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Author(s)	Murasugi, Kunio
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## *On the Alexander Polynomial of the Alternating Knot*

By Kunio MURASUGI

### §1 Introduction.

Let  $k$  be an alternating knot<sup>1)</sup>  $\subset S^3$  and let  $K$  be an image of a regular projection of  $k$  into the 2-sphere  $S^2 \subset S^3$ , and let  $K$  be oriented by the orientation induced by that of  $k$ .

In [2] Bankwitz has shown that  $8_9$  in Alexander-Brigg's table is not alternating, by proving that the minimum of crossing points in  $K$  is not larger than the product of the 2nd torsion number of  $k$ . We have obtained the same result by means of the Alexander polynomial  $\Delta_k(x)$  of  $k$  in [5]. In this paper we shall prove, moreover, the following:

**Theorem 1.1.** *All coefficients of  $\Delta_k(x)$  are distinct from zero.*

From this theorem, we can show that almost all parallel knots, in particular, "Schlauchknoten"<sup>2)</sup> of order  $\geq 2$ , are not alternating. Throughout this paper we shall use the same notations as in [4] and [5].

### §2 Proof of Theorem 1.1.

Let  $K$  have  $n$  crossing points  $c_1, \dots, c_n$ .  $K$  divides  $S^2$  into  $n+2$  regions  $r_0, \dots, r_{n+1}$ . At each crossing point  $c_i$ , four corners of four different regions  $r_j, r_k, r_l$  and  $r_m$ ,<sup>3)</sup> let us say, meet. We fix a point  $a_i$ , called a *center*, in each region  $r_i$ .

Now,  $K$  is divided into some standard loops which are classified into 2 classes of the first kind and of the second kind. Let  $C_1, C_2, \dots, C_m$  be loops of the second kind. These are modified to  $m$  disjoint loops in the same way as in [5]. These  $m$  loops divide  $S^2$  into  $m+1$  domains  $E_0, E_1, \dots, E_m$ .<sup>4)</sup> We arrange  $E_0, \dots, E_m$  and  $C_1, \dots, C_m$  in the same way as in §3 in [5].  $C_i$  are *outer boundaries* of  $E_i$  ( $i=1, \dots, m$ ), and the signs of  $E_0, \dots, E_d$  are positive, and those of  $E_{d+1}, \dots, E_m$  are negative ( $d \geq 0$ ). The regions contained in  $E_j$  are classified in black and white and in such a way that the region having some sides in common with

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1) A knot means a polygonal oriented closed curve.

2) See [6].

3) See footnote (5) in [4].

4) A domain means a connected open subset of  $S^2$ . If  $m=0$ , then  $E_0 = S^2$ .

$\dot{E}_j$  is white. Let  $\bar{E}_i \cap K = K_i$ .  $K_i$  may be regarded as an image of regular projection of a link.<sup>5)</sup> We define the crossing points contained in  $K$  and the sides of  $K$  in the same way as in §6 in [5]. Since all crossing points in  $K_i$  have constant incidence numbers,<sup>6)</sup> each white region in  $E_i$  has even crossing points on its boundary and a corner adjacent to a dotted (or undotted) corner is undotted (or dotted). (Cf. Lemma 6.3 in [5])

We have proved in [5] that there exist the  $L^{\bar{t}}$ -correspondence  $\sigma$  and the  $L^{\bar{t}}$ -correspondence corresponding to  $x^{2t}$  and a constant term in  $\Delta_k(x)$  respectively, where  $\bar{t} = \sum_i w_i - m + d - 2$ ,  $\bar{t} = d - 1$ ,  $w_i$  being the number of white regions in  $E_i$ .

