Harada, M.
Osaka J. Math.
2 (1965), 153–161

ON SEMI-PRIMARY PP-RINGS

MANABU HARADA

(Received March 20, 1965)

This paper is a supplement to the author [2]. Let Λ be a ring with identity. If every principal left ideal in Λ is Λ-projective, then Hattori and Nakano called Λ a left PP-ring in [3], [4]. Nakano and Chase ([1], [4]) showed that if Λ is a semi-primary left PP-ring, then Λ is a generalized triangular matrix ring. The author has defined a generalized triangular matrix ring over semi-simple rings with bi-linear mappings $\varphi_{i,k}^{j}$ in [2] and found a criterion of semi-primary hereditary ring.

In §1 we shall use the same argument and give a similar criterion of a semi-primary left PP-ring. Using this criterion we shall show that Λ is a left PP-ring if and only if Λ is a right PP-ring, provided Λ is semi-primary.

As we see in [2], some results were obtained from monomorphic mapping $\varphi_{i,k}^{j}$ in a hereditary ring. Thus, in §2, we define a partially PP-ring, which is a ring with property that $\varphi_{i,k}^{j}$ is monomorphic and show that if Λ is a semi-primary partially PP-ring with nilpotency n, then Λ is isomorphic to a generalized triangular matrix ring over semi-simple rings with degree n and each component of it is uniquely determined up to isomorphism. From this fact we note that some results in [2] are generalized in a case of partially PP-ring.

In this paper we only consider semi-primary rings and semi-simple rings with minimum conditions.

1. PP-rings

We recall the definition of a *generalized taiangular matrix ring* (briefly *g.t.a.matrix ring*).

Let $\{R_1, R_2, \cdots, R_n\}$ be a set of semi-simple rings and $\{M_{i,j}$ for $i>j\}$ a set of R_i, R_j-modules. With a bi-linear mapping $\varphi_{i,k}^{j}: M_{i,j}\otimes_{R_j} M_{j,k}\to M_{i,k}$ we define a *g.t.a.*matrix ring by the usual way. We denote it by $T_n(R_i; M_{i,j})$ and n is called the *degree* of it:

$$\Lambda = \begin{pmatrix} R_1 & 0 \cdots\cdots & & \\ M_{2,1} & R_2 & 0 \cdots & \\ \cdots\cdots\cdots & & & \\ M_{n,n-1} & M_{n,n-2} & \cdots & R_n \end{pmatrix} = T_n(R_i\,;\,M_{i,j})\,,$$

(see [2], §2).

The following lemmas were given in [1] and [4]. We shall give here simple proofs, one of which is the same as in [1], Theorem 4.2 and will be used later.

Lemma 1. *Let Λ be a ring and M a left Λ-module. If Λm is Λ-projective and $em=m$ for $m\in M$ and an idempotent e in Λ, then there exists an idempotent f in Λ such that $ef=fe=f$, $\Lambda f\approx\Lambda m$ and $fm=m$. Especially, $e\Lambda em$ is $e\Lambda e$-projective.*

Proof. Since $\Lambda e\to\Lambda m\to(0)$ splits, we obtain $\Lambda e=\Lambda f\oplus\Lambda f'$ and $\Lambda f\approx \Lambda m$, $fm=m$. Hence, $e\Lambda em=e\Lambda m\approx e\Lambda f=e\Lambda efe$ is a direct summand of $e\Lambda e$.

Lemma 2. 1) *Every semi-primary left PP-ring is a g.t.a.matrix ring.*

2) *Let $\Lambda=\begin{pmatrix} R & 0\\ M & S\end{pmatrix}$ be a g.t.a.matrix ring, where R is semi-simple, S is a semi-primary left PP-ring and M is a S,R-module. If every principal left S-module in M is S-projective, then Λ is a left PP-ring.*

Proof. 1) Let $N=N(\Lambda)$ be the radical of Λ. We assume $N^s\neq(0)$, $N^{s+1}=(0)$. Then N^s is completely reducible and hence, N^s is Λ-projective by the assumption; $N^s\approx\sum\oplus\Lambda e_i$, where e_i is a primitive idempotent. Then $(0)=\sum\oplus Ne_i$. Let $\{e_i\}$ be a complete set of mutually orthogonal primitive idempotents such that $1=\sum_{i=1}^{n}e_i$ and $Ne_t=(0)$ for $i\leq t\leq n$ and $Ne_j\neq(0)$ for $j<i$. Let $E=e_1+\cdots+e_{i-1}$. Since e_j , e_t for $j<i\leqslant t$, $E\Lambda(1-E)=EN(1-E)=(0)$. It is clear that $\Lambda=T_2(E\Lambda E, (1-E)\Lambda(1-E)\,;(1-E)\Lambda E))$ and that $(1-E)\Lambda(1-E)$ is semi-simple and $E\Lambda E$ is a left PP-ring by Lemma 1. Hence, we can prove 1) by induction on number of idempotents n.

