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CODEGREE OF SIMPLE LIE GROUPS-II

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0. Introduction

In [19] the n -th codegree (number) $\text{cdg}(X, n) \in \mathbb{Z}$ and its stable version ${}^s\text{cdg}(X, n) \in \mathbb{Z}$ were defined for every pair of a path-connected space X and a positive integer n . In [18], ${}^s\text{cdg}_p(G, 3)$, the exponent of a prime p in ${}^s\text{cdg}(G, 3)$, was determined for some simply connected simple Lie groups G . The purpose of this paper is to continue computing ${}^s\text{cdg}(G, n)$ for some (G, n) . We use notations in [19] and [18]. Our results are the following.

Theorem 1. *If $r \geq 3$, then*

$$\begin{aligned} r &\leq {}^s\text{cdg}_2(Spin(n), 3) \leq r+1 \quad \text{for } 2^r \leq n \leq 2^r+6, \\ {}^s\text{cdg}_2(Spin(n), 3) &= r+1 \quad \text{for } 2^r+7 \leq n \leq 2^{r+1}-1. \end{aligned}$$

Theorem 2. ${}^s\text{cdg}_3(E_6, 3) = {}^s\text{cdg}_3(F_4, 3) = 2$.

Theorem 3. (1) $\text{cdg}(SU(3), 3) = 2^2$ and $\text{cdg}(SU(3), 5) = {}^s\text{cdg}(SU(3), n) = 2$ for $n = 3, 5$; $[SU(3), S^3] = \mathbb{Z} \oplus \mathbb{Z}_2$; $\{SU(3), S^3\} = \mathbb{Z}$; $[SU(3), S^5] = \{SU(3), S^5\} = \mathbb{Z}$.

(2) ${}^s\text{cdg}(SU(4), 3) = 2^2 \cdot 3 \mid \text{cdg}(SU(4), 3) \mid 2^5 \cdot 3^2$; $\text{cdg}(SU(4), 5) = {}^s\text{cdg}(SU(4), 5) = 2$; $\text{cdg}(SU(4), 7) = {}^s\text{cdg}(SU(4), 7) = 2 \cdot 3$.

(3) $\text{cdg}(G_2, 11) = {}^s\text{cdg}(G_2, 11) = {}^s\text{cdg}(G_2, 3) = 2^3 \cdot 3 \cdot 5 \mid \text{cdg}(G_2, 3) \mid 2^5 \cdot 3^2 \cdot 5$.

(4) $\text{cdg}(Spin(n), 11) = \text{cdg}(SO(n), 11) = 2^3 \cdot 3 \cdot 5$ for $n = 7, 8$; $\text{cdg}(SO(7)/SO(5), 11) = 2^3$.

(5) $2^5 \cdot 3 \mid {}^s\text{cdg}(Sp(3), 7) \mid \text{cdg}(Sp(3), 7) \mid 2^8 \cdot 3$.

Proposition 4.

$$\begin{aligned} &2^3 \cdot 3^2 \cdot 5 \mid \text{cdg}(Spin(9), 7), \\ &2^4 \cdot 3^2 \cdot 5 \mid \text{cdg}(Spin(9), 11), \\ &2^2 \cdot 3^2 \cdot 5 \cdot 7 \mid \text{cdg}(Spin(9), 15), \\ &2^3 \cdot 3 \mid \text{cdg}(SU(5), 5), \\ &2^2 \cdot 3 \mid \text{cdg}(SU(5), 7), \\ &2^7 \cdot 3 \cdot 5 \cdot 7 \mid \text{cdg}(F_4, n) \quad \text{for } n = 11, 15, \\ &2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \mid \text{cdg}(F_4, 23). \end{aligned}$$

1. Proof of Theorem 1

Let $g: V_{2n-1} = SO(2n+1)/SO(2) \times SO(2n-1) \rightarrow \Omega \text{Spin}(2n+1)$ be the generating map for $\text{Spin}(2n+1)$ ($n \geq 3$) (see [2]). Let $g': \Sigma \Omega \text{Spin}(2n+1) \rightarrow \text{Spin}(2n+1)$ be the canonical map. Then $(g' \circ \Sigma g)_*: \pi_3(\Sigma V_{2n-1}) \cong \pi_3(\text{Spin}(2n+1))$, hence

$$(1.1) \quad \begin{aligned} {}^s\text{cdg}(V_{2n-1}, 2) &| {}^s\text{cdg}(\text{Spin}(2n+1), 3), \\ {}^s\text{cdg}^K(V_{2n-1}, 2) &| {}^s\text{cdg}^K(\text{Spin}(2n+1), 3). \end{aligned}$$

We will calculate the 2-components of these numbers.

