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# NON-HOMOGENEOUS KÄHLER-EINSTEIN METRICS ON COMPACT COMPLEX MANIFOLDS II

Dedicated to Professor Shingo Murakami on his sixtieth birthday

NORIHITO KOISO AND YUSUKE SAKANE

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In the previous paper K-S [12] we have considered  $P^{1}(C)$ -bundles over compact Kähler-Einstein manifolds to obtain non-homogeneous Kähler-Einstein manifolds with positive Ricci tensor. The purpose of this paper is to give more examples of non-homogeneous compact Kähler-Einstein manifolds, more precisely, compact almost homogeneous Kähler-Einstein manifolds with disconnected exceptional set. By [1] and [8], the structure of orbits of almost homogeneous projective algebraic manifolds with disconnected exceptional set have been investigated, but no explicit examples were given in [1] and [8] except complex projective spaces. To construct these examples, we start again with  $P^{1}(C)$ -bundles over Kähler C-spaces and consider compact complex manifolds obtained from these  $P^{1}(C)$ -bundles by blowing down. Note that compact complex manifolds obtained from projective algebraic manifolds by blowing down are not Kähler in general as an example of Moisezon [14] Chap. 3, section 3 shows. We construct our compact complex manifolds in section 3 and prove that our compact almost homogeneous complex manifolds are Kähler and have positive first Chern class (Theorem 4.1). But in general these almost homogeneous manifolds may be homogeneous. We give a sufficient condition for these Kähler manifolds being non-homogeneous (Theorem 5.1). In section 6 we show that for each positive integer d there are compact Kähler-Einstein manifolds which have cohomogeneity d. We follow the notation in Kobayashi-Nomizu [11] which is slightly different from the one in [12].

## 1 Kähler C-spaces and Dynkin diagrams

We recall known facts on compact simply connected homogeneous Kähler manifolds, called Kähler C-spaces (cf. Takeuchi [18]).

Let  $\Pi$  be a Dynkin diagram and  $\Pi_0$  a subdiagram of  $\Pi$ . The pair ( $\Pi$ ,  $\Pi_0$ ) is said to be *effective* if  $\Pi_0$  does not contain any irreducible component of  $\Pi$ . Let  $\Sigma$  be the root system with the fundamental root system  $\Pi$ . Choose a lexicographic order > on  $\Sigma$  such that the set of simple roots with respect to > coincides with  $\Pi$ . Take a compact semi-simple Lie algebra  $g_u$  with the root system  $\Sigma$  and let t be a maximal abelian subalgebra of  $\mathfrak{g}_{\mathfrak{u}}$ . Denote by  $\mathfrak{g}$  and  $\mathfrak{h}$  the complexification of  $\mathfrak{g}_{\mathfrak{u}}$  and t respectively. We identify a weight of  $\mathfrak{g}$  relative to the Cartan subalgebra  $\mathfrak{h}$  with an element of  $\sqrt{-1}\mathfrak{t}$  by the duality defined by the Killing form (,) of  $\mathfrak{g}$ . In particular, the root system  $\Sigma$  of  $\mathfrak{g}$  relative to  $\mathfrak{h}$  is a subset of  $\sqrt{-1}\mathfrak{t}$ . Let  $\{\Lambda_{\mathfrak{a}}\}_{\mathfrak{u}\in\Pi}\subset\sqrt{-1}\mathfrak{t}$  be the fundamental weights of  $\mathfrak{g}$  corresponding to  $\Pi$ :

(1.1) 
$$\frac{2(\Lambda_{\alpha},\beta)}{(\beta,\beta)} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

Let  $\Sigma^+$  be the set of all positive roots and  $\{\Pi_0\}_Z$  the subgroup of  $\sqrt{-1}t$  generated by  $\Pi_0$ . Put  $\Sigma_0 = \Sigma \cap \{\Pi_0\}_Z$ . We define a subalgebra  $\mathfrak{u}$  of  $\mathfrak{g}$  by

(1.2) 
$$\mathfrak{u} = \mathfrak{h} + \sum_{\alpha \in \Sigma_0 \cup \Sigma^+} \mathfrak{g}_{\alpha}$$

where  $\mathfrak{g}_{\omega}$  is the root space of  $\mathfrak{g}$  for  $\alpha \in \Sigma$ . Let G be a simply connected complex Lie group whose Lie algebra is  $\mathfrak{g}$ , and let U be the connected (closed) complex subgroup of G generated by  $\mathfrak{u}$ . Put M=G/U. Then it is known that the complex manifold M=G/U is compact, simply connected and admits a homogeneous Kähler metric. Let  $G_u$  be the compact connected subgroup of G generated by  $\mathfrak{g}_u$ . Put  $K=G_u \cap U$ . Then K is connected,  $G_u$  acts on M transitively and  $M=G/U=G_u/K$  as a smooth manifold. This homogeneous complex manifold M is s said to be associated to the pair  $(\Pi, \Pi_0)$  of Dynkin diagrams.

We define a subspace **c** of  $\sqrt{-1}$ **t** by

(1.3) 
$$\mathbf{c} = \sum_{\alpha \in \Pi - \Pi_0} \mathbf{R} \Lambda_{\alpha} \, .$$

Then  $\sqrt{-1}\mathfrak{c}$  coincides with the center of the Lie algebra  $\mathfrak{k}$  of K. We also define lattices Z of  $\sqrt{-1}\mathfrak{k}$  and  $Z_{\mathfrak{c}}$  of  $\mathfrak{c}$  by

(1.4) 
$$Z = \{\lambda \in \sqrt{-1} \mathfrak{t} | 2(\lambda, \alpha) / (\alpha, \alpha) \text{ is an integer for each } \alpha \in \Sigma \}$$

and

$$(1.5) Z_{c} = Z \cap c.$$

Let  $\mathfrak{m}$  be the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}_u$  with respect to the Killing form  $(,); \mathfrak{g}_u = \mathfrak{k} + \mathfrak{m}$ . The subspace  $\mathfrak{m}$  is K-invariant under the adjoint action and identified with the tangent space  $T_o(M)$  of M at the origin  $o \in M$ . Put

(1.6) 
$$\Sigma_{\mathfrak{m}}^{+} = \Sigma^{+} - \Sigma_{\mathfrak{0}}, \quad \Sigma_{\mathfrak{m}}^{-} = -\Sigma_{\mathfrak{m}}^{+}.$$

We define K-invariant subspaces  $\mathfrak{m}^{\pm}$  of  $\mathfrak{g}$  by

(1.7) 
$$\mathfrak{m}^{\pm} = \sum_{\alpha \in \Sigma_{\mathfrak{m}}^{\pm}} \mathfrak{g}_{-\alpha} \,.$$

Then the complexification  $\mathfrak{m}^{c}$  of  $\mathfrak{m}$  is the direct sum:

$$\mathfrak{m}^{c} = \mathfrak{m}^{+} + \mathfrak{m}^{-}$$
.

We denote by  $X \to \overline{X}$  the complex conjugation of  $\mathfrak{g}$  with respect to the real form  $\mathfrak{g}_{\mathfrak{u}}$ . Then  $\mathfrak{m}^{\mp} = \overline{\mathfrak{m}^{\pm}}$ . We choose  $E_{\mathfrak{o}} \in \mathfrak{g}_{\mathfrak{o}}$  for  $\alpha \in \Sigma$  with the following properties and fix them from now on:

(1.8) 
$$[E_{\alpha}, E_{-\alpha}] = -\alpha$$
,  $(E_{\alpha}, E_{-\alpha}) = -1$ ,  $\overline{E}_{\alpha} = E_{-\alpha}$  for  $\alpha \in \Sigma$ .

Let  $\{\omega^{\alpha}\}_{\alpha \in \Sigma}$  be the linear forms on g dual to  $\{E_{\alpha}\}_{\alpha \in \Sigma}$ , that is, linear forms defined by

(1.9) 
$$\begin{cases} \omega^{\boldsymbol{\sigma}}(\mathfrak{h}) = \{0\} \\ \\ \omega^{\boldsymbol{\sigma}}(E_{\boldsymbol{\beta}}) = \begin{cases} 1 & \text{if } \boldsymbol{\alpha} = \boldsymbol{\beta} \\ 0 & \text{if } \boldsymbol{\alpha} \neq \boldsymbol{\beta} \end{cases}$$

Let T be the toral subgroup of  $G_u$  generated by t. The tangent space  $T_e(T)$  of T at the identity element e is identified with t. Let  $\mathcal{I}^1(T)$  denote the space of T-invariant real 1-forms on T. Then we have natural linear isomorphisms:

(1.10) 
$$\mathbf{t} \to \mathbf{t}^* = T^*_{\boldsymbol{e}}(T) \to \mathcal{J}^1(T) \to H^1(T, \boldsymbol{R}).$$

We identify t with  $H^{1}(T, \mathbf{R})$ . Then we have

(1.11) 
$$\frac{1}{2\pi\sqrt{-1}}Z = H^{1}(T, \mathbf{Z}).$$

It is known that the inclusion  $\iota: T \to K$  induces an injective linear map  $\iota^*: H^1(K, \mathbb{R}) \to H^1(T, \mathbb{R})$  with  $\iota^*H^1(K, \mathbb{R}) = 1/(2\pi\sqrt{-1})\mathbf{c}$  and  $\iota^*H^1(K, \mathbb{Z}) = 1/(2\pi\sqrt{-1})Z_c$ , and that the transgression for the principal bundle  $K \to G \to M$  defines a linear isomorphism  $\tau: H^1(K, \mathbb{R}) \to H^2(M, \mathbb{R})$  with  $\tau(H^1(K, \mathbb{Z})) = H^2(M, \mathbb{Z})$ . We define a linear map  $\underline{\tau}: \mathbf{c} \to H^2(M, \mathbb{R})$  by

$$\underline{\tau}(\lambda) = -\tau(\lambda/(2\pi\sqrt{-1}))$$
 for  $\lambda \in \mathfrak{c}$ ,

where  $H^1(K, \mathbf{R})$  is identified with  $1/(2\pi\sqrt{-1})c$  through  $\iota^*$ . Then  $\underline{\tau}(Z_c)$  coincides with  $H^2(M, \mathbf{Z})$  (cf. Borel-Hirzebruch [4]).

We define a cone c<sup>+</sup> in c by

(1.12) 
$$c^{+} = \{\lambda \in c \mid (\lambda, \alpha) > 0 \text{ for each } \alpha \in \Pi - \Pi_{0}\}$$

and put  $Z_{\mathfrak{c}}^+ = Z \cap \mathfrak{c}^+$ . Then we have

N. KOISO AND Y. SAKANE

(1.13) 
$$\mathbf{c}^{+} = \sum_{\boldsymbol{\alpha} \in \Pi - \Pi_{\alpha}} \boldsymbol{R}^{+} \Lambda_{\boldsymbol{\alpha}},$$

(1.14) 
$$Z_{c}^{+} = \sum_{\alpha \in \Pi - \Pi_{0}} \mathbf{Z}^{+} \Lambda_{\sigma} .$$

Moreover, the cone c<sup>+</sup> is characterized by

 $\mathfrak{c}^{+} = \{\lambda \in \mathfrak{c} | (\lambda, \alpha) > 0 \text{ for each } \alpha \in \Sigma_{\mathfrak{m}}^{+} \}$ .

**Lemma 1.1** (Takeuchi [18]). Let  $\mathcal{J}^2_{G_u}(M)$  be the space of closed  $G_u$ -invariant real 2-forms on M and  $\mathcal{H}^2(M, g)$  the space of real harmonic 2-forms on M with respect to a  $G_u$ -invariant Riemannian metric g on M. Then  $\mathcal{J}^2_{G_u}(M) = \mathcal{H}^2(M, g)$ .