2) Let $\Lambda=T_2(R, S\,;M)$ and $m\in M$, $r\in R$. We put $Rr=Re$, $e^2=e$. Then $Mr+Sm=Me\oplus Sm(1-e)$. Let $x=T_2(r, s\,;m)$. Then $\Lambda x=T_2(Rr, Rs\,; Mr+Sm)=\Lambda T_2(e, 0\,;\,0)\oplus T_2((0), (0)\,;\,Sm(1-e))\oplus T_2((0), (0)\,;\,Ss)$. Since $Sm(1-e)$, Ss are S-projective, the last two modules are Λ-projective by [2] Lemma 4. Hence Λx is Λ-projective.

Proposition 1. *Λ is a semi-primary left PP-ring if and only if Λ*

is a g.t.a.matrix ring $T_n(R_i; M_{i,j})$ over semi-simple rings R_i with the following conditions;

1) $\varphi_{i,k}^{j}: M_{i,j}\otimes_{R_j} R_j x_{j,k}\to M_{i,j}x_{j,k}$ *is monomorphic for all* $i>j>k$ *and* $x_{j,k}\in M_{j,k}$.

2) *For any system* $\{x_{j+1,j}\cdots, x_{i,j}; x_{k,j}\in M_{k,j}, i>j\}$ $M_{i,j+1}C(_{j+1,j})+\cdots+M_{i,i-1}C(x_{i-1,j})$ *is a direct sum in* $M_{i,j}$, *where* $R_k x_{k,j}=C(x_{k,j})\oplus(\sum_{s=j+1}^{k-1} M_{k,s}x_{s,j})\cap R_k x_{k,j}$ *as a left* R_k*-module.*

Proof. We use the same argument as in the proof of [2], Theorem 1. By induction argument and Lemma 2 it is sufficient to show that every principal submodule of

$$\begin{pmatrix} 0 \\ M_{2,1} \\ \vdots \\ M_{n,1} \end{pmatrix}$$

is $\Gamma\ (=T_{n-1}(R_2,\cdots,R_n; M_{i,j}, j\neq 1))$-projective.

Let

$$x=\begin{pmatrix} x_{2,1} \\ \vdots \\ x_{n,1} \end{pmatrix}.$$

Then

$$\Gamma x=\begin{pmatrix} R_2x_{2,1} \\ M_{3,2}x_{2,1}+R_3x_{3,1} \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \\ M_{n,2}x_{2,1}+\cdots+R_nx_{n,1} \end{pmatrix}\supset N'x=\begin{pmatrix} 0 \\ 0 \\ M_{3,2}x_{2,1} \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \\ M_{n,2}x_{2,1}+\cdots+M_{n,n-1}x_{n-1,1} \end{pmatrix}$$

and

$$x\Gamma/N'x=\begin{pmatrix} C(x_{2,1}) \\ C(x_{3,1}) \\ \vdots \\ C(x_{n,1}) \end{pmatrix},$$

where $N'=N(\Gamma)$, $C(x_{2,1})=R_2x_{2,1}$. Hence, Γx is Γ-projective if and only if

$$\Gamma x\approx\Gamma\begin{pmatrix} x_{2,1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}\oplus\Gamma\begin{pmatrix} 0 \\ C(x_{3,1}) \\ 0 \\ \vdots \\ 0 \end{pmatrix}\cdots\oplus\Gamma\begin{pmatrix} 0 \\ 0 \\ \vdots \\ C(x_{n,1}) \end{pmatrix}$$

and

$$\Gamma \otimes \begin{pmatrix} 0 \\ 0 \\ \vdots \\ C(x_{i,1}) \\ \vdots \\ 0 \end{pmatrix} \to \Gamma$$

is monomorphic. Which is equivalent to 2) and 1′) $\varphi_{i,1}^{j}M_{i,j}\otimes C(x_{j,1})\to M_{i,1}$ is monomorphic. However if we replace $\{x_{2,1}, \cdots, x_{n,1}\}$ by $\{0, 0, \cdots, x_{i,1}, \cdots, x_{n,1}\}$, the $C(x_{i1})=R_i x_{i,1}$. Hence, we have 1).