The inclusions $U(n) \subset SO(2n) = SO(2n) \times I_1 \subset SO(2n+1)$, $SO(2n+1) = SO(2n+1) \times I_2 \subset SO(2n+3)$, and $U(n) = U(n) \times I_1 \subset U(n+1)$ induce maps:

$$\begin{aligned} \sigma_n: P(\mathbf{C}^n) &= U(n)/U(1) \times U(n-1) \rightarrow V_{2n-1}, \\ \tau_n: V_{2n-1} &\rightarrow V_{2n+1}, \\ \tau'_n: P(\mathbf{C}^n) &\rightarrow P(\mathbf{C}^{n+1}) \end{aligned}$$

such that $\tau_n \circ \sigma_n = \sigma_n \circ \tau'_n$. Let L_n be the canonical complex line bundle over the complex projective $(n-1)$ -space $P(\mathbf{C}^n)$, and let $a_n \in H^2(P(\mathbf{C}^n); \mathbf{Z})$ be the first Chern class of L_n . Then

$$\tau'_n*(a_{n+1}) = a_n.$$

As is easily seen (e.g., [2]), we have

$$\begin{aligned} H^*(V_{2n-1}; \mathbf{Z}) &= \mathbf{Z}[x_n, y_n]/(x_n^n - 2y_n, y_n^2), \\ \dim(x_n) &= 2, \quad \dim(y_n) = 2n, \\ \sigma_n^*(x_n) &= a_n, \\ \tau_n^*(x_{n+1}) &= x_n. \end{aligned}$$

Hence

$$(1.2) \quad \begin{aligned} \sigma_n^*: H^i(V_{2n-1}; \mathbf{Z}) &\cong H^i(P(\mathbf{C}^n); \mathbf{Z}) \quad \text{for } i \leq 2n-2, \\ \tau_n^*: H^i(V_{2n+1}; \mathbf{Z}) &\cong H^i(V_{2n-1}; \mathbf{Z}) \quad \text{for } i \leq 2n-2, \\ H^*(V_{2n-1}; \mathbf{Q}) &= \mathbf{Q}[x_n]/(x_n^{2n}), \end{aligned}$$

$$(1.3) \quad \tau_n^*: H^i(V_{2n+1}; \mathbf{Q}) \cong H^i(V_{2n-1}; \mathbf{Q}) \quad \text{for } i \leq 4n-2.$$

Recall from Clarke [4] that

$$(1.4) \quad K(V_{2n-1}) = \mathbf{Z}[X_n, Y_n]/(X_n^n - 2Y_n - X_n Y_n, Y_n^2).$$

Hence

$$K(V_{2n-1}) \otimes \mathbf{Q} = \mathbf{Q}[X_n]/(X_n^{2n}).$$

By the construction of X_n ([4]), we have

$$\begin{aligned}\sigma_n^*(X_n) &= L_n - 1, \\ \tau_n^*(X_{n+1}) &= X_n.\end{aligned}$$

The Chern character of X_n is given by

Lemma 1.5. $ch(X_n) = \exp(x_n) - 1$.

Proof. We have

$$\begin{aligned}\sigma_{2n}^*(ch(X_{2n})) &= ch(\sigma_{2n}^*(X_{2n})) \\ &= ch(L_{2n} - 1) = \exp(a_{2n}) - 1 = \sigma_{2n}^*(\exp(x_{2n}) - 1).\end{aligned}$$

Hence $ch(X_{2n}) \equiv \exp(x_{2n}) - 1 \pmod{x_{2n}^{2n}}$ by (1.2), thus $ch(X_n) = \exp(x_n) - 1$ by (1.3). This proves 1.5.

Proposition 1.6. (1) ${}^s\text{cdg}_2^K(V_{2n-1}, 2) = r \quad \text{if } 2^{r-1} < n \leq 2^r$.
 (2) ${}^s\text{cdg}_2(V_{2n-1}, 2) = r \quad \text{if } 2^{r-1} < n < 2^r$,
 $r \leq {}^s\text{cdg}_2(V_{2n-1}, 2) \leq r+1 \quad \text{if } n = 2^r$.

Proof. Put $D = {}^s\text{cdg}(V_{2n-1}, 2)$. Let $f: V_{2n-1} \rightarrow S^2$ be a stable map such that the induced homomorphism $f_*: {}^s\pi_2(V_{2n-1}) = \mathbf{Z} \rightarrow {}^s\pi_2(S^2) = \mathbf{Z}$ is multiplication by D . Let $\beta \in \tilde{K}(S^2) = \mathbf{Z}$ be a generator. For simplicity, we set $X = X_n$, $Y = Y_n$ and $x = x_n$. Set

$$f^*(\beta) = \sum_{1 \leq i < n} a_i X^i + Y \cdot \sum_{0 \leq i < n} b_i X^i = \sum_{1 \leq i < n} a_i X^i$$

in $\tilde{K}(V_{2n-1}) \otimes \mathbf{Q}$, where $a_i \in \mathbf{Z}$ ($1 \leq i < n$), $b_i \in \mathbf{Z}$ ($0 \leq i < n$), and $a_i \in \mathbf{Q}$ ($n \leq i < 2n$). Then

$$\begin{aligned}D \cdot \sum_{i \geq 1} ((-1)^{i-1}/i) (e^x - 1)^i &= D \cdot \log(e^x - 1 + 1) \\ &= D \cdot x = f^* ch(\beta) = ch(f^*(\beta)) = \sum_{i \geq 1} a_i (e^x - 1)^i.\end{aligned}$$

