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Osaka J. Math.
13 (1976), 99-108

PIVOTAL MEASURES IN THE CASE OF WEAK DOMINATION

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(Received December 17, 1974)
(Rivised August 30, 1975)

Weakly dominated statistical structures were introduced by T.S. Pitcher ([6]) and studied by various authors. Notably, D. Mussmann ([5]) proved a generalization of Neyman factorization theorem for sufficient σ-fields.

In this paper we give a construction of the "pivotal measures" based on whose existence his proof is developed. Mussmann's method does not provide for an explicit form of the measure but only an existence-proof of it, as is discussed in detail in the beginning of Section 3. As a result of our method, not only the whole process of arriving at the factorization theorem has been greatly simplified, but simple proofs of some additional results are furnished by making use of this concrete definition of pivotal measures. Moreover, we give a characterization of pivotal measures.

1. Definitions and notations

Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a measure space consisting of a set $\mathcal{X}$, a σ-algebra $\mathcal{A}$ of subsets of $\mathcal{X}$ and a measure μ on $(\mathcal{X}, \mathcal{A})$. We define that $\mathcal{A}_e(\mu)\equiv\{E\in\mathcal{A}\,|\,\mu(E)<\infty\}$, $\mathcal{A}_{e\sigma}(\mu)\equiv\{A\in\mathcal{A}\,|\,A=\bigcup_{n=1}^{\infty}E_n,\ E_n\in\mathcal{A}_e(\mu)\ \textit{for all}\ n\}$ and $\mathcal{A}_l(\mu)\equiv\{A\subset\mathcal{X}\,|\,A\cap E\in\mathcal{A}\ \textit{for all}\ E\in\mathcal{A}_e(\mu)\}$, μ is said to concentrate on a set T in $\mathcal{A}$ if $\mu(A)=\mu(A\cap T)$ for all A in $\mathcal{A}$.

A family of $\mathcal{A}$-measurable real functions $\{g_E\,|\,E\in\mathcal{A}_e(\mu)\}$ is called a *μ-cross section* ([8]), if each $g_E(x)=0$ outside of E and $I_{E_1\cap E_2}g_{E_1}=I_{E_1\cap E_2}g_{E_2}[\mu]$ for all E_1 and E_2 in $\mathcal{A}_e(\mu)$. Here I_E is the indicator function of E.

μ is called a *localizable* measure if for any family $\mathcal{F}\subset\mathcal{A}_e(\mu)$ there exists an essential supremum of $\mathcal{F}$ in $\mathcal{A}$ with respect to μ, written ess-sup $\mathcal{F}(\mu)$, which is a set in $\mathcal{A}$ satisfying the following two conditions:

(1) $\mu(E-\text{ess-sup}\ \mathcal{F}(\mu))=0$ for all E in $\mathcal{F}$,

(2) $\mu(\text{ess-sup}\ \mathcal{F}(\mu)-A)=0$ for any A in $\mathcal{A}$ satisfying $\mu(E-A)=0$ for all E in $\mathcal{F}$.

μ is called *D-localizable* ([2]) if for any μ-cross section $\{g_E\,|\,E\in\mathcal{A}_e(\mu)\}$

there exists a real $\mathcal{A}_l(\mu)$-measurable function g satisfying $I_E g = g_E[\mu]$ for all E in $\mathcal{A}_e(\mu)$. Diepenbrock ([2]) showed by an example that D-localizable measure and localizable measure do not coincide with each other.

μ is said to have the *finite subset property* if, for any A in $\mathcal{A}$ such that $\mu(A)>0$, there exists a set B in $\mathcal{A}$ such that $B \subset A$ and $0<\mu(B)<\infty$.

Let $\mathcal{P}$ be a family of probability measures on $(\mathcal{X}, \mathcal{A})$. Then triplet $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is often called a statistical structure. A statistical structure $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is dominated, if there exists a σ-finite measure μ on $(\mathcal{X}, \mathcal{A})$ such that $\mathcal{P} \sim \mu$ (that is, $\mathcal{P}$ is equivalent to μ). A statistical structure $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is *weakly dominated* ([4]) by μ, if there exists a localizable measure μ on $(\mathcal{X}, \mathcal{A})$ such that $\mathcal{P} \sim \mu$ and each element in $\mathcal{P}$ has a density with respect to μ. Equivalence of Pitcher's compact statistical structure and weakly dominated statistical structure was proved by Theorem 9.1 in [2].

