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<th>Codegree of simple Lie groups-II</th>
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<tr>
<td>Author(s)</td>
<td>Ōshima, Hideaki; Kono, Akira</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 28(1) P.129-P.139</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1991</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/3562">https://doi.org/10.18910/3562</a></td>
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<td>DOI</td>
<td>10.18910/3562</td>
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Osaka University
0. Introduction

In [19] the $n$-th codegree (number) $\text{cdg}(X, n) \in \mathbb{Z}$ and its stable version $\text{cdg}(X, n) \in \mathbb{Z}$ were defined for every pair of a path-connected space $X$ and a positive integer $n$. In [18], $\text{cdg}_p(G, 3)$, the exponent of a prime $p$ in $\text{cdg}(G, 3)$, was determined for some simply connected simple Lie groups $G$. The purpose of this paper is to continue computing $\text{cdg}(G, n)$ for some $(G, n)$. We use notations in [19] and [18]. Our results are the following.

**Theorem 1.** If $r \geq 3$, then
\[ r \leq \text{cdg}_2(\text{Spin}(n), 3) \leq r+1 \quad \text{for} \quad 2 \leq n \leq 2r+6, \]
\[ \text{cdg}_2(\text{Spin}(n), 3) = r+1 \quad \text{for} \quad 2r+7 \leq n \leq 2r+1-1. \]

**Theorem 2.** $\text{cdg}_3(E_6, 3) = \text{cdg}_3(F_4, 3) = 2$.

**Theorem 3.** (1) $\text{cdg}(SU(3), 3) = 2^2$ and $\text{cdg}(SU(3), 5) = \text{cdg}(SU(3), 11) = \mathbb{Z} \oplus \mathbb{Z}$; $[SU(3), S^3] = \mathbb{Z}$; $[SU(3), S^5] = \{SU(3), S^5\}$.

(2) $\text{cdg}(SU(4), 3) = 2^2 \cdot 3 \cdot \text{cdg}(SU(4), 3) | 2^5 \cdot 3^2$; $\text{cdg}(SU(4), 5) = \text{cdg}(SU(4), 7) = 2 \cdot 3$.

(3) $\text{cdg}(G_2, 11) = \text{cdg}(G_2, 3) = \text{cdg}(G_2, 3) = 2^2 \cdot 3 \cdot 5 \cdot \text{cdg}(G_2, 3) | 2^5 \cdot 3^2 \cdot 5$.

(4) $\text{cdg}(\text{Spin}(n), 11) = \text{cdg}(SO(n), 11) = 2^4 \cdot 3 \cdot 5$ for $n = 7, 8$; $\text{cdg}(SO(7)|SO(5), 11) = 2^4$.

(5) $2^5 \cdot 3 | \text{cdg}(Sp(3), 7) | \text{cdg}(Sp(3), 7) | 2^4 \cdot 3$.

**Proposition 4.**

\[ 2^3 \cdot 3 \cdot 5 | \text{cdg}(\text{Spin}(9), 7), \]
\[ 2^4 \cdot 3 \cdot 5 | \text{cdg}(\text{Spin}(9), 11), \]
\[ 2^3 \cdot 3 \cdot 5 \cdot 7 | \text{cdg}(\text{Spin}(9), 15), \]
\[ 2^3 \cdot 3 | \text{cdg}(SU(5), 5), \]
\[ 2^3 \cdot 3 | \text{cdg}(SU(5), 7), \]
\[ 2^3 \cdot 3 \cdot 5 \cdot 7 | \text{cdg}(F_4, n) \quad \text{for} \quad n = 11, 15, \]
\[ 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 | \text{cdg}(F_4, 23). \]
1. Proof of Theorem 1

Let \( g : V_{2n-1} = SO(2n+1)/SO(2) \times SO(2n-1) \to \Omega Spin(2n+1) \) be the generating map for \( Spin(2n+1) \) (\( n \geq 3 \)) (see [2]). Let \( g' : \Sigma \Omega Spin(2n+1) \to Spin(2n+1) \) be the canonical map. Then \( (g' \circ \Sigma g)_* : \pi_3(\Sigma V_{2n-1}) \cong \pi_3(\text{Spin}(2n+1)) \), hence

\[
\text{cdg}(V_{2n-1}, 2) = \text{cdg}(\text{Spin}(2n+1), 3),
\]

\[
\text{cdg}(V_{2n-1}, 2) = \text{cdg}(\text{Spin}(2n+1), 3).
\]

We will calculate the 2-components of these numbers.