Hence $a_i = D \cdot (-1)^{i-1}/i$ ($1 \leq i < 2n$). We then have

$$\begin{aligned}f^*(\beta) &= \sum_{1 \leq i < n} a_i X^i + (2Y + XY) \sum_{n \leq i < 2n} a_i X^{i-n} \\ &= \sum_{1 \leq i < n} (D(-1)^{i-1}/i) X^i + Y \cdot 2D(-1)^{n-1}/n \\ &\quad + Y \cdot \sum_{n \leq i \leq 2n-2} \{D(-1)^{i-1}/i + 2D(-1)^i/(i+1)\} X^{i-n+1}.\end{aligned}$$

Thus

(1.7) D/i ($1 \leq i < n$), $2D/n$, and $D/i - 2D/(i+1)$ ($n \leq i \leq 2n-2$) are in \mathbf{Z} .

Let $r \geq 1$ be an integer such that $2^{r-1} < n \leq 2^r$. Then the relation $D/2^r - 2D/(2^r + 1) \in \mathbf{Z}$ implies that $2^r \mid D$. Conversely, if $2^r \mid D$, then (1.7) with \mathbf{Z} replaced by its 2-localized ring $\mathbf{Z}_{(2)}$ holds. Therefore ${}^s\text{cdg}_2^K(V_{2n-1}, 2) = r$. This proves (1).

A map $V_{2n-1} \rightarrow K(\mathbf{Z}, 2)$ which represents x_n factorizes as $V_{2n-1} \rightarrow P(\mathbf{C}^{2n}) \subset K(\mathbf{Z}, 2)$. Hence ${}^s\text{cdg}(V_{2n-1}, 2) \leq {}^s\text{cdg}(P(\mathbf{C}^{2n}), 2)$ so that $r = {}^s\text{cdg}_2(V_{2n-1}, 2) \leq {}^s\text{cdg}_2(V_{2n-1}, 2) \leq {}^s\text{cdg}_2(P(\mathbf{C}^{2n}), 2)$. By [18], we have ${}^s\text{cdg}_2(P(\mathbf{C}^{2n}), 2) = {}^s\text{cdg}_2(SU(2n), 3)$ which is $r+1$ or r according as $n=2^r$ or $2^{r-1} < n < 2^r$. Hence we have (2).

Corollary 1.8. ${}^s\text{cdg}_2(Spin(2n+1), 3) \geq r$ if $2^{r-1} < n \leq 2^r$.

Proof. This follows from 1.1 and 1.6.

Proof of Theorem 1. The complexification induces isomorphisms of representation rings:

$$RO(Spin(m)) \cong R(Spin(m)) \quad \text{if } m \equiv 0, 1, 7 \pmod{8}$$

(see [7, p. 193]). By the proof of [18, 4.4], we then have

$${}^s\text{cdg}^{KO}(Spin(m), 3) = 2 \cdot {}^s\text{cdg}^K(Spin(m), 3) \quad \text{if } m \equiv 0, 1, 7 \pmod{8}.$$

Thus, by 1.8, we have

$${}^s\text{cdg}_2(Spin(2n+1), 3) \geq r+1 \quad \text{if } n \equiv 0, 3 \pmod{4} \text{ and } 2^{r-1} < n \leq 2^r.$$

On the other hand, if $n \geq 2$, then the canonical homomorphism

$$\mathbf{Z} = \pi_3(Spin(2n+1)) \rightarrow \pi_3(SO(2n+1)) \rightarrow \pi_3(SU(2n+1)) = \mathbf{Z}$$

is multiplication by 2, so that

$${}^s\text{cdg}_2(Spin(2n+1), 3) \leq 1 + {}^s\text{cdg}_2(SU(2n+1), 3).$$

The latter number is $r+2$ or $r+1$ according as $n=2^r$ or $2^{r-1} < n < 2^r$, by [18]. Hence

$${}^s\text{cdg}_2(Spin(2n+1), 3) = r+1 \quad \text{if } n \equiv 0, 3 \pmod{4} \text{ and } 2^{r-1} < n < 2^r.$$

In particular, if $r \geq 3$, then ${}^s\text{cdg}_2(Spin(2^r-1), 3) = r$ and ${}^s\text{cdg}_2(Spin(2^r+7), 3) = {}^s\text{cdg}_2(Spin(2^{r+1}-1), 3) = r+1$. Hence, if $r \geq 3$, then

$$\begin{aligned} r &\leq {}^s\text{cdg}_2(Spin(n), 3) \leq r+1 \quad \text{for } 2^r \leq n \leq 2^r+6, \\ {}^s\text{cdg}_2(Spin(n), 3) &= r+1 \quad \text{for } 2^r+7 \leq n \leq 2^{r+1}-1. \end{aligned}$$

This proves Theorem 1.