REMARK 1. Let e be a sum of the set of non-isomorphic primitive idempotents in Λ. From [3], Corollary 1 we know that Λ is hereditary if and only if so is $e\Lambda e$. However, it is not true for a left PP-ring as we see in the following example. (Only if part is true by Lemma 1).

EXAMPLE. Let K be the field of real numbers. M, N and L be K-vector spaces with basis (u, v), (a, b) and (t, s), respectively. We define a bi-linear mapping $\varphi: M\otimes N\to L$. $M\otimes N=u\otimes(Ka+Kb)\oplus v\otimes(Ka+Kb)$. $\varphi(u(ax+by)+v(ax'+by'))=t(x+y')+s(y-x')$. Then we can easily check that φ is monomorphic on $M\otimes Kn$ for any $n\neq 0$ in N. However, $\varphi(u(a+b)+v(a-b))=0$ and $u(a+b)+v(a-b)\neq 0$.

Let $\Lambda=\begin{pmatrix} K & 0 & 0 \\ \begin{pmatrix} N \\ N \end{pmatrix} & K_2 & 0 \\ L & (M, M) & K \end{pmatrix}$ and $e=\begin{pmatrix} 1 & & 0 \\ & e_{1,1} & \\ 0 & & 1 \end{pmatrix}$. Then $e\Lambda e=\begin{pmatrix} K & 0 & 0 \\ N & K & 0 \\ L & M & K \end{pmatrix}$.

From Proposition 1 and the above observation we know that $e\Lambda e$ is a left PP-ring. Let

$$x=\begin{pmatrix} 0 & 0 & 0 \\ \begin{pmatrix} a \\ b \end{pmatrix} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in Λ. $(M, M)\underset{K_2}{\otimes}K_2\begin{pmatrix} a \\ b \end{pmatrix}=(M, M)\otimes\begin{pmatrix} N \\ N \end{pmatrix}$. Since φ is not monomorphic on $M\otimes N$, $\bar{\varphi}: (M, M)\underset{K_2}{\otimes}K_2\begin{pmatrix} a \\ b \end{pmatrix}\to L$ is not monomorphic, and hence, Λ is not a left PP-ring.

REMARK 2. The above example shows that the endomorphism ring of finitely generated projective module over a PP-ring is not, in general, a PP-ring, (cf. [2], Theorem 7). Because $\Lambda=\mathrm{Hom}^{r}_{e\Lambda e}(\Lambda e, \Lambda e)$ and Λe is a finitely generated $e\Lambda e$-projective module.

Lemma 2. *Let Λ be a g.t.a.matrix ring with property* 1) *and $M_{i,j}\ni y$. We assume $R_i y\approx R_i e$ by a correspondence $y\leftrightarrow e$ and $e^2=e$. If $xy=0$ for $x\in M_{k,i}$, then $xe=0$.*

Proof. $M_{k,i}\underset{R_i}{\otimes}R_i y\approx M_{k,i}\underset{R_i}{\otimes}R_i e$ is a direct summand of $M_{k,i}\approx M_{k,i}\underset{R_i}{\otimes}R_i$. Hence, $x\otimes y\approx x\otimes e=xe\otimes 1\approx xe$. Since $M_{k,i}\underset{R_i}{\otimes}R_i y\to M_{k,j}$ is monomorphic, $xe=0$.

From the example given in [1] we know that a left PP-ring is not necessarily to be a right PP-ring. However, we have

Theorem 1. *Let Λ be semi-primary. If Λ is a left PP-ring, then Λ is a right PP-ring.*

Proof. We shall show that Λ satisfies the following conditions 1′) and 2′) which are replaced left by right in Proposition 1.

1′) $\varphi^{j}_{i,k}$: $x_{i,j}R_j\underset{R_j}{\otimes}M_{j,k}\to M_{i,k}$ is monomorphic.

2′) For any system $(x_{j,i}, x_{j,i+1}, \cdots, x_{j,j-1})$ $D(x_{j,j-1})M_{j-1,i}+D(x_{j,j-2})M_{j-2,i}+\cdots+D(x_{j,i+1})M_{i+1,i}$ is a direct sum in $M_{j,i}$, where $x_{j,k}R_k=D(x_{j,k})\oplus(\sum_{t=k+1}^{j-1}x_{j,t}M_{t,k})\cap x_{j,k}R_k$ as a right R_k-module.