The inclusions \( U(n) \subset SO(2n) = SO(2n) \times I_1 \subset SO(2n+1) \), \( SO(2n+1) = SO(2n+1) \times I_2 \subset SO(2n+3) \), and \( U(n) = U(n) \times I_1 \subset U(n+1) \) induce maps:

\[
\sigma_n : P(C^n) = U(n)/U(1) \times U(n-1) \to V_{2n-1},
\]

\[
\tau_n : V_{2n-1} \to V_{2n+1},
\]

\[
\tau_n^* : P(C^n) \to P(C^{n+1})
\]

such that \( \tau_n \circ \sigma_n = \sigma_n \circ \tau_n'. \) Let \( L_n \) be the canonical complex line bundle over the complex projective \((n-1)-\text{space} P(C^n)\), and let \( a_n \in H^2(P(C^n); \mathbb{Z}) \) be the first Chern class of \( L_n \). Then

\[
\tau_n^*(a_{n+1}) = a_n.
\]

As is easily seen (e.g., [2]), we have

\[
H^*(V_{2n-1}; \mathbb{Z}) = \mathbb{Z}[x_n, y_n]/(x_n^2 - 2y_n, y_n^2),
\]

\[
\dim(x_n) = 2, \quad \dim(y_n) = 2n,
\]

\[
\sigma_n^*(x_n) = a_n,
\]

\[
\tau_n^*(x_{n+1}) = x_n.
\]

Hence

\[
\text{cdg}(V_{2n-1}, 2) = \text{cdg}(\text{Spin}(2n+1), 3),
\]

\[
\text{cdg}(V_{2n-1}, 2) = \text{cdg}(\text{Spin}(2n+1), 3).
\]

Recall from Clarke [4] that

\[
K(V_{2n-1}) = \mathbb{Z}[X_n, Y_n]/(X_n^2 - 2Y_n - X_n Y_n, Y_n^2).
\]

Hence

\[
K(V_{2n-1}) \otimes \mathbb{Q} = \mathbb{Q}[X_n]/(X_n^2).
\]
By the construction of $X_n$ ([4]), we have
\[ \sigma^*_n(X_n) = L_n - 1, \]
\[ \tau^*_n(X_{n+1}) = X_n. \]

The Chern character of $X_n$ is given by

**Lemma 1.5.** $c_h(X_n) = \exp(x_n) - 1$.

**Proof.** We have
\[ \sigma^*_n(ch(X_{2n})) = ch(\sigma^*_n(X_{2n})) = ch(L_{2n} - 1) = \exp(a_{2n}) - 1 = \sigma^*_n(\exp(x_{2n}) - 1). \]

Hence $ch(X_{2n}) \equiv \exp(x_{2n}) - 1 \mod x_{2n}^2$ by (1.2), thus $ch(X_n) = \exp(x_n) - 1$ by (1.3). This proves 1.5.

**Proposition 1.6.** (1) $\cdg^2(V_{2n-1}, 2) = r$ if $2^{r-1} < n \leq 2^r$.

(2) $\cdg^2(V_{2n-1}, 2) = r$ if $2^{r-1} < n < 2^r$,
\[ r \leq \cdg^2(V_{2n-1}, 2) \leq r + 1 \text{ if } n = 2^r. \]

**Proof.** Put $D = \cdg^2(V_{2n-1}, 2)$. Let $f: V_{2n-1} \to S^2$ be a stable map such that the induced homomorphism $f_*: \pi_2(V_{2n-1}) = \mathbb{Z} \to \pi_2(S^2) = \mathbb{Z}$ is multiplication by $D$. Let $\beta \in R(S^2) = \mathbb{Z}$ be a generator. For simplicity, we set $X = X_n, Y = Y_n$ and $x = x_n$. Set
\[ f^*(\beta) = \sum_{i \leq i < n} a_i X^i + Y \cdot \sum_{i \leq i < n} b_i X^i = \sum_{i \leq i < n} a_i X^i \]
in $K(V_{2n}) \otimes \mathbb{Q}$, where $a_i \in \mathbb{Z}(1 \leq i < n), b_i \in \mathbb{Z}(0 \leq i < n)$, and $a_i \in \mathbb{Q}(n \leq i < 2n)$. Then
\[ D \cdot \sum_{i \leq i < n} e^i = f^* ch(\beta) = ch(f^*(\beta)) = \sum_{i \leq i < n} a_i (e^i - 1). \]

Hence $a_i = D \cdot (-1)^{i-1} i (1 \leq i < 2n)$. We then have
\[ f^*(\beta) = \sum_{i \leq i < n} a_i X^i + (2Y + XY) \sum_{n \leq i < 2n} a_i X^{i-n} \]
\[ = \sum_{i \leq i < n} (D - 1)^{i-1} X^i + Y \cdot 2D(-1)^{n-1}/n \]
\[ + Y \cdot \sum_{n \leq i \leq 2n-2} (D - 1)^{i-1}/(i+1) X^{i-n+1}. \]