2. Proof of Theorem 2

The relations ${}^s\text{cdg}_3(E_6, 3) = {}^s\text{cdg}_3(F_4, 3) \geq 2$ were proved in [18]. We will prove ${}^s\text{cdg}_3(F_4, 3) \leq 2$. By [6] and [12], there exist a mod 3 H -space X of dimension 26 and a mod 3 homotopy equivalence

$$F_4 \simeq_3 X \times B_5(3)$$

where $B_5(3)$ is the total space of an S^{11} -bundle over S^{15} [15]. It follows from [3] that the top cell of the localized space $X_{(3)}$ splits off stably, that is, $X \simeq_3 X^{(23)} \vee S^{26}$ (stably), where $X^{(23)}$ is the 23-skeleton of X , and it follows from [5] that $X^{(23)}_{(3)}$ is stably homotopy equivalent to $X_1 \vee X_2$ where X_2 is 17-connected and $H^*(X_1; \mathbb{Z}_3) = \mathbb{Z}_3\{1, x_3, x_7, x_8, x_{18}, x_{19}, x_{23}\}$ such that $\dim(x_i) = i$, $\mathcal{P}^1 x_3 = x_7$, $\beta x_7 = x_8$, $\beta x_{18} = x_{19}$, and $\mathcal{P}^1 x_{19} = x_{23}$.

Lemma 2.1. $X_1 = S^3_{(3)} \cup e^7_{(3)} \cup e^8_{(3)} \cup e^{18}_{(3)} \cup e^{19}_{(3)} \cup e^{23}_{(3)}$.

In the rest of this section we work in the stable homotopy category of mod 3 local spaces. For simplicity we omit the notation “(3)”.

Proof of 2.1. Let

$$S^3 \rightarrow X_1 \rightarrow Y_1 \rightarrow \Sigma S^3 \rightarrow \Sigma X_1$$

be a cofibre sequence such that Y_1 is 6-connected. Then $X_1 = S^3 \cup C\Sigma^{-1} Y_1$. Inductively we have cofibre sequences

$$\begin{aligned} S^7 &\rightarrow Y_1 \rightarrow Y_2 \rightarrow \Sigma S^7 \rightarrow \Sigma Y_1, \\ S^8 &\rightarrow Y_2 \rightarrow Y_3 \rightarrow \Sigma S^8 \rightarrow \Sigma Y_2, \\ S^{18} &\rightarrow Y_3 \rightarrow Y_4 \rightarrow \Sigma S^{18} \rightarrow \Sigma Y_3, \\ S^{19} &\rightarrow Y_4 \rightarrow Y_5 \rightarrow \Sigma S^{19} \rightarrow \Sigma Y_4 \end{aligned}$$

and

$$\begin{aligned} Y_1 &= S^7 \cup C\Sigma^{-1} Y_2, \\ Y_2 &= S^8 \cup C\Sigma^{-1} Y_3, \\ Y_3 &= S^{18} \cup C\Sigma^{-1} Y_4, \\ Y_4 &= S^{19} \cup C\Sigma^{-1} Y_5 = S^{19} \cup e^{23} \end{aligned}$$

where the last equality follows from the fact that $Y_5 = S^{23}$. Therefore we have

$$X_1 = S^3 \cup C(S^6 \cup C(S^6 \cup C(S^{15} \cup C(S^{15} \cup e^{19})))) .$$

This proves 2.1.

Proof of Theorem 2. Put $Y = S^3 \cup e^7 \cup e^8 \cup e^{18} \cup e^{19} \cup e^{23}$. Then ${}^s\text{cdg}_3(F_4, 3) = {}^s\text{cdg}_3(X, 3) = {}^s\text{cdg}_3(Y, 3)$. Let $\alpha_i \in {}^s\pi_{4i-1}(S^0)$ ($1 \leq i \leq 5$) be the element of order 3 defined in [20, p. 178]. Let $\alpha'_i: S^{4i-1} \cup_3 e^{4i} \rightarrow S^0$ and $\alpha''_i: S^4 \rightarrow S^0 \cup_3 e^1$ be an extension of α_i and a coextension of α_1 respectively. The 8-skeleton $Y^{(8)}$ of Y is equivalent to the mapping cone $C(\Sigma^3 \alpha'_1) = S^3 \cup C(S^6 \cup_3 e^7)$ and $Y/Y^{(8)}$ is equivalent to $C(\Sigma^{15} \alpha''_1) = S^{18} \cup_3 e^{19} \cup e^{23}$. Hence we have a cofibre sequence

$$C(\Sigma^3 \alpha'_1) \rightarrow Y \rightarrow C(\Sigma^{15} \alpha'_1) \xrightarrow{h} C(\Sigma^4 \alpha'_1) \xrightarrow{k} \Sigma Y.$$

