ORDINARY INDUCTION FROM A SUBGROUP
AND FINITE GROUP BLOCK THEORY

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Abstract

The first step in the fundamental Clifford Theoretic Approach to General Block Theory of Finite Groups reduces to: $H$ is a subgroup of the finite group $G$ and $b$ is a block of $H$ such that $b^{(g)b} = 0$ for all $g \in G - H$. We extend basic results of several authors in this situation and place these results into current categorical and character theoretic equivalences frameworks.

1. Introduction and statements of results

Let $G$ be a finite group, let $p$ be a prime integer and let $(O, K, k)$ be a $p$-modular system for $G$ that is “large enough” for all subgroups of $G$ (i.e., $O$ is a complete discrete valuation ring, $k = O/J(O)$ is an algebraically closed field of characteristic $p$ and the field of fractions $K$ of $O$ is of characteristic zero and is a splitting field for all subgroups of $G$).

Let $N$ be a normal subgroup of $G$ and let $\gamma$ be a block (a primitive) idempotent of $Z(O N)$. Set $H = \text{Stab}_G(\gamma)$ so that $N \leq H \leq G$. Also let $\text{Bl}(OH|\gamma)$ and $\text{Bl}(OG|\gamma)$ denote the set of blocks of $OH$ and $OG$ that cover $\gamma$, resp. Then it is well-known that if $b \in \text{Bl}(OH|\gamma)$, then $b^{(g)b} = 0$ for all $g \in G - H$ and the trace map from $H$ to $G$, $\text{Tr}_H^G$, induces a bijection $\text{Tr}_H^G : \text{Bl}(OH|\gamma) \to \text{Bl}(OG|\gamma)$ such that corresponding blocks are “equivalent.” This basic analysis pioneered by P. Fong and W. Reynolds (cf. [5, V, Theorem 2.5]) is the first step in the fundamental Clifford theoretic approach to general block theory: the reduction to the case of a stable block of a normal subgroup.

Consider the more general situation: (P) $H$ is a subgroup of $G$ and $e$ is an idempotent of $Z(OH)$ such that $e^{(g)e} = 0$ for all $g \in G - H$.

Note that if $\beta$ is an idempotent of $Z(OH)$ such that $e\beta = \beta$, then $\beta^{(g)\beta} = 0$ for all $g \in G - H$.

Fundamental contributions to this context appear in [9, Theorem 1] and in [11, Theorem 1].

The purpose of this paper is to put the significant results of [9, Theorem 1] and [11, Theorem 1] into current categorical and character theoretic equivalences context and to extend these basic results in this context.

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It is also well-known that if $H$ is a subgroup of $G$ and if $\chi \in \text{Irr}_K(H)$ is such that $\text{Ind}_H^K(\chi) \in \text{Irr}_K(G)$ and if $e_\chi = (\chi(1)/|H|)(\sum_{h \in H} \chi(h^{-1})h)$ denotes the primitive idempotent of $Z(KH)$ associated to $\chi$, then $e_\chi^2 = e_\chi$ and $\text{Tr}_H^G(e_\chi)$ is the primitive idempotent of $Z(KG)$ associated to $\text{Ind}_H^K(\chi)$ (cf. Corollary 1.5).

In this article, we shall generally follow the (standard) notation and terminology of [5] and [10].

All rings have identities and are Noetherian and all modules over a ring are unitary and finitely generated left modules. If $R$ is a ring, then $R\text{-mod}$ will denote the category of left $R$-modules and $R^0$ denotes the ring opposite to $R$.

The required proofs of the following main results will be presented in Section 3. Section 2 contains basic results that are needed in our proofs. We shall assume that $H$ is a subgroup of the finite group $G$ in the remainder of this section and we shall let $T$ be a left transversal of $H$ in $G$ with $1 \in T$. Thus $G = \bigcup_{t \in T} tH$ is disjoint.

For our first three results, $\mathcal{O}$ will denote a commutative Noetherian ring.

Our first two results are well-known and easy to prove (cf. [10, Sections 9 and 16]).

**Lemma 1.1.** Let $B$ be a unitary $\mathcal{O}$-algebra that is an interior $H$-algebra (as in [10, Section 16]). Then:

(a) $$\text{Ind}_H^K(B) = \mathcal{O}G \otimes_{\mathcal{O}H} B \otimes_{\mathcal{O}H} \mathcal{O}G = \bigoplus_{s,t \in T}(s(\mathcal{O}H) \otimes_{\mathcal{O}H} B \otimes_{\mathcal{O}H} (\mathcal{O}H)t^{-1})$$

\[ \cong \bigoplus_{s,t \in T}(s \otimes_{\mathcal{O}} B \otimes_{\mathcal{O}} t^{-1}) \]

is a unitary interior $G$-algebra with $1_{\text{Ind}_H^K(B)} = \sum_{t \in T}(t \otimes_{\mathcal{O}} 1_B \otimes_{\mathcal{O}} t^{-1})$ and with $\phi: G \rightarrow \text{Ind}_H^K(B)^\times$ such that $g \mapsto \sum_{t \in T}(gt \otimes_{\mathcal{O}} 1_B \otimes_{\mathcal{O}} t^{-1})$ for all $g \in G$. Moreover $\{t \otimes_{\mathcal{O}} 1_B \otimes_{\mathcal{O}} t^{-1} \mid t \in T\}$ is a set of orthogonal idempotents of $\text{Ind}_H^K(B)$; and

(b) The map $\alpha: Z(B) \rightarrow Z(\text{Ind}_H^K(B))$ such that $z \mapsto \sum_{t \in T}(t \otimes_{\mathcal{O}} z \otimes_{\mathcal{O}} t^{-1})$ for all $z \in Z(B)$ is an $\mathcal{O}$-algebra isomorphism.