Thus
\[ (1.7) \quad D|i (1 \leq i < n), 2D/n, \text{ and } D|i - 2D/(i+1) (n \leq i \leq 2n - 2) \text{ are in } \mathbb{Z}. \]

Let $r \geq 1$ be an integer such that $2^{r-1} < n \leq 2^r$. Then the relation $D|2^r - 2D/(2^r + 1) \in \mathbb{Z}$ implies that $2^r \mid D$. Conversely, if $2^r \mid D$, then (1.7) with $\mathbb{Z}$ replaced by its 2-localized ring $\mathbb{Z}(2)$ holds. Therefore $\cdg^2(V_{2n-1}, 2) = r$. This proves (1).
A map \( V_{2n-1} \rightarrow K(\mathbb{Z}, 2) \) which represents \( x_n \) factorizes as \( V_{2n-1} \rightarrow P(\mathbb{C}^2) \subset K(\mathbb{Z}, 2) \). Hence \( ^*\text{cdg}(V_{2n-1}, 2) \leq ^*\text{cdg}(P(\mathbb{C}^2), 2) \), so that \( r = ^*\text{cdg}(V_{2n-1}, 2) \leq ^*\text{cdg}(P(\mathbb{C}^2), 2) \). By \([18]\), we have \( ^*\text{cdg}(P(\mathbb{C}^2), 2) = ^*\text{cdg}(SU(2n), 3) \) which is \( r+1 \) or \( r \) according as \( n=2' \) or \( 2'^{-1} < n < 2' \). Hence we have (2).

**Corollary 1.8.** \( ^*\text{cdg}(\text{Spin}(2n+1), 3) \geq r \) if \( 2'^{-1} < n \leq 2' \).

**Proof.** This follows from 1.1 and 1.6.

**Proof of Theorem 1.** The complexification induces isomorphisms of representation rings:

\[
RO(\text{Spin}(m)) \cong R(\text{Spin}(m)) \quad \text{if} \quad m \equiv 0, 1, 7 \mod 8
\]

(see [7, p. 193]). By the proof of \([18, 4.4]\), we then have

\[
^*\text{cdg}(\text{Spin}(m), 3) = 2 \cdot ^*\text{cdg}(\text{Spin}(m), 3) \quad \text{if} \quad m \equiv 0, 1, 7 \mod 8.
\]

Thus, by 1.8, we have

\[
^*\text{cdg}(\text{Spin}(2n+1), 3) \geq r+1 \quad \text{if} \quad n \equiv 0, 3 \mod 4 \text{ and } 2'^{-1} < n \leq 2'.
\]

On the other hand, if \( n \geq 2 \), then the canonical homomorphism

\[
Z = \pi_3(\text{Spin}(2n+1)) \rightarrow \pi_3(\text{SO}(2n+1)) \rightarrow \pi_3(\text{SU}(2n+1)) = Z
\]

is multiplication by 2, so that

\[
^*\text{cdg}(\text{Spin}(2n+1), 3) \equiv 1 + ^*\text{cdg}(\text{SU}(2n+1), 3).
\]

The latter number is \( r+2 \) or \( r+1 \) according as \( n=2' \) or \( 2'^{-1} < n < 2' \), by [18]. Hence

\[
^*\text{cdg}(\text{Spin}(2n+1), 3) = r+1 \quad \text{if} \quad n \equiv 0, 3 \mod 4 \text{ and } 2'^{-1} < n < 2'.
\]

In particular, if \( r \geq 3 \), then \( ^*\text{cdg}(\text{Spin}(2' - 1), 3) = r \) and \( ^*\text{cdg}(\text{Spin}(2' + 7), 3) = ^*\text{cdg}(\text{Spin}(2'^{-1} - 1), 3) = r+1 \). Hence, if \( r \geq 3 \), then

\[
 r \leq ^*\text{cdg}(\text{Spin}(n), 3) \leq r+1 \quad \text{for} \quad 2' \leq n \leq 2'+6,
\]

\[
^*\text{cdg}(\text{Spin}(n), 3) = r+1 \quad \text{for} \quad 2'+7 \leq n \leq 2'^{-1}-1.
\]

This proves Theorem 1.

