Classifications and Basis Enumerations in Many-Valued Logic Algebras

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Abstract

Let P_k be the set of k-valued logical functions. The functions in a closed subset F of P_k may be classified by their membership in the maximal subsets of F. This also divides all its bases into finite equivalence classes. This thesis presents classifications and basis enumerations in the following cases: various functional constructions in P_2, the set P_3 and its several maximal sets, the set P_{k2} of functions which map cartesian power of k-element set $\{0, 1, \ldots, k-1\}$ into the two values $\{0, 1\}$, and its 4 out of all 5 families of maximal sets.

The formulas for the numbers of n-ary Sheffer functions, functions Sheffer with constants, symmetric Sheffer functions, and symmetric functions Sheffer with constants, in various functional constructions of P_2, are given. The formulas for the number of bases consisting solely of n-ary symmetric functions in each of the constructions are also given.

Applications of a subset generating algorithm to efficient base enumeration, knapsack and minimal covering problems are also described.

Contents

Introduction

In the synthesis of large and complicated electronic instruments such as computers, a small number of basic primitives are used to compose logic networks in the instruments. These basic primitives should be, in general, able to compose an arbitrary network. For example, the NAND primitive is commonly used as one of such primitives. Let us see an example. A network $f(g(x,y,z),y,h(y,z))$ is composed of three-primitives f,g and h and has three inputs x,y and z. Note that neither delay nor synchronization is considered in this example, and no feed-back connection is allowed (a circuit with this restriction is called a combinatorial circuit). A set of basic primitives which can compose any logical network is called a *complete set* (or a *base*) of logical functions. There is a variety of compositions depending on the methods of constructing a network from gates, or on restrictions imposed by real circuit requirements. Accordingly, there are many notions for complete sets.

Recently, the concept of many-valued logic has been found to be useful in many areas, such as diagnosis of multiprocessor systems [But86], software (e.g. decision tables [Miy85b]), pattern recognition [Mic77], signal processing [LiR77], and optoelectronics [Hur86].

Main expectation in practice for many-valued logic in contrast to two-valued logic exists in its information density achievable without increasing the size or complexity of devices. It is well-known that one of the crucial problems in increasing information density in VLSI is the "pin" and "line" limitations associated with it (i.e. too large numbers of pins and lines to be arranged in a limited area). Many-valued logic allows each input pin to accept and each output pin to deliver more information, thereby making the total number of pins required in an integrated circuit chip much less than the case of binary elements. This eventually extends the line limitation, because the line density can be kept less in VLSI. A serious effort is being done for developing optical three-valued devices [WIS86a]. Optical devices are advantageous since they can avoid "interconnection delay" limitation; as the lines become very thin, their resistance increases and the propagation of voltage becomes delayed, so that this eventually limits the speed of VLSI [GLK84].

The synthesis problem of network can be divided into three major problems. The first one is to find an efficient criterion, *completeness criterion*, to determine whether a given set of functions is complete or not. The second one is to enumerate all bases. Finally, the third one is to investigate an optimum construction of a network from a given base. This thesis is mainly devoted to the second problem, and especially, we are interested in many-valued cases. To be precise, we treat two-valued, three-valued, and some of general k-valued cases. The enumeration of bases is useful when one needs to select an appropriate base. Such a situation often arises when by a specific device some logical functions are difficult to implement while others are easy. The selection of a simple, reliable and economic base implementable by physical devices is a fundamental problem in the construction of networks.

Historically, the completeness problem about Boolean functions was first studied. Although several complete systems were known earlier, a general and most natural criterion is expressed in terms of so-called *precomplete* or *maximal* sets. Such completeness criterion was given first by Post in [Pos21], which has been rediscovered many times, cf. [Jab52,INN63]. As the first step toward many-valued logic, Jablonskij gave the completeness criterion for 3-valued logic in [Jab58]. For general k-valued logic, it was given by Rosenberg [Ros65]. The criterion consists of a list of all maximal sets. Let P_k be the set of all k-valued logical functions. There are 5 and 18 maximal sets in P_2 and P_3, respectively, and 6 families of them in P_k. Some other studies of the completeness problem can be seen in [Mal76,Ros77,Pok79,Lau84b].

Further in [Jab52] Jablonskij showed a straightforward method for classifying the whole functions of P_2 into nonempty equivalence classes in order to determine all its bases (nonredundant complete sets): one has to investigate the intersections of the partitions by H_i and $P_2 \setminus H_i$, where H_i ($1 \leq i \leq m$) are P_2-maximal sets. This also divides all its bases into finite equivalence classes. This was done independently in [INN63] and [Krn65]. It is shown that P_2 is divided into 15 classes [Jab52], and there are 42 classes of bases in P_2 [INN63]. It is also shown that the maximal number of elements of a base of P_2 is 4 [Jab52]. The above classification is valid provided the considered set has a finite base.

Thus the classification and the basis enumeration became the second step of the functional completeness theory following the completeness criterion. However, this is often not so simple because of three reasons. Firstly, the number m of the maximal sets is usually rather large. The possible classes are only $2^5 = 32$ for $m = 5$ (P_2 case), while for $m = 18$ (P_3 case) it is $2^{18} = 262144$, and the number m grows very rapidly when k increases. The second reason is that the descriptions of maximal sets are usually not easy to handle. Owing to the development of many-valued logic algebras we can now describe most maximal sets in terms of *relations* which the functions in the maximal sets "preserve". However, the relations are often complex. Lastly, the enumeration of bases is equivalent to the minimum cover problem, a famous NP-complete problem, which makes the enumeration extremely difficult in some cases. One has to invent an efficient algorithm to make the enumeration feasible. We have developed an efficient algorithm, but even with it, the enumeration of bases which involves about 600 classes required about 17 hours by FACOM M380 computer (about 16 MIPS).

The first step for the classification and base enumeration of P_3 was done by the author in [Miy71] and [Miy79], respectively. There are 406 classes of functions and 6,239,721 classes of bases of P_3 (the original classification counted some classes twice; this was corrected in [Sto84a]). The author showed in [Miy79] that the maximal number of elements of a base of P_3 is 6, which answered the long-standing problem posed early in [Jab58] about the bases of P_3. Since there exists an incomplete nonredundant set with 7 elements, the maximal number of elements of a base of P_3 had been conjectured to be greater than or equal to 7. The above answer disproved the conjecture (this result is confirmed later by another method, not through enumeration, in [Vuk84]).

Recently, Machida [Mac79], Lau [Lau82b] and others determined all submaximal sets of P_3. The author [Miy82,Miy83,Miy84] and Stojmenović [Sto86a,Sto86b] determined their classes and bases (this was jointly reported in [MiS87a]). There are few classification results about closed sets in P_k. The set L of linear functions for the case k prime number is classified in [Sto86c]. The set of functions P_{k2} which maps k-values $\{0, 1, ..., k-1\}^n$ to the two values $\{0, 1\}$ and its maximal sets were classified jointly by the author and Stojmenović.

The present thesis describes the classifications and basis enumerations done by the author. We now give a detailed description for each chapter (we also indicate the papers where the given results were reported).

In Chapter 1 we give basic definitions. From Chapter 2 through Chapter 4 we treat Boolean cases. We consider 7 different kinds of functional constructions in P_2: ordinary composition, 2-line fixed coding construction, r-line coding construction, uniform composition, its Ibuki variation, its Inagaki variation, and sequential circuit construction.

In Chapter 2 we give classes of functions and classes of bases of Boolean functions under each of these functional constructions [MIS85]. In Chapter 3 we give the formulas for the numbers of bases consisting solely of symmetric n-ary functions (so called *s-bases*) for each construction. And in Chapter 3 we give formulas for the numbers of Sheffer, symmetric Sheffer, "Sheffer with constants", and "symmetric Sheffer with constants" functions of n-ary functions [MSH87].

In Chapter 5 we show that the set P_3 of three-valued logical functions is divided into 406 classes and that the number of its classes of bases is 6,239,721. We also show that, despite the existence of noncomplete independent sets with 7 elements, the maximal number of functions of a base of P_3 is 6. We also give some example of bases and nonredundant incomplete sets [Miy71,Miy79].

In Chapter 6 we present classes and bases for several maximal sets of P_3: T, L, S [Miy83], B [Miy82], and T_0 [Miy84] (also cf. [MiS87a]).

In Chapter 7 we show that the problem of base enumeration is equivalent to the minimal cover problem (an NP-complete problem). We give an algorithm which enumerates all bases in lexicographic order. We demonstrate its efficiency on some examples of real data. We also show that our base enumeration algorithm is applicable with slight modifications to minimal covering and knapsack problems [Miy85a,StM86a,StM87].

In Chapter 8 we present classifications of $P_{k,2}$. We show that the number of classes is $13A_k - 11A_{k-1}$, where A_k is the number of equivalence relations on the set of k elements. The maximal rank of $P_{k,2}$ is proved to be $k+2$ [MiS87b].

In Chapter 9 we present classifications of 4 families of maximal sets out of all its 5 families, namely Z_{it}, T_0', L' and S'. We also prove that their maximal ranks are $2k-2, k+1, k+1$ and less than $2k$, respectively [MSL87]. We also give the numerical data of the numbers of bases and s-bases for $2 \leq k \leq 10$.

In Chapter 10 we state several open problems. All the above mentioned results about classifications and basis enumerations are also included in the survey paper [MSLR87].

Chapter 1

Definitions and Preliminaries

1.1. Functional completeness problem and classification in P_k

As a motivation we shall consider the following situation arising in the synthesis of switching functions. We have certain basic elements called *gates*. Each gate has one or several *inputs* and a single *output*. The gate receives signals on the inputs and transform them into the output signal. For simplicity's sake we assume that all the input and output signals belong to the same finite set (called *alphabet*) whose elements (called *letters*) are denoted by 0,1,...,$k-1$. Note that it does not matter how the letters are denoted; the first k natural numbers are as convenient as any other symbols. We are to describe synthesis of *networks* constructed from gates by connecting outputs of certain gates to inputs of other gates. Variable x_i is used to denote the signals feeded in the input of a gate (or network).

Let k be a fixed positive integer ($k > 1$), and $E_k := \{0, 1, \dots, k-1\}$ be the set of k integers. An ordered n-tuple of elements from E_k (an element of cartesian product E_k^n) is called a *vector* and denoted by $(a_1, a_2, \dots, a_n)$. We may delete the commas between the coordinates as well as parenthesis of the vector when there is no confusion, i.e. a vector may be represented by $\boldsymbol{a} = a_1 \dots a_n \in E_k^n$. An *$n$-ary k-valued function* f is a map from E_k^n to E_k, i.e. f is a function of n variables ranging in E_k with values in E_k. The functioning of a gate can be described by assigning an output letter $fa_1 \dots a_n$ to every vector $\boldsymbol{a}$=$a_1 \dots a_n$. Thus the gate realizes a function f. The number n of inputs corresponds to the *arity* of the function f. For our purposes the function f completely describes the functioning of the gate. A function f can be represented by a table shown in Table 1.1.

Definition 1.1.1. The set of k-valued logical function of n variables is denoted by $P_k^{(n)}$, i.e.

$$P_k^{(n)} := \{f(x_1, \ldots, x_n) \mid f : E_k^n \to E_k\}.$$

Put $P_k := \bigcup_{n=1}^{\infty} P_k^{(n)}$, the set of k-valued logical functions.

The elements of P_2 (a special case $k = 2$) is called *Boolean functions.*

Two functions f and g ($f, g \in P_k$) is *equal,* in symbol $f = g$, if the arities of both functions are equal (n) and $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$ for all $(a_1, \ldots, a_n) \in E_k^n$.

Definition 1.1.2. $f(\boldsymbol{x})$ *depends* on x_i iff there exist $a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b, c \in E_k, b \neq c$, such that

$$f(a_1, a_2, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \neq f(a_1, a_2, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_n).$$

If f depends on x_i then x_i is said to be an *essential* variable of f. Otherwise it is a *nonessential* (*fictitious* or *dummy*) variable.

Table 1.1.

x_1	x_2	$\ldots$	x_{n-1}	x_n	$f(x_1$	x_2	$\ldots$	x_{n-1}	$x_n)$
0	0	$\ldots$	0	0	$f(0$	0	$\ldots$	0	$0)$
0	0	$\ldots$	0	1	$f(0$	0	$\ldots$	0	$1)$
		$\ldots$					$\ldots$		
0	0	$\ldots$	0	$k-1$	$f(0$	0	$\ldots$	0	$k-1)$
0	0	$\ldots$	1	0	$f(0$	0	$\ldots$	1	$0)$
		$\ldots$					$\ldots$		
a_1	a_2	$\ldots$	a_{n-1}	a_n	$f(a_1$	a_2	$\ldots$	a_{n-1}	$a_n)$
		$\ldots$					$\ldots$		
$k-1$	$k-1$	$\ldots$	$k-1$	$k-1$	$f(k-1$	$k-1$	$\ldots$	$k-1$	$k-1)$

Suppose that we have a collection of gates $\{G_i\}$ realizing functions $f_i \in P_k$. These gates can be combined into *combinatorial switching network* by attaching outputs of certain gates to inputs of certain gates so that the resulting network has a single output and no feedback is created. This means that the single output of the network defines a unique $f \in P_k$ of inputs of the network which is nothing else than a "composition" of the f_i's. Note that we automatically assume that we are allowed to reorder or identify the inputs. Thus, having a gate $f \in P_k^{(2)}$ we have at our disposal the gates realizing both $g \in P_k^{(2)}$ and $h \in P_k^{(1)}$ defined by $ga_1a_2 := fa_2a_1$ and $ha_1 := fa_1a_1$ for every $a_1, a_2 \in E_k$.

The above composition of functions needs a more precisely definition. Operations over P_k means (1) renaming variables of a function (especially, this includes permuting variables and equating variables) and (2) substituting a function into an argument (variable) of a function. This can be defined more formally by introducing the following elementary operations over P_k (represented in basic universal algebra terminology after Mal'cev [Mal76]).

Definition 1.1.3. The three unary operations $\zeta, \tau, \triangle, \nabla$ and a binary operation $*$ we define by the following equations. Let $f \in P_k^{(n)}$ and $g \in P_k^{(m)}$. Then $\zeta f \in P_k^{(n)}, \tau f \in P_k^{(n)}, \triangle f \in P_k^{(max(n-1,1))}, \nabla f \in P_k^{(n+1)}$ and $f * g \in P_k^{(n+m-1)}$:

$$\tau f = \zeta f = \triangle f = f \ for\ n = 1$$

$$\begin{aligned}
(\zeta f)(x_1 \dots x_n) &:= f(x_2 \dots x_n x_1), \\
(\tau f)(x_1 \dots x_n) &:= f(x_2 x_1 \dots x_n), \\
(\triangle f)(x_1 \dots x_{n-1}) &:= f(x_1 x_1 \dots x_{n-1}), \\
(\nabla f)(x_1 \dots x_{n+1}) &:= f(x_2 x_3 \dots x_{n+1}), \\
(f * g)(x_1 \dots x_{n+m-1}) &:= f(g(x_1 \dots x_m) x_{m+1} \dots x_{n+m-1}),
\end{aligned}$$

for every $x_1, \dots, x_{n+m-1} \in E_k$.

The algebra $< P_k; \tau, \zeta, \triangle, \nabla, * >$ is called *iterative algebra.* A function h is called a *superposition* over a set F of functions if it is obtained from the elements of F by applying the above operations $\zeta, \tau, \triangle, \nabla$ and $*$ finite times. Note that the operation ∇ serves to introduce new variables as well as to identify two functions which are different only in fictitious (nonessential) variables.

Example 1.1.1. A composition $h(x_1, x_2) := f(x_1, g(x_1, x_2))$ can be represented by the following elementary operations to $f(x_1, x_2)$ and $g(x_1, x_2)$; $h(x_1, x_2) = \triangle(\zeta(((\tau f) * g)(x_1, x_2, x_3)))$. Indeed, $f_1(x_1, x_2) := \tau f(x_1, x_2) = f(x_2, x_1)$; $h_1(x_1, x_2, x_3) := (f_1 * g)(x_1, x_2, x_3) = f_1(g(x_1, x_2), x_3) = f(x_3, g(x_1, x_2))$; $h_2(x_1, x_2, x_3) := \tau h_1(x_1, x_2, x_3) := h_1(x_2, x_3, x_1) = f(x_1, g(x_2, x_3))$. Finally, $h(x_1, x_2) := \triangle h_2(x_1, x_1, x_2) = f(x_1, g(x_1, x_2))$. □

Definition 1.1.4. A subset of P_k is said to be *closed* if it contains all superpositions of its members. For $F \subseteq P_k$ we define its *closure* $[F]$ as the least set which is generated by superpositions from F.

Thus $F \subseteq P_k$ is closed if $F = [F]$.

Additionally, we introduce the following simple n-ary operation e_i^n $(1 \leq i \leq n)$ called *projections* which are defined by $e_i^n(x_1, \ldots, x_n) = x_i$ (i-th coordinate) for every $\boldsymbol{x} \in E_k^n$. Thus e_1^1 is the identity map on E_k. Let $E := \{e_i^n | 1 \leq i \leq n, n = 1, 2, \ldots\}$ be the set of all projections. Usually all the projections are also allowed as a basic operation of "composition" since projection functions are directly obtained from the inputs of network in practice. A closed set containing the set of projection is called *clone* in the terminology of universal algebra. Most of the closed sets treated in this thesis are clones.

Definition 1.1.5. For closed sets F and H such that $F \subset H$ (proper inclusion), F is *H-maximal* if there is no closed set G such that $F \subset G \subset H$.

Equivalently, a subset F is H-maximal if and only if $[F \cup \{f\}] = H$ for every $f \in H \setminus F$.

Definition 1.1.6. A subset $F \subseteq H$ is *complete* in H if H is the least closed set containing F.

Again, equivalently, a subset F is H-complete if and only if $[F] = H$.

In the sequel we always assume that H has the following property: each proper closed subset of H extends to an H-maximal set, i.e. for each proper closed subset there is an H-maximal set containing it (this property need not hold in general, in fact there is an example of such P_8-maximal set [Mik86,Tar86]). Then, it is known that then there are finitely many H-maximal sets, say $H_1, \ldots, H_m$. The following theorem due to Kuznecov is well-known [Jab58].

Theorem 1.1.1. (Completeness theorem in a general form) [Jab58] Suppose the number m of H-maximal sets is finite. Then a subset of functions in H is complete in H if and only if it is contained in no H-maximal set.

This theorem reduces the completeness problem to giving all maximal sets. Investigations of completeness and related topics, usually called *the functional completeness* *problems*, are mathematically important, and have a wide range of applications including their direct relationship to logical circuit design.

Example 1.1.2. Let T_i be the set of functions such that $f(i) = i$ for $i = 0, 1$, S be the set of self-dual functions, L be the set of linear functions and M be the set of monotone functions in P_2 (see Example 2.1 below for a more detailed description). The five sets T_0, T_1, L, S, M are all the P_2-maximal sets: a subset F is P_2-complete if and only if F is not contained in each of the five sets. □

Definition 1.1.7. An H-complete set F is a *base* of H if no proper subset of F is complete in H.

Note that F is a base of H if and only if 1) F is H-complete, i.e. $[F] = H$ and 2) F is not redundant, i.e. $[F \setminus f] \neq H$ for every $f \in F$. The *rank* of a base is the number of its elements.

Example 1.1.3. In view of the disjunctive normal form expansion of Boolean functions, the set $\{AND, OR, NOT\}$ is P_2-complete but is not a base. It is well-known that $\{AND, NOT\}$ and $\{OR, NOT\}$ are bases. □

Definition 1.1.8. A function f is *Sheffer* for H if $\{f\}$ is a base (of rank 1) of H.

A function f is Sheffer for H if and only if every $g \in H$ is a composition of a finite number of copies of f. Clearly f is Sheffer for H if and only if it belongs to no H-maximal sets. Typical examples of Boolean two-variable functions that are Sheffer for P_2 are the Sheffer (or better Nicode's) strokes NAND and NOR of the algebra of logic. A Sheffer stroke describes the "operation" of a two-input one-output gate (or element) G such that every Boolean function $f(x_1, \ldots, x_n)$ may be represented by the output of a combinatorial (i.e. feedback-free) network with inputs $x_1, \ldots, x_n$ and built solely from copies of G (however, the number of the gates needed for the representation may be large).

A comprehensive survey on Sheffer functions can be found in [Ros77]. A variation of the definition of completeness is the concept of "*complete with constants*", abbreviated *c-complete*, which assumes that for composition besides f one can freely utilize constant-valued functions. More precisely, let Q denote the set of unary constant functions from

H. A subset F of H is c-complete in H if $F \cup Q$ is complete in H. This makes sense in real combinatorial circuits, since the constant-valued functions (i.e. constant signals) are usually obtained with no extra cost. In particular, f is *c-Sheffer* for H means $\{f\}$ is c-complete in H.

Classification of P_k [Jab52,INN63,Krn64,Miy71]

There is a straightforward method for enumerating all H-bases. The functions from H may be classified by their membership in the H-maximal sets. Let $H_1, \ldots, H_m$ be the H-maximal sets. As mentioned above, a subset F of H is complete in H if and only if for each $1 \leq i \leq m$ there is $f_i \in F \cap (H \setminus H_i)$ (the f_i's need not be distinct). This leads to the following:

Definition 1.1.9. Define the map $\varphi : H \to \{0,1\}^m$ by setting $\varphi(f) := a_1 \ldots a_m$ where $a_i = 0$ if $f \in H_i$ and $a_i = 1$ if $f \notin H_i$ (here $a_1 \ldots a_m$ stands for the more customary $(a_1, \ldots, a_m)$ or $< a_1, \ldots, a_m >$). We call $\varphi(f)$ the *characteristic vector* of f. We put $f \equiv g$ if $f, g \in H$ have the same characteristic vector, i.e. if $\varphi(f) = \varphi(g)$.

Clearly $\equiv$ is an equivalence relation on H (it is the standard kernel of φ) and so it partitions H into pairwise disjoint nonempty sets called *(equivalence) classes*. Note that for $f \equiv g$ we have either $f, g \in H_i$ or $f, g \notin H_i$ for all $i = 1, \ldots, m$. We write AB for $A \cap B$, A^1 for A and A^0 for $H \setminus A$ (A, B subsets of H). Clearly each class is of the form $H_1^{a_1} \ldots H_m^{a_m}$ where $(1 - a_1) \ldots (1 - a_m)$ is a characteristic vector (i.e. it is a non-empty set of the form $H_1^{a_1} \ldots H_m^{a_m}$ with $a_1 \ldots a_m \in \{0,1\}^m$).

Example 1.1.4. The set $T_0 \overline{T}_1 L \overline{S} M$ is a P_2-class, which consists only of the n-ary constant functions c_0^n for $n = 1, 2, \ldots$. □

If $f \in F \subseteq H$ and $f \equiv g$, then clearly F is complete (base) in H if and only if $(X \setminus \{f\}) \cup \{g\}$ is complete (base) in H. Thus it suffices to study the completeness in H up to the equivalence $\equiv$. In other words, we can discuss the completeness in H in terms of these classes instead of individual functions. If there are m maximal sets, then the number of possible classes of functions is 2^m, each of which being associated with a unique characteristic vector. However, as we will see throughout this thesis, most of the classes are empty depending on the structure of the set H.

If to $a_1 \dots a_m \in \{0,1\}^m$ we associate $A = \{i : a_i = 1\}$ and if $A_1, \dots, A_l$ are the subsets of $\{1, \dots, m\}$ corresponding to the characteristic vectors, the completeness problem is reduced to the listing of subsets of $\{A_1, \dots, A_l\}$ covering $\{1, \dots, m\}$ and the basis problem to the listing of such coverings which are irredundant (no proper subset covers $\{1, \dots, m\}$).

As we have already seen, a set $F = \{f_1, \dots, f_r\} \subseteq H$ is a base of H if and only if it is complete and nonredundant. It is easy to see that these conditions, respectively, can be represented in terms of characteristic vectors as follows (from Theorem 1.1.1 and Definition 1.1.7):

$$\sum_{f \in F} \varphi(f) = 1 \cdots 1 \text{ (i.e. has all coordinates} = 1), \tag{1.1}$$

$$\sum_{f \in F \setminus f_i} \varphi(f) \neq \sum_{f \in F} \varphi(f) \text{ for all } i = 1, \dots, r, \tag{1.2}$$

where sum is the component-wise logical OR of Boolean m-vectors.

Definition 1.1.10. A set F of functions is *pivotal* if it satisfies the condition (1.2).

A *pivotal incomplete set* is simply called pivotal in case of no confusion.

Once we know all the characteristic vectors of a set, we can find all complete sets, pivotal sets and all bases by a direct combinatorial check (which may be done by a simple computer program, provided m is not large).

For a given set $F \subseteq H$ the *classes* of F is the set of classes of functions belonging to F. All bases and pivotals consisting of the same classes of functions form a *class of bases* (or *aggregate*) and *classes of pivotals*. The enumeration algorithm of all classes of bases and pivotals for moderately large m (the number of all maximal sets for H) and for large number of classes by efficiently checking the above conditions of completeness and nonredundancy (pivotalness) for all combinations of the characteristic vectors will be discussed in chapter 7.

The study of classes also provides information on the closed sets which are the intersections of families of H-maximal sets, which is of independent interest (e.g. for $H = P_3$ with one exception the least nontrivial intersections are all minimal clones [Ros87: private communication]). The characteristic vectors can also be applied to seek the set of classes of functions which makes a given incomplete set complete.

1.2. Functions preserving a relation

For the description of closed sets containing all projections (i.e. clones), we need the following essential concept of "functions preserving a relation" (cf. [Ros77]).

Let $h \geq 1$. An h-ary *relation* ρ on E_k is a subset of E_k^h (i.e. a set of h-tuples over E_k) whose elements are written as column vectors. Given h row n-vectors $\boldsymbol{a}_i = (a_{i1}, \ldots, a_{in})$ $(i = 1, \ldots, h)$ we write $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_h)^T \in \rho^n$ to indicate that $(a_{1j}, \ldots, a_{hj})^T \in \rho$ for all $j = 1, \ldots, n$, where T denotes the transpose (this means that the $h \times n$ matrix with rows $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_h$ has all columns in ρ). We say that an n-ary $f \in P_k$ *preserves* ρ if

$$(f(\boldsymbol{a}_1), \ldots, f(\boldsymbol{a}_h))^T \in \rho \text{ whenever } (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_h)^T \in \rho^n.$$

The set of functions *preserving* ρ is denoted by Pol ρ.

For a special case h=2, we write $\boldsymbol{a}\rho\boldsymbol{b} \Leftrightarrow (a_i, b_i) \in \rho$ for all $1 \leq i \leq n$. Several examples are given below in Theorem 2.1. It is known that each $Pol\ \rho$ is a clone, and conversely that to each clone C there are relations $\rho_1, \rho_2, \ldots$ such that $Pol\ \rho_1 \supseteq Pol\ \rho_2 \supseteq \ldots \supseteq C$ and $C = \bigcap_{i=1}^{\infty} Pol\ \rho_i$. In particular, if H is a clone, then all H-maximal sets are of the form $Pol\ \rho$ for some relation ρ.

Throughout this chapter by $x + y$ and xy we mean $x + y\ (mod\ k)$ and $xy\ (mod\ k)$, respectively. Intersection of sets $X_1, ..., X_r$ will be denoted by $X_1 ... X_r$. Finally, let x^r denote $x \ldots x$ (r times) whenever x is a component of a vector.

Example 1.2.1. The P_2-maximal sets can be represented as follows.

$$\begin{aligned}
T_0 &= Pol(0) = \{f \mid f(0, ..., 0) = 0\} \textit{(set of functions preserving 0)},\\
T_1 &= Pol(1) = \{f \mid f(1, ..., 1) = 1\} \textit{(set of functions preserving 1)},\\
S &= Pol\begin{pmatrix} 01 \\ 10 \end{pmatrix} = \{f \mid f(x_1 + 1, ..., x_n + 1) \neq f(x_1, ..., x_n) \textit{ for each } x_i \in \{0,1\}, 1 \leq i \leq n\}\\
&\quad \textit{(set of selfdual functions)},\\
L &= Pol(\{(a, b, c, d)^T \in E_2^4 \mid a + b = c + d\})\\
&= \{f \mid f(x_1, \ldots, x_n) = a_0 + a_1x_1 + \ldots + a_nx_n \textit{ for some } a_i \in E_2, 0 \leq i \leq n\}\\
&\quad \textit{(set of linear functions)}.\\
M &= Pol\begin{pmatrix} 010 \\ 011 \end{pmatrix} = \{f \mid x_1 \leq y_1 \wedge \ldots \wedge x_n \leq y_n \Rightarrow f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)\}\\
&\quad \textit{(set of monotone non-decreasing functions)},
\end{aligned}$$

1.3. Operations over relations

In the classification we have to use many inclusion relations between functions preserving relations, such as $T_{01}T_{12} \subseteq T_1$ (see §5.1, Chapter 5). The following binary operations over relations provide methods to prove such inclusions by showing directly that the relation on the right may be built from the relations on the left by applying them finite times.

We define the unary relation ζ, τ and binary relations $\circ$ (relational product), $\times$ (cartesian product) and $\cap$ (inclusion) as follows.

$$\rho \circ \rho' = \{(a_1, \ldots, a_{h-1}, a_h, \ldots, a_{h+h'-2} \mid \exists u : (a_1, \ldots, a_{h-1}, u) \in \rho \wedge (u, a_h, \ldots, a_{h+h'-2}) \in \rho'\}$$
$$\rho \times \rho' = \{(a_1, \ldots, a_{h+h'}) \mid (a_1, \ldots, a_h) \in \rho \wedge (a_{h+1}, \ldots, a_{h+h'}) \in \rho'\}$$
$$\rho \cap \rho' = \{(a_1, \ldots, a_h) \mid (a_1, \ldots, a_h) \in \rho \wedge (a_1, \ldots, a_h) \in \rho'\},$$
$$\zeta\rho := \{(a_1, \ldots, a_h) | (a_2, \ldots, a_h, a_1) \in \rho\},$$
$$\tau\rho := \{(a_1, \ldots, a_h) | (a_2, a_1, \ldots, a_h) \in \rho\}$$

The following lemma holds [Pok79].

Lemma 1.3.1.

$$Pol\rho Pol\rho' \subseteq Pol\rho * \rho'$$

*where * is any of* $\circ$, $\times$ *and* $\cap$ *operations.*

Lemma 1.3.2. *Let the* inverse relation *of* ρ *be* $\rho^{-1} = \{(a_h, \ldots, a_1) | (a_1, \ldots, a_h) \in \rho\}$. *Then* $Pol\rho = Pol\rho^{-1}$.

We also note that permuting and duplicating columns of a relation does not change the set of functions preserving it, i.e. $Pol\rho = Pol\rho'$, where ρ' is a permuted columns of the relation ρ.

In addition to these operations, we also use a more general operation, which produce a relation from a given set of relations.

Definition 1.3.1. [Ros70] Let $\mathbf{C} = [c_{ij}]$ be an $m \times h$ matrix with elements from E_{mh} $(h, m, p \geq 1)$. An *m-ary operation over relations* $O^p_{\mathbf{C}}(\rho_0, \ldots, \rho_{m-1})$ is a map, which associate any h-ary $\rho_0, \ldots, \rho_{m-1}$ on E_k the following p-ary relation σ on E_k:

$$(a_0, \ldots, a_{p-1}) \in \sigma \Leftrightarrow \exists a_p, a_{p+1}, \ldots, a_{mh-1}$$

such that, for all $i = 0, \ldots, m-1$, $(a_{c_{i0}}, \ldots, a_{c_{i,h-1}}) \in \rho_i$.

Example 1.3.1. Let $h = m = p = 2$ and

$$\mathbf{C} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}.$$

Then $\sigma = O^2_{\mathbf{C}}(\rho_0, \rho_1)$ is a binary relation σ on E_k: $(a_0, a_1) \in \sigma \Leftrightarrow \exists a_2$ such that $(a_0, a_2) \in \rho_0, (a_2, a_1) \in \rho_1$. Thus $O^2_{\mathbf{C}}(\rho_0, \rho_1) = \rho_0 \circ \rho_1$ (a relational product). An intersection of relations can be expressed as an operation over the relations.

Theorem 1.3.1. [Ros70] *Let $\rho_0, \ldots, \rho_{m-1}$ be h-ary relations on E_k, $\sigma = O^p_{\mathbf{C}}(\rho_0, \ldots, \rho_{m-1})$ be an operation over the relations. Then*

$$\bigcap_{i=0}^{h-1} Pol\rho_i \subseteq Pol\sigma.$$

1.4. Homomorphism and similarity

Definition 1.4.1. Let $A, B \subset P_k$ be closed sets. A and B is *homomorphic* if there exists a mapping $\alpha : A \to B$ $(f \to f^\alpha)$ satisfying

$$\begin{array}{ll} (\zeta f)^\alpha = \zeta f^\alpha, & (\tau f)^\alpha = \tau f^\alpha, \\ (\triangle f)^\alpha = \triangle f^\alpha, & (\nabla f)^\alpha = \nabla f^\alpha \quad \textit{and} \end{array}$$

$$(f * g)^\alpha = f^\alpha * g^\alpha.$$

If the mapping α is one to one, then A and B is *isomorphic* (in symbol $A \cong B$).

Definition 1.4.2. Let S_k be the *permutation group* (*symmetric group*) over E_k and let $\sigma = \begin{pmatrix} 0 \ldots k-1 \\ a_1 \ldots a_{k-1} \end{pmatrix} \in S_k$. Let ϵ be the identity permutation. We write permuted value by σ as $a_i = \sigma i$. Define the product of permutations by $\alpha\beta(x) := \alpha(\beta(x))$ for each $\alpha, \beta \in S_k$. For $f \in P_k$ and $\sigma \in P_k$ we define a *similar* function of f by $f^\sigma := g(a_1 \ldots a_n) := \sigma^{(-1)} f(\sigma a_1 \ldots a_n)$ for any $\boldsymbol{a} \in E^n_k$. For a set A, its *σ-similar* is defined by

$$A^\sigma := \{f^\sigma | f \in A\}.$$

The mapping σ-similar is one-to-one mapping, since S_k is a group. An iteration of σ-similar transformations is represented by a product of permutations as follows.

$$(f^{\sigma_1})^{\sigma_2}(\boldsymbol{a}) = f^{\sigma_1\sigma_2}(\boldsymbol{a}) = (\sigma_1\sigma_2)^{-1} f(\sigma_1(\sigma_2 \boldsymbol{a})).$$

The set of σ-transformations for $\sigma \in S_k$ with the iteration operations is a group which is isomorphic to S_k over P_k. Hence, properties of permutation group S_k are preserved by σ-similar transformations. In general, $f^{\alpha\beta} \neq f^{\beta\alpha}$ since the symmetric group is not commutative when $k > 3$.

Lemma 1.4.1. [Jab58]

$$F \cong F^\sigma \textit{ for } \sigma \in S_k.$$

Corollary 1.4.1.

$$A \subseteq B \Rightarrow A^\sigma \subseteq B^\sigma, \quad (A \cup B)^\sigma = A^\sigma \cup B^\sigma,$$
$$(AB)^\sigma = A^\sigma B^\sigma, \quad (A \setminus B)^\sigma = A^\sigma \setminus B^\sigma,$$
$$(\overline{A})^\sigma = \overline{A^\sigma} \textit{ where } \overline{A} := P_k \setminus A.$$

Thus, when an inclusion relation holds, for example, $AB \subseteq C$, then its dual $A^\sigma B^\sigma \subseteq C^\sigma$ holds for each $\sigma \in S_k$. The latter inclusion relation is a *σ-similar* of the former. The notion of σ-similar is used also for proof procedures. It is an extension of the notion of "dual" in the usual Boolean logic.

Corollary 1.4.2. *The following properties of sets of P_k are preserved by σ-similar:*

1) closed, 2) maximal, 3) complete and 4) base.

Corollary 1.4.3.

$$f \in M_{i_1} \dots M_{i_j} \overline{M}_{i_{j+1}} \dots \overline{M}_{i_m} \Leftrightarrow f^\sigma \in M^\sigma_{i_1} \dots M^\sigma_{i_j} \overline{M}^\sigma_{i_{j+1}} \dots \overline{M}^\sigma_{i_m},$$

where M_{i_j}, $1 \leq j \leq m$ are maximal sets of P_k and $\overline{M}_{i_j}$ is the complement of M_{i_j}.

Thus σ-transformation induces an automorphism of the sets of all classes. This means that if the class χ_i exists then the class χ_i^σ exists for each $\sigma \in S_k$. However, χ_i and χ_i^σ coincide when χ_i is invariant under σ-similar. Corollary 1.4.3 greatly reduces the search of possible classes.

The next lemma provides a method to find a corresponding σ-similar set when a given set is characterized by a relation.

Lemma 1.4.2. [Miy71] $(Pol\rho)^\sigma = Pol\sigma^{-1}\rho$, *where* $\sigma^{-1}\rho = \{(\sigma^{-1}a_1, \dots, \sigma^{-1}a_h) | (a_1, \dots, a_h) \in \rho\}$.

Corollary 1.4.4. *Let $\rho = R_\sigma = \{(0, \sigma 0), \ldots, (k-1, \sigma(k-1)\}$ be an induced binary relation by a permutation $\sigma \in S_k$. Then $(PolR_\sigma)^\sigma = PolR_{\sigma^{-1}}$. Hence, if $\sigma^2 = \varepsilon$, i.e $\sigma = \sigma^{-1}$, $PolR_\sigma$ is σ-invariant.*

Especially, we note that for $k = 3$, $\sigma_0 = (12), \sigma_1 = (02)$ and $\sigma_2 = (01)$ are idempotent, where (ij) denote the transposition of i and j.

Example 1.4.1. Assume $k = 3$ and $\sigma_3 = \begin{pmatrix} 012 \\ 120 \end{pmatrix}, \sigma_4 = \begin{pmatrix} 012 \\ 201 \end{pmatrix}$. Let $\rho := R_{\sigma_3} = \{(0,1),(1,2),(2,0)\}$. The set of functions $S = Pol\rho$ is a maximal set of P_3. We have $\sigma_3^{-1}\rho = \rho$, $\sigma_4^{-1}\rho = \rho$, and $\sigma_i^{-1} = \sigma_i$ for $i = 0,1,2$. Hence from Lemma 1.4.2 and Corollary 1.4.4 S is σ-invariant for any $\sigma \in S_3$.

Example 1.4.2. Let $\rho := \{0,1\}$ be a unary relation and $T_{01} = Pol(01)$. Since $\sigma_2^{-1}\rho = \rho$, $T_{01}^{\sigma_2} = T_{01}$, i.e. T_{01} is σ_2-invariant. While $\sigma_1\rho = \{2,1\}$. Hence $T_{01}^{\sigma_1} = T_{12}$, where T_{12} is the set of functions preserving a unary relation $\{1,2\}$.

Chapter 2

Functional Constructions and their Bases in P_2

The notion of completeness of a set of logical functions depends on the construction method of a network from a given set of logical primitives. The delay caused by functioning of gates which we ignored in the previous definitions also poses restrictions on the composition of functions and on the logical function the network is intended to realize. Besides ordinary composition, we consider six ways of various functional constructions in this chapter. Our purpose is to present classes of functions and classes of bases (aggregates) for each of these constructions. Throughout Chapters 2 through 4 we consider in the set of all Boolean functions P_2.

2.1. Introduction

We are given certain basic elements (primitives) called gates which are realizations of certain logical functions. These gates can be combined into a switching circuit called network. For each network we distinguish *inputs* and an *output* (if necessary, *primary inputs* and *primary output* will be used to distinguish from those of the gates). Thus the network can be represented by $f(x_1, \ldots, x_n)$, which defines output $y = f(x_1, \ldots, x_n)$ as a function of the primary input $x_1, \ldots, x_n$.

We briefly describe seven different ways of the construction of networks arising in practical switching circuit designs, giving classes of bases for each of them.

In the next section we give short preliminaries for some subsets of Boolean functions P_2 to be used in the completeness criteria described in the later sections. In Section 2.3 we summarize classical Post completeness. In Section 2.4 we treat completeness under r-line coding, in Section 2.5 completeness under 2-line fixed coding (both with primitives without delay), in Section 2.6 three completeness under composition with unit delay primitives (uniform composition and its 2 modifications), and in Section 2.7 sequential circuit completeness (with unit delay primitives).

2.2. Preliminaries on subsets of Boolean functions

For a set F we denote the number of its elements by $|F|$. $|F(n)|$ denotes the number of n-ary functions contained in F. We denote the complement set of F by $\overline{F}$, i.e. $\overline{F} = P_2 \setminus F$. Let c_0^n and c_1^n be the constant-valued functions of n-variables assuming the values 0 and 1, respectively. The set of constant functions which takes 0 (1) for arities $n = 1, 2, \ldots$ we denote simply by 0 (1).

We give definitions of several subsets of P_2 which we use for the classifications of P_2 [MSH87].

1) Functions preserving zero.

$$T_0 = \{f | f(0, ..., 0) = 0\},$$
$$|T_0(n)| = 2^{2^n - 1}.$$

2) Functions preserving one.

$$T_1 = \{f | f(1, ..., 1) = 1\},$$
$$|T_1(n)| = 2^{2^n - 1}.$$

3) Monotone increasing functions.

$$M = \{f | f(x_1, ..., x_n) \leq f(y_1, ..., y_n) \text{ if } x_i \leq y_i \text{ for all } i\}.$$
$$|M(n)| = \Psi(n).$$

4) Selfdual functions.

$$S = \{f | \overline{f(x_1, ..., x_n)} = f(\overline{x_1}, ..., \overline{x_n})\},$$
$$|S(n)| = 2^{2^{n-1}}.$$

5) Linear functions.

$$L = \{f | f(x_1, ..., x_n) = a_0 + a_1 x_1 + ... + a_n x_n \text{ for some } a_i \in E\}.$$
$$|L(n)| = 2^{n+1}.$$

6) Conjunctions.

$$C = \{0, 1\} \cup \{x_{i_1} ... x_{i_l}\},$$

$|C(n)| = 2^n + 1.$

7) Disjunctions.