**Proposition 1.2.** Let $e$ be an idempotent of $Z(\mathcal{O}H)$ such that $e(e^*e) = 0$ for all $g \in G - H$ and set $E = \text{Tr}_H^G(e) = \sum_{t \in T}(t^*e)$, so that $E$ is an idempotent of $Z(\mathcal{O}G)$. Then:

(a) $$\mathcal{O}GE = (\mathcal{O}G)e(\mathcal{O}G), \quad e(\mathcal{O}G)e = e(\mathcal{O}G)Ee = (\mathcal{O}H)e$$

and the $\mathcal{O}$-linear map

$$f: \text{Ind}_H^K((\mathcal{O}H)e) \rightarrow (\mathcal{O}G)E$$
such that $x \otimes_{OH} b \otimes_{OH} y \mapsto xby$ for all $x, y \in G$ and all $b \in (OH)e$ is an interior $G$-algebra isomorphism. Also the $O$-linear map

$$\phi: Z((OH)e) \to Z(\text{Ind}^G_H((OH)e))$$

such that $z \mapsto \sum_{t \in T} (t \otimes_O z \otimes_O t^{-1})$ for all $z \in Z((OH)e)$ is an $O$-algebra isomorphism;

(b) The inclusion map $i: (OH)e \to (OG)E$ is an embedding of interior $H$-algebras;

(c) The functors

$$\text{Ind}^G_H(\ast) = (OG)e \otimes_{(OH)e} \ast: (OH)e\text{-mod} \to (OG)E\text{-mod}$$

and

$$e \cdot \text{Res}^{OG}_{OH}(\ast) = e(OG) \otimes_{(OG)E} \ast: (OG)E\text{-mod} \to (OH)e\text{-mod}$$

exhibit a Morita equivalence between the Abelian categories $(OH)e\text{-mod}$ and $(OG)E\text{-mod}$ with associated $((OH)e, (OG)E)$-bimodule $e(OG)$; and

(d) Let $M$ be an $(OH)e$-module. Then

$$\text{Ind}^G_H(M) = (OG)e \otimes_{(OH)e} M = \bigoplus_{t \in T}(t \otimes_O M)$$

and

$$\alpha(g \otimes_{(OH)e} m) = \begin{cases} 0 & \text{if } g \notin H \\ 1 \otimes_O (\alpha g)m & \text{if } g \in H, \text{ for all } \alpha \in (OH)e, \text{ all } m \in M \text{ and all } g \in G. \end{cases}$$

Let $e$ be an idempotent of $Z(OH)$.

**Remark 1.3.** Let $g \in G$. The following three conditions are equivalent:

(i) $e(O(HgH))e = (0)$;

(ii) $e(\mathcal{E}) = 0$; and

(iii) $e(O(HgH) \otimes_{OH} V) = (0)$ for any module $V$ in $(OH)e$-mod.

Indeed, it is clear that (i) implies (ii) and (iii). Let $h_1, h_2 \in H$. Then $e(h_1gh_2)e = h_1e(\mathcal{E})gh_2$, so that (ii) implies (i). Also if $V = (OH)e$ in (iii), then

$$e(O(HgH) \otimes_{OH} (OH)e) \cong e(O(HgH)e)$$

in $(OH)e$-mod and so (iii) implies (i).
Lemma 1.4 (E.C. Dade [4]). Let \( K \) be a field and let \( e \) be an idempotent in \( Z(KH) \). Suppose that

\[
\dim(\text{Hom}_K(\text{Ind}_H^G(X), \text{Ind}_H^G(Y))/K) = \dim(\text{Hom}_{KH}(X, Y)/K)
\]

for any irreducible modules \( X, Y \) in \((KH)e\mod\). Then \( e(\ {}^g e) = 0 \) for all \( g \in G - H \).

An immediate implication of Lemma 1.4 is:

Corollary 1.5. Assume that \( K \) is a splitting field for \( G \) and \( H \) and that \( e \) is an idempotent of \( Z(KH) \) such that \( \text{Ind}_H^G \) defines an injective map \( \text{Ind}_H^G : \text{Irr}_K(e) \rightarrow \text{Irr}_K(G) \). Then \( e(\ {}^g e) = 0 \) for all \( g \in G - H \).

For the remainder of this section, we assume that \((O, \mathcal{K}, k)\) is a \( p \)-modular system that is “large enough” for all subgroups of \( G \). As is standard, the natural ring epimorphism \(- : O \rightarrow k = O/J(O)\) induces an epimorphism on all \( O\)-algebras that is also denoted by \(- \). Similarly for \( O\)-modules.

Theorem 1.6 (cf. [5, V, Theorem 2.5], [9, Proposition 1] and [11, Theorem 1]). Assume that \( b \in \text{Bl}(OH) \) is such that \( b(\ {}^g b) = 0 \) for all \( g \in G - H \) (as in Proposition 1.2) and let \( D \) be a defect group of \( b \) in \( H \). Then:

(a) Proposition 1.2 applies (with \( R = O \), \( B = \text{Tr}_H^G(b) \in \text{Bl}(OG) \) and \( D \) is a defect group of \( B \) in \( G \);

(b) The functors \( \text{Ind}_H^G(\ {}^* b) = (OG) \otimes_O H(\ {}^* b) = (OG) b \otimes (O_H b) (\ {}^* b) : (OG)b\mod \rightarrow (OG)B\mod \) and \( b \cdot \text{Res}_H^G(\ {}^* b) : (OG)B\mod \rightarrow (OG)b\mod \)

exhibit a Morita equivalence between the Abelian categories \((OG)b\mod\) and \((OG)B\mod\). On the character level, this Morita equivalence induces the bijections:

\[
\text{Ind}_H^G : \text{Irr}_K(b) \rightarrow \text{Irr}_K(B), \quad \text{Ind}_H^G : \text{Irr}_k(b) \rightarrow \text{Irr}_k(B)
\]

and

\[
\text{Ind}_H^G : \text{Irr}_K(b) \rightarrow \text{Irr}_K(B), \quad \text{Ind}_H^G : \text{Irr}_k(b) \rightarrow \text{Irr}_k(B).
\]