Let $g: C(\Sigma^4 \alpha'_1) \rightarrow S^4$ be an extension of $3: S^4 \rightarrow S^4$. As is easily seen, we have an exact sequence

$$\begin{aligned} {}^s\pi^3(S^{18} \cup_3 e^{19}) &= \mathbf{Z}_3 \{\alpha'_1\} \xrightarrow{\alpha'^{*}} {}^s\pi^3(S^{22}) = \mathbf{Z}_3 \{\alpha_5\} \\ &\rightarrow {}^s\pi^4(S^{18} \cup_3 e^{19} \cup e^{23}) \rightarrow {}^s\pi^4(S^{18} \cup_3 e^{19}) = \mathbf{Z}_3 \rightarrow 0. \end{aligned}$$

Since $\alpha'^{*}(\alpha'_1) \in \langle \alpha_4, 3, \alpha_1 \rangle = \alpha_5$, it follows that ${}^s\pi^4(S^{18} \cup_3 e^{19} \cup e^{23}) = \mathbf{Z}_3$ and $3g \circ h = 0$. Hence there exists a map $r: \Sigma Y \rightarrow S^4$ such that $r \circ k = 3g$, so that r has degree 9 on the bottom sphere S^4 and ${}^s\text{cdg}_3(Y) \leq 2$. Hence ${}^s\text{cdg}(F_4, 3) = 2$ as desired.

3. Proof of Theorem 3

Lemma 3.1. *Given an integer $n \geq 2$ and a connected finite CW-complex X such that*

X and its $(n-1)$ -skeleton Y are simply connected;

$$\pi_{n-1}(X) = 0;$$

$$\pi_{n-1}(Y) = \mathbf{Z}_m;$$

$$\text{rank}(\pi_n(X/Y)) = \text{rank}(\pi_n(X)) = 1,$$

then we have $\text{Cdg}(X, n) \subset m \cdot \text{Hom}(\pi_n(X), \pi_n(S^n))$. If moreover Cdg is surjective for $(X/Y, n)$, then $\text{Cdg}(X, n) = m \cdot \text{Hom}(\pi_n(X), \pi_n(S^n))$.

Proof. By a theorem of Blakers-Massey, the collapsing map induces an isomorphism $c_*: \pi_n(X, Y) \cong \pi_n(X/Y)$. From the assumptions and the homotopy exact sequence of the pair (X, Y) , it follows that $c_*: \pi_n(X)/\text{Tor} = \mathbf{Z} \rightarrow \pi_n(X/Y) = \mathbf{Z}$ is multiplication by m . Hence the assertion follows from the commutative diagram

$$\begin{array}{ccc} [X/Y, S^n] & \rightarrow & \text{Hom}(\pi_n(X/Y), \pi_n(S^n)) = \mathbf{Z} \\ c^* \downarrow & & \downarrow c_*^* = m \\ [X, S^n] & \rightarrow & \text{Hom}(\pi_n(X), \pi_n(S^n)) = \mathbf{Z} \end{array}$$

since c^* is surjective.

Proof of Theorem 3(1). We shall use notations and results in [20]. The group $SU(3)$ has a cell structure: $SU(3) = S^3 \cup_{\eta_3} e^5 \cup_{\eta_7} e^8$. As noticed in [14, p. 475], we have $\Sigma f = j_*(\nu_4 \circ \eta_7)$ from [9, 3.1], where $j: S^4 \subset S^4 \cup e^6$. Let $h: S^3 \cup e^5 \rightarrow S^3$ be a map having degree 2 on S^3 . We have

$$\begin{aligned} 2\iota_4 \circ \nu_4 &= 2\nu_4 + [\iota_4, \iota_4], \quad \text{by [22, XI],} \\ &= 4\nu_4 - \Sigma \nu', \quad \text{by [20, p. 43].} \end{aligned}$$

Hence $\Sigma(h \circ f) = 2\iota_4 \circ \nu_4 \circ \eta_7 = -\Sigma \nu' \circ \eta_7 = \Sigma(\nu' \circ \eta_6)$, so $h \circ f = \nu' \circ \eta_6$, since Σ is injective.

tive on $\pi_7(S^3) = \mathbb{Z}_2\{\nu' \circ \eta_6\}$. It follows that $2\iota_3 \circ h$ can be extended to $SU(3)$ and that $\text{cdg}(SU(3), 3) = 2^2$. By the same method as [18, 4.3(1)], we can prove that $\text{cdg}(SU(3), 5) = {}^s\text{cdg}(SU(3), n) = 2$ for $n = 3, 5$. By applying the functor $[\ , S^3]$ to the cofibre sequence $S^7 \rightarrow S^3 \cup e^5 \subset SU(3)$, we have $[SU(3), S^3] = \mathbb{Z} \oplus \mathbb{Z}_2$. Other assertions of (1) can be proved easily.

Proof of Theorem 3(2). By [13] there is a stable homotopy equivalence $SU(4) \simeq \Sigma P(\mathbf{C}^4) \vee Y$, where Y is a 7-connected finite CW -complex. Hence, for $n = 3, 5, 7$, we have ${}^s\text{cdg}(SU(4), n) = {}^s\text{cdg}(P(\mathbf{C}^4), n-1)$ which can be easily determined by using the cell structure of $P(\mathbf{C}^4)$ (see [17, 1.15]).

