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0. Introduction

In [19] the $n$-th codegree (number) $\text{cdg}(X, n) \in \mathbb{Z}$ and its stable version $\widehat{\text{cdg}}(X, n) \in \mathbb{Z}$ were defined for every pair of a path-connected space $X$ and a positive integer $n$. In [18], $\text{cdg}_p(G, 3)$, the exponent of a prime $p$ in $\widehat{\text{cdg}}(G, 3)$, was determined for some simply connected simple Lie groups $G$. The purpose of this paper is to continue computing $\widehat{\text{cdg}}(G, n)$ for some $(G, n)$. We use notations in [19] and [18]. Our results are the following.

Theorem 1. If $r \geq 3$, then

\[ r \leq \text{cdg}_2(\text{Spin}(n), 3) \leq r + 1 \quad \text{for} \quad 2^r \leq n \leq 2^r + 6, \]

\[ \text{cdg}_2(\text{Spin}(n), 3) = r + 1 \quad \text{for} \quad 2^r + 7 \leq n \leq 2^{r+1} - 1. \]

Theorem 2. $\text{cdg}_3(E_6, 3) = \text{cdg}_3(F_4, 3) = 2$.

Theorem 3. (1) $\text{cdg}(SU(3), 3) = 2^2$ and $\text{cdg}(SU(3), 5) = \text{cdg}(SU(3), 3) | 2^5 \cdot 3^2$; $\text{cdg}(SU(3), 5) = \text{cdg}(SU(4), 5) = 2^2 \cdot 3$.

(2) $\text{cdg}(SU(4), 3) = 2^2 \cdot 3 | \text{cdg}(SU(4), 3) | 2^5 \cdot 3^2$; $\text{cdg}(SU(4), 5) = \text{cdg}(SU(4), 3) | 2^5 \cdot 3^2$;

(3) $\text{cdg}(G_2, 11) = \text{cdg}(G_2, 11) = \text{cdg}(G_2, 3) = 2^2 \cdot 3 \cdot 5 | \text{cdg}(G_2, 3) | 2^5 \cdot 3^2 \cdot 5$.

(4) $\text{cdg}(\text{Spin}(n), 11) = \text{cdg}(SO(7), 11) = 2^4 \cdot 3 \cdot 5$ for $n = 7, 8$; $\text{cdg}(SO(7), 11) = 2^4$.

(5) $2^5 \cdot 3 \cdot \text{cdg}(Sp(3), 7) | \text{cdg}(Sp(3), 7) | 2^8 \cdot 3$.

Proposition 4.

\[ 2^3 \cdot 3^2 \cdot 5 | \text{cdg}(\text{Spin}(9), 7), \]

\[ 2^4 \cdot 3^2 \cdot 5 | \text{cdg}(\text{Spin}(9), 11), \]

\[ 2^4 \cdot 3^2 \cdot 5 \cdot 7 | \text{cdg}(\text{Spin}(9), 15), \]

\[ 2^2 \cdot 3 | \text{cdg}(SU(5), 5), \]

\[ 2^2 \cdot 3 | \text{cdg}(SU(5), 7), \]

\[ 2^2 \cdot 3 \cdot 5 \cdot 7 | \text{cdg}(F_4, n) \text{ for } n = 11, 15, \]

\[ 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 | \text{cdg}(F_4, 23). \]
1. Proof of Theorem 1

Let \( g: V_{2n-1} \rightarrow SO(2n+1) \) be the generating map for \(\text{Spin}(2n+1) \) (see [2]). Let \( g': \Sigma \text{Spin}(2n+1) \rightarrow \text{Spin}(2n+1) \) be the canonical map. Then \((g' \circ \Sigma g)_* : \pi_3(\Sigma V_{2n-1}) \approx \pi_3(\text{Spin}(2n+1)),\)

\[
\text{cdg}(V_{2n-1}, 2) \mid \text{cdg}(	ext{Spin}(2n+1), 3),
\]

\[
\text{cdg}^*(V_{2n-1}, 2) \mid \text{cdg}^*(\text{Spin}(2n+1), 3).
\]

We will calculate the 2-components of these numbers.