Let  $\lambda \in \mathfrak{c}$ . Regarding each  $\omega^{\sigma}$  a  $G_u$ -invariant C-valued 1-form on  $G_u$ , we define a  $G_u$ -invariant C-valued 2-form  $\eta(\lambda)$  on  $G_u$  by

(1.15) 
$$\eta(\lambda) = \frac{1}{2\pi\sqrt{-1}} \sum_{\alpha \in \Sigma_{\mathfrak{m}}^+} (\lambda, \alpha) \omega^{-\omega} \wedge \overline{\omega^{-\omega}} \, .$$

We define a complex linear form  $\tilde{\lambda}$  on  $g_{\mu}$  by

$$\tilde{\lambda}(X) = (\lambda, X)$$
 for  $X \in \mathfrak{g}_u$ ,

and regard  $\tilde{\lambda}$  as a  $G_u$ -invariant C-valued 1-form on  $G_u$ . Thus  $1/(2\pi\sqrt{-1})\tilde{\lambda}$  is regarded as a  $G_u$ -invariant R-valued 1-form on  $G_u$ . Then we have

$$\eta(\lambda) = -d(1/(2\pi\sqrt{-1})\tilde{\lambda})$$
 ,

and  $\eta(\lambda)$  can be pulled down to a unique form in  $\mathcal{J}^2_{G_u}(M)$ . Thus the correspondence  $\lambda \to \eta(\lambda)$  defines a linear map  $\eta: \mathfrak{c} \to \mathcal{J}^2_{G_u}(M)$ .

**Lemma 1.2** (Takeuchi [18]). Let  $\psi$  be the natural map assigning  $\omega \in \mathcal{J}^2_{G_u}(M)$  to the de Rham class  $[\omega]$  in  $H^2(M, \mathbb{R})$ . Then we have the following commutative diagram consisting of linear isomorphisms:

(1.16) 
$$c \xrightarrow{\underline{\tau}} H^{2}(M, \mathbf{R})$$
$$\eta \bigvee_{\mathcal{J}_{G_{u}}^{2}} / \psi$$

We define elements  $\delta_{m}$ ,  $\delta$  of  $\sqrt{-1}t$  by

(1.17) 
$$\delta_{\mathfrak{m}} = \frac{1}{2} \sum_{\alpha \in \Sigma_{\mathfrak{m}}^+} \alpha, \quad \delta = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha$$

respectively. It is known that  $2\delta_m \in Z_c^+$  and  $\delta = \sum_{\alpha \in \Pi} \Lambda_{\alpha}$ . Now we recall the following facts.

FACT 1 (cf. Borel-Hirzebruch [4], Takeuchi [18]). Let  $M = G/U = G_{*}/K$  be the compact homogeneous complex manifold associated to an effective pair  $(\Pi, \Pi_0)$  of Dynkin diagrams. Then we have the followings.

1) For  $\lambda \in \mathfrak{c}$ ,

(1.18) 
$$g(\lambda) = \frac{1}{2\pi} \sum_{\alpha \in \Sigma_{\mathfrak{m}}^+} (\lambda, \alpha) \omega^{-\alpha} \cdot \overline{\omega^{-\alpha}}$$

defines a  $G_u$ -invariant real covariant symmetric tensor field of degree 2 on M, and the correspondence  $\lambda \rightarrow g(\lambda)$  gives a bijection from  $c^+$  to the set of  $G_u$ -invariant Kähler metrics on M.

2) The first Chern class  $c_1(M)$  of M is given by  $c_1(M) = \underline{\tau}(-2\delta_m)$ . For the Kähler metric g corresponding to  $\lambda \in c^+$ , the Kähler form  $\omega$  (defined by  $\omega(X, Y) = g(X, JY)$ , where J is the almost complex structure of M), the Ricci tensor r and the Ricci form  $\rho$  are given by

(1.19) 
$$\omega = \eta(\lambda) = -(\sqrt{-1}/(2\pi)) \sum_{\alpha \in \Sigma_{\mathfrak{m}}^+} (\lambda, \alpha) \omega^{-\omega} \wedge \overline{\omega^{-\omega}},$$

(1.20) 
$$r = 4\pi g(2\delta_{\mathfrak{m}}) = 2 \sum_{\alpha \in \Sigma_{\mathfrak{m}}^{+}} (2\delta_{\mathfrak{m}}, \alpha) \omega^{-\alpha} \cdot \overline{\omega^{-\alpha}},$$

(1.21) 
$$\rho = 4\pi\eta(2\delta_{\mathfrak{m}}) = -2\sqrt{-1}\sum_{\alpha\in\Sigma_{\mathfrak{m}}^+} (2\delta, \alpha)\omega^{-\sigma}\wedge\overline{\omega^{-\sigma}}.$$

FACT 2 (cf. Ise [9]). For each  $\Lambda \in \mathbb{Z}_c$ , there is a unique holomorphic character  $\mathfrak{X}_{\Lambda}$  of U such that

$$\chi_{\Lambda}(\exp H) = \exp(\Lambda, H)$$
 for each  $H \in \mathfrak{h}$ .

Let  $L_{\Lambda}$  denote the holomorphic line bundle on M associated to the principal bundle  $U \rightarrow G \rightarrow M$  by the character  $\chi_{\Lambda}$ . The correspondence  $\Lambda \rightarrow L_{\Lambda}$  induces an isomorphism from  $Z_{c}$  onto the group  $H^{1}(M, \theta^{*})$  of all holomorphic line bundles on M. Moreover, under this isomorphism the subset  $-Z_{c}^{+}$  corresponds to the set of all very ample holomorphic line bundles on M. The first Chern class  $c_{1}(L_{\Lambda})$  of  $L_{\Lambda}$  contains a unique  $G_{u}$ -invariant 2-form

(1.22) 
$$\eta(\Lambda) = -\frac{\sqrt{-1}}{2\pi} \sum_{\alpha \in \Sigma_{\mathfrak{m}}^{+}} (\Lambda, \alpha) \omega^{-\omega} \wedge \overline{\omega^{-\omega}}$$

on M.

### 2 Kähler C-spaces as projective bundles

Let E be a holomorphic vector bundle of rank r over a complex manifold N. The complex projective bundle P(E) associated to E is defined as follows. Let  $C^*$  act freely on E-(0-section) by scalar multiplication. Then P(E) is the quotient complex manifold

$$P(E) = E - (0 - \operatorname{section})/C^*$$
.

Thus a point of P(E) over  $x \in N$  represents a complex line in the fiber  $E_x$  of E at x. We organize various spaces and maps by the following diagram:

$$(2.1) \qquad P(E) \xrightarrow{\pi} E - (0 \operatorname{-section}) \subseteq E$$

Using the projection  $\varphi: P(E) \to N$ , we pull back the bundle E to obtain the vector bundle  $\varphi^*E$  of rank r over P(E). We define the tautological line bundle L(E) over P(E) as a subbundle of  $\varphi^*E$  as follows. The fiber  $L(E)_{\xi}$  at  $\xi \in P(E)$  is the complex line in  $E_{\varphi(\xi)}$  represented by  $\xi$ . Note also that if L is a holomorphic line bundle over N, then P(E) is canonically identified with  $P(E \otimes L)$  as complex manifolds and  $L(E \otimes L) = L(E) \otimes \varphi^*L$  as holomorphic line bundles.

Let  $\Pi$  be a Dynkin diagram and  $\Pi_0$  a subdiagram of  $\Pi$  such that  $\Pi_0$  is of type  $A_{l-1}$  ( $\Pi_0 = \emptyset$  if l=1). Consider also a subdiagram  $\Pi_1$  such that  $\Pi_1$  contains  $\Pi_0$  as a subdiagram and  $\Pi_1$  is of type  $A_l$ . Put  $\Sigma_1 = \Sigma \cap {\{\Pi_1\}_Z}$ . We define a Lie subalgebra  $\mathfrak{p}$  of  $\mathfrak{g}$  by

$$(2.2) \qquad \qquad \mathfrak{p} = \mathfrak{h} + \sum_{\alpha \in \Sigma_1 \cup \Sigma^+} \mathfrak{g}_{\alpha}$$

as in section 1. We denote by G/U, G/P the Kähler C-spaces associated to the pairs  $(\Pi, \Pi_0), (\Pi, \Pi_1)$  of Dynkin diagrams respectively. Put  $\{\alpha_0\} = \Pi_1 - \Pi_0$  and  $\Lambda_0 = \Lambda_{\alpha_0} \in \mathbb{Z}$ . We define a subalgebra  $\mathfrak{g}(1)$  of  $\mathfrak{p}$  by

(2.3) 
$$g(1) = \mathfrak{h} + \sum_{\alpha \in \Sigma_1} \mathfrak{g}_{\alpha},$$

and let G(1) be the complex subgroup of G generated by  $\mathfrak{g}(1)$ . Then there is an irreducible representation  $\rho_{\Lambda_0}: G(1) \to GL(V_{\Lambda_0})$  of G(1) with the highest weight  $\Lambda_0$ . The representation  $\rho_{\Lambda_0}$  can be uniquely extended to an irreducible representation of P, which is also denoted by  $\rho_{\Lambda_0}: P \to GL(V_{\Lambda_0})$ . Note also that  $\dim_{\mathfrak{C}} V_{\Lambda_0} = l+1$ .

We denote by  $E_{\Lambda_0}$  the homogeneous vector bundle over G/P defined by the representation  $\rho_{\Lambda_0}: P \to GL(V_{\Lambda_0})$  and by  $P(E_{\Lambda_0})$  the complex projective bundle over G/P associated to the vector bundle  $E_{\Lambda_0}$ . Then G acts on  $E_{\Lambda_0}$  and  $P(E_{\Lambda_0})$  in natural ways. We denote by [g, v] the element of  $E_{\Lambda_0}$  defined by  $(g, v) \in G \times V_{\Lambda_0}$  and let  $p: E_{\Lambda_0} - (0\text{-section}) \to P(E_{\Lambda_0})$  be the projection. Take a highest weight vector  $v_{\Lambda_0}$  of  $\rho_{\Lambda_0}: P \to GL(V_{\Lambda_0})$ , that is,  $v_{\Lambda_0}$  is a non-zero vector of  $V_{\Lambda_0}$  such that

(2.4) 
$$\begin{cases} \rho_{\Lambda_0}(H)v_{\Lambda_0} = (\Lambda_0, H)v_{\Lambda_0} & \text{for } H \in \mathfrak{h} \\ \rho_{\Lambda_0}(E_{\alpha})v_{\Lambda_0} = 0 & \text{for } E_{\alpha} \in \mathfrak{g}_{\alpha}, \ \alpha \in \Sigma^+ \end{cases}$$

and fix it.

**Lemma 2.1.** We have an identification:  $P(E_{\Lambda_0}) = G/U$ .

Proof. At first note that G acts on  $P(E_{\Lambda_0})$  transitively, since G(1) acts on  $P(V_{\Lambda_0})$  transitively. Put  $o = p([e, v_{\Lambda_0}])$ . Consider the isotropy subgroup  $G_o$  of G at  $o \in P(E_{\Lambda_0})$ . Then we have  $G_o = \{g \in G \mid g \in P, \rho_{\Lambda_0}(g)v_{\Lambda_0} = \lambda(g)v_{\Lambda_0} \text{ for some } \lambda(g) \in C - (0)\}$ . Thus the Lie algebra  $\mathfrak{g}_o$  of  $G_o$  is given by

(2.5) 
$$\mathfrak{g}_{o} = \{X \in \mathfrak{p} | \rho_{\Lambda_{0}}(X) v_{\Lambda_{0}} \in \mathbb{C} v_{\Lambda_{0}} \}$$
$$= \mathfrak{h} + \sum_{(\Lambda_{0}, \alpha) \geq 0, \alpha \in \Sigma_{1} \cup \Sigma^{+}} \mathfrak{g}_{\sigma} = \mathfrak{h} + \sum_{\alpha \in \Sigma_{0} \cup \Sigma^{+}} \mathfrak{g}_{\sigma} .$$

Hence  $\mathfrak{g}_{\mathfrak{o}} = \mathfrak{u}$ . Since the normalizer of the parabolic subgroup U coincides with U, we see that  $U = G_{\mathfrak{o}}$  and  $P(E_{\Lambda_0}) = G/G_{\mathfrak{o}} = G/U$ . q.e.d.