Now, to prove Theorem 1.1, it is sufficient to show the following

**Lemma 2.1.** *For any  $s$  ( $d - 1 \leq s \leq \sum_i w_i - m + d - 2$ ), there exists  $L^s$ -correspondence  $\tau$ , and  $\sigma, \tau$  have the property (P).<sup>7)</sup>*

Let  $G_i$  be the graph<sup>8)</sup> of  $K_i$  and let  $P_{i,p}$  be the semi-graph with respect to the correspondence of the white regions by any  $L^p$ -correspondence  $\sigma_p$ , hereafter  $P_{i,p}$  will be called simply the semi-graph with respect to  $\sigma_p$ .  $P_{i,p}$  is a tree<sup>9)</sup> in  $G_i$ . Let us orient the sides of  $P_{i,p}$  as follows: Let  $s_{ij}$  be a side connecting the centers  $a_i$  and  $a_j$  of  $r_i$  and  $r_j$ , and let a crossing point  $c_k$  be in  $s_{ij}$ . If  $r_i$  corresponds to  $c_k$  by  $\sigma_p$ , then  $s_{ij}$  will be oriented in the direction from  $a_i$  to  $a_j$ . We shall call, then, that  $s_{ij}$  is an oriented side joining  $a_i$  to  $a_j$ , and  $a_j$  and  $a_i$  are the initial and the end of  $s_{ij}$  respectively.

Next, we shall classify all oriented sides of  $P_{i,p}$  into 2 classes  $\mathfrak{A}$  and  $\mathfrak{B}$  as follows. Let  $s_{ij}$  be an oriented side joining  $a_i$  to  $a_j$ , and let  $r_i$  and  $r_j$  be regions with centers  $a_i$  and  $a_j$  respectively, and let  $c_i$  be a crossing point in  $s_{ij}$ . If  $c_i$ -corner of  $r_i$  is dotted, then  $s_{ij}$  belongs to  $\mathfrak{A}$ , denoted by  $s_{ij} \in \mathfrak{A}$ . Otherwise,  $s_{ij}$  belongs to  $\mathfrak{B}$ . Then

(1) A semi-graph  $P_{i,\bar{t}}$  with respect to an  $L^{\bar{t}}$ -correspondence  $\sigma_{\bar{t}}$  is a tree and all sides of it  $\in \mathfrak{A}$ .

(2) A semi-graph  $P_{i,\bar{t}}$  with respect to an  $L^{\bar{t}}$ -correspondence is a tree and all sides of it  $\in \mathfrak{B}$ .

(3) Each center of all white regions except one is the end of one

5) A link means a figure composed of a finite number of disjoint knots in  $S^3$ . See footnote (7) in [5].

6) For the definition, see p. 95 in [4].

7) See [5].

8) For the definition and properties of a graph, see [8].

9) A tree means a connected set consisting of some segments and containing no loop.

side of a semi-graph  $P_{i,p}$  with respect to an  $L^p$ -correspondence for any  $p$  ( $\bar{t} \leq p \leq \bar{t}$ ).

Now, let  $r_i$  be the white region in  $E_i$  corresponding to a crossing point  $c_i$ , say, not contained in  $K_i$ , by an  $L^{\bar{t}}$ -correspondence  $\sigma_{\bar{t}}$ .<sup>10)</sup> Then we can prove the following lemma in the same way as in §5 in [4].

**Lemma 2.2.** *Let  $P$  be a set consisting of some sides in  $G_i$  satisfying the following conditions:*

- (1) *Each side of  $P$  is oriented.*
- (2) *Each center of all white regions except  $r_i$  is the end of one and only one side of  $P$ .*
- (3)  *$P$  is a tree.*
- (4) *The center  $a_i$  of the region  $r_i$  is the initial of at least one side of  $P$ .*

*Then we can construct an  $L$ -correspondence  $\sigma$  such that each of all black regions except one adjacent to  $r_i$  corresponds to a crossing point on its boundary. If  $t$  sides of  $P$  belong to  $\mathfrak{A}$  and the others to  $\mathfrak{B}$ , then  $\sigma$  is an  $L^t$ -correspondence.*

Now, to prove Lemma 2.1, it is sufficient to prove the following

**Lemma 2.3.** *For any  $q$  ( $0 \leq q \leq w_i - 1$ ), there exists an  $L^q$ -correspondence  $\tau$  such that  $r_i$ , mentioned in Lemma 2.2, corresponds to nothing by  $\tau$ .*

Let us prove Lemma 2.3. By Lemma 2.2, it is sufficient to prove that there exists a subset  $P$  of  $G_i$  satisfying the conditions (1)–(4).