**2. Proof of Theorem 2**

The relations \( ^*\text{cdg}(E_6, 3) = ^*\text{cdg}(F_4, 3) \geq 2 \) were proved in [18]. We will prove \( ^*\text{cdg}(F_4, 3) \leq 2 \). By [6] and [12], there exist a mod 3 \( H \)-space \( X \) of dimension 26 and a mod 3 homotopy equivalence
where $B_5(3)$ is the total space of an $S^{15}$-bundle over $S^{26}$ [15]. It follows from [3] that the top cell of the localized space $X_{(5)}$ splits off stably, that is, $X_{(5)}^\infty = X^{(23)} \vee S^{26}$ (stably), where $X^{(23)}$ is the 23-skeleton of $X$, and it follows from [5] that $X^{(23)}_{(5)}$ is stably homotopy equivalent to $X_{1} \vee X_{2}$ where $X_{3}$ is 17-connected and $H^*(X_{1}; \mathbb{Z}_3) = \mathbb{Z}_3 \{1, x_2, x_3, x_5, x_{18}, x_{19}, x_{22}\}$ such that $\dim(x_i) = i$, $S_{1} x_3 = x_7$, $\beta x_7 = x_8$, $\beta x_{18} = x_{19}$, and $S_{1} x_{19} = x_{23}$.

**Lemma 2.1.** $X_{1} = S^{3}_{(5)} \cup e^{2}_{(5)} \cup e^{8}_{(5)} \cup e^{18}_{(5)} \cup e^{19}_{(5)} \cup e^{23}_{(5)}$.

In the rest of this section we work in the stable homotopy category of mod 3 local spaces. For simplicity we omit the notation "(3)."

**Proof of 2.1.** Let

$$S^3 \rightarrow X_1 \rightarrow Y_1 \rightarrow \Sigma S^3 \rightarrow \Sigma X_1$$

be a cofibre sequence such that $Y_1$ is 6-connected. Then $X_1 = S^3 \cup C\Sigma^{-1} Y_1$. Inductively we have cofibre sequences

$$S^7 \rightarrow Y_1 \rightarrow Y_2 \rightarrow \Sigma S^7 \rightarrow \Sigma Y_1,$$

$$S^8 \rightarrow Y_2 \rightarrow Y_3 \rightarrow \Sigma S^8 \rightarrow \Sigma Y_2,$$

$$S^{18} \rightarrow Y_3 \rightarrow Y_4 \rightarrow \Sigma S^{18} \rightarrow \Sigma Y_3,$$

$$S^{19} \rightarrow Y_4 \rightarrow Y_5 \rightarrow \Sigma S^{19} \rightarrow \Sigma Y_4$$

and

$$Y_1 = S^7 \cup C\Sigma^{-1} Y_2,$$

$$Y_2 = S^8 \cup C\Sigma^{-1} Y_3,$$

$$Y_3 = S^{18} \cup C\Sigma^{-1} Y_4,$$

$$Y_4 = S^{19} \cup C\Sigma^{-1} Y_5 = S^{23} \cup e^{23}$$

where the last equality follows from the fact that $Y_5 = S^{23}$. Therefore we have

$$X_1 = S^3 \cup C(S^6 \cup C(S^6 \cup C(S^{15} \cup C(S^{15} \cup C(S^{15} \cup e^{19}))))).$$

This proves 2.1.

**Proof of Theorem 2.** Put $Y = S^3 \cup e^7 \cup e^8 \cup e^{18} \cup e^{19} \cup e^{23}$. Then $\text{cdg}_3(F_4, 3) = \text{cdg}_3(X, 3) = \text{cdg}_3(Y, 3)$. Let $\alpha_i \in \pi_i(F_4) (1 \leq i \leq 5)$ be the element of order 3 defined in [20, p. 178]. Let $\alpha'_i : S^{0} \rightarrow S^0$ and $\alpha''_i : S^4 \rightarrow S^0 \cup e^4$ be an extension of $\alpha_i$ and a coextension of $\alpha_i$ respectively. The 8-skeleton $Y^{(8)}$ of $Y$ is equivalent to the mapping cone $C(\Sigma^3 \alpha'_i) = S^3 \cup C(S^{0} \cup e^i)$ and $Y/Y^{(8)}$ is equivalent to $C(\Sigma^5 \alpha''_i) = S^{18} \cup e^{19} \cup e^{23}$. Hence we have a cofibre sequence

$$F_4 \rightarrow X \times B_6(3)$$
Let \( g: C(\Sigma^3 \alpha') \rightarrow S^4 \) be an extension of \( \alpha': S^4 \rightarrow S^4 \). As is easily seen, we have an exact sequence

\[
\begin{align*}
\iota^* \pi^2(S^{18} \cup_3 e^{19}) &= \mathbb{Z}_3 \{\alpha'_i\} \\
\rightarrow \iota^* \pi^4(S^{18} \cup_3 e^{19} \cup e^{23}) \rightarrow \iota^* \pi^4(S^{18} \cup_3 e^{19}) &= \mathbb{Z}_3 
\end{align*}
\]