Now we consider the homogeneous vector bundle  $E_{\Lambda_0}$  over G/P. Then  $E_{\Lambda_0}$ —(0-section) is a  $C^*$ -bundle over  $P(E_{\Lambda_0})$ . Let  $L(E_{\Lambda_0})$  be the tautological line bundle over  $P(E_{\Lambda_0})$  associated to the vector bundle  $E_{\Lambda_0}$  over G/P. Then we have an identification:  $E_{\Lambda_0}$ —(0-section)= $L(E_{\Lambda_0})$ —(0-section).

**Lemma 2.2.** The tautological line bundle  $L(E_{\Lambda_0})$  is the holomorphic line bundle  $L_{\Lambda_0}$  over  $P(E_{\Lambda_0}) = G/U$  associated to the principal bundle  $U \rightarrow G \rightarrow G/U$  by the character  $\chi_{\Lambda_0}$  of U.

Proof. Since  $(\Lambda_0, \alpha) = 0$  for each  $\alpha \in \Sigma_0$ ,  $\rho_{\Lambda_0}(E_{\sigma})v_{\Lambda_0} = 0$  for each  $\alpha \in \Sigma_0$ . Thus  $\rho_{\Lambda_0}$  induces a representation  $\rho_{\Lambda_0}: U \to GL(Cv_{\Lambda_0})$ , which is identified with the character  $\chi_{\Lambda_0}$  of U, since  $\rho_{\Lambda_0}(\exp H)v_{\Lambda_0} = \exp(\Lambda_0, H)v_{\Lambda_0}$  for  $H \in \mathfrak{h}$ . Note that by Lemma 2.1 each element of  $L(E_{\Lambda_0})$  can be written as  $[g, \lambda v_{\Lambda_0}] (g \in G, \lambda \in C)$ . Now let  $[g, \lambda v_{\Lambda_0}], [g', \mu v_{\Lambda_0}]$  be elements of  $L(E_{\Lambda_0})$ . Then  $[g, \lambda v_{\Lambda_0}] = [g', \mu v_{\Lambda_0}]$  in  $L(E_{\Lambda_0})$  if and only if  $g' = gu (u \in U)$  and  $\rho_{\Lambda_0}(u)\mu v_{\Lambda_0} = \lambda v_{\Lambda_0}$ . Thus we get our claim.

Now we recall the following general formula for the canonical line bundle of a projective bundle. Let  $\varphi: E \to N$  be a holomorphic vector bundle of rank r over a complex manifold N and let  $K_{P(E)}$ ,  $K_N$  denote the canonical line bundle on P(E), N respectively. Then

(2.7) 
$$K_{P(E)} = \varphi^*(K_N \otimes \det E^*) \otimes L(E)'$$

where det  $E^*$  denotes the holomorphic line bundle  $\wedge E^*$ .

We apply this formula to compute the first Chern class of  $P(E_{\Lambda_0}) = G/U$ .

**Lemma 2.3.** The element  $-2\delta_{\mathfrak{m}} \in Z_{\mathfrak{c}}^+$  corresponding to the first Chern class  $c_1(P(E_{\Lambda_0}))$  of  $P(E_{\Lambda_0})=G/U$  is given by

(2.8) 
$$-2\delta_{\mathfrak{m}} = -(l+1)\Lambda_{\mathfrak{o}} + \sum_{\alpha \in \Pi - \Pi_{1}} -n_{\mathfrak{o}}\Lambda_{\mathfrak{o}} \quad \text{for some} \quad n_{\mathfrak{o}} \in \mathbb{N}.$$

Proof. Since  $2\delta_{\mathfrak{m}} \in Z_{\mathfrak{c}}^+$ , it is of the form

$$2\delta_{\mathfrak{m}} = \sum_{\alpha \in \Pi - \Pi_0} n_{\alpha} \Lambda_{\omega} \qquad (n_{\omega} \in N) .$$

Since  $K_{G/P} \otimes \det E^*_{\Lambda_0}$  is a holomorphic line bundle over G/P, the Chern class of  $\varphi^*(K_{G/P} \otimes \det E^*_{\Lambda_0})$  contains a unique  $G_u$ -invariant 2-form  $\eta(\Lambda_1)$  with

$$\Lambda_1 = \sum_{\alpha \in \Pi - \Pi_1} m_{\sigma} \Lambda_{\sigma} \qquad (m_{\sigma} \in \mathbf{Z}) \,.$$

By Lemma 2.2 and Fact 2, the first Chern class  $c_1(L(E_{\Lambda_0}))$  of  $L(E_{\Lambda_0})$  contains a unique  $G_u$ -invariant 2-form  $\eta(\Lambda_0)$ . Since  $E_{\Lambda_0}$  is a holomorphic vector bundle of rank l+1, we see that

$$2\delta_{\mathfrak{m}} = \Lambda_1 + (l+1)\Lambda_0$$

by the formula (2.7) and Fact 2, and hence we get our claim. q.e.d.

REMARK. We may prove Lemma 2.3 by a computation on root systems as follows. Put  $\Pi_0 = \{\alpha_{i_1}, \dots, \alpha_{i_{l-1}}\}$ . Since  $\Pi_0$  is of type  $A_{l-1}$ , we have

$$\sum_{\alpha \in \Sigma_0^+} \alpha = (l-1)\alpha_{i_1} + \dots + j(l-j)\alpha_{i_j} + \dots + (l-1)\alpha_{i_{l-1}},$$

where  $\Sigma_0^+ = \Sigma^+ \cap \Sigma_0$ . Since  $\Pi_i$  is of type  $A_i$ 

Since  $\Pi_1$  is of type  $A_l$  and  $\Pi_1 - \Pi_0 = \{\alpha_0\}$ , we may assume that

$$\frac{2(\alpha_0, \alpha_{i_1})}{(\alpha_0, \alpha_0)} = -1, \quad \frac{2(\alpha_0, \alpha_{i_j})}{(\alpha_0, \alpha_0)} = 0 \quad \text{for} \quad 2 \le j \le l - 1.$$

Thus we see that

$$\sum_{\alpha \in \Sigma_0^+} \alpha = -(l-1)\Lambda_0 + \sum_{\alpha \in \Pi - \{\alpha_0\}} m_{\alpha} \Lambda_{\alpha} \qquad (m_{\alpha} \in \mathbb{Z})$$

Hence we have

$$\begin{split} 2\delta_{\mathfrak{m}} &= 2\delta - \sum_{\alpha \in \Sigma_{0}^{+}} \alpha = 2(\sum_{\alpha \in \Pi} \Lambda_{\boldsymbol{\sigma}}) + (l-1)\Lambda_{0} - \sum_{\alpha \in \Pi - \{\alpha_{0}\}} m_{\boldsymbol{\sigma}} \Lambda_{\boldsymbol{\sigma}} \\ &= (l+1)\Lambda_{0} + \sum_{\alpha \in \Pi - \{\alpha_{0}\}} n_{\boldsymbol{\sigma}} \Lambda_{\boldsymbol{\sigma}} = (l+1)\Lambda_{0} + \sum_{\alpha \in \Pi - \Pi_{1}} n_{\boldsymbol{\sigma}} \Lambda_{\boldsymbol{\sigma}} , \end{split}$$

where  $n_s \in N$  for each  $\delta \in \Pi - \Pi_1$ , since  $2\delta_m \in Z_c^+$ .

#### 3 $P^{1}(C)$ -bundles over Kähler C-spaces and blowing down

Let  $N_1$ ,  $N_2$  be compact complex manifolds and consider holomorphic vector

bundles  $E_1$  of rank  $l+1\geq 2$  over  $N_1$ ,  $E_2$  of rank  $k+1\geq 2$  over  $N_2$ . We also assume that the total spaces  $P(E_1)$  and  $P(E_2)$  of projective bundles coincide as complex manifolds, which is denoted by M, and that there are holomorphic line bundles  $L'_1$  over  $N_1$  and  $L'_2$  over  $N_2$  such that the tautological line bundles  $L(E_1 \otimes L'_1^{-1})$  over  $P(E_1 \otimes L'_1^{-1})$  and  $L(E_2 \otimes L'_2^{-1})$  over  $P(E_2 \otimes L'_2^{-2})$  satisfy  $L(E_1 \otimes L'_1^{-1})^{-1} = L(E_2 \otimes L'_2^{-1})$ , more precisely, there is a holomorphic bundle isomorphism  $L(E_1 \otimes L'_1^{-1})^{-1} \to L(E_2 \otimes L'_2^{-1})$  compatible with the identification:  $P(E_1 \otimes L'_1^{-1}) \cong P(E_1) \cong P(E_2) \cong P(E_2 \otimes L'_2^{-1})$ . We also consider the  $P^1(C)$ -bundle  $P(1 \oplus L(E_1 \otimes L'_1^{-1})) = P(L(E_2 \otimes L'_2^{-1}) \oplus 1)$  over  $M = P(E_1 \otimes L'_1^{-1}) = P(E_2 \otimes L'_2^{-1})$ , whose total space is denoted by X. Note that complex submanifolds  $M_1$ ,  $M_2$  of X defined by the 0-section of  $L(E_1 \otimes L'_1^{-1})$  and 0-section of  $L(E_2 \otimes L'_2^{-1})$  are identified with  $M = P(E_1 \otimes L'_1^{-1})$  and  $M = P(E_2 \otimes L'_2^{-1})$  respectively.

We organize various spaces and maps by the following diagram:

Now the following lemma is a special case of Nakano [16], Fujiki-Nakano [6] (cf. Moisezon [14]).

**Lemma 3.1.** There exists a complex manifold Y containing  $N_1$ ,  $N_2$  as complex submanifolds and a holomorphic map  $\Phi: X \to Y$  in such a way that  $(X, \Phi)$  is a composition of monoidal transforms from Y with centers  $N_1$ ,  $N_2$  and  $M_1 = \Phi^{-1}(N_1)$ ,  $M_2 = \Phi^{-1}(N_2)$ , that is, Y is a complex manifold obtained from X by blowing down  $M_1 = P(E_1 \otimes L_1^{\prime -1})$  to  $N_1$  and  $M_2 = P(E_2 \otimes L_2^{\prime -1})$  to  $N_2$ .

Proof. Note that the normal bundle of  $P(E_1 \otimes L_1^{\prime^{-1}})$  is the line bundle  $L(E_1 \otimes L_1^{\prime^{-1}})$ . Thus the condition in Fujiki-Nakano [6] is satisfied. q.e.d.

REMARK. Note that the tautological line bundle L(E) over a projective bundle P(E) is obtained from E by blowing up the 0-section of E to P(E). Note also that  $P(1 \oplus L(E_1 \otimes L_1^{(-1)}))$  is a union of complex submanifolds  $L(E_1 \otimes L_1^{(-1)})$ and  $L(E_2 \otimes L_2^{(-1)})$  with the intersection  $L(E_1 \otimes L_1^{(-1)}) \cap L(E_2 \otimes L_2^{(-1)}) = L(E_1 \otimes L_1^{(-1)}) (0\text{-section}) = L(E_2 \otimes L_2^{(-1)}) - (0\text{-section})$ . Thus Y is a union of the canonically imbedded complex submanifolds  $E_1 \otimes L_1^{(-1)}$  and  $E_2 \otimes L_2^{(-1)}$  with the intersection  $E_1 \otimes L_1^{-1} \cap E_2 \otimes L_2^{(-1)} = E_1 \otimes L_1^{(-1)} - (0\text{-section}) = E_2 \otimes L_2^{(-1)} - (0\text{-section})$ , which is also  $L(E_1 \otimes L_1^{(-1)}) - (0\text{-section}) = L(E_2 \otimes L_2^{(-1)}) - (0\text{-section})$ .