Now, let  $P_i$  be the semi-graph with respect to an  $L^{w_i-1}$ -correspondence.  $P_i$  is a tree and all sides  $\in \mathfrak{A}$ . Suppose that  $P_i$  has  $p$  tree-tops  $e_1, \dots, e_p$ , where we call  $e_j$  of  $P_i$  a tree-top, if there is no side of  $P_i$  whose initial is  $e_j$ . Consider a path  $T^j$  connecting  $a_i$  with  $e_j$  by way of some sides of  $P_i$  provided that any side is never passed in  $T^j$  again. Since  $P_i$  is a tree,  $T^j$  is uniquely determined by  $e_j$ . Then we can write  $P_i = T^1 \cup \dots \cup T^p$ , where, of course,  $T^i$  and  $T^j$  may have some sides in common.

Let us assume that  $a_i$  is the initials of  $h$  sides  $\in \mathfrak{A}$  of  $P_i$ . Then  $r_i$  contains at least  $2h$  crossing points, of which at least  $h$  crossing points are not contained in  $P_i$ , and  $r_i$  has dotted corners at these points. Let the ends of sides of  $G_i$ , whose initials are the same  $a_i$  and which contain these  $h$  crossing points, be denoted by  $b_1, \dots, b_h$ . Now let us denote the points different from the other among these by  $b_{i_1}, \dots, b_{i_{h_0}}$  ( $h_0 \geq 1$ ).

Let  $m_{j_1} \in \mathfrak{A}$  be a side of  $P_i$  such that it has  $b_{i_1}$  as the end and let  $m'_{j_1}$

<sup>10)</sup> The existences of such a region  $r_i$  and a crossing point  $c_i$  are assured by Lemma 7.3 in [5].

$\in \mathfrak{B}$  be an oriented side of  $G_i - P_i$  joining  $b_{i_1}$  to  $a_i$ . Construct a subset  $P^1 = (P_i - m_{j_1}) \cup m'_{j_1}$  of  $G_i$ . Hereafter we shall call the operation the "replacing of a side  $m_{j_1} \in \mathfrak{A}$  by a side  $m'_{j_1} \in \mathfrak{B}$ ."

Let  $m_{j_1}$  be contained in some path, say  $T^1$ . Let  $T_0^1 = m_1 \cup m_2 \cup \dots \cup m_{j_1}$  be a sub-path of  $T^1$  connecting  $a_i$  with  $b_{i_1}$  in such an order that the end of  $m_i$  is the initial of  $m_{i+1}$ . Then we construct a subset  $P^k$  of  $G_i$  by replacing the side  $m_{j_1-k+1} \in \mathfrak{A}$  in  $P^{k-1}$  by the side  $m_{j_1-k+2}^{-1} \in \mathfrak{B}$  in succession for  $k=2, \dots, j_1$ , where  $m_j^{-1}$  denotes the side obtained by reversing the orientation of  $m_j$ . It is clear that  $P^k$  is a tree and it satisfies all conditions in Lemma 2.2. Hence we can construct  $L^{w_{i-1-k}}$ -correspondence by means of  $P^k$ .

Next let  $\bar{m}_{j_2} \in \mathfrak{A}$  be a side of  $P_i$  such that it has  $b_{i_2}$  as the end and let  $\bar{m}_{j_2}' \in \mathfrak{B}$  be a side of  $G_i - P_i$  joining  $b_{i_2}$  to  $a_i$ . Let  $\bar{m}_{j_2} \in T^2$ , say. Let  $T_0^2 = \bar{m}_1 \cup \bar{m}_2 \cup \dots \cup \bar{m}_{j_2}$  be a sub-path of  $T^2$  connecting  $a_i$  with  $b_{i_2}$ . Let us suppose that  $\bar{m}_1 = m_1, \dots, \bar{m}_{s-1} = m_{s-1}$ . Then we construct a subset  $P^{j_1+k}$  of  $G_i$  by replacing  $\bar{m}_{j_2-k+1} \in \mathfrak{A}$  by  $\bar{m}_{j_2-k+2}^{-1} \in \mathfrak{B}$  in succession for  $k=1, \dots, j_2-s+1$ , (we set  $\bar{m}_{j_2+1}^{-1} = \bar{m}'_{j_2}$ ). Since  $P^l$  clearly satisfies all conditions in Lemma 2.2, we can construct  $L^{w_{i-1-l}}$ -correspondence by means of  $P^l$ . In the same way, we replace sides  $\in \mathfrak{A}$  of  $P_i$  by sides joining  $b_{i_j}$  to  $a_i$  ( $1 \leq j \leq i_0$ ), if possible. Hence we obtain a subset  $P^\lambda$  of  $G_i$ .  $P^\lambda$  is a tree and by means of this we can construct an  $L^{w_{i-1-\lambda}}$ -correspondence. Moreover we construct  $P^\mu$  from  $P^\lambda$  as follows. Let  $t_i \in \mathfrak{A}$  be a side of  $P^\lambda$  joining a center  $a_i$  of a region  $r_i$  to the other center. If there is a side  $t_i' \in \mathfrak{B}$  joining  $a_i$  to the other center such that  $P_1^\lambda = (P^\lambda - t_i) \cup t_i'$  is a tree, then we replace  $t_i$  by  $t_i'$ . Thus we obtain  $P^\mu$ . If there is no such side  $t_i'$ , then we set  $P^\mu = P^\lambda$ .