2. A relation between weak domination and sufficiency

Let $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be a statistical structure and let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$. Then $\mathcal{B}$ is sufficient for $(\mathcal{X}, \mathcal{A}, \mathcal{P})$, for short for $\mathcal{P}$, if for any bounded $\mathcal{A}$-measurable function f, there exists a $\mathcal{B}$-measurable function $E(f \mid \mathcal{B})$ such that

$$\int_B f dP = \int_B E(f \mid \mathcal{B}) dP \mid \mathcal{B}$$

for all B in $\mathcal{B}$ and P in $\mathcal{P}$. Here $P \mid \mathcal{B}$ is the restriction of P on $\mathcal{B}$. We write $\mathcal{P} \mid \mathcal{B} \equiv \{P \mid \mathcal{B} \mid P \in \mathcal{P}\}$. We shall begin with a lemma concerning localizable measures.

Lemma 2.1. *Let μ and ν be measures on $(\mathcal{X}, \mathcal{A})$ with the finite subset property and let $\mu \sim \nu$. Then μ is localizable if and only if ν is localizable.*

Proof. Diepenbrock ([2] Theorem 3.2) showed this lemma in his unpublished doctoral thesis for D-localizable measures. In the present case, his proof works if we replace D-localizability with localizability. Namely,

(1) ([8] p. 264 Theorem 2)[1] Let μ be a measure on $(\mathcal{X}, \mathcal{A})$ with the finite subset property. Then μ is localizable if and only if for any μ-cross section $\{g_E \mid E \in \mathcal{A}_e(\mu)\}$ there exists an $\mathcal{A}$-measurable function g such that $I_E g = g_E[\mu]$ for all $E \in \mathcal{A}_e(\mu)$.

(2) ([2] Lemma 3.1) Let μ and ν be measures on $(\mathcal{X}, \mathcal{A})$ with the finite subset property and let $\mu \sim \nu$. Then $\mathcal{A}_{e\sigma}(\mu) = \mathcal{A}_{e\sigma}(\nu)$.

(3) Let μ be a measure on $(\mathcal{X}, \mathcal{A})$. In order that for any μ-cross section

1) Zaanen ([8]) proves this theorem under a context in which it is implied that $\mathcal{A}_l(\mu) = \mathcal{A}$ and μ is complete. However, on perusal of his proof one finds that these additional conditions are not involved in it.

$\{g_E \mid E \in \mathcal{A}_e(\mu)\}$ there exists an $\mathcal{A}$-measurable function g such that $I_E g = g_E[\mu]$ for all E in $\mathcal{A}_e(\mu)$, it is necessary and sufficient that for any family $\{g_B \mid B \in \mathcal{A}_{e\sigma}(\mu)\}$ of $\mathcal{A}$-measurable real functions such that each $g_B(x)=0$ outside of B and $I_{B_1 \cap B_2} g_{B_1} = I_{B_1 \cap B_2} g_{B_2}[\mu]$ for all B_1 and B_2 in $\mathcal{A}_{e\sigma}(\mu)$ there exists an $\mathcal{A}$-measurable function g such that $I_B g = g_B[\mu]$ for all B in $\mathcal{A}_{e\sigma}(\mu)$.

Lemma 2.2. ([4] Lemma (2.9) (1)) *Let $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be a statistical structure. And let each P in $\mathcal{P}$ has a density with respect to a measure μ, with $\mathcal{P} \sim \mu$, on $(\mathcal{X}, \mathcal{A})$. Then μ has the finite subset property.*

Theorem 2.1. *Let $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be a statistical structure and let $\mathcal{B}$ be sufficient for $\mathcal{P}$. For the statistical structure $(\mathcal{X}, \mathcal{B}, \mathcal{P} \mid \mathcal{B})$ we assume the following condition* (R): *there exists a measure μ on $(\mathcal{X}, \mathcal{B})$ such that,*

(R1) $\mathcal{P} \mid \mathcal{B} \sim \mu$,

(R2) *each $P \mid \mathcal{B}$ in $\mathcal{P} \mid \mathcal{B}$ has a density with respect to μ, written* $\dfrac{dP \mid \mathcal{B}}{d\mu}$.