Since \( \alpha'_i(\alpha') \in \langle \alpha_i, 3, \alpha_i \rangle = \mathbb{Z}_3 \), it follows that \( \iota^* \pi^4(S^{18} \cup_3 e^{19} \cup e^{23}) = \mathbb{Z}_3 \) and \( 3g \circ h = 0 \). Hence there exists a map \( r: \Sigma Y \rightarrow S^4 \) such that \( r \circ k = g > \), so that \( r \) has degree 9 on the bottom sphere \( S^4 \) and \( \text{cdg}_3(Y) \leq 2 \). Hence \( \text{cdg}(F_4, 3) = 2 \) as desired.

3. Proof of Theorem 3

Lemma 3.1. Given an integer \( n \geq 2 \) and a connected finite CW-complex \( X \) such that

- \( X \) and its \((n-1)\)-skeleton \( Y \) are simply connected;
- \( \pi_{n-1}(X) = 0 \);
- \( \pi_n(Y) = \mathbb{Z}_m \);
- \( \text{rank}(\pi_n(X/Y)) = \text{rank}(\pi_n(X)) = 1 \),

then we have \( \text{Cdg}(X, n) \subseteq m \cdot \text{Hom}(\pi_n(X), \pi_n(S^n)) \). If moreover \( \text{Cdg} \) is surjective for \((X/Y, n)\), then \( \text{Cdg}(X, n) = m \cdot \text{Hom}(\pi_n(X), \pi_n(S^n)) \).

Proof. By a theorem of Blakers-Massey, the collapsing map induces an isomorphism \( c_\ast: \pi_n(X, Y) \Rightarrow \pi_n(X/Y) \). From the assumptions and the homotopy exact sequence of the pair \((X, Y)\), it follows that \( c_\ast: \pi_n(X) / \text{Tor} = \mathbb{Z} \Rightarrow \pi_n(X/Y) = \mathbb{Z} \) is multiplication by \( m \). Hence the assertion follows from the commutative diagram

\[
\begin{align*}
[X/Y, S^n] &\rightarrow \text{Hom}(\pi_n(X/Y), \pi_n(S^n)) = \mathbb{Z} \\
c_\ast \downarrow &\quad \downarrow c_\ast = m \\
[X, S^n] &\rightarrow \text{Hom}(\pi_n(X), \pi_n(S^n)) = \mathbb{Z}
\end{align*}
\]

since \( c_\ast \) is surjective.

Proof of Theorem 3(1). We shall use notations and results in [20]. The group \( SU(3) \) has a cell structure: \( SU(3) = S^3 \cup_3 \mathbb{S}^0 \cup_2 \mathbb{S}^{0} \). As noticed in [14, p. 475], we have \( \Sigma f = j_\ast (\nu_4 \circ \eta_4) \) from [9, 3.1], where \( j: S^4 \subset S^4 \cup \mathbb{S}^0 \). Let \( h: S^3 \cup \mathbb{S}^0 \rightarrow S^9 \) be a map having degree 2 on \( S^3 \). We have

\[
2\iota_4 \circ \nu_4 = 2\nu_4 + [\iota_4, \iota_4], \quad \text{by [22, XI]}, \\
= 4\nu_4 - \Sigma \nu', \quad \text{by [20, p. 43].}
\]

Hence \( \Sigma (h \circ f) = 2\iota_4 \circ \nu_4 \circ \eta_4 = -\Sigma \nu' \circ \eta_4 = \Sigma (\nu' \circ \eta_6), \) so \( h \circ f = \nu' \circ \eta_6 \), since \( \Sigma \) is injec-
tive on \( \pi_7(S_3) = \mathbb{Z}_2 \{ \eta_6 \} \). It follows that \( 2\eta_3h \) can be extended to \( SU(3) \) and that \( \text{cdg}(SU(3), 3) = 2^2 \). By the same method as [18, 4.3(1)], we can prove that \( \text{cdg}(SU(3), 5) = \text{cdg}(SU(3), n) = 2 \) for \( n = 3, 5 \). By applying the functor \([ \cdot, S^3]\) to the cofibre sequence \( S^7 \to S^5 \cup \varphi \subset SU(3) \), we have \([SU(3), S^3]\) = \( \mathbb{Z} \oplus \mathbb{Z}_2 \). Other assertions of (1) can be proved easily.