The inclusions \( U(n) \subset SO(2n) = SO(2n) \times I, SO(2n+1) = SO(2n+1) \times I \subset SO(2n+3), \) and \( U(n) = U(n) \times I \subset U(n+1) \) induce maps:

\[
\begin{align*}
\sigma_n &: P(C^n) = U(n)/U(1) \times U(n-1) \rightarrow V_{2n-1}, \\
\tau_n &: V_{2n-1} \rightarrow V_{2n+1}, \\
\tau'_n &: P(C^n) \rightarrow P(C^{n+1})
\end{align*}
\]

such that \( \tau_n \circ \sigma_n = \sigma_n \circ \tau'_n. \) Let \( L_n \) be the canonical complex line bundle over the complex projective \((n-1)\)-space \( P(C^n) \), and let \( a_n \in H^2(P(C^n); \mathbb{Z}) \) be the first Chern class of \( L_n \). Then

\[
\tau'^*(a_{n+1}) = a_n.
\]

As is easily seen (e.g., [2]), we have

\[
H^*(V_{2n-1}; \mathbb{Z}) = \mathbb{Z}[x_n, y_n]/(x_n^2 - 2y_n, y_n^2),
\]

\[
\dim(x_n) = 2, \quad \dim(y_n) = 2n,
\]

\[
\sigma_n^*(x_n) = a_n, \\
\tau_n^*(x_n+1) = x_n.
\]

Hence

\[
\begin{align*}
\sigma_n^* &: H^i(V_{2n-1}; \mathbb{Z}) \approx H^i(P(C^n); \mathbb{Z}) \quad \text{for } i \leq 2n-2, \\
\tau_n^* &: H^i(V_{2n+1}; \mathbb{Z}) \approx H^i(V_{2n-1}; \mathbb{Z}) \quad \text{for } i \leq 2n-2, \\
H^*(V_{2n-1}; \mathbb{Q}) &= \mathbb{Q}[x_n]/(x_n^{2n}),
\end{align*}
\]

\[
\begin{align*}
\tau_n^* &: H^i(V_{2n+1}; \mathbb{Q}) \approx H^i(V_{2n-1}; \mathbb{Q}) \quad \text{for } i \leq 4n-2.
\end{align*}
\]

Recall from Clarke [4] that

\[
K(V_{2n-1}) = \mathbb{Z}[X_n, Y_n]/(X_n^2 - 2Y_n, X_nY_n, Y_n^2).
\]

Hence

\[
K(V_{2n-1}) \otimes \mathbb{Q} = \mathbb{Q}[X_n]/(X_n^{2n}).
\]
By the construction of $X_n$ ([4]), we have
\[ \sigma^*(X_n) = L_n - 1, \]
\[ \tau^*(X_{n+1}) = X_n. \]

The Chern character of $X_n$ is given by

**Lemma 1.5.** $ch(X_n) = \exp(x_n) - 1$.

**Proof.** We have
\[ \sigma^*_n(ch(X_{2n})) = ch(\sigma^*_n(X_{2n})) = ch(L_{2n} - 1) = \exp(a_{2n}) - 1 = \sigma^*_n(\exp(x_{2n}) - 1). \]

Hence $ch(X_{2n}) \equiv \exp(x_{2n}) - 1 \mod x_{2n}$ by (1.2), thus $ch(X_n) = \exp(x_n) - 1$ by (1.3). This proves 1.5.

**Proposition 1.6.**

1. $^{\text{cdg}}_2(V_{2n-1}, 2) = r$ if $2^{r-1} \leq n \leq 2^r$.

2. $^{\text{cdg}}_2(V_{2n-1}, 2) = r$ if $2^r \leq n < 2^r$,
   \[ r \leq _{\text{cdg}}(V_{2n-1}, 2) \leq r + 1 \text{ if } n = 2^r. \]

**Proof.** Put $D = _{\text{cdg}}(V_{2n-1}, 2)$. Let $f: V_{2n-1} \to S^2$ be a stable map such that the induced homomorphism $f_*: \pi_2(V_{2n-1}) = \mathbb{Z} \to \pi_2(S^2) = \mathbb{Z}$ is multiplication by $D$. Let $\beta \in \mathcal{K}(S^2) = \mathbb{Z}$ be a generator. For simplicity, we set $x = x_n$ and $y = y_n$. Set
\[ f*(\beta) = \sum_{1 \leq i < n} a_i X^i + \sum_{0 \leq i < n} b_i X^i = \sum_{1 \leq i < n} a_i X^i \]
in $\mathcal{K}(V_{2n-1}) \otimes \mathbb{Q}$, where $a_i \in \mathbb{Z}(1 \leq i < n)$, $b_i \in \mathbb{Z}(0 \leq i < n)$, and $a_i \in \mathbb{Q}(n \leq i < 2n)$. Then
\[ D \cdot \sum_{1 \leq i < n} \left((1)_{i-1}^i \right) (e^i - 1)^i = D \cdot \log(e^i - 1 + 1) = D \cdot x = f^*ch(\beta) = ch(f*(\beta)) = \sum_{1 \leq i < n} a_i (e^i - 1)^i. \]