We define a set function $\tilde{\mu}$ on $\mathcal{A}$ by

$$(*) \qquad \tilde{\mu}(A) \equiv \int_{\mathcal{X}} E(I_A \mid \mathcal{B}) d\mu$$

for each A in $\mathcal{A}$. Then we have:

(1) *$\tilde{\mu}$ is a measure in $(\mathcal{X}, \mathcal{A})$ such that $\mathcal{P} \sim \tilde{\mu}$ and each P in $\mathcal{P}$ has a density with respect to $\tilde{\mu}$.*

(2) *$(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is weakly dominated if and only if $\tilde{\mu}$ is localizable.*

(3) *$(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is dominated if and only if $\tilde{\mu}$ is σ-finite.*

Proof. (1): By condition (R1) and the definition of $\tilde{\mu}$ it follows that the set function $\tilde{\mu}$ is a measure on $(\mathcal{X}, \mathcal{A})$ satisfying $\mathcal{P} \sim \tilde{\mu}$ and is an extension of the measure μ to $(\mathcal{X}, \mathcal{A})$. By condition (R2) it is easy to see that each $P \mid \mathcal{B}$ in $\mathcal{P} \mid \mathcal{B}$ concentrates on the set $T_{P\mid\mathcal{B}} \equiv \left\{ x \in \mathcal{X} \mid \dfrac{dP \mid \mathcal{B}}{d\mu}(x) > 0 \right\}$ which belongs to $\mathcal{B}_{e\sigma}(\mu)$. Therefore each P in $\mathcal{P}$ concentrates on the set $T_{P\mid\mathcal{B}}$ which belongs to $\mathcal{A}_{e\sigma}(\tilde{\mu})$. We define a function f_P by

$$\begin{aligned} f_P(x) &= \frac{dP \mid T_{P\mid\mathcal{B}}\mathcal{A}}{d\tilde{\mu} \mid T_{P\mid\mathcal{B}}\mathcal{A}}(x) \qquad && \text{if} \quad x \in T_{P\mid\mathcal{B}}, \\ &= 0 && \text{if} \quad x \in \mathcal{X} - T_{P\mid\mathcal{B}}. \end{aligned}$$

This function is a density of P with respect to $\tilde{\mu}$. We write it $\dfrac{dP}{d\tilde{\mu}}$. By Lemma 2.2 $\tilde{\mu}$ has the finite subset property.

(2): By (1) if $\tilde{\mu}$ is localizable $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is weakly dominated by $\tilde{\mu}$. Conversely let $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be weakly dominated by a localizable measure ν. Then, again by Lemma 2.2, ν has the finite subset property. By $\nu \sim \mathcal{P} \sim \tilde{\mu}$ and Lemma

2.1, $\tilde{\mu}$ is localizable.

(3): If $\tilde{\mu}$ is σ-finite $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is dominated by $\tilde{\mu}$. Conversely let $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be dominated by a σ-finite measure ν. As σ-finiteness implies the finite subset property and $\tilde{\mu}\sim\nu$, $\mathcal{A}_{e\sigma}(\tilde{\mu})=\mathcal{A}_{e\sigma}(\nu)$ by (2) in the proof of Lemma 2.1. Hence the mesure $\tilde{\mu}$ is σ-finite. This completes the proof.

REMARK 2.1. This measure $\tilde{\mu}$ with a minimal sufficient σ-field as $\mathcal{B}$ turns out to be a pivotal measure.

REMARK 2.2. If $(\mathcal{X}, \mathcal{B}, \mathcal{P}|\mathcal{B})$ is a weakly dominated statistical structure it satisfies the condition (R).

REMARK 2.3. In the "if" part in Theorem 2.1 (2) we cannot replace the localizable measure with a D-localizable measure. For example let $\mathcal{X}$ be the real line and $\mathcal{A}$ be the Borel field of subsets of $\mathcal{X}$. And let $\mathcal{P}$ be the totality of discrete probability measures on $(\mathcal{X}, \mathcal{A})$. $\mathcal{B}\equiv\mathcal{A}$ is clearly sufficient for $\mathcal{P}$. On $(\mathcal{X}, \mathcal{A})$ we take the counting measure μ. Then the measure μ and $(\mathcal{X}, \mathcal{B}, \mathcal{P}|\mathcal{B})$ satisfy the condition (R) in Theorem 2.1. As $\mathcal{A}_l(\mu)$ defined above equals $2^{\mathcal{X}}$, the measure $\tilde{\mu}\equiv\mu$ is D-localizable. But $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is not weakly dominated by Theorem 1.1 in [3], which states that in our situation $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is weakly dominated if and only if $\mathcal{A}=2^{\mathcal{X}}$.

