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On the Alexander Polynomial of the Alternating Knot

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§1 Introduction.

Let k be an alternating knot[1] $\subset S^3$ and let K be an image of a regular projection of k into the 2-sphere $S^2 \subset S^3$, and let K be oriented by the orientation induced by that of k.

In [2] Bankwitz has shown that 8_{19} in Alexander-Brigg's table is not alternating, by proving that the minimum of crossing points in K is not larger than the product of the 2nd torsion number of k. We have obtained the same result by means of the Alexander polynomial $\Delta_k(x)$ of k in [5]. In this paper we shall prove, moreover, the following:

Theorem 1.1. *All coefficients of $\Delta_k(x)$ are distinct from zero.*

From this theorem, we can show that almost all parallel knots, in particular, "Schlauchknoten"[2] of order $\geqq 2$, are not alternating. Throughout this paper we shall use the same notations as in [4] and [5].

§2 Proof of Theorem 1.1.

Let K have n crossing points $c_1, \cdots, c_n$. K divides S^2 into $n+2$ regions $r_0, \cdots, r_{n+1}$. At each crossing point c_i, four corners of four different regions r_j, r_k, r_l and r_m,[3] let us say, meet. We fix a point a_i, called a *center*, in each region r_i.

Now, K is divided into some standard loops which are classified into 2 classes of the first kind and of the second kind. Let $C_1, C_2, \cdots, C_m$ be loops of the second kind. These are modified to m disjoint loops in the same way as in [5]. These m loops divide S^2 into $m+1$ domains $E_0, E_1, \cdots, E_m$.[4] We arrange $E_0, \cdots, E_m$ and $C_1, \cdots, C_m$ in the same way as in §3 in [5]. C_i are *outer boundaries* of E_i $(i=1, \cdots, m)$, and the signs of $E_0, \cdots, E_d$ are positive, and those of $E_{d+1}, \cdots, E_m$ are negative $(d \geqq 0)$. The regions contained in E_j are classified in black and white and in such a way that the region having some sides in common with

1) A knot means a polygonal oriented closed curve.
2) See [6].
3) See footnote (5) in [4].
4) A domain means a connected open subset of S^2. If $m=0$, then $E_0=S^2$.

$\dot{E}_j$ is white. Let $\bar{E}_i \cap K = K_i$. K_i may be regarded as an image of regular projection of a link.[5] We define the crossing points contained *in* K and the *sides* of K in the same way as in §6 in [5]. Since all crossing points in K_i have constant incidence numbers,[6] each white region in E_i has even crossing points on its boundary and a corner adjacent to a dotted (or undotted) corner is undotted (or dotted). (Cf. Lemma 6. 3 in [5])

We have proved in [5] that there exist the $L^{\bar{t}}$-correspondence σ and the $L^{\bar{\bar{t}}}$-correspondence corresponding to x^{2t} and a constant term in $\Delta_k(x)$ respectively, where $\bar{t} = \sum_i w_i - m + d - 2$, $\bar{\bar{t}} = d - 1$, w_i being the number of white regions in E_i.

Now, to prove Theorem 1. 1, it is sufficient to show the following

Lemma 2. 1. *For any* s $(d-1 \leqq s \leqq \sum_i w_i - m + d - 2)$, *there exists* L^s-*correspondence* τ, *and* σ, τ *have the property* (P).[7]

Let G_i be the *graph*[8] of K_i and let $P_{i,p}$ be the semi-graph with respect to the correspondence of the white regions by any L^p-correspondence σ_p, hereafter $P_{i,p}$ will be called simply the semi-graph *with respect to* σ_p. $P_{i,p}$ is a tree[9] in G_i. Let us orient the sides of $P_{i,p}$ as follows: Let s_{ij} be a side connecting the centers a_i and a_j of r_i and r_j, and let a crossing point c_k be in s_{ij}. If r_i corresponds to c_k by σ_p, then s_{ij} will be oriented in the direction from a_i to a_j. We shall call, then, that s_{ij} is an oriented side *joining* a_i *to* a_j, and a_j and a_i are the *initial* and the *end* of s_{ij} respectively.