Moreover, this Morita equivalence has associated bimodules:

\[
(OG)b \text{ in } (OG)B\mod-(OH)b \text{ and } b(OG) \text{ in } (OH)b\mod-(OG)B.
\]

Here \((OG)b\) when viewed as an \( O(G \times H)\)-module is indecomposable with \( \Delta D = \{ (d, d) \mid d \in D \} \) and trivial \( \Delta D\)-source and a similar fact holds for \( b(OG) \);
Let $M$ be an indecomposable $(\mathcal{O}H)b$-module with vertex $Q$ and $Q$-source $V$. Then $\text{Ind}^G_H(M) = \mathcal{O}G \otimes_{\mathcal{O}H} M = (\mathcal{O}G)b \otimes_{\mathcal{O}H} M$ in $(\mathcal{O}G)B$-mod is an indecomposable $(\mathcal{O}G)$-module with vertex $Q$ and $Q$-source $V$.

(d) The above conditions hold over $k$ for $B \in \text{Bl}(kH)$ and $\overline{B} = \text{Tr}^G_H(\overline{b}) \in \text{Bl}(kG)$, etc.

(e) The inclusion map $i : (\mathcal{O}H)b \rightarrow (\mathcal{O}G)B$ is an embedding of interior $H$-algebras so that $i$ induces injective maps ([10, Proposition 15.1])

$$i_* : \mathcal{P}\mathcal{G}((\mathcal{O}H)b) \rightarrow \mathcal{P}\mathcal{G}((\mathcal{O}G)B) \quad \text{and} \quad i_* : \mathcal{L}\mathcal{P}\mathcal{G}((\mathcal{O}H)b) \rightarrow \mathcal{L}\mathcal{P}\mathcal{G}((\mathcal{O}G)B).$$

Let $D_\gamma$ be a defect pointed group of $(\mathcal{O}H)b$ as an $H$-algebra. Thus $i_*(D_\gamma) = D_{i(\gamma)}$, where $i(\gamma) = \{\gamma(\mathcal{O}G)^{\gamma}\}$, is a defect pointed group of $(\mathcal{O}G)B$ as a $G$-algebra. Thus if $j \in \gamma$, then $j \in i(\gamma)$ and $j(\mathcal{O}G)B_j = jb(\mathcal{O}G)bj = j(\mathcal{O}H)bj$, so that these source algebras of $b$ and $B$ are equal as interior $D$-algebras; and

(f) The Puig category of local pointed groups of $b$ in $\mathcal{O}H$ and of $B$ in $\mathcal{O}G$ are equivalent.

The next result illuminates the hypothesis of [11, Theorem 1].

**Proposition 1.7.** Let $b$ be a block idempotent of $Z(\mathcal{O}H)$. The following four conditions are equivalent:

(a) $\text{Ind}^G_H$ induces an injective map of $\text{Irr}_K(\overline{b}) \rightarrow \text{Irr}_K(G)$;

(b) $\text{Ind}^G_H$ induces an injective map of $\text{Irr}_K(b) \rightarrow \text{Irr}_K(G)$; and

(c) $\text{Ind}^G_H$ induces an injective map of $\text{Irr} \text{Br}_K(b) \rightarrow \text{Irr} \text{Br}_K(G)$; and

(d) $b(xb) = 0$ for all $g \in G - H$.

In which case, Theorem 1.6 applies so that $B = \text{Tr}^G_H(b) \in \text{Bl}(\mathcal{O}G)$, the functor

$$\text{Ind}^G_H = (\mathcal{O}G)b \otimes_{(\mathcal{O}H)b} (\mathcal{O}H)b \rightarrow (\mathcal{O}G)B$$

induces a (Morita) categorical equivalence, the maps of (a), (b) and (c) are bijections, etc.

In our final result, (a), (b), (c) and (d) are presented in [9, Theorem 1] without proof. For the convenience of the reader, we shall include a proof of these items.

**Theorem 1.8** (cf. [9, Theorem 1]). Assume that $b \in \text{Bl}(\mathcal{O}H)$ is such that $b(xb) = 0$ for all $g \in G - H$ (as in Theorem 1.6). Set $\Omega = \{xb \mid g \in G\}$ so that $B = (\sum_{o \in \Omega} o) \in \text{Bl}(\mathcal{O}G)$, etc.

(a) Let $(P, \overline{b})$ be a $b$-subpair of $H$. Then $\overline{b}P(\overline{b}) = 0$ for all $x \in C_G(P) - C_H(P)$. Theorem 1.6 (d) applies $s(\overline{b}) = \text{Tr}_{C_H(P)}^G(\overline{b}) \in \text{Bl}(kC_G(P))$, $(P, s(\overline{b}))$ is a $B$-subpair of $G$ and the $k$-linear map

