of Theorem 4 and Case 1 in this step, we can prove that the pair $(\#_2(F_1), \#_3(F_1))$ is either $(s - 2, t - 1)$ or $(s - 3, t - 2)$. Thus, we obtain the following statement (*).

(*) $(\#_2(Q), \#_3(Q)) = (k, k)$ if and only if $(\#_2(F_1), \#_3(F_1)) = (s - 1 - k, t - k)$, for $k = 1, 2$.

Step 2. Suppose $t = 3$ (the 1-st step of induction).

Then, $\nu(P) = t - 2 = 1$. Then, by Proposition 6, we can write $P = P_1 \oplus P_2$.

Since $\nu(P) = 1$ implies $\nu(P_i) = 0$ for both $i = 1, 2$, we obtain the two possibility. That is, if $\#_2(A) = 0$, then $A$ is a 2-ball by Lemma 4 or Lemma 5. And hence $P_i$ belongs to $\mathcal{E}(i, i)$ for $i = 1, 2$. And if $\#_2(A) \neq 0$, we see $A = G_0$ by Step 1 (Case 1), because $\nu(A) = 0$. Hence, we can write $P = P_1 \oplus P_2$, and $P_i$ belongs to $\mathcal{E}(2, 2)$ for both $i = 1, 2$, by Proposition 6.

Step 3. We deal with the case $t \geq 4$.

Case 1. Suppose that $Q$ is acyclic.

In this case, take $A = Q$. Then, $Q = G_0$ by Case 1 of Step 1, and hence, $P = P_1 \oplus P_2$ and $P_i$ belongs to $\mathcal{E}(s_i, t_i)$, $i = 1, 2$. By Proposition 6, we obtain $s_i + s_2 = s$ and $s \geq 2$ and $t_i + t_2 = t + 1$ and $t_i \geq 2$. Put $s_i = 2t_i - u_i$, $i = 1, 2$. Then, we obtain $u_1 + u_2 = 4$, because
\[ 2t - (u_1 + u_2 - 2) = (2t_1 - u_1) + (2t_2 - u_2) \]
\[ = s_1 + s_2 \]
\[ = s \]
\[ = 2t - 2. \]

Since \( u_i \geq 1 \) by Theorem 1, for both \( i = 1, 2 \), we see that the pair \( (u_1, u_2) \) is either \((1, 3)\) or \((2, 2)\). Suppose \( u_1 = 1 \). Then, \( P_1 \) must be an element of \( \mathcal{E}(2t_1 - 1, t_1) \). But, for any integer \( t_1 \geq 1 \), it is clear, from Theorem 4, that no element of \( \mathcal{E}(2t_1, -1, t_1) \) contains \( T_0 \) as a subpolyhedron. Thus, \( (u_1, u_2) \) must be \((2, 2)\), and hence \( s_i = 2t_i - 2 \) for both \( i = 1, 2 \). This completes the proof of this case.

Case 2. Suppose that \( Q \) is not acyclic.

In this case, the construction of \( P_1 \) and \( P_2 \) highly resembles to the last Case 2 in the proof of Theorem 4. We use the statement (*) in Case 2 in Step 1. When \( k = 1 \), we can write \( F_1 = F_2 \oplus F_3 \) by the inductive hypothesis. And if \( k = 2 \), we can write \( F_1 = F_2 \oplus F_3 \) by Theorem 4. And we obtain \( P_1 \) and \( P_2 \) as required in Theorem 5.

Thus, Theorem 5 is established.

When we define the set \( \mathcal{C} \) of acyclic normal spines obtained from \( \mathcal{E}(1, 1) \) and \( \mathcal{E}(2, 2) \) using \( P_1 \oplus P_2 \) and \( P_1 \oplus P_2 \) as the set \( \mathcal{C} \) defined in §9 in [1], we have the following proposition by the similar reason to that of Proposition 8 [1].

**Proposition 7.** Let \( P \) be an element of \( \mathcal{C} \). Then, \( P \times \mathcal{I} \) is collapsible.

And we have the following as a corollary to Theorem 5, because \( \mathcal{E}(2t-2, t) \) is contained in \( \mathcal{C} \) by Theorem 5.