Now, let us denote the number of sides of  $P^\mu$  belonging to  $\mathfrak{A}$  by  $n(P^\mu)$ .

If  $n(P^\mu) = 0$ , Lemma has been proved. Therefore we assume that  $n(P^\mu) > 0$ . Then the proof of Lemma will be complete, if it is proved that we can construct a tree  $P^\nu$  of  $G_i$  from  $P^\mu$  such that  $n(P^\nu) = n(P^\mu) - 1$ .

Let  $t \in \mathfrak{A}$  be a side of  $P^\mu$ . If  $t$  is an end side of  $P^\mu$ , the proof is complete, where an *end side* means a side whose end is an initial of no side of  $P^\mu$ . In fact, a boundary of a region  $r_u$ , say, whose center is an end of  $t$ , contains at least one crossing point  $c_\lambda$  except a crossing point lying on  $t$  such that the  $c_\lambda$ -corner of  $r_u$  is undotted. Hence if we construct  $P^\nu$  by replacing  $t$  in  $P^\mu$  by a side  $t' \in \mathfrak{B}$  containing  $c_\lambda$ , then  $P^\nu$  is clearly a tree and  $n(P^\nu) = n(P^\mu) - 1$ . Therefore we assume that  $t$  is not an end side. We may assume, moreover, that  $t \in \mathfrak{A}$  is a side of  $P^\lambda$  such that any sub-one of any path in  $P^\lambda$ , which contains  $t$  and connects the end  $a_j$  of  $t$  with any tree-top, contains no side of  $\mathfrak{A}$ .

Now, let us assume that  $a_j$  is a common initial of  $p(\geq 1)$  sides  $\in \mathfrak{B}$ . Then a region  $r_j$  with a center  $a_j$  has at least  $p+1$  undotted corners. Let  $P_0$  be a subset of  $G_i$  obtained by replacing  $t$  by a side  $t' \in \mathfrak{B}$ , say, through one of these  $p+1$  undotted corners.

(a) If  $P_0$  is connected, then  $P_0$  is a tree and we have  $n(P_0) = n(P^\mu) - 1$ . Thus the proof is complete.

(b) If  $P_0$  is disconnected, we have  $P_0 = P_1' \cup P_2'$ ,  $P_1' \cap P_2' = \phi$ , where  $P_1', P_2'$  are subsets of  $G_i$ . Let  $t' \in P_1'$ . Then  $P_1'$  contains a loop  $L$  and  $P_2'$  is a tree. In fact, if  $P_1', P_2'$  are both trees, then we have  $\chi(P_1') = \chi(P_2') = 1$ .<sup>11)</sup> On the other hand  $1 = \chi(P_0) = \chi(P_1' \cup P_2') = \chi(P_1') + \chi(P_2') = 2$ , which is a contradiction. If  $P_1', P_2'$  are not trees, then we have  $\chi(P_1') \leq 0$ ,  $\chi(P_2') \leq 0$ . From this a contradiction is deduced. Thus, at least one of  $P_1'$  and  $P_2'$  must contain a loop. Let  $P_2'$  contain a loop  $L$ . Then, if  $L \not\ni t$ , it follows  $P^\mu \supset L$ , which is a contradiction. If  $L \ni t$ , then  $P_1' \cup P_2' = P_0$  is connected, which contradicts the assumption.