$D = \{0,1\} \cup \{x_{i_1} \vee ... \vee x_{i_l}\},$

$|D(n)| = 2^n + 1.$

8) Notbut-like functions.

$N_0 = \{f|$ if $f(x_1, ..., x_n) = f(y_1, ..., y_n) = 1$ then $x_i = y_i = 1$ for some $i\}.$

$|N_0(n)| = \Theta(n).$

9) If-like functions.

$N_1 = \{f|$ if $f(x_1, ..., x_n) = f(y_1, ..., y_n) = 0$ then $x_i = y_i = 0$ for some $i\}.$

$|N_1(n)| = \Theta(n).$

10) Functions exchanging zero and one.

$X = \{f | f(x, ..., x) = \overline{x}\},$

$|X(n)| = 2^{2^n - 2}.$

11) Monotone decreasing functions.

$M' = \{f | f(x_1, ..., x_n) \geq f(y_1, ..., y_n)$ if $x_i \leq y_i$ for all $i\}.$

$|M'(n)| = \Psi(n).$

12) Functions uniting zero and one.

$K = \{f | f(0, ..., 0) = f(1, ..., 1)\}.$

$|K(n)| = 2^{2^n - 1}.$

Note 2.2.1. We give a representation of the sets by relations. $T_0 = Pol(0)$, $T_1 = Pol(1)$, $M = Pol\begin{pmatrix} 010 \\ 011 \end{pmatrix}$, $S = Pol\begin{pmatrix} 01 \\ 10 \end{pmatrix}$, $L = Pol\{(a,b,c,d) | a+b = c+d \pmod 2\}$, $N_0 = Pol\begin{pmatrix} 001 \\ 010 \end{pmatrix}$, $N_1 = Pol\begin{pmatrix} 101 \\ 110 \end{pmatrix}$. The functions $\Psi(n)$ and $\Theta(n)$ we explain in Section 4.1.

We list several useful inclusion relations for the classification. We omit the proofs.

Lemma 2.2.1. $M(n) \cap L(n) = L(n) \cap C(n) = L(n) \cap D(n) = C(n) \cap D(n)$ $= L(n) \cap C(n) \cap D(n) = \{c_0^n, c_1^n, p_i^n\}.$

$L(n) \cap M'(n) = \{c_0^n, c_1^n, \overline{p_i^n}\}$, $M(n) \cap M'(n) = \{c_0^n, c_1^n\}.$

Lemma 2.2.2. $S \subseteq N_0 N_1 \cup \overline{N_0}\,\overline{N_1}$, $N_0 N_1 \subseteq S.$

Lemma 2.2.3. $SL = \{x+1$ *(only for* $n = 1$*)*, $a + x_1 + \ldots + x_{2m+1}, a \in \{0,1\}$, m=$1,2,\ldots,\}$,

$LN_0 = \{0, x_i\}$, $LN_1 = \{1, x_i\}$, $SN_0 \subseteq M$ *and* $SM \subseteq N_0$.

Lemma 2.2.4. $L(1)M(1) = \{0, 1, x_1\}$, $L(1)M'(1) = \{0, 1, x_1 + 1\}$.

Also we note that

n-ary linear functions (except constants) are selfdual for n odd
(no selfdual function exists for n even).

2.3. Bases under ordinary composition

The first completeness is one under ordinary composition which we defined in Chapter 1. The composition is defined as an operation of either renaming variables of a function (permuting variables and equating variables) or substituting a function into an argument of a function. One can construct a new function from a given set of primitives applying the composition any finite times. Additionally one is allowed to use any projection functions p_i^n in the construction.

The following Post's theorem on the P_2-completeness under this composition and the classification of Boolean functions are most fundamental facts. This is well-known.

Theorem 2.3.1. [Pos21] *P_2 has exactly the following 5 maximal sets: T_0, T_1, L, S, M.*

Theorem 2.3.2. [Jab52] *There are 15 classes of functions of P_2.*

We presents them by their characteristic vectors in Table 2.1. Components of characteristic vectors are given in the order T_0, T_1, S, L, M of P_2-maximal sets. For instance, class 6 represents the set $\overline{T}_0 T_1 \overline{S} L \overline{M}$, where $\overline{X}$ denotes $P_2 \setminus X$. The class 9 (10) consists only of the constant functions 1 (0), and the class 15 only of the set of all projection functions $p_i^n(x_1, \ldots, x_n) = x_i$, $i = 1, 2, \ldots, n$, $n = 1, 2, \ldots$, which is often denoted simply by x.

Theorem 2.3.3. [INN63] *There are 42 classes of bases of P_2.*

Table 2.1: P_2-classes under ordinary composition.

1. 11111	2. 11011	3. 01111	4. 10111	5. 11001
6. 10101	7. 01101	8. 00111	9. 10100	10. 01100
11. 00110	12. 00011	13. 00010	14. 00001	15. 00000

1	*class of rank 1:*	(1),
17	*classes of rank 2:*	$2 \times \{3,4,6,7,8,9,10,11\}$,
		$3 \times \{4,5,6,9\}, 4 \times \{5,7,10\}, 5 \times \{8,11\}$,
22	*classes of rank 3:*	$5 \times \{6,7,9,10\} \times \{12,13\}, \{6 \times \{7,10\}, (9,7)\} \times \{8,11,12,13\}$,
		$(9,10) \times \{8,12\}$,
2	*classes of rank 4:*	$(9,10,14) \times \{11,13\}$.

Note that there are only four classes of bases containing constant functions: (8,9,10), (9,10,12), (9,10,11,13) and (9,10,13,14).

The number of n-ary functions included in each of the 15 class are given in [Krn65] (for some classes it is given in terms of $\Psi(n)$: the number of monotone Boolean functions). There are 51 pivotals (13, 31 and 7 with ranks 1,2 and 3, respectively).

2.4. Bases under r-line coding

Freivalds [Fre68] introduced the notion of completeness under r-line coding (which he called up to coding completeness). In this construction every input and output of the outermost network consists of "r-lines" and signals 0 or 1 are feeded to each input or taken out from the output as a length r binary code. While internally these input lines are treated as usual binary input. So in the internal networks every composition is done according to ordinary composition. In Fig. 2.1 we show examples of networks of **AND** and **NAND** constructed with AND and OR primitives with the coding $0 \to 01$ and $1 \to 10$. Note that in this coding negation of the outermost network is realized simply by exchanging the output lines, so if f is realizable then its negation is also realizable in this composition.

Assume a coding

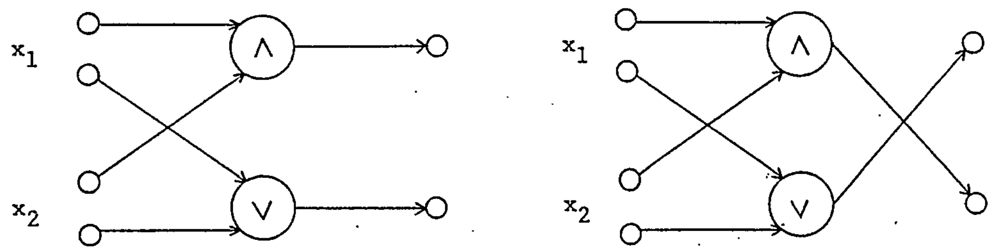

Figure 2.1: **AND** and **NAND** in double-line logic

$$0 \rightarrow \alpha_{01} \dots \alpha_{0r},$$

$$1 \rightarrow \alpha_{11} \dots \alpha_{1r},$$

where $\alpha_{ij} \in \{0,1\}$, $0 \leq i \leq 1$, $1 \leq j \leq r$.

We shall say that a network compute $f(x_1, \dots, x_n)$ with the coding, if , to each argument x_i there is associated the r inputs a_{ij} ($j = 1, \dots, r$), the network has r output b_l ($l = 1, \dots, r$) and operates as follows: for the computation of $f(m_1, \dots, m_n)$ one feed in signals $\alpha_{m_i j}$ (0 or 1) at input line a_{ij}, $1 \leq i \leq n, 1 \leq j \leq n$ and the network produces as output b_l the results $\beta_l = \alpha_{f(m_1,\dots,m_n)l}$, $1 \leq l \leq r$. We shall say that $F \subseteq P_2$ is complete under a fixed coding if every $f \in P_2$ is computable with the coding by a network on F. We say that a set of function is *complete under fixed r-line coding* if every function is computable by some network of r-lines under this coding using the functions in the set. A set of functions is *complete under r-line coding* (in original term, complete up to coding) if for every function there exists an r-line coding of 0 and 1 (depending on the function) under which the function is realizable by the functions in the sets.

Theorem 2.4.1. [Fre68] *A set of function is complete under r-line coding if and only if it is not included in each of the three sets: L, C and D.*

We note that the original presentation of the above theorem is not quite correct (the sets C and D are correct to include the constant functions, while in the original description they are excluded from the sets C and D, cf. [MSH87]).

Theorem 2.4.2. *There exists exactly 5 classes of functions under r-line coding completeness.*

Proof. We have $LD \subseteq C, LC \subseteq D$ and $CD \subseteq L$, i.e. $LCD(n) = \{0, 1, x_i, 1 \leq i \leq n\}$ (Lemma 2.2.1). The classes are shown in Table 2.2. □

Table 2.2: Classes of functions under r-line coding completeness

class	L C D	representatives (symmetric)
1.	0 0 0	$0, 1, x$
2.	0 1 1	$a + x_1 + \ldots + x_n, a = 0$ or 1, for $n > 1; 1 + x$ for $n = 1$
3.	1 0 1	$x_1 \ldots x_n, n > 1$
4.	1 1 0	$x_1 \vee x_2 \ldots \vee x_n, n > 1$
5.	1 1 1	all remaining symmetric functions, e.g. $\overline{x}_1\overline{x}_2$

Theorem 2.4.3. *There are 4 classes of bases: rank 1: (5); rank 2 : (2,3),(2,4),(3,4). There are 3 classes of pivotals: rank 1: (2),(3) and (4).*

Example 2.4.1. We give all bases for 2-ary functions under r-line coding:

$$\{x + y(+1), xy\},\ \{x + y(+1), x \vee y\},\ \{xy, x \vee y\},\ \{NAND(x, y)\},\ \{NOR(x, y)\}.$$

The following bases include a unary function $\overline{x}$: $\{\overline{x}, xy\}$, $\{\overline{x}, x \vee y\}$. As we show in Fig. 2.1, $AND(x, y)$ can be composed of $\{x \vee y, xy\}$ under the coding $0 \to 01, 1 \to 10$.

2.5. Bases under 2-line fixed coding

The completeness problem under a fixed coding $0 \to 01$ and $1 \to 10$ (this is so called double rail logic [Neu56]) was solved by Ibuki [Ibu68]. Karunanithi and Friedman [KaF78] also considered this completeness independently, and gave the condition which are stated in somewhat complex terms but equivalent to the following. This notion coincides with SP-algebra described in [Gin85]. The classification is done by Ibuki [Ibu68].

Theorem 2.5.1. *A set of functions is complete under 2-line fixed coding if and only if it is not contained in each of the following 6 sets: N_0, N_1, S, L, C and D.*

Theorem 2.5.2. [Ibu68] *There are 12 classes of functions, 28 classes of bases (1 for rank 1, 22 for rank 2 and 5 for rank 3) and 20 classes of pivotals (10 classes for each of ranks 1,2).*

The characteristic vectors of these classes are given in Section 3.5.

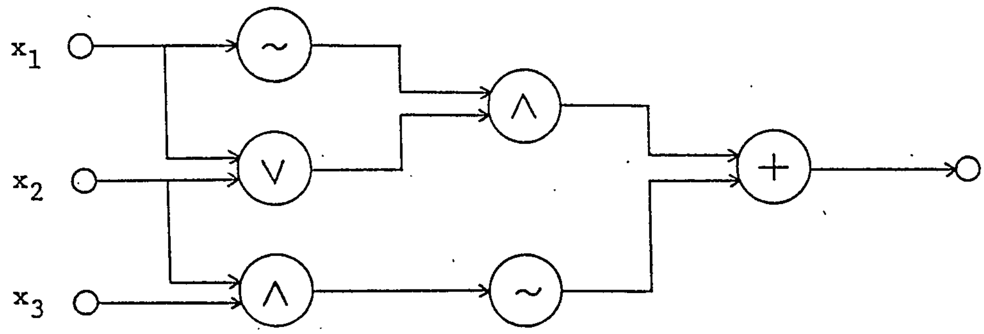

Figure 2.2: uniform composition

2.6. Bases under compositions with delayed functions

Usually a gate needs some duration of time to give an output. So it is natural to assume that each primitive function has certain delay time. In this section we assume that all primitives have uniform delay (a unit time). Taking the delay time into consideration various compositions have been proposed. We consider three constructions proposed by Kudrjavcev, Ibuki and Inagaki, respectively. These are closely related each other.

Uniform composition

The theory of uniform delay composition was initiated by Kudrjavcev [Kud60]. In this construction every composition is to be done so that for each gate the delays along all paths from the primary inputs to the inputs of the gate are equal. This means that the composition should be synchronized. This is imposed even on primitives of constant-valued functions. Projections can be used freely (which can be used in the first layer of the composition as primitives with delay zero). Furthermore in this composition it is assumed that (1) all initial input signals are given only once and simultaneously and (2) no feedback connections are allowed in compositions.

A set $F \subseteq P_2$ is *complete under uniform composition* if one can realize every function in some delay (which depends on the realized function) by a network on F using uniform composition.

For example, the network in Fig. 2.6 is synchronized, but one in Fig. 2.7 is not synchronized and have a feedback connection.

The following theorem is proved in [Kud60], but explicit statement in this form is due to Nozaki [Noz78].

Theorem 2.6.1. [Kud60] *A set of functions is complete under uniform composition if and only if it is not contained in each of the 8 sets:* T_0, T_1, S, L, M, M', X *and* K.

Table 2.3: Classes of functions under uniform delay compositions.

	$T_0T_1SLMM'XK$	representative
1.	1 1 1 1 1 1 0 1	$x_1\overline{x}_2x_3 \vee \overline{x}_1\overline{x}_2\overline{x}_3$
2.	1 1 1 1 1 0 0 1	$\overline{x}_1\overline{x}_2$
3.	1 1 0 1 1 1 0 1	$x_1\overline{x}_2 \vee \overline{x}_2\overline{x}_3 \vee \overline{x}_3x_1$
4.	1 1 0 1 1 0 0 1	$\overline{x}_1\overline{x}_2 \vee \overline{x}_2\overline{x}_3 \vee \overline{x}_3\overline{x}_1$
5.	0 1 1 1 1 1 1 0	$x_1\overline{x}_2$
6.	1 0 1 1 1 1 1 0	$x_1 \vee \overline{x}_2$
7.	1 1 0 0 1 1 0 1	$x_1 + x_2 + \overline{x}_3$
8.	1 1 0 0 1 0 0 1	$\overline{x}_1$
9.	1 0 1 0 1 1 1 0	$x_1 + \overline{x}_2$
10.	0 1 1 0 1 1 1 0	$x_1 + x_2$
11.	0 0 1 1 1 1 1 1	$x_1x_2x_3 + x_1\overline{x}_2\overline{x}_3$
12.	1 0 1 0 0 0 1 0	1
13.	0 1 1 0 0 0 1 0	0
14.	0 0 1 1 0 1 1 1	x_1x_2
15.	0 0 0 1 1 1 1 1	$x_1x_2 \vee x_2\overline{x}_3 \vee \overline{x}_3x_1$
16.	0 0 0 1 0 1 1 1	$x_1x_2 \vee x_2x_3 \vee x_3x_1$
17.	0 0 0 0 1 1 1 1	$x_1 + x_2 + x_3$
18.	0 0 0 0 0 1 1 1	x_1

Theorem 2.6.2. *There are 18 classes under uniform delay composition and they coincide with those under Ibuki's (Inagaki's) composition.*

Proof. Characteristic vector for these classes has 8 coordinates, which is constructed by adding K coordinate to Ibuki's coordinates. We have $K = T_0\overline{T}_1 \cup \overline{T}_0T_1$ (disjoint), because $f \in K \Leftrightarrow$ either $f(0) = f(1) = 0$ or $f(0) = f(1) = 1$. Therefore the values for the coordinate K is determined by those for T_0 and T_1. □

In Table 2.3 we give the classes and their representatives. Symmetric representatives for the classes 1,3,5,6,11 and 15 we mention in Section 3.6.

Theorem 2.6.3. *There are exactly 118 classes of bases and 115 classes of pivotals under uniform delay compositions. They are given below.*

Note that no Sheffer class exists in our case as well as in Ibuki's one.

Ibuki composition

Ibuki [Ibu68] defined a slightly different composition independently, and gave all 7 maximal set, which coincide with above sets except K. The only difference of this

Table 2.4: Classes of bases under uniform delay compositions.

rank 1 (0):	none;
rank 2 (44):	{1,2,3,4} × {5,6,9,10,11}, {1,2} × {15,16,17,18}, {1,3} × {12,13}, {1,2,3,4} × 14, {5,6,11,14} × {7,8},
rank 3 (72):	(2,7) × {12,13}, 4 × {12,13} × {7,15,16,17,18}, {5 × {6,9,12},(6,10)} × {11,14,15,16,17,18}, (6,11) × {13,14,15,16,17,18}, {7,8} × {9,10,12,13} × {15,16}, {9 × {10,13},(10,12)} × {11,14,15,16}, (12,13) × {11,15},
rank 4 (2):	(12,13,17) × {14,16}.

Table 2.5: Classes of pivotals under uniform delay compositions.

rank 1 (18):	(1) - (18);
rank 2 (79):	(2,3),{2,4} × {7,12,13}, {3,4,5,6,7,8,9,10,12,13} × {15,16,17,18}, {5,6} × {11,14}, {6,9,12} × {5,10,13}, {7,8,11,14} × {9,10,12,13}, 14 × {15,17}, (16,17)
rank 3 (18):	{8×{12,13},{9,12}×{10,13}}× {17,18}, {{12,13}×17,(12,13)}× {14,16}.

construction from the uniform composition consists in that the constant valued function with delay *zero* can be freely used. Thus, for example, a composition $f(x, c_0^1(y))$ is allowed.

Inagaki composition

Yet another modification was done by Inagaki, who gave 6 maximal sets which coincides with above sets except X and K. He weakened Ibuki's construction in the following points: the input paths of the constant valued functions may have non-uniform delays. He showed an example of such a realization of a constant valued function using $NAND$ primitives. Feedback loops are still prohibited. However, it is necessary to feed input signals in some span of time in order to obtain stable output; thus, for example, feeding oscillating signals like 0101... to inputs are prohibited.

It turns out that the uniform construction is the most restrictive construction among the three constructions. That is, if f is complete under uniform composition, then it is complete in the other constructions. Their classifications are closely related to ours.

Classes and bases of Ibuki and Inagaki compositions

Classes of functions in these cases are the same 18 classes as in the former case

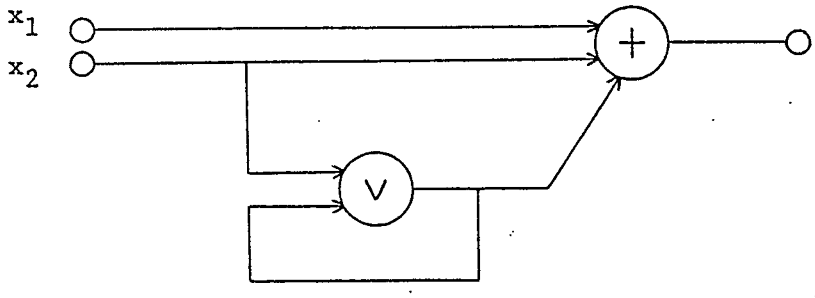

Figure 2.3: Sequential circuit composition

[Ibu68,Ina82]. The last component and the last two components must be eliminated (K and X, K are not maximal sets respectively in these cases).

Although the classes of uniform delay case coincides with those under Ibuki's and Inagaki's case, the bases and pivotals are different due to the extra coordinate. There are 93 classes of bases (49, 42 and 2 with ranks 2,3 and 4 respectively) [Ibu68], and 88 pivotals (18, 58 and 12 with ranks 1, 2 and 3 respectively). There are 82 classes of bases in Inagaki case (1, 39, 40, and 2 with ranks 1, 2, 3 and 4) [Ina82], and 77 pivotals (17, 48 and 12 with ranks 1, 2 and 3 respectively). Only in Inagaki case there exist Sheffer class.

2.7. Bases under sequential circuit composition

A composition allowing loops by using unit delay primitives is considered by Nozaki [Noz82]. He introduced the notion of s-completeness (s for sequential circuit). In Fig. 2.7 we show an example of the network. Note that we don't require uniform delay any more. We briefly explain the construction. Assume that in our network there are m primitives whose output is denoted by u_i $(1 \leq i \leq m)$ and n primary inputs denoted by $x_1, \ldots, x_n$. The output of the first primitive u_1 is assumed to be the primary output of the network. Now output of a primitive is determined by the previous states (outputs) of all the primitives as well as primary inputs. Thus the output of the primitive u_i after unit delay (denoted by u_i^*) is expressed by

$$\begin{aligned} u_1^* &= D_1(u_1, \ldots, u_m, x_1, \ldots, x_n), \\ &\ldots \\ u_m^* &= D_m(u_1, \ldots, u_m, x_1, \ldots, x_n). \end{aligned}$$

For example, in Fig. 2.7 we have $u_1^* = add(x_1, x_2, u_2)$ and $u_2^* = or(x_2, u_2)$. Let $q = \{0,1\}^m$ and $y = \{0,1\}^n$ correspond to the sets of *states* of the primitives and *inputs* of the network respectively. Then the network is described by a function

$$D : Q \times Y \to Q \tag{2.1}$$

and the first element of Q is the output of the network. For example, in Fig. 2.7 $D((1,0),(1,0)) = (1,0)$. The state transition of D under feeding $\boldsymbol{x}(1)\ldots\boldsymbol{x}(t)$ to an initial state $\boldsymbol{s}(1)$ is determined successively by $\boldsymbol{s}(2) = D(\boldsymbol{s}(1), \boldsymbol{x}(1))$, $\boldsymbol{s}(3) = D(\boldsymbol{s}(2), \boldsymbol{x}(2)), \ldots$, $\boldsymbol{s}(t+1) = D(\boldsymbol{s}(t), \boldsymbol{x}(t))$. The last state $\boldsymbol{s}(t+1)$ is denoted by $D^*(\boldsymbol{s}(1), \alpha)$ and called *final state* corresponding to the input sequence $\alpha = \boldsymbol{x}(1)\ldots.\boldsymbol{x}(t)$, and the first component of $\boldsymbol{s}(t+1)$ is *final output* denoted by $D^{final}(\boldsymbol{s}(1), \alpha)$. The notion of realization of function f by a network D is defied as follows.

(1) There exists an initial state $\mathbf{s(1)}$ called *good state* such that there exist some delay D such that the output $y(t)$ of the network at time t is the function value corresponding to the inputs at time $t-d$, i.e. $y(t) = f(\boldsymbol{x}(t-d))$.

(2) For any state $\boldsymbol{s}$ there is an input sequence α called an *initialize sequence* such that $D^*(\boldsymbol{s}, \alpha)$ is a good state.

In Fig. 2.7 D realizes x+y+1 with initialize sequence (0,1) or (1,0) with delay 1.

We denote the set of all functions realizable with some delay by a network on F by $[F]_s$. Now F is called *s-complete* if $[F]_s = P_2$.

Theorem 2.7.1. [Noz82] *There are exactly 6 maximal sets under s-completeness. They are N_0, N_1, S, L, M and M'.*

From this and the completeness criteria for Ibuki composition, we have that if F is complete under Ibuki construction, then it is s-complete.

Theorem 2.7.2. *There are exactly 16 classes of functions under s-completeness. They are indicated in Table 2.6.*

Proof. To have these classifications we use classes with respect to T_0, T_1, S, L, M and M' given in Table 2.3. Since $N_0 \subset T_0$ and $N_1 \subset T_1$, for example, the case T_0T_1 splits into

Table 2.6: Classes of functions under s-completeness.

	N_0N_1SLMM'	symmetric functions
1.	1 1 1 1 1 1	$x_1\overline{x}_2 \vee \overline{x}_1x_2 \vee \overline{x}_3$,
2.	1 1 1 1 1 0	$\overline{x}_1\overline{x}_2$,
3.	1 1 1 0 1 1	$x_1 + x_2$,
4.	1 1 0 1 1 1	$x_1\overline{x}_2 \vee \overline{x}_1\overline{x}_3 \vee \overline{x}_3x_1$,
5.	1 0 1 1 1 1	$x_1 \vee \overline{x}_2$,
6.	0 1 1 1 1 1	$x_1\overline{x}_2$,
7.	1 1 0 1 1 0	$\overline{x}_1\overline{x}_2 \vee \overline{x}_2\overline{x}_3 \vee \overline{x}_3\overline{x}_1$,
8.	1 1 0 0 1 1	$x_1 + x_2 + x_3$,
9.	1 0 1 1 0 1	$x_1 \vee x_2$,
10.	0 1 1 1 0 1	x_1x_2,
11.	1 1 0 0 1 0	$1 + x$
12.	1 0 1 0 0 0	1
13.	0 1 1 0 0 0	0
14.	0 0 0 1 0 1	$x_1x_2 \vee x_2x_3 \vee x_3x_1$,
15.	0 0 0 0 0 1	x_1
16.	1 1 1 1 0 1	$x_1x_1 \vee x_3x_4$

the four cases: $\overline{N}_0\overline{N}_1, \overline{N}_0N_1, N_0\overline{N}_1$ and N_0N_1 (the other three cases are similar). Thus it suffices to check each of these classes for each class in Table 2.3. We briefly give how the above classes are derived from Table 2.3, Chapter 2. Let the class number in Table 2.3 be denoted by prefixing # before the number, e.g. #15 is the class 000111 in the order of T_0T_1SLMM' coordinates. The classes 1,2,4,5,6,7,8,11,12 and 13 were derived from #1,#2,#3,#5,#6,#4,#7,#8,#12 and #13, respectively. The class 3 comes from #9,#10 and #11 jointly. Class #17 give also the class 8 (we have two possibility: $\overline{N}_0\overline{N}_1$ and N_0N_1 from Lemma 2.2.2 but the first case gives the class 8 and the second does not occur from $N_0N_1 \subseteq S$ of the same Lemma). The class #15 gives also only the class 4 (the other cases do not occur from Lemma 2.2.2). Finally the class #14 gives three classes 9, 10 and 16 because $N_0N_1 \subseteq S$ prohibit $N_0N_1\overline{S}$ case.

Only the class 16 has no symmetric representative (this will be discussed in detail in Chapter 3). □

Theorem 2.7.3. *There are exactly 58 classes of bases and 39 classes of pivotals under s-completeness. They are indicated in Table 2.7 and 2.8 respectively.*

Table 2.7: Classes of bases under s-completeness.

rank 1 (1):	(1);
rank 2 (47):	{2,3}× {4,5,6,9,10,14}, 2×{3,8,15},(3,7), {4,5} × {8,10,13}, (4,5), {4,6} × {9,12}, 5 × {6,7,11},{6,9,10} × {7,8,11}, {2,3,4,5,6,7,8,11}× 16,
rank 3 (10):	{7× {8+{14,15}} × {12,13}, {8,11}×{12,13}× 14.

Table 2.8: Classes of pivotals under s-completeness.

rank 1 (15):	2 - 16;
rank 2 (20):	(7) × {8,12,13,14,15}, (8) × {12,13,14}, (9) × {10,13}, (10,12), (11) × {12,13,14,15}, (12) × {13,14,15}, (13) × {14,15};
rank 3 (4):	(11,15)×{12,13}, (12,13) × {14,15}.

2.8. Concluding remarks

We have described several functional constructions and presented classes of bases for each of them, using the corresponding classification of P_2. They are summarized in Table 2.9. Another modification of the composition can be found in algebra Φ^o proposed by Cejtlin [Cej70]. Classifications and base consideration was done for this case by Tosić [Tos81]. Several other modifications of propositional algebras are considered in [Gin85].

Table 2.9: Maximal sets, classes and bases for the 7 constructions in this chapter.

	maximal sets	classes	bases	min rank	max rank	pivotals
ordinary composition	5	15	42	1	4	51
r-line	3	5	4	1	2	3
2-line fix	6	12	28	1	3	10
uniform composition	8	18	118	2	4	115
Ibuki composition	7	18	93	2	4	88
Inagaki composition	6	18	82	1	4	77
sequential	6	16	58	1	3	39

Chapter 3

Bases Consisting of Symmetric Functions

As an application of the enumeration of classes of bases we give formulas N^n for the number of bases of P_2 consisting solely of n-ary symmetric functions for each functional construction described in Chapter 2.

3.1. Introduction

Usually primitives are selected from symmetric functions in practice; nonsymmetry of the input variables complicates the situation, for example, by involving nonsymmetry of delays. Indeed, almost all bases are symmetric functions in practice. The symmetry of functions simplifies the synthesis of switching functions. Connecting an output of a gate to any input of another gate gives shorter length of geometrical connections and avoids extra intersection of the lines. Both are important issues in VLSI design. Moreover, symmetric functions have algebraic properties which make it desirable to treat them as a separate class. Thus we call bases (pivotals) consisting only of symmetric functions *s-bases* (*s-pivotals*). We show that there exists a symmetric representative in each class under the 7 constructions described in Chapter 2, except one class in sequential completeness. This gives the following theorem.

Theorem 3.1.1. *Classes of bases and classes of s-bases coincide under each of the 6 out of all 7 constructions described in Chapter 2 (the only exception is the sequential circuit construction). In other words there is a base consisting only of symmetric functions for each class of bases under each construction.*

It is ~~worth~~ mentioning that there are several classes having no symmetric representative in P_3. We are going to give formulas for the exact numbers of n-ary and up to n-ary symmetric functions included in each of the classes. By this we can calculate the formula for N^n and $N^{\leq n}$ (the number of bases consisting solely of up to n-ary symmetric functions). Indeed the number of bases consisting solely of n-ary symmetric functions in a class of bases can be calculated as a product of the numbers of corresponding functions in each class of functions belonging to the class of base. Summing these numbers for all classes of bases for rank i we obtain corresponding data N_i^n for bases of rank i and finally summing them for all ranks we have N^n. Similarly we can calculate $N^{\leq n}$. Our results in this chapter are the number N^n for each construction.

3.2. Preliminaries on subsets of symmetric Boolean functions

A function $f(x_1, \ldots, x_n)$ is said to be *symmetric* if

$$f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

holds for all $x_1, \ldots, x_n \in E_k$ and every permutation π on $\{1, \ldots, n\}$.

A *fundamental* symmetric function s_r^n is determined by the number of its variables n and the number r such that s_r^n takes the value 1 if and only if r of its arguments assume the value 1.

For given n, there exist exactly $n+1$ fundamental symmetric functions: $s_0^n, s_1^n, \ldots, s_n^n$. Each symmetric function can be uniquely represented as a disjunction of the fundamental symmetric functions [Sha49]. Hence the number of n-ary symmetric functions in P_2 is 2^{n+1}. The above property provides a suitable notation for symmetric functions, setting

$$s_{r_1,\ldots,r_l}^n := s_{r_1}^n \vee \ldots \vee s_{r_l}^n \ (n \geq 1).$$

The constants 0 and 1 are symmetric functions which correspond to s_ϕ^n and $s_{0,1,\ldots,n}^n$, respectively. Assume that $0 \leq r_1 < \ldots < r_l \leq n$. Let $R := \{r_1, \ldots, r_l\}$ and $s_R^n := s_{r_1,\ldots,r_l}^n$. Thus $s_R^n(x_1, \ldots, x_n) = 1 \Leftrightarrow x_1 + \ldots + x_n \in R$, where $x_1 + x_2 + \ldots + x_n$ denote the number of 1's in the vector $(x_1, \ldots, x_n)$.

We give representation of symmetric functions for each of the subsets described in Section 2, Chapter 2. The indicated number of symmetric functions is easily obtained from this. The set of symmetric functions from F we denote by F^s. Let c_0^n and c_1^n be the constant-valued functions of n-variables assuming the values 0 and 1, respectively. Let $p_i^n(x_1, ..., x_n) = x_i$ be the *projection* function of n variables that returns the value of the i-th argument; also let $\overline{p}_i^n$ be the function that returns the dual value of the i-th argument.

1) $T_0^s = \{s_R^n | 0 \notin R\}$.
$|T_0^s(n)| = 2^n$.

2) $T_1^s = \{s_R^n | n \in R\}$.
$|T_1^s(n)| = 2^n$.

3) $M^s = \{s_\phi^n, s_n^n, s_{n-1,n}^n, \ldots, s_{1,2,\ldots,n}^n, s_{0,1,\ldots,n}^n\}$.
$|M^s(n)| = n + 2$.

4) $S^s = \{s_R^n | i \in R$ if and only if $n - i \notin R$ for all $i = 0, \ldots, (n-1)/2,\ n$ odd$\}$.
$|S^s(n)| = 2^{(n+1)/2}$ for n odd and 0 for n even [ArH63].

5) $L^s = \{c_0^n, c_1^n, x_1 + \ldots + x_n (= s_{\{1,3,\ldots,n\}}^n$ for n odd and $= s_{\{1,3,\ldots,n-1\}}^n$ for n even$)$,
$1 + x_1 + \ldots + x_n (= s_{\{0,2,\ldots,n-1\}}^n$ for n odd and $= s_{\{0,2,\ldots,n\}}^n$ for n even$)\}$,
$|L^s(n)| = 4$.

6) $C^s = \{c_0^n, c_1^n, s_{\{n\}}^n (= x_1 \ldots x_n)\}$.
$|C^s(n)| = 3$.

7) $D^s = \{c_0^n, c_1^n, s_{1,2,\ldots,n\}}^n (= x_1 \vee \ldots \vee x_n)\}$.
$|D^s(n)| = 3$.

8) $N_0^s = \{s_R^n | 2r_1 > n,$ where r_1 is the smallest in $R\}$.
$|N_0^s(n)| = 2^{n/2}$ for n even and $2^{(n+1)/2}$ for n odd.

9) $N_1^s = \{s_R^n | 2r < n$ where r is the greatest in $\{0, 1, \ldots, n\} \setminus R\}$.
$|N_1^s(n)| = 2^{n/2}$ for n even and $2^{(n+1)/2}$ for n odd.

10) $X^s = \{s_R^n | 0 \in R, n \notin R\}$.
$|X^s(n)| = 2^{n-1}$.

11) $M'^s = \{0, s_0^n, \ldots, s_{0,1,\ldots,n-1}^n, 1\}$.
$|M'^s(n)| = n + 2$.

12) $K^s = \{s_R^n | 0, n \in R$ or $0, n \notin R\}$.
$|K^s(n)| = 2^n$.

Example 3.2.1.

$$S^s(3) = \{s^3_{\{0,1\}}, s^3_{\{0,2\}}, s^3_{\{1,3\}}, s^3_{\{2,3\}}\},$$
$$N^s_0(3) = \{s^3_\phi, s^3_{\{2\}}, s^3_{\{3\}} = \wedge^n_{i=1}x_i, s^3_{\{2,3\}}\},$$
$$N^s_1(3) = \{s^3_{\{2,3\}}, s^3_{\{0,2,3\}}, s^3_{\{1,2,3\}} = \vee^n_{i=1}x_i, s^3_{\{0,1,2,3\}}\}.$$

Table 3.1: Intersections of the subsets of symmetric functions.

	N_1	S	L	M	M'
N_0	$\{x\ (n=1)\}$	$\{x\ (n=1),$ $s^n_{(n+1)/2,\ldots,n}\}$ (n odd)	$\{x\ (n=1), 0\}$	$\{x\ (n=1), 0$ $s^n_{(n+1)/2,\ldots,n}\}$ (n odd)	$\{0\}$
N_1		$\{x\ (n=1),$ $s^n_{0,\ldots,n/2}\}$ (n odd)	$\{x\ (n=1),$ $1\}$	$\{x\ (n=1),$ $1\}$	$\{1,$ $s^n_{0,\ldots,n/2}\}$ (n odd)
S			$\{s^n_{1,3,\ldots,n}, s^n_{0,2,\ldots,n-1}\}$ (n odd)	$\{s^n_{(n+1)/2,\ldots,n}\}$ (n odd)	$\{s^n_{0,\ldots,(n-1)/2}\}$ (n odd)
L				$\{x\ (n=1),$ $0,1\}$	$\{x+1\ (n=1),$ $0,1\}$
M					$\{0,1\}$

In the next lemmas we summarize without proofs several results on the sets of symmetric functions expressed as intersections of the subsets defined previously. These results will be used in the argument in the succeeding sections.

Lemma 3.2.1. $M^s(n) \subseteq N^s_0 \cup N^s_1$.
For $n \geq 2$,

$M^s(n) \cap L^s(n) = L^s(n) \cap C^s(n) = L^s(n) \cap D^s(n) = C^s(n) \cap D^s(n) = M^s(n) \cap M'^s(n) =$
$= L^s(n) \cap M'^s(n) = \{c^n_0, c^n_1\}.$

And

$M'^s(n) \cap N^s_0(n) = L^s(n) \cap N^s_0(n) = c^n_0,$
$M'^s(n) \cap N^s_1(n) = L^s(n) \cap N^s_1(n) = c^n_1.$

Lemma 3.2.2. *For n even* $S^s(n) = \phi$. *For n odd,*

$S^s(n) \cap L^s(n) = \{a + x_1 + \ldots + x_n | a = 0, 1\},$
$S^s(n) \cap M^s(n) = S^s(n) \cap N^s_0(n) = S^s(n) \cap N^s_1(n) = N^s_0(n) \cap N^s_1(n) = N^s_0(n) \cap M^s(n)$
$= N^s_1(n) \cap M^s(n) = S^s(n) \cap M^s(n) \cap N^s_0(n) \cap N^s_1(n)$
$= \{s^n_{(n+1)/2,\ldots,n}\},$

and

$$S^s(n) \cap M'^s(n) = \{s^n_{0,1,\ldots,(n-1)/2}\}.$$

Lemma 3.2.3. $N^s_0(n) \cap C^s(n) = \{0, x_1 \wedge \cdots \wedge x_n\}$, $N^s_1(n) \cap C^s(n) = \{1, x_1 \vee \cdots \vee x_n\}$

In Table 3.1 we summarize the intersections of the sets.

3.3. S-bases under the ordinary composition

In [Tos72] Tosić characterized the n-ary symmetric functions contained in each of the 15 classes under ordinary composition.

Theorem 3.3.1. [Tos72] *The number of n-ary symmetric functions in each class under ordinary composition is given in Table 3.2.*

Table 3.2: Number of n-ary symmetric functions in each class under ordinary composition.

T_0T_1SLM	$n=1$	n even	$n>1$ odd
1. 11111	0	2^{n-1}	$2^{n-1}-2^{(n-1)/2}$
2. 11011	0	0	$2^{(n-1)/2}-1$
3. 01111 (4. 10111)	0	$2^{n-1}-2$	$2^{n-1}-1$
5. 11001	1	0	1
6. 10101 (7. 01101)	0	1	0
8. 00111	0	$2^{n-1}-n$	$2^{n-1}-2^{(n-1)/2}-n+1$
9. 10100 (10. 01100)	1	1	1
11. 00110	0	n	$n-1$
12. 00011	0	0	$2^{(n-1)/2}-2$
13. 00010	0	0	1
14. 00001	0	0	1
15. 00000	1	0	0

We briefly summarize it because our classification uses this. In all expression below we assume $\{r_1, \ldots, r_l\} \subseteq \{1, \ldots, n-1\}$ and $1 \leq l \leq n-1$.

1. (Sheffer class) $s^n_{0,r_1,\ldots,r_l}$, $n > 1$, except the case n odd, $l = (n-1)/2$ and $r_i \in \{i, n-i\}$ for all $i, 1 \leq i \leq (n-1)/2$; NOR function $s^2_0 = \overline{xy}$ and NAND function $s^2_{0,1} = \overline{x} \vee \overline{y}$.
2. (linear class) $s^n_{0,r_1,\ldots,r_l}$ for n odd, $n > 1$ and $r_i \in \{i, n-i\}$ for all i, $1 \leq i \leq (n-1)/2$, except the function $x_1 + \ldots + x_n + 1 = s^n_{0,2,4,\ldots,n-1}$; $s^3_{0,1} = \overline{xy} \vee \overline{yz} \vee \overline{zx}$.
3. (preserving 0 class) $s^n_{r_1,\ldots,r_l}$, $n > 2$, except the constant function 0. For n even the function $x_1 + \ldots + x_n + 1 = s_{0,2,\ldots,n}$ is also excluded; $s^3_{1,2} = \neg(xyz \vee \overline{xyz})$.