Hence $a_i = D \cdot (-1)^{i-1}/i (1 \leq i < 2n)$. We then have
\[ f*(\beta) = \sum_{1 \leq i < n} a_i X^i + (2Y + XY) \sum_{1 \leq i < 2n} a_i X^{i-n} \]
\[ = \sum_{1 \leq i < n} (D(-1)^{i-1}/i)(X^i + Y \cdot 2D(-1)^{n-1}/n) \]
\[ + Y \cdot \sum_{1 \leq i \leq 2n-2} (D(-1)^{i-1}/i + 2D(-1)^i/(i+1)) X^{i-n+1}. \]

Thus
\[ (1.7) \ D/i (1 \leq i < n), 2D/n, \text{ and } D/i - 2D/(i+1) (n \leq i \leq 2n - 2) \text{ are in } \mathbb{Z}. \]

Let $r \geq 1$ be an integer such that $2^{r-1} \leq n \leq 2^r$. Then the relation $D/2^r - 2D/(2^r + 1) \in \mathbb{Z}$ implies that $2^r | D$. Conversely, if $2^r | D$, then (1.7) with $\mathbb{Z}$ replaced by its 2-localized ring $\mathbb{Z}(2)$ holds. Therefore $^{\text{cdg}}_2(V_{2n-1}, 2) = r$. This proves (1).
A map $V_{2n-1} \rightarrow K(Z, 2)$ which represents $x_n$ factorizes as $V_{2n-1} \rightarrow P(C^2) \subset K(Z, 2)$. Hence $^*\text{cdg}(V_{2n-1}, 2)|^*\text{cdg}(P(C^2), 2)$ so that $r = ^*\text{cdg}_2(V_{2n-1}, 2) \leq ^*\text{cdg}_2(V_{2n-1}, 2) \leq ^*\text{cdg}_2(P(C^2), 2)$. By [18], we have $^*\text{cdg}_2(P(C^2), 2) = ^*\text{cdg}_2(SU(2n), 3)$ which is $r+1$ or $r$ according as $n=2^r$ or $2^{r-1}<n<2^r$. Hence we have (2).

**Corollary 1.8.** $^*\text{cdg}_2(\text{Spin}(2n+1), 3) \geq r$ if $2^{r-1}<n \leq 2^r$.

**Proof.** This follows from 1.1 and 1.6.

**Proof of Theorem 1.** The complexification induces isomorphisms of representation rings:

$$RO(\text{Spin}(m)) \simeq R(\text{Spin}(m)) \quad \text{if} \quad m \equiv 0, 1, 7 \mod 8$$

(see [7, p. 193]). By the proof of [18, 4.4], we then have

$$^*\text{cdg}^G(\text{Spin}(m), 3) = 2^*^*\text{cdg}(\text{Spin}(m), 3) \quad \text{if} \quad m \equiv 0, 1, 7 \mod 8.$$  

Thus, by 1.8, we have

$$^*\text{cdg}_2(\text{Spin}(2n+1), 3) \geq r+1 \quad \text{if} \quad n \equiv 0, 3 \mod 4 \text{ and } 2^{r-1}<n \leq 2^r.$$  

On the other hand, if $n \geq 2$, then the canonical homomorphism

$$Z = \pi_3(\text{Spin}(2n+1)) \rightarrow \pi_3(\text{SO}(2n+1)) \rightarrow \pi_3(\text{SU}(2n+1)) = Z$$

is multiplication by 2, so that

$$^*\text{cdg}_2(\text{Spin}(2n+1), 3) \leq 1+^*\text{cdg}_2(SU(2n+1), 3).$$

The latter number is $r+2$ or $r+1$ according as $n=2^r$ or $2^{r-1}<n<2^r$, by [18]. Hence

$$^*\text{cdg}_2(\text{Spin}(2n+1), 3) = r+1 \quad \text{if} \quad n \equiv 0, 3 \mod 4 \text{ and } 2^{r-1}<n<2^r.$$  

In particular, if $r \geq 3$, then $^*\text{cdg}_2(\text{Spin}(2^r+1), 3)=r$ and $^*\text{cdg}_2(\text{Spin}(2^r+7), 3)=^*\text{cdg}_2(\text{Spin}(2^{r+1}-1), 3)=r+1$. Hence, if $r \geq 3$, then

$$r \leq ^*\text{cdg}_2(\text{Spin}(n), 3) \leq r+1 \quad \text{for} \quad 2^{r-1}<n<2^r,$$

$$^*\text{cdg}_2(\text{Spin}(n), 3) = r+1 \quad \text{for} \quad 2^r+7 \leq n \leq 2^{r+1}-1.$$  

This proves Theorem 1.