Next, we shall classify all oriented sides of $P_{i,p}$ into 2 classes $\mathfrak{A}$ and $\mathfrak{B}$ as follows. Let s_{ij} be an oriented side joining a_i to a_j, and let r_i and r_j be regions with centers a_i and a_j respectively, and let c_i be a crossing point in s_{ij}. If c_i-corner of r_i is dotted, then s_{ij} belongs to $\mathfrak{A}$, denoted by $s_{ij} \in \mathfrak{A}$. Otherwise, s_{ij} belongs to $\mathfrak{B}$. Then

(1) A semi-graph $P_{i,\bar{t}}$ with respect to an $L^{\bar{t}}$-correspondence $\sigma_{\bar{t}}$ is a tree and all sides of it $\in \mathfrak{A}$.

(2) A semi-graph $P_{i,\bar{\bar{t}}}$ with respect to an $L^{\bar{\bar{t}}}$-correspondence is a tree and all sides of it $\in \mathfrak{B}$.

(3) Each center of all white regions except one is the end of one

5) A link means a figure composed of a finite number of disjoint knots in S^3. See footnote (7) in [5].

6) For the definition, see p. 95 in [4].

7) See [5].

8) For the definition and properties of a graph, see [8].

9) A tree means a connected set consisting of some segments and containing no loop.

side of a semi-graph $P_{i,p}$ with respect to an L^p-correspondence for any p $(\bar{\bar{t}} \leqq p \leqq \bar{t})$.

Now, let r_i be the white region in E_i corresponding to a crossing point c_i, say, not contained in K_i, by an $L^{\bar{t}}$-correspondence $\sigma_{\bar{t}}$.[10] Then we can prove the following lemma in the same way as in §5 in [4].

Lemma 2.2. *Let P be a set consisting of some sides in G_i satisfying the following conditions*:

(1) *Each side of P is oriented.*

(2) *Each center of all white regions except r_i is the end of one and only one side of P.*

(3) *P is a tree.*

(4) *The center a_i of the region r_i is the initial of at least one side of P.*

Then we can construct an L-correspondence σ such that each of all black regions except one adjacent to r_i corresponds to a crossing point on its boundary. If t sides of P belong to $\mathfrak{A}$ and the others to $\mathfrak{B}$, then σ is an L^t-correspondence.

Now, to prove Lemma 2.1, it is sufficient to prove the following

Lemma 2.3. *For any q $(0 \leqq q \leqq w_i - 1)$, there exists an L^q-correspondence τ such that r_i, mentioned in Lemma 2.2, corresponds to nothing by τ.*

Let us prove Lemma 2.3. By Lemma 2.2, it is sufficient to prove that there exists a subset P of G_i satisfying the conditions (1)-(4).

Now, let P_i be the semi-graph with respect to an L^{w_i-1}-correspondence. P_i is a tree and all sides $\in \mathfrak{A}$. Suppose that P_i has p tree-tops $e_1, \cdots, e_p$, where we call e_j of P_i a tree-top, if there is no side of P_i whose initial is e_j. Consider a path T^j connecting a_i with e_j by way of some sides of P_i provided that any side is never passed in T^j again. Since P_i is a tree, T^j is uniquely determained by e_j. Then we can write $P_i = T^1 \cup \cdots \cup T^p$, where, of course, T^i and T^j may have some sides in common.

Let us assume that a_i is the initials of h sides $\in \mathfrak{A}$ of P_i. Then r_i contains at least $2h$ crossing points, of which at least h crossing points are not contained in P_i, and r_i has dotted corners at these points. Let the ends of sides of G_i, whose initials are the same a_i and which contain these h crossing points, be denoted by $b_1, \cdots, b_h$. Now let us denote the points different from the other among these by $b_{i_1}, \cdots, b_{i_{h_0}}$ $(h_0 \geqq 1)$.

Let $m_{j_1} \in \mathfrak{A}$ be a side of P_i such that it has b_{i_1} as the end and let m'_{j_1}

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10) The existences of such a region r_i and a crossing point c_i are assured by Lemma 7.3 in [5].

$\in \mathfrak{B}$ be an oriented side of $G_i - P_i$ joining b_{i_1} to a_i. Construct a subset $P^1 = (P_i - m_{j_1}) \cup m'_{j_1}$ of G_i. Hereafter we shall call the operation the "replacing of a side $m_{j_1} \in \mathfrak{A}$ by a side $m'_{j_1} \in \mathfrak{B}$."