4. (preserving 1 class) $s^n_{0,r_1,\dots,r_l,n}$, $n > 2$, except the constant function 1. For n even also the function $1 + x_1 + \dots + x_n = s_{0,2,\dots,n}$ is also excluded; $s^3_{0,3} = (xyz \vee \overline{xyz})$.
5. (linear selfdual class) Only $1 + x_1 + \dots + x_n = s^n_{0,2,4,\dots,n-1}$ for $n > 1$ odd.
6. (linear preserving 1 class) $1+x_1+\dots+x_n s^n_{0,2,4,\dots,n}$ for n even, $n > 1$; $s^2_{0,2} = x+y+1$.
7. (linear preserving 0 class) $x_1 + \dots + x_n = s^n_{1,3,\dots,n-1}$ for n even, $n > 1$; $s^2_1 = x + y$.
8. (preserving constants class) $s^n_{r_1,\dots,r_l,n}$, $n > 3$, except the functions $s^n_{j,j+1,\dots,n}$ for $1 \leq j \leq n$. For n odd, $n > 1$, the functions $s^n_{r_1,\dots,r_l,n}$ are also excluded if they satisfy the selfdual condition $r_i \in \{i, n-i\}$ for all i, $1 \leq i \leq (n-1)/2$; $s^4_{1,4}$.
9. (constant 1 class) Only the constant function 1.
10. (constant 0 class) Only the constant function 0.
11. (monotone preserving constants class) $s^n_{j,j+1,\dots,n}$ for $n > 1$, $1 \leq j \leq n$ and $j \neq (n+1)/2$ if n is odd;$s^2_2 = xy$.
12. (selfdual preserving constants class) $s^n_{r_1,\dots,r_l,n}$ for $r_i \in \{i, n-i\}$, $1 \leq i \leq (n-1)/2$ and n odd, $n > 4$. The functions $s^n_{(n+1)/2,\dots,n}$ and $x_1+\dots+x_n = s_{1,3,\dots,n}$ are excluded; $s^5_{1,5}$.
13. (monotone selfdual class) $s^n_{(n+1)/2,\dots,n}$ for n odd, $n > 1$; $s^3_{2,3} = xy \vee yz \vee zx$.
14. (linear selfdual class) $x_1 + \dots + x_n = s_{1,3,\dots,n}$ for n odd, $n > 1$; $s^3_{1,3} = x + y + z$.
15. (identity class) Only the function $f(x) = x = s^1_1$.

The number of s-bases of P_2 consisting of n-ary ($n > 1$) functions is $N(n) = 2^n + 4^{n-1} - n - 4$ if n is even and $N(n) = 2^{(n-1)/2} + 4^{n-1} + 3 \cdot 8^{(n-1)/2} + 2^{n-1} - 6$ otherwise [Tos72]. The formulas for $N(\leq)$ are also given there.

3.4. S-bases under r-line coding

Theorem 3.4.1. *The numbers of n-ary and up to n-ary symmetric functions in each class under 2-line fixed coding are given in Table 3.3.*

The proof is obvious from Table 2.2, Chapter 2.

Theorem 3.4.2 *S-bases consisting of n-ary functions ($n \geq 2$) is: rank 1: $N^n_1 = 2^{n+1}-6$ (Sheffer symmetric functions), rank 2: $N^n_2 = 2 \cdot 1 + 2 \cdot 1 + 1 \cdot 1 = 5$. Thus there are*

$$N^n = 2^{n+1} - 1$$

Table 3.3: Number of symmetric functions in each class under r-line coding.

Number of n-ary functions

class	LCD	n= 1	$n>1$
1.	000	3	2
2.	011	1	2
3.	101	0	1
4.	110	0	1
5.	111	0	$2^{n+1}-6$
sum		4	2^{n+1}

Number of up to n-ary functions

class	
1.	$2n+1$
2.	$2n-1$
3.	$n-1$
4.	$n-1$
5.	$2^{n+2}-6n-2$
sum	$2^{n+2}-4$

s-bases under r-line coding. Similarly we have the number of s-bases consisting of up to n-ary functions

$$N^{\leq n} = 2^{n+2} + 5n^2 - 14n + 1.$$

3.5. S-bases under 2-line fixed coding

We give the classes [Ibu68], where the components are in the order of $LDCSN_1$ and N_0.

Table 3.4: Classes under 2-line fixed coding

1. 000110	2. 000101	3. 011011	4. 011111	5. 101101	6. 110110
7. 111000	8. 111011	9. 111101	10. 111110	11. 111111	12. 000000

Theorem 3.5.1. [Sto85] *The number of symmetric functins in the classes under 2-line fixed coding are given in Table 3.5.*

Symmetric representatives in each of the above classes are given in [Sto85]. We explain it briefly, because the original counting is slightly incorrect. It is easy to see that only $0, 1, a+\sum_{i=0}^{n} x_i$ $(n = 2m+1, m \geq 1, a = 0,1; a = 1$ for $n = 1)$, $a+\sum_{i=0}^{n} x_i$ $(n = 2m, m > 0, a = 0,1)$, $\vee_{i=1}^{n} x_i$, $\wedge_{i=1}^{n} x_i$ belong to the first 6 classes, respectively. From Lemma 3.2.2 only $s^n_{(n+1)/2,\ldots,n}$ for n odd and $n \geq 2$ is in the class 7. The class 8 contains the selfdual functions except the intersection with each of the other sets. From Lemma 3.2.2 only the three functions belong to these intersections: $s^n_{(n+1)/2,\ldots,n} \in SMN_0N_1$ and $s_{1,3,\ldots,n} = \sum_{i=1}^{n} x_i$ and $s_{1,3,\ldots,n} = 1 + \sum_{i=1}^{n} x_i$ belong to SL for n odd. The classes 9 and 10 consist of N_1 and N_0, respectively, except the intersection with each of the

Table 3.5: Numbers of n-ary symmetric functions under 2-line fixed coding.

class	$n=1$	$n=2m>1$	$n=2m+1$
1,2	1	1	1
3	1	0	2
4	0	2	0
5,6	0	1	1
7	0	0	1
8	0	0	$2^{(n+1)/2}-3$
9,10	0	$2^{n/2}-2$	$2^{(n+1)/2}-3$
11	0	$2^{n+1}-2^{n/2+1}-2$	$2^{n+1}-3\cdot 2^{(n+1)/2}+2$
12	1	0	0
sum	4	2^{n+1}	2^{n+1}

Table 3.6: Numbers of up to n-ary symmetric functions under 2-line fixed coding.

class	$n=1$	$n>1$
1,2	1	n
3	1	$2[(n-1)/2]+1$
4	0	$2[n/2]$
5,6	0	$n-1$
7	0	$[(n-1)/2]$
8	0	$2^{[(n+3)/2]}-3[(n-1)/2]-4$
9,10	0	$(3+(1+(-1)^n)/2)2^{[(n+1)/2]}-2n-[(n-1)/2]-4$
11	0	$2^{n+2}-2^2(2^{[n/2]}+3\cdot 2^{[(n-1)/2]})+8-(1+(-1)^n)$
12	1	1
sum	4	$2^{n+2}-4$

other sets. From Lemma 3.2.3 only the following functions belong to these intersections: $s_{0,1,\dots,n}=1$, $s^n_{1,2,\dots,n}=\vee^n_{i=1}x_i\in N_1D$; further $s^n_{(n+1)/2,\dots,n}\in N_1N_0MS$ when n odd. The class 10 is similar; $s^n_\phi=0$, $s^n_n=\wedge^n_{i=1}x_i\in N_0C$ and further $s^n_{(n+1)/2,\dots,n}\in N_0N_1MS$ when $n>1$ odd. The class 11 contains all the remaining functions (the Sheffer class to be considered in the next Chapter 4). The class 12 contains only the identity function.

We show the numbers of up to n-ary symmetric functions in each class in Table 3.6 (note that Table 3.5 and 3.6 are corrected slightly: classes 8,9,10 case n odd).

Theorem 3.5.2. *The number of s-bases consisting solely of n-ary functions under 2-line fixed coding is given in Table 3.7.*

Table 3.7: Number of s-bases consisting of n-ary functions under 2-line fixed coding.

	n even	n odd
N^n	$3 \cdot 2^n + 2 \cdot 2^{n/2} - 9$	$2^{n+3} - 9 \cdot 2^{(n+1)/2} + 5$
N_1^n	$2^{n+1} - 2 \cdot 2^{n/2} - 2$	$2^{n+1} - 3 \cdot 2^{(n+1)/2} + 2$
N_2^n	$2^n + 4 \cdot 2^{n/2} - 7$	$3 \cdot 2^{n+1} + 6 \cdot 2^{(n+1)/2} - 4$
N_3^n	0	7

Table 3.8: Number of symmetric functions in each class under uniform compositions.

	$n = 1$	n even	n odd
1,11	0	$2^{n-1} - n$	$2^{n-1} - 2^{(n-1)/2} - n + 1$
2,14	0	n	$n - 1$
3,15	0	0	$2^{(n-1)/2} - 2$
4,7,16,17	0	0	1
5,6	0	$2^{n-1} - 2$	$2^{n-1} - 1$
8,18	1	0	0
9,10	0	1	0
12,13	1	1	1
sum	4	2^{n+1}	2^{n+1}

3.6. S-bases under the uniform composition and its variations

Theorem 3.6.1. *The number of symmetric functions in each class under uniform composition are given in Table 3.8*

Proof. Our classification is a subclassification of P_2-classes under ordinary composition (cf. Table 2.1) described in Section 3, Chapter 2 since 5 sets T_0, T_1, S, L and M are common to both cases. The only difference between the two classifications consists in dividing the classes 1,2 and 5 in Table 3.2 into the classes 1,2; 3,4; 7,8 respectively, so that functions of the set M' belongs to the classes 2,4,8 and functions from $\overline{M}'$ to the classes 1,3,7. Let us use a prefix # to denote the classes in Table 3.2 (also in Section 3, Chapter 2). We divide the classes #1, #2 and #5 by M'.

1. Classification of #1. Case n even. Only the n functions: $s^n_{0,1,\dots,n-1}, s^n_{0,1,\dots,n-2}, \dots, s^n_0$ belong to M'; the remaining belong to $\overline{M}$. Case n odd. Among the functions described in the case n even, only one $s^n_{0,1,\dots,(n-1)/2} \in SM'$ should be deleted from Lemma 3.2.2. Thus we have $n-1$ functions for $f \in M'$; the other $2^{(n-1)/2} - n + 1$ functions belong to $\overline{M}'$.

Table 3.9: The number of up to n-ary functions in each class under uniform delay composition.

Class	Number of up to n-ary symmetric functions
1, 11	$2^n - 2^{[(n+1)/2]} - [n^2/2]$
2, 14	$[n^2/2]$
3,15	$2^{[(n+1)/2]} - 2[(n-1)/2] - 2$
4,7,16,17	$[(n-1)/2]$
5, 6	$2^n - n - 1 - [n/2]$
8,18	1
9,10	$[n/2]$
12,13	n
sum	$2^{n+2} - 4$

Table 3.10: Number of s-bases consisting of n-ary functions under uniform composition.

	n even	n odd
N^n	$2^{3(n-1)} + (n+3)2^{2n-2}$ $-n(n+3)2^{n-1} + 2n^2 - n$	$2^{3(n-1)} + 2^{5(n-1)/2} + (n+1)2^{2n-2} - n \cdot 2^{(3n-1)/2}$ $+(1-n^2)2^{n-1} + (2n+3)2^{(n-1)/2} + n^2 - n - 5$
N_1^n	0	0
N_2^n	$3 \cdot 2^{2n-2} - 2n$	$3 \cdot 2^{2n-2} - 2^{n-1} - 2n - 2$
N_3^n	$2^{3(n-1)} + n \cdot 2^{2n-2}$ $-n(n+3)2^{n-1} + 2n^2 + n$	$2^{3(n-1)} + 2^{5(n-1)/2} + (n-2)2^{2n-2} - n \cdot 2^{(3n-1)/2}$ $+(2-n^2)2^{n-1} + (2n+3)2^{(n-1)/2} + n^2 - 3$
N_4^n	0	n

2. Classification of #2. Only one function $s^n_{0,1,\ldots,(n-1)/2}$ belongs to SM' from Lemma 3.2.2; the other belong to $\overline{M}'$.

3. Classification of #5. For n even no function exists. Consider n odd. Only one function $s^1_0 = x_1 + 1$ belongs to M' for $n = 1$. For n odd > 1 only one function $s^n_{0,2,4,\ldots,n-1} = 1 + x_1 + \cdots x_n$ belongs to $\overline{M}'$. □

In Table 3.9 the number of up to n-ary functions is given for each class which is easily verified from the result in [Tos72] and Table 3.8.

Theorem 3.6.2. *The number of symmetric functions consisitng solely of n-ary function is given in Table 3.10.*

Ibuki and Inagaki constructions

We give the formula for the number of s-bases consisting of solely n-ary symmetric functions for each case in Tables 3.11 and 3.12.

Table 3.11: Number of s-bases consisting of n-ary functions (Ibuki construction).

	n even	n odd
N^n	$2^{2n}+2^{n+1}-3n-4$	$2^{2n}+2\cdot 2^{(n+1)/2}-6$
N_1^n	0	0
N_2^n	$2^{2n}-2n-4$	$2^{2n}-2^{n-1}-2n-3$
N_3^n	$2^{n+1}-n$	$2^{n-1}+2\cdot 2^{(n+1)/2}+n-3$
N_4^n	0	n

Table 3.12: Number of s-bases consisting of n-ary functions (Inagaki construction).

	n even	n odd
N^n	$2^{2n-2}+(3n+5)2^{n-1}-4n-4$	$2^{2n-2}+3\cdot 2^{3(n-1)/2}+(3n-2)2^{n-1}$ $+(n+2)2^{(n-1)/2}-4n-2$
N_1^n	$2^{n-1}-n$	$2^{n-1}-2^{(n-1)/2}-n+1$
N_2^n	$2^{2n-2}+3n\cdot 2^{n-1}-2n-4$	$2^{2n-2}+3\cdot 2^{3(n-1)/2}+(3n-2)2^{n-1}$ $+(n+2)2^{(n-1)/2}-4n-2$
N_3^n	$2^{n+1}-n$	$2^{n-1}-2\cdot 2^{(n+1)/2}-n-1$
N_4^n	0	n

3.7. S-bases under sequential circuit composition

In Table 3.13 we show symmetric functions included in each class of s-completeness.

Lemma 3.7.1. *There is no symmetric representative in the class 16.*

Proof. Assume $f \in M$, i.e. $f = s^n_{m,m+1,\dots,n}$, $0 \le m \le n$ (we exclude the constant $s^n_\phi = 0$ from the consideration). From $f \notin N_0$ we have $2m \le n$ and from $f \notin N_1$ we have $2(m-1) \ge n$. That is, $m \le n/2$ and $m \ge n/2+1$, a contradiction. □

This give the following.

Theorem 3.7.1. *There are exactly 50 classes of s-bases and 38 classes of s-pivotals under s-completeness.*

They are given by deleting the classes of bases and pivotals including the class 16 from those indicated in Tables 2.7 and 2.8, Chapter 2 respectively (we simply delete the last line of rank 2 bases and one pivotal consisting solely of the class 16).

Theorem 3.7.2. *The number of n-ary symmetric functions in each of the 16 classes under sequential completeness are given in Table 3.14.*

Table 3.13: Symmetric functions in the classes of functions under s-completeness.

	N_0N_1SLMM'	symmetric functions
1.	1 1 1 1 1 1	the remaining symmetric functions
2.	1 1 1 1 1 0	$s^n_{0,1,\ldots,m}, m \neq n$. Exclude $m \neq (n-1)/2$ for n odd
3.	1 1 1 0 1 1	$a + x_1 + \ldots + x_{2m}$ $(m \geq 1, a \in \{0,1\})$
4.	1 1 0 1 1 1	$s^n_{r_0,\ldots r_m}$ for n odd > 1: $m = (n-1)/2, r_i \in \{i, n-i\}$ except $a + x_1 \ldots + x_{2m+1}$; $s^n_{(n+1)/2,\ldots,n}$; $s^n_{0,1,\ldots,(n-1)/2}$
5.	1 0 1 1 1 1	$s^n_R, 2r < n$ and $s^n_R \notin M$
6.	0 1 1 1 1 1	$s^n_R, 2r_1 > n$ and $s^n_R \notin M$
7.	1 1 0 1 1 0	$s^n_{0,1,\ldots,(n-1)/2} : n$ odd
8.	1 1 0 0 1 1	$a + x_1 + \ldots + x_{2m+1} (m \geq 1, a \in \{0,1\})$
9.	1 0 1 1 0 1	$s^n_{m,m+1,\ldots,n}; m \leq n/2, m > 0$
10.	0 1 1 1 0 1	$s^n_{m,m+1,\ldots,n}; m > n/2$ n even; $m > (n+1)/2$ n odd
11.	1 1 0 0 1 0	$1 + x$
12.	1 0 1 0 0 0	1
13.	0 1 1 0 0 0	0
14.	0 0 0 1 0 1	$s^n_{(n+1)/2,\ldots,n} : n$ odd
15.	0 0 0 0 0 1	x
16.	1 1 1 1 0 1	ϕ

Proof. We describe symmetric functions contained in in each class (cf. Table 3.13).

The class 1 is Sheffer class described in Section 5, Chapter 3. It is easy to see the classes 3,7,8,11,12,12,13,14 and 15 since they are linear functions and SM and SM' and other special functions. The class 2 consists of monotone decreasing functions except one function SM'; the intersections of the other sets and monotone decreasing functions are constants or the unary function $x + 1$. Class 3,8: $L^s(n) \subseteq (S^s \cup \overline{S}^s)\overline{N_0}^s\overline{N_1}^s\overline{M}^s\overline{M'}^s$. Class 4: we are to exclude SL, SM=SN_1=SN_0 and SM' from S. Class 5,6: we consider class 6 (the class 5 is similar). From $N_0S \subseteq N_0M$, $N_0L \subseteq N_0M$ and $N_0M' \subseteq N_0M$ the class 6 equals to $N_0 \setminus N_0M$. We have $s_{m,m+1,\ldots,n} \in N_0 \Leftrightarrow m > n/2$, i.e. $m \geq n/2 + 1$ for n even and $m \geq (n+1)/2$ for n odd. Thus $|N_0^sM^s(n)| = n + 2 - (n/2 + 1) = n/2 + 1$ for n even and $= n + 2 - ((n-1)/2 + 1) = (n+3)/2$ for n odd. Finally, class 9,10: From $f \in M\overline{M}'$ we have $f = s^n_{m,m+1,\ldots,n}, m > 0$ (if $m = 0$ then $f \in MM'$). These do not belong to L for $n > 1$. We have $f \in N_0 \Leftrightarrow m > n/2$ and $f \in N_1 \Leftrightarrow m < n/2 + 1$. Consider the class 9. Then $m \leq n/2$ and $m < n/2 + 1$. For n even this means $m \leq n/2$ and there are all $n/2$ such functions. For n odd this means $m \leq (n-1)/2$ and there are all $(n-1)/2$ such functions. None of functions in both cases belong to S. Class 10

Table 3.14: Number of n-ary symmetric functions in each class under sequential completeness.

$class\backslash$	$n=1$	n even	n odd >1
1	0	$2^{n+1}-2^{n/2+1}-n-2$	$2^{n+1}-3\cdot 2^{(n+1)/2}-n+3$
2	0	n	$n-1$
3	0	2	0
4	0	0	$2^{(n+1)/2}-4$
5,6	0	$2^{n/2}-n/2-1$	$2^{(n+1)/2}-(n+1)/2-1$
7,14	0	0	1
8	0	0	2
9, 10	0	$n/2$	$(n-1)/2$
11,15	1	0	0
12,13	1	1	1
sum	4	2^{n+1}	2^{n+1}

Table 3.15: Numbers of up to n-ary symmetric functions in each class.

$class\backslash$	$n\geq 1$
1	$2^{n+2}-(9+(-1)^n)2^{[(n+1)/2]}-[n^2/2]+2n-4[n/2]+6$
2	$[n^2/2]$
3	$2[n/2]$
4	$2^{[(n-1)/2]+2}-4[(n-1)/2]-4$
5,6	$(7+(-1)^n)2^{[(n-1)/2]}-(1/2)[n^2/2]-n-[(n-1)/2]-5$
7,14	$[(n-1)/2]$
8	$2[(n-1)/2]$
9, 10	$[n^2/2]/2$
11,15	1
12,13	n
sum	$2^{n+2}-4$

is similar. □

The number of up to n-ary symmetric functions in each class is given in Table 3.15.

Theorem 3.7.3. *The number N^n of bases consisting solely of n-ary symmetric functions under sequential completeness is given in Table 3.16.*

3.8. Concluding remarks

We have given the numbers of symmetric functions in each class for each construction described in Chapter 2. By this we have given formulas for the number of bases consisting solely of n-ary functions. The numerical data for the small numbers of n are given

Table 3.16: Number of s-bases consisting of n-ary functions under sequential completeness.

	n even	n odd
N^n	$3\cdot 2^n+(n+1)2^{n/2+1}$ $-n^2/4-2n-7$	$3\cdot 2^{n+1}+(7n-9)2^{(n-1)/2}$ $-n^2/4-17n/2+3/4$
N_1^n	$2^{n+1}-2^{n/2+1}-n-2$	$2^{n+1}-3\cdot 2^{(n+1)/2}-n+3$
N_2^n	$2^n+(n+2)2^{n/2+1}$ $-n^2/4-n-5$	$2^{n+2}+(7n-3)2^{(n-1)/2}$ $-n^2/4-15n/2-49/4$
N_3^n	0	10

Table 3.17: Numbers of bases consisting solely of n-ary symmetric functions.

n	2	3	4	5	6	7	8	9	10
ordinary composition	2	36	72	446	1,078	5,634	16,628	77,834	263,154
r-line	7	15	31	63	127	255	511	1,023	2,047
2-line fix	7	33	47	189	199	885	791	3,813	3,127
uniform composition	14	99	764	5,699	40,322	317,613	2,266,232	18,387,347	137,559,230
Ibuki composition	14	66	272	1,034	4,202	16,410	66,020	262,202	1,050,590
Inagaki composition	14	64	180	662	1,732	6,890	20,060	84,362	280,020
sequential	12	45	69	248	276	1,017	1,017	3,840	3,724

in Table 3.17. By the given data for the number of up to n-ary functions contained in each class we can calculate the formula for the number of bases consisting of up to n-ary functions.

Chapter 4

Sheffer and Symmetric Sheffer functions in P_2

In this chapter we give the four formulas for the numbers of Sheffer functions, Sheffer with constant functions, Sheffer symmetric functions and Sheffer symmetric with constant functions under each functional construction which we have seen in the previous chapters.

4.1. Introduction

A Sheffer Boolean function is a well-known notion which means that it can produce by itself all Boolean functions through composition. A typical example of such function is the NAND operation. A variation of the notion of Sheffer functions is that of Sheffer with constants (in this chapter abbreviated to *c-Sheffer*), which assumes that one can freely utilize constant-valued functions (0 and 1). This is a more suitable assumption in real circuit design, since the constant-valued functions are usually obtained with no extra cost. A comprehensive survey on the topic of completeness can be found in [Ros77].

We show the formulas for the number of n-variable Sheffer functions, for the four cases: Sheffer, c-Sheffer, symmetric Sheffer, symmetric c-Sheffer. Here some previous results by other researchers are included in order to achieve completeness of the presentation. The derivations for the formulas are always by the so-called inclusion exclusion principle (cf. [Vil71]) using the inclusion relations of the sets we have seen in lemmas 2.2.1–2.2.4, Chapter 2 and 3.2.1–3.2.3, Chapter 3 freely. Thus the proofs are not described in detail.

The subsets of Boolean functions which we have seen in the previous chapters are used in describing the conditions for Shefferness. From Section 2 through Section 8 we present the explicit formulas for Sheffer and symmetric Sheffer functions. Finally in Section 9 tables are shown which exhibit the calculated numbers of Sheffer functions in each case.

We must note that some cases still remain unsolved because we don't know the formula for the numbers of the two subsets of Boolean functions. An explicit formula for the number $\Psi(n)$ of monotone increasing Boolean functions is not known (the Dedekind problem), but a good asymptotic formula has been obtained [Kor81] (see also [Hro85,Kle69]). The first few values of the function are shown in Table 4.1. Also we could not find an explicit formula for the function $\Theta(n)$ (only the shape of the formula is known very recently [PMN88]), which shows that the number $\Theta(n)$ increases very rapidly comparing even with $\Psi(n)$). We calculated first a few values, which are shown in Table 4.2 (the calculation is possible only up to $n = 4$ by naive enumeration using computer).

Table 4.1. Values of $\Psi(n)$.

n	1	2	3	4	5	6	7
$\Psi(n)$	3	6	20	168	7,581	7,828,354	2,414,682,040,998

Table 4.2. Values of $\Theta(n)$.

n	1	2	3	4	5
$\Theta(n)$	2	6	40	1,376	1,314,816

4.2. Sheffer functions under ordinary composition

Our first construction method is the ordinary one. In this construction functions from a given set of primitives are combined by composition of functions, together with identification and permutation of variables. Thus the projection functions p_i^n are freely used in composition.

In this section only the last theorem is new. The first theorem is well-known (see [Ros77]) and is easily obtained from the Post completeness theorem 2.3.1, Chapter 2 : *a* *set of functions is complete under ordinary composition if and only if it is not included in each of the 5 sets T_0, T_1, M, S and L.*

Theorem 4.2.1. *A function f is Sheffer if and only if $f \notin T_0 \cup T_1 \cup S$.*

Theorem 4.2.2. *The number of n-ary Sheffer functions is $\Sigma(n) = 2^{2^n-2} - 2^{2^{n-1}-1}$.*

Theorem 4.2.3. [Tos72] *The number $\Sigma^s(n)$ of n-ary symmetric Sheffer functions is 2^{n-1} for n even and $2^{n-1} - 2^{(n-1)/2}$ for n odd.*

Theorem 4.2.4. [Jab52] *A function f is c-Sheffer if and only if $f \notin M \cup L$.*

Theorem 4.2.5. [Hik82] *The number of n-ary c-Sheffer functions is $\Sigma^c(n) = 2^{2^n} - 2^{n+1} + n + 2 - \Psi(n)$.*

Theorem 4.2.6. *The number of n-ary symmetric c-Sheffer functions is $\Sigma^{cs}(n) = 2^{n+1} - n - 4$.*

Proof. $\Sigma^{cs}(n) = |P^s(n)| - |M^s(n)| - |L^s(n)| + |M^s(n) \cap L^s(n)| = 2^{n+1} - (n+2) - 4 + 2$. □

Thus, when n is large, almost all symmetric Boolean functions are Sheffer with constants.

4.3. Sheffer functions under r-line coding

All the result about r-line coding completeness is derived from the following Theorem 2.4.1, Chapter 2:

Theorem 4.3.1. *A set of functions is complete under r-line coding if and only if it is not contained in each of the 3 sets L, C and D.*

Theorem 4.3.2. *A function f is Sheffer and c-Sheffer under r-line coding if and only if $f \notin L \cup C \cup D$.*

Proof. The second assertion comes from the fact that $\{c_0^n, c_1^n\} \subseteq L \cap C \cap D$. □

Thus, the notions of Sheffer and c-Sheffer coincide under r-line coding completeness.

Theorem 4.3.3. *The numbers of n-ary functions Sheffer and c-Sheffer under r-line coding are* $\Sigma_{rlc}(n) = \Sigma^c_{rlc}(n) = 2^{2^n} - 2^{n+2} + 2n + 2$.

Proof. $\Sigma_{rlc}(n) = \Sigma^c_{rlc}(n) = 2^{2^n} - 2^{n+1} - 2(2^n + 1) + 3(n+2) - (n+2)$. □

Theorem 4.3.4. *The numbers of n-ary symmetric functions Sheffer and c-Sheffer under r-line coding are* $\Sigma^s_{rlc}(n) = \Sigma^{cs}_{rlc}(n) = 2^{n+1} - 6$.

Proof. $\Sigma^s_{rlc}(n) = \Sigma^{cs}_{rlc}(n) = 2^{n+1} - 4 - 3 - 3 + 2 + 2 + 2 - 2$. □

4.4. Sheffer functions under 2-line fixed coding

The theorems about Sheffer functions in this section are derived from the following Theorem 2.5.1, Chapter 2:

A set of functions is complete under the 2-line fixed coding if and only if it is not contained in each of the 6 sets S, L, C, D, N_0 and N_1.

Theorem 4.4.1. *A function f is Sheffer under the 2-line fixed coding if and only if* $f \notin S \cup L \cup C \cup D \cup N_0 \cup N_1$.

We could not find an explicit formula for the number of functions in the above case.

Theorem 4.4.2. *A symmetric function f is Sheffer under the 2-line fixed coding if and only if* $f \notin S^s \cup L^s \cup N^s_0 \cup N^s_1$.

Proof. $C^s \cup D^s \subseteq N^s_0 \cup N^s_1$. □

Theorem 4.4.3. *The number $\Sigma^s_{2lfc}(n)$ of n-ary symmetric Sheffer functions under the 2-line fixed coding is $2^{n+1} - 2^{n/2+1} - 2$ for n even and $2^{n+1} - 3 \cdot 2^{(n+1)/2} + 2$ for n odd.*

Proof. When n even, $S^s(n) = N^s_0(n) \cap N^s_1(n) = \phi$. Thus $\Sigma^s_{2lfc}(n) = |P^s(n)| - |L^s(n)| - |N^s_0(n)| - |N^s_1(n)| + |L^s(n) \cap N^s_0(n)| + |L^s(n) \cap N^s_1(n)| = 2^{n+1} - 2^{n/2} - 2^{n/2} - 2$. When n odd, $\Sigma^s_{2lfc}(n) = 2^{n+1} - 2^{(n+1)/2} - 2^{(n+1)/2} - 4 + 5 + 2 - 2^{(n+1)/2} - 1$. □

Theorem 4.4.4. *A function f is c-Sheffer under the 2-line fixed coding if and only if* $f \notin L \cup C \cup D$.

Proof. Because $c_0 \notin S, c_1 \notin N_0$ and $c_0 \notin N_1$. □

Thus, from Theorem 4.3..2, the sets of c-Sheffer functions under r-line coding and the 2-line fixed coding coincide. Hence from Theorems 4.3..3 and 4.3..4 we immediately have the following theorems.

Theorem 4.4.5. *The number of n-ary c-Sheffer functions under the 2-line fixed coding is* $\Sigma^c_{2lfc}(n) = 2^{2^n} - 2^{n+2} + 2n + 2$.

Theorem 4.4.6. *The number of n-ary symmetric c-Sheffer functions under the 2-line fixed coding is* $\Sigma^{cs}_{2lfc}(n) = 2^{n+1} - 6$.

4.5. Sheffer functions under uniform delay composition

The following Theorem 2.6.1, Chapter 2 is fundamental for this section.

A set of functions is complete under uniform delay composition if and only if it is not contained in each of the 8 sets: T_0, T_1, M, S, L, X, M' *and* K.

There is no Sheffer function under this construction [Kud60], because $\overline{T}_0 \cap \overline{T}_1 \subseteq X$. However, in the case of Shefferness with constants, we have the following:

Theorem 4.5.1. [Hik82] *A function f is c-Sheffer under uniform delay composition if and only if* $f \notin M \cup M' \cup L \cup K$.

Theorem 4.5.2. [Hik82] *The number of n-ary functions Sheffer with constants is* $\Sigma^c_{uni}(n) = 2^{2^n-1} - 2^n + 2n + 4 - 2\Psi(n)$.

Theorem 4.5.3. *The number of n-ary symmetric c-Sheffer functions under uniform delay composition is* $\Sigma^{cs}_{uni}(n) = 2^n - 2n - 1 + (-1)^n$.

Proof. Consider the functions outside K^s. Note that $|M^s(n) \cap \overline{K^s(n)}| = |M'^s(n) \cap \overline{K^s(n)}| = n$; and also note that $L^s(n) \cap \overline{K^s(n)} = \{a + x_1 + ... + x_n\}$ when n is odd, and is ϕ when n is even. □

4.6. Sheffer functions under Ibuki construction

Another construction method for unit delay primitives is proposed independently in Ibuki [Ibu68]. He allows non-uniform composition in some case. His completeness theorem is.

Theorem 4.6.1. [Ibu68] *A set of functions is complete under Ibuki construction if and only if it is not contained in each of the 7 sets:* T_0, T_1, M, S, L, X *and* M'.

There is no Sheffer function in this construction by the same reason as the previous section. From Theorems 4.5.1 and 4.6.1 the following corollary is immediate.

Corollary 4.6.1. *If a set of functions is complete under uniform delay composition then it is complete under Ibuki construction.*

Theorem 4.6.2. *A function* f *is c-Sheffer under Ibuki construction if and only if* $f \notin M \cup L \cup M'$.

Proof. $c_1 \notin T_0$, $c_0 \notin T_1$, $c_0 \notin S$, and $c_0 \notin X$. □

Theorem 4.6.3. *The number of n-ary c-Sheffer functions under Ibuki construction is*

$$\Sigma^{c}_{Ibuki}(n) = 2^{2^n} - 2^{n+1} + 2n + 4 - 2\Psi(n).$$

Proof. Note that $|M(n) \cup M'(n)| = 2\Psi(n) - 2$. Also use Lemma 2.1. □

Theorem 4.6.4. *The number of n-ary symmetric c-Sheffer functions under Ibuki construction is*

$$\Sigma^{cs}_{Ibuki}(n) = 2^{n+1} - 2n - 4.$$

Proof. From lemmas in Section 3 we have $|M^s(n) \cup L^s(n) \cup M'^s(n)| = 2n + 4$. □

4.7. Sheffer functions under Inagaki construction

Still another modification to Kudrjavcev's construction is treated in [Ina82] by Inagaki. He further weakened Ibuki's restriction, assuming that one is allowed to construct constant-valued functions with their inputs being nonuniform delays. In this construction one feeds input signals in some span of time so that one can maintain stable output. Thus, for example, oscillating input sequence like 0101... is prohibited.

Theorem 4.7.1. [Ina82] *A set of Boolean functions is complete under Inagaki construction if and only if it is not contained in each of the 6 sets:* T_0, T_1, M, S, L *and* M'.

From Theorems 4.6.1 and 4.7.1 the following corollary is immediate.

Corollary 4.7.1. *If a set of functions is complete under Ibuki construction then it is complete under Inagaki construction.*

There exist Sheffer functions in contrast to the former two cases.

Theorem 4.7.2. *A function f is Sheffer under Inagaki construction if and only if $f \notin T_0 \cup T_1 \cup S \cup M'$.*

Proof. $f \notin T_0 \cup T_1 \cup S$ implies $f \notin M \cup L$. □

We could not find an explicit formula for the number $\Sigma_{Inagaki}(n)$. But for the symmetric case we have the following.

Theorem 4.7.3. *The number $\Sigma^s_{Inagaki}(n)$ of n-ary symmetric Sheffer functions under Inagaki construction is $2^{n-1} - n$ for n even, and $2^{n-1} - 2^{(n-1)/2} - n + 1$ for n odd.*

Proof. Only the rough sketch. When n is even, note that the number is $|\overline{T^s_0(n)} \cap \overline{T^s_1(n)}| - |M'^s(n)| + |(T^s_0(n) \cup T^s_1(n)) \cap M'^s(n)|$. When n is odd, note that the number is $|\overline{T^s_0(n)} \cap \overline{T^s_1(n)}| - |\overline{T^s_0(n)} \cap \overline{T^s_1(n)} \cap S^s(n)| - |\overline{T^s_0(n)} \cap \overline{T^s_1(n)} \cap M'^s(n)| + |\overline{T^s_0(n)} \cap \overline{T^s_1(n)} \cap S^s(n) \cap M'^s(n)|$. □

Theorem 4.7.4. *A function f is c-Sheffer under Inagaki construction if and only if $f \notin M \cup L \cup M'$.*

Hence, from Theorem 4.6.2, the sets of c-Sheffer functions under Ibuki construction and Inagaki construction coincide. Following theorems are immediately obtained from Theorems 4.6.3 and 4.6.4.

Theorem 4.7.5. *The number of n-ary c-Sheffer functions under Inagaki construction is*

$$\Sigma^c_{Inagaki}(n) = 2^{2^n} - 2^{n+1} + 2n + 4 - 2\Psi(n).$$

Theorem 4.7.6. *The number of n-ary symmetric c-Sheffer functions under Inagaki construction is*

$$\Sigma^{cs}_{Inagaki}(n) = 2^{n+1} - 2n - 4.$$

4.8. Sheffer functions under sequential circuit construction

We present the result about Sheffer functions based on the following Theorem 2.7.1, Chapter 2:

Theorem 4.8.1. *A set of functions is complete under sequential circuit construction if and only if it is not contained in each of the 6 sets:* M, S, L, N_0, N_1 *and* M'.

Since $N_0 \subseteq T_0$ and $N_1 \subseteq T_1$, the following corollary is immediate from Theorems 4.7.1 and 4.8.1.

Corollary 4.8.1. *If a set of functions is complete under Inagaki construction then it is complete under sequential circuit construction.*

Theorem 4.8.2. *A function* f *is Sheffer under sequential circuit construction if and only if* $f \notin M \cup S \cup L \cup N_0 \cup N_1 \cup M'$.

We could not find an explicit formula for $\Sigma_{seq}(n)$. But in the symmetric case we have the following.

Theorem 4.8.3. *The number* $\Sigma^s_{seq}(n)$ *of* n*-ary symmetric Sheffer functions under sequential circuit construction is* $2^{n+1}-2^{n/2+1}-n-2$ *for* n *even and* $2^{n+1}-3\cdot2^{(n+1)/2}-n+3$ *for* n *odd.*

Proof. When n is even, $M^s(n) \subseteq N^s_0(n) \cup N^s_1(n)$. When n is odd, note that $N^s_0(n) \cap N^s_1(n) \subseteq S^s(n)$, $L^s(n) \cap N^s_0(n) = M'^s(n) \cap N^s_0(n) = \{c^n_0\}$, and $L^s(n) \cap N^s_1(n) = M'^s(n) \cap N^s_1(n) = \{c^n_1\}$. Details omitted. □

Theorem 4.8.4. *A function* f *is c-Sheffer under sequential circuit construction if and only if* $f \notin M \cup L \cup M'$.

From Theorem 4.6.2 and Theorem 4.7.5, c-Shefferness coincides under Ibuki, Inagaki and sequential. Thus we have the following.

Theorem 4.8.5. *The number of* n*-ary c-Sheffer functions under sequential circuit construction is*

$$\Sigma^c_{seq}(n) = 2^{2^n} - 2^{n+1} + 2n + 4 - 2\Psi(n).$$

Theorem 4.8.6. *The number of* n*-ary symmetric c-Sheffer functions under sequential circuit construction is* $\Sigma^{cs}_{seq}(n) = 2^{n+1} - 2n - 4$.

4.9. Concluding remarks

As is well-known, the condition for completeness is conveniently expressed by listing all maximal incomplete sets under each construction. In Table 4.3 the maximal incomplete sets under the constructions treated in this chapter are summarized. In Tables 4.4 are shown consditions of Shefferness and c-Shefferness (Table 4.5 presents the same conditions for symmetric functions). Table 4.6 presents essentially 2-ary Sheffer functions. In Tables 4.7 and 4.8 are shown n-ary functions Sheffer and Sheffer with constants, respectively, for $2 \leq n \leq 4$, for each case of the constructions. In Tables 4.9 and 4.10 are shown n-ary *symmetric* functions Sheffer and Sheffer with constants, respectively, for $2 \leq n \leq 6$. All the values in the tables are calculated by the formulas given in the paper, except those marked by (*) in Table 4.7 which are obtained by naive enumeration.

Table 4.3: Maximal incomplete sets under various constructions.

	T_0	T_1	M	S	L	C	D	N_0	N_1	X	M'	K
ordinary composition	x	x	x	x	x							
r-line coding					x	x	x					
2-line fixed coding				x	x	x	x	x	x			
uniform	x	x	x	x	x					x	x	x
Ibuki construction	x	x	x	x	x					x	x	
Inagaki construction	x	x	x	x	x						x	
sequential construction			x	x	x			x	x		x	
ordinary with consts.			x		x							
r-line with consts.					x	x	x					
2-line fixed with consts.					x	x	x					
uniform with consts.			x		x						x	x
Ibuki with consts.			x		x						x	
Inagaki with consts.			x		x						x	
sequential with consts.			x		x						x	

Table 4.4: Conditions of Shefferenss and c-Sheferness under various constructions.

	T_0	T_1	M	S	L	C	D	N_0	N_1	X	M'	K
ordinary composition	x	x		x								
r-line coding					x	x	x					
2-line fixed coding				x	x	x	x	x	x			
Inagaki construction	x	x		x							x	
sequential construction			x	x	x			x	x		x	
ordinary with consts.			x		x							
r-line with consts.					x	x	x					
2-line fixed with consts.					x	x	x					
uniform with consts.			x		x						x	x
Ibuki with consts.			x		x						x	
Inagaki with consts.			x		x						x	
sequential with consts.			x		x						x	

Table 4.5: Conditions of symmetric function Shefferenss and c-Sheferness.

	T_0^s	T_1^s	M^s	S^s	L^s	C^s	D^s	N_0^s	N_1^s	X^s	M'^s	K^s
ordinary composition	x	x		x								
r-line coding					x	x	x					
2-line fixed coding				x	x			x	x			
Inagaki construction	x	x		x							x	
sequential construction			x	x	x			x	x		x	
ordinary with consts.			x		x							
r-line with consts.					x	x	x					
2-line fixed with consts.					x	x	x					
uniform with consts.			x		x						x	x
Ibuki with consts.			x		x						x	
Inagaki with consts.			x		x						x	
sequential with consts.			x		x						x	

Table 4.6: Essentially 2-ary Sheffer functions under various constructions.