$$\mu : \text{Ind}_{C_H(P)}^G(\overline{b})_{C_H(P)} = kC_G(P) \otimes_{kC_H(P)} kC_H(P) \overline{b} \otimes_{kC_H(P)} kC_G(P) \rightarrow kC_G(P)s(\overline{b})$$
such that \( x \otimes_{kC_H(P)} \alpha \otimes_{kC_H(P)} y \to x\alpha y \) for all \( x, y \in C_G(P) \) and all \( \alpha \in kC_H(P)\bar{b}_P \) is an interior \( C_G(P) \)-algebra isomorphism. Also \( \text{Ind}^G_H \colon \text{Irr}_k(\bar{b}_P) \to \text{Irr}_k(s(\bar{b}_P)) \) is a bijection;
(b) The map \((P, \bar{b}_P) \mapsto (P, s(\bar{b}_P))\) from the set of \( b \)-subpairs of \( H \) into the set of \( \text{B-subpairs of } G \) is injective;
(c) Let \((Q, \bar{b}_Q)\) and \((P, \bar{b}_P)\) be \( b \)-subpairs of \( H \). Then:
(i) \( \{ g \in G \mid s(Q, s(\bar{b}_Q)) = (P, s(\bar{b}_P)) \} = C_G(P)\{ h \in H \mid h(Q, \bar{b}_Q) = (P, \bar{b}_P) \} \) so that \((Q, \bar{b}_Q)\) and \((P, \bar{b}_P)\) are conjugate in \( H \) if and only if \((Q, s(\bar{b}_Q))\) and \((P, s(\bar{b}_P))\) are conjugate in \( G \), and
(ii) \((Q, \bar{b}_Q) \leq (P, \bar{b}_P)\) in \( H \) if and only if \((Q, s(\bar{b}_Q)) \leq (P, s(\bar{b}_P))\) in \( G \);
(d) For any \( B \)-subpair \((P', \bar{b}'_P)\) of \( G \) there is an \( x \in G \) and a \( b \)-subpair \((P, \bar{b}_P)\) of \( H \) such that \( s(P', \bar{b}'_P) = (P, \bar{b}_P) \); consequently the Brauer category of \( b \) in \( H \) is equivalent to the Brauer category of \( B \) in \( G \);
(e) Let \((Q, \bar{b}_Q)\) be a \( b \)-subpair of \( H \). The injective map \( i_s \colon \mathcal{LPG}(\langle \mathcal{O}H \rangle b) \to \mathcal{LPG}(\langle \mathcal{O}G \rangle B) \) of Theorem 1.6 induces a bijection

\[
i_s(Q, \bar{b}_Q) : \{ Q_\gamma \in \mathcal{LPG}(\langle \mathcal{O}H \rangle b) \mid Q_\gamma \text{ is associated with } (Q, \bar{b}_Q) \} \]

\[ \to \{ Q_\delta \in \mathcal{LPG}(\langle \mathcal{O}G \rangle B) \mid Q_\delta \text{ is associated with } (Q, s(\bar{b}_Q)) \} \]

in which \( Q_\gamma \mapsto Q_{i_s(\gamma)} \) for all \( Q_\gamma \in \mathcal{LPG}(\langle \mathcal{O}H \rangle b) \) such that \( Q_\gamma \) is associated with \((Q, \bar{b}_Q)\);
(f) Let \((P, \bar{b}_P)\) be a \( b \)-subpair of \( H \) and let \((P, s(\bar{b}_P))\) be the corresponding \( \text{B-subpair of } G \). Let \( b_P \) be the unique block idempotent of \( Z(\text{OC}_H(P)) \) that “lifts” \( \bar{b}_P \). Then \( b_P (s b_P) = 0 \) for all \( x \in C_G(P) - C_H(P), s(b_P) = \text{Tr}^{\text{C}_G(P)}_{\text{C}_H(P)}(b_P) \) is a block idempotent of \( C_G(P) \) that “lifts” \( s(\bar{b}_P) \) and Theorem 1.6 applies to \( b_P \in \text{Bl}(OC_H(P)) \) where \( C_H(P) \leq C_G(P) \), and
(g) Let \((D, b_D)\) be a maximal \( b \)-subpair of \( H \). Let \( P \leq D \) and let \((P, \bar{b}_P)\) be the unique \( b \)-subpair of \( H \) such that \((P, \bar{b}_P) \leq (D, \bar{b}_D)\). Then

\[
\text{Ind}^{\text{C}_G(P)}_{\text{C}_H(P)}(\ast) : R_K(C_H(P), b_P) \to R_K(C_G(P), s(b_P))
\]

is a perfect isometry and consequently induces the linear map

\[
\text{Ind}^{\text{C}_G(P)}_{\text{C}_H(P)}(\ast)_{\mathfrak{p}'} : \mathcal{C}_{\mathfrak{p}'}(C_H(P), b_P, K) \to \mathcal{C}_{\mathfrak{p}'}(C_G(P), s(b_P), K).
\]

Let \( u \in D \) and set \( P = \langle \mu \rangle \). Then

\[
\Phi_G^{(\mu, s(b_P))} \circ \text{Ind}_H^G(\ast) = \text{Ind}^{\text{C}_G(P)}_{\text{C}_H(P)}(\ast)_{\mathfrak{p}'} \circ \Phi_H^{(\mu, b_P)} : \mathcal{C}(H, b, K) \to \mathcal{C}_{\mathfrak{p}'}(C_G(P), s(b_P), K).
\]

Consequently the perfect isometry \( \text{Ind}^G_H(\ast) : R_K(H, b) \to R_K(G, B) \) is part of an isotopy between \( b \) and \( B \) with local system the family \( \{ \text{Ind}^{\text{C}_G(P)}_{\text{C}_H(P)}(\ast) \mid P \leq D, P \text{ cyclic} \} \).
Remark 1.9. In the situation of Theorem 1.8 and after Theorem 1.6 (a) has been established, the more general investigations of [6] apply (cf. [6, Remark 1.3 (a)]).

2. Preliminary results

Let $G$ be a finite group and let $(\mathcal{O}, K, k = \mathcal{O}/J(\mathcal{O}))$ be a $p$-modular system that is "large enough" for all subgroups of $G$. We shall, as in [3], set $CF_p(G, K) = \{ f \in CF(G, K) \mid f(G - G_p') = (0) \}$.