Let m_{j_1} be contained in some path, say T^1. Let $T_0^1 = m_1 \cup m_2 \cup \cdots \cup m_{j_1}$ be a sub-path of T^1 connecting a_i with b_{i_1} in such an order that the end of m_i is the initial of m_{i+1}. Then we construct a subset P^k of G_i by replacing the side $m_{j_1-k+1} \in \mathfrak{A}$ in P^{k-1} by the side $m^{-1}_{j_1-k+2} \in \mathfrak{B}$ in succession for $k=2, \cdots, j_1$, where m^{-1}_j denotes the side obtained by reversing the orientation of m_j. It is clear that P^k is a tree and it satisfies all conditions in Lemma 2.2. Hence we can construct $L^{w_{i-1-k}}$-correspondence by means of P^k.

Next let $\bar{m}_{j_2} \in \mathfrak{A}$ be a side of P_i such that it has b_{i_2} as the end and let $\bar{m}_{j_2}' \in \mathfrak{B}$ be a side of $G_i - P_i$ joining b_{i_2} to a_i. Let $\bar{m}_{j_2} \in T^2$, say. Let $T_0^2 = \bar{m}_1 \cup \bar{m}_2 \cup \cdots \cup \bar{m}_{j_2}$ be a sub-path of T^2 connecting a_i with b_{i_2}. Let us suppose that $\bar{m}_1 = m_1, \cdots, \bar{m}_{s-1} = m_{s-1}$. Then we construct a subset P^{j_1+k} of G_i by replacing $\bar{m}_{j_2-k+1} \in \mathfrak{A}$ by $\bar{m}^{-1}_{j_2-k+2} \in \mathfrak{B}$ in succession for $k=1, \cdots, j_2-s+1$, (we set $\bar{m}^{-1}_{j_2+1} = \bar{m}'_{j_2}$). Since P^l clearly satisfies all conditions in Lemma 2.2, we can construct $L^{w_{i-1-l}}$-correspondence by means of P^l. In the same way, we replace sides $\in \mathfrak{A}$ of P_i by sides joining b_{i_j} to a_i $(1 \leqq j \leqq j_0)$, if possible. Hence we obtain a subset P^λ of G_i. P^λ is a tree and by means of this we can construct an $L^{w_{i-1-\lambda}}$-correspondence. Moreover we construct P^μ from P^λ as follows. Let $t_l \in \mathfrak{A}$ be a side of P^λ joining a center a_l of a region r_l to the other center. If there is a side $t_l' \in \mathfrak{B}$ joining a_l to the other center such that $P_1^\lambda = (P^\lambda - t_l) \cup t_l'$ is a tree, then we replace t_l by t_l'. Thus we obtain P^μ. If there is no such side t_l', then we set $P^\mu = P^\lambda$.

Now, let us denote the number of sides of P^μ belonging to $\mathfrak{A}$ by $n(P^\mu)$.

If $n(P^\mu) = 0$, Lemma has been proved. Therefore we assume that $n(P^\mu) > 0$. Then the proof of Lemma will be complete, if it is proved that we can construct a tree P^ν of G_i from P^μ such that $n(P^\nu) = n(P^\mu) - 1$.

Let $t \in \mathfrak{A}$ be a side of P^μ. If t is an end side of P^μ, the proof is complete, where *an end side* means a side whose end is an initial of no side of P^μ. In fact, a boundary of a region r_u, say, whose center is an end of t, contains at least one crossing point c_λ except a crossing point lying on t such that the c_λ-corner of r_u is undotted. Hence if we construct P^ν by replacing t in P^μ by a side $t' \in \mathfrak{B}$ containing c_λ, then P^ν is clearly a tree and $n(P^\nu) = n(P^\lambda) - 1$. Therefore we assume that t is not an end side. We may assume, moreover, that $t \in \mathfrak{A}$ is a side of P^λ such that any sub-one of any path in P^λ, which contains t and connects the end a_j of t with any tree-top, contains no side of $\mathfrak{A}$.

Now, let us assume that a_j is a common initial of $p(\geqq 1)$ sides $\in \mathfrak{B}$. Then a region r_j with a center a_j has at least $p+1$ undotted corners. Let P_0 be a subset of G_i obtained by replacing t by a side $t' \in \mathfrak{B}$, say, through one of these $p+1$ undotted corners.

(a) If P_0 is connected, then P_0 is a tree and we have $n(P_0) = n(P^{\mu}) - 1$. Thus the proof is complete.