Now we consider the triples  $(\Pi, \Pi_1^1, \Pi_0)$ ,  $(\Pi, \Pi_1^2, \Pi_0)$  of Dynkin diagrams which are one of the followings.

(a) The Dynkin diagram  $\Pi$  is connected,  $\Pi_0$  is a subdiagram of  $\Pi$  and

of type  $A_{I-1}$ , and subdiagrams  $\Pi_1^1$ ,  $\Pi_1^2$  of  $\Pi$  are of type  $A_I$  and contain  $\Pi_0$  as a subdiagram.

(b) The Dynkin diagram  $\Pi$  has two connected components  $\Pi(1)$  and  $\Pi(2)$ , and  $\Pi_0$  is a subdiagram of  $\Pi$  which has also two connected components  $\Pi_0(1)$  of type  $A_{l-1}$  and  $\Pi_0(2)$  of type  $A_{k-1}$ . Subdiagrams  $\Pi_1^1$ ,  $\Pi_1^2$  of  $\Pi$  have also two connected components  $\Pi_1^1(1)$  and  $\Pi_1^1(2)$ ,  $\Pi_1^2(1)$  and  $\Pi_1^2(2)$  respectively, and we assume they satisfy the following conditions:

(1)  $\Pi_1^1(1)$  is a subdiagram of  $\Pi(1)$ , of type  $A_1$  and contains  $\Pi_0(1)$  as a subdiagram, and  $\Pi_1^1(2)$  conicides with  $\Pi_0(2)$ .

(2)  $\Pi_1^2(2)$  is a subdiagram of  $\Pi(2)$ , of type  $A_k$  and contains  $\Pi_0(2)$  as a subdiagram, and  $\Pi_1^2(1)$  conicides with  $\Pi_0(1)$ .

EXAMPLES 3.1. The vertices contained in  $\Pi_0$ ,  $\Pi_1^i - \Pi_0$ ,  $\Pi - \Pi_1^i$  of a Dynkin diagram  $\Pi$  are denoted by  $\bigcirc$ ,  $\Box$ ,  $\times$  for i=1, 2 respectively.

(a)	$(\Pi, \Pi^1_1, \Pi_0)$	×		
$(\Pi, \Pi_1^2, \Pi_0)$		x x 0 0 0 0 x		
(b)	(Π, Π <sup>1</sup> <sub>1</sub> , Π <sub>0</sub> )	×	×00	
	$(\Pi, \Pi_1^2, \Pi_0)$	×	×0	

Put  $\{\alpha_0(i)\}=\Pi_1^i-\Pi_0$  and  $\Lambda_0(i)=\Lambda_{\alpha_0(i)}$  for i=1, 2.

We consider Kähler C-spaces associated to pairs of Dynkin diagrams and  $P^{I}(C)$ -bundles over Kähler C-spaces.

Case (a). We denote by G/U,  $G/P_1$ ,  $G/P_2$  the Kähler C-spaces associated to the pairs (II, II<sub>0</sub>), (II, II<sub>1</sub><sup>1</sup>), (II, II<sub>1</sub><sup>2</sup>) respectively, and by  $E_1$ ,  $E_2$  the homogeneous vector bundles  $E_{\Lambda_0(1)}$ ,  $E_{\Lambda_0(2)}$  over  $G/P_1$ ,  $G/P_2$  respectively. By Lemma 2.1, we have  $M=P(E_1)=P(E_2)=GU$ , and  $L(E_1)=L_{\Lambda_0(1)}$ ,  $L(E_2)=L_{\Lambda_0(2)}$  by Lemma 2.2. Put  $L_1H=L_{\Lambda_0(1)}$  and  $L_2=L_{\Lambda_0(2)}$ . Note that there is a holomorphic line bundle  $L'_1$  (resp.  $L'_2$ ) over  $N_1=G/P_1$  (resp. over  $N_2=G/P_2$ ) such that  $\varphi_1^*L'_1=L_2$  (resp.  $\varphi_2^*L'_2=L_1$ ), where  $\varphi_1: M=G/U \rightarrow N_1=G/P_1$  (resp.  $\varphi_2: M=G/U \rightarrow N_2=G/P_2$ ) is the projection. We thus have  $L(E_1 \otimes L'_1^{-1})=L_1 \otimes L_2^{-1}=L(E_2 \otimes L'_2^{-1})^{-1}$ . Note also that the  $P^1(C)$ -bundle X is given by  $P(1 \oplus L_1 \otimes L_2^{-1})$ .

Case (b). We denote by  $G_1/U_1$ ,  $G_1/P_1$ ,  $G_2/U_2$ ,  $G_2/P_2$  the Kähler C-spaces associated to the pairs ( $\Pi(1)$ ,  $\Pi_0(1)$ ), ( $\Pi(1)$ ,  $\Pi_1^1(1)$ ), ( $\Pi(2)$ ,  $\Pi_0(2)$ ), ( $\Pi(2)$ ,  $\Pi_1^2(2)$ ) respectively and by  $E_1$ ,  $E_2$  the homogeneous vector bundles  $E_{\Lambda_0(1)}$ ,  $E_{\Lambda_0(2)}$  over  $G_1/P_1$ ,  $G_2/P_2$  respectively. We regard the vector bundle  $E_1$  over  $G_1/P_1$  (resp.  $E_2$ over  $G_2/P_2$ ) as a vector bundle over  $N_1 = G_1/P_1 \times G_2/U_2$  (resp.  $N_2 = G_1/U_1 \times G_2/P_2$ ), which is also denoted by  $E_1$  (resp.  $E_2$ ). By Lemma 2.1, we have  $M = P(E_1) =$  $P(E_2) = G_1/U_1 \times G_2/U_2$ , and  $L(E_1) = L_{\Lambda_0(1)}$  and  $L(E_2) = L_{\Lambda_0(2)}$  by Lemma 2.2. Put  $L_1 = L_{\Lambda_0(1)}$  and  $L_2 = L_{\Lambda_0(2)}$ . Note that there is a holomorphic line bundle  $L'_1$ (resp.  $L'_2$ ) over  $N_1$  (resp. over  $N_2$ ) such that  $\varphi_1^*L'_1 = L_2$  (resp.  $\varphi_2^*L'_2 = L_1$ ), where

Non-Homogeneous Kähler-Einstein Metrics on Compact Complex Manifolds II 943

 $\varphi_1: M \to N_1$  (resp.  $\varphi_2: M \to N_2$ ) is the natural projection. We thus have  $L(E_1 \otimes L_1^{(-1)}) = L_1 \otimes L_2^{(-1)} = L(E_2 \otimes L_2^{(-1)})^{-1}$ . Note also that the  $P^1(C)$ -bundle X is given by  $P(1 \oplus L_1 \otimes L_2^{(-1)})$ . Put  $G = G_1 \times G_2$  and  $U = U_1 \times U_2$ .

In case (a) and (b), we call X the  $P^{1}(C)$ -bundle associated to the triples  $(\Pi, \Pi_{1}^{1}, \Pi_{0}), (\Pi, \Pi_{1}^{2}, \Pi_{0})$  of Dynkin diagrams. We also call Y obtained as in Lemma 3.1 the compact complex manifold obtained from X by blowing down associated to the triples  $(\Pi, \Pi_{1}^{1}, \Pi_{0}), (\Pi, \Pi_{1}^{2}, \Pi_{0})$  of Dynkin diagrams. Note that in this case Y is almost homogeneous with respect to the complex Lie group G, since  $E_{1} \otimes L_{1}^{\prime^{-1}} - (0$ -section) =  $L_{1} \otimes L_{2}^{-1} - (0$ -section) is an open G-orbit in Y, and Y has a disconnected exceptional set which consists of two G-orbits  $N_{1}, N_{2}$ . Note also that  $N_{1}, N_{2}$  are Kähler C-spaces associated to the pairs  $(\Pi, \Pi_{1}^{1}), (\Pi, \Pi_{1}^{2})$  respectively.

#### 4 Almost homogeneous Fano manifolds

A compact complex manifold is called *Fano* if its first Chern class is positive. In this section we prove the following.

**Theorem 4.1.** Let  $(\Pi, \Pi_1^1, \Pi_0)$  and  $(\Pi, \Pi_1^2, \Pi_0)$  be triples of Dynkin diagrams, as in section 3, X the  $P^1(C)$ -bundle associated to these triples of Dynkin diagrams and Y the compact complex manifold obtained from X by blowing down associated these triples of Dynkin diagrams. Then Y is a Kähler manifold with positive first Chern class.

First we recall the notation of K-S [12]. Let  $\pi: L \to M$  be a holomorphic line bundle over a compact Kähler manifold M with a hermitian metric h. Denote by  $\mathring{L}$  the open set L—(0-section) of L. Let t be a function on  $\mathring{L}$  which depends only on the norm s of h and increases for the norm. Then the horizontal lift  $\tilde{X}$  of a vector field X of M to  $\mathring{L}$  with respect to the canonical hermitian connection of L is characterized by

(4.1) 
$$\pi_* \tilde{X} = X, \quad \tilde{X}[t] = (\tilde{J}\tilde{X})[t] = 0$$

where  $\tilde{J}$  is the almost complex structure of the total space of L. We decompose the group  $C^*$  into  $S^1 \times R^+$  and define holomorphic vector fields S, H on  $\mathring{L}$ generated by  $S^1$ -action,  $R^+$ -action respectively so that

(4.2) 
$$\exp 2\pi S = id$$
,  $H = -\tilde{J}S$ ,  $H[t] > 0$ .

If we denote by  $\rho_L$  the Ricci form of L, then we have

(4.3) 
$$[\tilde{X}, \tilde{Y}] - \widetilde{[X, Y]} = -\rho_L(X, Y)S.$$

Define a hermitian 2-form B on M, the Ricci tensor of L, by

N. KOISO AND Y. SAKANE

$$(4.4) B(X, Y) = \rho_L(X, JY),$$

where J is the almost complex structure of M.

We also cinsider a riemannian metric  $\tilde{g}$  on L of the form

where  $\{g_t\}$  is a one-parameter family of riemannian metrics on M. Define a positive function u on  $\mathring{L}$  depending only on t by

$$(4.6) u(t)^2 = \tilde{g}(H, H) \, .$$

Then, by Lemma 1.1 of K-S [12], the metric  $\tilde{g}$  on  $\tilde{L}$  is a Kähler metric if and only if each  $g_t$  is a Kähler metric on M and  $\frac{d}{dt}g_t = -u(t)B$ . We also assume that the range of t contains 0. Put

$$(4.7) U(t) = \int_0^t u(t) dt ,$$

then we have

$$(4.8) g_t = g_0 - U(t)B.$$

We put

(4.9) 
$$u(t) = a \cos \frac{t}{a}$$
 with  $t \in \left(-\frac{\pi}{2}a, \frac{\pi}{2}a\right)$  for  $a > 0$ ,

and define U(t) by (4.7). Take a Kähler metric  $g_0$  on M and assume that each  $g_t$  defined by (4.8) is positive definite. We condider the Kähler metric  $\tilde{g}$  on  $\mathring{L}$  of the form (4.5) satisfying (4.6). Then we have

(4.10) 
$$U(t) = a^2 \sin \frac{t}{a} + b$$
.

We may assume that the range of U is (-(l+1), k+1) for given positive integers k and l, by changing the origin of U and a>0 if necessary. Thus we have

(4.11) 
$$a^2 = \frac{1}{2}(l+k+2), \quad b = \frac{1}{2}(k-l).$$

**Lemma 4.2.** Let s be the norm of the hermitian line bundle  $\pi: L \rightarrow M$ . Then on  $\mathring{L}$ 

(4.12) 
$$U(t) = \frac{(a^2+b)s^2+(b-a^2)}{s^2+1},$$

by replacing t(s) by t(cs) for a positive constant c if necessary.