	$\overline{x \vee y}$	$x \vee y$	$y\overline{x}$	$y \to x$	$x\overline{y}$	$x \to y$	$x \not\equiv y$	$x \equiv y$	$\overline{x}\overline{y}$	xy
ordinary composition	x								x	
r-line coding	x		x	x	x	x			x	
2-line fixed coding	x								x	
uniform										
Ibuki construction										
Inagaki construction										
sequential construction										
ordinary with consts.	x		x	x	x	x			x	
r-line with consts.	x		x	x	x	x			x	
2-line fixed with consts.	x		x	x	x	x			x	
uniform with consts.										
Ibuki with consts.			x	x	x	x				
Inagaki with consts.			x	x	x	x				
sequential with consts.			x	x	x	x				

Table 4.7: The number of n-ary Sheffer functions.

n	2	3	4	ratio when $n \to \infty$
total	16	256	65,536	
ordinary composition	2	56	16,256	1/4
r-line coding	6	232	65,482	1
2-line fixed coding(*)	2	162	62,538	?
uniform delay	-	-	-	0
Ibuki	-	-	-	0
Inagaki(*)	0	42	16,102	1/4
sequential composition(*)	0	148	62,366	?

Table 4.8: The number of n-ary functions Sheffer with constants.

n	2	3	4	ratio when $n \to \infty$
total	16	256	65,536	
ordinary composition	6	225	65,342	1
r-line coding	6	232	65,482	1
2-line fixed coding	6	232	65,482	1
uniform delay	0	90	32,428	1/2
Ibuki	4	210	65,180	1
Inagaki	4	210	65,180	1
sequential composition	4	210	65,180	1

Table 4.9: The number of n-ary symmetric Sheffer functions.

n	2	3	4	5	6	ratio when $n \to \infty$
total	8	16	32	64	128	
ordinary composition	2	2	8	12	32	1/4
r-line coding	2	10	26	58	122	1
2-line fixed coding	2	6	22	42	110	1
uniform delay	-	-	-	-	-	0
Ibuki	-	-	-	-	-	0
Inagaki	0	0	4	8	26	1/4
sequential composition	0	4	18	38	104	1

Table 4.10: The number of n-ary symmetric functions Sheffer with constants.

n	2	3	4	5	6	ratio when $n \to \infty$
total	8	16	32	64	128	
ordinary composition	2	9	24	55	118	1
r-line coding	2	10	26	58	122	1
2-line fixed coding	2	10	26	58	122	1
uniform delay	0	0	8	20	52	1/2
Ibuki	0	6	20	50	112	1
Inagaki	0	6	20	50	112	1
sequential composition	0	6	20	50	112	1

Chapter 5

Classification of P_3

In this chapter we classify P_3, the set of all three-valued logical functions. In the first section we state the completeness criterion for P_3 which gives 18 P_3-maximal sets. Then we present inclusion relations of intersections of the maximal sets as lemmas. These lemmas are useful not only for the classification of P_3 but also for understanding the basic structure of P_3. The study of classes also provides information on the closed sets which are the intersections of families of maximal sets. This is of independent interest relating to a further study toward describing all closed sets of P_3. In Section 5.2 we explain a strategy of the classification briefly. After that we proceed to the classification of P_3.

5.1. Basic structure of P_3

In this section intersections of the P_3-maximal sets are investigated. Operations over relations introduced in Chapter 1 are used to prove basic inclusion relations among them. In some cases we present the results from [Miy71] omitting the proofs.

The investigation of this chapter is based on the following fundamental result due to Jablonskij. In the following theorem T is the Słupecki clone (of all essentially unary or non-surjective functions); L is the clone of all linear or affine (mod 3) functions; S of all functions selfdual with respect to the cyclic permutation (012); M_0, M_1, M_2 are determined by linear orders (chains) on E_3; U_0, U_1, U_2 by the nontrivial equivalence relations on E_3; B_0, B_1, B_2 by the so called central relations and $T_0, \ldots, T_{12}$ by unary relations (i.e. subsets of E_3). Throughout this chapter $x + y$ and xy denote the element of E_3 congruent (mod 3) to $x + y$ and xy, respectively.

Theorem 5.1.1. [Jab58] *P_3 has exactly the following 18 maximal sets:*

$$
\begin{aligned}
T &= Pol(\{(a,b,c)^T \in E_3^3 \mid a=b \text{ or } a=c \text{ or } b=c\}),\\
L &= Pol(\{(a,b,c)^T \in E_3^3 \mid c=2(a+b)\}),\\
S &= Pol\begin{pmatrix} 012 \\ 120 \end{pmatrix},
\end{aligned}
$$

$$
\begin{array}{lll}
M_0 = Pol\begin{pmatrix} 012220 \\ 012011 \end{pmatrix}, & M_1 = Pol\begin{pmatrix} 012001 \\ 012122 \end{pmatrix}, & M_2 = Pol\begin{pmatrix} 012112 \\ 012200 \end{pmatrix},\\
U_0 = Pol\begin{pmatrix} 01212 \\ 01221 \end{pmatrix}, & U_1 = Pol\begin{pmatrix} 01202 \\ 01220 \end{pmatrix}, & U_2 = Pol\begin{pmatrix} 01201 \\ 01210 \end{pmatrix},\\
B_0 = Pol\begin{pmatrix} 0120102 \\ 0121020 \end{pmatrix}, & B_1 = Pol\begin{pmatrix} 0120112 \\ 0121021 \end{pmatrix}, & B_2 = Pol\begin{pmatrix} 0120212 \\ 0122021 \end{pmatrix},\\
T_0 = Pol(0), & T_1 = Pol(1), & T_2 = Pol(2),\\
T_{01} = Pol(01), & T_{02} = Pol(02), & T_{12} = Pol(12).
\end{array}
$$

Let us call the functions of $D := \{f | W(f) \neq E_k\}$ *degenerate* functions, where $W(f)$ denotes the sets of values of f (range of f).

Theorem 5.1.2. [Slu39]

$$T = D \cup P_k^{(1)}.$$

Another characteristic of T is the set of functions, substituting any degenerate functions in its all arguments results in a degenerate function (T may be called *semi-degenerate* functions). L is the set of functions which can be expressed as a linear function of its variables. The set of linear functions is maximal only if k is a prime. S is the set of functions preserving the mapping $\phi : E_3 \to E_3; \phi(x) = x + 1$.

If a binary relation $R \subset E_3 \times E_3$ contains $\{(0,0),(1,1),(2,2)\}$ then R is called *reflexive.* The sets M_i, U_i, B_i are reflexive.

M_1, M_0, M_2 are the set of functions preserving the three order relations $2 \leq_0 0 \leq_0 1$, $0 \leq_1 1 \leq_1 2, 1 \leq_2 2 \leq_2 0$ respectively. They are called nondecreasing functions, respectively with respect to the three orders $\leq_0, \leq_1, \leq_2$. $f \in U_2$ is equivalent to : if $(f(\boldsymbol{a}), f(\boldsymbol{b})) = (0,2)$ or $(1,2)$ then there is i such that $(a_i, b_i) = (0,2)$ or $(1,2)$. In the same manner, $f \in B_1$ is equivalent to : if $(f(\boldsymbol{a}), f(\boldsymbol{b})) = (0,2)$ then there is i such that $(a_i, b_i) = (0,2)$.

T_a and T_{ab} is the set of functions *preserving* a and $\{a, b\}$, respectively. That is, for $f \in T_a$ we have $f(\boldsymbol{a}) = a$, and for $f \in T_{ab}$ we have $f(\boldsymbol{x}) \in \{a, b\}$ for any $\boldsymbol{x} \in \{a, b\}^n$.

Let the permutation group (symmetric group) over $\{0, 1, 2\}$ be $S_3 = \{\varepsilon, \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$; $\sigma_0 = (12), \sigma_1 = (02), \sigma_2 = (01), \sigma_3 = (012), \sigma_4 = (210)$, where $\varepsilon, (p, q)$ and (p, q, r) denote identity, transposition of p and q, and cyclic permutation of p, q, r, respectively. In Table 5.1 we presents the multiplications of the elements of the permutation, where $\gamma = \alpha\beta$ means $\gamma(x) = \alpha(\beta(x))$. Note that $\sigma_i^2 = \varepsilon$ $(i = 0, 1, 2)$.

Since similarity plays an important role in our discussion we present the table of σ-similar of each maximal set for all $\sigma \in S_3$ in Table 5.2. These σ-similar can be calculated by Lemma 1.3.2 and Lemma 1.4.2 (Chapter 1).

Table 5.1: $S_3 \times S_3$.

$\alpha \setminus \beta$	σ_0	σ_1	σ_2	σ_3	σ_4
σ_0	ε	σ_3	σ_4	σ_1	σ_2
σ_1	σ_4	ε	σ_3	σ_2	σ_0
σ_2	σ_3	σ_4	ε	σ_0	σ_1
σ_3	σ_2	σ_0	σ_1	σ_4	ε
σ_4	σ_1	σ_2	σ_0	ε	σ_3

Table 5.2: σ-similar of maximal sets, where - denotes invariance.

ε	σ_0	σ_1	σ_2	σ_3	σ_4
T	–	–	–	–	–
L	–	–	–	–	–
S	–	–	–	–	–
M_1	M_2	–	M_0	M_0	M_2
M_2	M_1	M_0	–	M_1	M_0
M_0	–	M_2	M_1	M_2	M_2
U_2	U_1	U_0	–	U_1	U_0
U_0	–	U_2	U_1	U_2	U_1
U_1	U_2	–	U_0	U_0	U_2
B_0	–	B_2	B_1	B_2	B_1
B_1	B_2	–	B_0	B_0	B_2
B_2	B_1	B_0	–	B_1	B_0
T_0	–	T_2	T_1	T_2	T_1
T_1	T_2	–	T_0	T_0	T_2
T_2	T_1	T_0	–	T_1	T_0
T_{01}	T_{02}	T_{12}	–	T_{20}	T_{12}
T_{12}	–	T_{01}	T_{02}	T_{01}	T_{02}
T_{20}	T_{01}	–	T_{12}	T_{12}	T_{01}

We now proceed to investigate the intersections of the maximal sets.

Theorem 5.1.3.

$K := M_1M_2M_0 = \{0, 1, 2$ (*constant functions*), x_i $(i = 1, 2, \ldots;$ *projection functions*$)\}$.

We give the proof after two lemmas.

Lemma 5.1.1. *If $f \in K$ then for any i and for any a_j $(j = 1, \ldots, n; j \neq i)$,*

$$g(x_i) \equiv f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) = x_i \text{ or constant.}$$

Proof. Let $\hat{a} = (a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)$, then $\hat{0} \leq_1 \hat{1} \leq_1 \hat{2}, \hat{1} \leq_2 \hat{2} \leq_2 \hat{0}, \hat{2} \leq_0 \hat{0} \leq_0 \hat{1}$. Hence, if $f \in K$ then $f(\hat{0}) = 0, f(\hat{1}) = 1, f(\hat{2}) = 2$ or $f(\hat{0}) = f(\hat{1}) = f(\hat{2}) =$ constant. □

Lemma 5.1.2. *If $f(x, y) \in P_3^{(2)}$ depends both on x and y, then $f \notin K$.*

Proof. Assume $f(x, y) \in K$ and $f(x, y)$ depends both on x and y. Then there are c, c' $(c \neq c')$ and a such that $f(c, a) \neq f(c', a)$. From Lemma 5.1.1 $f(x, a)$ must be x or constant, therefore $f(x, a) \equiv x$. Analogously it should be $f(b, y) \equiv y$ for some b. Hence $f(b, a) = b = a$ follows. Assume $b = a = 0$. Then $f(x, y)$ is represented by the following table:

$x \setminus y$	0	1	2
0	0	1	2
1	1	*	*
2	2	*	*

Again by Lemma 5.1.1 $f(2, y) \equiv$ constant or y. On the other hand $f(2, 0) = 2$ from the above table. Hence $f(2, y) \equiv 2$. Analogously we conclude $f(x, 1) \equiv 1$. Accordingly we have $f(2, 1) = 2$ and $f(2, 1) = 1$, a contradiction. The case $b = a = 1$ or 2 is similar. □

Proof of the theorem. It is easy to see that only the constant functions and projection functions belong to K among all functions of $P_k^{(1)}$. Therefore it suffice to show that, if $f \in K$ then f depends only one variable. We show that if $f(x_1 \ldots x_n) \in K$ depends on at least two variables, say x_j and x_i, then there is a function in K which depends just two variables. Since this contradict to Lemma 5.1.2, any f contained in K must depends at most only one variable.

For simplicity put $j := 1$. From Lemma 5.1.1,

$$f(x_1 a_2 \ldots a_n) = x_i \text{ and } f(b_1 \ldots b_{i-1} x_i b_{i+1} \ldots b_n) = x_i, \tag{5.1}$$

for some $a_2, \ldots, a_n, b_1, \ldots, b_n$. Put $x_i = c$ for any $c \in E_3$. Suppose

$$f(b_1 b_2 \ldots c \ldots b_n) \neq f(b_1 b_2' \ldots c \ldots b_n), \tag{5.2}$$

for some b_2 and b_2' $(b_2 \neq b_2')$. Then $h(x, y) := f(b_1 x b_3 \ldots b_{i-1} y b_{i+1} \ldots b_n)$ depends on x and y. In fact, $f(b_2, c) \neq h(b_2', c)$ and $f(b_2, c) \neq h(b_2, c')$ for $c \neq c'$ from (5.1) and (5.2).

Table 5.3:

ε	σ_0	σ_1	σ_2	σ_3	σ_4
$D(0,1)$	$D(2,0)$	$D(1,2)$	–	$D(2,0)$	$D(1,2)$
$D(1,2)$	–	$D(0,1)$	$D(2,0)$	$D(0,1)$	$D(2,0)$
$D(2,0)$	$D(0,1)$	–	$D(1,2)$	$D(1,2)$	$D(0,1)$

Since K contains constants and K is closed, we have $h(x,y) \in K$. This contradicts to Lemma 5.1.2. Thus varying the value of x_2 does not vary the value of f. Repeating the same procedure for $x_3, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ we conclude $f(b_1 a_2 \ldots a_{i-1} c a_{i+1} \ldots a_n) = c$. Since c is arbitrary, this indicate

$$f(b_1 a_2 \ldots a_{i-1} x_i a_{i+1} \ldots a_n) \equiv x_i. \tag{5.3}$$

Let $g(x,y) := f(x a_2 \ldots a_{i-1} y a_{i+1} \ldots a_n)$, then $g(x,y) \in K$ depends on x and y from (5.1) and (5.3). This contradicts to Lemma 5.1.2 and completes the proof. □

Lemma 5.1.3. [Miy71]

$$M_1 M_0 \subseteq U_2,\ \ M_2 M_1 \subseteq U_0,\ \ \textit{and}\ M_0 M_2 \subseteq U_1.$$

Corollary 5.1.1.

$$M_1 M_2 M_0 \subseteq U_2 U_0 U_1.$$

Lemma 5.1.4. [Miy71]

$$U_2 U_0 U_1 \subseteq M_1 M_2 M_0.$$

Note 5.1.1. Let $D(0,1) := \{W(f) \subseteq \{0,1\}\}$ and let $D(1,2), D(2,0)$ be analogously defined. σ-similar of $D(p,q)$ is indicated in Table 5.3.

We can show the following relations [Miy71]:

$$D(0,1) U_2 U_0 \subseteq M_1, \tag{5.4}$$

$$D(2,0) U_2 U_1 \subseteq M_1. \tag{5.5}$$

Taking σ_2 and σ_0-similar of (5.4) and (5.5), respectively, we have

$$D(0,1) U_0 U_1 \ \subseteq\ M_0, \tag{5.6}$$

$$D(0,1) U_1 U_0 \ \subseteq\ M_2. \tag{5.7}$$

By (5.4), (5.6) and (5.7),

$$D(0,1)U_1U_0 \subseteq M_0M_1M_2 = K. \tag{5.8}$$

Hence considering Theorem 5.1.3, we have

$$D(0,1)U_0U_1 = \{0,1\}\ , D(1,2)U_1U_2 = \{1,2\} \text{ and } D(2,0)U_2U_0 = \{2,0\}.$$

Lemma 5.1.5.

$$M_1M_2 \subseteq B_0,\ M_2M_0 \subseteq B_1 \textit{ and } M_0M_1 \subseteq B_2.$$

Proof. The right-hand side relation can be obtained by the left-hand side relations by an operation

$$\mathbf{C} = \begin{vmatrix} 01 \\ 10 \end{vmatrix}.$$

□

Corollary 5.1.2. $M_0M_1M_2 \subseteq B_0B_1B_2$.

Lemma 5.1.6. [Miy71] $U_2U_0 \subseteq B_1,\ U_0U_1 \subseteq B_2$ *and* $U_1U_2 \subseteq B_0$.

Corollary 5.1.3. $U_0U_1U_2 \subseteq B_0B_1B_2$.

Lemma 5.1.7. [Miy71] $B_0B_1 \subseteq U_2,\ B_1B_2 \subseteq U_0$ *and* $B_2B_0 \subseteq U_1$.

Corollary 5.1.4. $B_0B_1B_2 \subseteq U_0U_1U_2$.

Theorem 5.1.4. $K = M_0M_1M_2 = B_0B_1B_2 = U_0U_1U_2 = \{0,1,2$ *(constant functions),* x_i $(i = 1, 2, \ldots)$ *(projections)*$\}$.

Proof. By Corollaries 5.1.1,5.1.3,5.1.4, Lemma 5.1.4 and Theorem 5.1.3. □

Lemma 5.1.8. $T_{01}T_{12} \subseteq T_1,\ T_{12}T_{20} \subseteq T_2$ *and* $T_{20}T_{01} \subseteq T_0$.

Proof. From the relational intersection we have $T_{01} \cap T_{12} = T_1$. □

Corollary 5.1.5. $T_{01}T_{12}T_{20} \subseteq T_0T_1T_2$.

Lemma 5.1.9. $M_1 \cup M_2 \cup M_0 \subseteq T_{01} \cup T_{12} \cup T_{20}$.

Proof. Let $f \in T_{01}T_{12}T_{20}$ then there are $\boldsymbol{a} \in \{0,1\}^n, \boldsymbol{b} \in \{1,2\}^n, \boldsymbol{c} \in \{2,0\}^n$ such that $f(\boldsymbol{a}) = 2, f(\boldsymbol{b}) = 0, f(\boldsymbol{c}) = 1$. This implies $f \in \overline{M}_1\overline{M}_2\overline{M}_0$. □

Note 5.1.2.

$$U_0 \cup U_1 \cup U_2 \;\not\subseteq\; T_{01}T_{12}T_{20},$$
$$B_0 \cup B_1 \cup B_2 \;\not\subseteq\; T_{01}T_{12}T_{20}.$$

Counterexamples of $U_2 \subseteq T_{01}T_{12}T_{20}$ and $B_0 \subseteq T_{01}T_{12}T_{20}$ are any functions in the classes #10 and #26, respectively (they are given later).

Lemma 5.1.10.

$$B_0B_1 \subseteq T_{01},\; M_0M_1 \subseteq T_{01} \quad \textit{except constant function} \quad f = 2,$$
$$B_1B_2 \subseteq T_{12},\; M_1M_2 \subseteq T_{12} \quad \textit{except constant function} \quad f = 0,$$
$$B_2B_0 \subseteq T_{01},\; M_2M_0 \subseteq T_{20} \quad \textit{except constant function} \quad f = 1.$$

Proof. 1). Assume $f \notin B_0B_1\overline{T}_{01}$. Then $f(\{0,1\}^n) = 2$. Because, since $(\boldsymbol{a}, \boldsymbol{b}) \in B_0B_1$ for any $\boldsymbol{a}, \boldsymbol{b} \in \{0,1\}^n$, assuming $f(\boldsymbol{c}) \in \{0,1\}$ for some $\boldsymbol{c} \in \{0,1\}$ leads to $f(\{0,1\}^n) \in \{0,1\}$. We show $f(\boldsymbol{a}) = 2$ for any $\boldsymbol{a} \in E^n \setminus \{0,1\}^n$. Put $u_i := 0, v_i := 1$ if $a_i = 2$, $u_i = v_i = a_i$ otherwise. Then $\boldsymbol{u}, \boldsymbol{v} \in \{0,1\}^n$ and $(\boldsymbol{a}, \boldsymbol{u}) \in B_0$ and $(\boldsymbol{a}, \boldsymbol{v}) \in B_1$. From $f \in B_0B_1$ we have $(f(\boldsymbol{a}), f(\boldsymbol{u})) \in B_0$ and $(f(\boldsymbol{a}), f(\boldsymbol{v})) \in B_1$. Since $f(\boldsymbol{u}) = f(\boldsymbol{v}) = 2$ we have $f(\boldsymbol{a}) = 2$.

2). Assume $f \in M_1M_0\overline{T}_{01}$. There is $\boldsymbol{a} \in \{0,1\}^n$ such that $f(\boldsymbol{a}) = 2$, which implies $f(\boldsymbol{1}) = f(\boldsymbol{2}) = 2$ from $f \in M_1$ and $\boldsymbol{a} \leq \boldsymbol{1} \leq \boldsymbol{2}$. On the other hand $\boldsymbol{2}$ and $\boldsymbol{1}$ are respectively the minimal and maximal elements according to the order $2 \leq_0 0 \leq_0 1$. This implies $f \equiv 2$. □

Lemma 5.1.11.

$$U_0 = M_2 \;\; on \;\; D(0,1)M_1, \tag{5.9}$$
$$U_2 = M_0 \;\; on \;\; D(2,0)M_1. \tag{5.10}$$

Proof. First we prove (5.9). 1) Let us show $D(0,1)M_1U_0 \subseteq M_2$. Assume $f \in D(0,1)M_1U_0M_2$ then there are $\boldsymbol{a} \leq_2 \boldsymbol{b}$ such that $f(\boldsymbol{a}) \not\leq_2 f(\boldsymbol{b})$. From $f \in D(0,1)$ we have $(f(\boldsymbol{a}), f(\boldsymbol{b})) = (0,1)$. Define $\boldsymbol{a}'$ by

$$a'_i = \begin{cases} 2, & \text{if } (a_i, b_i) = (1,2) \text{ or } (1,0), \\ a_i, & \text{otherwise.} \end{cases}$$

If $a = a'$, then we have $b \le a$ and $f(b) = 1 \not\le f(a) = 0$, contradicting $f \in M_1$. If $a \ne a'$, then $(a, a') \in U_0$ and hence $f(a') = 0$. On the other hand, $b \le a'$ from the construction of a'. Again $f(b) = 1 \not\le f(a') = 0$ contradicts to $f \in M_1$.

2) Converse $D(0,1)M_1M_2 \subseteq U_0$ is from Lemma 5.1.3. The proof of (5.10) can be done analogously (note that (5.9) and (5.10) are not σ-simislar). □

Corollary 5.1.6.

$$U_0 = M_2 \quad on \quad D(0,1)M_1 \tag{5.11}$$

$$U_0 = M_1,\ U_1 = M_0 \quad on \quad D(0,1)M_2 \tag{5.12}$$

$$U_1 = M_2 \quad on \quad D(0,1)M_0 \tag{5.13}$$

$$U_1 = M_0 \quad on \quad D(1,2)M_2 \tag{5.14}$$

$$U_1 = M_2,\ U_2 = M_1 \quad on \quad D(1,2)M_0 \tag{5.15}$$

$$U_2 = M_0 \quad on \quad D(1,2)M_1 \tag{5.16}$$

$$U_2 = M_1 \quad on \quad D(2,0)M_0 \tag{5.17}$$

$$U_2 = M_0,\ U_0 = M_2 \quad on \quad D(2,0)M_1 \tag{5.18}$$

$$U_0 = M_1 \quad on \quad D(2,0)M_2. \tag{5.19}$$

Proof. Equation (5.11) is (5.9). The first and the second equations of (5.12) are σ_4 and σ_0-similar of (5.10), respectively. (5.13) is σ_2-similar of (5.11). The equations (5.14),(5.15),(5.16) and (5.17),(5.18),(5.19) are σ_4 and σ_3-similar of (5.11),(5.12),(5.13), respectively. □

Note 5.1.3. From Lemma 1.4.2 (Chapter 1) we have the following equations.

$$M_r^\sigma = M_{\sigma^{-1}(r)},\ U_r^\sigma = U_{\sigma^{-1}(r)},\ B_r^\sigma = B_{\sigma^{-1}(r)} \ and\ D(p,q)^\sigma = D(\sigma^{-1}p, \sigma^{-1}q).$$

5.2. Strategy of the classification

The final classification result of P_3 is indicated in the Appendix 1, where *no (number preceded by *) denotes serial identification number of the class (according its order of appearance), while #no (number preceded by #) denotes the sorted according to the "degree of completeness" number of the class. All the representatives of the classes are indicated in Appendix 2 separately.

The process of classification is as follows. First we classify the functions of T. Then $\overline{T}(L \cup S)$ is classified, and after this the remaining functions $\overline{TLS}$ are classified by M type, U type, B type, T_p type and finally T_{pq} type maximal sets. It is clear that we consider the functions which are not yet classified in each stage of the process. After above process it remains only one class, namely the class of functions which belong to none of 18 maximal sets. This class consists of so called Sheffer functions or complete functions.

The process is straightforward and we will identify all 406 classes of P_3 among $2^{18} = 262,144$ possible classes. The classification process reveals the finite structure of P_3.

We show the correspondence of sections and the functions to be classified.

Section 5.3	T
Section 5.4	$\overline{T}(L \cap S)$
Section 5.5	$M := \overline{TLS}(M_0 \cup M_1 \cup M_2)$
Section 5.6	$U := \overline{TLSM}(U_0 \cup U_1 \cup U_2)$
Section 5.7	$B := \overline{TLSMU}(B_0 \cup B_1 \cup B_2)$
Section 5.8	$\overline{TLSMUB}$.

5.3. Classification of T

Let $P^{(1)}_{onto} := \{f | f \in P^{(1)}_3 \text{ and } f \text{ is onto}\}$ and $D'(0,1) := D(0.1) \setminus \{0,1\}$, $D'(1,2) := D(1,2) \setminus \{1,2\}$, $D'(2,0) := D(2,0) \setminus \{2,0\}$. Then from Theorem 5.1.2 and σ-similar, $T = P^{(1)}_{onto} + \{0,1,2\} + D'(0,1) + D'(1,2) + D'(2,0) = P^{(1)}_{onto} + \{0,1,2\} + D'(0,1) + D'(0,1)^{\sigma_1} + D'(0,1)^{\sigma_0}$, where "+" denotes direct sum and "{0,1,2}" denotes all constant functions. As for each function in $P^{(1)}_{onto}$ and {0,1,2}, its class is immediately known (Table 5.4). Hence it is sufficient to consider $D'(0,1)$. Note that we must pay attention to the classes in $D'(0,1)$ that are invariant under σ_1 and σ_0 similar in counting the total number of classes of T.

First we prepare some lemmas for the classification of $D'(0,1)$.

Lemma 5.3.1. $D'(0,1) \subseteq \overline{S}$.

By σ-similar we have the following.

Corollary 5.3.1. $D \subseteq \overline{S}$.

Thus by Theorem 5.1.2 and Corollary 5.3.1 we have the following.

Table 5.4: Classes of $P^{(1)}_{onto} + \{0,1,2\}$.

$*no$	$functions$	TLS	$M_1M_2M_0$	$U_2U_0U_1$	$B_0B_1B_2$	$T_0T_1T_2$	$T_{01}T_{12}T_{20}$	$\#no$
$*1$	x	000	000	000	000	000	000	#406
$*2$	$x+1, x+2$	000	111	111	111	111	111	# 83
$*3$	$2x$	001	111	101	011	011	101	#259
$*4$	$2x+1$	001	111	011	110	110	011	#260
$*5$	$2x+2$	001	111	110	101	101	110	#258
$*6$	0	001	000	000	000	011	010	#405
$*7$	1	001	000	000	000	101	001	#404
$*8$	2	001	000	000	000	110	100	#403

Corollary 5.3.2. $TS = \{x_i, x_i+1, x_i+2\ (i=1,2,\ldots)\}$.

Lemma 5.3.2. $D'(0,1) \subseteq \overline{L}$.

Proof. Prove that if $f \in L \setminus \{0,1,2\}$ then f is onto. Assume such f. Then $f(\boldsymbol{x}) = c_0 + \sum c_i x_i$ and there is at least a $c_i \neq 0$. Put $d = f(\boldsymbol{x}+\boldsymbol{1}) - f(\boldsymbol{x}) = \sum c_i = 0, 1$ or 2. If $d \neq 0$ then f is onto, because $f(\boldsymbol{x}), f(\boldsymbol{x}+\boldsymbol{1})$ and $f(\boldsymbol{x}+\boldsymbol{2})$ differ one another. Assume $d = 0$. Since $c_i \neq 0$ there are $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{a}}'$ such that $f(\hat{\boldsymbol{a}}) \neq f(\hat{\boldsymbol{a}}')$, where two vectors $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{a}}'$ differs only at i-th coordinate $(a_i \neq a_i')$. Let a'' be the remaining element of $E_3 \setminus \{a, a'\}$. Then we have $f(\hat{\boldsymbol{a}}') = f(\hat{\boldsymbol{a}}) - c_i(a - a')$ and $f(\hat{\boldsymbol{a}}'') = f(\hat{\boldsymbol{a}}) - c_i(a - a'')$. This implies that f is onto, since $f(\hat{\boldsymbol{a}}') \neq f(\hat{\boldsymbol{a}}), f(\hat{\boldsymbol{a}}'') \neq f(\hat{\boldsymbol{a}})$ and $f(\hat{\boldsymbol{a}}'') \neq f(\hat{\boldsymbol{a}}')$. □

By σ-similar we have the following corollary.

Corollary 5.3.3. $D \setminus \{0,1,2\} \subseteq \overline{L}$.

Thus by Theorem 5.1.2 and Corollary 5.3.3 we have:

Corollary 5.3.4. $TL = P^{(1)}_{onto} = \{0,1,2\}$.

Lemma 5.3.3. $D'(0,1) \subseteq \overline{B}_2 B_0 B_1 \overline{T}_2 T_{01} U_2$.

Proof. Suppose $f \in D'(0,1)B_2$, then there are $\boldsymbol{a}, \boldsymbol{b} \in E_3^n$ such that $f(\boldsymbol{a}) = 0$ and $f(\boldsymbol{b}) = 1$ from $f \in D'(0,1)$. Considering that $(\boldsymbol{a}, \boldsymbol{2}) \in B_2$ we have $f(\boldsymbol{2}) = 0$ since $f(\boldsymbol{a}) = 0$ and $f \in D'(0,1)$. On the other hand. $(\boldsymbol{b}, \boldsymbol{2}) \in B_2$ leads to $f(\boldsymbol{2}) = 1$ in the analogous manner. A contradiction. The remaining assertions are obvious from the definitions. □

Note 5.3.1. From this lemma we see that the classes of $D'(0,1)$ are neither σ_0- nor σ_1-invariant.

Now let us introduce a new notation to represent a partition of $D'(0,1)$. We set

$$A(i,j,k) := \{f | f(\mathbf{0}) = i, f(\mathbf{1}) = j, f(\mathbf{2}) = k\}.$$

Then $D'(0,1)$ can be represented as

$$D'(0,1) = \sum_{i,j,k=0}^{1} A(i,j,k),$$

where the summation is direct sum of sets. It is easy to see that

$$A(1,1,1) = A(0,0,0)^{\sigma_2}, \quad A(1,1,0) = A(0,0,1)^{\sigma_2},$$
$$A(0,1,1) = A(0,1,0)^{\sigma_2}, \quad A(1,0,1) = A(1,0,0)^{\sigma_2}.$$

Therefore it suffices to investigate only the four sets (again we must be careful about that the same class may be included in the different sets of $A(i,j,k)$). We prepare some preliminary lemmas on $A(i,j,k)$.

Lemma 5.3.4. $(A(0,0,0) + A(0,1,0))U_1 \subseteq T_{20}(M_2M_0 + \overline{M}_2\overline{M}_0)$.

Proof. First we show $f \in T_{20}$. Let $f \in (A(0,0,0) + A(0,1,0))U_1$ then $f(\{2,0\}^n) = 0$, hence $f \in T_{20}$. This is because $f(\mathbf{0}) = f(\mathbf{2}) = 0$, $(\boldsymbol{a}, \mathbf{0}) \in U_1$, $(\boldsymbol{a}, \mathbf{2}) \in U_1$ for any $\boldsymbol{a} \in \{2,0\}^n$ and $f \in D'(0,1)$.

Next we show $D'(0,1)U_1T_{20}M_2 \subseteq M_0$. Assume $f \in D'(0,1)U_1T_{20}M_2\overline{M}_0$. There are $\boldsymbol{a}, \boldsymbol{b}$ such that $f(\boldsymbol{a}) \leq_0 f(\boldsymbol{b})$ and $f(\boldsymbol{a}) = 1$, $f(\boldsymbol{b}) = 0$ from $f \in D'(0,1)$. Define $\boldsymbol{b}'$ in the following: $b'_i = 2$ if $(a_i, b_i) = (2,0)$, $b'_i = b_i$ otherwise for each i. Since, $(\boldsymbol{b}, \boldsymbol{b}') \in U_1$ we have $f(\boldsymbol{b}') = 0$ (including the case $\boldsymbol{b} = \boldsymbol{b}'$). From the definition we have $\boldsymbol{b} \leq_2 \boldsymbol{a}$, and we have $f(\boldsymbol{b}') = 0 \geq_2 f(\boldsymbol{a}) = 1$, a contradiction. Analogously we can prove $D'(0,1)U_1T_{20}M_0 \subseteq M_2$. Thus $f \in M_2$ and $f \in M_0$ is equivalent under $f \in (A(0,0,0) + A(0,1,0))U_1$. □

Lemma 5.3.5. $D'(0,1)U_0\overline{U}_1 \subseteq \overline{T}_{20}$.

Proof. Assume $f \in D'(0,1)U_0\overline{U}_1$. Then there are $(\boldsymbol{a}, \boldsymbol{b}) \in U_1$ such that $(f(\boldsymbol{a}), f(\boldsymbol{b})) = (0,1)$. If $\boldsymbol{b} \in \{2,0\}^n$ then $f \in \overline{T}_{20}$. Otherwise construct $\boldsymbol{b}'$ by putting $b'_i = 2$ if $b_i = 1$, $b'_i = b_i$ otherwise. Then $\boldsymbol{b}' \in \{2,0\}^n$ and $f(\boldsymbol{b}') = 1$ from $(\boldsymbol{b}, \boldsymbol{b}') \in U_0$. Hence $f \in \overline{T}_{20}$. □

Lemma 5.3.6. $A(0,1,0)\overline{T}_{20} \subseteq \overline{M}_2\overline{M}_0$.

Proof. There is $\boldsymbol{a}$ such that $f(\boldsymbol{a}) = 1$. The results follow from $f \in A(0,1,0)$, $\mathbf{2} \leq_2 \boldsymbol{a} \leq_2 \mathbf{o}$ and $\mathbf{2} \leq_2 \boldsymbol{a} \leq_2 \mathbf{o}$. □

Lemma 5.3.7.

$$
\begin{aligned}
A(0,0,0) &\subseteq T_0\overline{T}_1\overline{T}_{12}\overline{M}_1\overline{M}_2\overline{M}_0, \\
A(0,0,1) &\subseteq T_0\overline{T}_1\overline{T}_{12}\overline{T}_{20}\overline{M}_2\overline{M}_0\overline{U}_0\overline{U}_1, \\
A(0,1,0) &\subseteq T_0T_1\overline{T}_{12}\overline{M}_1\overline{U}_0, \\
A(1,0,0) &\subseteq \overline{T}_0\overline{T}_1\overline{T}_{12}\overline{T}_{20}\overline{M}_1\overline{M}_2\overline{M}_0\overline{U}_1.
\end{aligned}
$$

Proof. The right hand side terms are implied from the definition of $A(i,j,k)$. □

We now proceed to classify $A(0,0,0)$, $A(0,0,1)$, $A(0,1,0)$ and $A(1,0,0)$ in this order.

(1) Classification of $A(0,0,0)$**.** From Lemmas 5.3.3 and 5.3.7 the remaining sets are U_0, U_1 and T_{20}. Since U_0U_1 is impossible from Lemmas 5.1.5 and 5.3.6. We have the following classifications:

$$
A(0,0,0) = \begin{cases} 1)\ \overline{U}_0U_1T_{20} & f3.1 \quad (*\,9 = \#349) \\ 2)\ U_0\overline{U}_1\overline{T}_{20} & f3.2 \quad (*10 = \#303) \\ 3)\ \overline{U}_0\overline{U}_1 \begin{cases} T_{20} \\ \overline{T}_{20} \end{cases} & \begin{matrix} f3.3 \quad (*11 = \#298) \\ f3.4 \quad (*12 = \#244) \end{matrix} \end{cases}
$$

Proof. 1) and 2) are derived from Lemmas 5.3.4 and 5.3.5, respectively. □

(2) Classification of $A(0,0,1)$**.** From Lemmas 5.3.3 and 5.3.7 the remaining sets is M_1 only. Hence we have the following.

$$
A(0,0,1) = \begin{cases} 1)\ M_1 & f3.5\ (*17 = \#321) \\ 2)\ \overline{M}_1 & \text{(same class as *12)} \end{cases}
$$

(3) Classification of $A(0,1,0)$**.** From the same lemmas the remaining sets are M_2, M_0, T_{20} and U_1. We have the following.

$$A(0,1,0)=\begin{cases} 1)\ U_1T_{20} & \begin{cases} \overline{M_2}\overline{M_0} & f3.6 & (*19=\#367) \\ M_2M_0 & f3.7 & (*20=\#392) \end{cases} \\ 2)\ \overline{U_1} & \begin{cases} T_{20} & \begin{cases} \overline{M_2}\overline{M_0} & f3.8 & (*21=\#346) \\ \overline{M_2}M_0 & f3.9 & (*22=\#373) \\ M_2\overline{M_0} & f3.10 & (*23=\#376) \end{cases} \\ \overline{T}_{20} & \overline{M_2}\overline{M_0} \quad f3.11 \quad (*24=\#299) \end{cases} \end{cases}$$

Lemma *Proof.* The terms of 1) are derived from Lemma 5.3.4 and the last term of 2) is derived from Lemma 5.3.6. In 2) the term M_2M_0 is impossible from Lemma 5.1.3. □

Note 5.3.2. The class *24 is σ_2-invariant.

(4) Classification of $A(1,0,0)$**.** From Lemmas 5.3.3 and 5.3.7 the remaining set is U_0 only. Hence we have the following.

$$A(1,0,0)=\begin{cases} 1)\ U_0 & f3.12 & (*30=\#248) \\ 2)\ \overline{U}_0 & f3.13 & (*31=\#190) \end{cases}$$

Note 5.3.3. The class *31 is σ_2-invariant.

Conclusions: Thus we have completed the classification of $D'(0,1)$ and hence of T. Let $|D'(0,1)|$ denote the number of classes of $D'(0,1)$. Paying attention to the two σ_2-invariant classes (*24 and *31), we have $|D'(0,1) = 2(|A(0,0,0)| + |A(0,0,1)| + |A(0,1,0)| + |A(1,0,0) - 2 = 2(4+1+6+2) - 2 = 24$.

Since the classes of $D'(0,1)$ is neither σ_0 nor σ_1 invariant, we have $|T| = |P^{(1)}_{onto}| + |\{0,1,2\} + |D'(0,1)| + |D'(1,2)| + |D'(2,0)| = 5 + 3 + 3 \times 24 = 80$, of which 4+13=17 classes are σ-similar-free.

5.4. Classification of $L \cup S$

In this section the structure of $L \cup S$ is investigate and the set $\overline{T}(L \cup S)$ is classified.

First some lemmas will be proved. For the summation ($\sum_i$) which appears in a linear function we always omit indicating the variable i when no confusion is evident.

Lemma 5.4.1. [Ros70] $f \in L \Leftrightarrow f(\boldsymbol{a}+\boldsymbol{b}) = f(\boldsymbol{a}) + f(\boldsymbol{b}) - f(\mathbf{o})$, *where* $\boldsymbol{a}, \boldsymbol{b} \in E_3^n$ *and* $\mathbf{o}$ *is the identity of the field* {*0,1,2*}.

This lemma is useful to certify whether a function f belongs to L or not.

Lemma 5.4.2.

$$f \in S \Rightarrow f \in F_R \textit{ if and only if } f \in F_{R+1},$$

where $R+1 = \sigma_3 R = \{(a_i+1, b_i+1) | (a_i, b_i) \in R\}$.

Proof. First we note that $R+1 = \sigma_3 R$ and $\sigma_4(R+1) = R$ and further that any function $f \in S$ is both σ_3 and σ_4-invariant. Thus $f(\sigma_3 \boldsymbol{a}) = \sigma_3 f(\boldsymbol{a})$ and $f(\sigma_4 \boldsymbol{a}) = \sigma_4 f(\boldsymbol{a})$ for $f \in S$. Since $f^{\sigma_3}(\boldsymbol{x}) = \sigma_3^{-1} f(\sigma_3 \boldsymbol{x})$ and $f^{\sigma_4}(\boldsymbol{x}) = \sigma_4^{-1} f(\sigma_4 \boldsymbol{x})$ we have $f^{\sigma_3} = f^{\sigma_4} = f$. Thus if $f \in S$ belongs to F_R then $f^{\sigma_4} = f \in (F_r)^{\sigma_4} = F_{R+1}$ from Lemma 1.4.2. □

Note 5.4.1. The relation R and $R+1$ is σ_3-similar. Lemma 5.4.2 asserts that if $f \in S$ belong to F_R then f belongs to F^{σ_3} (equivalently F^{σ_4} simultaneously).

Lemma 5.4.3. $\overline{T}S \subseteq \tilde{M}\tilde{U}\tilde{B}$,
where $\tilde{M} = \overline{M}_0\overline{M}_1\overline{M}_2$, $\tilde{U} = \overline{U}_0\overline{U}_1\overline{U}_2$, *and* $\tilde{B} = \overline{B}_0\overline{B}_1\overline{B}_2$.