**2. Proof of Theorem 2**

The relations $^*\text{cdg}_2(E_6, 3) = ^*\text{cdg}_2(F_4, 3) \geq 2$ were proved in [18]. We will prove $^*\text{cdg}_2(F_4, 3) \leq 2$. By [6] and [12], there exist a mod 3 $H$-space $X$ of dimension 26 and a mod 3 homotopy equivalence
where $B_5(3)$ is the total space of an $S^{15}$-bundle over $S^{26}$ [15]. It follows from [3] that the top cell of the localized space $X_{(3)}$ splits off stably, that is, $X = X^{(23)} \vee S^{26}$ (stably), where $X^{(23)}$ is the 23-skeleton of $X$, and it follows from [5] that $X^{(23)}$ is stably homotopy equivalent to $X_1 \vee X_2$ where $X_5$ is 17-connected and $H^*(X_1; \mathbb{Z}_3) = \mathbb{Z}_3 \{1, x_3, x_7, x_8, x_{18}, x_{19}, x_{23}\}$ such that $\dim(x_i) = i, \mathcal{P}^{13} x_3 = x_7, \beta x_7 = x_8, \beta x_{18} = x_{19}$, and $\mathcal{P}^{13} x_{19} = x_{23}$.

**Lemma 2.1.** $X_1 = S^3(3) \cup e_5(3) \cup e_8(3) \cup e_{18}(3) \cup e_{19}(3) \cup e_{23}(3)$.

In the rest of this section we work in the stable homotopy category of mod 3 local spaces. For simplicity we omit the notation "(3)".

**Proof of 2.1.** Let

$$S^3 \to X_1 \to Y_1 \to \Sigma S^3 \to \Sigma X_1$$

be a cofibre sequence such that $Y_1$ is 6-connected. Then $X_1 = S^3 \cup C_{(3)}^{-1} Y_1$. Inductively we have cofibre sequences

$$S^7 \to Y_1 \to Y_2 \to \Sigma S^7 \to \Sigma Y_2,$$

$$S^8 \to Y_2 \to Y_3 \to \Sigma S^8 \to \Sigma Y_3,$$

$$S^{18} \to Y_3 \to Y_4 \to \Sigma S^{18} \to \Sigma Y_4,$$

$$S^{19} \to Y_4 \to Y_5 \to \Sigma S^{19} \to \Sigma Y_5$$

and

$$Y_1 = S^7 \cup C_{(3)}^{-1} Y_7,$$

$$Y_2 = S^8 \cup C_{(3)}^{-1} Y_8,$$

$$Y_3 = S^{18} \cup C_{(3)}^{-1} Y_{18},$$

$$Y_4 = S^{19} \cup C_{(3)}^{-1} Y_{19} = S^{23} \cup e_{23}$$

where the last equality follows from the fact that $Y_5 = S^{23}$. Therefore we have

$$X_1 = S^3 \cup C(S^6 \cup C(S^6 \cup C(S^{15} \cup C(S^{15} \cup e^{19})))) .$$

This proves 2.1.

**Proof of Theorem 2.** Put $Y = S^3 \cup e^7 \cup e^8 \cup e^{18} \cup e^{19} \cup e^{23}$. Then $\text{cdg}_3(F_4, 3) = \text{cdg}_3(X, 3) = \text{cdg}_3(Y, 3)$. Let $\alpha_i \in \pi_{4i-1}(S^6) (1 \leq i \leq 5)$ be the element of order 3 defined in [20, p. 178]. Let $\alpha'_i: S^{4i-1} \cup e^i \to S^6$ and $\alpha''_i: S^i \to S^6 \cup e^i$ be an extension of $\alpha_i$ and a coextension of $\alpha_i$ respectively. The 8-skeleton $Y^{(8)}$ of $Y$ is equivalent to the mapping cone $C(\Sigma^3 \alpha'_i) = S^3 \cup C(S^6 \cup e^i)$ and $Y/Y^{(8)}$ is equivalent to $C(\Sigma^{15} \alpha''_i) = S^{18} \cup e^{19} \cup e^{23}$. Hence we have a cofibre sequence
Let \( g: C(\Sigma^4 \alpha') \to S^4 \) be an extension of \( 3: S^4 \to S^4 \). As is easily seen, we have an exact sequence

\[
\pi^4(S^{18} \cup_3 e^{19}) = \mathbb{Z}_3 \{\alpha_i\} \to \pi^4(S^{22}) = \mathbb{Z}_3 \{\alpha_i\} \\
\to \pi^4(S^{18} \cup_3 e^{19} \cup e^{20}) = \pi^4(S^{18} \cup_3 e^{19}) = \mathbb{Z}_3 \to 0.
\]

Since \( \alpha_i' \ast (\alpha_i) \in \langle \alpha_i, 3, \alpha_i \rangle = \mathbb{Z}_3 \), it follows that \( \pi^4(S^{18} \cup_3 e^{19} \cup e^{20}) = \mathbb{Z}_3 \) and \( 3g \circ h = 0 \). Hence there exists a map \( r: \Sigma Y \to S^4 \) such that \( r \circ k = 3g \), so that \( r \) has degree 9 on the bottom sphere \( S^4 \) and \( \text{cdg}_4(Y) \leq 2 \). Hence \( \text{cdg}(F, 3) = 2 \) as desired.