By using a cell structure of $SU(4)$ and known structures of $\pi_*(S^3)$ and ${}^s\pi_{14}(S^3)$ (see [20]), we can construct a map $SU(4) \rightarrow S^3$ which has degree $2^5 \cdot 3^2$ on S^3 , hence $\text{cdg}(SU(4), 3) \mid 2^5 \cdot 3^2$. On the other hand we have $2^2 \cdot 3 = {}^s\text{cdg}(SU(4), 3) \mid \text{cdg}(SU(4), 3)$ by [18, 3.3] and [19, 3.4(3)].

Taking $(X, n) = (SU(4), 5)$ in 3.1, we have $\text{Cdg}(SU(4), 5) \subset 2\mathbb{Z}$. By the homotopy exact sequence of the principal $\text{Sp}(2)$ -bundle $q: SU(4) \rightarrow S^5$, we have $\text{Cdg}(q) = 2$, so $\text{cdg}(SU(4), 5) = 2$. By [19, 4.2(1)], $\text{cdg}(SU(4), 7) = 6$. This completes the proof of (2).

Proof of Theorem 3(3). Using a stable duality map $S^{14} \rightarrow G_{2+} \wedge G_{2+}$ (see [3]), we can prove ${}^s\text{cdg}(G_2, 11) = {}^s\text{cdg}(G_2, 3)$ which is $2^3 \cdot 3 \cdot 5$ as proved in [18]. The group G_2 has a cell structure: $G_2 = S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}$. Hence $G_2/Y = S^{11} \vee S^{14}$, where Y is the 9-skeleton of G_2 , so Cdg is surjective for $(G_2/Y, 11)$. Consider the exact sequence:

$$\pi_{11}(G_2) = \mathbb{Z} \oplus \mathbb{Z}_2 \rightarrow \pi_{11}(G_2, Y) = \mathbb{Z} \rightarrow \pi_{10}(Y) \rightarrow \pi_{10}(G_2) = 0.$$

By [16, 4.2], $\pi_{10}(Y) = \mathbb{Z}_{120}$. Taking $(X, n) = (G_2, 11)$ in 3.1, we have $m = 2^3 \cdot 3 \cdot 5$ and $\text{cdg}(G_2, 11) = 2^3 \cdot 3 \cdot 5$. Since Sq^2 is trivial on $H^6(G_2; \mathbb{Z}_2)$ by [1], the attaching map of the 8-dimensional cell of G_2 factorizes as $S^7 \rightarrow S^3 \cup e^5 \subset S^3 \cup e^5 \cup e^6$. Using this fact and the additive structures of $\pi_*(S^3)$ and ${}^s\pi_{13}(S^3)$ (see [20]), we can construct a map $G_2 \rightarrow S^3$ which has degree $2^5 \cdot 3^2 \cdot 5$ on S^3 , hence $\text{cdg}(G_2, 3) \mid 2^5 \cdot 3^2 \cdot 5$. This proves (3).

Proof of Theorem 3(4). Applying $\pi_*(\)$ to the following commutative diagram

$$\begin{array}{ccc} SO(5) & \subset & SO(6) \\ \downarrow & & \downarrow \\ SO(7) & = & SO(7) \\ p \downarrow & & \downarrow \\ S^5 & \subset & V_{7,2} = SO(7)/SO(5) \rightarrow S^6 \end{array}$$

we have $p_* = 15: \pi_{11}(SO(7))/\text{Tor} = \mathbb{Z} \rightarrow \pi_{11}(V_{7,2})/\text{Tor} = \mathbb{Z}$. Hence

$$\mathrm{cdg}(SO(7), 11) | 3 \cdot 5 \cdot \mathrm{cdg}(V_{7,2}, 11).$$

The space $V_{7,2}$ has a cell structure: $V_{7,2} = S^5 \cup e^6 \cup e^{11}$. Let $q: V_{7,2} \rightarrow S^{11}$ be the collapsing map. By [8, 2.8 and 2.9], we have an exact sequence $\pi_9(S^5) = \mathbb{Z}_2 \rightarrow \pi_{10}(S^5 \cup e^6, S^5) \rightarrow \pi_5(S^5 \cup e^6) = \mathbb{Z}_2 \rightarrow 0$, hence the order of $\pi_{10}(S^5 \cup e^6, S^5)$ is at most 4, so that the order of $\pi_{10}(S^5 \cup e^6)$ is at most 8 because $\pi_{10}(S^6) = \mathbb{Z}_2$ by [20]. Since $\pi_{10}(V_{7,2}) = 0$, it then follows from the homotopy exact sequence of the pair $(V_{7,2}, S^5 \cup e^6)$ that $\mathrm{Coker}[q_*: \pi_{11}(V_{7,2}) \rightarrow \pi_{11}(S^{11})] \cong \pi_{10}(S^5 \cup e^6)$, so that the last group is cyclic. Hence by the commutative diagram

$$\begin{array}{ccc} [S^{11}, S^{11}] & \rightarrow & \mathrm{Hom}(\pi_{11}(S^{11}), \pi_{11}(S^{11})) = \mathbb{Z} \\ q^* \downarrow & & \downarrow q_*^* \\ [V_{7,2}, S^{11}] & \rightarrow & \mathrm{Hom}(\pi_{11}(V_{7,2}), \pi_{11}(S^{11})) = \mathbb{Z} \end{array}$$