1′): We assume $\varphi^{k}_{j,s}: x_{j,k}R_k\otimes M_{k,s}\to x_{j,k}M_{k,s}$ is not monomorphic. Then there were elements $x=x_{j,k}$ and $m\in M_{k,s}$ such that $x\otimes m\neq 0$ in $x_{j,k}R_k\otimes M_{k,s}$ and $xm=0$. $M_{k,s}=Rm\oplus N$ and $M_{j,k}=xR_k\oplus N'$ as left and right R_k-module, respectively. Then $M_{j,k}\otimes M_{k,s}=xR_k\otimes R_k m\oplus N'\otimes R_k m\oplus xR_k\otimes N\oplus N'\otimes N$. Hence $x\otimes m\neq 0$ in $M_{j,k}\otimes R_k m$, which contradicts 1) of Proposition 1. Next, we shall show that Λ satisfies 2′). Let $\Lambda=T_n(R_i\,;M_{i,j})$; R_i simple ring. We prove 2′) by induction on degree n. If $n=1$, then Λ is simple, and hence, it is clear. From Lemma 1 and the induction hypothesis $\Gamma_i=T_{n-1}(R_1,\cdots\overset{i}{\vee},R_n\,;M_{k,j},\ k\neq i,\ j\neq i)$ satisfies 2′). Hence it is sufficient to consider a system

$$\{x_{n,2}, x_{n,3}, \cdots, x_{n,n-1}\,;\ x_{n,j}\in M_{n,j}\}\ .$$

We shall show a sum $x_{n,n-1}M_{n-1,1}+D(x_{n,n-2})M_{n-2,1}\cdots+D(x_{n,2})M_{2,1}$ is a direct sum. If $D(x_{n,i})=0$ for some i, then for $\{x_{n,2},\cdots\overset{i}{\vee}\cdots,x_{n,n-1}\}$ in Γ_i $x_{n,n-1}M_{n-1,1}\oplus D'(x_{n,n-2})M_{n-2,1}\oplus\cdots\overset{i}{\vee}\cdots\oplus D'(x_{n,2})M_{2,1}$, where $D'(x_{n,j})$ is a direct summand in $M_{n,j}$ as in 2′). Since $D'(x_{n,j})\supseteq D(x_{n,j})$, we obtain 2′). Hence, we assume $D(x_{n,i})\neq(0)$ for all i. If the above sum were not a direct sum then there were an element $0=x'_{n,n-1}m_{n-1,1}+\cdots+x'_{n,2}m_{2,1}$ such that some $x'_{n,j}m_{j,1}\neq 0$, where $x'_{n,j}\in D(x_{n,j})$. By the same reason as above, we may assume $x'_{n,j}m_{j,1}\neq 0$ for all j.

Let $x'_{n,i}R_i\approx e_iR_i$ by a correspondence $x'_{n,i}\leftrightarrow e_i$. Then $x'_{n,i}e_i=x'_{n,i}$. Hence, we may assume $0\neq m_{i,1}=e_im_{i,1}$. We have, from Lemma 1, an idempotent f_i such that $R_im_{i,1}\approx R_if_i$ and $f_ie_if_i=f_ie_i$, $f_im_{i,1}=m_{i,1}$. Since $x'_{n,i}R_i\approx e_iR_i$, $x'_{n,1}f_iR_i\approx f_iR_i$. Thus, we may assume that $0\neq x'_{n,i}f_i=x'_{n,i}$, $f_im_{i,1}=m_{i,1}$ and $x'_{n,i}R_i\approx f_iR_i$, $R_im_{i,1}\approx R_if_i$. Hence, the right annihilator

$r(x'_{n,i})$ of $x'_{n,i}$ in R_i is equal to $(1-f_i)R_i$. We consider a system $\{m_{2,1}, m_{3,1}, \cdots, m_{n-1,1}\}$ as in 2). If $C(m_{i,1})=(0)$, then there exist elements $m_{i,j}\in M_{i,j}$ such that $m_{i,1}=m_{i,2}m_{2,1}+\cdots+m_{i,i-1}m_{i-1,1}$ $(m_{i,j}=m_{i,j}f_j)$. Hence,

$$(*)\qquad 0 = x'_{n,n-1}m_{n-1,1}+\cdots+x'_{n,i+1}m_{i+1,1}\overset{i}{\vee}+(x'_{n,i}m_{i,i-1}+x'_{n,i-1})m_{i-1,1} + (x'_{n,i}m_{i,2}+x'_{n,2})m_{2,1}\,.$$