3. Proof of Theorem 3

Lemma 3.1. Given an integer \( n \geq 2 \) and a connected finite CW-complex \( X \) such that

- \( X \) and its \((n-1)\)-skeleton \( Y \) are simply connected;
- \( \pi_{n-1}(X) = 0 \);
- \( \pi_n(Y) = \mathbb{Z}_m \);
- \( \text{rank}(\pi_n(X/Y)) = \text{rank}(\pi_n(X)) = 1 \);

then we have \( \text{Cdg}(X, n) \subset m \cdot \text{Hom}(\pi_n(X), \pi_n(S^n)) \). If moreover \( \text{Cdg} \) is surjective for \((X/Y, n)\), then \( \text{Cdg}(X, n) = m \cdot \text{Hom}(\pi_n(X), \pi_n(S^n)) \).

Proof. By a theorem of Blakers-Massey, the collapsing map induces an isomorphism \( c_*: \pi_n(X, Y) \approx \pi_n(X/Y) \). From the assumptions and the homotopy exact sequence of the pair \((X, Y)\), it follows that \( c_*: \pi_n(X)/\text{Tor} = \mathbb{Z} \to \pi_n(X/Y) = \mathbb{Z} \) is multiplication by \( m \). Hence the assertion follows from the commutative diagram

\[
[X/Y, S^n] \to \text{Hom}(\pi_n(X/Y), \pi_n(S^n)) = \mathbb{Z} \\
c_* \downarrow \downarrow c_* = m \\
[X, S^n] \to \text{Hom}(\pi_n(X), \pi_n(S^n)) = \mathbb{Z}
\]
since \( c_* \) is surjective.

Proof of Theorem 3(1). We shall use notations and results in [20]. The group \( SU(3) \) has a cell structure: \( SU(3) = S^3 \cup_3 e^6 \cup_6 e^6 \). As noticed in [14, p. 475], we have \( \Sigma f = j_3(\nu \circ \eta) \) from [9, 3.1], where \( j: S^4 \subset S^4 \cup e^6 \). Let \( h: S^3 \cup e^6 \to S^9 \) be a map having degree 2 on \( S^3 \). We have

\[
2\nu_1 \circ \nu_4 = 2\nu_1 + [\nu_4, h], \text{ by } [22, XI], \\
= 4\nu_1 - \Sigma \nu', \text{ by } [20, p. 43].
\]

Hence \( \Sigma (h \circ f) = 2\nu_1 \circ \nu_4 \circ \eta_1 = -\Sigma \nu' \circ \eta_1 = \Sigma (\nu' \circ \eta_6) \), so \( h \circ f = \nu' \circ \eta_6 \), since \( \Sigma \) is injec-
tive on $\pi_7(S_3) = \mathbb{Z}_2 \{\eta\}$. It follows that $2\eta h$ can be extended to $SU(3)$ and that $cdg(SU(3), 3) = 2$. By the same method as [18, 4.3(1)], we can prove that $cdg(SU(3), 5) = \cdg(SU(3), n) = 2$ for $n=3, 5$. By applying the functor $[ , S^3]$ to the cofibre sequence $S^7 \to S^5 \cup e \subset SU(3)$, we have $[SU(3), S^3] = \mathbb{Z} \oplus \mathbb{Z}_2$. Other assertions of (1) can be proved easily.

Proof of Theorem 3(2). By [13] there is a stable homotopy equivalence $SU(4) = \Sigma P(C^4) \vee Y$, where $Y$ is a 7-connected finite CW-complex. Hence, for $n=3, 5, 7$, we have $\cdg(S(U(4)), n) = \cdg(P(C^4), n-1)$ which can be easily determined by using the cell structure of $P(C^4)$ (see [17, 1.15]).