The explicit form of the pivotal measure (*) enables us to prove:

Corollary 2.1. ([7]) *Let $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be a statistical structure and $\mathcal{B}$ be sufficient for $\mathcal{P}$. Then $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is dominated if $(\mathcal{X}, \mathcal{B}, \mathcal{P}|\mathcal{B})$ is dominated.*

Proof. If $(\mathcal{X}, \mathcal{B}, \mathcal{P}|\mathcal{B})$ is dominated there exists a σ-finite measure μ on $(\mathcal{X}, \mathcal{B})$ such that $\mathcal{P}|\mathcal{B}\sim\mu$. Then $(\mathcal{X}, \mathcal{B}, \mathcal{P}|\mathcal{P})$ and μ satisfy the condition (R) in Theoerem 2.1. Since the measure $\tilde{\mu}$ in Theorem 2.1 is an extension of μ to $(\mathcal{X}, \mathcal{A})$ it follows that $\tilde{\mu}$ is σ-finite. Hence $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is dominated.

Corollary 2.2. ([5]) *Let $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be a weakly dominated statistical structure and let $\mathcal{B}$ be sufficient for $\mathcal{P}$. Then there exists a measure $\tilde{\mu}$ on $(\mathcal{X}, \mathcal{A})$ such that $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is weakly dominated by $\tilde{\mu}$ and $(\mathcal{X}, \mathcal{B}, \mathcal{P}|\mathcal{B})$ is weakly dominated by $\tilde{\mu}|\mathcal{B}$.*

Proof. By Theorem 2.2 in [3], $(\mathcal{X}, \mathcal{B}, \mathcal{P}|\mathcal{B})$ is weakly dominated. Let it be weakly dominated by μ. Then $(\mathcal{X}, \mathcal{B}, \mathcal{P}|\mathcal{B})$ and μ satisfy the condition (R) in Theorem 2.1. By (2) in Theorem 2.1 the measure $\tilde{\mu}$ is localizable and $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is weakly dominated by $\tilde{\mu}$.

3. A construction of pivotal measures

A "pivotal measure" $\tilde{\mu}$ with the following property plays a central role in the proof of Neyman factorization theorem: $\mathcal{B}$ is sufficient if and only if each P

in $\mathcal{P}$ has a $\mathcal{B}$-measurable density. Mussmann ([5]) starts his proof of a similar factorization for any sufficient $\mathcal{B}$ with an existence-proof of a "maximal decomposition" of the sample space with certain properties by making use of Zorn's lemma. He then defines a measure $\tilde{\mu}$ with respect to which each P in $\mathcal{P}$ has a $\mathcal{B}$-measurable density on the basis of this maximal decomposition. This inspires us importance of constructing explicitly a measure with the same property for all sufficient σ-fields. In view of this we give the following:

DEFINITION. Let $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be weakly dominated. A measure $\tilde{\mu}$ on $(\mathcal{X}, \mathcal{A})$ is *pivotal* if $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is weakly dominated by $\tilde{\mu}$ and for each sufficient σ-field $\mathcal{B}$ each P in $\mathcal{P}$ has a $\mathcal{B}$-measurable density with respect to $\tilde{\mu}$.

REMARK 3.1. The foregoing definition naturally applies to the dominated case, in which, however, our "pivotal measures" include some more variety of measures than "$\sum_{n=1}^{\infty} c_n P_n$" ([1]), which are commonly called pivotal measures.