Again we consider a system $\{x'_{n,n-1}, \cdots, x'_{n,i-1}, (x'_{n,i-1}+x'_{n,i}m_{i,i-1}), \cdots, (x'_{n,2}+x'_{n,i}m_{i,2})\}$ in Γ_i. Since $x'_{n,t}R_t\cap(\sum_{k=t+1}^{n-1}x'_{n,k}M_{k,t})\subseteq x'_{n,t}R_t\cap(\sum_{k=t+1}^{n-1}x_{n,k}M_{k,t})$ $=(0)$ for $t>i$, $D'(x'_{n,t})=x'_{n,t}R_t$ for $t>i$. For $(x'_{n,t}+x'_{n,i}m_{i,t})r\in (x'_{nt}+x'_{n,i}m_{i,t})R_t\cap((x'_{n,t+1}+x'_{n,i}m_{i,t+1})M_{t+1,t}+\cdots+(x'_{n,i-1}+x'_{n,i}m_{i,i-1})M_{i-1,t}+ x'_{n,i+1}M_{i+1,t}+\cdots+x'_{n,n-1}M_{n-1,t})$ $(t<i)$, we have $x'_{n,t}r\in D(x_{n,t})\cap(\sum_{k=t+1}^{n-1}x_{n,k}M_{k,t})$ $=(0)$. Hence, $r\in r(x'_{n,t})=(1-f_t)R_t$. Therefore, $x'_{n,i}m_{i,t}r=0$, which means $D'(x'_{n,t}+x'_{n,i}m_{i,t})=(x'_{n,t}+x'_{n,i}m_{i,t})R_t$. From the induction hypothesis we know that (*) is a direct sum. Hence, $x'_{n,n-1}m_{n-1,1}=0$ or $(x'_{n,i}m_{i,2}+x'_{n,2})m_{2,1}=0$. From the latter and Lemma 3 we obtain $0=x'_{n,i}m_{i,2}f_2+x'_{n,2}f_2=x'_{n,i}m_{i,2}+x'_{n,2}$, which contradicts $D(x'_{n,2})\neq(0)$. In either case we have a contradiction and hence, $C(m_{i,1})\neq(0)$ for all i. Therefore, we have from 2)

$$M_{n,2}m_{2,1}+M_{n,3}C(m_{3,1})+\cdots+M_{n,n-1}C(m_{n-1,1})$$

is a direct sum. Hence,

$$\begin{aligned}0 &= x'_{n,2}m_{2,1}+\cdots+x'_{n,n-1}m_{n-1,1}\\ &= (x'_{n,2}+x'_{n,n-1}t_{n-1,2}+\cdots+x'_{n,3}t_{3,2})m_{2,1}\\ &\quad +(x'_{n,3}+\cdots)m'_{3,1}\\ &\quad \cdots\cdots\cdots\\ &\quad \cdots\cdots\cdots\\ &\quad \cdots\cdots\cdots\\ &\quad +x'_{n,n-1}m'_{n-1,1}\,,\end{aligned}$$

where $m_{i,1}=m'_{i,1}+t_{i,2}m_{2,1}+\cdots+t_{i,i-1}m_{i-1,1}$ and $t_{i,j}e_j=t_{i,j}$, $m'_{i,1}\in C(m_{i,1})$. Hence, $x'_{n,2}=-(x'_{n,n-1}t_{n-1,2}+\cdots+x'_{n,3}t_{3,2})$, which contradicts the fact $D(x'_{n,2})\neq(0)$. We have proved the theorem.

2. Partially PP-rings

We found a criterion of semi-primary PP-rings in Proposition 1. However, we need only the condition 1) in this section.

Let Λ be a semi-primary ring such that $\Lambda/N=\sum\oplus S_i$; S_i is a simple ring. Let $1=\sum_i E_i$, $E_i^2=E_i$ and E_i is the identity is S_i modulo N. We assume for idempotents E'_i, E'_j that $E_i\approx E'_i$, $E_j\approx E'_j$. Let x be in $E_i\Lambda E_j$.