Proof. Note that, in terms of polar coordinates  $(r, \theta)$  on  $C^*$ , the natural complex structure  $\tilde{J}$  on  $C^*$  is given by

$$ilde{J} rac{\partial}{\partial r} = rac{1}{r} rac{\partial}{\partial heta} \,, \ \ ilde{J} rac{\partial}{\partial heta} = -r rac{\partial}{\partial r}$$

and that if  $s = \tilde{c}r$  for a constant  $\tilde{c} > 0$ ,  $\tilde{f} \frac{\partial}{\partial \theta} = -s \frac{\partial}{\partial s}$ . Note also that the restriction to a fiber  $C^*$  of the  $C^*$ -action on  $\mathring{L}$  coincides with the group action of  $C^*$ . Thus the vector field H restricted to a fiber  $C^*$  satisfies

$$H = -\tilde{J}S = s\frac{\partial}{\partial s} = s\frac{dt}{ds}\frac{\partial}{\partial t}$$

and thus

$$s\frac{dt}{ds}=u(t)=a\cos\frac{t}{a}.$$

Since  $\int \sec x \, dx = \log \left| \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) \right|$ , we see that

$$cs = \tan\left(\frac{\pi}{4} + \frac{t}{2a}\right) = \frac{1 + \tan\left(t/2a\right)}{1 - \tan\left(t/2a\right)} \qquad \left(-\frac{\pi a}{2} < t < \frac{\pi a}{2}\right)$$

for some positive constant c, and  $\tan(t/2a) = (cs-1)/(cs+1)$ . Thus we have

$$U(t) = a^{2} \sin \frac{t}{a} + b = a^{2} \frac{2 \tan(t/2a)}{1 + \tan^{2}(t/2a)} + b = a^{2} \frac{(cs)^{2} - 1}{(cs)^{2} + 1} + b. \quad \text{q.e.d.}$$

In general, let  $p: E \to N$  be a holomorphic vector bundle over a compact complex manifold  $N, \varphi: P(E) \to N$  the associated projective bundle over N and  $\pi: L(E) \to P(E)$  the tautological line bundle over P(E). Denote by  $\mathring{E}$  the open set E-(0-section) of E. Let  $h_1$  be a hermitian metric on E. Since  $\mathring{E}=\mathring{L}(E)$ =L(E)-(0-section), a metric  $h_1$  on E defines a hermitian metric h on L(E): for  $x \in P(E)$  and  $v, w \in \mathring{L}(E) = \mathring{E}$  with  $\pi(v) = \pi(w) = x, h_x(v, w) = (h_1)_{\varphi(x)}(v, w)$ .

REMARK. In general a fiber metric on L(E) does not define a hermitian metric on E. There is a natural one-to-one correspondence between complex Finsler structures in E and hermitian structures in L(E). See Kobayashi [10].

**Corollary 4.3.** Let  $N_1$ ,  $N_2$ ,  $E_1$ ,  $E_2$ ,  $L'_1$  and  $L'_2$  be as in (3.1) with M Kähler. Assume that there are hermitian metrics  $h_1$  on  $E_1 \otimes {L'_1}^{-1}$  and  $h_2$  on  $E_2 \otimes {L'_2}^{-1}$  with the following properety: If we denote the hermitian metric on  $L=L(E_1 \otimes {L'_1}^{-1})$ induced from  $h_1$  by h and the norm of h by s, the norm  $s_2$  of the hermitian metric on  $L(E_2 \otimes {L'_2}^{-1})$  induced from  $h_2$  depends only on s, under the identification:  $\mathring{L}(E_1 \otimes {L'_1}^{-1}) = \mathring{L}(E_2 \otimes {L'_2}^{-1})$ . Assume further that we can construct a Kähler metric  $\tilde{g}$  on  $\tilde{L}$  in the above way, that is, each  $g_t$  in (4.8) is positive definite. We choose the function t in such a way that the range (-(l+1), k+1) of U is  $l+1=\operatorname{rank} E_1=$ codimension  $N_1$  in Y and  $k+1=\operatorname{rank} E_2=$  codimension  $N_2$  in Y. Then the function U on the open set  $\tilde{L}$  of the compact complex manifold Y is extended to a smooth function U on Y such that the range of U on the complex submanifold  $E_1 \otimes {L'_1}^{-1}$  is [-(l+1), k+1) and the range of U on  $E_2 \otimes {L'_2}^{-1}$  is (-(l+1), k+1].

In general, for a Kähler metric g the corresponding Kähler form is denoted by  $\omega_g$ . We now seek the condition that the metric  $\tilde{g}$  on  $\mathring{L} = \mathring{L}(E_1 \otimes L_1^{-1}) =$  $\mathring{L}(E_2 \otimes L_2^{-1})$  can be written as

$$\omega_{\widetilde{g}} = (\varphi_1 \circ \pi)^* \underline{\omega}_1 - 2\sqrt{-1}d'd'' f_0 = (\varphi_2 \circ \pi)^* \underline{\omega}_2 - 2\sqrt{-1}d'd'' f_{\infty}$$

where  $\underline{\omega}_i$  is a Kähler form of a Kähler metric  $\underline{g}_i$  on  $N_i$  for i=1, 2 and  $f_0, f_{\infty}$  are smooth functions on  $\mathring{L}$  depending only on t.

**Lemma 4.4.** Under the assumptions in Corollary 4.3, if the Kähler metric  $g_0$  on  $M = P(E_1 \otimes L'_1^{-1}) = P(E_2 \otimes L'_2^{-1})$  and the hermitian form B on M satisfy that  $g_0 + (l+1)B = \varphi_1^* g_1$  where  $g_1$  is a Kähler metric on  $N_1$  and  $g_0 - (k+1)B = \varphi_2^* g_2$  where  $g_2$  is a Kähler metric on  $N_2$ , then there are smooth functions  $f_0: E_1 \otimes L'_1^{-1} \rightarrow \mathbf{R}$  and  $f_\infty: E_2 \otimes L'_2^{-1} \rightarrow \mathbf{R}$  such that on  $\mathring{L}$ 

$$(4.13) \qquad \omega_{\tilde{g}} = (\varphi_1 \circ \pi)^* \underline{\omega}_1 - 2\sqrt{-1} d' d'' f_0 = (\varphi_2 \circ \pi)^* \underline{\omega}_2 - 2\sqrt{-1} d' d'' f_{\infty} .$$

Proof. We use the notation  $\hat{\partial}_{\alpha}$ ,  $\hat{\partial}_{\bar{\alpha}} (0 \le \alpha \le n)$  used in K-S [12]. We may assume that  $\hat{\partial}_{\alpha}t = \hat{\partial}_{\bar{\alpha}}t = 0$   $(1 \le \alpha \le n)$  on a fiber. First we consider a function f on  $\hat{L}$  satisfying  $\omega_{\tilde{s}} = (\varphi_1 \circ \pi)^* \underline{\omega}_1 - 2\sqrt{-1}d'd''f$ . Since  $\tilde{g}_{\bar{0}0} = \hat{\partial}_{\bar{0}}\hat{\partial}_0 f$ , we have

(4.14) 
$$2u^2 = u \frac{d}{dt} \left( u \frac{df}{dt} \right)$$

by Lemmas 1.2 and 1.3 of K-S [12]. As (2.15) in K-S [12], we put  $\varphi(U)=u^2$ . Then the equation (4.14) is given by

(4.15) 
$$2 = \frac{d}{dU} \left( \varphi(U) \frac{df}{dU} \right), \text{ since } \frac{d}{dt} = u \frac{d}{dU}.$$

By solving this equation, we have

(4.16) 
$$\frac{df}{dU} = \frac{2U+C}{\varphi(U)}$$
 for some constant  $C \in \mathbf{R}$ .

Now  $\varphi(U) = u^2 = a^2 \cos^2 \frac{t}{a} = a^2 \left( 1 - \sin^2 \left( \frac{t}{a} \right) \right)$ . By (4.10) and (4.11), we see that

(4.17) 
$$\varphi(U) = \frac{1}{a^2}(a^2 + b - U)(a^2 - b + U) = \frac{1}{a^2}(k + 1 - U)(l + 1 + U)$$

Let  $f_0$  denote a solution of (4.16) with C=2(l+1). Then the equation (4.16) is given by

$$\frac{df_0}{dU} = \frac{2a^2}{k+1-U}$$

and hence  $f_0 = -2a^2 \log (k+1-U) + C'$  ( $C' \in \mathbb{R}$ ) and  $f_0$  is extended to a smooth function on  $E_1 \otimes L'_1^{-1}$ . Similarly we have a solution

$$f_{\infty} = -2a^2 \log(l+1+U) + C'' \qquad (C'' \in \mathbf{R})$$

of (4.16) with C = -2(k+1), which is a smooth function on  $E_2 \otimes L_2^{\prime^{-1}}$ . By K-S [12] Lemma 1.3, we have

(4.19) 
$$\begin{cases} \hat{\partial}_{\bar{o}} \hat{\partial}_{\beta} f = 0\\ \hat{\partial}_{\bar{a}} \hat{\partial}_{\beta} f = -\frac{1}{2} u \frac{df}{dt} B_{\bar{a}\beta} = -\frac{1}{2} \varphi(U) \frac{df}{dU} B_{\bar{a}\beta} \,.$$

Since 
$$\frac{1}{2}\varphi(U)\frac{df_0}{dU} = U + l + 1$$
 by (4.17) and (4.18), we have  
 $(g_0 - UB) + \frac{1}{2}\varphi(U)\frac{df_0}{dU}B = g_0 + (l+1)B = \varphi_1^* \frac{g_1}{2}$ .

Thus  $\omega_{\tilde{s}} = (\varphi_1 \circ \pi)^* \omega_1 - 2\sqrt{-1} d' d'' f_0$  on  $\mathring{L}$ . Similarly,  $\omega_{\tilde{s}} = (\varphi_2 \circ \pi)^* \underline{\omega}_2 - 2\sqrt{-1} d' d'' f_{\infty}$  on  $\mathring{L}$ . q.e.d.

**Corollary 4.5.** Under the same assumption of Corollary 4.3 and Lemma 4.4, the Kähler metric  $\tilde{g}$  on  $\mathring{L}$  can be extended to a Kähler metric on the complex manifold Y.

Proof. Note that by (4.11) and (4.12) we have  $k+1-U=\frac{l+k+2}{s^2+1}$  where  $s^2$  is the square of the norm of the hermitian metric  $h_1$  on  $E_1 \otimes L_1^{\prime -1}$ . Thus we have

(4.20) 
$$f_0 = 2a^2 \log(1+s^2) - 2a^2 \log(k+l+2) + C'$$

Let  $p_1: E_1 \otimes {L'_1}^{-1} \to N_1$  be the projection. It is easy to see that

(4.21) 
$$p_1^* \underline{\omega}_1 - 4a^2 \sqrt{-1} d' d'' \log(1+s^2)$$

is the Kähler form of a Kähler metric on a neighborhood of 0-section of  $p_1: E_1 \otimes L_1^{\prime^{-1}} \to N_1$ . Since  $p_1 = \varphi_1 \circ \pi$  on  $E_1 \otimes L_1^{\prime^{-1}} - (0 \operatorname{-section}) = \mathring{L}$ , the metric  $\tilde{g}$  on  $\mathring{L}$  can be extended to a Kähler metric

N. KOISO AND Y. SAKANE

$$p_1^* \omega_1 - 4a^2 \sqrt{-1} d' d'' \log(1+s^2)$$

on  $E_1 \otimes L'_1^{-1}$ . Similarly the metric  $\tilde{g}$  on  $\mathring{L}$  can be extended to a Kähler metric on  $E_2 \otimes L'_2^{-1}$  and hence to a Kähler metric on Y. q.e.d.

**Corollary 4.6.** Under the same assumption of Theorem 4.1, the compact complex manifold Y is Kähler. More precisely a Kähler metric  $\tilde{g}$  on  $\mathring{L} = L_1 \otimes L_2^{-1} - (0$ -section) can be extended to a Kähler metric on Y, which is also denoted by  $\tilde{g}$ .