Lemma 3.1 ([5] Theorem (4.2)). *Let $(\mathcal{X}, \mathcal{A}, \tilde{\mu})$ be a measure space and $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$ and $\mu \equiv \tilde{\mu} \mid \mathcal{B}$ be a localizable measure on $(\mathcal{X}, \mathcal{B})$ with the finite subset property. Then for any $\mathcal{A}$-measurable f for which the integral $\int_B f d\tilde{\mu}$ exists and is finite for all B in $\mathcal{B}_e(\mu)$, there exists a $\mathcal{B}$-measurable function $E(f \mid \mathcal{B}, \tilde{\mu})$, which is μ-unique, such that*

$$\int_B f d\tilde{\mu} = \int_B E(f \mid \mathcal{B}, \tilde{\mu}) d\mu$$

for all B in $\mathcal{B}_e(\mu)$.

The function $E(f \mid \mathcal{B}, \tilde{\mu})$ is called a conditional $\tilde{\mu}$-expectation function of f with respect to $\mathcal{B}$. The function $E(f \mid \mathcal{B}, \tilde{\mu})$ behaves like a usual conditional expectation function. For example, if f is $\mathcal{A}$-measurable, g is $\mathcal{B}$-measurable and $E(g \cdot f \mid \mathcal{B}, \tilde{\mu})$ and $E(f \mid \mathcal{B}, \tilde{\mu})$ exist, then

$$E(g \cdot f \mid \mathcal{B}, \tilde{\mu}) = g \cdot E(f \mid \mathcal{B}, \tilde{\mu}) \, [\mu] \quad ((4.3) \, [5])$$

When $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is weakly dominated then there exists a minimal sufficient sub-σ-field $\mathcal{B}_0$ for $\mathcal{P}$ (Theorem 2.15 in [4]), that is for any other sufficient sub-σ-field $\mathcal{B}$ for $\mathcal{P}$ $\mathcal{B}_0 \subset \mathcal{B}[\mathcal{P}]$ holds. Here $\mathcal{B}_0 \subset \mathcal{B}[\mathcal{P}]$ means that for any $\mathcal{B}_0$-measurable function f there exists a $\mathcal{B}$-measurable function g such that $f = g[P]$ for all P in $\mathcal{P}$.

Theorem 3.1. *Let $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be a weakly dominated statistical structure and $\mathcal{B}_0$ be a minimal sufficient sub-σ-field for $\mathcal{P}$. Then the measure $\tilde{\mu}$ defined in $(*)$ in terms of $\mathcal{B}_0$, namely*

$$\tilde{\mu}(A) \equiv \int_{\mathcal{X}} E(I_A \mid \mathcal{B}_0) d\mu$$

for any A in $\mathcal{A}$, where μ is a measure on $(\mathcal{X}, \mathcal{B}_0)$ by which $(\mathcal{X}, \mathcal{B}_0, \mathcal{P}|\mathcal{B}_0)$ is weakly dominated, is a pivotal measure. (*Existence of such a measure μ is guaranteed by Corollary* 2.2.)

Proof. We can define $E(I_A|\mathcal{B}_0, \tilde{\mu})$ for all A in $\mathcal{A}$ since

$$\int_B I_A d\tilde{\mu} = \tilde{\mu}(A\cap B) \leqq \tilde{\mu}(B) = \mu(B) < \infty$$

for all B in $\mathcal{B}_{0e}(\mu)$, and $\mu = \tilde{\mu}|\mathcal{B}_0$ is a localizable measure on $(\mathcal{X}, \mathcal{B}_0)$ with the finite subset property. Also we note that for each A in $\mathcal{A}$ and B in $\mathcal{B}_0$,

$$E(I_{A\cap B}|\mathcal{B}_0) = I_B E(I_A|\mathcal{B}_0)\ [\mu] \tag{3.1}$$

for

$$E(I_{A\cap B}|\mathcal{B}_0) = I_B E(I_A|\mathcal{B}_0)\ [P|\mathcal{B}_0]$$

for all $P|\mathcal{B}_0$ in $\mathcal{P}|\mathcal{B}_0$ and $\mathcal{P}|\mathcal{B}_0 \sim \mu$.
Then we have

$$\begin{aligned}\int_B E(I_A|\mathcal{B}_0)d\mu &= \int_{\mathcal{X}} E(I_{A\cap B}|\mathcal{B}_0)d\mu \qquad \text{(by (3.1))}\\ &\equiv \tilde{\mu}(A\cap B)\\ &= \int_B I_A d\tilde{\mu}\\ &= \int_B E(I_A|\mathcal{B}_0, \tilde{\mu})d\mu\end{aligned}$$

for all A in $\mathcal{A}$ and B in $\mathcal{B}_{0e}(\mu)$. Since μ has the finite subset property we have

$$E(I_A|\mathcal{B}_0) = E(I_A|\mathcal{B}_0, \tilde{\mu})\ [\mu] \tag{3.2}$$

for all A in $\mathcal{A}$.