Let $u$ be a $p$-element of $G$ and set $P = \langle u \rangle$. Let $\chi \in \text{Irr}_K(G)$ and let $\phi \in \text{Irr} Br_K(C_G(P)) \subseteq CF_{p'}(C_G(P), K)$. We shall let $d_u(\chi, \phi)$ denote the generalized decomposition number associated to $u \in G_p$, $\chi \in \text{Irr}_K(G)$ and $\phi \in \text{Irr} Br_K(C_G(P))$, cf. [5, IV, Section 6]. Thus $d_u^G(\chi)(\ast) \in CF_{p'}(C_G(P), K)$ where $d_u^G(\chi)(s) = \chi(u s) = \sum_{\phi \in \text{Irr} Br_K(C_G(P))} d_u(\chi, \phi) \phi(s)$ for all $s \in C_G(P)_{p'}$. Moreover, as in [3, Section 4A], if $b \in Bl(\mathcal{O}G)$ and $b_p \in Bl(\mathcal{O}C_G(P))$, then $d_{G}^{(u, b_p)}: CF(G, b, K) \rightarrow CF_{p'}(C_G(P), b_p, K)$ is defined by: if $\alpha \in CF(G, b, K)$ and $s \in C_G(P)_{p'}$, then $(d_{G}^{(u, b_p)}(\alpha))(s) = (b_p \cdot d_u^G(\alpha))(s) = \alpha(u s b_p)$.

Since $\text{Irr}_K(b)$ is a basis of $CF(G, b, K)$, the $K$-linear map $d_{G}^{(u, b_p)}$ is characterized by the well-known:

Lemma 2.1. Let $\chi \in \text{Irr}_K(b)$. If $Br_p(b)b_p = 0$, then $d_{G}^{(u, b_p)}(\chi) = 0$. If $Br_p(b)b_p = b_p$, then $d_{G}^{(u, b_p)}(\chi) = \sum_{\phi \in \text{Irr} Br_K(b_p)} d_u(\chi, \phi) \phi$

Proof. With the notation and hypotheses of this lemma, the first statement is a consequence of Brauer’s Second Main Theorem on Blocks ([5, IV, Theorem 6.1]) and the second statement is a consequence of [2, Theorem A2.1].

Remark 2.2. As above, if $\phi \in \text{Irr} Br_K(C_G(P))$ corresponds to $\gamma \in \mathcal{L}P(\mathcal{O}G^P)$ (i.e., $\phi$ is the irreducible Brauer character obtained from the irreducible $kC_G(P)$-module $kC_G(P)Br_p(j)/J(kC_G(P)Br_p(j))$ for any $j \in \gamma$), then, by [10, Theorem 43.4] $d_u(\chi, \phi) = \chi(u j)$ for any $j \in \gamma$.

3. Proofs

As noted above, Lemma 1.1 and Proposition 1.2 are well-known and easy to prove.

Proof of Lemma 1.4. Assume the hypotheses of Lemma 1.4. Let $S$ be a set of double $(H, H)$-coset representatives in $G$ such that $1 \in S$ and let $X, Y$ be irreducible modules in $(\mathcal{K}H)e$-mod. Here

$$\text{Hom}_{\mathcal{K}G}(\text{Ind}_G^H(X), \text{Ind}_G^H(Y)) \cong \text{Hom}_{\mathcal{K}H}(X, \bigoplus_{s \in S} (\mathcal{K}(Hs H) \otimes_{\mathcal{K}H} Y))$$

$$\cong \bigoplus_{s \in S} \text{Hom}_{\mathcal{K}H}(X, \mathcal{K}(Hs H) \otimes_{\mathcal{K}H} Y)$$
in $\mathcal{K}$-mod. Thus $\text{Hom}_{\mathcal{K}}(X, \mathcal{K}(HsH) \otimes_{\mathcal{K}} Y) = (0)$ for all $1 \neq s \in S$.

Fix $1 \neq s \in S$ and an irreducible module $X$ in $(\mathcal{K}e)$-mod.

We assert: (*) $\text{Hom}_{\mathcal{K}}(X, \mathcal{K}(HsH) \otimes_{\mathcal{K}} V) = (0)$ for all $V$ in $(\mathcal{K}e)$-mod.

Indeed, we may assume that $V$ is reducible in $(\mathcal{K}e)$-mod and we proceed by induction on $\dim(V / \mathcal{K})$. Let $V_1$ be a maximal submodule of $V$. Then

$$(0) \to V_1 \to V \to V / V_1 \to (0)$$

is a short exact sequence in $(\mathcal{K}e)$-mod. Thus, since $\mathcal{K}(HsH) \otimes_{\mathcal{K}} V$ is irreducible, a short exact sequence in $(\mathcal{K}e)$-mod. Consequently

$$\text{Hom}_{\mathcal{K}}(X, \mathcal{K}(HsH) \otimes_{\mathcal{K}} V_1) \to \text{Hom}_{\mathcal{K}}(X, \mathcal{K}(HsH) \otimes_{\mathcal{K}} V) \to \text{Hom}_{\mathcal{K}}(X, \mathcal{K}(HsH) \otimes_{\mathcal{K}} (V / V_1))$$

is exact in $\mathcal{K}$-mod and we conclude from the induction hypothesis that

$$\text{Hom}_{\mathcal{K}}(X, \mathcal{K}(HsH) \otimes_{\mathcal{K}} V) = (0).$$

This establishes (*).