Proof. Suppose $f \in \overline{T}SM_1$. Then $f \in \overline{T}SM_1M_2M_0 \subseteq K$ from Lemma 5.4.2 and Theorem 5.1.4. However, from Theorem 5.1.2 $\overline{T}K = \phi$, a contradiction. With respect to the remaining U and B the proofs are analogous. □

Lemma 5.4.4. $S\overline{T}_0\overline{T}_1\overline{T}_2 \subseteq \overline{T}_{01}\overline{T}_{12}\overline{T}_{20}$.

Proof. Let $f \in S\overline{T}_0$ then we have only two cases: either $f(\mathbf{0}) = 1,\ f(\mathbf{1}) = 2,\ f(\mathbf{2}) = 0$ or $f(\mathbf{0}) = 2,\ f(\mathbf{1}) = 0,\ f(\mathbf{2}) = 1$. In both cases $f \in \overline{T}_{01}\overline{T}_{12}\overline{T}_{20}$. □

Lemma 5.4.5. $\overline{T}L \subseteq \overline{T}_{01}\overline{T}_{12}\overline{T}_{20}$.

Proof. Let $f = c_0 + \sum c_i x_i$. First we show that if $f \in \overline{T}LT_{01}$ then at least there are $c_i = 1$ and $c_j = 2$ in the coefficients of f. From $f \in T_{01}$ we have $c_0 = 0$ or 1. Again from $f \in \overline{T}$, f depends on at least two variables. So, for simplicity, assume that f depends both on x_1 and x_2. First, suppose $c_k = 1$ for all nonzero c_k. Then

$$f(1,1,0,\ldots,0) = c_0 + c_1 + c_2 = 2, \text{ if } c_0 = 0,$$
$$f(1,0,\ldots,0) = c_0 + c_1 = 2, \text{ if } c_0 = 1.$$

These contradict to $f \in T_{01}$. Analogously, assuming the other case: $c_k = 2$ for all nonzero c_k, leads to a contradiction. Thus assume $f \in \overline{T}LT_{01}$ and assume that $c_1 = 1,\ c_2 = 2$ for simplicity. Then

$$\begin{aligned} f(0,1,0,\ldots,0) &= c_0 + c_2 = 2, \text{ if } c_0 = 0, \\ f(1,0,\ldots,0) &= c_0 + c_1 = 2, \text{ if } c_0 = 1. \end{aligned}$$

These contradict to $f \in T_{01}$. With respect to T_{12} and T_{20} the proofs are similar. □

For convenience we divide L and S into several subsets:

$$L = L_0 + L_1 + L_2,$$

where $L_a := \{f | f = c_0 + \sum c_i x_i \text{ and } \sum_{i=1}^{n} = a\}$.

Again we divide each L_a into 3 subsets:

$$L_a = L_{a0} + L_{a1} + L_{a2},$$

where $L_{ab} := \{f | f \in L_a \text{ and } f(\mathbf{o}) = c_0 = b\}$.

Similarly, we divide S into the following 3 subsets:

$$S = S_0 + S_1 + S_2,$$

where $S_a := \{f | f \in S \text{ and } f(\mathbf{o}) = a\}$.

σ-similar of each of these subsets is indicated in the Table 5.5. The next Lemma 5.4.6 is used to calculate this table.

Lemma 5.4.6. [Miy71] $(L_{ab})^{\sigma} = L_{a\ lb+m'+lma}$, *where* $\sigma \in S_3,\ \sigma(x) = lx + m,\ \sigma^{-1}(x) = lx + m'$.

Lemma 5.4.7. $LS = L_1$.

Proof. Suppose $f \in LS$, then $f(\boldsymbol{x} + \mathbf{1}) = f(\boldsymbol{x}) + 1$. Hence $\sum c_i = 1$, i.e. $f \in L_1$. The converse is analogous. □

Due to Lemma 5.4.7, previous lemmas concerning S are also applicable to L_1. Next lemma provides a property of the remaining set of L, i.e. $L \setminus L_1 = L_0 + L_2 = L_{00} + L_{20} + L_{01} + L_{22} + L_{02} + L_{21}$.

Table 5.5: σ-similar of L_{ab} and S_a.

ε	σ_0	σ_1	σ_2	σ_3	σ_4
L_{00}	–	L_{02}	L_{01}	L_{02}	L_{01}
L_{01}	L_{02}	–	L_{00}	L_{00}	L_{02}
L_{02}	L_{01}	L_{00}	–	L_{01}	L_{00}
L_{10}	–	–	–	–	–
L_{11}	L_{12}	L_{12}	L_{12}	–	–
L_{12}	L_{11}	L_{11}	L_{11}	–	–
L_{20}	–	L_{21}	L_{22}	L_{21}	L_{22}
L_{21}	L_{22}	L_{20}	–	L_{22}	L_{20}
L_{22}	L_{21}	–	L_{20}	L_{20}	L_{21}
S_0	–	–	–	–	–
S_1	–	–	–	–	–
S_2	–	–	–	–	–

Lemma 5.4.8.

$$\begin{aligned} 1)\ L_{00}+L_{20} &\subseteq T_0\overline{T}_1\overline{T}_2, \\ 2)\ L_{01}+L_{22} &\subseteq \overline{T}_0T_1\overline{T}_2, \\ 3)\ L_{02}+L_{21} &\subseteq \overline{T}_0\overline{T}_1T_2. \end{aligned}$$

Proof. Assume $f \in L_{00}+L_{20}$. Then $f(\mathbf{0}) = c_0 = 0$ and $f(\mathbf{1}) = \sum c_i = 0$, $f(\mathbf{2}) = 2\sum c_i = 0$ (in case $f \in L_{00}$) or $f(\mathbf{1}) = \sum c_i = 2$, $f(\mathbf{2}) = 2\sum c_i = 1$ (in case $f \in L_{20}$). Hence $f \in T_0\overline{T}_1\overline{T}_2$. The cases 2) and 3) are similar of 1). □

Now by a series of lemmas we will prove Corollary 5.4.1 which is an analog inclusion of Lemma 5.4.3 (we have L in place of S). Lemma 5.4.2 simplified the proof of Lemma 5.4.3. However, we have no corresponding one with respect to L. Thus we must consider M, U and B separately, although it suffices to consider L_0 and L_2 owing to Lemma 5.4.7. In fact it is sufficient to consider L_{00} and L_{20} from Table 5.5.

Lemma 5.4.9. $\overline{T}L \subseteq \tilde{M}$.

Proof. Let us prove $\overline{T}(L_0+L_2) \subseteq \overline{M}_1$. Note that $f \in L_a$ implies $f(\mathbf{x}+\mathbf{1}) = f(\mathbf{x})+a$. If $a=0$ then $f(\mathbf{0}) = f(\mathbf{1}) = f(\mathbf{2})$. Hence $f \notin M_1$ because f should be an onto function from $f \notin T$. If $a=2$, we have $f(\mathbf{1}) = f(\mathbf{0})+2$, $f(\mathbf{2}) = f(\mathbf{0})+1$. Hence $f \notin M_1$ whichever $f(\mathbf{0}) = 0$, 1 or 2. By σ-similar we have $\overline{T}L \subseteq \tilde{M}$. □

In the following proofs we use the "modular operation" +1 which maps $\boldsymbol{a} \in \{0,1\}^n$ onto $\boldsymbol{a}+\mathbf{1} \in \{1,2\}^n$ and $\boldsymbol{a}+\mathbf{2} \in \{0,2\}^n$. Hence $f(\boldsymbol{a}+\mathbf{2}) = c_0 + \sum c_i a_i + 2\sum c_i = f(\boldsymbol{a})$ if $f \in L_0$ and $f(\boldsymbol{a}+\mathbf{1}) = f(\boldsymbol{a}) + 2$ if $f \in L_2$.

Lemma 5.4.10. $\overline{T}L \subseteq \tilde{U}$.

Proof. The lemma follows from $\overline{T}(L_0 + L_2) \subseteq U_2$ and Lemmas 5.4.3 and 5.4.7. Assume $f \in \overline{T}U_2$ then f is onto, hence there is $\boldsymbol{a} \in E_3^n n$ such that $f(\boldsymbol{a}) = 2$. Define $\boldsymbol{a}'$ as follows: $a_i' = 0$ if $a_i = 1$, $a_i' = a_i$ otherwise for all i. Then $\boldsymbol{a}' \in \{2,0\}^n$ and $f(\boldsymbol{a}') = 2$ since $(\boldsymbol{a}, \boldsymbol{a}') \in U_2$. Hence $\boldsymbol{a}' + \mathbf{1} =\in \{0,1\}^n$ and $f(\{0,1\}^n) = 2$ since $(\boldsymbol{a}'+\mathbf{1}, \boldsymbol{b}) \in U_2$ for any $\boldsymbol{b} \in \{0,1\}^n$. Thus if $f \in L_0$ then $f(\{0,1\}^n = f(\{1,2\}^n = 2$, and if $f \in L_2$ then $f(\{1,2\}^n) = f(\{0,1\}^n) + 1 = 1$. In both cases $f \in T_{12}$, contradicting Lemma 5.4.5. □

Lemma 5.4.11. $\overline{T}L \subseteq \tilde{B}$.

Proof. Let us prove $\overline{T}L \subseteq \overline{B}_0$. Suppose $f \in B_0$. Then f cannot take the values 1 and 2 on $\{0,1\}^n$, because $(1,2) \in B_0$ and $(\boldsymbol{a}, \boldsymbol{b}) \in B_0$ foe any $\boldsymbol{a}$, $\boldsymbol{b} \in \{0,1\}^n$. Therefore, either 1) $f(\{0,1\}^n) \subseteq \{0,1\}$ or 2) $f(\{0,1\}^n) \subseteq \{0,2\}$ holds exclusively for all $f \in B_0$. In case 1) $f \in T_{01}$ results. In case 2), if $f \in L_0$ then from $f(\{0,1\}^n) = f(\{0,2\}^n)$ we have $f \in T_{02}$; if $f \in L_2$ then we have $f \in T_{12}$, since $f(\{0,1\}^n) \subseteq \{0,2\}$ leads to $f(\{1,2\}^n) \subseteq \{1,2\}$ by modular operation. These contradict to Lemma 5.4.5. Thus $\overline{T}L \subseteq B_0$, and hence $\overline{T}L \subseteq \tilde{B}$ by σ-similar. □

Corollary 5.4.1. $\overline{T}L \subseteq \tilde{M}\tilde{U}\tilde{B}\tilde{T}_{pq}$.

Proof. From Lemmas 5.4.5, 5.4.9, 5.4.10, and 5.4.11. □

We now proceed to classify $\overline{T}(L \cup S) = \overline{T}(L\overline{S} + LS + \overline{L}S)$, considering each subset separately in this order.

(1) Classification of $L\overline{S} = L_0 + L_2$. The remaining sets are T_p type only from Lemma 5.4.11. Since $L_0 + L_2 = L_{00} + L_{20} + L_{01} + L_{22} + L_{02} + L_{21}$, possible classes are restricted to the following 3 classes by Lemma 5.4.8. Only an example to class *81 is sufficient.

$$\overline{T}LS\tilde{M}\tilde{U}\tilde{B}\tilde{T}_{pq} \begin{cases} 1)\ T_0\overline{T}_1\overline{T}_2 = L_{00} + L_{20} & f4.1 \quad (*81 = \#41) \\ 2)\ \overline{T}_0 T_1 \overline{T}_2 = L_{01} + L_{22} & f^{\sigma_2}4.1 \quad (*\underline{82} = \#40) \\ 3)\ \overline{T}_0\overline{T}_1 T_2 = L_{02} + L_{21} & f^{\sigma_1}4.1 \quad (*\underline{83} = \#39) \end{cases}$$

(2) Classification of $LS = L_0 + L_2$. The remaining sets are T_p type only from Lemma 5.4.1. Further from Lemma 5.4.2 only two cases:$\underline{T}_p$ or $\tilde{T}_p$ are possible, where $\underline{T}_p = T_0T_1T_2$. Thus L_1 is divided into the following 2 classes.

$$\overline{T}LS\tilde{M}\tilde{U}\tilde{B}\tilde{T}_{pq} \begin{cases} 1)\ \tilde{T}_p & f4.2 \quad (*84 = \#42) \\ 2)\ \underline{T}_p & f4.3 \quad (*85 = \#187) \end{cases}$$

(3) Classification of $\overline{L}S$. The remaining sets are T_p and T_{pq} types from Lemma 5.4.3. By Lemmas 5.4.2,5.4.4 and 5.1.5 possible classes are restricted to the following 3 classes, where $\underline{T}_{pq}$ denotes the intersection $T_{01}T_{12}T_{20}$.

$$\overline{T}LS\tilde{M}\tilde{U}\tilde{B} \begin{cases} 1)\ \tilde{T}_p\tilde{T}_{pq} = S_1 + S_2 & f4.4 \quad (*86 = \#11) \\ \left.\begin{array}{l} 2)\ \underline{T}_p\underline{T}_{pq} \\ 3)\ \underline{T}_p\tilde{T}_{pq} \end{array}\right\} = S_0 & \begin{array}{l} f4.5 \quad (*87 = \#297) \\ f4.6 \quad (*88 = \#135) \end{array} \end{cases}$$

Conclusions of Section 5.4.

We have completed the classification of $\overline{T}(L \cup S)$. $|\overline{T}(L \cup S)|$=8, and 6 classes out of them are σ-similar free (underline of the class number preceded by * denotes σ-similar class).

5.5. Classification of M

In this section the set $M := \overline{TSL}(M_1 \cup M_2 \cup M_0)$ is classified. For simplicity, we abbreviate $\overline{TSL}$ to $\overline{N}$. The set M is divide into subsets and they can be represented by using σ-similar as follows:

$$M = M^1 + (M^1)^{\sigma_1} + (M^1)^{\sigma_2} + M^2 + (M^2)^{\sigma_0} + (M^2)^{\sigma_2},$$

where $M^1 := \overline{N}M_1M_2\overline{M}_0$ and $M^2 := \overline{N}M_1\overline{M}_2\overline{M}_0$. Thus it is sufficient to consider M^1 and M^2. Note that no classes from M^1 (M^2)are σ_1 and σ_2 (σ_0 and σ_1) invariant.

5.5.1. Classification of M^1

First we prepare a lemma for M^1. We follow a convention that a suffix pqr represents any of 012, 120 and 201.

Lemma 5.5.1.

$$\begin{aligned} &1)\ M_q\overline{T} &&\subseteq T_pT_r,\\ &2)\ M_qT_pT_q &&\subseteq T_{pq},\\ &3)\ M_qT_qT_r &&\subseteq T_{qr}. \end{aligned}$$

Proof. 1) Suppose $f(\boldsymbol{p}) \neq p$ and $f \in M_q\overline{T}$. Then $f(\boldsymbol{p}) = q$ or r. If $f(\boldsymbol{p}) = q$ then $f(\boldsymbol{a}) \in D(q, r)$ for any $\boldsymbol{a} \in E_3^n$, because $\boldsymbol{p} \leq_q \boldsymbol{a}$ implies $f(\boldsymbol{p}) = q \leq_q f(\boldsymbol{a})$. Thus $f \in T$, a contradiction. If $f(\boldsymbol{p}) = r$ then analogously $f \equiv r$, again contradicting to $f \in T$. With respect to T_r the proof is similar. 2) and 3) are obvious. □

From Lemma 5.5.1 we have the following.

Corollary 5.5.1. $M_1M_2\overline{T} \subseteq T_0T_1T_2T_{01}T_{12}T_{20}$.

Classification of M^1. From Lemma 5.1.3 and from Lemma 5.1.5 we have

$$M^1 \subseteq B_0U_0. \tag{5.20}$$

Considering Corollary 5.5.1 the remaining sets are now restricted to U_2, U_1, B_1 and B_2. Let us consider U type first. Following four classes are possible (we call such trivial classification *induced* classes): (1) U_2U_1, (2) $U_2\overline{U}_1$, (1) $\overline{U}_2U_1$ and (1) $\overline{U}_2\overline{U}_1$. For each of these subsets we consider the classification by B type maximal sets subsequently.

(1) U_2U_1: Assume $f \in M_1U_2U_1$. Then from (5.20) $f \in M^1U_2U_1U_0 = M^1K$. While $M^1K \subseteq \overline{T}K = \phi$ from Theorem 5.1.4.

(2) $U_2\overline{U}_1$: Assume $f \in M_1U_2\overline{U}_1$. Then from (5.20) and Lemma 5.1.6, $f \in B_1$ is derived. Next we conclude $f \notin B_2$, because assuming $f \in B_2$ results $f \in K$, a contradiction. So this case gives one class.

(3) $\overline{U}_2U_1$: This is the σ_0-similar of the above (2).

(4) $\overline{U}_2\overline{U}_1$: We conclude $f \in \overline{B}_2\overline{B}_1$ from (5.20) and Lemma 5.1.7.

Thus M^1 is divided into the following three classes.

$$M^1 = \left\{ \begin{array}{lll} 1)\ U_2\overline{U}_1B_1\overline{B}_2 & f5.1 & (*89 = \#402) \\ 2)\ \overline{U}_2U_1\overline{B}_1B_2 & f^{\sigma_0}5.1 & (*\underline{90} = \#\underline{401}) \\ 3)\ \overline{U}_2\overline{U}_1\overline{B}_1\overline{B}_2 & f5.2 & (*91 = \#390) \end{array} \right.$$

From above considerations we note that the structure of U type maximal sets determines the structure of the B type maximal sets in M^1. Hence we have the following.

Corollary 5.5.2. *$U_2 = B_1$ and $U_1 = B_2$ in $M_1M_2\overline{T}$.*

5.5.2. Classification of M^2

We divide $M^2 = M_1\overline{M}_2\overline{M}_0$ into subsets using σ-similar as follows.

$$M^2 = N_0 + N_1 + N_2 = N_0 + N_1 + N_0^{\sigma_1},$$

where $N_i = \{f | f \in M^2 \text{ and } f(\mathbf{1}) = i\}$. We classify N_0 and N_1 in the following subsections separately.

5.5.2.1. N_0

We prove the following lemma for N_0.

Lemma 5.5.2. $N_0 \subseteq T_0\overline{T}_1T_2T_{01}\overline{T}_{12}\overline{U}_0\overline{U}_1\overline{B}_1\overline{B}_2$.

Proof. Assume $f \in N_0$. Then Lemma 5.5.1 implies $f \in T_0T_2$. We have $f \in \overline{T}_1\overline{T}_{12}T_{01}$ because $f \in M_1$ implies $f(\{0,1\}^n) = 0$ since $f(\mathbf{1}) = 0$. We have $f \in \overline{T}_1\overline{T}_{12}T_{01}$ from $(\mathbf{1}, \mathbf{2}) \in U_0B_1$ and $(f(\mathbf{1}), f(\mathbf{2})) = (0, 2)$. Finally, Let us show $f \in \overline{U}_1\overline{B}_2$. By $f \notin T$ there is $\boldsymbol{a} \notin \{01\}^n$ such that $f(\boldsymbol{a}) = 1$. Define $\boldsymbol{a}'$ as follows: $a_i' = 0$ if $a_i = 2$, $a_i' = a_i$ otherwise for each i. Obviously $(\boldsymbol{a}, \boldsymbol{a}') \in U_1B_2$ and $\boldsymbol{a}' \in \{0,1\}^n$, hence $f \in \overline{U}_1\overline{B}_2$, because $(f(\boldsymbol{a}), f(\boldsymbol{a}')) = (1, 0)$. □

We divide N_0 into two subsets by T_{20} as follows:

$$N_0 = N_0T_{20} + N_0\overline{T}_{20}.$$

Each subset we classify by the remaining sets U_2 and B_0.

Classification of N_0T_{20}

There is a representative in each induced set by the remaining U_2 and B_0. Thus, N_0T_{20} is divided into the following 4 classes.

$$N_0T_{20} = \begin{cases} 1)\ U_1B_0 & f5.3 \quad (*\ 98 = \#287) \\ 2)\ \overline{U}_1B_0 & f5.4 \quad (*\ 99 = \#234) \\ 3)\ U_2\overline{B}_0 & f5.5 \quad (*100 = \#239) \\ 4)\ \overline{U}_2\overline{B}_0 & f5.6 \quad (*101 = \#184) \end{cases}$$

Lemma 5.5.3. $N_0\overline{T}_{20} \subseteq \overline{B}_0$

Proof. Let $f \in N_0\overline{T}_{20}$. Then there is $\boldsymbol{a} \in \{0,2\}^n$ such that $f(\boldsymbol{a}) = 1$. From Lemma 5.5.2 we have $f(\boldsymbol{2}) = 2$. Thus $f \notin B_0$ from $(\boldsymbol{a}, \boldsymbol{2}) \in B_0$ and $(f(\boldsymbol{a}), f(\boldsymbol{2})) = (1,2) \notin B_0$. □

Classification of $N_0\overline{T}_{20}$

There is a representative in each induced set by the remaining U_2. Thus, $N_0\overline{T}_{20}$ is divided into the following 2 classes.

$$N_0\overline{T}_{20} = \begin{cases} 1)\ U_2\overline{B}_0 & f5.7 \quad (*102 = \#186) \\ 2)\ \overline{U}_2 B_0 & f5.8 \quad (*103 = \#134) \end{cases}$$

Thus $|N_0| = |N_0T_{20}| + |N_0\overline{T}_{20}| = 4 + 2 = 6$, all of which are σ-similar free.

5.5.2.2 N_1

The classification of N_1 is not so simple as that of N_0.

Lemma 5.5.4. $N_1 \subseteq T_0T_1T_2T_{01}T_{12}$

Proof. From $\{01\}^n \leq \boldsymbol{1} \leq \{1,2\}^n$ and $f \in M_1$ we have $f(\{01\}^n) \leq f(\boldsymbol{1}) \leq f(\{1,2\}^n)$. Thus $f(\{01\}^n) \subseteq \{0,1\}$, $f(\{1,2\}^n) \subseteq \{1,2\}$ since $f(\boldsymbol{1}) = 1$. From $f \in \overline{T}$ there are $\boldsymbol{a}$, $\boldsymbol{b}$ such that $f(\boldsymbol{a}) = 0$, $f(\boldsymbol{b}) = 2$. Hence $f(\boldsymbol{0}) = 0$, $f(\boldsymbol{2}) = 2$. □

Thus the remaining sets are T_{20} and U type and B type sets. Let us divide N_1 into two subsets by T_{20}: $N_1 = N_1\overline{T}_{20} + N_1T_{20}$. Consider the classification of each subset by the U and B type sets. There exists a simple structure in the case of $N_1\overline{T}_{20}$. However, in the other case we must consider the structure of the set $M_1\overline{M}_2\overline{M}_0$.

Classification of $N_1\overline{T}_{20}$

Lemma 5.5.5. $N_1\overline{T}_{20} \subseteq \overline{U}_1\overline{U}_0\overline{U}_2$.

Proof. Let $f \in N_1\overline{T}_{20}$. Then there is $\boldsymbol{a} \in \{2,0\}^n$ such that $f(\boldsymbol{a}) = 1$. From $(\boldsymbol{a}, \boldsymbol{2}) \in B_0U_1$, $f \in T_2$ and $(f(\boldsymbol{a}), f(\boldsymbol{2})) = (1,2) \in \overline{B}_0\overline{U}_1$. By σ_1-similar we have $N_1\overline{T}_{20} \subseteq \overline{B}_2\overline{U}_1$. □

Note 5.5.1. [Miy71] $M_1\overline{T}_{20}\overline{T} \subseteq \overline{M}_2\overline{M}_0$.

Thus $f \in M_1$ belongs to $\overline{M}_2\overline{M}_0$ if $f \in \overline{T}_{20}\overline{T}$.

As for the 8 induced classes by the remaining sets B_1, U_2 and U_0, the class $U_2U_0\overline{B}_1$ is empty from Lemma 5.1.6. There are representatives in all the other classes. Thus $N_1\overline{T}_{20}$ is divided into the following 7 classes.

$$N_0\overline{T}_{20} = \begin{cases} 1)\ U_2U_0B_1 & f5.\ 9 & (*110 = \#363) \\ 2)\ U_2\overline{U}_1B_1 & f5.10 & (*111 = \#339) \\ 3)\ \overline{U}_2U_1B_1 & f^{\sigma_1}5.10 & (*\underline{112} = \#\underline{337}) \\ 4)\ \overline{U}_2\overline{U}_1B_1 & f5.11 & (*113 = \#283) \\ 5)\ U_2\overline{U}_1\overline{B}_1 & f5.12 & (*114 = \#286) \\ 6)\ \overline{U}_2U_1\overline{B}_1 & f^{\sigma_1}5.12 & (*\underline{115} = \#\underline{284}) \\ 7)\ \overline{U}_2\overline{U}_1B_1 & f5.13 & (*116 = \#232) \end{cases}$$

The remaining part of this section is devoted to the classification of N_1T_{20} by U type and B type maximal sets.

We call two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are *neighbors* by the order relation $\leq$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ differs only one coordinate i and there is no c such that $a < c < b$. Let us introduce the following notation to represent neighboring vectors (suffix i may be omitted):

$$\begin{aligned} \boldsymbol{a}0_i &= (a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) \\ \boldsymbol{a}1_i &= (a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n) \\ \boldsymbol{a}2_i &= (a_1, \ldots, a_{i-1}, 2, a_{i+1}, \ldots, a_n). \end{aligned}$$

Neighboring vectors are useful because of the following lemma.

Lemma 5.5.6. [Miy71] *If $f \in \overline{M}_q$ then there are neighboring two vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ such that $\boldsymbol{b} <_q \boldsymbol{c}$ and $f(\boldsymbol{b}) \not\leq_q f(\boldsymbol{c})$.*

Now we prepare a lemma which plays an important role in the classification of N_1T_{20}.

Lemma 5.5.7. *If $f \in M_1\overline{M}_2\overline{M}_0$ then there are sets (set) of neighboring vectors $\boldsymbol{u}0$, $\boldsymbol{u}1$, $\boldsymbol{u}$ and $\boldsymbol{v}0$, $\boldsymbol{v}1$, $\boldsymbol{v}$ corresponding to at least one of the cases indicated in the following Table 5.6.*

Table 5.6:

cases	$\backslash x$	$u0$	$u1$	$u2$	$v0$	$v1$	$v2$	*class*
I	$f(x)$	0	0	2	0	2	2	$U_2U_0\overline{B}_1$
II	$f(x)$	0	0	2	0	1	1	$\overline{U}_1\overline{U}_0\overline{B}_2$
III	$f(x)$	1	1	2	0	2	2	$\overline{U}_2\overline{U}_1\overline{B}_0\overline{B}_1$
IV	$f(x)$	1	1	2	0	1	1	$\overline{U}_1\overline{B}_0\overline{B}_2$
V	$f(x)$	0	0	1				$\overline{U}_0\overline{U}_1\overline{B}_2$
VI	$f(x)$	1	2	2				$\overline{U}_2\overline{U}_1\overline{B}_0$

Table 5.7:
Possible values of $f \in M_1\overline{M}_2$.

cases	$\backslash x$	$u0$	$u1$	$u2$
I	$f(x)$	0	0	1
II	$f(x)$	0	0	2
III	$f(x)$	1	1	2
IV	$f(x)$	1	2	2

Proof. Assume $f \in M_1\overline{M}_2\overline{M}_0$ and f depends on at least two variables. From $f \in M_1\overline{M}_2$, we show that f has at least one of the set of neighboring vectors indicated in Table 5.7. From $f \in \overline{M}_2$ there are neighboring $\boldsymbol{a}$ and $\boldsymbol{a}'$ such that $f(\boldsymbol{a}) \not\leq_2 f(\boldsymbol{a}')$ from Lemma 5.5.6. Putting $\boldsymbol{a} := \boldsymbol{u}1$ and $\boldsymbol{a}' := \boldsymbol{u}2$, $f(\boldsymbol{x})$ has values (1), (2) or (3) of Table 5.8, where $*$ may be any of 0,1 or 2. While the condition $f \in M_1$ requires $f(\boldsymbol{u}0) \leq f(\boldsymbol{u}1) \leq f(\boldsymbol{u}2)$. Hence the case (1) is impossible and $*$ must be 0 for the both cases (2) and (3). Thus the cases of I and II of Table 5.7 are necessary. The other case of the same table are derived by taking the neighboring vectors $\boldsymbol{a} := \boldsymbol{u}2$ and $\boldsymbol{a}' := \boldsymbol{u}0$.

Table 5.8:
Possible values of
$f \in \overline{M}_2$ and $f(\boldsymbol{u}1) \not\leq_2 f(\boldsymbol{u}2)$.

cases	$\backslash x$	$u0$	$u1$	$u2$
(1)	$f(x)$	$*$	2	1
(2)	$f(x)$	$*$	0	1
(3)	$f(x)$	$*$	0	2

Table 5.9:
Possible values of $f \in M_1\overline{M}_0$.

cases	$\backslash x$	$v0$	$v1$	$v2$
I	$f(x)$	0	0	1
II	$f(x)$	0	1	1
III	$f(x)$	0	2	2
IV	$f(x)$	1	2	2

In the same manner from $f \in M_1\overline{M}_0$ we conclude that f must have at least a construction out of the four cases in Table 5.9. From Table 5.7 and Table 5.9 the lemma follows. □

In the rightmost column of Table 5.6 the sets are shown to which the corresponding $f(\boldsymbol{x})$ obviously belongs to.

We show lemmas.

Lemma 5.5.8. $B_1T_{20}M_1 \subseteq U_2U_0$.

Proof. Suppose $f \in B_1 T_{20} M_1 \overline{U}_2$. There are $(\boldsymbol{a}, \boldsymbol{b}) \in U_2$ such that $(f(\boldsymbol{a}), f(\boldsymbol{b})) = (0,2)$ or (1,2). However $f \in B_1$ requires $(f(\boldsymbol{a}), f(\boldsymbol{b})) = (1,2)$ since $(\boldsymbol{a}, \boldsymbol{b}) \in U_2$ implies $(\boldsymbol{a}, \boldsymbol{b}) \in B_1$. From $f \in \overline{T}_{20}$ there is $a_i = 1$. Define $\boldsymbol{a}'$ as follows: $a'_i = 0$ if $a_i = 1$, $a'_i = a_1$ otherwise. Then $\boldsymbol{a}' \in \{2,0\}^n$ and $f(\boldsymbol{a}') = 0$ from $f \in T_{20}$. On the other hand, $(\boldsymbol{a}', \boldsymbol{b}) \in B_1$ since if $a_i = 1$ then $b_i = 0$ or 1 from $(a_i, b_i) \in U_2$. Thus $(f(\boldsymbol{a}', f(\boldsymbol{b})) = (0,2) \notin B_1$ leads to a contradiction. With respect to U_0 the proof is similar. □

Lemma 5.5.9. *$U_1 \subseteq \overline{B}_1$, $U_2 \subseteq \overline{B}_2$ and $U_0 \subseteq \overline{B}_0$ in $M_1 \overline{M}_2 \overline{M}_0$.*

Proof. If $f \in U_1$ then f have values corresponding to the case I of Table 5.6 from Lemma 5.5.7, hence $f \in \overline{B}_1$. If $f \in U_2$ then f corresponds to II, IV or V of the same table. Hence $f \in \overline{B}_2$. If $f \in U_0$ then f corresponds to III, IV or VI of the same table. Hence $f \in \overline{B}_0$. □

Lemma 5.5.10. *$\overline{B}_0 B_1 \overline{B}_2 = B_2 B_0 \overline{B}_1$ in $M_1 \overline{M}_2 \overline{M}_0 T_{20}$.*

Proof. 1) Suppose $f \in \overline{B}_0 B_1 \overline{B}_2$. Then f has a cases of IV, V or VI of Table 5.6 from Lemma 5.5.7. On the other hand, from $f \in B_1 T_{20} M_1$ and from Lemma 5.5.8, $f \in U_2 U_0$ is derived. Thus the cases V and VI are impossible. Hence $f \in \overline{U}_1 U_2 U_3$ from IV of the table. 2) The converse is obvious from Lemma 5.1.6 and from Lemma 5.5.9. □

Lemma 5.5.11. *$U_1 \subseteq B_2 B_0$ in M_1.*

Proof. We prove $M_1 U_1 \subseteq B_2$. The other is the σ_1-similar of this. Assume $f \in M_1 U_1 \overline{B}_2$. There are $(\boldsymbol{a}, \boldsymbol{b}) \in B_1$ such that $(f(\boldsymbol{a}), f(\boldsymbol{b})) = (0,1)$. Define $\boldsymbol{a}'$ as follows: $a'_i = 0$ if $(a_i, b_i) = (1,2)$, $a'_i = 1$ if $(a_i, b_i) = (2,1)$, $a'_i = a_i$ otherwise. Then $\boldsymbol{a}' \leq \boldsymbol{a}$, hence $f(\boldsymbol{a}') = 0$. While by above construction we have $(\boldsymbol{a}', \boldsymbol{b}) \in U_1$. However $(f(\boldsymbol{a}'), f(\boldsymbol{b})) = (0,1) \in \overline{U}_1$, a contradiction. □

Note 5.5.2. Combining Lemmas 5.1.7 and 5.1.11 we have

$$B_2 B_0 = U_1 \text{ in } M_1.$$

We now classify the concerning set $N_1 T_{20}$. First we decide possible classes by U maximal sets, then each of this class we divide by B maximal sets.

Consider 8 induced sets by U maximal sets. From Table 5.6, we will conclude neither class $U_2\overline{U}_0U_1$ nor $\overline{U}_2U_0U_1$ exists. First, let us confirm this. Assume $f \in U_2\overline{U}_0U_1$ then $f \in B_0$ from Lemma 5.1.6. Hence only cases I, II or V of Table 5.6 is possible for f. While in all these cases either $f \in \overline{U}_2$ or $f \in \overline{U}_1$, a contradiction. The discussion is analogous to the second case: $\overline{U}_2U_0U_1$. Further, the class $U_2U_0U_1N_1$ is empty from Theorem 5.1.3 and $K\overline{T} = \phi$. We classify the remaining 5 sets by B maximal sets.

Classification of N_1T_{20}

(1) $U_2U_0\overline{U}_1$ coincides with $\overline{B}_0B_1\overline{B}_2$ from Lemma 5.5.10.

$$U_2U_0\overline{U}_1\overline{B}_0B_1\overline{B}_2 \quad f5.14 \quad (*117 = \#388)$$

(2) $\overline{U}_2\overline{U}_0U_1$ coincides with $B_0\overline{B}_1B_2$ from Lemmas 5.5.10, 5.1.4 and 5.1.7

$$\overline{U}_2\overline{U}_0\overline{U}_1B_0\overline{B}_1B_2 \quad f5.15 \quad (*118 = \#399)$$

(3) $U_2\overline{U}_0\overline{U}_1$. From Lemmas 5.5.8 and 5.5.9 we have $\overline{B}_1\overline{B}_2$. There exist representatives for the two induced classes by the remaining set B_0.

$$U_2\overline{U}_0\overline{U}_1\left\{\begin{array}{c} B_0 \\ \overline{B}_0 \end{array}\right\}\overline{B}_1\overline{B}_2 \quad \begin{array}{ll} f5.16 & (*119 = \#362) \\ f5.17 & (*120 = \#338) \end{array}$$

(4) $\overline{U}_2U_0\overline{U}_1$ is the σ_1-similar of (3).

(5) $\overline{U}_2\overline{U}_0\overline{U}_1$. The possible classes by B maximal sets are restricted to the following 3 classes by Lemmas 5.5.8 and 5.1.7.

$$\overline{U}_2\overline{U}_0\overline{U}_1\left\{\begin{array}{c} B_0 \\ \overline{B}_0 \\ \overline{B}_0 \end{array}\right\}\overline{B}_1\left\{\begin{array}{lll} \overline{B}_2 & f5.18 & (*123 = \#335) \\ B_2 & f^{\sigma_1}5.18 & (*124 = \#346) \\ \overline{B}_2 & f5.19 & (*125 = \#282) \end{array}\right.$$

Conclusions of Section 5.5.

We have completed the classification of $M = \overline{TLS}(M_1 \cup M_2 \cup M_0)$. We summarize the classification as follows: $M = M^1 + (M^1)^{\sigma_1} + (M^1)^{\sigma_2} + M^2 + (M^2)^{\sigma_0} + (M^2)^{\sigma_2}$. M^1 is separated into the three classes. Further $M^2 = N_0 + N_1 + N_2 = N_0 + N_1 + N_0^{\sigma_1}$. $|N_0|$=6. $N_1 = N_1\overline{T}_{20} + N_1T_{20}$ and $|N_1\overline{T}_{20}| = 7$, $|N_1T_{20}| = 9$, hence $|N_1| = 16$, thus $|M^2| = 28$. We note that no class is common among N_0, N_1 and $N_0^{\sigma_1}$.

Therefore $|M| = 3|M_1| + 3|M_2| = 3 \times 3 + 3 \times 28 = 93$, of which σ-similar-free classes are 19.

5.6. Classification of U

In this section the set $U := \{f | f \in \overline{TLSM}(U_2 \cup U_0 \cup U_1)\}$ will be classified. Obviously we can write $U = U^1 + (U^1)^{\sigma_1} + (U^1)^{\sigma_2} + U^2 + (U^2)^{\sigma_1} + (U^2)^{\sigma_2}$, where

$$U^1 := \overline{TLSM} U_2 \overline{U}_0 U_1 \text{ and } U^2 := \overline{TLSM} \overline{U}_2 U_0 \overline{U}_1.$$

Thus it is sufficient to consider U^1 and U^2 in subsections 5.6.1 and 5.6.2, respectively. We note that any class in U^1 and U^2 is neither σ_1- nor σ_2-invariant.

5.6.1. U^1.

We prepare several lemmas. The symbol $\overline{D}$ is used to indicate that we are concerning onto functions.

Lemma 5.6.1. $U_2 U_1 \subseteq T_{01} T_{20} T_0 B_0$.

Proof. From Lemmas 5.1.6 and 5.1.8 it is sufficient to show $U_2 U_1 \subseteq T_{01} T_{20}$. Suppose $f \in U_2 U_1$. There is $\boldsymbol{a} \in \{0,1\}^n$ such that $f(\boldsymbol{a}) = 2$. We have $f(\{0,1\}^n) = 2$ since $(\boldsymbol{a}, \{0,1\}^n) \in U_2$. From $f \in \overline{D}$ there is $\boldsymbol{b}$ such that $f(\boldsymbol{b}) = 1$. Define $\boldsymbol{b}'$ as follows: $b'_i = 0$ if $b_i = 2$, $b'_i = b_i$ otherwise. Obviously $(\boldsymbol{b}, \boldsymbol{b}') \in U_1$ and hence $f(\boldsymbol{b}') = 1$ and $\boldsymbol{b}' \in \{0,1\}^n$, contradicting the above assertion. As for T_{20} the proof is similar. □

Lemma 5.6.2. *If $f \in \overline{M}_1 U_2 U_1$ then there are vectors $\boldsymbol{u}0$, $\boldsymbol{u}1$ and $\boldsymbol{u}2$ such that*

$$(f(\boldsymbol{u}0), f(\boldsymbol{u}1), f(\boldsymbol{u}2)) = (1,0,1) \text{ or } (2,2,0)$$

as shown in Table 5.10.

Table 5.10:
Possible values of $f \in \overline{M}_0 U_2 U_0$.

cases	$f \setminus \boldsymbol{x}$	$\boldsymbol{u}0$	$\boldsymbol{u}1$	$\boldsymbol{u}2$
I	$f(\boldsymbol{x})$	1	0	1
II	$f(\boldsymbol{x})$	2	2	0

Table 5.11:
Possible values of $f \in \overline{M}_0$.

cases	$f \setminus \boldsymbol{x}$	$\boldsymbol{u}0$	$\boldsymbol{u}1$	$\boldsymbol{u}2$
(1)	$f(\boldsymbol{x})$	2	*	0
(2)	$f(\boldsymbol{x})$	0	*	1
(3)	$f(\boldsymbol{x})$	2	*	1
(4)	$f(\boldsymbol{x})$	0	2	*
(5)	$f(\boldsymbol{x})$	1	0	*
(6)	$f(\boldsymbol{x})$	1	2	*

Proof. If $f \in \overline{M}_0$ then f has at least one of values out of 6 cases indicated in Table 5.11. This can be easily shown from $f \in \overline{M}_0$ analogously as Lemma 5.5.6. Then considering the additional condition of $f \in U_2 U_1$ leads to Table 5.10. □

Lemma 5.6.3. $\overline{M}_0 U_2 U_1 \subseteq \overline{T}_{12}$.

Proof. Assume $f \in \overline{M}_0 U_2 U_1 T_{12}$. Then there are vectors satisfying $(f(\boldsymbol{u}0), f(\boldsymbol{u}1), f(\boldsymbol{u}2)) = (1,0,1)$ or (2,2,0) from Lemma 5.6.2. Consider the first case. Define $\boldsymbol{v} \in E_3^{n-1}$ as follows: $v_i = 2$ if $u_i = 0$, $v_i = u_i$ otherwise (we may assume $n \geq 2$, since the assertion holds when $n = 1$). Obviously $(\boldsymbol{u}1, \boldsymbol{v}1) \in U_1$ and $\boldsymbol{v}1 \in \{1,2\}^n$. Hence $f(\boldsymbol{v}1) = 2$ from $f \in T_{12}$ and $f(\boldsymbol{u}1) = 0$. On the other hand, we have $f(\boldsymbol{v}0) = 1$ from $(\boldsymbol{u}0, \boldsymbol{v}0) \in U_1$. Thus $(f(\boldsymbol{v}0), f(\boldsymbol{v}1)) = (1,2) \notin U_2$, contradicting $f \in U_2$ since $(\boldsymbol{v}0, \boldsymbol{v}1) \in U_2$.

For the second case, the proof is similar. □

Lemma 5.6.4. $U_2 \overline{T}_{12} \subseteq \overline{B}_1$.