Since $\Lambda E_j \approx \Lambda E'_j$, there exists y' in $E_i \Lambda E'_j$ such that $\Lambda x \approx \Lambda y'$. Furthermore, since $E_i \Lambda \approx E'_i \Lambda$, we have $t \in E'_i \Lambda E_i$, $u \in E_i \Lambda E'_i$ such that $ut = E_i$. If we put $y = ty' \in E'_i \Lambda E'_j$, then $\Lambda y = \Lambda t y' = \Lambda E_i y' = \Lambda y' \approx \Lambda x$, since $\Lambda t \supseteq \Lambda E_i \supseteq \Lambda t$. Hence, if Λx is Λ-projective for every x in $E_i \Lambda E_j$, then Λy is Λ-projective for every y in $E'_i \Lambda E'_j$.

Thus, we can define a partially PP-ring as follows:

Let Λ and E_i be as above. If Λx is Λ-projective for all $x \in E_i \Lambda E_j$ $(i, j = 1, \cdots, n)$, then we call Λ a *partially PP-ring.*

From Lemma 1, we obtain

Lemma 4. *Let Λ be a partially PP-ring and e an idempotent. Then $e\Lambda e$ is a partilaly PP-ring.*

Proposition 2. *Λ is a partially PP-ring if and only if Λ is a g.t.a. matrix ring $T_n(S_i; M_{i,j})$ over simple rings S_i with property* 1) *in Proposition* 1).

Proof. First we shall show that a partially PP-ring is a *g.t.a.* matrix ring. Let $1 = \sum_{i,j=1}^{m,\rho_i} e_{i,j}$, where $\{e_{i,j}\}$ is a complete set of primitive idempotents. Let $N^n = (0)$, $N^{n-1} \neq (0)$. Then there exist primitive idempotents e, f such that $(0) \neq eN^{n-1}f \subset E_i \Lambda E_j$, where $e \leqslant E_i$, $f \leqslant E_j$. Hence, Λx is Λ-projective for $0 \neq x \in eN^{n-1}f$. Since $\Lambda x \approx \sum \oplus \Lambda e_{\kappa,1}$ and $Nx = (0)$, there exists a primitive idempotent $e_{\kappa,1}$ such that $Ne_{\kappa,1} = (0)$. Then we can prove similarly to the proof of Lemma 2 that $\Lambda \approx T(R_i; M_{i,j})$; R_i semi-simple. Hence, the proposition is an immediate consequence from the next lemma.

Lemma 5. *Let Λ be a g.t.a.matrix ring. $T_k(S_i; M_{i,j})$ over simple rings S_i. Λ is a partially PP-ring if and only if $\varphi_{i,k}^{j} | M_{i,j} \otimes S_j x_{j,k}$ is monomorphic for every i, j, k and $x \in M_{j,k}$.*

Proof. It is clear from the proof of Proposition 1.

REMARK 3. From the first half of the proof of Theorem 1, Λ is a partially PP-ring if $x\Lambda$ is Λ-projective for every $x \in E_i \Lambda E_j$.

REMARK 4. We can show by examples that the set of semi-primary hereditary ring $\subset$ that of PP-rings $\subset$ that of partially PP-rings.

Let Λ be a partially PP-ring and $1 = \sum_{i,j=1}^{n,\rho_i} e_{i,j}$ as in the proof of Proposition 2. Since we can find an idempotent $e_{n,1}$ such that $Ne_{n,1} = (0)$, we may assume $Ne_{p_1,1} = Ne_{p_1+1,1} = \cdots = Ne_{n,1} = (0)$ and $Ne_{i,1} \neq (0)$ for $i < p$. Then Λ is isomorphic to $\begin{pmatrix} S_1 & 0 \\ M_1 & R_1 \end{pmatrix}$ as in the proof of Lemma 2. S_1 is a partially PP-ring by Lemma 4. After rearraging primitive idempotents $e_{p_2,1}, \cdots, e_{p_1-1,1}$ such that $N(S_1)e_{p_2,1}=\cdots=N(S_1)e_{p_1-1,1}=(0)$ and $N(S_1)e_{i,1}\neq(0)$ for $i<p_2$, we have

$$\Lambda=\begin{pmatrix} S_2 & & 0 \\ M_1 & R_2 & \\ M_2 & M_3 & R_1 \end{pmatrix}; \; S_2 \text{ is a partially PP-ring and } R_1, \; R_2 \text{ are semi-simple.}$$

Furthermore, $M_2f\neq(0)$ for any primitive idempotent f in R_2. Repeating this argument we know that $\Lambda\approx T_{n'}(R_i\,;\,M_{i,j})$ over semi-simple rings R_i and $M_{i+1,i}f_i\neq(0)$ for any primitive idempotent f_i in R_i.