For each $P|\mathcal{B}_0$ in $\mathcal{P}|\mathcal{B}_0$, $P|\mathcal{B}_0$ concentrates on the set $T_{P|\mathcal{B}_0} \equiv \left\{x\in\mathcal{X} \,\middle|\, \frac{dP|\mathcal{B}_0}{d\mu}(x) > 0\right\}$ which is σ-finite with respect to μ. So we can write $T_{P|\mathcal{B}_0} = \sum_{n=1}^{\infty} E_n$, $E_n \cap E_m = \phi$ if $n \neq m$ and $E_n \in \mathcal{B}_{0e}(\mu)$, $n=1, 2, \cdots$. Here $\{E_n\}$ depends on $P|\mathcal{B}_0$, Then for any A in $\mathcal{A}$,

$$\begin{aligned}P(A) &= \int_{\mathcal{X}} E(I_A|\mathcal{B}_0)dP|\mathcal{B}_0\\ &= \int_{T_{P|\mathcal{B}_0}} E(I_A|\mathcal{B}_0)dP|\mathcal{B}_0\\ &= \int_{T_{P|\mathcal{B}_0}} E(I_A|\mathcal{B}_0, \tilde{\mu})dP|\mathcal{B}_0 \quad \text{(by } \mathcal{P}|\mathcal{B}_0\sim\mu \text{ and (3.2))}\\ &= \sum_{n=1}^{\infty}\int_{E_n} E(I_A|\mathcal{B}_0, \tilde{\mu})\frac{dP|\mathcal{B}_0}{d\mu}d\mu\end{aligned}$$

$$= \sum_{n=1}^{\infty} \int_{E_n} E(I_A \frac{dP \mid \mathcal{B}_0}{d\mu} \mid \mathcal{B}_0, \tilde{\mu}) d\mu$$

$$= \sum_{n=1}^{\infty} \int_{E_n} I_A \frac{dP \mid \mathcal{B}_0}{d\mu} d\tilde{\mu}$$

$$= \int_{T_P \mid \mathcal{B}_0} I_A \frac{dP \mid \mathcal{B}_0}{d\mu} d\tilde{\mu} = \int_A \frac{dP \mid \mathcal{B}_0}{d\mu} d\tilde{\mu} \tag{3.3}$$

Hence $\frac{dP \mid \mathcal{B}_0}{d\mu}$ is a $\mathcal{B}_0$-measurable density of P with respect to $\tilde{\mu}$.

Let $\mathcal{B}$ be any sufficient sub-σ-field for $\mathcal{P}$. By $\mathcal{B}_0 \subset \mathcal{B}[\mathcal{P}]$, there exists a $\mathcal{B}$-measurable function f_P such that $f_P = \frac{dP \mid \mathcal{B}_0}{d\mu}$ $[Q]$ for all Q in $\mathcal{P}$. Hence $f_P = \frac{dP \mid \mathcal{B}_0}{d\mu}$ $[\tilde{\mu}]$ because $\mathcal{P} \sim \tilde{\mu}$. The function f_P is clearly a $\mathcal{B}$-measurable density of P with respect to $\tilde{\mu}$. On the other hand, by Theorem 2.1 (1) and (2), $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is weakly dominated by $\tilde{\mu}$. Consequently $\tilde{\mu}$ is a pivotal measure. This completes the proof.

Theorem 3.2. *Let $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be weakly dominated. A measure $\tilde{\mu}$ on $(\mathcal{X}, \mathcal{A})$ is pivotal if and only if for each sufficient sub-σ-field $\mathcal{B}$ for $\mathcal{P}$ the following conditions are satisfied:*

(1) *$(\mathcal{X}, \mathcal{B}, \mathcal{P} \mid \mathcal{B})$ is weakly dominated by $\tilde{\mu} \mid \mathcal{B}$.*

(2) *$E(I_A \mid \mathcal{B}) = E(I_A \mid \mathcal{B}, \tilde{\mu})$ $[\tilde{\mu} \mid \mathcal{B}]$ holds for all A in $\mathcal{A}$.*