(b) If P_0 is disconnected, we have $P_0 = P_1' \cup P_2'$, $P_1' \frown P_2' = \phi$, where P_1', P_2' are subsets of G_i. Let $t' \in P_1'$. Then P_1' contains a loop L and P_2' is a tree. In fact, if P_1', P_2' are both trees, then we have $\chi(P_1') = \chi(P_2') = 1$.[11] On the other hand $1 = \chi(P_0) = \chi(P_1' \cup P_2') = \chi(P_1') + \chi(P_2') = 2$, which is a contradiction. If P_1', P_2' are not trees, then we have $\chi(P_1') \leqq 0$, $\chi(P_2') \leqq 0$. From this a contradiction is deduced. Thus, at least one of P_1' and P_2' must contain a loop. Let P_2' contain a loop L. Then, if $L \not\ni t$, it follows $P^{\mu} \supset L$, which is a contradiction. If $L \ni t$, then $P_1' \cup P_2' = P_0$ is connected, which contradicts the assumption.

Now, let $L = l_1 \cup l_2 \cup \cdots \cup l_s$, where we set $l_1 = t'$. Since an end a_j of t' is also an initial of l_s, there is at least one side $l_1' \in \mathfrak{B}$ joining a_j to some center. From the assumption, $P' = (P_0 - t') \cup l_1'$ contains a loop L'. Then $L' - l_1'$ consists of sides of P_1' belonging to $\mathfrak{B}$. Let $L' = l_1' \cup l_2' \cup \cdots \cup l_s'$. If $l_s' \neq l_s$, then there is at least another side l_1'' joining a_j to some center and $P'' = (P' - l_1') \cup l_1''$ contains a loop L''. Since the number of sides having a_j as a common end is finite, there are some positive integers n, m such that $l^{(n)}_{s(n)} = l^{(m)}_{s(m)}$, where $P^{(n)} = (P_1' - t) \cup l^{(n)}_{s(n)}$ and $P^{(m)} = (P_1' - t) \cup l^{(m)}_{s(m)}$ contain loops $L^{(n)}$ and $L^{(m)}$ respectively, and $L^{(n)} - l^{(n)}_1$ and $L^{(m)} - l^{(m)}_1$ consist of sides of P_2'. Let us assume, moreover, that $l^{(n)}_{t(n)} = l^{(m)}_{t(m)}$, $l^{(n)}_{t(n)+1} = l^{(m)}_{t(m)+1}$, $\cdots$, $l^{(n)}_{s(n)} = l^{(m)}_{s(m)}$. Then, an end a_λ of $l^{(n)}_{t(n)} = l^{(m)}_{t(m)}$ is contained in at least three sides, of which two sides have a_λ as an initial and the other as an end. Hence there is at least one side $m_1 \in \mathfrak{B}$ joining a_λ to some center. Then, we obtain P_1'' from P_1' by replacing t and $l^{(n)}_{t(n)}$ by $l^{(n)}_1$ and m_1 respectively. If $P^{\mu}_1 = P_1'' \cup P_2'$ is connected, then P^{μ}_1 is a tree and $n(P^{\mu}_1) = n(P^{\mu}) - 1$, which is the required result. If P^{μ}_1 contains a loop M, then the above operation will be performed again. Since the number of sides of P_1'' is finite, there must exist some integer l such that $P^{\mu}_l \cup P_2' = P^{\nu}$ is connected. Then it follows that P^{ν} is a tree and $n(P^{\nu}) = n(P^{\mu}) - 1$. Thus the proof of Lemma 2.3 is complete.

§ 3 Applications.

(I) A torus knot $k_{p,q}$ of type (p, q).

Let $\Delta_{p,q}(x)$ be the Alexander polynomial of $k_{p,q}$. As is well-known, then, we have ([3], [6])

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11) $\chi(P)$ denotes the Euler's characteristic of P.

$$\Delta_{p,q}(x) = \frac{x^{qp}-1}{x^p-1}\cdot\frac{x-1}{x^q-1}. \tag{3.1}$$

On the other hand, denoting $\Delta_{p,q}(x)=a_0+a_1x+a_2x^2+\cdots+a_{(p-1)(q-1)}x^{(p-1)(q-1)}$, it follows from (3.1) by a simple computation, for $i\leqq\frac{(p-1)(q-1)}{2}$, $(p>q,\ (p,q)=1)$,

(a) if $q=1$, then $a_0=1$, $a_i=0$ for $i\neq 0$,

(b) if $q=2$, then $a_i=(-1)^i$.

(c) The case $q\geqq 3$. Let $p=mq+s$, $1\leqq s\leqq q-1$.