Proof. Let  $g_0$  be the  $G_u$ -invariant Kähler metric on  $M = G/U = P(E_1 \otimes L_1'^{-1}) = P(E_2 \otimes L_2'^{-1})$  corresponding to  $8\pi\delta_{\rm m}$  as in Fact 1 in section 1 and h a  $G_u$ -invariant hermitian metric on the homogeneous line bundle  $L = L(E_1 \otimes L_1'^{-1}) = L(E_2 \otimes L_2'^{-1})^{-1}$  over M. Since we are in  $G_u$ -invariant situation, the first assumption in Corollary 4.3 is satisfied. And the hermitian form B on M is  $G_u$ -invariant and corresponds to  $4\pi(-\Lambda_0(1)+\Lambda_0(2)) \in c$  by Fact 2 in section 2. Thus  $g_t$  is  $G_u$ -invariant and corresponds to  $4\pi\{2\delta_{\rm m}+U(t)(\Lambda_0(1)-\Lambda_0(2))\}$ , which belongs to  $c^+$  by Lemma 2.3. So the second assumption in Corollary 4.3 is satisfied. In the same way we see that  $g_0+(l+1)B=\varphi_1^*g_1$  where  $g_1$  is a  $G_u$ -invariant Kähler metric on the Kähler C-space  $N_1$  associated to the pair  $(\Pi, \Pi_1^1)$  and  $g_0-(k+1)B=\varphi_2^*g_2$  where  $g_2$  is a  $G_u$ -invariant Kähler metric  $\tilde{g}$  on  $\tilde{L}$  can be extended to a Kähler metric on Y.

From now on we assume further that the eigenvalues of B, regarded as a hermitian form on a holomorphic tangent space of M, with respect to  $g_0$  are constant on M. Note that the assumption in Lemma 4.4 implies that  $\frac{-1}{l+1}\left(\operatorname{resp.} \frac{1}{k+1}\right)$  is an eigenvalue of B with respect to  $g_0$  with multiplicity l (resp. k) because  $\varphi_1^*g_1$  (resp.  $\varphi_2^*g_2$ ) is a positive semi-definite hermitian form of nullity l (resp. k). Thus the function det  $(g_0^{-1}g_t)=Q(U)$  on  $\mathring{L}$  is given by

(4.22) 
$$Q(U) = \det\left(1 - Ug_0^{-1}B\right) = \left(1 + \frac{U}{l+1}\right)^l \left(1 - \frac{U}{k+1}\right)^k Q_1(U)$$

where  $Q_1(U)$  is a polynomial of U such that  $Q_1(U) \neq 0$  on [-(l+1), k+1]. Here also  $g_0^{-1}g_t$  and  $g_0^{-1}B$  are regarded as endomorphisms on homolorphic tangent spaces of M.

**Theorem 4.7.** Under the assumption above, together with assumptions in Corollary 4.3 and Lemma 4.4, if the Ricci tensor  $r_0$  of the Kähler metric  $g_0$  on Mis equal to  $g_0$ , then the first Chern class  $c_1(Y)$  of Y is positive. More precisely, let  $\tilde{\rho}$  be the Ricci form of the Kähler metric  $\tilde{g}$  on Y, then there is a  $C^{\infty}$  function F(U)of U on [-(l+1), k+1] such that

$$\tilde{\rho} - \omega_{\tilde{g}} = -2\sqrt{-1}d'd''F$$

Proof. By Lemmas 1.2, 1.3 and 1.4 in K-S [12], we see that the equation (4.23) is equiavlent to the equation

(4.24) 
$$\varphi \frac{d}{dU} \log (\varphi Q) + 2U + \varphi \frac{dF}{dU} = 0.$$

By solving this equation,

(4.25) 
$$F = -\log(\varphi Q) - 2\int \frac{U}{\varphi} dU.$$

By (4.17) and (4.22),

(4.26)  $\log(\varphi Q) = (l+1)\log(l+1+U) + (k+1)\log(k+1-U) + \log Q_1 + C_1$ where  $C_1 \in \mathbb{R}$ .

By (4.11) and (4.17),

$$2\frac{U}{\varphi} = 2a^2 \frac{U}{(k+1-U)(l+1+U)} = \frac{k+1}{k+1-U} - \frac{l+1}{l+1+U}$$

and hence

(4.27) 
$$2\int \frac{U}{\varphi} dU = -(k+1)\log(k+1-U) - (l+1)\log(l+1+U).$$

Thus  $F = -\log Q_1 + C_2$  ( $C_2 \in \mathbb{R}$ ). Since  $Q_1(U) \neq 0$  on [-(l+1), k+1], F is a smooth function on [-(l+1), k+1]and hence, it is smooth on Y. q.e.d.

Proof of Theorem 4.1. Since  $g_0$  and B in Corollary 4.6 are  $G_u$ -invariant, the eigenvalues of B with respect to  $g_0$  are constant. By (1.20) we have  $r_0=g_0$ . Note that the assumptions in Corollary 4.3 and Lemma 4.4 are astisfied as in the proof of Corollary 4.6. Thus our theorem follows from Theorem 4.7. q.e.d.

REMARK. Note that, under the assumption in Theorem 4.1, by taking  $L = L_1 \otimes L_2^{-1}$ ,  $\hat{L} = Y$ ,  $M = P(E_1) = P(E_2) = G/U$  and the metric  $\tilde{g}$  on Y as in Corollray 4.6, the following assumptions A) and B) in K-S [12] are satisfied for a Kähler metric  $\tilde{g}$  on  $\hat{L}$  of the form (4.5).

Assumption A). Let  $(\min t, \max t)$  be the range of t. The function t extends to a continuous function on  $\hat{L}$  with range  $[\min t, \max t]$ , and the subset  $M_{\min}$  (resp.  $M_{\max}$ ) of  $\hat{L}$  defined by  $t = \min t$  (resp.  $t = \max t$ ) is a complex submanifold of  $\hat{L}$  with codimension  $D_{\min}$  (resp.  $D_{\max}$ ). Moreover the Kähler

metric  $\tilde{g}$  extends to a Kähler metric on  $\hat{L}$ , which is also denoted by  $\tilde{g}$ .

Assumption B). (1) The Kähler form of the metric  $\tilde{g}$  on  $\hat{L}$  is cohomologous to the Ricci form  $\tilde{\rho}$  of  $\tilde{g}$ . (2) The eigenvalues of the Ricci tensor  $r_0$  of  $g_0$  with respect to  $g_0$  are constant on M.

#### 5 Non-homogeneous Kähler-Einstein metrics

Let  $\pi: L \to M$  be a hermitian holomorphic line bundle over a compact Kahler manifold M. As above we consider a Kähler metric  $\tilde{g}$  on  $\mathring{L}$  of the form (4.5). We also assume that the eigenvalues of B with respect to a Kähler metric  $g_0$  on M are constant and a compactification  $\hat{L}$  of  $\mathring{L}$  satisfies the assumptions A) and B). By Lemma 2.2 of K-S [12], we may assume that the range of U is  $[-D_{\min}, D_{\max}]$ .

Now we give a necessary condition for a Kähler-Einstein metric on  $\hat{L}$  of the form (4.5) being homogeneous.

**Theorem 5.1.** Under the above situation, assume further that the Ricci tensor  $\tilde{r}$  of the Kähler metric  $\tilde{g}$  of  $\hat{L}$  of the form (4.5) is equal to  $\tilde{g}$ . If  $\tilde{g}$  is riemannian homogeneous, the followings hold.

(1) If the codimensions  $D_{\min}=D_{\max}=1$ , then B=0.

(2) If one of the codimensions  $D_{\min}$ ,  $D_{\max}$  is equal to 1 and the other >1, then the non-zero eigenvalues of  $g_0^{-1}B$  are all equal.

(3) If both codimensions  $D_{\min}$ ,  $D_{\max} > 1$ , then the number of distinct non-zero eigenvalues of  $g_0^{-1}B$  are 2.

First we recall the following.

**Lemma 5.2.** Every complete totally geodesic submanifold of a homogeneous riemannian manifold is homogeneous.

Proof. See K-N [11] Chap. 7, Corollary 8.10.

Proof of Theorem 5.1. Since the closure  $S^2$  of each fiber  $C^*$  is a totally geodesic submanifold of  $(\hat{L}, \tilde{g})$  and  $\tilde{g}$  is homogeneous, it is a riemannian homogeneous manifold by Lemma 5.2. We use the notations in K-S [12]. Note that the induced metric  $\tilde{g}_{\bar{0}0}=2u^2$  is an Einstein metric on  $S^2$ , since  $S^2$  is 2-dimensional. Thus we have

(5.1)  $-\hat{\partial}_{\bar{\partial}}\hat{\partial}_{0}(\log (2u^{2})) = c \cdot 2u^{2}$  where c is a constant.

Note that  $u^2 = \varphi$ ,  $u \frac{d}{dt} = \varphi \frac{d}{dU}$ . By Lemma 1.3 of K-S [12], we see that the equation (5.1) is given by

(5.2) 
$$-\varphi \frac{d}{dU}(\varphi \frac{d}{dU}(\log \varphi)) = c \cdot \varphi$$

and hence

(5.3) 
$$\frac{d\varphi}{dU} = -cU + \text{ constant.}$$

Thus  $\varphi$  is a quadric polynomial of U. On the other hand  $\varphi$  vanishes at  $U = -D_{\min}$ ,  $D_{\max}$ . Therefore  $\varphi$  is of the form

$$arphi = c'(U + D_{\min}) \left( U - D_{\max} 
ight)$$
 for some  $c' \in \mathbf{R}$ .

By (4.1.5) in K-S [12], the first term of Taylor expansion of  $\varphi(U)$  at  $U = -D_{\min}$  is given by  $2(U+D_{\min})$ . Thus  $\varphi$  is given by

(5.4) 
$$\varphi = \frac{-2}{D_{\min} + D_{\max}} (U + D_{\min}) (U - D_{\max}).$$

Since  $\tilde{r} = \tilde{g}$ , the polynomial Q of U satisfies the equation

(5.5) 
$$\frac{d}{dU}\varphi + 2U + \frac{\varphi}{Q} \cdot \frac{dQ}{dU} = 0$$

by Lemma 2.2 in K-S [12]. By (5.4) and (5.5), we have

$$\begin{split} \frac{d}{dU} \log Q &= -\frac{(2U + D_{\min} - D_{\max}) - U(D_{\max} + D_{\min})}{(U + D_{\min}) \left(U - D_{\max}\right)} \\ &= -\frac{1 - D_{\min}}{U + D_{\min}} - \frac{1 - D_{\max}}{U - D_{\max}} \,. \end{split}$$

Thus we have

$$\log Q = -(1 - D_{\min}) \log (U + D_{\min}) - (1 - D_{\max}) \log |U - D_{\max}| + c''$$

and thus we have

(5.6) 
$$Q = C(U+D_{\min})^{p_{\min}-1}(D_{\max}-U)^{p_{\max}-1}.$$

Since  $Q = \det(1 - Ug_0^{-1}B)$ , we get our claim.

Now we recall the following theorem in K-S [12].

**Theorem 5.3** (Theoerm 4.2 in K-S [12]). Let M be a compact Kähler-Einstein manifold whose Kähler form represents the first Chern class  $c_1(M)$  and La hermitian holomorphic line bundle over M. Assume that there is a Kähler metric  $\tilde{g}$  on a compactification  $\hat{L}$  of  $\hat{L}$  of the form (4.5) with  $g_0$  Kähler-Einstein, whose Kähler form is cohomologous to the Ricci form of  $\hat{L}$  and that the eigenvalues of the Ricci form B of L with respect to  $g_0$  are constant. Then the complex manifold  $\hat{L}$ admits a Kähler-Einstein metric if and only if the integral

q.e.d.

N. KOISO AND Y. SAKANE

(5.7) 
$$F(\hat{L}) = \int_{-p_{\min}}^{p_{\max}} UQ(U) dU$$

vanishes.