The following theorem and corollary are generalizations of [2], Theorem 4‴ and Proposition 5.

Theorem 2. *Let Λ be a semi-primary partially PP-ring and $N^{n-1}\neq(0)$, $N^n=(0)$. Then Λ is isomorphic to a g.t.a.matrix ring $T_n(R_i, M_{i,j})$ over semi-simple rings R_i with degree n. Furthermore, $M_{i,j}\supseteq M_{i,i-1}M_{i-1,i-2}\cdots M_{j+1,j}f_j\neq(0)$ for any idempotent f_j in R_j and for all i.*

Proof. From the above argument we have $\Lambda=T_m(R_i\,;\,M_{i,j})$ and $M_{i+1,i}f_i\neq(0)$ for all primitive idempotent f_i in R_i. Let $L=M_{i,i-1}\cdots M_{j+1,j}$. We assume that $Lf_j\neq(0)$ for any primitive idempotents f_j in R_j. There exist $m\in M_{j,j-1}$ and f_j such that $f_jmf_{j-1}\neq0$ and $R_jf_j\approx R_jf_jmf_{j-1}$. Since $Lf_j\neq(0)$, $LM_{j,j-1}\supseteq Lf_jmf_{j-1}\neq(0)$ by Lemma 3. Thus, we can prove by induction $M_{i',i'-1}\cdots M_{j,j-1}f_{j-1}\neq(0)$ for all i'. Therefore, $(0)\neq M_{m,m-1}\cdots M_{2,1}\subseteq N^{m-1}$. Hence, $m-1<n$. Since $N^n=(0)$, $m\geqslant n$. Hence, $n=m$.

Corollary. *Let Λ be as above. Then*

$$n=\text{gl.dim}\,(\Lambda/N^2)=l(\Lambda)\,,$$

where $l(\Lambda)$ is the maximal length of connected sequence of primitive idempotents (see [2]).

Proof. From [2], Proposition 4 we know $n\leqslant l(\Lambda)=\text{gl.dim}\,(\Lambda/N^2)$. On the other hand $\Lambda\approx T_n(R_i\,;\,M_{i,j})$ by Theorem 2. Hence $n\geqslant l(\Lambda)$.

REMARK 5. We know from Theorem 2 that $n(f_i)=n-i+1$, where $n(f)$ is an integer m such that $N^{m-1}f\neq(0)$, $N^mf=(0)$, (see [2]).

In the expression of Λ as a g.t.a.matrix ring in Theorem 2 the set of primitive idempotents in R_j consists of those f_i such than $n(f_i)=n-i+1$. Hence, R_i, $M_{k,j}$ in $T_n(R_i, M_{k,j})$ are uniquely determined up to isomorphism.

By making use of the same argument as in the proof of [2], Proposition 8 we have

Proposition 3. *Let Λ be an indecomposable semi-primary partially PP-ring. Then the center K of Λ is a field. If $\Lambda \otimes_K L$ is a semi-primary partially PP-ring for every extension field L of K, then Λ/N is separable over K.*

REMARK 6. The converse is not true in general. In the example after Proposition 1 we obtain Λ/N is separable over K. However if C is the field of complex numbers, then φ is not monomorphic on $(M\otimes C)\otimes_C(a+bi)$.

REMARK 7. Proposition 10 in [2] is valid for a semi-primary PP-ring from Theorem 2. Furthermore, all results in [2], §5 are true for a semi-primary PP-ring by a slight change of proof.

OSAKA CITY UNIVERSITY

References

[1] S. U. Chase, *A generalization of the ring of triangular matrices*, Nagoya Math. J. **18** (1961), 13–25.

[2] M. Harada, *Semi-primary hereditary rings and triangular matrix rings*, to appear in Nagoya Math. J.

[3] A. Hattori, *A foundation of torsion theory for modules over general rings*, Nagoya Math. J. **17** (1960) 147–158.

[4] T. Nakano, *A nearly semi-simple ring*, Commentarii Mathematici Universitatis Sancti Pauli **7** (1959) 27–33.