$\mathrm{cdg}(V_{7,2}, 11)$ is the order of $\pi_{10}(S^5 \cup e^6)$, since q^* is surjective. Thus $\mathrm{cdg}(V_{7,2}, 11) | 2^3$. On the other hand, $\mathrm{cdg}(G_2, 11) | \mathrm{cdg}(Spin(7), 11)$ by use of the principal G_2 -bundle $Spin(7) \rightarrow S^7$. Hence $2^3 \cdot 3 \cdot 5 = \mathrm{cdg}(G_2, 11) | \mathrm{cdg}(Spin(7), 11) | \mathrm{cdg}(SO(7), 11) | 3 \cdot 5 \cdot \mathrm{cdg}(V_{7,2}, 11) | 2^3 \cdot 3 \cdot 5$, therefore these numbers are equal and $\mathrm{cdg}(V_{7,2}, 11) = 2^3$. Then [19, 3.7(2) and 4.4] complete the proof of (4).

Proof of Theorem 3(5). The groups $Sp(2)$ and $Sp(3)$ have cell structures: $Sp(2) = S^3 \cup_g e^7 \cup e^{10}$ and $Sp(3) = S^3 \cup_g e^7 \cup e^{10} \cup e^{11} \cup e^{14} \cup e^{18} \cup e^{21}$. They contain quasi-projective spaces $Q_2 = S^3 \cup_g e^7$ and $Q_3 = S^3 \cup_g e^7 \cup e^{11}$ respectively. The inclusions induce isomorphisms $\pi_7(Q_2) \cong \pi_7(Q_3) \cong \pi_7(Sp(3))$. In $Q_{3,2} = Q_3 / S^3 = S^7 \cup_h e^{11}$, h has order 8 (see [10, p. 38]). Hence the homomorphisms $\{Q_{3,2}, S^7\} = \mathbb{Z} \rightarrow \{S^7, S^7\} = \mathbb{Z}$ and $\{Q_3, S^7\} = \mathbb{Z} \rightarrow \{Q_2, S^7\} = \mathbb{Z}$ induced by the inclusions are multiplications by 8. Let t be the order of the cokernel of the stabilization $\pi_7(Sp(3)) = \mathbb{Z} \rightarrow {}^s\pi_7(Sp(3)) = \mathbb{Z}$. Consider the following commutative diagram.

$$\begin{array}{ccc} [Sp(3), S^7] & \rightarrow & \mathrm{Hom}(\pi_7(Sp(3)), \pi_7(S^7)) = \mathbb{Z} \\ \downarrow & & \uparrow t \\ \{Sp(3), S^7\} & \rightarrow & \mathrm{Hom}({}^s\pi_7(Sp(3)), {}^s\pi_7(S^7)) = \mathbb{Z} \\ \downarrow & & \downarrow \cong \\ \{Q_3, S^7\} = \mathbb{Z} & \rightarrow & \mathrm{Hom}({}^s\pi_7(Q_3), {}^s\pi_7(S^7)) = \mathbb{Z} \\ 2^3 \downarrow & & \downarrow \cong \\ \{Q_2, S^7\} = \mathbb{Z} & \rightarrow & \mathrm{Hom}({}^s\pi_7(Q_2), {}^s\pi_7(S^7)) = \mathbb{Z} \end{array}$$

By the proof of [19, 4.3(1)], the bottom homomorphism is multiplication by $2^2 \cdot 3$. Hence

$$t \cdot 2^5 \cdot 3 | t \cdot {}^s\mathrm{cdg}(Sp(3), 7) | \mathrm{cdg}(Sp(3), 7).$$

Let $p: Sp(3) \rightarrow X_{3,2} = Sp(3)/Sp(1)$ be the canonical fibration. Since $\mathrm{Coker}[p_*: \pi_7(Sp(3)) \rightarrow \pi_7(X_{3,2})] \cong \pi_6(S^3) = \mathbb{Z}_{12}$, we have $\mathrm{cdg}(Sp(3), 7) | 2^2 \cdot 3 \cdot \mathrm{cdg}(X_{3,2}, 7)$. The space $X_{3,2}$ has a cell structure: $X_{3,2} = S^7 \cup_h e^{11} \cup_v e^{18}$. Since the order of h

is 8, there is a map $u: Q_{3,2} \rightarrow S^7$ which has degree 8 on S^7 . Since $u \circ v \in \pi_{17}(S^7) = \mathbf{Z}_{24} \oplus \mathbf{Z}_2$ (see [20]), $2^3 \cdot 3u$ can be extended to a map $X_{3,2} \rightarrow S^7$ which has degree $2^6 \cdot 3$ on S^7 . Therefore $\text{cdg}(X_{3,2}, 7) \mid 2^6 \cdot 3$ and so $\text{cdg}(Sp(3), 7) \mid 2^8 \cdot 3^2$. On the other hand, $\text{cdg}_3(Sp(3), 7) = \text{cdg}_3(\text{Spin}(7), 7) = 1$ by [19, 4.2(2) and 4.3(2)]. Hence $\text{cdg}(Sp(3), 7) \mid 2^8 \cdot 3$. This proves (5) and completes the proof of Theorem 3.