The conditions (1) *and* (2) *are quivalent to* (1) *and*

(2)′ *$\tilde{\mu}(A \cap B) = \int_B E(I_A \mid \mathcal{B}) d\tilde{\mu} \mid \mathcal{B}$ holds for all A in $\mathcal{A}$ and B in $\mathcal{B}_e(\tilde{\mu} \mid \mathcal{B})$.*

Proof. Let $\tilde{\mu}$ be a pivotal measure and let $\mathcal{B}$ be any sufficient sub-σ-field for $\mathcal{P}$. Then for each P in $\mathcal{P}$,

$$P \mid \mathcal{B}(B) = P(B) = \int_B \frac{dP}{d\tilde{\mu}} d\tilde{\mu} = \int_B \frac{dP}{d\tilde{\mu}} d\tilde{\mu} \mid \mathcal{B}$$

for all B in $\mathcal{B}$, where $\frac{dP}{d\tilde{\mu}}$ is a $\mathcal{B}$-measurable density of P with respect to $\tilde{\mu}$. Hence $P \mid \mathcal{B}$ has a ($\mathcal{B}$-measurable) density with respect to $\tilde{\mu} \mid \mathcal{B}$. By Lemma 2.2, $\tilde{\mu} \mid \mathcal{B}$ has the finite subset property. Since $\mathcal{B}$ is sufficient for $\mathcal{P}$, by Theorem 2.2 in [3], $(\mathcal{X}, \mathcal{B}, \mathcal{P} \mid \mathcal{B})$ is weakly dominated. Let it be weakly dominated by μ'. Then it follows that $\mu' \sim \mathcal{P} \mid \mathcal{B} \sim \tilde{\mu} \mid \mathcal{B}$. By Lemma 2.1, $\tilde{\mu} \mid \mathcal{B}$ is localizable. Hence (1) is satisfied.

Since $(\mathcal{X}, \mathcal{B}, \mathcal{P} \mid \mathcal{B})$ is weakly dominated by $\tilde{\mu} \mid \mathcal{B}$ each $P \mid \mathcal{B}$ has a ($\mathcal{B}$-measu rable) density $\frac{dP \mid \mathcal{B}}{d\tilde{\mu} \mid \mathcal{B}}$ with respect to $\tilde{\mu} \mid \mathcal{B}$. By the assumption it is easy to show that $\frac{dP \mid \mathcal{B}}{d\tilde{\mu} \mid \mathcal{B}}$ is also a $\mathcal{B}$-measurable density of P with respect to $\tilde{\mu}$.

Then we can define for each $P|\mathcal{B}$ in $\mathcal{P}|\mathcal{B}$, $T_{P|\mathcal{B}}$ and $\{E_n\}$ as in the proof of Theorem 3.1 by changing $\mathcal{B}_0$ and μ into $\mathcal{B}$ and $\tilde{\mu}|\mathcal{B}$ respectively.
Then for any A in $\mathcal{A}$ and P in $\mathcal{P}$,

$$\begin{aligned}
P(A) &= \int_A \frac{dP|\mathcal{B}}{d\tilde{\mu}|\mathcal{B}}\, d\tilde{\mu} \\
&= \int_{T_{P|\mathcal{B}}} I_A \frac{dP|\mathcal{B}}{d\tilde{\mu}|\mathcal{B}}\, d\tilde{\mu} \\
&= \sum_{n=1}^{\infty} \int_{E_n} E\Big(I_A \frac{dP|\mathcal{B}}{d\tilde{\mu}|\mathcal{B}} \,|\, \mathcal{B}, \tilde{\mu}\Big) d\tilde{\mu}|\mathcal{B} \\
&= \sum_{n=1}^{\infty} \int_{E_n} E(I_A|\mathcal{B}, \tilde{\mu}) \frac{dP|\mathcal{B}}{d\tilde{\mu}|\mathcal{B}}\, d\tilde{\mu}|\mathcal{B} \\
&= \sum_{n=1}^{\infty} \int_{E_n} E(I_A|\mathcal{B}, \tilde{\mu})\, dP|\mathcal{B} \\
&= \int_{T_{P|\mathcal{B}}} E(I_A|\mathcal{B}, \tilde{\mu})\, dP|\mathcal{B} \\
&= \int_{\mathcal{X}} E(I_A|\mathcal{B}, \tilde{\mu})\, dP|\mathcal{B}\,.
\end{aligned}$$