By using a cell structure of $SU(4)$ and known structures of $\pi^*(S^5)$ and $\pi_{14}(S^3)$ (see [20]), we can construct a map $SU(4) \to S^3$ which has degree $2^5 \cdot 3^2$ on $S^3$, hence $cdg(S(U(4), 3)|2^5 \cdot 3^2$. On the other hand we have $2^5 \cdot 3 = \cdg(S(U(4), 3)|cdg(S(U(4), 3)$ by [18, 3.3] and [19, 3.4(3)].

Taking $(X, n) = (SU(4), 5)$ in 3.1, we have $Cdg(S(U(4), 5) \subset 2\mathbb{Z}$. By the homotopy exact sequence of the principal $Sp(2)$-bundle $q: SU(4) \to S^5$, we have $Cdg(q) = 2$, so $cdg(S(U(4), 5) = 2$. By [19, 4.2(1)], $cdg(S(U(4), 7) = 6$. This completes the proof of (2).

Proof of Theorem 3(3). Using a stable duality map $S^{14} \to G_{2+} \wedge G_{2+}$ (see [3]), we can prove $\cdg(G_{2}, 11) = \cdg(G_{2}, 3)$ which is $2^5 \cdot 3$ as proved in [18]. The group $G_{2}$ has a cell structure: $G_{2} = S^3 \cup e \cup e \cup e \cup e \cup e$. Hence $G_{2}/Y = S^{11} \cup S^{14}$, where $Y$ is the 9-skeleton of $G_{2}$, so Cdg is surjective for $(G_{2}/Y, 11)$. Consider the exact sequence:

$$\pi_{11}(G_{2}) = \mathbb{Z} \oplus \mathbb{Z}_2 \to \pi_{11}(G_{2}, Y) = \mathbb{Z} \to \pi_{10}(Y) \to \pi_{10}(G_{2}) = 0.$$  

By [16, 4.2], $\pi_{10}(Y) = \mathbb{Z}_{20}$. Taking $(X, n) = (G_{2}, 11)$ in 3.1, we have $m=2^5 \cdot 3$ and $cdg(G_{2}, 11) = 2^5 \cdot 3$. Since $Sq^2$ is trivial on $H^6(G_{2}; \mathbb{Z}_2)$ by [1], the attaching map of the 8-dimensional cell of $G_{2}$ factorizes as $S^7 \to S^5 \cup e \subset S^5 \cup e \cup e$. Using this fact and the additive structures of $\pi_8(S^3)$ and $\pi_{13}(S^3)$ (see [20]), we can construct a map $G_{2} \to S^3$ which has degree $2^5 \cdot 3$ on $S^3$, hence $cdg(G_{2}, 3)|2^5 \cdot 3^2$. This proves (3).

Proof of Theorem 3(4). Applying $\pi_8( )$ to the following commutative diagram

$$
\begin{array}{c}
SO(5) \subset SO(6) \\
\downarrow \quad \downarrow \\
SO(7) = SO(7) \\
\downarrow p \downarrow \\
S^5 \subset V_{7,2} = SO(7)/SO(5) \to S^3 \\
\end{array}
$$

we have $p_8 = 15: \pi_{11}(SO(7))/Tor = \mathbb{Z} \to \pi_{11}(V_{7,2})/Tor = \mathbb{Z}$. Hence
The space $V_{7,2}$ has a cell structure: $V_{7,2} = S^8 \cup e^6 \cup e^{11}$. Let $q: V_{7,2} \to S^{11}$ be the collapsing map. By [8, 2.8 and 2.9], we have an exact sequence $\pi_9(S^8) = Z_2 \to \pi_{10}(S^8 \cup e^6, S^8) \to \pi_9(S^8 \cup e^6) = Z_2 \to 0$, hence the order of $\pi_{10}(S^8 \cup e^6, S^8)$ is at most 4, so that the order of $\pi_{10}(S^8 \cup e^6)$ is at most 8 because $\pi_{10}(S^8) = Z_2$ by [20]. Since $\pi_{10}(V_{7,2}) = 0$, it then follows from the homotopy exact sequence of the pair $(V_{7,2}, S^8 \cup e^6)$ that $\text{Coker}[q_*: \pi_{11}(V_{7,2}) \to \pi_{11}(S^{11})] = \pi_{10}(S^8 \cup e^6)$, so that the last group is cyclic. Hence by the commutative diagram

\[
\begin{array}{ccc}
[S^{11}, S^{11}] & \to & \text{Hom}(\pi_{11}(S^{11}), \pi_{11}(S^{11})) = Z \\
q^* \downarrow & & \downarrow q_*^* \\
[V_{7,2}, S^{11}] & \to & \text{Hom}(\pi_{11}(V_{7,2}), \pi_{11}(S^{11})) = Z
\end{array}
\]