Now let  $(\Pi, \Pi_0)$  be an effective pair of Dynkin diagrams as in section 1 and M = G/U the Kähler *C*-space associated to  $(\Pi, \Pi_0)$ . Consider the Kähler-Einstein metric  $g_0$  on G/U corresponding to  $8\pi\delta_{\rm m}\in c^+$  with  $r_0=g_0$  and a holomorphic line bundle  $L_{\Lambda}$  on G/U for  $\Lambda \in \mathbb{Z}_c$  with a  $G_u$ -invariant hermitian metric. Note that a unique  $G_u$ -invariant form in the first Chern class  $c_1(L_{\Lambda})$  is given by  $\eta(\Lambda)$  of (1.22). Let *B* be the Ricci tensor of  $L_{\Lambda}$  which is the  $G_u$ -invariant hermitian form on *M* corresponding to  $-4\pi\Lambda \in c$ .

**Lemma 5.4.** Under the assumption above, we have

(5.8) 
$$Q(x) = \det (1 - xg_0^{-1}B) = \prod_{\alpha \in \Sigma_{\mathfrak{m}}^+} \left(1 + \frac{(\Lambda, \alpha)}{(2\delta_{\mathfrak{m}}, \alpha)}x\right).$$

**Proof.** Straightforwards by (1.18).

Let  $\rho$  be an automorphism of Dynkin diagram  $\Pi$  such that  $\rho^2 = id$  and  $\rho \neq id$ . It is known that if  $\Pi$  is irreducible and it admits such an automorphism  $\rho$ , then  $\Pi$  is of type  $A_n$   $(n \geq 2)$ ,  $D_n$   $(n \geq 4)$  or  $E_6$  (cf. [5]). Note also that if  $\Pi$  has two connected components  $\Pi(1) = \{\alpha_1, \dots, \alpha_n\}$ ,  $\Pi(2) = \{\beta_1, \dots, \beta_n\}$  and  $\Pi(1), \Pi(2)$  are isomorphic by the map  $\alpha_i \rightarrow \beta_i$ , then the map  $\rho: \Pi \rightarrow \Pi$  defined by  $\rho(\alpha_i) = \beta_i$ ,  $\rho(\beta_i) = \alpha_i$  (for each i) is such an automorphism of  $\Pi$ , and from now on we consider this automorphism  $\rho$  exclusively in the case when a Dynkin diagram  $\Pi$  is reducible. A pair  $(\Pi, \Pi_0)$  of Dynkin diagram is said to be *admissible* for  $\rho$  if  $\rho(\Pi_0) = \Pi_0$ .

**Lemma 5.5.** Let  $(\Pi, \Pi_0)$  be an admissible pair of Dynkin diagrams for an automorphism  $\rho$  and assume that  $\Lambda \in \mathbb{Z}_c$  satisfies  $\rho(\Lambda) = -\Lambda$ . Then

$$Q(x) = \prod_{\alpha \in \Sigma_{\mathfrak{m}}^+} \left( 1 + \frac{(\Lambda, \alpha)}{(2\delta_{\mathfrak{m}}, \alpha)} x \right)$$

is an even function of x.

Proof. We use notation in section 1. Since  $\rho$  induces the bijections  $\rho: \Sigma^+ \to \Sigma^+$  and  $\rho: \Sigma_0^+ \to \Sigma_0^+$ , it also induces the bijection  $\rho: \Sigma_m^+ \to \Sigma_m^+$ . Since  $2\delta_m = \sum_{\alpha \in \Sigma_m^+} \alpha$ , we have  $\rho(2\delta_m) = 2\delta_m$ . Note that  $(\Lambda, \rho(\alpha)) = (\rho(\Lambda), \alpha) = -(\Lambda, \alpha)$ . Thus if  $\rho(\alpha) = \alpha$ ,  $(\Lambda, \alpha) = 0$ . For  $\alpha \in \Sigma_m^+$ ,  $\alpha \neq \rho(\alpha)$ ,

$$\left(1+\frac{(\Lambda, \alpha)}{(2\delta_{\mathfrak{m}}, \alpha)}x\right)\left(1+\frac{(\Lambda, \rho(\alpha))}{(2\delta_{\mathfrak{m}}, \rho(\alpha))}x\right)=1-\left(\frac{(\Lambda, \alpha)}{(2\delta_{\mathfrak{m}}, \alpha)}\right)^{2}\cdot x^{2}.$$

Thus we get our claim.

q.e.d.

Non-Homogeneous Kähler-Einstein Metrics on Compact Complex Manifolds II 953

**Corollary 5.6.** Let G/U be a Kähler C-space associated to an admissible pair  $(\Pi, \Pi_0)$  for an automorphism  $\rho$ . Put  $2\delta_{\mathrm{nt}} = \sum_{\alpha \in \Pi - \Pi_0} a_{\alpha} \Lambda_{\alpha}$ . Let  $L_{\Lambda}$  be a holomorphic line bundle over G/U such that  $\rho(\Lambda) = -\Lambda$  and  $\Lambda = \sum_{\alpha \in \Pi - \Pi_0} b_{\alpha} \Lambda_{\alpha}$  with  $|b_{\alpha}| < a_{\alpha}$  for each  $\alpha \in \Pi - \Pi_0$ . Then the  $P^1(C)$ -bundle  $P(1 \otimes L_{\Lambda})$  over G/U admits an Kähler-Einstein metric.

Proof. Note that by the assumption for  $\Lambda$  the absolute values of eigenvalues of B are less than 1. By Theorem 5.4 in K-S [12], it is sufficient to see that the integral

$$\int_{-1}^{1} UQ(U) dU = \int_{-1}^{1} U \cdot \det(1 - Ug_0^{-1}B) dU$$

vanishes. Since det $(1-Ug_0^{-1}B)$  is an even function of U by Lemma 5.5, we get our claim. q.e.d.

EXAMPLES 5.1. In the following cases the  $P^1(C)$ -bundle  $P(1 \oplus L_{\Delta})$  over a Kähler C-space G/U admits an Kähler-Einstein metric. The vertices contained in  $\Pi_0$ ,  $\Pi - \Pi_0$  of a Dynkin diagram  $\Pi$  are denoted by  $\bigcirc$ ,  $\times$  respectively.

(1) 
$$(\Pi, \Pi_0)$$
  $(\Pi, \Pi_0)$ 

 $2\delta_{\mathfrak{m}}=2(\Lambda_{\mathfrak{a}_1}+\Lambda_{\mathfrak{a}_2})$ . Put  $\Lambda=\Lambda_{\mathfrak{a}_1}-\Lambda_{\mathfrak{a}_2}$ . Then  $\rho(\Lambda)=-\Lambda$ . In this case the associated  $P^1(\mathcal{C})$ -bundle  $P(1\oplus L_{\Lambda})$  is the Example 5.10 in K-S [12].

 $2\delta_{\mathfrak{m}} = 2\Lambda_{\mathfrak{a}_{1}} + 4\Lambda_{\mathfrak{a}_{2}} + 4\Lambda_{\mathfrak{a}_{5}} + 2\Lambda_{\mathfrak{a}_{6}}. \text{ Put } \Lambda = \Lambda_{\mathfrak{a}_{1}} + \Lambda_{\mathfrak{a}_{2}} - \Lambda_{\mathfrak{a}_{5}} - \Lambda_{\mathfrak{a}_{6}}. \text{ Then } \rho(\Lambda) = -\Lambda. \text{ In this case } G = SL(7, \mathbb{C}) \text{ and } U \text{ is given by}$ 

(3)

 $2\delta_{\mathfrak{m}} = 2\Lambda_{\boldsymbol{\omega}_{1}} + 4\Lambda_{\boldsymbol{\omega}_{3}} + 4\Lambda_{\boldsymbol{\omega}_{5}} + 2\Lambda_{\boldsymbol{\omega}_{6}}. \quad \text{Put } \Lambda = \Lambda_{\boldsymbol{\omega}_{3}} - \Lambda_{\boldsymbol{\omega}_{5}}. \quad \text{Then } \rho(\Lambda) = -\Lambda.$ 

(5) 
$$(\Pi, \Pi_0)$$
  
 $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \circ & & \times \\ \uparrow & \uparrow & \\ \beta_1 & \beta_2 \end{pmatrix} \rho$ 

 $2\delta_{\mathfrak{m}}=4\Lambda_{\mathfrak{a}_2}+4\Lambda_{\mathfrak{p}_2}$ . Put  $\Lambda=\Lambda_{\mathfrak{a}_2}-\Lambda_{\mathfrak{p}_2}$ . Then  $\rho(\Lambda)=-\Lambda$ .

Now we consider triples  $(\Pi, \Pi_1^1, \Pi_0)$ ,  $(\Pi, \Pi_1^2, \Pi_0)$  of Dynkin diagrams as in section 3. These triples are said to be *admissible for an automorphism*  $\rho$  of Dynkin diagram  $\Pi$  if  $\rho(\Pi_0) = \Pi_0$ , and  $\rho(\alpha_0(1)) = \alpha_0(2)$ . Note that the holomorphic line bundle  $L_1 \otimes L_2^{-1} = L(E_1 \otimes L_1'^{-1}) = L(E_2 \otimes L_2'^{-1})^{-1}$  over M = G/U is given by  $L_{\Lambda}$ , where  $\Lambda = \Lambda_0(1) - \Lambda_0(2)$  and thus  $\rho(\Lambda) = -\Lambda$ . By Lemma 2.4  $\Lambda$  satisfies the assumption in Corollary 5.6. Recall that the  $P^1(C)$ -bundle X is given by  $X = P(1 \oplus L_{\Lambda})$ .

**Corollary 5.7.** Let  $(\Pi, \Pi_1^1, \Pi_0)$ ,  $(\Pi, \Pi_1^2, \Pi_0)$  be admissible triples of Dynkin diagrams for  $\rho$ . Then both the  $P^1(C)$ -bundle X associated to these triples of Dynkin diagrams and the compact complex manifold Y obtained from X by blowing down associated to these triples of Dynkin diagrams admit Kähler-Einstein metrics.

Proof. By the last Remark in section 4 and Theorem 5.3, it is enough to see that the integral (5.7) vanishes. Since  $D_{\max}=D_{\min}$  and Q(U) is an even function of U by Lemma 5.5, we get our claim. q.e.d.

REMARK. As in K-S [12], X and Y admit a Kähler-Einstein metric if and only if Futaki's integral F(H) of the holomorphic vector field H vanishes. We can explain Corollaries 5.6 and 5.7 as follows. The automorphism  $\rho$  of the Dynkin diagram induces automorphisms  $\gamma_x$  and  $\gamma_y$  of the complex manifolds X and Y respectively, such that  $\gamma_*H = -H$ . Thus Futaki's integral F(H)vanishes, because it is invariant under complex automorphisms (cf. Futaki [7] Theorem 2.1). However the existence of such an automorphism  $\gamma$  is not necessary to the existence of a Kähler-Einstein metric. See Example 5.3 (2) and Example 6.4 in [17].

**Corollary 5.8.** Under the same notation as in Corollary 5.7, if the number of elements in  $\Pi - \Pi_0 \ge 3$ , then the Kähler-Einstein metric on Y is non-homogeneous.