Since $X$ can be any irreducible $(\mathcal{K}e)$-module, (*) implies that $\text{Soc}(e \mathcal{K}(HsH) \otimes_{\mathcal{K}} V) = (0)$ for any module $V$ in $(\mathcal{K}e)$-mod. Thus $e \mathcal{K}(HsH)e \otimes_{\mathcal{K}} V = (0)$ for any module $V$ in $(\mathcal{K}e)$-mod and we are done.

Proof of Theorem 1.6. Assume the hypotheses of Theorem 1.6. Applying Proposition 1.2, [5, V, Lemma 1.2] and [5, III, Lemma 9.6], $D$ is contained in a defect group of $B \in Bl(\mathcal{O}G)$. Since $\text{Ind}^G_H: \text{Irr}_K(b) \to \text{Irr}_K(B)$ is bijective, [5, IV, Theorem 4.5] and degree considerations complete a proof of (a). Clearly $(\mathcal{O}G)b$ is indecomposable in $\mathcal{O}(\tilde{G} \times H)(B \otimes_{\mathcal{O}} b^0)$-mod and $D \times D$ is a defect group of $B \otimes_{\mathcal{O}} b^0 \in \mathcal{O}(\tilde{G} \times H)$. Also $(\mathcal{O}H)b \mid \text{Res}^G_{H \times H}(\mathcal{O}G)b$ in $\mathcal{O}(H \times H)$-mod and $(\mathcal{O}H)b$ is indecomposable in $\mathcal{O}(H \times H)$-mod with $D$ as a vertex and trivial $D$-source. Then [5, III, Lemma 4.6 (ii) and Corollary 6.8] implies the last part of (b). Thus (b) holds, [5, III, Corollary 4.7] yields (c) and (d) and (e) are clear. Finally (e) and [5, Theorem 47.10 (b)] yield (f).

Proof of Proposition 1.7. Assume the situation of this proposition. Let $V$ and $W$ be irreducible $(k\tilde{H})\tilde{b}$-modules with irreducible characters $\phi_V$, $\phi_W$ in $\text{Irr}_K(\tilde{b})$ and irreducible Brauer characters $\beta_V$, $\beta_W$ in $\text{Irr} Br_K(b)$.
Assume that (a) holds. Then $\text{Ind}_H^G(V)$ is an irreducible $kG$-module and $\text{Ind}_H^G(\beta_V) = \beta_{\text{Ind}_H^G(V)} \in \text{Irr} \, B_k(G)$. Similarly $\text{Ind}_H^G(W)$ is an irreducible $kG$-module and $\text{Ind}_H^G(\beta_W) = \beta_{\text{Ind}_H^G(W)} \in \text{Irr} \, B_k(G)$. Suppose that $\beta_{\text{Ind}_H^G(V)} = \beta_{\text{Ind}_H^G(W)}$. Then

$$\phi_{\text{Ind}_H^G(V)} = \beta_{\text{Ind}_H^G(V)} = \beta_{\text{Ind}_H^G(W)} = \phi_{\text{Ind}_H^G(W)}, \quad \text{Ind}_H^G(V) \cong \text{Ind}_H^G(W) \quad \text{in} \quad kG$$

and hence $\text{Ind}_H^G(\phi_V) = \text{Ind}_H^G(\phi_W)$. But then $\phi_V = \phi_W$, $V \cong W$ in $(kH)\overline{b}$-mod and $\beta_V = \beta_W$, so that (c) follows.

Assume that $\text{(c)}$ holds. Then $\overline{b} \beta = \phi_V \in \text{Irr}_k(\overline{b})$ and $\text{Ind}_H^G(\beta_V) = \beta_{\text{Ind}_H^G(V)} \in \text{Irr} \, B_k(G)$. Thus $\text{Ind}_H^G(\phi_V) = \phi_{\text{Ind}_H^G(V)} \in \text{Irr}_k(G)$. Similarly $\beta_{\text{Ind}_H^G(W)} \in \text{Irr} \, B_k(G)$ and $\text{Ind}_H^G(\phi_W) = \phi_{\text{Ind}_H^G(W)} \in \text{Irr}_k(G)$. Suppose that $\text{Ind}_H^G(\phi_V) = \text{Ind}_H^G(\phi_W)$. Then $\beta_{\text{Ind}_H^G(V)} = \beta_{\text{Ind}_H^G(W)} = \text{Ind}_H^G(\beta_V) = \text{Ind}_H^G(\beta_W)$, so that $\beta_V = \beta_W$, $\phi_V = \phi_W$ and (a) holds. Consequently (a) and (c) are equivalent.

That (d) implies (a), (b) and (c) is a consequence of Theorem 1.6 (b). Assume (a) and let $g \in G - H$. Then $M = bO(HgH)b$ is an $O$-lattice where $\overline{M} = \overline{b}k(HgH)\overline{b} = (0)$ by Corollary 1.5. Consequently $bO(HgH)b = (0)$ and (d) holds.

Assume (b) and for each $\chi \in \text{Irr}_k(b)$, let $e_{\chi} = (\chi(1)/|H|)(\sum_{h \in H}(\chi(h^{-1})h))$ be the primitive idempotent of $Z(KH)$ corresponding to $\chi$. Then $e_{\chi}(\overline{s} e_{\chi}) = 0$ for all $g \in G - H$ by Corollary 1.5 and $\text{Tr}_H^G(e_{\chi}) = e_{\text{Ind}_H^G(\chi)}$ is the primitive idempotent of $Z(KG)$ corresponding to $\text{Ind}_H^G(\chi)$. Let $\chi, \psi \in \text{Irr}_k(b)$ and let $g \in G - H$. Then $e_{\chi}(\overline{s} e_{\psi}) = (e_{\chi} \text{Tr}_H^G(e_{\chi}))(\text{Tr}_H^G(e_{\chi}))^{s} e_{\psi} = 0$, (d) holds and our proof is complete. 