$\text{cdg}(V_{7,2}, 11)$ is the order of $\pi_{10}(S^8 \cup e^6)$, since $q^*$ is surjective. Thus $\text{cdg}(V_{7,2}, 11)$ is $2^3$. On the other hand, $\text{cdg}(G_2, 11)|\text{cdg}(\text{Spin}(7), 11)$ by use of the principal $G_2$-bundle $\text{Spin}(7) \to S^7$. Hence $2^3 \cdot 3 = \text{cdg}(G_2, 11)|\text{cdg}(\text{Spin}(7), 11)|\text{cdg}(SO(7), 11)|3 \cdot 5 = \text{cdg}(F_7, 11)|2^3 \cdot 3$, therefore these numbers are equal and $\text{cdg}(V_{7,2}, 11) = 2^3$. Then [19, 3.7(2) and 4.4] complete the proof of (4).

Proof of Theorem 3(5). The groups $\text{Sp}(2)$ and $\text{Sp}(3)$ have cell structures: $\text{Sp}(2) = S^5 \cup e^2 \cup e^6 \cup e^{10}$ and $\text{Sp}(3) = S^5 \cup e^2 \cup e^{10} \cup e^{11} \cup e^{14} \cup e^{18} \cup e^{21}$. They contain quasi-projective spaces $Q_2 = S^5 \cup e^2$ and $Q_3 = S^5 \cup e^2 \cup e^{11}$ respectively. The inclusions induce isomorphisms $\pi_7(Q_2) = \pi_7(Q_3) = \pi_7(\text{Sp}(3))$. In $Q_{3,2} = Q_3/S^3 = S^7 \cup e^{11}$, $h$ has order 8 (see [10, p. 38]). Hence the homomorphisms $\{Q_{3,2}, S^7\} = \mathbb{Z} \to \{Q_2, S^7\} = \mathbb{Z}$ and $\{Q_3, S^7\} = \mathbb{Z} \to \{Q_2, S^7\} = \mathbb{Z}$ induced by the inclusions are multiplications by 8. Let $t$ be the order of the cokernel of the stabilization $\pi_7(\text{Sp}(3)) = \mathbb{Z} \to \pi_7(\text{Sp}(3)) = \mathbb{Z}$. Consider the following commutative diagram.

\[
\begin{array}{ccc}
[\text{Sp}(3), S^7] & \to & \text{Hom}(\pi_7(\text{Sp}(3)), \pi_7(S^7)) = Z \\
\downarrow & & \uparrow t \\
\{\text{Sp}(3), S^7\} & \to & \text{Hom}(\pi_7(\text{Sp}(3)), \pi_7(S^7)) = Z \\
\downarrow & & \downarrow \cong \\
\{Q_3, S^7\} = \mathbb{Z} & \to & \text{Hom}(\pi_7(\text{Q}_3), \pi_7(S^7)) = Z \\
\downarrow 2^3 & & \downarrow \cong \\
\{Q_2, S^7\} = \mathbb{Z} & \to & \text{Hom}(\pi_7(\text{Q}_2), \pi_7(S^7)) = Z
\end{array}
\]

By the proof of [19, 4.3(1)], the bottom homomorphism is multiplication by $2^3 \cdot 3$. Hence $t \cdot 2^3 \cdot 3 | t \cdot \text{cdg}(\text{Sp}(3), 7)|\text{cdg}(\text{Sp}(3), 7)$.

Let $p: \text{Sp}(3) \to X_{3,2} = \text{Sp}(3)/\text{Sp}(1)$ be the canonical fibration. Since $\text{Coker}[p_*: \pi_7(\text{Sp}(3)) \to \pi_7(X_{3,2})] = \pi_6(S^9) = Z_{12}$, we have $\text{cdg}(\text{Sp}(3), 7)|2^3 \cdot 3 \cdot \text{cdg}(X_{3,2}, 7)$. The space $X_{3,2}$ has a cell structure: $X_{3,2} = S^7 \cup e^{11} \cup e^{14}$. Since the order of $h$
is 8, there is a map $u: \mathbb{Q}/t^2 \to S^7$ which has degree 8 on $S^7$. Since $u \circ v \in \pi_7(S^7) = \mathbb{Z}_{2^k} \oplus \mathbb{Z}_2$ (see [20]), $2^k \cdot 3^3 u$ can be extended to a map $X_{3^3} \to S^7$ which has degree $2^k \cdot 3^3$ on $S^7$. Therefore $\text{cdg}(X_{3^3}, 7) \mid 2^k \cdot 3^3$ and so $\text{cdg}(\text{Sp}(3), 7) \mid 2^k \cdot 3^3$. On the other hand, $\text{cdg}(\text{Sp}(3), 7) = \text{cdg}(\text{Spin}(7), 7) = 1$ by [19, 4.2(2) and 4.3(2)]. Hence $\text{cdg}(\text{Sp}(3), 7) \mid 2^k \cdot 3^3$. This proves (5) and completes the proof of Theorem 3.