Proof. Since  $\Pi - (\Pi_1^1 \cup \Pi_1^2) \neq \emptyset$  by our assumption, we can take an element  $\alpha \in \Pi - (\Pi_1^1 \cup \Pi_1^2)$ . Note that  $\Pi_1^1 = \Pi_0 \cup \{\alpha_0(1)\}$  and  $\Pi_1^2 = \Pi_0 \cup \{\alpha_0(2)\}$ . We may assume that there is a connected subdiagram  $\Pi'$  of  $\Pi$  such that  $\alpha$  and  $\alpha_0(1)$  are terminal vertices of  $\Pi'$  and  $\alpha_0(2)$  is not a vertex of  $\Pi'$ , taking  $\rho(\alpha)$  instead of  $\alpha$  if necessary. Note that  $\gamma = \sum_{\beta \in \Pi'} \beta$  is a positive root (cf. Bourbaki [5] Chap. 6, Prop. 19 Cor. 3 b)) and hence  $\gamma \in \Sigma_{\Pi}^+$ . Put  $\Lambda = \Lambda_{\sigma_0(1)} - \Lambda_{\sigma_0(2)}$ . Since  $(\Lambda, \gamma) = (\Lambda, \alpha_0(1))$  and  $(2\delta_{\Pi}, \gamma) > (2\delta_{\Pi}, \alpha_0(1))$ , we see that

$$\frac{(\Lambda, \alpha_{0}(1))}{(2\delta_{\mathfrak{m}}, \alpha_{0}(1))} \! > \! \frac{(\Lambda, \gamma)}{(2\delta_{\mathfrak{m}}, \gamma)} \! > \! 0 \! > \! \frac{(\Lambda, \alpha_{0}(2))}{(2\delta_{\mathfrak{m}}, \alpha_{0}(2))},$$

and hence the number of distinct non-zero eigenvalues of  $g_0^{-1}B$  are greater than or equal to 3, by Lemma 5.4. Thus we get our claim by Theorem 5.1. q.e.d.

EXAMPLES 5.2. In the following cases the blowing down Y admits a nonhomogeneous Kähler-Einstein metric. The vertices contained in  $\Pi_0$ ,  $\Pi_1^i - \Pi_0$ ,  $\Pi - \Pi_1^i$  of a Dynkin diagram  $\Pi$  are denoted by  $\bigcirc$ ,  $\Box$ ,  $\times$  for i=1, 2 respectively as in section 3.

(1)  $(\Pi, \Pi_1^1, \Pi_0) \xrightarrow{\alpha_1 \alpha_2 \alpha_3}_{\rho} (\Pi, \Pi_1^2, \Pi_0) \xrightarrow{\alpha_1 \alpha_2 \alpha_3}_{\rho} \rho$ 

Note that  $\Lambda = \Lambda_{\alpha_1} - \Lambda_{\alpha_3}$ .



ρ

Note that  $\Lambda = \Lambda_{\alpha_4} - \Lambda_{\alpha_5}$ .

(3) 
$$(\Pi, \Pi_1^1, \Pi_0)$$
  $\alpha_1 \alpha_3 \alpha_4 \alpha_5 \alpha_6$   
 $\alpha_2$ 

$$(\Pi, \Pi_1^2, \Pi_0) \xrightarrow[]{\alpha_1} [\alpha_3]{\alpha_4} [\alpha_5]{\alpha_4} [\alpha_5]{\alpha_6} .$$

ρ

Note that  $\Lambda = \Lambda_{\alpha_3} - \Lambda_{\alpha_5}$ .

(4) 
$$(\Pi, \Pi_1^1, \Pi_0)$$
  $\overbrace{\alpha_1 \alpha_3 \alpha_4 \alpha_5 \alpha_6}^{\checkmark}$   
 $\alpha_1 \alpha_3 \alpha_4 \alpha_5 \alpha_6$   
 $\downarrow_{\times} \alpha_2$   
 $\rho$   
 $(\Pi, \Pi^2, \Pi)$   $\alpha \alpha \alpha \alpha \alpha \alpha \alpha$ 

Note that  $\Lambda = \Lambda_{\alpha_1} - \Lambda_{\alpha_6}$ .

Note that  $\Lambda = \Lambda_{\alpha_2} - \Lambda_{\beta_2}$ .

Now we give examples of Y being homogeneous.

Examples 5.3. (1) 
$$(\Pi, \Pi_1^1, \Pi_0) \xrightarrow[\rho]{\alpha_1 \ \alpha_2} (\Pi, \Pi_1^2, \Pi_0) \xrightarrow[\chi]{\alpha_1 \ \alpha_2} \rho$$

In this case  $\Lambda = \Lambda_{\alpha_1} - \Lambda_{\alpha_2}$ , compact Kähler manifolds  $N_1$ ,  $N_2$  are  $P^2(C)$ , M is the flag manifold SL(3, C)/B where B is a Borel subgroup of SL(3, C), X is the  $P^1(C)$ -bundle  $P(1 \oplus L_{\Lambda})$  over M and Y is the complex quadric  $Q^4(C)$ .

(2)  $(\Pi, \Pi_{1}^{1}, \Pi_{0}) \qquad \overset{\alpha_{1}}{\Box} \overset{\alpha_{2}}{\frown} \overset{\alpha_{n}}{\frown} \overset{\beta_{1}}{\bullet} \overset{\beta_{2}}{\times} \overset{\beta_{n}}{\frown} \overset{\beta_{n}}{\circ} \overset{\beta_{n}}{\bullet} \overset{\beta_{n}}{\bullet}$ 

In this case  $\Lambda = \Lambda_{\alpha_1} - \Lambda_{\beta_1}$ , compact Kähler manifolds  $N_1$ ,  $N_2$  are  $P^m(C)$ ,  $P^n(C)$  respectively, M is  $P^n(C) \times P^m(C)$ , X is the  $P^1(C)$ -bundle  $P(1 \oplus L_{\Lambda})$  over M and Y is the complex projective space  $P^{n+m+1}(C)$ .

#### 6. Remarks

A riemannian manifold N is said to have *cohomogeneity* d if the codimension of the principal orbits for the action of the isometry group is d, and d is denoted by cohomg(N). For a given positive integer d we give examples of Kähler-Einstein manifolds which have cohomogeneity d.

**Lemma 6.1.** Let  $M_1$ ,  $M_2$  be Fano manifolds of  $n_1$ -dimension and  $n_2$ -dimension  $(n_1, n_2 \ge 2)$  and let  $F_1$ ,  $F_2$  be holomorphic line bundles on  $M_1$ ,  $M_2$  respectively such that  $c_1(F_1) > 0$  and  $c_1(F_2) > 0$ . Then  $H^1(M_1 \times M_2, \operatorname{End}(1 \oplus F_1 \otimes F_2^{-1})) = (0)$ .

Proof. By Kunneth formula,  $H^{1}(M_{1} \times M_{2}, F_{1} \otimes F_{2}^{-1}) = \sum_{i=0,1} H^{i}(M_{1}, F_{1}) \otimes H^{1-i}(M_{2}, F_{2}^{-1})$ . Since  $c_{1}(F_{2}^{-1}) < 0, H^{j}(M_{2}, F_{2}^{-1}) = (0)$  for  $j < n_{2}$ . Thus  $H^{1}(M_{1} \times M_{2}, F_{1} \otimes F_{2}^{-1}) = (0)$  by the same way. Since  $\operatorname{End}(1 \oplus F_{1} \otimes F_{2}^{-1}) = 1 \oplus (F_{1} \otimes F_{2}^{-1}) \oplus (F_{1}^{-1} \otimes F_{2}) \oplus 1$  and  $M_{1}, M_{2}$  are simply conected, we get our claim.

In general, for a compact complex manifold X let  $\operatorname{Aut}_0(X)$  denote the identity component of the group of all holomorphic automorphisms of X. Let E be a holomorphic vector bundle of rank r over a compact complex manifold M and P(E) the associated projective bundle over M. By a theorem of Blanchard [3], we see that  $\operatorname{Aut}_0(P(E))$  coincides with the identity component of all fiber preserving automorphisms of P(E). Thus the projection  $\pi: P(E) \to M$  induces a homomorphism  $\pi: \operatorname{Aut}_0(P(E)) \to \operatorname{Aut}_0(M)$ . Note also that the group of all fiber preserving holomorphic automorphisms of P(E) is naturally isomorphic to the group of all fiber preserving holomorphic automorphic automorphisms of the principal fiber bundle  $P(M, PGL(r, C), \pi)$  associated to the bundle  $\pi: P(E) \to M$ .

**Lemma 6.2.** Under the assumption as in Lemma 6.1, the homomorphism  $\pi$ : Aut<sub>0</sub>( $P(1 \oplus F_1 \otimes F_2^{-1})$ )  $\rightarrow$  Aut<sub>0</sub>( $M_1 \times M_2$ ) is surjective.

Proof. By Proposition 2 in [15] and Proposition 9 in [2], it is enough to show that  $H^1(M_1 \times M_2, \operatorname{End}(1 \oplus F_1 \otimes F_2^{-1})) = (0)$ . Thus we get our claim by Lemma 6.1. q.e.d.

We consider a holomorphic line bundle L over a compact complex manifold M and the  $P^{1}(C)$ -bundle  $P(1 \oplus L)$  over M. We assume that M has a Kähler-Einstein metric  $g_{0}$  with  $r_{0}=g_{0}$  and that L has a hermitian fiber metric such that the eigenvalues of the Ricci tensor B are constant on M and their absolute values are less than 1. We also assume that N. KOISO AND Y. SAKANE

$$\int_{-1}^{1} U \cdot \det (1 - U g_0^{-1} B) dU = 0$$

Thus  $P(1 \oplus L)$  admits a Kähler-Einstein metric by Theorem 5.4 in K-S [12]. Now we recall the following.

**Proposition 6.3.** In the above situation, if the homomorphism  $\pi$ : Aut<sub>0</sub>( $P(1 \oplus L)$ ) $\rightarrow$ Aut<sub>0</sub>(M) is surjective and B is non-trivial on each irreducible factor of the Kähler manifold M, then the Kähler-Einstein manifold  $P(1 \otimes L)$  is irreducible and cohomg ( $P(1 \oplus L)$ )=cohomg (M)+1.

Proof. See K-S [12] Proposition 5.6. Note that the homomoprhism  $\pi$ : Isom<sub>0</sub>( $P(1 \oplus L)$ ) $\rightarrow$ Isom<sub>0</sub>(M) is surjective by a theorem of Matsushima [13]. q.e.d.

Let  $N_0 = P^n(C)$ , *H* the holomorphic line bundle over  $P^n(C)$  corresponding to a hyperplane and  $L_0 = H^m$  for  $1 \le m \le n$ . Then we have  $c_1(L_0) = (m/(n+1))c_1(N_0)$ and we get an almost homogeneous Kähler-Einstein manifold  $P(L_0 \oplus L_0)$  of cohomogeneity one. Let  $N' = P(L_0 \oplus L_0)$ ,  $\pi: N' \to N_0 \times N_0$  the projection and  $\xi = L(L_0 \oplus L_0)$  the tautological line bundle over N'. Then we have

$$c_1(N') = (n+1-m)\pi^*(c_1(H)\oplus c_1(H)) + 2c_1(\xi).$$

Thus, if n+1-m is even, there exists a holomorphic line bundle L' over N' such that  $c_1(L')=(1/2)c_1(N')$ .

Now we construct a Kähler-Eintein manifold of cohomogeneity d for each given positive integer d. If d is even, put d=2k, and if d is odd, put d=2k+1 (we may assume  $d\geq 2$ ). Consider the product  $M_1=N'\times\cdots\times N'$  of d-1 copies of N' and the product  $F_1=L'\otimes\cdots\otimes L'$  of d-1 holomorphic line bundles on  $M_1$  induced from L' on N'. Then  $c_1(F_1)=(1/2)c_1(M_1)$ . If d is even, consider the complex projective space  $M_2$  of (2n+1)(d-1) dimension. Then  $c_1(M_2)=((2n+1)(d-1)+1)c_1(H)$ . Put  $F_2=H'$  where l=((2n+1)(d-1)+1)/2. If d is odd, consider the complex quadric  $M_2$  of (2n+1)(d-1) dimension. Then  $c_1(M_2)=((M_2)=(2n+1)(d-1)c_1(H)$  for a holomorphic line bundle H over  $M_2$ . Put  $F_2=H'$  where l=((2n+1)(d-1)/2. Consider the  $P^1(C)$ -bundle  $P(1\oplus F_1\otimes F_2^{-1})$  over  $M_1\times M_2$ . Then, by Lemma 6.2 and Proposition 6.3, we see that  $P(1\oplus F_1\otimes F_2^{-1})$  has cohomogeneity d.

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