Proof of Theorem 3(2). By [13] there is a stable homotopy equivalence \( SU(4) = \Sigma P(C^4) \vee Y \), where \( Y \) is a 7-connected finite \( CW \)-complex. Hence, for \( n = 3, 5, 7 \), we have \( ' \text{cdg}(SU(4), n) = ' \text{cdg}(P(C^4), n-1) \) which can be easily determined by using the cell structure of \( P(C^4) \) (see [17, 1.15]).

By using a cell structure of \( SU(4) \) and known structures of \( \pi_n(S^3) \) and \( ' \pi_1(S^3) \) (see [20]), we can construct a map \( SU(4) \to S_3 \) which has degree \( 2^5 \cdot 3^2 \) on \( S_3 \), hence \( \text{cdg}(SU(4), 3) = 2^5 \cdot 3^2 \). On the other hand we have \( 2^5 \cdot 3 = ' \text{cdg}(SU(4), 3) \) by [18, 3.3] and [19, 3.4(3)].

Taking \( (X, n) = (SU(4), 5) \) in 3.1, we have \( \text{Cdg}(SU(4), 5) \subset 2\mathbb{Z} \). By the homotopy exact sequence of the principal \( Sp(2) \)-bundle \( q: SU(4) \to S_5 \), we have \( \text{Cdg}(q) = 2 \), so \( \text{cdg}(SU(4), 5) = 2 \). By [19, 4.2(1)], \( \text{cdg}(SU(4), 7) = 6 \). This completes the proof of (2).

Proof of Theorem 3(3). Using a stable duality map \( S^{14} \to G_{2+} \land G_{2+} \) (see [3]), we can prove \( ' \text{cdg}(G_2, 11) = ' \text{cdg}(G_2, 3) \) which is \( 2^2 \cdot 3^5 \) as proved in [18]. The group \( G_2 \) has a cell structure: \( G_2 = S^3 \cup e^5 \cup e^6 \cup e^9 \cup e^{11} \cup e^{14} \). Hence \( G_2/ \rho = S^3 \cup S^4 \), where \( Y \) is the 9-skeleton of \( G_2 \), so \( \text{Cdg} \) is surjective for \( (G_2/ \rho, Y) \).

Consider the exact sequence:

\[
\pi_{11}(G_2) = \mathbb{Z} \oplus \mathbb{Z}_2 \to \pi_{11}(G_2, Y) = \mathbb{Z} \to \pi_{10}(Y) \to \pi_{10}(G_2) = 0.
\]

By [16, 4.2], \( \pi_{10}(Y) = \mathbb{Z}_{20} \). Taking \( (X, n) = (G_2, 11) \) in 3.1, we have \( m = 2^1 \cdot 3^5 \) and \( \text{cdg}(G_2, 11) = 2^2 \cdot 3^5 \). Since \( Sq^2 \) is trivial on \( H^n(G_2; \mathbb{Z}_2) \) by [1], the attaching map of the 8-dimensional cell of \( G_2 \) factorizes as \( S^7 \to S^5 \cup \varphi \subset S^3 \cup e^8 \cup e^9 \). Using this fact and the additive structures of \( \pi_*(S^3) \) and \( ' \pi_1(S^3) \) (see [20]), we can construct a map \( G_2 \to S^3 \) which has degree \( 2^3 \cdot 3^5 \) on \( S^3 \), hence \( \text{cdg}(G_2, 3) = 2^5 \cdot 3^5 \). This proves (3).

Proof of Theorem 3(4). Applying \( \pi_*(\ ) \) to the following commutative diagram

\[
\begin{array}{ccc}
SO(5) & \subset & SO(6) \\
\downarrow & & \downarrow \\
SO(7) = SO(7) & & \\
\downarrow & & \downarrow \\
S^5 & \subset & V_{7,2} = SO(7)/SO(5) \to S^5
\end{array}
\]

we have \( p_*=15: \pi_{11}(SO(7))/\text{Tor}=\mathbb{Z} \to \pi_{11}(V_{7,2})/\text{Tor}=\mathbb{Z} \). Hence
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cdg(SO(7), 11) | 3 · 5 · cdg(V_{7,2}, 11).