Proof. Suppose $f \in U_2 \overline{T}_{12}$. Then there is $\boldsymbol{a} \in \{1,2\}^n$ such that $f(\boldsymbol{a}) = 0$. From $f \in \overline{D}$ there is $\boldsymbol{b}$ such that $f(\boldsymbol{b}) = 2$. Define $\boldsymbol{b}'$ as follows: $b'_i = 1$ if $(a_i, b_i) = (2,0)$, $b'_i = b_i$ otherwise for each i. Then $f(\boldsymbol{b}') = 2$ from $(\boldsymbol{b}, \boldsymbol{b}') \in U_2$. Thus we see that $(\boldsymbol{a}, \boldsymbol{b}') \in B_1$ and $(f(\boldsymbol{a}), f(\boldsymbol{b}')) \notin B_1$. Note that no $(a_i, b_i) = (0,2)$ occurs because $\boldsymbol{a} \in \{1,2\}^n$. □

Taking σ_0-similar of this we have the following.

Corollary 5.6.1. $U_2 U_1 \overline{D}\,\overline{T}_{12} \subseteq \overline{B}_1 \overline{B}_2$.

Corollary 5.6.2. $U^1 \subseteq T_{01} T_{20} \overline{T}_{12} T_0 B_0 \overline{B}_1 \overline{B}_2$.

Proof. From Lemmas 5.6.1, 5.6.3 and Corollary 5.6.1. □

Classification of U^1. Now the remaining sets are only T_1 and T_2 from Corollary 5.6.2. There are representatives in all 4 induced classes by these sets. Thus U^1 is divided into the following 4 classes.

$$U^1 = \begin{cases} 1)\ T_1 T_2 & f6.1 & (*182 = \#320) \\ 2)\ T_1 \overline{T}_2 & f6.2 & (*183 = \#267) \\ 3)\ \overline{T}_1 T_2 & f^{\sigma_0}6.2 & (*\underline{184} = \#\underline{266}) \\ 4)\ \overline{T}_1 \overline{T}_2 & f6.3 & (*185 = \#220) \end{cases}$$

5.6.2. U^2.

For convenience we again follow the convention that the suffix pqr represents 012, 120 and 201. We prepare several lemmas.

Lemma 5.6.5. *If* $f \in U_r T_{pq} \overline{D}$ *then* $f(\boldsymbol{a}) = r$ *and* $f(\boldsymbol{b}) = r$ *for some* $\boldsymbol{a} \in \{p, r\}^n$ *and* $\boldsymbol{b} \in \{p, r\}^n$.

Proof. From $f \in \overline{D} T_{pq}$ there is $\boldsymbol{u}$ such that $f(\boldsymbol{u}) = r$ and there is i such that $u_i = r$. Define $\boldsymbol{a}$ and $\boldsymbol{b}$ as follows: $a_i = p,\ b_i = q$ if $u_i \neq r$, otherwise $a_i = b_i = u_i\ (= r)$. Then $f(\boldsymbol{a}) = f(\boldsymbol{b}) = r$ follows from $(\boldsymbol{a}, \boldsymbol{u}) \in U_r$ and $(\boldsymbol{b}, \boldsymbol{u}) \in U_r$. □

Lemma 5.6.6.

$$\begin{aligned} U_r T_{pq} \overline{T}_{pr} \overline{D} &\subseteq \overline{B}_p, \\ U_r T_{pq} \overline{T}_{qr} \overline{D} &\subseteq \overline{B}_q. \end{aligned}$$

Proof. Assume $f \in U_r T_{pq} \overline{T}_{pr} \overline{D}$. Then $f(\boldsymbol{b}) = q$ for some $\boldsymbol{b} \in \{p, r\}^n$. On the other hand, $f(\boldsymbol{a}) = r$ for some $\boldsymbol{a} \in \{p, r\}^n$ from Lemma 5.6.5 ($\boldsymbol{a} \neq \boldsymbol{b}$). Then $f \notin B_p$ because $(\boldsymbol{b}, \boldsymbol{a}) \in B_p$. The second relation is similar. □

Lemma 5.6.7. $T_p \overline{T}_{pq} \subseteq \overline{B}_p$.

Proof. From $f \in T_p \overline{T}_{pq}$ we have $f(\boldsymbol{p}) = p$ and $f(\boldsymbol{a}) = r$ for some $\boldsymbol{a} \in \{p, q\}^n$. Then $f \in \overline{B}_q$ because $(\boldsymbol{p}, \boldsymbol{a}) \in B_q$. □

Corollary 5.6.3. $T_p T_q \overline{T}_r \subseteq \overline{B}_r$.

Proof. From $f \in \overline{T}_r$ and Lemma 5.1.8, either $f \in \overline{T}_{pr}$ or $f \in \overline{T}_{qr}$. From $f \in T_p T_q$ and Lemma 5.6.7 we have $f \in \overline{B}_r$ in both cases. □

Lemma 5.6.8. $\overline{T}_p T_{pq} \overline{D} \subseteq \overline{T}_{pr} \overline{B}_p$.

Proof. From $f \in \overline{T}_p T_{pq}$ we have $f(\boldsymbol{p}) = q$, hence $f \in \overline{T}_{pr}$. Further $f(\boldsymbol{a}) = r$ for some $\boldsymbol{a}$ from $f \in \overline{D}$. Thus we conclude $f \in \overline{B}_r$ from $(\boldsymbol{p}, \boldsymbol{a}) \in B_p$. □

Lemma 5.6.9. $\overline{T}_p \overline{T}_q \overline{T}_r T_{pq} \subseteq \overline{B}_r$.

Proof. From $f \in \overline{T}_p \overline{T}_q \overline{T}_r T_{pq}$ we have $f(\boldsymbol{p}) = q,\ f(\boldsymbol{q}) = p$, and $f(\boldsymbol{r}) = p$ or q. Hence $f \in \overline{B}_r$ from $(\boldsymbol{p}, \boldsymbol{r}) \in B_r$ and $(\boldsymbol{q}, \boldsymbol{r}) \in B_r$. □

We divide U^2 into two subsets by T_{12} as follows:

$$U^2 = U^2 T_{12} + U^2 \overline{T}_{12}.$$

Then we classify each subset separately in Subsections 5.6.2.1 and 5.6.2.2 by the remaining T_p, T_{pq} and B type maximal sets.

5.6.2.1. U^2T_{12}.

We divide U^2T_{12} further into the following 4 induced subsets by T_1 and T_2: (1) T_1T_2, (2) $T_1\overline{T}_2$, (3) $\overline{T}_1T_2$ and (4) $\overline{T}_1\overline{T}_1$. Each subset is classified by the remaining maximal sets in this order.

(1) $U^2T_{12}T_1T_2$.

We divide this set by T_0 into the two induced subsets, and consider each case separately as (1a) and (1b).

(1a) $U^2T_{12}T_0T_1T_2$: This set is divided into the following 10 classes by the remaining T_{01}, T_{20} and B type maximal sets.

$$U^2T_{12}T_0T_1T_2 = \left\{ \begin{array}{lllll} 1)\ \overline{T}_{01}\overline{T}_{20} & & \overline{B}_0\overline{B}_1\overline{B}_2 & f6.4 & (*194 = \#163) \\ 2)\ \overline{T}_{01}T_{20} & & \overline{B}_0\overline{B}_1 \left\{ \begin{array}{l} B_2 \\ \overline{B}_2 \end{array} \right. & \begin{array}{l} f6.5 \\ f6.6 \end{array} & \begin{array}{l} (*195 = \#259) \\ (*196 = \#203) \end{array} \\ 3)\ T_{01}\overline{T}_{20} & = & (\overline{T}_{01}T_{20})^{\sigma_0} & & \\ 4)\ T_{01}T_{20} & & \left\{ \begin{array}{l} B_0\overline{B}_1\overline{B}_2 \\ \overline{B}_0B_1B_2 \\ \overline{B}_0B_1\overline{B}_2 \\ \overline{B}_0\overline{B}_1B_2 \\ \overline{B}_0\overline{B}_1\overline{B}_2 \end{array} \right. & \begin{array}{l} f6.7 \\ f6.8 \\ f6.9 \\ f^{\sigma_0}6.9 \\ f6.10 \end{array} & \begin{array}{l} (*199 = \#315) \\ (*200 = \#353) \\ (*201 = \#314) \\ (*\underline{202} = \#\underline{313}) \\ (*203 = \#258) \end{array} \end{array} \right.$$

Proof. 1), 2). From $f \in T_0T_1\overline{T}_{01}$ and from Lemma 5.6.7 we conclude $f \in \overline{B}_0\overline{B}_1$. Further in 1) from $f \in \overline{T}_{20}$ we have $f \in \overline{B}_2$. 4). Among 8 induced classes by B_0, B_1 and B_2, three which include B_0B_2 and B_0B_1 are impossible from Lemma 5.1.7 and $f \in \overline{U}_2\overline{U}_1$. □

(1b) $U^2\overline{T}_0T_{12}T_1T_2$: This set is classified into the following 5 classes. Note that from Corollary 5.6.3 we derive $f \in \overline{B}_0$. And from Lemma 5.1.8 the class $T_{01}T_{20}$ is impossible.

$$U^2T_{12}\overline{T}_0T_1T_2\overline{B}_0 = \left\{ \begin{array}{lllll} 1)\ T_{01}\overline{T}_{20} & & \overline{B}_2 \left\{ \begin{array}{l} B_1 \\ \overline{B}_1 \end{array} \right. & \begin{array}{l} f6.11 \\ f6.12 \end{array} & \begin{array}{l} (*204 = \#207) \\ (*205 = \#160) \end{array} \\ 2)\ \overline{T}_{01}T_{20} & = & (T_{01}\overline{T}_{20})^{\sigma_0} & & \\ 3)\ \overline{T}_{01}\overline{T}_{20} & & \overline{B}_1\overline{B}_2 & f6.13 & (*206 = \#213) \end{array} \right.$$

Proof. 1), 2), 3). From Lemma 5.6.6 and $f \in U_0T_{21}\overline{T}_{20}\overline{D}$ results $f \in \overline{B}_2$. 3). Further from Lemma 5.6.7 we have $f \in \overline{B}_1$. □

(2) $U^2T_{12}T_1\overline{T}_2$.

From Lemma 5.6.8 we have $f \in \overline{T}_{20}\overline{B}_2$. Hence the remaining sets are T_0, T_{01}, B_0 and B_1. We divide this set into two subsets by T_0, and consider each case separately as (2a) and (2b).

(2a) $U^2T_{12}T_1\overline{T}_2T_0$: This set is divided into the following 4 classes.

$$U^2T_{12}T_0T_1\overline{T}_2 = \begin{cases} 1) \ \overline{T}_{01}\overline{B}_0\overline{B}_1 & f6.14 \ (*209 = \#116) \\ 2) \ T_{01}\begin{cases} B_0\overline{B}_1 & f6.15 \ (*210 = \#210) \\ \overline{B}_0B_1 & f6.16 \ (*211 = \#208) \\ \overline{B}_0\overline{B}_1 & f6.17 \ (*212 = \#162) \end{cases} \end{cases}$$

Proof. 1). From $f \in T_0T_1\overline{T}_{01}$ and from Lemma 5.6.7 we have $f \in \overline{B}_0\overline{B}_1$. 2). Among 4 induced classes by B_0 and B_1 we cannot have B_0B_1 from Lemma 5.1.7 and $f \notin U_2$. □

(2b) $U^2T_{12}T_1\overline{T}_2\overline{T}_0$: This set is divided into the following 3 classes.

$$U^2T_{12}\overline{T}_0T_1\overline{T}_2 = \begin{cases} 1) \ \overline{T}_{01}\overline{B}_0\overline{B}_1 & f6.18 \ (*213 = \# \ 72) \\ 2) \ T_{01}\overline{B}_0\begin{cases} B_1 & f6.19 \ (*214 = \#165) \\ \overline{B}_1 & f6.20 \ (*215 = \#112) \end{cases} \end{cases}$$

Proof. 1). From Lemma 5.6.6 we have $U_0T_{21}\overline{T}_{10}\overline{D} \subseteq \overline{B}_1$ and from Lemma 5.6.7 we have $T_1\overline{T}_{10} \subseteq \overline{B}_0$. 2). From Lemma 5.6.8 we have $\overline{T}_0T_{01} \subseteq \overline{B}_0$. □

(3) $U^2T_{12}\overline{T}_1T_2$.

This set is the σ_0-similar of the case (2).

(4) $U^2T_{12}\overline{T}_1\overline{T}_2$.

We have $\overline{B}_1\overline{B}_1\overline{T}_{01}\overline{T}_{20}$ from Lemma 5.6.8. Thus the remaining sets are T_0 and B_0. Hence this set is divided into the following 3 classes.

$$U^2T_{12}T_1\overline{T}_2 = \begin{cases} 1) \ T_0\begin{cases} B_0 & f6.21 \ (*223 = \#118) \\ \overline{B}_0 & f6.22 \ (*224 = \# \ 74) \end{cases} \\ 2) \ \overline{T}_0\overline{B}_0 & f6.23 \ (*225 = \# \ 32) \end{cases}$$

Proof. 2). From Lemma 5.6.9 we have $\overline{T}_0\overline{T}_1\overline{T}_2T_{12}\overline{D} \subseteq \overline{B}_0$. □

Conclusion of Section 5.6.2.1 We have considered 4 subsets: $U^2T_{12}(T_1T_2+T_1\overline{T}_2+\overline{T}_1T_2 +\overline{T}_1\overline{T})$. We have $|U^2T_{12}T_1T_2| = 15$, $|U^2T_{12}\overline{T}_1T_2|$=$|U^2T_{12}T_1\overline{T}_2| = 7$ and $|U^2T_{12}\overline{T}_1\overline{T}_2| = 3$. Hence $|U^2T_{12}| = 32$, of which σ-similar-free classes are 20.

5.6.2.2. $U^2\overline{T}_{12}$.

Now the remaining part of U^2 is $U^2\overline{T}_{12}$. First we show two lemmas with respect to the remaining B, T_p and T_{pq} type maximal sets.

Lemma 5.6.10. $T_{pq}U_r\overline{D} \subseteq \overline{T}_p\overline{T}_q\overline{B}_p\overline{B}_q$.

Proof. Assume $f \in T_{pq}U_r\overline{D}$. Then $f(\boldsymbol{a}) = r$ for some $\boldsymbol{a} \in \{p,q\}^n$. Hence $f(\{p,q\}^n) = r$ since $(\boldsymbol{a}, \boldsymbol{b}) \in U_r$ for any $\boldsymbol{b} \in \{p,q\}^n$. Thus $f \in \overline{T}_p\overline{T}_q$. Further $f(\boldsymbol{c}) = q$ from $f \in \overline{D}$. Since $(\boldsymbol{p}, \boldsymbol{c}) \in B_p$ and $(f(\boldsymbol{p}), f(\boldsymbol{c})) = (r,q) \in \overline{B}_p$, we conclude $f \in \overline{B}_p$. As for $\overline{B}_q$ the proof is similar. □

Lemma 5.6.11. $\overline{T}_p\overline{D} \subseteq \overline{B}_p$.

Proof. From $f \in \overline{D}$ we have $f(\boldsymbol{a}) = r,\ f(\boldsymbol{b}) = q$. From $f \in \overline{T}_p$ we have $f(\boldsymbol{p}) = q$ or r. Since $(\boldsymbol{p}, \boldsymbol{a}) \in B_p$ and $(\boldsymbol{p}, \boldsymbol{b}) \in B_p$ we conclude $f \in \overline{B}_p$. □

Now we are ready to classify U^2T_{12}. Since $U^2T_{12} \subseteq \overline{T}_1\overline{T}_2\overline{B}_1\overline{B}_2$, we divide U^2T_{12} by T_{01} and T_{20} into 4 induced subsets: (1) $T_{01}T_{20}$, (2) $T_{01}\overline{T}_{20}$, (3) $\overline{T}_{01}T_{20}$ and (4) $\overline{T}_{01}\overline{T}_{20}$. We classify them separately. We have only two remaining sets T_0 and B_0 for each of above cases.

(1) $U^2\overline{T}_{12}T_{01}T_{20}$.

From Lemma 5.1.8 we have $f \in T_0$. We have the following two classes.

$$U^2\overline{T}_{12}T_{01}T_{20} = T_0 \left\{ \begin{array}{lll} B_0 & f6.24 & (*226 = \#166) \\ \overline{B}_0 & f6.22 & (*227 = \#114) \end{array} \right.$$

(2) $U^2\overline{T}_{12}T_{01}\overline{T}_{20}$.

$$U^2\overline{T}_{12}T_{01}\overline{T}_{20} = \left\{ \begin{array}{llll} 1) & T_0 \left\{ \begin{array}{l} B_0 \\ \overline{B}_0 \end{array} \right. & \begin{array}{l} f6.26 \\ f6.27 \end{array} & \begin{array}{l} (*228 = \#119) \\ (*229 = \#\ \ 75) \end{array} \\ 2) & \overline{T}_0\overline{B}_0 & f6.28 & (*230 = \#\ \ 33) \end{array} \right.$$

Proof. 2). From Lemma 5.6.11 we have $\overline{T}_0\overline{D} \subseteq \overline{B}_0$. □

(3) $U^2\overline{T}_{12}\overline{T}_{01}T_{20}$.

This case is the σ_0-similar of the case (2).

(4) $U^2\overline{T}_{12}\overline{T}_{01}\overline{T}_{20}$.

This set is divided into the following 3 classes.

$$U^2\overline{T}_{12}\overline{T}_{01}\overline{T}_{20} = \left\{ \begin{array}{llll} 1) & T_0 \left\{ \begin{array}{l} B_0 \\ \overline{B}_0 \end{array} \right. & \begin{array}{l} f6.29 \\ f6.30 \end{array} & \begin{array}{l} (*234 = \#76) \\ (*235 = \#34) \end{array} \\ 2) & \overline{T}_0\overline{B}_0 & f6.31 & (*236 = \#\ \ 9) \end{array} \right.$$

Proof. 2). From Lemma 5.6.11 we have $\overline{T}_0\overline{D} \subseteq \overline{B}_0$. □

Conclusions of 5.6.2.2. Thus, summing all four cases we have $|U^2T_{12}| = |U^2T_{12}T_{01}T_{12}| + |U^2T_{12}T_{01}\overline{T}_{12}| + |U^2T_{12}\overline{T}_{01}T_{12}| + |U^2T_{12}\overline{T}_{01}\overline{T}_{12}| = 2 + 2 \times 3 + 3 = 11$, of which σ_0-free classes are 8. Thus $|U^2| = |U^2T_{12}| + |U^2\overline{T}_{12}| = 32 + 11 = 43$, of which σ-free classes are 28.

Conclusion of Sections 5.6. $|U| = 3|U^1| + 3|U^2| = 3 \times 4 + 3 \times 43 = 141$, of which σ-free classes are 3+28=31.

5.7. Classification of B

In this section the set $B := \overline{TLSMU}(B_0 \cup B_1 \cup B_2)$ will be classified. Put $\overline{G} := \overline{TLSMU}$ and $B := \overline{G}(B_0 \cup B_1)$ for simplicity. Obviously we can represent B as

$$B = \overline{G}(B_0B_1B_2 + B_0B_1\overline{B}_2 + B_0\overline{B}_1B_2 + \overline{B}_0B_1B_2 + B_0\overline{B}_1\overline{B}_2 + \overline{B}_0\overline{B}_1B_2 + \overline{B}_0B_1\overline{B}_2).$$

However, we have $\overline{G}B_0B_1B_2 = \phi$ from Lemma 5.1.4 and $\overline{G}B_pB_q \subseteq \overline{G}U_r = \phi$ from Lemma 5.1.7. Hence we have $B = B^1 + (B^1)^{\sigma_2} + (B^1)^{\sigma_1}$, where

$$B^1 = \overline{G}B_0\overline{B}_1\overline{B}_2.$$

Thus it is sufficient to consider only B^1. We prepare several lemmas.

Lemma 5.7.1. $B_p\overline{D} \subseteq T_p$.

Proof. . We show a contradiction assuming $f(\boldsymbol{p}) = q$. From $f \in \overline{D}$ we have $f(\boldsymbol{b}) = r$ for some $\boldsymbol{b}$. Hence $f \in \overline{B}_p$, since $(\boldsymbol{p}, \boldsymbol{b}) \in B_p$. When $f(\boldsymbol{p}) = r$, a similar contradiction results. □

Lemma 5.7.2. $T_pB_q \subseteq T_{pq}$.

Proof. Assume $f \in T_pB_q\overline{T}_{pq}$. Then $f(\boldsymbol{a}) = r$ for some $\boldsymbol{a} \in \{p, q\}^n$. This contradicts to $f \in B_q$ since $(\boldsymbol{p}, \boldsymbol{a}) \in B_q$ and $(f(\boldsymbol{p}), f(\boldsymbol{a})) \notin B_q$. □

Now we divide B^1 into the following 4 subsets by T_1 and T_2 and consider each case separately:

$$B^1 = B^1(T_1T_2 + T_1\overline{T}_2 + \overline{T}_1T_2 + \overline{T}_1\overline{T}_2).$$

(1) $B^1T_1T_2$.

From Lemma 5.7.2 we have $B^1T_1T_2 \subseteq T_{10}T_{20}$. Thus the remaining set is T_{12}.

$$B^1T_1T_2 = T_{01}T_{20}\left\{\begin{array}{lll} 1)\ T_{12} & f7.1 & (*323 = \#254) \\ 2)\ \overline{T}_{12} & f7.2 & (*324 = \#194) \end{array}\right.$$

(2) $B^1T_1\overline{T}_2$.

From Lemma 5.7.2 we have $B^1T_1 \subseteq T_{01}$, and with respect to the remaining T_{12} and T_{20}, the class $T_{12}T_{20}$ is impossible from Lemma 5.1.8 and $\overline{T}_2$. Thus there are 3 classes.

$$B^1T_1\overline{T}_2 = T_{01}\left\{\begin{array}{lll} 1)\ \overline{T}_{12}T_{20} & f7.3 & (*325 = \#149) \\ 2)\ T_{12}\overline{T}_{20} & f7.4 & (*326 = \#150) \\ 3)\ \overline{T}_{12}\overline{T}_{20} & f7.5 & (*327 = \#101) \end{array}\right.$$

(3) $B^1\overline{T}_1T_2$. This is the σ_0-similar of (2).

(4) $B^1\overline{T}_1\overline{T}_2$.

Among 8 induced classes by T_{01}, T_{12} and T_{20}, 3 classes which include $T_{01}T_{12}$ and $T_{20}T_{12}$ are impossible from $\overline{T}_1\overline{T}_2$ and Lemma 5.1.8.

$$B^1T_1\overline{T}_1\overline{T}_2 = \left\{\begin{array}{lll} 1)\ T_{01}\overline{T}_{12}T_{20} & f7.6 & (*331 = \#99) \\ 2)\ T_{01}\overline{T}_{12}\overline{T}_{20} & f7.7 & (*332 = \#64) \\ 3)\ \overline{T}_{01}\overline{T}_{12}T_{20} & f^{\sigma_0}7.7 & (*\underline{333} = \#\underline{62}) \\ 4)\ \overline{T}_{01}T_{12}\overline{T}_{20} & f7.8 & (*334 = \#63) \\ 5)\ \overline{T}_{01}\overline{T}_{12}T_{20} & f7.9 & (*335 = \#26) \end{array}\right.$$

Conclusion of Section 5.7. Thus $B = B^1 + (B^1)^{\sigma_2} + (B^1)^{\sigma_1}$ and $|B^1| = |B^1T_1T_2| + 2|B^1T_1\overline{T}_2| + |B^1\overline{T}_1\overline{T}_2| = 13$. Thus, $|B| = 3 \times 13 = 39$, of which 9 are σ-similar-free.

5.8. Classification of $\overline{TLSMUB}$

In this section all the functions (the complement set of the set of functions so far classified) will be classified. We put $I := \overline{TLSMUB}$. Obviously I can be represented as

$$I = T^0 + T^1 + (T^1)^{\sigma_1} + (T^1)^{\sigma_2} + (T^2)^{\sigma_1} + (T^2)^{\sigma_2} + T^3,$$

where $T^0 := IT_0T_1T_2$, $T^1 := I\overline{T}_0T_1T_2$, $T^2 := IT_0\overline{T}_1\overline{T}_2$ and $T^3 := I\overline{T}_0\overline{T}_1\overline{T}_2$. We are suffice to consider T^0, T^1, T^2 and T^3. Now the remaining sets are only T_{pq} type sets. In the following classification we will see that the condition $\overline{TLSMUB}$ does not influence the possible classes of T_p type maximal sets. We mean that possible classes by T_p maximal sets are restricted only by T_{pq} sets through Lemma 5.1.8.

Classification of T^0

T^0 is divided into the following 8 classes (all induced sets by T_{01}, T_{12} and T_{20}).

$$T^0 = \left\{ \begin{array}{lll} 1)\ T_{01}\overline{T}_{12}T_{20} & f8.1 & (*362 = \#191) \\ 2)\ T_{01}T_{12}\overline{T}_{20} & f8.2 & (*363 = \#138) \\ 3)\ T_{01}\overline{T}_{12}T_{20} & f^{\sigma_2}8.2 & (*\underline{364} = \#\underline{137}) \\ 4)\ T_{01}\overline{T}_{12}\overline{T}_{20} & f8.3 & (*365 = \ \#92) \\ 5)\ \overline{T}_{01}T_{12}T_{20} & f^{\sigma_0}8.2 & (*\underline{366} = \#\underline{136}) \\ 6)\ \overline{T}_{01}\overline{T}_{12}T_{20} & f^{\sigma_1}8.3 & (*\underline{367} = \#\ \underline{91}) \\ 7)\ \overline{T}_{01}T_{12}\overline{T}_{20} & f^{\sigma_0}8.3 & (*\underline{368} = \#\ \underline{90}) \\ 8)\ \overline{T}_{01}\overline{T}_{12}\overline{T}_{20} & f8.7 & (*\underline{369} = \#\ \underline{62}) \end{array} \right.$$

Classification of T^1

T^1 is divided into the following 6 classes. From Lemma 5.1.8 and $\overline{T}_0$ the classes which include $T_{01}T_{20}$ are impossible.

$$T^1 = \left\{ \begin{array}{lll} 1)\ T_{01}\overline{T}_{12}T_{20} & f8.5 & (*370 = \#85) \\ 2)\ T_{01}\overline{T}_{12}\overline{T}_{20} & f^{\sigma_0}8.5 & (*\underline{371} = \#\underline{92}) \\ 3)\ \overline{T}_{01}T_{12}T_{20} & f8.6 & (*372 = \#47) \\ 4)\ \overline{T}_{01}\overline{T}_{12}T_{20} & f8.7 & (*373 = \#46) \\ 5)\ \overline{T}_{01}T_{12}\overline{T}_{20} & f^{\sigma_0}8.6 & (*\underline{374} = \#\underline{45}) \\ 6)\ \overline{T}_{01}\overline{T}_{12}\overline{T}_{20} & f8.8 & (*375 = \#18) \end{array} \right.$$

Classification of T^2

T^2 is divided into the following classes. From Lemma 5.1.8 and $\overline{T}_1\overline{T}_2$, the 3 classes which include $T_{01}T_{12}$ and $T_{12}T_{20}$ are impossible.

$$T^2 = \left\{ \begin{array}{lll} 1)\ T_{01}\overline{T}_{12}T_{20} & f8.9 & (*388 = \#48) \\ 2)\ T_{01}\overline{T}_{12}\overline{T}_{20} & f8.10 & (*389 = \#21) \\ 3)\ \overline{T}_{01}T_{12}\overline{T}_{20} & f8.11 & (*390 = \#20) \\ 4)\ \overline{T}_{01}\overline{T}_{12}T_{20} & f^{\sigma_0}8.10 & (*\underline{391} = \#\underline{19}) \\ 5)\ \overline{T}_{01}\overline{T}_{12}\overline{T}_{20} & f8.12 & (*392 = \#\ 7) \end{array} \right.$$

Classification of T^3

T^3 is divided into the following classes. From Lemma 5.1.8 and $\overline{T}_0\overline{T}_1\overline{T}_2$, the 4 classes which include $T_{01}T_{12}$, and $T_{12}T_{20}$ and $T_{20}T_{01}$ are impossible.

$$T^3 = \left\{ \begin{array}{lll} 1)\ T_{01}\overline{T}_{12}\overline{T}_{20} & f8.13 & (*403 = \#4) \\ 2)\ \overline{T}_{01}T_{12}\overline{T}_{20} & f^{\sigma_1}8.13 & (*\underline{404} = \#\underline{3}) \\ 3)\ \overline{T}_{01}\overline{T}_{12}T_{20} & f^{\sigma_0}8.13 & (*\underline{405} = \#\underline{2}) \\ 4)\ \overline{T}_{01}\overline{T}_{12}\overline{T}_{20} & f8.14 & (*406 = \#1) \end{array} \right.$$

Table 5.12: Numbers of the classes of the subsets of P_3.

subsection	considered subset	classes	σ-similar-free classes
5.4	T	80	17
5.5	$L \cup S$	8	6
5.6	M	93	19
5.7	U	141	31
5.8	B	39	9
5.9	$\overline{TLSMUB}$	45	14
	Total	406	96

Conclusion of 4.8. Thus we have $|T^0| = 8$, $|T^1| = 6$, $|T^2| = 5$ and $|T^3| = 4$. Hence $|\overline{TLSMUB}| = |T^0| + 3|T^1| + 3|T^2| + |T^3| = 45$, of which 14 are σ-similar free.

5.9. The result of the classification of P_3

We have completed the classification of P_3, investigating the structure of the intersections of the 18 P_3-maximal sets. All the classes and representative functions are presented in Appendix 1 and 2, respectively. Every representative is chosen from the least arity functions [Miy71].

Thus we have the following theorem.

Theorem 5.9.1. *P_3 is divided into the 406 nonempty classes, of which 96 are σ-similar-free.*

In Table 5.12 we show the classes of the set considered in the corresponding subsections 5.4 - 5.9. We note that the origianl classification in [Miy71] counted a few characteristic vectors twice as different classes, consequently the number of classes reported in [Miy79] is not quite right; this was corrected in [Sto84a].

Further from the fact that all the representative functions of the classes shown in the Appendix 2 are of not greater than 3 arity, we have the following theorem.

Theorem 5.9.2. *Each class of P_3 has a representative function of not greater than 3 variables.*

Let a closed set $F \subseteq P_k$ be finitely generated. The minimal number r such that every base of F can be constructed by functions depending on at most r variables (i.e.

r arity) is called the *order* of F [Lau84b]. In case that F has no finite base the order of F is set to $\aleph_0$.

Theorem 5.9.2 states that

Corollary 5.9.1. *The order of* P_3 *is 3.*

We know that the order of P_2 (under ordinary composition) is also 3.

5.10. Enumerations of bases of P_3

The list of 406 characteristic vectors of P_3-classes tells many things. Especially, we will show that the maximal rank of a pivotal incomplete set is 7, while that of a base is 6. This is a rather unexpected result. Since a base corresponds to a minimum cover of $1\cdots1$ and a pivotal incomplete set corresponds to a minimal cover of some binary vector in which at least one coordinate should be 0, one may naturally assume that the maximal rank of a base is greater than or equal to that of a pivotal incomplete set. The reality is not like this. The number of classes of bases of P_3 is exactly 6,239,721 (recall that we have only 42 for P_2), in which the number of bases which contain constant functions is exactly 1,391 .

Let us call a characteristic vector simply a vector. Recall that a set of vector is a base if it satisfies the following two conditions: 1) bit-wise OR for all the vectors results in unit vector $1\cdots1$ (Equation (1.1)) and 2) for each vector of the set, bit-wise OR for all the remaining vectors of the set does not equal that for all the vectors (Equation (1.2)).

The last condition is equivalent to saying that for every class of the set there is at least one "pivot", a maximal set in which all the other classes of the set except the class are included. Also recall that a set is called pivotal if it satisfies the condition 2).

First, let us see how vectors can be used.

Example 5.10.1. In Table 5.13 we show vectors of the function $j_0(x), j_1(x)$ and $j_2(x)$, where $j_i(x)$ is defined by $j_i(i) = 2, j_i(x) = 0$ for $x \neq i$. Note that $\max(x,y) = \sigma_1$-sim. of $\min(x,y)$, $2 = \sigma_1$-, σ_3-sim. of 0 and $1 = \sigma_2$-,σ_4-sim. of 0, where sim. stand for similar. It is well-known that the set $F = \{0, 1, 2, j_0(x), j_1(x),\ j_2(x),\ \min(x,y),\ \max(x,y)\}$ is complete. By examining the vectors of these functions we see that F is complete but

Table 5.13: Characteristic vectors of $j_i(x)$, max, min and constants.

wt	$\#no$	TLS	$M_1M_2M_0$	$U_2U_0U_1$	$B_0B_1B_2$	$T_0T_1T_2$	$T_{01}T_{12}T_{20}$	$*no$	$representative$
12	#242	011	111	100	010	111	110	*78	$j_0(x)$
11	#306	011	101	110	010	011	110	*65	$j_1(x)$
7	#393	011	010	010	010	010	010	*68	$j_2(x)$
6	#400	111	010	001	100	000	000	*92	max (x, y)
6	#402	111	001	001	001	000	000	*89	min (x, y)
4	#403	001	000	000	000	110	100	* 8	2
4	#404	001	000	000	000	101	001	* 7	1
4	#405	001	000	000	000	011	010	* 6	0

not a base. It is easily verified that a base from F should contain $\min(x, y), \max(x, y), 1$, since these are only elements that cover B_2, B_0 and T_{20}-th coordinates, respectively. By the base criteria we see that the following two sets are only bases that can be composed from F:

$$\{\min(x, y), \max(x, y), 1, j_1(x)\} \text{ and } \{\min(x, y), \max(x, y), 1, j_0(x), j_2(x)\}.$$

□

The enumerationons of bases of P_3 can be done by examining the base criteria for all combinations of the classes. Although the procedure is quite simple, its direct application is far from feasibility due to combinatorial difficulty; it has required over 20 hours to examine the base criteria for 10^6 combinations of 6 tuples of vectors $\langle b_1, b_2, \ldots, b_6\rangle$ from P_3-vectors by a computer which has about 1 MIPS processing speed (Tosbac 5600 computer). The feasible algorithm to overcome this difficulty we present in Chapter 7.

Here we summarize the enumeration results.

An example of redundant incomplete (actually a pivotal) set with rank 7 is shown in [Jab58]. It has been a problem whether this is the maximum rank of a pivotal set. We show that it is true.

Theorem 5.10.1. *The maximal rank of a pivotal incomplete set of P_3 is 7.*

This means that maximal rank of a nonredundant incomplete set is greater than or equal to 7 (not every nonredundant incomplete set is pivotal incomplete set), and this tempts us to believe that the maximal rank of a base is also greater than or equal to 7. However, this does not hold.

Theorem 5.10.2. *The maximal rank of a base of P_3 is 6.*

In Example 5.10.2 we will see these situations in more detail.

Theorem 5.10.3. *The number of bases of P_3 is exactly 6,239,721.*

We note that the first report [Miy79] on the number of classes of base was not quite right and the above number is the corrected result by [Sto84a].

Theorem 5.10.4. *The number of bases which contain constant functions 0,1,2 is exactly 1,391.*

rank	1	2	3	4	5	6	*total*
bases	0	0	0	2	633	756	1,391

5.10.1. Examples of bases and pivotals

The situation which yields an interesting "gap" between Theorem 5.10.1 and Theorem 5.10.2 can be understood by the following example.

Example 5.10.2. In Table 5.14 and Table 5.15 we list 10 classes with the least degrees of completeness (i.e. weight) and their representative functions, respectively. By examining these vectors we can see that the set Y ={ σ_4-min, σ_2-min, max, min, 0,1,2} is pivotal incomplete set with maximal rank 7. Indeed, it is easy to see that Y is contained in the maximal set B and each class has at least a pivot. This example is essentially the same as one presented by Jablonskij [Jab58, p.136]. Joining σ_3-min or σ_0-min to Y yields a complete set, but in both cases the resulting sets are redundant (non-pivotal). More precisely, by examining the vectors we can see that joining σ_3-min to Y yields redundancy of σ_2-min and max, and joining σ_0-min results redundancy of σ_4-min and min. Thus we have only two bases of the maximal rank 6: {σ_4-min,σ_3-min, min, 2,1,0} and {σ_2-min, max,σ_0-min, 2,1,0} that can be constructed from these classes. □

Example 5.10.3. The following 9 sets are *all* pivotal incomplete sets with maximal rank 7 . Every permutations in $\{\sigma_0, \sigma_1, \sigma_2\}$ is with even length, while one from $\{\varepsilon, \sigma_3, \sigma_4\}$ is with odd length. The following list consists of taking every two functions from each of these categories and adding constant functions.

$$\begin{array}{llllllll}
1) & \{\sigma_0\text{-}min, & max, & min, & \sigma_3\text{-}min, & 0,1,2\} & \subset & M_1 \\
2) & \{\sigma_0\text{-}min, & \sigma_2\text{-}min, & min, & \sigma_4\text{-}min, & 0,1,2\} & \subset & M_2 \\
3) & \{max, & \sigma_2\text{-}min, & \sigma_3\text{-}min, & \sigma_4\text{-}min, & 0,1,2\} & \subset & M_0 \\
4) & \{max, & \sigma_2\text{-}min, & min, & \sigma_3\text{-}min, & 0,1,2\} & \subset & U_2 \\
5) & \{\sigma_0\text{-}min, & max, & min, & \sigma_4\text{-}min, & 0,1,2\} & \subset & U_0 \\
6) & \{\sigma_0\text{-}min, & \sigma_2\text{-}min, & \sigma_3\text{-}min, & \sigma_4\text{-}min, & 0,1,2\} & \subset & U_1 \\
7) & \{\sigma_0\text{-}min, & \sigma_2\text{-}min, & min, & \sigma_3\text{-}min, & 0,1,2\} & \subset & B_0 \\
8) & \{max, & \sigma_2\text{-}min, & min, & \sigma_4\text{-}min, & 0,1,2\} & \subset & B_1 \\
9) & \{\sigma_0\text{-}min, & max, & \sigma_3\text{-}min, & \sigma_2\text{-}min, & 0,1,2\} & \subset & B_2
\end{array}$$

□

Table 5.14: 10 classes of P_3 which have the least completeness degrees.

wt	*#no*	*TLS*	$M_1M_2M_0$	$U_2U_0U_1$	$B_0B_1B_2$	$T_0T_1T_2$	$T_{01}T_{12}T_{20}$	**no*	*representative*
6	#397	111	100	100	100	000	000	*96	σ_4-sim. min
6	#398	111	100	010	001	000	000	*95	σ_2-sim. min
6	#399	111	010	010	010	000	000	*93	σ_3-sim. min
6	#400	111	010	001	100	000	000	*92	σ_1-sim. min
6	#401	111	001	100	010	000	000	*90	σ_0-sim. min
6	#402	111	001	001	001	000	000	*89	$\min(x,y) = f.5.1$
4	#403	001	000	000	000	110	100	* 8	2 (constant)
4	#404	001	000	000	000	101	001	* 7	1 (constant)
4	#405	001	000	000	000	011	010	* 6	0 (constant)
0	#406	000	000	000	000	000	000	* 1	x (projections)

Table 5.15: Representatives functions.

$f \setminus xy$	00	01	02	10	11	12	20	21	22
σ_4-min	0	1	2	1	1	2	2	1	2
σ_2-min	0	1	0	1	1	1	1	1	2
σ_3-min	0	0	2	0	1	2	2	2	2
max = σ_1-min	0	1	2	1	1	2	2	2	2
σ_0-min	0	0	0	0	1	2	0	2	2
min	0	0	0	0	1	1	0	1	2

Example 5.10.4. In Table 5.16 and Table 5.17 we show three classes and their representative functions, respectively. The first one is a base with single function (a similar function of Webb function $\max(x, y) + 1$). The last two are all classes each of which is complete with constant functions (c-complete). It may have a practical significance

Table 5.16:

wt	$\#no$	TLS	$M_1M_2M_0$	$U_2U_0U_1$	$B_0B_1B_2$	$T_0T_1T_2$	$T_{01}T_{12}T_{20}$	$*no$	*representative*
18	#1	111	111	111	111	111	111	*406	$f8.14$ (Sheffer)
12	#191	111	111	111	111	000	000	*362	$f8.1$
11	#288	110	111	111	111	000	000	*87	$f4.5$

Table 5.17: Representatives functions.

$f \setminus xy$	00	01	02	10	11	12	20	21	22
$f8.14$	1	0	1	0	2	0	1	0	0
$f8.1$	0	1	0	0	1	2	0	2	2
$f4.5$	0	0	2	0	1	1	2	1	2

that these two representatives depend on two variables, while in two-valued case there exist only three-variable representatives in the corresponding classes (there exist also all two classes which are complete with constants in two-valued case). □

5.10.2. Conclusive discussions

We have enumerated all the bases of three-valued logical functions. Now it has become known that three-valued case is far much complex than two-valued case. The classification approach, originally due to [Jab52], has been proved to be useful also for three-valued case, but it will be hard to apply for the cases with greater than three.

In the base enumeration a peculiar structure of P_3 is revealed: the maximal rank of a base is 6, while that of a pivotal incomplete set is 7. There are a few investigation on the maximal rank of a base of P_3 [Krn73]. Another proof that the maximal rank of bases of P_3 is 6 is presented recently [Vuk84], which does not resort to enumeration of whole bases directly. It is known that for P_k $(k \geq 3)$ there is a set which has a base with infinite rank, and a set with no base [JaM59]. Thus a family of the closed sets each of which is spanned by a pivotal incomplete set is merely a special family of all closed sets of P_k.