(i) If $1<s<q-1$, then

$a_{hq}=1$, $a_{hq+1}=-1$, $a_{p+hq}=1$, $a_{p+hq+1}=-1$, $(h=0,1,2,\cdots)$.

The others are 0.

(ii) If $s=1$, then

$a_{hq}=1$, $a_{p+hq+1}=-1$ $(h=0,1,2,\cdots)$,

$a_{h'q}=-1$ $(h'=0,1,2,\cdots,m-1)$.

The others are 0.

(iii) If $s=q-1$, then

$a_{h'q}=1$ $(h'=0,1,2,\cdots,m)$, $a_{hq+1}=-1$, $a_{hq+p}=1$ $(h=0,1,2,\cdots)$.

The others are 0.

From these, we have immediately

Theorem 3.2. *$k_{p,q}$ $(p>q)$ is not alternating for $q\geqq 3$.*

(II) A parallel knot k of type (p,q) with the carrier knot k_0, where $p\geqq 2$, $q\geqq 1$, $(p,q)=1$.

Denoting the Alexander polynomials of k and k_0 by $\Delta(x)$ and $\Delta_{k_0}(x)$ respectively, we have, as is well-known, that

$$\Delta(x) = \Delta_{k_0}(x^p)\Delta_{r,p}(x), \tag{3.3}$$

where r is defined as follows: We denote the incidence number at each crossing point c_i in an image of a regular projection of k_0 by $I(c_i)$[12] and we set $v=\sum_i I(c_i)$. We may assume that $v\geqq 0$, by reversing the classification into *black and white*, if necessary. Then we define $r=pv+q$. It is obvious that $(r,p)=1$.

Now, let us denote $\Delta_{k_0}(x)=a_0+a_1x+\cdots a_{2t}x^{2t}$ $(a_0>0)$. Then, since $\Delta_{k_0}(1)=\pm 1$ and $a_i=a_{2t-i}$, we have

$$2(a_0+\cdots+a_{t-1})+a_t=\pm 1. \tag{3.4}$$

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12) See p. 95 in [4].

(A) The case where $v>0$. Then $r>p, q$.

(a) The case $p\geqq 3$, q: arbitrary ($\geqq 1$).

Since from (I), $\Delta_{r,p}(x)=1-x+x^p\pm\cdots$, we have

$$\Delta(x)=a_0-a_0x+(a_0+a_1)x^p\pm\cdots.$$

Thus, by Theorem 1.1, k is not alternating.

(b) The case $p=2$, q: arbitrary, and $\Delta_{k_0}(x)\neq 1$.

Let us assume that k is alternating.

(i) If $2t\leqq r-1$, then the coefficients of x^{2t-2} and x^{2t} are distinct from 0 by Theorem 1.1. Hence

$$(3.5)_{t-1}\quad a_0+a_1+\cdots+a_{t-1}>0,$$
$$(3.5)_t\quad a_0+a_1+\cdots+a_{t-1}+a_t>0.$$

Therefore we have $2(a_0+\cdots+a_{t-1})+a_t>1$, which contradicts (3.4).

(ii) If $2t>r-1=2u$, ($u\geqq 1$), then the following inequalities must hold:

$$(3.6)_0\quad a_0>0,$$
$$(3.6)_1\quad a_0+a_1>0,$$
$$\vdots$$
$$(3.6)_u\quad a_0+a_1+\cdots+a_u>0,$$
$$(3.6)_{u+1}\quad a_0+a_1+\cdots+a_u+a_{u+1}>0,$$
$$\vdots$$
$$(3.6)_t\quad a_0+a_1+\cdots+a_{t-u}+\cdots+a_t>0.$$

Now, there is an integer μ such that

(α) $t=\mu(u+1)$, ($\mu>0$) or

(β) $t>\mu(u+1)$ and $t\leqq\mu(u+1)+u$, ($\mu\geqq 0$).