4. Proof of Proposition 4

By [19, 2.4], we have

Proposition 4.1 (cf., [19, 3.15(2)]). *If n is odd, $\pi_n(X) = \mathbf{Z}\{s\} \oplus \text{Tor}$ and $H^n(X; \mathbf{Z}) = \mathbf{Z}\{x_n\} \oplus \text{Tor}$, then there exists a map $f: X \rightarrow S^n$ such that $\text{cdg}(X, n) = \deg(f_* s) = |AB|$, where integers A and B are defined by $s^*(x_n) = A[S^n]$ and $f^*[S^n] \equiv Bx_n \pmod{\text{Tor}}$ respectively. (Here $[S^n]$ is a generator of $H^n(S^n; \mathbf{Z})$.) Stable version also holds.*

Hence the next result proves Proposition 4.

Lemma 4.2. *In 4.1, $(|A|, |B|)$ is equal to*

$$\begin{array}{ll} (3, 2^3 \cdot 3 \cdot 5 \cdot y_1) & \text{for } (\text{Spin}(9), 7), \\ (2^3 \cdot 3 \cdot 5, 2 \cdot 3 \cdot y_2) & \text{for } (\text{Spin}(9), 11), \\ (2^2 \cdot 3^2 \cdot 5 \cdot 7, y_3) & \text{for } (\text{Spin}(9), 15), \\ (2, 2^2 \cdot 3 \cdot y_4) & \text{for } (\text{SU}(5), 5), \\ (2 \cdot 3, 2 \cdot y_5) & \text{for } (\text{SU}(5), 7), \\ (2^3 \cdot 5, 2^4 \cdot 3 \cdot 7 \cdot y_6) & \text{for } (F_4, 11), \\ (2^3 \cdot 3 \cdot 7, 2^4 \cdot 5 \cdot y_7) & \text{for } (F_4, 15), \\ (2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11, y_8) & \text{for } (F_4, 23) \end{array}$$

for some integers y_i . In these cases, $B/y_i \mid {}^s\text{cdg}(G, n)$.

Proof. We prove the assertion only for $(F_4, 11)$, because others can be proved similarly. Assertion about A has been known (for example, see [11]). Recall that the type of F_4 is $(3, 11, 15, 23)$. Let $x_n \in H^n(F_4; \mathbf{Z})$ be a generator of the free part for $n \in \{3, 11, 15, 23\}$ (see [1]). Consider the commutative diagram:

$$\begin{array}{ccc} K^1(S^{11}) & \xrightarrow{ch} & H^{11}(S^{11}; \mathbf{Z}) \subset H^*(S^{11}; \mathbf{Q}) \\ f^* \downarrow & & \downarrow f^* \\ K^1(F_4) & \xrightarrow{ch} & H^*(F_4; \mathbf{Q}) \end{array}$$

As is well-known, $K^*(F_4)$ is an exterior algebra generated by some elements

$\beta_1, \dots, \beta_4 \in K^1(F_4)$ whose Chern characters were determined in [21]. Let g be a generator of $K^1(S^{11})$. Let ${}^K\text{Dec}$ and ${}^H\text{Dec}$ be the groups of decomposable elements with respect to $\{\beta_1, \dots, \beta_4\}$ and $\{x_3, x_{11}, x_{15}, x_{23}\}$, respectively. Express $f^*(g) \equiv \Sigma a_i \beta_i \pmod{{}^K\text{Dec}}$ and $\Sigma a_i \text{ch}(\beta_i) = P_3 x_3 + P_{11} x_{11} + P_{15} x_{15} + P_{23} x_{23}$, where P_i is a polynomial of a_1, \dots, a_4 with rational coefficients. Then $\text{ch}(f^*(g)) \equiv \Sigma P_i x_i \pmod{{}^H\text{Dec}}$. On the other hand, $\text{ch}(f^*(g)) = f^* \text{ch}(g) = \pm f^*[S^{11}] \in H^{11}(F_4; \mathbf{Z})$. Hence $P_{11} = \pm B$ and $P_3 = P_{15} = P_{23} = 0$, then $B = |P_{11}| = 2^4 \cdot 3 \cdot 7 \cdot |a_3|$ by elementary calculation. This argument can also be applied to stable case. By setting $y_6 = |a_3|$, we have the desired result.

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