4. Proof of Proposition 4

By [19, 2.4], we have

**Proposition 4.1** (cf., [19, 3.15(2)]). If $n$ is odd, $\pi_n(X) = \mathbb{Z} \{s\} \oplus \text{Tor}$ and $H^n(X; \mathbb{Z}) = \mathbb{Z} \{s_n\} \oplus \text{Tor}$, then there exists a map $f: X \to S^n$ such that $\text{cdg}(X, n) = \deg(f \ast s) = |AB|$, where integers $A$ and $B$ are defined by $s \ast (s_n) = A[S^n]$ and $f \ast [S^n] \equiv Bx_n \mod \text{Tor}$ respectively. (Here $[S^n]$ is a generator of $H^n(S^n; \mathbb{Z})$.) Stable version also holds.

Hence the next result proves Proposition 4.

**Lemma 4.2.** In 4.1, $(|A|, |B|)$ is equal to

$$(3, 2^2 \cdot 3 \cdot 5 \cdot y_1) \quad \text{for} \quad (\text{Spin}(9), 7),$$

$$(2^2 \cdot 3 \cdot 5, 2 \cdot 3 \cdot y_2) \quad \text{for} \quad (\text{Spin}(9), 11),$$

$$(2^2 \cdot 3 \cdot 3 \cdot 7, y_3) \quad \text{for} \quad (\text{Spin}(9), 15),$$

$$(2, 2^2 \cdot 3 \cdot y_4) \quad \text{for} \quad (SU(5), 5),$$

$$(2 \cdot 3, 2 \cdot y_5) \quad \text{for} \quad (SU(5), 7),$$

$$(2^2 \cdot 5, 2^2 \cdot 3 \cdot 7 \cdot y_6) \quad \text{for} \quad (F_4, 11),$$

$$(2^2 \cdot 3 \cdot 7, 2^4 \cdot 5 \cdot y_7) \quad \text{for} \quad (F_4, 15),$$

$$(2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11, y_8) \quad \text{for} \quad (F_4, 23)$$

for some integers $y_1$. In these cases, $B|y_1|^\ast \text{cdg}(G, n)$.

Proof. We prove the assertion only for $(F_4, 11)$, because others can be proved similarly. Assertion about $A$ has been known (for example, see [11]). Recall that the type of $F_4$ is $(3, 11, 15, 23)$. Let $x_9 \in H^n(F_4; \mathbb{Z})$ be a generator of the free part for $n \in \{3, 11, 15, 23\}$ (see [1]). Consider the commutative diagram:

$$
\begin{array}{c}
K^1(S^3) \overset{ch}{\cong} H^1(S^3; \mathbb{Z}) \subset H^*(S^3; \mathbb{Q}) \\
\downarrow f^* \quad \downarrow f^* \\
K^1(F_4) \overset{ch}{\longrightarrow} H^*(F_4; \mathbb{Q})
\end{array}
$$

As is well-known, $K^*(F_4)$ is an exterior algebra generated by some elements
\[ \beta_1, \ldots, \beta_6 \in K^*(S^n) \] whose Chern characters were determined in [21]. Let \( g \) be a generator of \( K^*(S^n) \). Let \( \kappa \text{Dec} \) and \( \mu \text{Dec} \) be the groups of decomposable elements with respect to \( \{ \beta_1, \ldots, \beta_6 \} \) and \( \{ x_3, x_{11}, x_{15}, x_{29} \} \), respectively. Express \( f^*(g) \equiv \Sigma a_i \beta_i \mod \kappa \text{Dec} \) and \( \Sigma a_i ch(\beta_i) = P_3 x_3 + P_1 x_{11} + P_{15} x_{15} + P_{29} x_{29} \), where \( P_i \) is a polynomial of \( a_1, \ldots, a_6 \) with rational coefficients. Then \( ch(f^*(g)) \equiv \Sigma P_i x_i \mod \mu \text{Dec} \). On the other hand, \( ch(f^*(g)) = f^* ch(g) = \pm f^*[S^{12}] \in H^{12}(F_4; \mathbb{Z}) \). Hence \( P_{11} = \pm B \) and \( P_3 = P_{15} = P_{29} = 0 \), then \( B = |P_{11}| = 2^4 \cdot 3 \cdot 7 \cdot |a_3| \) by elementary calculation. This argument can also be applied to stable case. By setting \( y_6 = |a_3| \), we have the desired result.

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