Now, let  $L = l_1 \cup l_2 \cup \dots \cup l_s$ , where we set  $l_1 = t'$ . Since an end  $a_j$  of  $t'$  is also an initial of  $l_s$ , there is at least one side  $l'_1 \in \mathfrak{B}$  joining  $a_j$  to some center. From the assumption,  $P' = (P_0 - t') \cup l'_1$  contains a loop  $L'$ . Then  $L' - l'_1$  consists of sides of  $P_1'$  belonging to  $\mathfrak{B}$ . Let  $L' = l'_1 \cup l'_2 \cup \dots \cup l'_s$ . If  $l'_s \neq l_s$ , then there is at least another side  $l''_1$  joining  $a_j$  to some center and  $P'' = (P' - l'_1) \cup l''_1$  contains a loop  $L''$ . Since the number of sides having  $a_j$  as a common end is finite, there are some positive integers  $n, m$  such that  $l_s^{(n)} = l_s^{(m)}$ , where  $P^{(n)} = (P_1' - t) \cup l_s^{(n)}$  and  $P^{(m)} = (P_1' - t) \cup l_s^{(m)}$  contain loops  $L^{(n)}$  and  $L^{(m)}$  respectively, and  $L^{(n)} - l_s^{(n)}$  and  $L^{(m)} - l_s^{(m)}$  consist of sides of  $P_2'$ . Let us assume, moreover, that  $l_{t(n)}^{(n)} = l_{t(m)}^{(m)}$ ,  $l_{t(n)+1}^{(n)} = l_{t(m)+1}^{(m)}, \dots, l_s^{(n)} = l_s^{(m)}$ . Then, an end  $a_\lambda$  of  $l_{t(n)}^{(n)} = l_{t(m)}^{(m)}$  is contained in at least three sides, of which two sides have  $a_\lambda$  as an initial and the other as an end. Hence there is at least one side  $m_1 \in \mathfrak{B}$  joining  $a_\lambda$  to some center. Then, we obtain  $P_1''$  from  $P_1'$  by replacing  $t$  and  $l_{t(n)}^{(n)}$  by  $l_1^{(m)}$  and  $m_1$  respectively. If  $P_1^\mu = P_1'' \cup P_2'$  is connected, then  $P_1^\mu$  is a tree and  $n(P_1^\mu) = n(P^\mu) - 1$ , which is the required result. If  $P_1^\mu$  contains a loop  $M$ , then the above operation will be performed again. Since the number of sides of  $P_1''$  is finite, there must exist some integer  $l$  such that  $P_1^\mu \cup P_2' = P^\nu$  is connected. Then it follows that  $P^\nu$  is a tree and  $n(P^\nu) = n(P^\mu) - 1$ . Thus the proof of Lemma 2.3 is complete.

§ 3 Applications.

(I) A torus knot  $k_{p,q}$  of type  $(p, q)$ .

Let  $\Delta_{p,q}(x)$  be the Alexander polynomial of  $k_{p,q}$ . As is well-known, then, we have ([3], [6])

11)  $\chi(P)$  denotes the Euler's characteristic of  $P$ .

$$(3.1) \quad \Delta_{p,q}(x) = \frac{x^{qp}-1}{x^p-1} \cdot \frac{x-1}{x^q-1}.$$

On the other hand, denoting  $\Delta_{p,q}(x) = a_0 + a_1x + a_2x^2 + \dots + a_{(p-1)(q-1)}x^{(p-1)(q-1)}$ , it follows from (3.1) by a simple computation, for  $i \leq \frac{(p-1)(q-1)}{2}$ , ( $p > q$ ,  $(p, q) = 1$ ),

- (a) if  $q=1$ , then  $a_0=1, a_i=0$  for  $i \neq 0$ ,
- (b) if  $q=2$ , then  $a_i = (-1)^i$ .
- (c) The case  $q \geq 3$ . Let  $p = mq + s, 1 \leq s \leq q-1$ .

(i) If  $1 < s < q-1$ , then

$$a_{hq} = 1, a_{hq+1} = -1, a_{p+hq} = 1, a_{p+hq+1} = -1, (h=0, 1, 2, \dots).$$

The others are 0.