Proof of Theorem 1.8. For (a), note that $Br_P(b)\overline{b}_P = \overline{b}_P$. Let $x \in C_G(P) - C_H(P)$; then

$$\overline{b}_P(x \overline{b}_P) = \overline{b}_P Br_P(b) Br_P(x \overline{b}_P) = \overline{b}_P Br_P(b \overline{b}_P) = 0.$$

Thus $\text{Stab}_{C_G(P)}(\overline{b}_P) = C_H(P)$ and, since $Br_P(B)\overline{b}_P = \overline{b}_P$, we conclude that

$$Br_P(B) \text{Tr}_{C_G(P)}(\overline{b}_P) = \text{Tr}_{C_G(P)}(\overline{b}_P).$$

Then Proposition 1.2 and Theorem 1.6 yield (a). Since $Br_P(b)\overline{b}_P = \overline{b}_P$, (b) holds.

Let $(Q, \overline{b}_Q)$ and $(P, \overline{b}_P)$ be $b$-subpairs of $H$ and let $S$ be a left transversal of $C_H(Q)$ in $C_G(Q)$ with $1 \in S$, so that $C_G(Q) = \bigcup_{S} sC_H(Q)$ is disjoint. Let $h \in H$ be such that $h \in (Q, \overline{b}_Q) = (P, \overline{b}_P)$. Then

$$\overline{h}_s(\overline{b}_Q) = \sum_{s \in S} (hs) \overline{b}_Q = \sum_{s \in S} (\overline{b}_Q) = s(\overline{b}_Q)$$

and hence

$$C_G(P) \{ h \in H \mid \overline{h}_s(\overline{b}_Q) = (P, \overline{b}_P) \} \leq \{ g \in G \mid \overline{s}(Q, s(\overline{b}_Q)) = (P, \overline{s}(\overline{b}_P)) \}.$$
Conversely, let \( g \in G \) be such that \( \tilde{s}(Q, s(\bar{b}_Q)) = (P, s(\bar{b}_P)) \). Then \( \tilde{s}B = B \) and \( \tilde{s}\Omega = \Omega \). Let \( U \) be a left transversal of \( C_H(P) \) in \( C_G(P) \) with \( 1 \in U \), so that \( C_G(P) = \bigcup_{u \in U} uC_H(P) \) is disjoint. Here \( \tilde{s}(\bar{b}_Q) = s(\bar{b}_P) = \sum_{u \in U} Br_P(\tilde{s}(\bar{b}_P)) \), \( Br_Q(b)(\bar{b}_P) = \bar{b}_Q \) and \( Br_P(\tilde{s}(\bar{b}_P)) = \tilde{b}_P \) for all \( u \in U \). Thus

\[
0 \neq \tilde{s}(\bar{b}_Q) = Br_P(\tilde{s}(\bar{b}_P)) = \sum_{u \in U} Br_P(\tilde{s}(\bar{b}_P)).
\]

We conclude that \( \tilde{s}b = \tilde{b}u \) for some \( u \in U \) and so \( g = uh \) for some \( h \in H \). But then \( \tilde{s}(Q, s(\bar{b}_Q)) = \tilde{s}(hQ, h\bar{b}_Q) = (P, s(\bar{b}_P)) \) and \( \tilde{s}(hQ, h\bar{b}_Q) = (P, s(\bar{b}_P)) \). Since \( Br_Q(b)(\bar{b}_Q) = \bar{b}_Q \), we have \( Br_P(b)(\bar{b}_P) = \bar{b}_Q \), and then \( \bar{b}_Q = \tilde{b}_P \), which completes a proof of (c) (i).

For a proof of (c) (ii), it suffices to assume that \( Q \leq P \). First suppose that \( (Q, \bar{b}_Q) \leq (P, \bar{b}_P) \). Thus \( \bar{b}_Q \) is \( P \)-stable and \( Br_P(b)(\bar{b}_P) = \tilde{b}_P \). As \( P \leq N_G(Q) \), we conclude that \( P \) stabilizes \( s(\bar{b}_Q) \). Let \( U \) be a left transversal of \( C_H(P) \) in \( C_G(P) \) with \( 1 \in U \). Here

\[
Br_P(s(\bar{b}_Q))(\bar{b}_P) = Br_P(s(\bar{b}_Q))Br_P(\bar{b}_Q)\bar{b}_P = Br_P(\bar{b}_Q)\bar{b}_P = \tilde{b}_P
\]

and, since \( C_G(P) \leq C_G(Q) \), we have \( Br_P(s(\bar{b}_Q)) = \tilde{b}_P \) for all \( u \in U \). Thus \( Br_P(s(\bar{b}_Q)) = \tilde{b}_P \). Conversely, suppose that \( (Q, s(\bar{b}_Q)) \leq (P, s(\bar{b}_P)) \). Then \( s(\bar{b}_Q) \in (kC_G(Q))^P \) and \( Br_P(s(\bar{b}_Q)) = s(\bar{b}_P) \). Utilizing [10, Lemma 40.2],

\[
s(\bar{b}_P)Br_P(b) = \tilde{b}_P = Br_P(s(\bar{b}_Q))s(\bar{b}_P)Br_P(b)
\]

\[
= Br_{P/Q}(s(b_Q)Br_Q(b))\tilde{b}_P = Br_{P/Q}(\bar{b}_Q)\tilde{b}_P = Br_P(\bar{b}_Q)\tilde{b}_P.
\]

Since \( s(\bar{b}_Q)Q(\bar{b}_Q) = \bar{b}_Q \), and \( \bar{b}_Q \) is \( P \)-stable and so \( (Q, \bar{b}_Q) \leq (P, \bar{b}_P) \) which completes a proof of (c) (ii).