**Corollary to Theorem 5.** For any element \( P \) of \( \mathcal{E}(2t-2, t) \) with \( t \geq 2 \), the Zeeman's conjecture is true, that is, \( P \times \mathcal{I} \) is collapsible.

We prepare the following lemmas to prove Theorem 6.

**Lemma 12.** \( \mathcal{E}(1, t) \) contains a spine of a 3-ball, for any integer \( t \geq 1 \).

Proof. Suppose that \( t \) is odd, that is, \( t = 2r + 1 \). When \( r = 0 \), there is nothing to prove, because the unique element \( F_{3,1}^1 \) (abalone) of \( \mathcal{E}(1, 1) \) is a spine of a 3-ball by Theorems 3 and 4 [1]. We construct a normal spine of a 3-ball in \( \mathcal{E}(1, t) \) inductively. Let \( P \) be an element of \( \mathcal{E}(1, 2(r-1)+1) \) which is a spine of a 3-ball \( V \). Then, we can apply an elementary deformation of type \( I \) to \( P \) in \( V \), and we obtain a normal spine \( P(1) \) of \( V \), by Lemma 8. Then, by Lemma 9, it is clear that \( P(1) \) belongs to \( \mathcal{E}(1, t) \). When \( t \) is even, we obtain a spine of a 3-ball in \( \mathcal{E}(1, t) \) by the same way as above from an element of \( \mathcal{E}(1, 2) \) which is non-empty by Proposition 5 and it is known, by Theorem 3, that any
element of $\mathcal{E}(1, 2)$ is a spine of a 3-ball.

**Lemma 13.** Suppose that $G_o$ is embedded in a 3-ball $V$ properly, that is, $G_o \cap \hat{V} = G_o$. Then, $V$ collapses to $G_o$.

Proof. Let $N$ be the regular neighborhood of $G_o$ in $V$ meeting the boundary regularly, that is, $N \cap \hat{V}$ is a regular neighborhood of $G_o$ in $\hat{V}$. Since $G_o$ is collapsible and $G_o$ is a 1-sphere, $N$ is a 3-ball and $N \cap \hat{V}$ is an annulus. Then, $\hat{V} - N$ is the disjoint union of two 3-balls $V_1$ and $V_2$. And, clearly, $N \cap V_i = \hat{N} \cap \hat{V}_i = F_i$ is a 2-ball for $i = 1, 2$. Then, $V$ collapses to $N$ by collapsing each $V_i$ to $F_i$ and $N$ collapses to $G_o$. Thus, $V$ collapses to $G_o$.

**Lemma 14.** Let $P$ be a normal spine of a 3-manifold $W$, that is, $W$ collapses to $P$. Then, $G \circ P$ is also a spine of $W$, where the connected sum is taken with respect to $B_G$.

Proof. Let $B_P$ be the 2-ball of $P$ used in the connected sum $G \circ P$, and let $N$ be the 2-nd derived neighborhood of $B_P$ in $W$. Note that we can expand $P$ to $P \cup N$ in $W$. It is possible to replace $B_P$ by $G_o$ in $N$ to satisfy $G_o \cap \hat{N} = \hat{G}_o = \hat{B}_P$, because $N$ is a 3-ball and $G_o$ and $\hat{B}_P$ are 1-spheres. Then, by Lemma 13, $N$ collapses to $G_o$, and hence $P \cup N$ collapses to $(P - B_P) \cup G_o$ which is clearly $G \circ P$. Thus, $G \circ P$ is a spine of $W$.

**Theorem 6.** $\mathcal{E}(s, t)$ contains a spine of a 3-ball for any pair $(s, t)$ with $1 \leq s \leq 2t - 1$.

Proof. By Lemma 12 and Corollary to Theorem 4, each of $\mathcal{E}(1, t)$ and $\mathcal{E}(2t - 1, t)$ contains a spine of a 3-ball for any integer $t \geq 1$. So, assuming $2 \leq s \leq 2t - 2$, we construct a spine $Q$ of a 3-ball in $\mathcal{E}(s, t)$ inductively. Suppose that $P$ is a spine of a 3-ball in $\mathcal{E}(s - 1, t - 1)$. Define $Q = G \circ P$. Then, by Lemma 14, $Q$ is also a spine of a 3-ball and clearly $Q$ belongs to $\mathcal{E}(s, t)$.

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**References**