5.11. Classifications and base enumeration results for P_3 and its all maximal sets

In this last section we are going to presents classification and enumeration results of all bases for the set P_3 and all 18 P_3-maximal sets. First we give some historical remarks. First attempt to derive classes of functions of P_3 was done in [Miy71]. This paper also give the notion of pivotal sets as necessary conditions for a set to be base. However, as we noted before, it counted a few characteristic vectors twice as different classes, consequently the number of bases reported in [Miy79] was not quite right; this was corrected in [Sto84a]. The following Table 5.18 presents the numbers of maximal sets and the numbers of classes of functions for the sets P_2, P_3 and all P_3-maximal sets.

The numbers of classes of bases and pivotal incomplete sets for the same sets as in the former table are shown in the following two Tables 5.19 and 5.20.

In the table we abbreviated references as follows: [P] for [Pos21], [J1,J2] for [Jab52,Jab58], [L] for [Lau82b], [Ma] for [Mac79], [B1,B2] for [BaD78,BaD80], [JIK] for [Jab52,INN63,Krn65], [M1,M2,M3,M4,M5] for [Miy71,Miy79,Miy82,Miy83,Miy84], [S1] for [Sto84a] and [S2,S3,S4] for [Sto84b,Sto86a,Sto86b].

Table 5.18: Numbers of maximal sets and numbers of classes of functions for P_3 and its maximal sets.

	P_2	P_3	B_1	M_1	T_0	U_0	T_{01}	T	L	S
maximal	5	18	7	13	12	13	15	5	5	2
sets	[P]	[J2]	[L]	[Ma]	[L]	[L]	[L]	[L]	[B1]	[B2]
classes of	15	406	54	88	253	383	607	6	10	4
functions	[JIK]	[M1,S1]	[M3]	[S2]	[M5]	[S3]	[S4]	[M4]	[M4]	[M4]

Table 5.19: Classes of bases of P_3 and of its all maximal sets.

	P_2	P_3	B_1	M_1	T_0	U_0	T_{01}	T	L	S
rank	[I,K]	[S1,M2]	[M3]	[S2]	[M5]	[S3]	[S4]	[M4]	[M4]	[M4]
1	1	1	-	-	1	1	1	-	-	1
2	17	8,265	28	-	4,492	4,344	12,259	-	18	1
3	22	794,256	999	1,514	234,031	680,285	2,580,026	6	6	-
4	2	4,612,601	2,831	40,104	552,927	7,300,491	38,508,259	-	-	-
5	-	810,474	724	75,209	91,377	7,627,060	53,641,851	-	-	-
6	-	14,124	17	1,916	892	944,257	7,545,748	-	-	-
7	-	-	-	1	-	15,804	35,616	-	-	-
Σ	42	6,239,721	4,599	118,744	883,720	16,572,242	102,323,760	6	24	2

Table 5.20: Classes of pivotal incomplete sets of P_3 and of its all maximal sets.

	P_2	P_3	B_1	M_1	T_0	U_0	T_{01}	T	L	S
1	13	404	53	87	251	381	605	5	9	2
2	31	60,335	931	3,153	21,363	57,284	147,266	10	10	-
3	7	1,418,970	3,678	37,946	202,689	1,594,342	6,385,808	-	-	-
4	-	2,677,899	2,240	96,323	149,804	5,057,975	32,278,690	-	-	-
5	-	176,187	168	15,087	6,595	1,911,408	18,947,380	-	-	-
6	-	1,368	1	55	8	96,464	1,198,502	-	-	-
7	-	9	-	-	-	240	648	-	-	-
Σ	51	4,335,172	7,071	152,651	38,0710	8,718,094	58,958,899	15	19	2

Chapter 6

Classifications of maximal sets of P_3

In this chapter we classify the maximal sets of P_3 : T (semi-degenerate or Słupecki set), L (linear functions) and S (self-dual functions), B and T_0 (the set of functions preserving a constant 0). We also presents enumerations of bases and pivotal incomplete sets for each case.

6.1. T (Słupecki functions or semi-degenerate functions)

In this section we will classify the P_3-maximal clone $T = D \cup [P_3^{(1)}]$, which we call *semi-degenerate* functions or Słupecki functions.

For a unary function $f \in P_3^{(1)}$ we denote it by $s_{f(0)f(1)f(2)}$; for example, identity function is denoted by s_{012}; for simplicity we use x for identity function also, and also put $c_0 = s_{000}$, $c_1 = s_{111}$ and $c_2 = s_{222}$.

The classification is based on the following theorem. In presenting the theorem we introduce our notations for the submaximal sets.

Theorem 6.1.1. [Lau82b]

T has exactly the following 5 maximal clones.

(1) $S_0 := D \cup [s_{021}]$.

(2) $S_1 := D \cup [s_{210}]$.

(3) $S_2 := D \cup [s_{102}]$.

(4) $S_+ := D \cup [s_{120}, s_{201}]$.

(5) $S_b := [P_3^{(1)}] \cup \bigcup_{n=1}^{\infty}\{f^{(n)} \in P_3 | \exists f_i \in P_3^{(1)}$ *such that* $f(x_1, ..., x_n) = f_0(f_1(x_1) + f_2(x_2) + ... + f_n(x_n) \bmod 2)\}$

We recall that the notation $[F]$ denotes the clone generated from F. For simplicity we omit set notation; thus $[s_{021}]$ means $[\{s_{021}\}]$. The identity function s_{012} is always included in all sets by definition. Also note that onto functions that can be generated by each of the above first four maximal sets are only its own unary onto functions.

Let S_i and S_j $(i \neq j)$ be any of S_0, S_1, S_2 and S_+.

Lemma 6.1.1. *We have* $S_iS_j = D + [s_{012}]$, *hence* $S_0S_1S_2S_+ = D + [s_{012}]$.

Proof. $[D + s_{012}] \subset S_iS_j$ is obvious. Converse. Suppose $f \in S_iS_j$ and f is onto. Then there exists an onto function $f_0 \in S_iS_j$ such that $f(x_1, \ldots, x_n) = f_0(\ldots)$. As we noted above there exist no such onto function except s_{012}. □

Lemma 6.1.2. *Let* S_i *and* S_j *be as above. Then the set* $S_i\overline{S}_j$ *consists exactly of those onto functions in* S_i *excluding* $[s_{012}]$.

Proof. Obviously every onto function contained in S_i does not belongs to S_j except s_{012}.
□

Example 6.1.1. $S_+\overline{S}_0 = \{s_{120}, s_{201}\}$. □

Lemma 6.1.3. $T = S_0 \cup S_1 \cup S_2 \cup S_+$

Proof. If $f \in T$ is an onto function then f belongs to the right hand side. □

Classification. From Lemma 6.1.1 we have the following 4 classes as for S_0, S_1, S_2 and S_+.

	S_0	S_1	S_2	S_+	set
1)	0	0	0	0	$= \{D + s_{012}\}$
2)	0	1	1	1	$= \{s_{021}\}$
3)	1	0	1	1	$= \{s_{210}\}$
4)	1	1	0	1	$= \{s_{102}\}$
5)	1	1	1	0	$= \{s_{120}, s_{201}\}$

We combine the above classes and the remaining maximal set S_b. The class formed by combining $\overline{S}_b$ and each of the above 2)- 5) is empty from Lemma 6.1.2, because combining $\overline{S}_b$ means to exclude all $P_3^{(1)}$, while only unary onto functions exist in the above classes. Thus we have the following theorem.

Theorem 6.1.2. *T has the following 6 classes.*

Class	S_0	S_1	S_2	S_+	S_b	representatives
1)	1	1	1	0	0	$\{s_{120}, s_{201}\}$
2)	1	1	0	1	0	$\{s_{102}\}$
3)	1	0	1	1	0	$\{s_{210}\}$
4)	0	1	1	1	0	$\{s_{021}\}$
5)	0	0	0	0	1	$g_{1.1}$
6)	0	0	0	0	0	$s_{012}, 0, 1, 2$

where $g_{1.1} := g(x, y) = 1$ if $x = y = 2$, otherwise $g(x, y) = 0$.

Note 6.1.1. The class 5) includes functions which depend on $2n$ variables (we can easily extend $g_{1.1}$. to such functions), and the class 6) which contains DS_b also includes functions which depend on n variables, e.g., $f(x_1, \ldots, x_n) := s_{011}(s_{001}(x_1) + s_{001}(x_2) + \ldots + s_{001}(x_n) \bmod 2)$.

Since the proof of $g_{1.1} \notin S_b$ is a bit lengthy, we put it separately in the end of this subsection. We first give bases and pivotals of T.

Theorem 6.1.3. *T has exactly the following 6 bases whose rank = 3:*

{1,2,5}, {1,3,5}, {1,4,5}, {2,3,5}, {2,4,5}, {3,4,5}.

Thus any base of T consists exactly of three elements.

Theorem 6.1.4. *T has exactly the following 15 pivotal incomplete sets.*

rank = 1: each of 5 classes except null class.

rank = 2: {1,5}, {2,5}, {3,5}, {4,5}, {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4}.

Now we give the *proof of* $g_{1.1} \notin S_b$.

Put $g(x, y) := g_{1.1}$. Recall $g(2, 2) = 1$, and $g(x, y) = 0$ for the other values of x and y. Assume $g(x, y) = f_0(f_1(x) + f_2(y) \bmod 2)$ for some $f_i \in P_3^{(1)}$. We show a contradiction.

Since range $g = \{0, 1\}$, f_0 should map the subdomain {0,1} onto {0,1}, i.e. f_0 should be either s_{01*} or s_{10*}, where * denote 0,1 or 2 .

1) Case of $f_0 = s_{01*}$.

We have $g(x,y) = s_{01*}(f_1(x) + f_2(y) \bmod 2) = f_1(x) + f_2(y) \bmod 2$. Hereafter till the end this section $x+y$ and xy denote the element of E_3 congruent $(\bmod 3)$ $x+y$ and xy, respectively. From $g(2,2) = f_1(2)+f_2(2) = 1$, we have $(f_1(2), f_2(2)) = (0,1), (1,0), (1,2)$ or (2,1) . From the symmetry of f_1 and f_2, it suffices to consider that $(f_1(2), f_2(2)) = (0,1)$ or (1,2) .

1.1) Case of $f_1(2) = 0$ and $f_2(2) = 1$.

We note that $f_2(2) = 1$ and that $1+a = 0$ leads to $a = 1$. Then $g(0,2) = f_1(0)+f_2(2) = 0$ leads to $f_1(0) = 1$. Thus $g(0,0) = f_1(0) + f_2(0) = 0$ leads to $f_2(0) = 1$. Hence $g(2,0) = f_1(2) + f_2(0) = 0$ leads to $f_1(2) = 1$. But this contradicts to the assumption $f_1(2) = 0$.

1.2) Case of $f_1(2) = 1$ and $f_2(2) = 2$.

Then $g(2,0) = f_1(2) + f_2(0) = 0$ leads to $f_2(0) = 1$. Thus $g(0,0) = f_1(0) + f_2(0) = 0$ leads to $f_1(0) = 1$. Hence $g(0,2) = f_1(0) + f_2(2) = 0$ leads to $f_2(2) = 1$. But this contradicts to the assumption $f_2(2) = 2$.

2) Case of $f_0 = s_{10*}$.

We have

$$g(x,y) = s_{10*}(f_1(x) + f_2(y) \bmod 2) = (f_1(x) + f_2(y) \bmod 2) + 1 = f_1(x) + f_2(y) + 1.$$

From $g(0,0) = f_1(0) + f_2(0) + 1 = 0$ we have $f_1(0) + f_2(0) = 1$. Thus from the symmetry of f_1 and f_2, just like as we already saw for Case 1), it suffices to consider that $(f_1(0), f_2(0)) = (0,1)$ or (1,2) .

2.1) Case of $f_1(0) = 0$ and $f_2(0) = 1$.

We note that $f_1(0) = 0$ and that $1 + a = 0$ leads to $a = 1$. Then $g(0,2) = f_1(0) + f_2(2) + 1 = 0$ leads to $f_2(2) = 1$. Thus $g(2,2) = f_1(2) + f_2(2) + 1 = 1$ leads to $f_1(2) = 1$. Hence $g(2,0) = f_1(2) + f_2(0) + 1 = 0$ leads to $f_2(0) = 0$. But this contradicts to the assumption $f_2(0) = 1$.

2.2) Case of $f_1(0) = 1$ and $f_2(0) = 2$.

Then $g(2,0) = f_1(2)+f_2(0)+1 = 0$ leads to $f_1(2) = 1$. Thus $g(2,2) = f_1(2)+f_2(2)+1 = 1$ leads to $f_2(2) = 1$. Hence $g(0,2) = f_1(0) + f_2(2) + 1 = 0$ leads to $f_1(0) = 0$. But this contradicts to the assumption $f_1(0) = 1$. □

6.2. L (Linear functions)

We will classify the P_3-maximal set $L := \{f | f(\boldsymbol{x}) = \sum_{i=1}^{n} c_i x_i + c_0\}$, which is called linear functions. All maximal sets of L are given by the following theorem. In [BaD78] they showed all the closed sets of L for prime valued k. Their notations are slightly different from ours in the following theorem: they use L_α for LT_α ($\alpha = 0, 1, 2$), $L\triangle$ for LS, and $L^{(1)}$ is the same.

Theorem 6.2.1. [BaD78] *L has exactly the following 5 maximal sets.*

(1) $LT_0 = \{f | f \in L \text{ and } f(\mathbf{o}) = 0)\}$

(2) $LT_1 = \{f | f \in L \text{ and } f(\mathbf{1}) = 1)\}$

(3) $LT_2 = \{f | f \in L \text{ and } f(\mathbf{2}) = 2)\}$

(4) $LS = \{f | f \in L \text{ and } f(\boldsymbol{x} + \mathbf{1}) = f(\boldsymbol{x}) + 1\}$

(5) $L^{(1)} = [0, 1, 2, x, x+1, x+2, 2x, 2x+1, 2x+2]$.

Classification goes in the following manner.
First we will classify $L^{(1)}$ (5 classes), then $\overline{L}^{(1)} \cap LS$ (2 classes) ,and finally the remaining set (3 classes). Thus we will find total 10 classes.

Lemma 6.2.1. *Obviously $L^{(1)}$ is classified by the other maximal sets into the following 5 classes.*

LT_0	LT_1	LT_2	LS	representatives
0	0	0	0	x
1	1	1	0	$x+1, x+2$
0	1	1	1	$0, 2x$
1	0	1	1	$1, 2x+2$
1	1	0	1	$2, 2x+1$

Now we divide L into subsets as we did in the previous chapter (Chapter 5).
Put $L := L_0 + L_1 + L_2$, where $L_a := \{f | f(\boldsymbol{x}) = c_0 + \sum_{i=1}^{n} c_i x_i,\ \sum_{i=1}^{n} c_i = a\}$. Further each L_a is divided into the following three subsets:

$$L_a = L_{a0} + L_{a1} + L_{a2}, \text{ where } L_{ab} = \{f | f \in L_a \text{ and } f(\mathbf{o}) = a\}.$$

Then we have $LS = L_1$ from Lemma 5.4.7, Chapter 5.

Lemma 6.2.2. *From the property of $f(\mathbf{x}+\mathbf{1}) = f(\mathbf{x})+1$, the set $\overline{L}^{(1)} \cap LS(\subset L_1)$ is divided into the following 2 classes.*

LT_0	LT_1	LT_2	representatives
0	0	0	$2x+2y = f4.3$
1	1	1	$2x+2y+1 = f4.2$

where $f.4.3$ and $f4.4$ are from the previous chapter (they are given in Appendix 2).

Lemma 6.2.3. *$\overline{L}^{(1)}(L_0+L_2)$ is divided into the following 3 classes.*

LT_0	LT_1	LT_2	representatives
0	1	1	$x+2y = f4.1$
1	0	1	σ_2-similar of $x+2y$ $(= x+2y+1)$
1	1	0	σ_1-similar of $x+2y$ $(= x+2y+2)$

where $f4.1$ is from the previous chapter.

Proof. This is in fact Lemma 5.4.5. And this can be easily seen Also from the properties: $L_0 = L_{00}+L_{12}+L_{21}$ and $L_2 = L_{02}+L_{11}+L_{20}$, and $f(\mathbf{0}) = b$, $f(\mathbf{1}) = a+b$ and $f(\mathbf{2}) = 2a+b$ for $f \in L_{ab}$. □

From Lemmas 6.2.1,6.2.2 and 6.2.3 we have the following theorem.

Theorem 6.2.2. *L is divided into the following 10 classes.*

	L_1	LS	LT_0	LT_1	LT_2	representatives
1)	1	1	1	1	0	$x+y+1, x+2y+2$
2)	1	1	1	0	1	$x+y+2, x+2y+1$
3)	1	1	0	1	1	$x+y, x+2y$
4)	1	0	1	1	1	$2x+2y+1, 2x+2y+2$
5)	0	1	1	1	0	$2, 2x+1$
6)	0	1	1	0	1	$1, 2x+2$
7)	0	1	0	1	1	$0, 2x$
8)	0	0	1	1	1	$x+1, x+2$
9)	1	0	0	0	0	$2x+2y$
10)	0	0	0	0	0	x

In the above table we listed all n-ary $(n \leq 2)$ linear functions as representatives.

Theorem 6.2.3. *L has exactly the following 24 bases.*

rank = 1 : none.

rank = 2 : 1 $\times\{2,3,4,6,7,8\}$, *2* $\times\{3,4,5,7,8\}$, *3* $\times\{4,5,6,8\}$, *4* $\times\{5,6,7\}$.

rank = 3 : $\{5,6,9\},\{5,7,9\},\{5,8,9\},\{6,7,9\},\{6,8,9\},\{7,8,9\}$.

Theorem 6.2.4. *L has exactly the following 19 pivotal incomplete sets.*

rank = 1: each of 9 classes except null class.

rank = 2: {5,6}, {5,7},{5,8}, {5,9}, {6,7}, {6,8}, {6,8}, {7,8}, {7 ,9}, {8,9}.

In the linear functions one can see most clearly the relation between pivotal and nonredundant sets. A nonpivotal incomplete set can be redundant as is seen in the following example.

Example 6.2.1. 1. $F_1 := \{0\}$ is pivotal, and hence nonredundant.

2. $F_2 := \{0, 2x\}$ is nonredundant, and not pivotal; as we have seen these functions have the same characteristic vector (hence F_2 is not a minimal cover).

3. $F_3 := \{x+1, x+2\}$ is not pivotal and is redundant.

4. $F_4 := \{2x, 2x+2y\}$ is pivotal and noncomplete.

5. $F_5 := \{2x, 2x+2y, x+1\}$ is pivotal and complete, i.e. it is a base. □

A nonpivotal incomplete set can also be nonredundant.

Example 6.2.2. $F = \{0, f(x,y) = x+2y\}$ is not pivotal and incomplete. F is redundant; indeed $f(x,x) = x+2x \equiv 0$. □

Example 6.2.3. The set F of constants and any linear function of two variables, i.e., $F = \{0,1,2,l(x,y) = ax+by+c\ (a \neq 0, b \neq 0)\}$ is complete, but it is redundant; one or two of constants (depending on $l(x,y)$) is not necessary to be a base. □

6.3. S (Self-dual functions)

We will classify the set $S = \{f | f(\mathbf{x}+\mathbf{1}) = f(\mathbf{x})+1\}$ which are called self-dual functions. All the submaximal sets of S is given by the following theorem.

Theorem 6.3.1. [DHM80a] *S has exactly the following 2 maximal sets.*

$$\begin{aligned} (1)\ SL &= \{f | f \in S \text{ and } f \in L\}. \\ (2)\ ST_0 &= \{f | f \in S \text{ and } f(\mathbf{o}) = 0\}. \end{aligned}$$

Thus S is divided into the following four classes, and immediately we have the following classes of bases.

class	SL	ST_0	representative
1)	1	1	$f4.4$
2)	1	0	$f4.5$
3)	0	1	$x+1$
4)	0	0	$2x+2y$

where $f4.4$ and $f4.5$ are from the previous chapter.

Theorem 6.3.2. *S has exactly the following 2 bases and 2 pivotal incomplete sets.*

bases: 1 (rank = 1), {2,3} (rank = 2).

pivotals: 2, 3 (rank = 1).

It is interesting to note that such a non-trivial function as $2x+2y$ belongs to the null class; thus no incomplete set exists adding to which $2x+2y$ becomes complete in S. For functions in null class no incomplete set of functions can be added so that the joined set become complete. Null class containing non-trivial functions is seen in T, S and B (to be described in the next section).

6.4. Classification of B_1

In this section we classify a P_3-maximal set $B_1 = Pol\begin{pmatrix} 0120112 \\ 0121021 \end{pmatrix}$, which is the set of functions preserving a so called central relation. We will show 54 classes and prove that B_1 has 4,599 classes of bases. We also show that there is no Sheffer function in B_1.

The maximal set B_1 is the set of functions f: if $f\begin{pmatrix} a \\ b \end{pmatrix} \in \begin{pmatrix} 02 \\ 20 \end{pmatrix}$ then there is i such that $\begin{pmatrix} a_i \\ b_i \end{pmatrix} \in \begin{pmatrix} 02 \\ 20 \end{pmatrix}$.

First we show a completeness theorem for B_1 due to Lau.

Theorem 6.4.1. [Lau82b] B_1 *has exactly the following 7 maximal sets:*

$$\begin{array}{llll}
(1) & T_1 & = & B_1 \cap Pol(1), \\
(2) & T_{01} & = & B_1 \cap Pol(01), \\
(3) & T_{12} & = & B_1 \cap Pol(12), \\
(4) & T_{20} & = & B_1 \cap Pol(20), \\
(5) & M_5 & = & Pol\begin{pmatrix} 01211 \\ 01202 \end{pmatrix}, \\
(6) & M_6 & = & Pol\begin{pmatrix} 01210122 \\ 01201210 \end{pmatrix}, \\
(7) & M_7 & = & Pol\begin{pmatrix} 1210121122120110010 \\ 2101112121221010100 \\ 0022111212221101000 \end{pmatrix}.
\end{array}$$

Now, we give a few explanations for each submaximal set. M_5 has the following property:$f \in M_5 \Leftrightarrow$ if $f\begin{pmatrix} a \\ b \end{pmatrix} \in \begin{pmatrix} 01 \\ 12 \end{pmatrix}$ then there is i such that $\begin{pmatrix} a_i \\ b_i \end{pmatrix} \in \begin{pmatrix} 01 \\ 12 \end{pmatrix}$. M_6 has the following property: $f \in M_6 \Leftrightarrow$ if $f\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ then there is i such that $\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. M_7 is the set of functions preserving the relation $\rho :=$ 3-ary universal relation$\backslash\rho'$, where $\rho' = \begin{pmatrix} 0202 \\ 2002 \\ **20 \end{pmatrix}$. Since M_7 is a subset of B_1, we have the following property: $f \in M_7 \Leftrightarrow$ if $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \begin{pmatrix} 02 \\ 02 \\ 20 \end{pmatrix}$ then there is i such that $\begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} \in \begin{pmatrix} 02 \\ 02 \\ 20 \end{pmatrix}$. The reason of not occurring the first and second columns is that, otherwise f does not belong to B_1. Finally, we note the following inclusion

$$D(0,1) \cup D(1,2) \subset M_6 M_7.$$

We recall some lemmas from Chapter 5. The following lemma is a corollary of Lemma 5.7.1.

Lemma 6.4.1. $f \in B_1 \Rightarrow f \in T_1 \cup D(0,1) \cup D(1,2)$.

Corollary 6.4.1. $f \in \overline{T}_1 B_1 \Rightarrow f \in D(0,1) \cup D(1,2)$.

The following is the Lemma 5.1.8.

$$f \in T_{01}T_{12} \Rightarrow f \in T_1 \ (T_{01}T_{12}\overline{T}_1 \text{ is impossible}).$$

The following is the corollary of Lemma 5.7.2.

Corollary 6.4.2. $f \in B_1\overline{T}_{01}\overline{T}_{12} \Rightarrow f \in T_1$ ($\overline{T}_{01}\overline{T}_{12}\overline{T}_1$ is impossible in B_1).

We consider all the possible subsets and classify them separately in the following subsections: $M_7M_6M_5$, $M_7M_6\overline{M}_5$, $M_7\overline{M}_6M_5$, $M_7\overline{M}_6\overline{M}_5$, $\overline{M}_7M_6(M_5 \cup \overline{M}_5)$, $\overline{M}_7\overline{M}_6M_5$, and $\overline{M}_7\overline{M}_6\overline{M}_5$. Since B_1 is σ_1-similar invariant, we say simply similar for σ_1-similar in this section. Recall that σ_1, σ_2 and σ_4 are (20), (01) transposition and (210) cyclic permutation, respectively.

6.4.1. $M_7M_6M_5$.

Lemma 6.4.2. $f \in \overline{T}_1\overline{T}_{20}B_1 \Rightarrow f \in \overline{M}_5$.

Lemma 6.4.3. $f \in \overline{T}_{01}\overline{T}_{12}B_1 \Rightarrow f \in \overline{M}_6$.

From these two lemmas the classes $\overline{T}_1\overline{T}_{20}$ and $\overline{T}_{01}\overline{T}_{12}$ are impossible. We have the following 8 classes (cf. Lemma 5.1.8).

*no	T_1	T_{01}	T_{12}	T_{20}	
*1	0	0	0	0	σ_2-min, σ_4-min
*2	0	0	0	1	s_{012},1
*3	0	0	1	0	s_{010}
*4	0	0	1	1	s_{110}
*5	1	0	1	1	0
*6	0	1	0	0	similar of *3
*7	0	1	0	1	similar of *4
*8	1	1	0	0	similar of *5

Recall that max = σ_1-min. These min, max and σ_i-min functions are given in the previous chapter.

6.4.2. $M_7M_6\overline{M}_5$.

From Lemma 6.4.3 the class $\overline{T}_{01}\overline{T}_{12}$ is impossible. we have the following 10 classes (cf. Lemma 5.1.8):

*no	T_1	T_{01}	T_{12}	T_{20}	representative
*9	0	0	0	0	σ_2-min, σ_4-min
*10	0	0	0	1	$f2.1$
*11	0	0	1	0	$f2.2$
*12	0	0	1	1	$f2.3$
*13	1	0	1	0	$f2.4$
*14	1	0	1	1	$f2.5$
*15	0	1	0	0	similar of *11
*16	0	1	0	1	similar of *12
*17	1	1	0	0	similar of *13
*18	1	1	0	1	similar of *14

6.4.3. $M_7\overline{M}_6M_5$.

Lemma 6.4.4.

$$f \in B_1\overline{M}_6 \Rightarrow f \in T_1$$

Proof. Suppose $f(\mathbf{1}) = 0$. From $f \in \overline{M}_6$ there is $f\begin{pmatrix} 01201212 \\ 01210120 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Since $f \in B_1$, $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ exists in the arguments. Then we have $f\begin{pmatrix} 11111111 \\ 01210120 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \in \overline{B}_1$, a contradiction. If $f(\mathbf{1}) = 2$ the proof is similar. □

By this lemma we can delete all classes of $\overline{T}_1$ in $\overline{M}_6$. This leads to the following 8 classes ($f \in T_1$).

*no	T_{01}	T_{12}	T_{20}	representative
*19	0	0	0	$f3.1$
*20	0	0	1	$f3.2$
*21	0	1	0	$f3.3$
*22	0	1	1	$f3.4$
*23	1	0	0	similar of *21
*24	1	0	1	similar of *22
*25	1	1	0	s_{210}
*26	1	1	1	$f3.5$

6.4.4. $M_7\overline{M}_6\overline{M}_5$.

Similarly as the previous case we can delete all classes of $f \in \overline{T}_1$. We have the following 8 classes ($f \in T_1$).

*no	T_{01}	T_{12}	T_{20}	representative
*27	0	0	0	$f4.1$
*28	0	0	1	$f4.2$
*29	0	1	0	$f4.3$
*30	0	1	1	$f4.4$
*31	1	0	0	similar of *29
*32	1	0	1	similar of *30
*33	1	1	0	$f4.5$
34	1	1	1	$f4.6$

6.4.5. $\overline{M}_7 M_6 (M_5 \cup \overline{M}_5)$.

Lemma 6.4.5. $f \in \overline{M}_7 M_6 B_1 \Rightarrow f \in T_{12} T_{01}$.

We omit a rather complicate proof of this lemma (cf. [Miy82]). This lemma together with Lemma 5.1.8 and Lemma 6.4.3 reduces the number of classes remarkably. For $\overline{M}_7 M_6 M_5$ we have only two classes:

*no	T_1	T_{01}	T_{12}	T_{20}	representative
*35	0	0	0	0	min, max
*36	0	0	0	1	$f5.1$

Similarly, for $\overline{M}_7 M_6 \overline{M}_5$ we have only two classes.

*no	T_1	T_{01}	T_{12}	T_{20}	representative
*37	0	0	0	0	$f5.2$
*38	0	0	0	1	$f5.3$

6.4.6. $\overline{M}_7 \overline{M}_6 M_5$.

From Lemma 6.4.4 we have $f \in T_1$ and 8 classes are possible. There exists a representative function in each class.

*no	T_{01}	T_{12}	T_{20}	representative
*39	0	0	0	$f6.1$
*40	0	0	1	$f6.2$
*41	0	1	0	$f6.3$
*42	0	1	1	$f6.4$
*43	1	0	0	similar of *41
*44	1	0	1	similar of *42
*45	1	1	0	$f6.5$
*46	1	1	1	$f6.6$

6.4.7. $\overline{M}_7\overline{M}_6\overline{M}_5$.

By the same reason as the former subsection we have the following 8 classes ($f \in T_1$).

	T_{01}	T_{12}	T_{20}	representative
*47	0	0	0	$f7.1$
*48	0	0	1	$f7.2$
*49	0	1	0	$f7.3$
*50	0	1	1	$f7.4$
*51	1	0	0	similar of *49
*52	1	0	1	similar of *50
*53	1	1	0	$f7.5$
*54	1	1	1	$f7.6$

And this complete our classification of B_1. The complete classes are shown in Table 6.1.

6.4.8. Results and conclusive discussions

It is a bit surprising that such nontrivial functions as σ_2-min or σ_4-min have a null characteristic vector, since this indicates that these functions joined to any subset of B_1 effect null concerning generation of a function by superposition. We summarize the results as the theorems.

Theorem 6.4.2. *B_1 is divided into 54 nonempty classes.*

Since there exists a representative with at least four arguments in every class, we have:

Theorem 6.4.3. *For every base of B_1 there exist an equivalent base consisting of at most 4-ary functions, i.e. the order of B_1 is 4.*

The classes of bases and pivotals of B_1 are enumerated.

Theorem 6.4.4. *The numbers of classes of bases and pivotals of B_1 are 4,599 and 7,071 respectively.*

Corollary 6.4.3. *Maximal rank of bases or pivotals of B_1 is 6 (there are 17 bases with the maximal rank) and there is no Sheffer function in B_1.*

We give several illustrative examples.

Example 6.4.1. We list all 28 bases of B_1 with rank 2.

$1 \times \{17,18,30,31,44,45\}$, $2 \times \{17,18\}$, $3 \times \{18,31,45\}$, $4 \times \{17,30,44\}$, $5 \times \{17,18,30,31\}$, 7×18, 8×17, $10 \times \{17,18\}$, $11 \times \{18,31\}$, $12 \times \{17,30\}$, 17×21, 18×20. □

Example 6.4.2. There is only one base containing all constants functions among 17 bases with maximal rank 6. One such example is $\{2, 1, 0, \min, f3.1, f2.1\}$. □

Example 6.4.3. There is only one pivotal with the maximal rank 6. One such example is $\{\min, f3.1, f2.1, s_{212}, s_{010}, 1\}$. □

Example 6.4.4. The following set is P_3 pivotal with a maximal rank 7 [Jab58]: $\{\max, \sigma_2\text{-min}, \min, \sigma_4\text{-min}, 0, 1, 2\} \subset B_1$. It may seem that this set span some maximal set of B_1, however actually this spans a smaller set. We show characteristic vectors of these functions. Thus this set spans some subset of M_5M_6.

wt	*#no*	M_7	M_6	M_5	T_1	T_{01}	T_{12}	T_{20}	**no*	*representative*
1	#48	1	0	0	0	0	0	0	*35	*max, min*
0	#54	0	0	0	0	0	0	0	*1	σ_2-*min*, σ_4-*min*
2	#45	0	0	0	1	0	1	0	*5	0
1	#53	0	0	0	0	0	0	1	*2	1
2	#44	0	0	0	1	1	0	0	*8	2

□

Table 6.1: Classes of B_1.

wt	#no	$M_7M_6M_5$	T_1	$T_{01}T_{12}T_{2O}$	*no	representative
6	(#1)	111	0	111	*54	$f7.6$
5	(#2)	111	0	110	*53	$f7.5$
5	(#3)	111	0	101	*52	similar of $f7.4$
5	(#4)	111	0	011	*50	$f7.4$
5	(#5)	110	0	111	*46	$f6.6$
5	(#6)	011	0	111	*34	$f4.6$
4	(#7)	111	0	100	*51	similar of $f7.3$
4	(#8)	111	0	010	*49	$f7.3$
4	(#9)	111	0	001	*48	$f7.2$
4	(#10)	110	0	110	*45	$f6.5$
4	(#11)	110	0	101	*44	similar of $f6.4$
4	(#12)	110	0	011	*42	$f6.4$
4	(#13)	011	0	110	*33	$f4.5$
4	(#14)	011	0	101	*32	similar of $f4.4$
4	(#15)	011	0	011	*30	$f4.4$
4	(#16)	010	0	111	*26	$f3.5$
4	(#17)	001	1	101	*18	$s_{121}, s_{122}, s_{221}$
4	(#18)	001	1	011	*14	$s_{001}, s_{100}, s_{101}$
3	(#19)	111	0	000	*47	$f7.1$
3	(#20)	110	0	100	*43	similar of $f6.3$
3	(#21)	110	0	010	*41	$f6.3$
3	(#22)	110	0	001	*40	$f6.2$
3	(#23)	101	0	001	*38	$f5.3$
3	(#24)	011	0	100	*31	similar of $f4.3$
3	(#25)	011	0	010	*29	$f4.3$
3	(#26)	011	0	001	*28	$f4.2$
3	(#27)	010	0	110	*25	s_{210}
3	(#28)	010	0	101	*24	similar of $f3.4$
3	(#29)	010	0	011	*22	$f3.4$
3	(#30)	001	1	100	*17	similar of $f2.5$
3	(#31)	001	1	010	*13	$f2.5$
3	(#32)	001	0	101	*16	similar of $f2.4$
3	(#33)	001	0	011	*12	$f2.4$
2	(#34)	110	0	000	*39	$f6.1$
2	(#35)	101	0	000	*37	$f5.2$
2	(#36)	100	0	001	*36	$f5.1$
2	(#37)	011	0	000	*27	$f4.1$
2	(#38)	010	0	100	*23	similar of $f3.3$
2	(#39)	010	0	010	*21	$f3.3$
2	(#40)	010	0	001	*20	$f3.2$
2	(#41)	001	0	100	*15	similar of $f2.3$
2	(#42)	001	0	010	*11	$f2.3$
2	(#43)	001	0	001	*10	$f2.2$
2	(#44)	000	1	100	*8	2
2	(#45)	000	1	010	*5	0
2	(#46)	000	0	101	*7	s_{211}
2	(#47)	000	0	011	*4	s_{110}
1	(#48)	100	0	000	*35	min, max
1	(#49)	010	0	000	*19	$f3.1$
1	(#50)	001	0	000	*9	$f2.1$
1	(#51)	000	0	100	*6	s_{212}
1	(#52)	000	0	010	*3	s_{010}
1	(#53)	000	0	001	*2	$s_{011}, 1$
0	(#54)	000	0	000	*1	σ_2- and σ_4-similar of min

Table 6.2: Representatives of classes of B_1.

$f \setminus xy$	00	01	02	10	11	12	20	21	22
$f2.2$	1	0	1	1	1	1	1	1	1
$f2.4$	1	1	1	1	1	1	1	0	1
$f3.3$	0	1	0	1	1	1	2	1	0
$f3.4$	1	1	1	1	1	1	2	1	0
$f3.5$	2	1	0	1	1	1	1	1	0
$f4.4$	1	1	1	1	1	0	2	1	1

$f \setminus xy$	00	01	02	10	11	12	20	21	22
$f4.6$	1	1	1	2	1	1	1	1	0
$f6.3$	0	0	0	1	1	0	2	1	0
$f6.5$	2	1	0	1	1	0	0	0	0
$f7.4$	1	0	1	1	1	0	2	1	1
$f7.6$	1	2	1	2	1	1	1	1	0

$f2.1$	00	01	10	11	12	21	22	20	02
0	0	1	1	1	1	1	2	2	2
1	1	1	1	1	1	2	1	1	1
2	2	1	1	1	1	1	2	2	2

$f2.3$	00	01	10	11	12	21	22	20	02
0	0	1	1	1	1	1	0	0	0
1	1	1	1	1	1	1	1	1	1
2	0	1	1	0	1	1	0	0	0

$f2.5$	00	01	10	11	12	21	22	20	02
0	0	1	1	1	1	1	0	0	0
1	1	1	1	0	1	1	1	1	1
2	0	1	1	0	1	1	0	0	0

$f3.1$	00	01	10	11	12	21	22	20	02
0	0	1	1	1	1	1	0	2	0
1	1	1	1	1	1	1	1	1	1
2	2	1	2	1	1	1	2	2	0

$f3.2$	00	01	10	11	12	21	22	20	02
0	0	1	1	1	1	1	0	2	0
1	1	1	1	1	1	1	1	1	1
2	2	1	2	1	1	1	1	2	0

$f4.2$	00	01	10	11	12	21	22	20	02
0	1	1	1	1	1	1	0	1	1
1	1	1	1	1	1	1	1	1	2
2	1	1	1	1	1	1	1	1	1

$f4.3$	00	01	10	11	12	21	22	20	02
0	0	1	1	1	1	1	0	0	2
1	1	1	1	1	1	0	1	1	1
2	0	1	1	1	1	1	0	0	0

$f4.5$	00	01	10	11	12	21	22	20	02
0	2	1	1	1	1	1	0	0	2
1	1	1	1	1	1	0	1	1	1
2	2	1	1	1	1	1	0	0	2

$f5.1$	00	01	10	11	12	21	22	20	02
0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	1	1	1	0
2	0	0	1	1	1	1	2	1	0

$f5.3$	00	01	10	11	12	21	22	20	02
0	0	1	1	0	1	1	1	1	1
1	1	0	1	1	1	1	1	1	1
2	1	1	1	1	1	2	1	1	1

$f6.1$	00	01	10	11	12	21	22	20	02
0	0	1	1	1	1	1	2	2	2
1	0	1	1	1	1	1	1	1	1
2	0	1	0	1	1	1	2	0	2

$f6.2$	00	01	10	11	12	21	22	20	02
0	0	1	1	1	1	1	2	2	2
1	0	1	1	1	1	1	1	1	1
2	0	1	0	1	1	1	2	0	1

Representatives of classes of B_1 (continued).

$f6.4$	00	01	10	11	12	21	22	20	02
0	1	1	1	1	2	1	2	1	2
1	1	1	1	1	1	1	1	1	2
2	1	1	1	1	1	1	0	1	2

$f6.6$	00	01	10	11	12	21	22	20	02
0	2	1	1	1	2	1	2	1	2
1	1	1	1	1	1	1	1	1	2
2	1	1	1	1	1	1	0	1	2

$f7.1$	00	01	10	11	12	21	22	20	02
0	0	1	1	1	2	1	2	0	2
1	1	1	1	1	1	2	1	1	1
2	0	0	1	1	1	1	2	2	0

$f7.2$	00	01	10	11	12	21	22	20	02
0	0	1	1	1	2	1	2	0	2
1	1	1	1	1	1	2	1	1	1
2	0	0	1	1	1	1	2	1	0

$f7.3$	00	01	10	11	12	21	22	20	02
0	0	1	1	1	1	1	0	2	0
1	1	1	1	1	1	2	1	1	1
2	2	1	1	1	0	1	1	2	0

$f4.1$	00	01	10	11	12	21	22	20	02
00	0	1	1	1	1	1	0	2	2
01	1	1	1	1	1	1	1	1	1
10	1	1	1	1	1	1	1	1	1
11	1	1	1	1	1	1	1	1	2
12	1	1	1	1	1	1	1	1	1
21	1	1	1	1	1	1	1	1	1
22	2	1	1	1	1	1	2	2	2
20	2	1	1	1	1	1	2	2	2
02	2	1	1	1	1	1	2	2	2

$f5.2$	00	01	10	11	12	21	22	20	02
00	0	1	1	1	1	1	0	0	0
01	1	1	1	1	1	1	1	1	1
10	1	1	1	0	1	1	1	1	1
11	1	0	1	1	1	1	1	1	1
12	1	1	1	1	1	2	1	1	1
21	1	1	1	1	1	1	1	1	1
22	0	1	1	1	1	1	2	2	0
20	0	1	1	1	1	1	0	0	0
02	0	1	1	1	1	1	2	2	0

$f7.5$	00	01	10	11	12	21	22	20	02
00	2	1	1	1	1	1	0	2	2
01	1	1	1	1	1	1	1	1	1
10	1	1	2	1	1	1	1	1	1
11	1	2	1	1	1	1	0	2	2
12	1	1	1	1	1	1	1	1	1
21	1	1	1	1	1	1	1	1	1
22	2	1	1	1	1	1	0	2	2
20	2	1	1	1	1	1	0	2	2
02	2	1	1	1	1	1	0	2	2

6.5 Classification of T_0

The set T_0 of three-valued logical functions preserving 0 is classified into 253 classes using the known classification of P_3 (the whole set of three-valued logical functions).