The space $V_{7,2}$ has a cell structure: $V_{7,2} = S^8 \cup e^8 \cup e^{11}$. Let $q: V_{7,2} \to S^{11}$ be the collapsing map. By [8, 2.8 and 2.9], we have an exact sequence $\pi_9(S^9) = \mathbb{Z}_{2^n} \to \pi_9(S^8 \cup e^8) = \mathbb{Z}_{2^n} \to 0$, hence the order of $\pi_9(S^9)$ is at most 4, so that the order of $\pi_9(S^8 \cup e^8)$ is at most 8 because $\pi_9(S^9) = \mathbb{Z}_2$ by [20]. Since $\pi_{10}(V_{7,2}) = 0$, it then follows from the homotopy exact sequence of the pair $(V_{7,2}, S^8 \cup e^8)$ that $\text{Coker}[q_*: \pi_{11}(V_{7,2}) \to \pi_{11}(S^{11})] \neq \pi_{10}(S^8 \cup e^8)$, so that the last group is cyclic. Hence by the commutative diagram

$[S^{11}, S^{11}] \to \text{Hom}(\pi_{11}(S^{11}), \pi_{11}(S^{11})) = \mathbb{Z}$

$\qquad q^* \downarrow$\hspace{2cm}$q_*^*$

$[V_{7,2}, S^{11}] \to \text{Hom}(\pi_{11}(V_{7,2}), \pi_{11}(S^{11})) = \mathbb{Z}$

cdg($V_{7,2}, 11$) is the order of $\pi_{10}(S^8 \cup e^8)$, since $q^*$ is surjective. Thus cdg($V_{7,2}, 11$) $\mid 2^3$. On the other hand, cdg($G_2, 11$) $\mid$ cdg($Spin(7), 11$) by use of the principal $G_2$-bundle $Spin(7) \to S^7$. Hence $2^7 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ $\mid$ cdg($SO(7), 11$) $\mid$ cdg($Spin(7), 11$), therefore these numbers are equal and cdg ($V_{7,2}, 11$) $\mid 2^3$. Then [19, 3.7(2) and 4.4] complete the proof of (4).

Proof of Theorem 3(5). The groups Sp(2) and Sp(3) have cell structures: Sp(2) = $S^9 \cup e^9 \cup e^{10}$ and Sp(3) = $S^9 \cup e^9 \cup e^{10} \cup e^{11} \cup e^{14} \cup e^{18} \cup e^{21}$. They contain quasi-projective spaces $Q_2 = S^9 \cup e^9$ and $Q_3 = S^9 \cup e^9 \cup e^{11}$ respectively. The inclusions induce isomorphisms $\tau_f(Q_2) = \tau_f(Q_3) = \tau_f(Sp(3))$. In $Q_{3,2} = Q_3/S^3 = S^7 \cup e^{11}$, $h$ has order 8 (see [10, p. 38]). Hence the homomorphisms $\{Q_{3,2}, S^7\} \to \mathbb{Z} \to \{Q_2, S^7\} = \mathbb{Z} \to \{Q_2, S^7\} = \mathbb{Z}$ induced by the inclusions are multiplications by 8. Let $t$ be the order of the cokernel of the stabilization $\tau_f(Sp(3)) = \mathbb{Z} \to \tau_f(Sp(3)) = \mathbb{Z}$. Consider the following commutative diagram.

$[Sp(3), S^7] \to \text{Hom}(\pi_f(Sp(3)), \pi_f(S^7)) = \mathbb{Z}$

$\qquad \downarrow$

$\{Sp(3), S^7\} \to \text{Hom}(\pi_f(Sp(3)), \pi_f(S^7)) = \mathbb{Z}$

$\{Q_2, S^7\} = \mathbb{Z} \to \text{Hom}(\pi_f(Q_2), \pi_f(S^7)) = \mathbb{Z}$

$\qquad t \downarrow$

$\{Q_3, S^7\} = \mathbb{Z} \to \text{Hom}(\pi_f(Q_3), \pi_f(S^7)) = \mathbb{Z}$

By the proof of [19, 4.3(1)], the bottom homomorphism is multiplication by $2^7 \cdot 3$. Hence

$t \cdot 2^7 \cdot 3 \cdot t \cdot \text{cdg}(Sp(3), 7) \cdot \text{cdg}(Sp(3), 7) .

$Let $p: Sp(3) \to X_{3,2} = Sp(3)/Sp(1)$ be the canonical fibration. Since Coker[$p_*: \pi_f(Sp(3)) \to \pi_f(X_{3,2})] = \pi_6(S^9) = \mathbb{Z}_{12}$, we have cdg($Sp(3), 7$) $\mid 2^7 \cdot 3 \cdot \text{cdg}(X_{3,2}, 7)$. The space $X_{3,2}$ has a cell structure: $X_{3,2} = S^7 \cup e^{11} \cup e^{14}$. Since the order of $h$
is 8, there is a map $u : Q_3 \to S$ which has degree 8 on $S$. Since $u \circ v \in \pi_7(S) = Z_2 \oplus Z_2$ (see [20]), $2^3 \cdot 3u$ can be extended to a map $X_3 \to S$ which has degree $2^3 \cdot 3$ on $S$. Therefore $cdg(X_3, 7) = 2^3 \cdot 3$ and so $cdg(S_p(3), 7) = 2^6 \cdot 3$. On the other hand, $cdg(S_p(3), 7) = cdg(S(3), 7) = 1$ by [19, 4.2(2) and 4.3(2)]. Hence $cdg(S(3), 7) = 2^6 \cdot 3$. This proves (5) and completes the proof of Theorem 3.