(ii) If  $s=1$ , then

$$a_{hq} = 1, a_{p+hq+1} = -1 (h=0, 1, 2, \dots),$$

$$a_{h'q} = -1 (h'=0, 1, 2, \dots, m-1).$$

The others are 0.

(iii) If  $s=q-1$ , then

$$a_{h'q} = 1 (h'=0, 1, 2, \dots, m), a_{hq+1} = -1, a_{hq+p} = 1 (h=0, 1, 2, \dots).$$

The others are 0.

From these, we have immediately

**Theorem 3.2.**  $k_{p,q} (p > q)$  is not alternating for  $q \geq 3$ .

(II) A parallel knot  $k$  of type  $(p, q)$  with the carrier knot  $k_0$ , where  $p \geq 2, q \geq 1, (p, q) = 1$ .

Denoting the Alexander polynomials of  $k$  and  $k_0$  by  $\Delta(x)$  and  $\Delta_{k_0}(x)$  respectively, we have, as is well-known, that

$$(3.3) \quad \Delta(x) = \Delta_{k_0}(x^p) \Delta_{r,p}(x),$$

where  $r$  is defined as follows: We denote the incidence number at each crossing point  $c_i$  in an image of a regular projection of  $k_0$  by  $I(c_i)$ <sup>12)</sup> and we set  $v = \sum_i I(c_i)$ . We may assume that  $v \geq 0$ , by reversing the classification into *black and white*, if necessary. Then we define  $r = pv + q$ . It is obvious that  $(r, p) = 1$ .

Now, let us denote  $\Delta_{k_0}(x) = a_0 + a_1x + \dots + a_{2t}x^{2t}$  ( $a_0 > 0$ ). Then, since  $\Delta_{k_0}(1) = \pm 1$  and  $a_i = a_{2t-i}$ , we have

$$(3.4) \quad 2(a_0 + \dots + a_{t-1}) + a_t = \pm 1.$$

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12) See p. 95 in [4].

(A) The case where  $v > 0$ . Then  $r > p, q$ .

(a) The case  $p \geq 3$ ,  $q$ : arbitrary ( $\geq 1$ ).

Since from (I),  $\Delta_{r,p}(x) = 1 - x + x^p \pm \dots$ , we have

$$\Delta(x) = a_0 - a_0x + (a_0 + a_1)x^p \pm \dots$$

Thus, by Theorem 1.1,  $k$  is not alternating.

(b) The case  $p=2$ ,  $q$ : arbitrary, and  $\Delta_{k_0}(x) \neq 1$ .

Let us assume that  $k$  is alternating.

(i) If  $2t \leq r-1$ , then the coefficients of  $x^{2t-2}$  and  $x^{2t}$  are distinct from 0 by Theorem 1.1. Hence

$$(3.5)_{t-1} \quad a_0 + a_1 + \dots + a_{t-1} > 0,$$

$$(3.5)_t \quad a_0 + a_1 + \dots + a_{t-1} + a_t > 0.$$

Therefore we have  $2(a_0 + \dots + a_{t-1}) + a_t > 1$ , which contradicts (3.4).

(ii) If  $2t > r-1 = 2u$ , ( $u \geq 1$ ), then the following inequalities must hold:

$$(3.6)_0 \quad a_0 > 0,$$

$$(3.6)_1 \quad a_0 + a_1 > 0,$$

$\vdots$

$$(3.6)_u \quad a_0 + a_1 + \dots + a_u > 0,$$

$$(3.6)_{u+1} \quad a_0 + a_1 + \dots + a_u + a_{u+1} > 0,$$

$\vdots$

$$(3.6)_t \quad a_0 + a_1 + \dots + a_{t-u} + \dots + a_t > 0.$$

Now, there is an integer  $\mu$  such that

( $\alpha$ )  $t = \mu(u+1)$ , ( $\mu > 0$ ) or

( $\beta$ )  $t > \mu(u+1)$  and  $t \leq \mu(u+1) + u$ , ( $\mu \geq 0$ ).