Let \( (P', \bar{b}_{P'}) \) be a \( B \)-subpair of \( G \). Let \( (D, b_D) \) be a maximal \( b \)-subpair of \( H \); thus \( (D, s(\bar{b}_D)) \) is a maximal \( B \)-subpair of \( G \). Then there is an \( x \in G \) such that \( \tilde{s}(P', \bar{b}_{P'}) \leq (D, s(\bar{b}_D)) \). Thus \( \tilde{s}P = (D, s(\bar{b}_D)) \) and setting \( Q = \tilde{s}P' \), we have \( (Q, \bar{b}_Q) \leq (D, \bar{b}_D) \) for a unique \( \bar{b}_Q \in Bl(kC_H(Q)) \). But then \( (Q, s(\bar{b}_Q)) \) (d) \( \tilde{s}P = (D, s(\bar{b}_D)) \); consequently \( \tilde{s}\bar{b}_P = s(\bar{b}_Q) \) and \( \tilde{s}(P', \bar{b}_{P'}) = (Q, s(\bar{b}_Q)) \), which completes a proof of (d).

For (e), let \( (Q, \bar{b}_Q) \) be a \( b \)-subpair of \( H \). By (a), \( kC_H(Q)\bar{b}_Q \)-mod and \( kC_G(Q)\bar{b}_Q \)-mod are Morita equivalent. Thus \( |\mathcal{P}(kC_H(Q)\bar{b}_Q)| = |\mathcal{P}(kC_G(Q)\bar{b}_Q)| \).

\[
|\{Q_\gamma \in \mathcal{L}_P\mathcal{G}(\mathcal{O}H)B \mid Q_\gamma \text{ is associated with } (Q, \bar{b}_Q)\}| = |\mathcal{P}(kC_H(Q)\bar{b}_Q)|
\]

and

\[
|\{Q_\delta \in \mathcal{L}_P\mathcal{G}(\mathcal{O}G)B \mid Q_\delta \text{ is associated with } (Q, s(\bar{b}_Q))\}| = |\mathcal{P}(kC_G(Q)s(\bar{b}_Q))|.\]
Also if \( Q_\gamma \in \mathcal{L} \mathcal{P} \mathcal{G}((\mathcal{O} \mathcal{H})b) \) and \( Q_\gamma \) is associated with \((Q, \overline{b}_Q)\) and \( j \in \gamma \), then

\[
Br_Q(j) \overline{b}_Q = Br_Q(j) = Br_Q(j) \overline{b}_Q s(\overline{b}_Q) = Br_Q(j) s(\overline{b}_Q).
\]

Thus \( i_*(Q_\gamma) \in \mathcal{L} \mathcal{P} \mathcal{G}((\mathcal{O} \mathcal{G})B) \) and \( i_*(Q_\gamma) \) is associated with \((Q, s(\overline{b}_Q))\). The desired conclusion now follows from Theorem 1.6 (e).

Let \( (P, \overline{b}_P), (P, s(\overline{b}_P)) \) and \( b_P \) be as in (f). Note that \( \text{Ind}^G_E : \text{Irr}_K(\overline{b}_P) \to \text{Irr}_K(s(\overline{b}_P)) \) is a bijection by (a). Then Proposition 1.2 and Theorem 1.6 yield (f).

Let \( (D, \overline{b}_D) \) and \( (P, b_P) \) be as in (g). Then Theorem 1.6 (b) and [3, Proposition 1.2] imply that \( \text{Ind}^{C_G(P)}_{C_H(P)}(\ast) : \mathcal{R}_K(C_H(P), b_P) \to \mathcal{R}_K(C_G(P), s(b_P)) \) is a perfect isometry that induces the linear map

\[
\text{Ind}^{C_G(P)}_{C_H(P)}(\ast)_p : \mathcal{C} F_{\mathcal{F}'}(C_H(P), b_P, K) \to \mathcal{C} F_{\mathcal{F}'}(C_G(P), s(b_P), K).
\]

Let \( u \in D \), set \( P = \langle u \rangle \) and let \( \psi \in \text{Irr}_K(b) \). Then, by Lemma 2.1,

\[
\text{Ind}^{C_G(P)}_{C_H(P)}(d^H(u, b_P)(\psi)) = \sum_{\phi \in \text{Irr}_K(b_P)} d_u(\psi, \phi)(\text{Ind}^{C_G(P)}_{C_H(P)}(\phi))
\]

and

\[
d_G^{(d, s(b_P))}(\text{Ind}_H^G(\psi)) = \sum_{\phi \in \text{Irr}_K(b_P)} (d_u(\text{Ind}_H^G(\psi), \text{Ind}^{C_G(P)}_{C_H(P)}(\phi))) \text{Ind}^{C_G(P)}_{C_H(P)}(\phi).
\]

The desired conclusion now follows from [11, Theorem 1 (iv)]. An alternate proof can be obtained from [10, Theorem 43.4]. Indeed, let \( \phi \in \text{Irr}_K(b_P) \) and let \( \gamma \in \mathcal{L} \mathcal{P}((\mathcal{O} \mathcal{H})b)^\mathcal{F} \) correspond as in Remark 2.2. It is easy to see that \( \text{Ind}^{C_G(P)}_{C_H(P)}(\phi) \in \text{Irr}_K(s(b_P)) \) corresponds \( i(\gamma) \in \mathcal{L} \mathcal{P}((\mathcal{O} \mathcal{G})B)^\mathcal{F} \). Let \( j \in \gamma \). Here Proposition 1.2 (d) implies that \( \text{Ind}_H^G(\psi)(u j) = \psi(u j) \) and the desired conclusion follows from Remark 2.2.

\[\square\]

References


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