4. Proof of Proposition 4

By [19, 2.4], we have

Proposition 4.1 (cf., [19, 3.15(2)]). If $n$ is odd, $\pi_n(X) = Z \{s\} \oplus \text{Tor}$ and $H^n(X; Z) = Z \{x_n\} \oplus \text{Tor}$, then there exists a map $f : X \to S^n$ such that $cdg(X, n) = \deg(f^*s) = |AB|$, where integers $A$ and $B$ are defined by $s^*(x_n) = A[S^n]$ and $f^*[S^n] = Bx_n$ mod Tor respectively. (Here $[S^n]$ is a generator of $H^n(S^n; Z)$.)

Stable version also holds.

Hence the next result proves Proposition 4.

Lemma 4.2. In 4.1, $(|A|, |B|)$ is equal to

$$(3, 2^2 \cdot 3 \cdot 5 \cdot y_1) \quad \text{for} \quad (\text{Spin}(9), 7),$$

$$(2^3 \cdot 3 \cdot 5, 2 \cdot 3 \cdot y_2) \quad \text{for} \quad (\text{Spin}(9), 11),$$

$$(2^3 \cdot 3 \cdot 7, y_3) \quad \text{for} \quad (\text{Spin}(9), 15),$$

$$(2, 2^2 \cdot 3 \cdot y_4) \quad \text{for} \quad (SU(5), 5),$$

$$(2 \cdot 3, 2 \cdot y_5) \quad \text{for} \quad (SU(5), 7),$$

$$(2^5 \cdot 3 \cdot 7 \cdot y_6) \quad \text{for} \quad (F_4, 11),$$

$$(2^3 \cdot 7, 2^4 \cdot 5 \cdot y_7) \quad \text{for} \quad (F_4, 15),$$

$$(2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11, y_8) \quad \text{for} \quad (F_4, 23)$$

for some integers $y_i$. In these cases, $B|y_i|^cdg(G, n)$.

Proof. We prove the assertion only for $(F_4, 11)$, because others can be proved similarly. Assertion about $A$ has been known (for example, see [11]). Recall that the type of $F_4$ is $(3, 11, 15, 23)$. Let $x \in H^n(F_4; Z)$ be a generator of the free part for $n \in \{3, 11, 15, 23\}$ (see [1]). Consider the commutative diagram:

$$\begin{array}{ccc} K^1(SU) & \overset{ch}{\longrightarrow} & H^1(SU; Z) \subset H^*(SU; Q) \\
\uparrow f^* & & \downarrow f^* \\
K^1(F_4) & \longrightarrow & H^*(F_4; Q) \\
\end{array}$$

As is well-known, $K^*(F_4)$ is an exterior algebra generated by some elements
$\beta_1, \cdots, \beta_4 \in K^i(F_4)$ whose Chern characters were determined in [21]. Let $g$ be a generator of $K^i(S^4)$. Let $^k$Dec and $^\mu$Dec be the groups of decomposable elements with respect to $\{\beta_1, \cdots, \beta_4\}$ and $\{x_3, x_{11}, x_{15}, x_{23}\}$, respectively. Express $f^*(g) \equiv \Sigma a_i \beta_i \mod ^k$Dec and $\Sigma a_i ch(\beta_i) = P_3 x_3 + P_{11} x_{11} + P_{15} x_{15} + P_{23} x_{23}$, where $P_i$ is a polynomial of $a_1, \cdots, a_4$ with rational coefficients. Then $ch(f^*(g)) \equiv \Sigma P_i x_i \mod ^\mu$Dec. On the other hand, $ch(f^*(g)) = f^*ch(g) = \pm f^*[S^{11}] \in H^{11}(F_4; \mathbb{Z})$. Hence $P_{11} = \pm B$ and $P_3 = P_{15} = P_{23} = 0$, then $B = |P_{11}| = 2^4 \cdot 3 \cdot 7 \cdot |a_3|$ by elementary calculation. This argument can also be applied to stable case. By setting $y_6 = |a_3|$, we have the desired result.

References