If t satisfies (α), then, adding the inequalities $(3.6)_0$, $(3.6)_u$, $(3.6)_{u+1}$, $\cdots$, $(3.6)_{(\mu-1)(u+1)-1}$, $(3.6)_{(\mu-1)(u+1)}$, $(3.6)_{\mu(u+1)-1}$ and $(3.6)_{\mu(u+1)}$, we have

$$(3.7)\qquad 2(a_0+\cdots+a_{t-1})+a_t\geqq 2\mu+1\geqq 3.$$

On the other hand, if t satisfies (β), then, adding the inequalities $(3.6)_{t-\mu u-(\mu+1)}$, $(3.6)_{t-\mu u-u}$, $(3.6)_{t-(\mu-1)u-\mu}$, $(3.6)_{t-(\mu-1)u-(\mu-1)}$, $\cdots$, $(3.6)_{t-u-2}$, $(3.6)_{t-u-1}$, $(3.6)_{t-1}$ and $(3.6)_t$, we have

$$(3.8)\qquad 2(a_0+\cdots+a_{t-1})+a_t\geqq 2(\mu+1)\geqq 2.$$

(3.7) and (3.8) contradict (3.4). Therefore, k is not alternating.

(B) The case where $v=0$. Then, $r=q$.

(a) The case $q=1$, p: arbitrary ($\geqq 2$)

Since $\Delta_{r,p}(x)=\Delta_{1,p}(x)=1$ by (I), it follows $\Delta(x)=\Delta_{k_0}(x^p)$. Thus, if $\Delta_{k_0}(x)\neq 1$, then k is not alternating.

(b) The case $q=2$, p: arbitrary ($\geqq 3$).

By a simple computation, it follows that if $\Delta_{k_0}(x)$ is of non-alternating form[13], k is not alternating.

(c) The case $q\geqq 3$.

(i) If $p=2$, we can show in the same way as in the case (II) (A) (b) that if $\Delta_{k_0}(x)\neq 1$, then k is not alternating.

(ii) If $p\geqq 3$, we have $\Delta(x)=a_0-a_0x+a_0x^h\pm\cdots$, where $h=\min(p,q)\geqq 3$. Thus k is not alternating by Theorem 1.1.

Thus, collecting these results, we have the following

Theorem 3.9. *A parallel knot k of type (p,q) with a carrier knot k_0 is not alternating in the following cases*:

I. *For arbitrary carrier knot k_0,*

(1) *if $v>0$, the case where $p\geqq 3$ and q: arbitrary,*

(2) *if $v=0$, the case where $p\geqq 3$ and $q\geqq 3$.*

II. *For a carrier knot k_0 such that $\Delta_{k_0}(x)\neq 1$,*

(1) *if $v>0$, the case where $p=2$, and q: arbitrary,*

(2) *if $v=0$, the case where $p=2$, and q: arbitrary or $p\geqq 3$ and $q=1$*

III. *For a carrier knot k_0 such that $\Delta_{k_0}(x)$ is of non-alternating form,*

(1) *if $v=0$, the case where $p\geqq 3$, and $q=2$.*

(III) A "Schlauchknoten" k_s of order s ($\geqq 2$) defined by s pairs (p_1, q_1), (p_2, q_2), $\cdots$, (p_s, q_s), where $p_i\geqq 2$, $q_i\geqq 1$, $(p_i, q_i)=1$ ([3], [6]).

Then, by computing the Alexander polynomial of k_s, it follows from Theorem 3.9 that k is not alternating. Hence we have

Theorem 3.10. *All "Schlauchknoten" of order $\geqq 2$ are not alternating.*

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References

[1] J. W. Alexander: Topological invariants of knots and links. Trans. Amer. Math. Soc. **30** (1928), 275–306.

[2] C. Bankwitz: Über die Torsionzahlen der alternierenden Knoten, Math. Ann. **103** (1930), 145–161.

13) We shall say that a polynomial $\Delta(x)=a_0+a_1x+a_2x^2+\cdots+a_mx^m$ is of alternating form, if $a_0, a_2, \cdots, a_{2i}, \cdots\geqq 0$, $a_1, a_3, \cdots, a_{2i-1}, \cdots\leqq 0$. See [5].

[3] W. Burau: Kennzeichnung der Schlauchknoten, Abh. Math. Univ. Hamburg. **9** (1932), 125–133.

[4] K. Murasugi: On the genus of the alternating knot, I. J. Math. Soc. Japan. **10** (1958), 94–105.

[5] ————: On the genus of the alternating knot, II. ibid. **10** (1958), 235–248.

[6] K. Reidemeister: Knotentheorie, Berlin, 1932.

[7] H. Seifert: Über das Geschlecht von Knoten, Math. Ann. **110** (1935), 571–592.

[8] T. Yajima and S. Kinoshita: On the graphs of knots. Osaka Math. J. **9** (1957), 155–163.