If  $t$  satisfies ( $\alpha$ ), then, adding the inequalities (3.6)<sub>0</sub>, (3.6)<sub>u</sub>, (3.6)<sub>u+1</sub>,  $\dots$ , (3.6)<sub>( $\mu-1$ )( $u+1$ )-1</sub>, (3.6)<sub>( $\mu-1$ )( $u+1$ )</sub>, (3.6) <sub>$\mu$ ( $u+1$ )-1</sub> and (3.6) <sub>$\mu$ ( $u+1$ )</sub>, we have

$$(3.7) \quad 2(a_0 + \dots + a_{t-1}) + a_t \geq 2\mu + 1 \geq 3.$$

On the other hand, if  $t$  satisfies ( $\beta$ ), then, adding the inequalities

(3.6) <sub>$t-\mu u-(\mu+1)$</sub> , (3.6) <sub>$t-\mu u-u$</sub> , (3.6) <sub>$t-(\mu-1)u-\mu$</sub> , (3.6) <sub>$t-(\mu-1)u-(\mu-1)$</sub> ,  $\dots$ , (3.6) <sub>$t-u-2$</sub> , (3.6) <sub>$t-u-1$</sub> , (3.6) <sub>$t-1$</sub>  and (3.6) <sub>$t$</sub> , we have

$$(3.8) \quad 2(a_0 + \dots + a_{t-1}) + a_t \geq 2(\mu+1) \geq 2.$$

(3.7) and (3.8) contradict (3.4). Therefore,  $k$  is not alternating.

(B) The case where  $v=0$ . Then,  $r=q$ .

(a) The case  $q=1$ ,  $p$ : arbitrary ( $\geq 2$ )

Since  $\Delta_{r,p}(x) = \Delta_{1,p}(x) = 1$  by (I), it follows  $\Delta(x) = \Delta_{k_0}(x^p)$ . Thus, if  $\Delta_{k_0}(x) \neq 1$ , then  $k$  is not alternating.



(b) The case  $q=2$ ,  $p$ : arbitrary ( $\geq 3$ ).

By a simple computation, it follows that if  $\Delta_{k_0}(x)$  is of non-alternating form<sup>13)</sup>,  $k$  is not alternating.

(c) The case  $q \geq 3$ .

(i) If  $p=2$ , we can show in the same way as in the case (II) (A) (b) that if  $\Delta_{k_0}(x) \neq 1$ , then  $k$  is not alternating.

(ii) If  $p \geq 3$ , we have  $\Delta(x) = a_0 - a_0x + a_0x^h \pm \dots$ , where  $h = \min(p, q) \geq 3$ . Thus  $k$  is not alternating by Theorem 1.1.

Thus, collecting these results, we have the following

**Theorem 3.9.** *A parallel knot  $k$  of type  $(p, q)$  with a carrier knot  $k_0$  is not alternating in the following cases:*

I. *For arbitrary carrier knot  $k_0$ ,*

(1) *if  $v > 0$ , the case where  $p \geq 3$  and  $q$ : arbitrary,*

(2) *if  $v = 0$ , the case where  $p \geq 3$  and  $q \geq 3$ .*

II. *For a carrier knot  $k_0$  such that  $\Delta_{k_0}(x) \neq 1$ ,*

(1) *if  $v > 0$ , the case where  $p = 2$ , and  $q$ : arbitrary,*

(2) *if  $v = 0$ , the case where  $p = 2$ , and  $q$ : arbitrary or  $p \geq 3$  and  $q = 1$*

III. *For a carrier knot  $k_0$  such that  $\Delta_{k_0}(x)$  is of non-alternating form,*

(1) *if  $v = 0$ , the case where  $p \geq 3$ , and  $q = 2$ .*

(III) A "Schlauchknoten"  $k_s$  of order  $s$  ( $\geq 2$ ) defined by  $s$  pairs  $(p_1, q_1), (p_2, q_2), \dots, (p_s, q_s)$ , where  $p_i \geq 2, q_i \geq 1, (p_i, q_i) = 1$  ([3], [6]).

Then, by computing the Alexander polynomial of  $k_s$ , it follows from Theorem 3.9 that  $k$  is not alternating. Hence we have

**Theorem 3.10.** *All "Schlauchknoten" of order  $\geq 2$  are not alternating.*

Hōsei University

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13) We shall say that a polynomial  $d(x) = a_0 + a_1x + a_2x^2 + \dots + a_mx^m$  is of alternating form, if  $a_0, a_2, \dots, a_{2i}, \dots \geq 0, a_1, a_3, \dots, a_{2i-1}, \dots \leq 0$ . See [5].

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