On the other hand,

$$P(A) = \int_{\mathcal{X}} E(I_A|\mathcal{B})\, dP|\mathcal{B}\,.$$

Hence

$$\int_{\mathcal{X}} E(I_A|\mathcal{B})\, dP|\mathcal{B} = \int_{\mathcal{X}} E(I_A|\mathcal{B}, \tilde{\mu})\, dP|\mathcal{B}$$

holds for each A in $\mathcal{A}$ and P in $\mathcal{P}$. Therefore for any A in $\mathcal{A}$ and B in $\mathcal{B}$,

$$\begin{aligned}
\int_B E(I_A|\mathcal{B})\, dP|\mathcal{B} &= \int_{\mathcal{X}} E(I_{A\cap B}|\mathcal{B})\, dP|\mathcal{B} \\
&= \int_{\mathcal{X}} E(I_{A\cap B}|\mathcal{B}, \tilde{\mu})\, dP|\mathcal{B} \\
&= \int_B E(I_A|\mathcal{B}, \tilde{\mu})\, dP|\mathcal{B}\,.
\end{aligned}$$

Thus we have

$$E(I_A|\mathcal{B}) = E(I_A|\mathcal{B}, \tilde{\mu})\ [P|\mathcal{B}]$$

for all A in $\mathcal{A}$ and $P|\mathcal{B}$ in $\mathcal{P}|\mathcal{B}$. And finally we have

$$E(I_A|\mathcal{B}) = E(I_A|\mathcal{B}, \tilde{\mu})\ [\tilde{\mu}|\mathcal{B}]$$

for all A in $\mathcal{A}$ bacause $\mathcal{P}|\mathcal{B} \sim \tilde{\mu}|\mathcal{B}$. Hence (2) is satisfied.

Let (1) and (2) be satisfied. Carrying out the calculation leading to (3.3) we have that $\frac{dP|\mathcal{B}_0}{d\tilde{\mu}|\mathcal{B}_0}$ is a $\mathcal{B}_0$-measurable density of P with respect to $\tilde{\mu}$, where $\mathcal{B}_0$ is a minimal sufficient σ-field for $\mathcal{P}$. Then, just as in the final paragraph of the proof of Theorem 3.1, $\tilde{\mu}$ is a pivotal measure.

We next prove the equivalence of (1), (2) and (1), (2)$'$. Let $\tilde{\mu}$ satisfy (1) and (2). For any A in $\mathcal{A}$ and B in $\mathcal{B}_e(\tilde{\mu}|\mathcal{B})$,

$$\begin{aligned}\tilde{\mu}(A\cap B) &= \int_B I_A d\tilde{\mu} \\ &= \int_B E(I_A|\mathcal{B}, \tilde{\mu})d\tilde{\mu}|\mathcal{B} \\ &= \int_B E(I_A|\mathcal{B})d\tilde{\mu}|\mathcal{B}\,.\end{aligned}$$

Conversely let (1) and (2)$'$ be satisfied. For any A in $\mathcal{A}$ and B in $\mathcal{B}_e(\tilde{\mu}|\mathcal{B})$,

$$\begin{aligned}\int_B E(I_A|\mathcal{B})d\tilde{\mu}|\mathcal{B} &= \tilde{\mu}(A\cap B) \\ &= \int_B E(I_A|\mathcal{B}, \tilde{\mu})d\tilde{\mu}|\mathcal{B}\,,\end{aligned}$$

Since $\tilde{\mu}|\mathcal{B}$ has the finite subset property,

$$E(I_A|\mathcal{B}) = E(I_A|\mathcal{B}, \tilde{\mu})\ [\tilde{\mu}|\mathcal{B}]$$

for all A in $\mathcal{A}$. This completes the proof.

REMARK 3.2. Using Theorem 3.2, it is easily verified that the measure "$\sum_{n=1}^{\infty} c_n P_n$" in the case of domination ([1]) is a pivotal measure.

Acknowledgement. The author wishes to thank Professor T. Kusama for many useful advices and continuing encouragements. Moreover, he is also thankful to the referee for careful reading and valuable suggestions.

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