Recall that T_0 is the set of all 3-valued logical functions f such that $f(0,\ldots,0) = 0$. In [Miy84] the classes of functions and bases for T_0 are given. In this sections we give much simpler description of it using the classification of P_3. We recall:

Theorem 6.5.1. [Lau82b] *T_0 has exactly the following 12 maximal sets.*

Group I.

(1) $K_{10} = Pol\begin{pmatrix} 012 \\ 021 \end{pmatrix}$.

(2) $K_{11} = Pol\begin{pmatrix} 00102 \\ 01020 \end{pmatrix}$.

(3) $K_{12} = Pol\begin{pmatrix} 0010212 \\ 0102021 \end{pmatrix}$.

Group II.

(4) $T_0M_1 = Pol(0)Pol\begin{pmatrix} 012001 \\ 012122 \end{pmatrix}$.

(5) $T_0M_2 = Pol(0)Pol\begin{pmatrix} 012121 \\ 012200 \end{pmatrix}$.

(6) $T_0U_{12} = Pol(0)Pol\begin{pmatrix} 01212 \\ 01221 \end{pmatrix}$.

(7) $T_0B_0 = Pol(0)Pol\begin{pmatrix} 0120012 \\ 0121200 \end{pmatrix}$.

Group III.

(8) $T_0T_1 = Pol(0)Pol(1)$.

(9) $T_0T_2 = Pol(0)Pol(2)$.

(10) $T_0T_{01} = Pol(0)Pol(01)$.

(11) $T_0T_{12} = Pol(0)Pol(12)$.

(12) $T_0T_{20} = Pol(0)Pol(20)$.

Note that only the three sets K_{10}, K_{11} and K_{12} are not P_3-maximal. In Section 2 we need the following 14 technical lemmas which are of independent interest (as statements about the lattice of closed sets ordered by $\subseteq$). First we list them together (as Lemmas 1.1–1.14) and then proceed with their proofs.

Lemma 6.5.1. $K_{10}K_{12} \subseteq K_{11}$.

Lemma 6.5.2. $T_1K_{10} \subseteq T_2$, $T_2K_{10} \subseteq T_1$.

Lemma 6.5.3. $T_{01}K_{10} \subseteq T_{02}$.

Lemma 6.5.4. $U_0K_{12} \subseteq K_{10}$.

Lemma 6.5.5. $T_1K_{12} \subseteq T_{02}$, $T_2K_{12} \subseteq T_{01}$.

Lemma 6.5.6. $B_0T_{01}T_{02}U_0 \subseteq K_{11}$.

Lemma 6.5.7. $K_{10}K_{12} \subseteq B_0$.

Lemma 6.5.8. $U_0K_{12} \subseteq B_0$.

Lemma 6.5.9. $M_1K_{10} \subseteq M_2$.

Lemma 6.5.10. $M_1K_{10} \subseteq U_0$.

Lemma 6.5.11. $B_0K_{12} \subseteq K_{11}$.

Lemma 6.5.12. $K_{12}T_{12} \subseteq B_0$.

Lemma 6.5.13. $K_{10}B_0 \subseteq K_{12}$.

Lemma 6.5.14. $M_1T_{02}K_{12} \subseteq K_{11}$.

Proofs. We must prove inclusions of the form $Pol\rho_1 \cdots Pol\rho_i \subseteq Pol\rho_0$ (where $i = 4$ in Lemma 6.5.6, $i = 3$ in Lemma 6.5.14 and $i = 2$ otherwise). The inclusion holds if we can express ρ_0 by a logical formula based on $\exists, \&, =$ and membership in ρ_j $(1 \leq j \leq i)$.

We show what we mean by an example. Let

$$\kappa_{10} := \begin{pmatrix} 012 \\ 021 \end{pmatrix}, \ \kappa_{12} := \begin{pmatrix} 0010212 \\ 0102021 \end{pmatrix}, \ \kappa_{11} := \begin{pmatrix} 00102 \\ 01020 \end{pmatrix}.$$

Put

$$\lambda := \{(x,y) : (x,y) \in \kappa_{12}, (x,u) \in \kappa_{10}, (u,y) \in \kappa_{12} \text{ for some } u\}.$$

This may be written as $\lambda = \kappa_{12} \cap (\kappa_{10} \circ \kappa_{12})$ where $\circ$ denotes the relational (de Morgan) product or composition.

We prove $\kappa_{11} = \lambda$ by a direct check. First clearly $\lambda \subseteq \kappa_{12}$. We have $(0,0), (0,1), (0,2)$ $\in \kappa_{10} \circ \kappa_{12}$ (choose $u = 0$ in all 3 cases), $(2,0) \in \kappa_{10}$ (choose $u = 1$) and $(1,0) \in \kappa_{10} \circ \kappa_{12}$

(choose $u = 2$) and so $\kappa_{11} \subseteq \lambda \subseteq \kappa_{12}$. Next $(1,2) \notin \kappa_{10} \circ \kappa_{12}$ (if it were we would need $u = 2$ but $(2,2) \notin \kappa_{12}$) and similarly $(2,1) \notin \kappa_{10} \circ \kappa_{12}$ (we need $u = 1$ but $(1,1) \notin \kappa_{12}$). It follows that $\kappa_{11} = \lambda$.

The above fact $Pol\rho_1 \cdots Pol\rho_i \subseteq Pol\rho_0$ is well known ([Ros70, §4], for more information cf. [Pok79, §1.1, ch. 2]), and may be proved directly (it has also an interesting and basic converse called Galois polytheory, cf. ibid).

In the sequel κ_{ij} denotes the relation in $K_{ij} = Pol\ \kappa_{ij}$ (see Theorem 1.2, group I), similarly $U_i = Pol\ \nu_i, M_i = Pol\ \mu_i$, and $B_0 = Pol\ \beta_0$.

Lemma 6.5.1.

$\kappa_{11} = \{(x,y)|(x,y) \in \kappa_{12}, (x,u) \in \kappa_{10}$ and $(y,u) \in \kappa_{12}$ for some $u\}$ (see above). □

Lemma 6.5.2.

$\{2\} = \{x|(x,u) \in \kappa_{10}$ for some $u \in \{1\}\}$ (as $T_i = Pol\{i\}$ where $\{i\}$ is a unary relation; of course $u \in \{1\}$ means $u = 1$). Similarly $\{1\} = \{x|(x,2) \in \kappa_{10}\}$. □

Lemma 6.5.3.

$\{0,2\} = \{x|(x,u) \in \kappa_{10}$ for some $u \in \{0,1\}\}$. □

Lemma 6.5.4.

$\kappa_{10} = \nu_0 \cap \kappa_{12}$. □

Lemma 6.5.5.

$\{0,2\} = \{x|(x,1) \in \kappa_{12}\}$, $\{0,1\} = \{x|(x,2) \in \kappa_{12}\}$.

Lemma 6.5.6.

$\kappa_{11} = \{(x,y)|(x,y) \in \beta_0, (x,u) \in \mu_0, (u,v) \in \beta_0, (v,y) \in \nu_0$ for some $u \in \{0,1\}$ and $v \in \{0,2\}\}$. To see $\subseteq$ consider the following (x,u,v,y) : $(0,0,2,1),(1,1,0,0)$, $(0,0,2,2)$,$(2,1,0,0)$ and (0,0,0,0). The inclusion $\supseteq$ is obtained as follows. If $(1,u) \in \mu_0$ and $(v,1) \in \nu_0$ for some $u \in \{0,1\}$ and $v \in \{0,2\}$, then $u = 1$ and $v = 2$ and hence $(u,v) \notin \beta_0$ proving $(1,1)$ does not belong to the right side. The proof for $(2,2)$ is similar. As the right side is a subrelation of β_0 this complete the proof.

Lemma 6.5.7.

$\beta_0 = \{(x,y)|(x,u),(v,y) \in \kappa_{10}, (x,v),(u,y) \in \kappa_{12}$ for some u and $v\}$. □

Lemma 6.5.8.

Combine Lemmas 6.5.4 and 6.5.7. □

Lemma 6.5.9.

$\mu_2 = \{(x,y)|(x,u),(v,y) \in \kappa_{10} \text{ for some } u \geq v\}$, □

Lemma 6.5.10.

$\nu_0 = \{(x,y)|(u,v),(w,t) \in \kappa_{10}, u \leq x \leq t, w \leq y \leq v\}$. □

Lemma 6.5.11.

Let $f \in \overline{K}_{11}K_{12}$. From $f \in \overline{K}_{11}$ there are $\begin{pmatrix} a \\ b \end{pmatrix} \in \kappa_{11}$ such that $\begin{pmatrix} f(a) \\ f(b) \end{pmatrix} \in \begin{pmatrix} 1212 \\ 1221 \end{pmatrix}$. However, from $f \in \kappa_{12}$ and $\kappa_{11} \subseteq \kappa_{12}$ we have $\begin{pmatrix} f(a) \\ f(b) \end{pmatrix} \notin \begin{pmatrix} 12 \\ 12 \end{pmatrix}$. Hence we have $f \notin B_0$. □

Lemma 6.5.12.

$\beta_0 = \{(x,y)|(x,u),(u,y) \in \kappa_{12} \text{ for some } u \in \{1,2\}\}$.

Lemma 6.5.13.

$\kappa_{12} = \{(x,y) : (x,u),(v,y) \in \kappa_{10}, (x,v),(u,y) \in \beta_0 \text{ for some } u \text{ and } v\}$. To prove $\subseteq$ we take the following quadruples $(x,u,v,y) : (0,0,0,0),(0,0,2,1),(0,0,1,2)$ and $(1,2,1,2)$ (the right side is obviously symmetric). For $\supseteq$ note that neither $(1,1)$ nor $(2,2)$ belong to the right side (if $(1,1)$ would then $u = 2$ in contradiction to $(2,1) \notin \beta_0$ and similarly for (2,2)).

Lemma 6.5.14.

$\kappa_{11} = \{(x,y) \in \kappa_{12} : x \leq u, v \geq y, (x,v),(u,y) \in \kappa_{12} \text{ for some } u,v \in \{0,2\}\}$. To see $\subseteq$ note that the right side is symmetric and take the quadruples (x,u,v,y) : $(0,0,0,0),(0,2,2,1)$ and $(0,0,2,2,)$. For $\supseteq$ note the following. First the right side is symmetric. If $(1,2)$ belongs to the right side then $u \geq 1, u \in \{0,2\}$ means $u = 2$ in contradiction to $(2,2) \notin \kappa_{12}$.

Lemma 6.5.15. $U_0B_0 \subseteq T_{01} \cup T_{02} \cup K_{11}$.

Proof. Suppose there exists an n-ary $f \in U_0B_0\overline{T}_{01}\overline{T}_{02}\overline{K}_{11}$. Then there are $\begin{pmatrix} a \\ b \end{pmatrix} \in \kappa_{11}^n$ such that $\begin{pmatrix} f(a) \\ f(b) \end{pmatrix} \notin \kappa_{11}$, i.e $\in \begin{pmatrix} 1212 \\ 1221 \end{pmatrix}$. Were $\begin{pmatrix} f(a) \\ f(b) \end{pmatrix} \in \begin{pmatrix} 12 \\ 21 \end{pmatrix}$, in view of $\kappa_{11} \subseteq \beta_0$ we would have $f \notin B_0$. Next suppose $f(a) = f(b) = 1$. Define a vector c so that $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \begin{pmatrix} 01020 \\ 00102 \\ 01010 \end{pmatrix}^n$. Now $\begin{pmatrix} a \\ c \end{pmatrix} \in \nu_0^n$ and $f \in U_0$ imply $f(c) \neq 0$. Next $\begin{pmatrix} b \\ c \end{pmatrix} \in \beta_0^n$ and $f \in B_0$ imply $\begin{pmatrix} 1 \\ f(c) \end{pmatrix} \in \beta_0$ and therefore together we have $f(c) \neq 2$ and $f(c) \neq 1$. Since $f \notin T_{01}$, there is a vector $d \in \{0,1\}^n$ such that $f(d) = 2$. From

$f(c) = 1, f(d) = 2$ and $\begin{pmatrix} c \\ d \end{pmatrix} \in \begin{pmatrix} 0110 \\ 0011 \end{pmatrix}$ we conclude $f \notin B_0$, a contradiction. Finally if $f(a) = f(b) = 2$ the proof is quite similar. □

Lemma 6.5.16. *The set* $M_1\overline{T}_2T_{02}$ *consists of constant functions with value 0 only and so* $M_1\overline{T}_2T_{02} \subseteq K_{10}K_{11}K_{12}$.

Proof. From $f \in \overline{T}_2T_{02}$ follows $f(2) \in \{0, 2\}$ and $f(2) \neq 2$ i.e. $f(2) = 0$. From $f \in M_1$ and $y \leq 2$ for all $y \in E$ we get $f \equiv 0$ $f(x) \leq 0$ for all $x \in E^n$ i.e. which is an element of $K_{10}K_{11}K_{12}$. □

6.5.1. Classification of T_0

The sets $T_1, T_2, T_{01}, T_{02}, T_{12}, U_0, B_0, M_1$ and M_2 are P_3-maximal sets. Among the 406 classes of P_3 exactly 248 classes are subsets of T_0. However, only 93 classes are obtained from the above nine P_3-maximal sets (as intersections of the sets or their complements). The interchange 1 and 2 in the definition of each maximal set $T_1, T_2, T_{01}, T_{02}, T_{12}, U_0$, B_0, M_1, M_2, K_{10}, K_{11} and K_{12} yields $T_2, T_1, T_{02}, T_{01}, T_{12}$, U_0, B_0, M_2, M_1, K_{10}, K_{11} and K_{12} respectively. The class T_0 is mapped onto itself. Two classes are *similar* if the characteristic vectors are obtained by one from the other by applying the above mapping to all coordinates of the vector, i.e., $a'_i = a_{i'}$, where $'$ denote the above mapping of maximal sets. Among the 93 classes (the sum of the fourth column in Table 6.3), 58 are pairwise nonsimilar.

The complete classification of T_0 is obtained by checking all 8 possible cases with respect to the sets K_{10}, K_{11} and K_{12} for each of the above 93 classes. From Lemmas 1 - 16 we can show that many classes are empty. In Table 6.3 for each of the 58 nonsimilar classes with respect to the first 9 maximal sets we give the ordinal number of one of the corresponding classes of P_3 from [Sto84a,Miy71] (the second and the third column of the table). In the next to the last column we give the number of corresponding classes of the set T_0 obtained by concatenating the characteristic vectors corresponding to K_{10}, K_{11} and K_{12}. In the last column we indicate the lemmas, on the basis of which some of the 8 cases do not occur.

For each of the remaining 169 (the sum of the numbers of the next to the last column) classes, a representative function is shown in Table 6.5 (163 representatives, the 6 representatives are unary, which are shown in the table). Counting the similarity (summing s-column multiplied by c-column for all rows), we have:

Theorem 6.5.2. [Miy84] *The number of the classes of T_0 is 253.*

The classes are listed in Table 6.4 and there representatives in Table 6.5.

6.5.2. Enumeration of bases of T_0

Using the list of 353 characteristic vectors the T_0-bases and T_0-pivotal incomplete sets are computed [Miy84]: they are 883,720 and 380,710, respectively. The maximal rank of a base of T_0 is 6. The detailed data are shown in Chapter 5.

6.6. Concluding remarks

The classifications for the other maximal sets of P_3 were done by Stojmenović as we have seen in Chapter 5. The maximal sets M_1, U_0 and T_{01} have 88, 383 and 607 classes and their bases are 118,744, 16,572,242 and 102,323,760. Maximal rank of a base of each set is 7. All the results were reported in [MiS87b] jointly with Stojmenović.

Table 6.3:

no	P_3-class	sim.	M_1M_2	U_0	B_0	T_1T_2	$T_{01}T_{12}T_{20}$	gen. classes	lemma
1	7	1	11	1	1	11	111	6	$L7$
2	20	1	11	1	1	11	101	4	$L12$
3	21	2	11	1	1	11	011	4	$L3$
4	23	2	11	1	1	01	111	2	$L2, 5$
5	26	1	11	1	0	11	111	4	$L11, 13$
6	34	1	11	0	1	11	111	4	$L8$
7	48	1	11	1	1	11	010	6	$L7$
8	52	2	11	1	1	01	110	4	$L2$
9	53	2	11	1	1	01	101	2	$L2, 5$
10	54	2	11	1	1	01	011	2	$L2, 5$
11	55	1	11	1	1	00	111	4	$L5$
12	63	1	11	1	0	11	101	4	$L11, 13$
13	64	2	11	1	0	11	011	3	$L3, 11$
14	74	1	11	0	1	11	101	4	$L8$
15	75	2	11	0	1	11	011	2	$L3, 8$
16	76	1	11	0	0	11	111	2	$L4, 13, 15$
17	88	2	11	1	1	01	010	4	$L2$
18	89	2	11	1	1	01	001	2	$L2, 5$
19	91	1	11	1	1	00	101	4	$L5$
20	92	2	11	1	1	00	011	2	$L3, 5$
21	99	1	11	1	0	11	010	4	$L11, 13$
22	101	2	11	1	0	01	011	2	$L2, 5$
23	114	1	11	0	1	11	010	4	$L8$
24	116	2	11	0	1	01	101	2	$L2, 5$
25	118	1	11	0	0	11	101	2	$L4, 13, 15$
26	119	2	11	0	0	11	011	2	$L3, 4$
27	133	2	01	1	1	10	101	2	$L2, 5$
28	134	2	01	1	1	10	011	4	$L2$
29	137	1	11	1	1	00	010	6	$L7$
30	138	2	11	1	1	00	001	2	$L3, 5$
31	149	2	11	1	0	01	010	3	$L2, 11$
32	150	2	11	1	0	01	001	2	$L2, 5$
33	162	2	11	0	1	01	001	2	$L2, 5$
34	163	1	11	0	1	00	101	4	$L5$
35	166	1	11	0	0	11	010	2	$L4, 6, 13$
36	183	2	01	1	1	10	100	2	$L2, 5$
37	184	2	01	1	1	10	010	3	$L2, 14$
38	185	2	01	0	1	10	101	2	$L2, 5$
39	191	1	11	1	1	00	000	4	$L12$
40	194	1	11	1	0	00	010	4	$L11, 13$
41	204	2	11	0	1	00	001	2	$L3, 5$
42	210	2	11	0	0	01	001	2	$L2, 5$
43	232	2	01	1	1	00	001	2	$L3, 5$
44	234	2	01	1	0	10	010	3	$L2, 11$
45	235	2	01	0	1	10	100	2	$L2, 5$
46	254	1	11	1	0	00	000	4	$L11, 13$
47	258	1	11	0	1	00	000	4	$L8$
48	282	2	01	1	1	00	000	2	$L10, 12$
49	284	2	01	0	1	00	001	2	$L3, 5$
50	309	2	01	1	0	11	011	3	$L3, 11$
51	315	1	11	0	0	00	000	2	$L4, 6, 13$
52	335	2	01	1	0	00	000	3	$L10, 11$
53	336	2	01	0	1	00	000	2	$L8, 9$
54	378	2	01	1	0	10	100	2	$L2, 5$
55	381	2	01	1	0	01	001	2	$L2, 5$
56	390	1	00	0	0	00	000	2	$L4, 6, 13$
57	396	2	00	0	0	01	001	2	$L2, 5$
58	405	1	00	0	0	11	010	1	$L16$

Table 6.4: Classes of T_0

coordinates are: $K_{10}K_{11}K_{12}M_1M_2U_0B_0T_1T_2T_{01}T_{12}T_{20}$.

wt	*no*		*similar*
12	1	111111111111	
11	2	111111111110	$g'4$
11	3	111111111101	
11	4	111111111011	
11	5	111111110111	
11	6	111111101111	
11	7	111111011111	
11	8	111110111111	
11	9	110111111111	
11	10	101111111111	
11	11	011111111111	
10	12	111111111010	
10	13	111111110110	
10	14	111111110101	
10	15	111111110011	$g'16$
10	16	111111101110	
10	17	111111101101	
10	18	111111101011	$g'13$
10	19	111111100111	
10	20	111111011110	
10	21	111111011101	
10	22	111111011011	$g'20$
10	23	111110111110	
10	24	111110111101	
10	25	111110111011	$g'23$
10	26	110111111110	
10	27	110111111011	$g'26$
10	28	101111111110	
10	29	101111111101	
10	30	101111111011	$g'28$
10	31	101111110111	
10	32	101111101111	
10	33	101111011111	
10	34	101110111111	
10	35	100111111111	
10	36	011111111101	
10	37	011110111111	
10	38	001111111111	
9	39	111111110100	$g'42$
9	40	111111110010	
9	41	111111101010	
9	42	111111101001	
9	43	111111100110	
9	44	111111100101	
9	45	111111100011	$g'43$
9	46	111111011010	
9	47	111111010110	
9	48	111111001011	$g'47$
9	49	111110111010	
9	50	111110110101	
9	51	111110101101	
9	52	111110011110	
9	53	111110011011	$g'52$
9	54	111101101110	$g'58$
9	55	111101101101	$g'57$
9	56	111101011110	$g'59$
9	57	111011110101	
9	58	111011110011	
9	59	111011011011	
9	60	110111111010	
9	61	110111110011	$g'62$
9	62	110111101110	
9	63	101111111010	
9	64	101111110110	
9	65	101111110101	
9	66	101111110011	$g'67$
9	67	101111101110	
9	68	101111101101	
9	69	101111101011	$g'64$
9	70	101111100111	
9	71	101111011110	
9	72	101111011101	
9	73	101111011011	$g'71$
9	74	101110111110	
9	75	101110111101	
9	76	101110111011	$g'74$
9	77	101110011111	
9	78	100111111110	
9	79	100111111011	$g'78$
9	80	100111011111	
9	81	011111111010	
9	82	011111100111	
9	83	011110111101	
9	84	001111111101	
9	85	001110111111	
8	86	111111100100	$g'88$
8	87	111111100010	
8	88	111111100001	
8	89	111111010100	$g'92$
8	90	111111010010	
8	91	111111001010	
8	92	111111001001	
8	93	111110110100	$g'94$
8	94	111110101001	
8	95	111110100101	
8	96	111101101010	$g'100$
8	97	111101101001	$g'99$
8	98	111100101101	$g'101$
8	99	111011110100	
8	100	111011110010	

wt	no		similar
8	101	111010110101	
8	102	110111110010	g'103
8	103	110111101010	
8	104	110101101110	
8	105	110011110011	g'104
8	106	101111110100	g'109
8	107	101111110010	
8	108	101111101010	
8	109	101111101001	
8	110	101111100110	
8	111	101111100101	
8	112	101111100011	g'110
8	113	101111011010	
8	114	101111010110	
8	115	101111001011	g'114
8	116	101110111010	
8	117	101110110101	
8	118	101110101101	
8	119	101110011110	
8	120	101110011101	
8	121	101110011011	g'119
8	122	101101101110	g'126
8	123	101101101101	g'125
8	124	101101011110	g'127
8	125	101011110101	
8	126	101011110011	
8	127	101011011011	
8	128	100111111010	
8	129	100111110011	g'130
8	130	100111101110	
8	131	100111011110	
8	132	100111011101	
8	133	100111011011	g'131
8	134	011111100101	
8	135	011110111010	
8	136	001111111010	
8	137	001111100111	
8	138	001110111101	
8	139	000111011111	
7	140	111111100000	
7	141	111111000010	
7	142	111110100100	g'142
7	143	111110100001	
7	144	111110010100	g'145
7	145	111110001001	
7	146	111101100100	g'151
7	147	111101010100	g'154
7	148	111101001010	g'153
7	149	111101001001	g'152
7	150	111100101001	g'155

wt	no		similar
7	151	111011100001	
7	152	111011010100	
7	153	111011010010	
7	154	111011001001	
7	155	111010110100	
7	156	110111100010	
7	157	101111100100	g'159
7	158	101111100010	
7	159	101111100001	
7	160	101111010100	g'163
7	161	101111010010	
7	162	101111001010	
7	163	101111001001	
7	164	101110110100	g'165
7	165	101110101001	
7	166	101110100101	
7	167	101110011010	
7	168	101101101010	g'172
7	169	101101101001	g'171
7	170	101100101101	g'173
7	171	101011110100	
7	172	101011110010	
7	173	101010110101	
7	174	100111110010	g'175
7	175	100111101010	
7	176	100111011010	
7	177	100101101110	
7	178	100101011110	s_{020}
7	179	100011110011	g'177
7	180	100011011011	s'_{020}
7	181	011111100010	
7	182	011110100101	
7	183	001111100101	
7	184	001110111010	
7	185	000111011101	
7	186	000110011111	
6	187	111111000000	
6	188	111110100000	
6	189	111101100000	g'191
6	190	111100100100	g'192
6	191	111011100000	
6	192	111010100001	
6	193	101111100000	
6	194	101111000010	
6	195	101110100100	g'196
6	196	101110100001	
6	197	101110010100	g'198
6	198	101110001001	
6	199	101101100100	g'204
6	200	101101010100	g'207

wt	*no*	$K_{10}K_{11}K_{12}M_1M_2U_0B_0T_1T_2T_{01}T_{12}T_{20}$	*similar*
6	201	101101001010	$g'206$
6	202	101101001001	$g'205$
6	203	101100101001	$g'208$
6	204	101011100001	
6	205	101011010100	
6	206	101011010010	
6	207	101011001001	
6	208	101010110100	
6	209	100111100010	
6	210	100111010010	$g'211$
6	211	100111001010	
6	212	100101101010	
6	213	100011110010	$g'212$
6	214	011111100000	
6	215	001111100010	
6	216	001110100101	
6	217	000111011010	
6	218	000110011101	s_{021}
5	219	111101000000	$g'221$
5	220	111100100000	$g'222$
5	221	111011000000	
5	222	111010100000	
5	223	111000010100	$g'224$
5	224	111000001001	
5	225	101111000000	
5	226	101110100000	
5	227	101101100000	$g'229$
5	228	101100100100	$g'230$
5	229	101011100000	
5	230	101010100001	
5	231	100111000010	
5	232	100101001010	s_{010}
5	233	100011010010	s'_{010}
5	234	011110100000	
5	235	001111100000	
5	236	000110011010	
4	237	101110000000	
4	238	101101000000	$g'240$
4	239	101100100000	$g'241$
4	240	101011000000	
4	241	101010100000	
4	242	101000010100	s'_{011}
4	243	101000001001	s_{011}
4	244	100111000000	
4	245	001110100000	
4	246	000111000010	
3	247	100101000000	$g'248$
3	248	100011000000	
3	249	000111000000	
3	250	000000011010	0
2	251	101000000000	
2	252	000110000000	
0	253	000000000000	s_{012}

Table 6.5: Representatives of classes of T_0 (163 functions).

$f \setminus xy$	00	01	02	10	11	12	20	21	22
$g1$	0	1	2	1	2	0	2	1	1
$g3$	0	2	0	2	2	1	0	1	1
$g4$	0	1	2	1	0	0	0	2	1
$g6$	0	2	1	2	1	0	1	0	1
$g7$	0	2	0	2	0	2	0	2	1
$g8$	0	2	1	1	0	0	1	0	0
$g10$	0	1	2	0	2	0	0	1	1
$g11$	0	2	1	2	0	2	1	1	0
$g12$	0	1	2	1	0	1	0	0	0
$g13$	0	1	0	1	2	0	0	0	2
$g16$	0	2	0	2	1	2	0	0	0
$g17$	0	2	1	2	1	1	0	1	1
$g19$	0	2	1	2	1	0	1	1	2
$g20$	0	2	2	0	0	0	2	0	0
$g21$	0	0	1	0	2	1	1	1	1
$g23$	0	1	2	2	0	0	2	0	0
$g24$	0	1	2	1	2	1	2	1	1
$g26$	0	2	0	1	2	0	0	2	0
$g28$	0	1	0	0	2	1	0	0	0
$g29$	0	2	0	0	2	1	0	1	1
$g32$	0	2	1	0	1	0	0	0	1
$g33$	0	2	0	0	0	2	0	0	1
$g35$	0	0	1	0	2	2	0	0	0
$g37$	0	2	1	2	0	0	1	0	0
$g38$	0	1	2	0	2	0	0	0	1
$g41$	0	1	0	1	1	2	0	2	0
$g42$	0	1	2	0	1	1	2	2	1
$g43$	0	2	0	2	1	2	0	0	2
$g44$	0	0	1	2	1	1	1	1	2
$g46$	0	1	0	1	0	1	0	1	0
$g47$	0	0	2	0	2	0	2	0	2
$g51$	0	2	1	2	1	1	1	1	1
$g52$	0	2	2	2	0	0	2	0	0
$g57$	0	2	2	0	2	2	1	2	2
$g58$	0	0	1	0	0	1	1	1	2
$g59$	0	0	1	0	0	1	1	1	1
$g62$	0	2	0	1	1	0	0	0	0
$g63$	0	1	2	0	0	1	0	0	0
$g64$	0	1	0	0	2	0	0	0	2
$g67$	0	2	0	0	1	2	0	0	0

$f \setminus xy$	00	01	02	10	11	12	20	21	22
$g68$	0	2	0	0	1	2	0	1	1
$g70$	0	2	1	0	1	0	0	1	2
$g71$	0	0	0	0	2	1	0	1	0
$g72$	0	0	1	0	2	1	0	1	1
$g75$	0	1	2	0	2	1	0	1	1
$g78$	0	1	0	0	2	0	0	2	0
$g80$	0	0	1	0	2	0	0	0	1
$g81$	0	1	2	1	0	1	2	2	0
$g82$	0	2	1	1	1	0	2	0	2
$g83$	0	2	1	1	2	2	2	1	1
$g87$	0	1	2	0	1	0	2	0	2
$g88$	0	1	1	1	1	1	0	1	2
$g91$	0	0	2	0	1	2	2	2	0
$g92$	0	1	0	0	1	1	1	1	1
$g94$	0	1	1	1	1	2	2	1	1
$g95$	0	2	1	2	1	1	1	1	2
$g99$	0	1	2	0	2	2	2	2	2
$g101$	0	1	1	1	2	2	1	2	2
$g104$	0	2	0	1	1	1	0	2	0
$g108$	0	1	0	0	1	2	0	2	0
$g109$	0	1	0	0	1	1	0	2	1
$g110$	0	2	0	0	1	2	0	0	2
$g113$	0	0	0	0	0	1	0	1	0
$g114$	0	0	0	0	2	0	0	0	2
$g118$	0	2	1	0	1	1	0	1	1
$g119$	0	2	2	0	0	0	0	0	0
$g120$	0	0	0	0	2	1	0	1	1
$g125$	0	0	0	0	2	2	1	2	2
$g126$	0	0	1	0	0	1	0	1	2
$g127$	0	0	1	0	0	1	0	1	1
$g128$	0	0	0	0	0	0	2	1	0
$g130$	0	2	0	0	1	0	0	0	0
$g131$	0	2	0	0	2	0	0	0	0
$g132$	0	0	0	2	2	2	0	1	1
$g135$	0	1	2	1	0	0	2	0	0
$g136$	0	1	2	0	0	1	0	2	0
$g137$	0	2	1	0	1	0	0	0	2
$g138$	0	2	1	0	2	2	0	1	1
$g139$	0	0	0	0	2	0	0	0	1
$g140$	0	1	0	1	1	1	0	2	2

Representatives of classes of T_0 (continued).

$f \setminus xy$	00	01	02	10	11	12	20	21	22
$g141$	0	0	2	0	1	0	2	0	2
$g151$	0	0	1	0	1	1	1	1	2
$g152$	0	0	2	0	2	2	2	2	2
$g153$	0	0	2	0	0	2	2	2	2
$g154$	0	0	1	0	1	1	1	1	1
$g155$	0	1	2	1	2	2	2	2	2
$g162$	0	0	2	0	1	0	0	2	0
$g163$	0	0	0	0	1	2	1	1	1
$g166$	0	0	0	2	1	1	1	1	2
$g173$	0	1	1	0	2	2	0	2	2
$g175$	0	0	2	0	1	1	0	0	0
$g176$	0	0	0	0	0	0	0	1	0
$g177$	0	2	0	0	1	0	0	2	0
$g182$	0	2	1	1	1	2	2	1	2
$g186$	0	2	1	0	0	0	0	0	0
$g191$	0	0	2	1	1	2	2	2	2
$g192$	0	1	1	1	1	1	1	1	2
$g193$	0	1	0	0	1	1	0	2	2
$g204$	0	0	1	0	1	1	0	1	2
$g205$	0	0	2	0	2	2	0	2	2

$f \setminus xy$	00	01	02	10	11	12	20	21	22
$g206$	0	0	2	0	0	2	0	2	2
$g207$	0	0	1	0	1	1	0	1	1
$g208$	0	1	2	0	2	2	0	2	2
$g209$	0	0	2	0	1	1	0	0	2
$g211$	0	1	0	0	1	0	0	0	0
$g216$	0	2	1	0	1	1	0	2	2
$g217$	0	0	0	0	0	2	0	1	0
$g221$	0	0	2	0	1	2	2	2	2
$g222$	0	1	2	1	1	2	2	2	2
$g224$	0	1	1	1	1	1	1	1	1
$g230$	0	1	1	0	1	1	0	1	2
$g231$	0	0	2	0	1	0	0	0	2
$g234$	0	1	2	1	1	2	2	1	2
$g236$	0	1	2	0	0	0	0	0	0
$g240$	0	0	2	0	1	2	0	2	2
$g241$	0	1	2	0	1	2	0	2	2
$g245$	0	1	2	0	1	1	0	2	2
$g246$	0	0	0	0	1	0	0	0	2
$g248$	0	0	2	0	1	2	0	1	2
$g251$	0	0	0	0	1	1	0	1	2

$g9$	$g34$	$g36$	$g49$	$g60$	$g74$
000200000	001211110	000200100	011111222	000100000	002000020
000000000	000000000	100212100	000000000	000000000	001000020
100000000	000000000	200212100	000000000	200000000	001000020

$g77$	$g84$	$g85$	$g100$	$g103$	$g111$
000211100	000012000	002121210	000011200	000100000	000200000
000000000	000211100	000000000	000022211	000100000	000121100
000000000	000222100	000000000	222222222	200000000	000111210

$g116$	$g134$	$g143$	$g145$	$g156$	$g158$
010000002	000200100	010000001	010000001	000100000	000020000
010000001	100112100	001111110	001111110	000100000	000101000
010000002	200212200	001111210	001111110	200000200	000010200

Representatives of classes of T_0 (continued).

$g159$	$g165$	$g167$	$g171$	$g172$	$g181$
000100000	001112110	000000000	000010200	000000000	011000022
000121100	000111100	010000001	000222200	000022200	000100000
000111210	000111100	010000002	000222200	001222220	000000200

$g183$	$g184$	$g185$	$g187$	$g188$	$g194$
002012010	010112202	000012000	000020000	000111200	000000000
000122100	000000000	000212100	000121100	100111100	000101000
000211200	000000000	000212100	200222200	200111200	000010200

$g196$	$g198$	$g212$	$g214$	$g215$	$g225$
001112110	001000010	000120000	001021020	000112200	000100000
000111100	000111100	001111000	000111101	000101200	000121100
000111200	000111100	000120000	020222200	000120200	000111200

$g226$	$g229$	$g235$	$g237$	$g244$	$g249$
001000020	000020200	000021000	000000000	000010000	001100220
000111100	001121210	000112111	000112200	000112100	000112200
000111200	001122220	022212200	000111200	000212200	000112200

$g252$
001000020
000112200
000112200

Chapter 7

Applications of a Subset Generating Algorithm to Base Enumeration, Knapsack and Minimal Covering Problems

On the basis of a backtrack procedure for lexicographic enumeration of all subsets of a set of n elements we give an algorithm for both determining of all bases consisting of functions from a given complete set in a considered subset of the set of k-valued logical functions and for enumeration of all classes of bases in the subset. We use the lexicographic algorithm also for solving knapsack and minimal covering problems. A cut technique is described which is used in these algorithms to reduce the number of examined subsets of $\{1, \ldots, n\}$. Some computational data upon the classes of P_3 are also given.

7.1. Generating all subsets of $\{1, \ldots, n\}$ in lexicographic order

In this Section we consider the problem of generating all r-subsets (subsets containing r elements) of the set $\{1, 2, \ldots, n\}$ for $1 \leq r \leq n$ and for $1 \leq r \leq m \leq n$. We assume that each subset will be represented as a sequence $a_1 a_2 \ldots a_r$ where $1 \leq a_1 < \ldots < a_r \leq n$.

Recall definition of lexicographic order of subsets. For two subsets $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_q)$, $a < b$ is satisfied if and only if there exists i $(1 \leq i \leq q)$ such that $a_j = b_j$ for $1 \leq j < i$ and either $a_i < b_i$ or $p = i - 1$. This order has an important property that enables simple calculation with r-subsets. Ehrlich [Ehr73] described a loopless procedure for generating of subsets of a set of n elements. A procedure based on Gray code for the same problem is given in [NiW78]. Also, in [NiW78] an algorithm for generating all r-subsets ($1 \leq r \leq m \leq n$) in lexicographic order is proposed. Semba [Sem84] improved the efficiency of the algorithm. We will modify his algorithm by presenting it in PASCAL-like notation without goto statements. Application of the algorithm for minimal covering problem results in another modification of the algorithm in the case $1 \leq r \leq m \leq n$.

The lexicographic enumeration of r-subsets goes in the following manner (for example, let $n = 5$):

```
1, 12, 123, 1234, 12345,
                  1235,
       124, 1245,
       125,
   13, 134, 1345,
       135,
   14, 145,
   15,
2, 23, 234, 2345,
       235,
   24, 245,
   25,
3, 34, 345,
   35,
4, 45,
5.
```

The algorithm is in "extend" phase when it goes from "left" to "right" staying in a row. If the last element of a subset is n then algorithm shifts to the next row. We call this phase "reduce" phase. Every subset of $\{1, \ldots, n\}$ is represented in the algorithm below by a sequence $j_1, \ldots, j_r$, $1 \leq r \leq n$, $1 \leq j_1 < \ldots < j_r \leq n$.

First we give an algorithm for generating all r-subsets for $1 \leq r \leq n$. This algorithm will be used in base enumerations.

begin
    read(n); $r := 0$; $j_r := 0$;
    repeat
        if $j_r < n$ **then** extend **else** reduce;
        print out $j_1, \ldots, j_r$

```
   until j_1 = n
end;
extend≡ begin j_{r+1} := j_r + 1; r := r + 1 end
reduce≡ begin r := r − 1; j_r := j_r + 1 end .
```

Note that between any two printed subsets exactly two conditions are checked: $j_r < n$ and $j_1 = n$.

The algorithm for generating all r-subsets for $1 \leq r \leq m \leq n$ we modify with respect to its use in minimal covering problem.

```
begin
   read(n); r := 0; j_r := 0;
   repeat
     if j_r < n and r < m then extend else cut;
     print out j_1, ..., j_r
   until j_1 = n
end;
extend≡ begin j_{r+1} := j_r + 1; r := r + 1 end
reduce≡ begin r := r − 1; j_r := j_r + 1 end
cut≡ if j_r < n then j_r := j_r + 1 else reduce .
```

Besides "extend" and "reduce" phases we use in the algorithm a new phase called "cut" phase. The phase will be used when algorithm goes from some subset to some subset in a lower row (not necessarily in the subsequent row) skipping several subsets (when the number r of elements in these subsets is greater than m).

7.2. Functional completeness and enumeration of bases

In this Section we describe an application of our lexicographic algorithm to base enumeration for a subset of the set of k-valued logical functions.

We call nonredundant incomplete sets simply *addable*. The *rank* of a base (addable set) is the number of its elements. Here we recall some definitions. The *characteristic vector* of $f \in H$ is $c_1 \ldots c_d$, where $c_i = 0$ if $f \in H_i$ and $c_i = 1$ otherwise $(1 \leq i \leq d)$. Whenever it is possible to avoid confusion we call characteristic vectors simply vectors. All functions $f \in H$ with the same (characteristic) vector form a *class of functions*. For a base its *class of bases* is the set of classes of functions for functions belonging to the base.

The conditions of completeness and nonredundancy of a set of (classes of) functions F can be conveniently expressed by using characteristic vectors of (classes of) functions belonging to F. We can say that a base corresponds to a minimal cover of $1\ldots1$ (unit vector), and nonredundant set corresponds to a minimal cover of some non-unit vector (in which some 0's may occur; we except null vector).

We define bitwise OR operation $\vee$ for characteristic vectors in the following way:

$$(a'_1,\ldots,a'_d)\vee(a''_1,\ldots,a''_d)=(a'_1\vee a''_1,\ldots,a'_d\vee a''_d).$$

Criteria for the completeness and nonredundancy of a set $a_1,\ldots,a_r$ of characteristic vectors are respectively in the following (the two equations are shown in Chapter 1):

$$a_1\vee\ldots\vee a_r = 1\ldots1 \quad \text{(completeness)} \qquad (1.1)$$

$$a_1\vee\ldots\vee a_{j-1}\vee a_{j+1}\vee\ldots\vee a_r \neq a_1\vee\ldots\vee a_r \quad \text{for each } j=1,\ldots,r \quad \text{(nonredundancy).} \qquad (1.2)$$

Thus any set containing null class (whose vector is $0\ldots0$) is redundant. Addable sets are nonredundant, but not conversely.

If we have a complete list of characteristic vectors for nonempty classes of functions of a set, we can enumerate all its classes of bases.

As an example, assume a set M contains 4 maximal sets M_1, M_2, M_3, M_4 and 6 classes of functions:

1.0011 2.0100 3.1000 4.0010 5.0001 6.0000 .

For instance, class 1 is the set $M_1M_2\overline{M}_3\overline{M}_4$, where $\overline{X}=M\setminus X$ (complement set).

M has exactly two classes of bases: {1,2,3} and {2,3,4,5}. We consider the class {1,2,3}. Bitwise OR for the set results 1111 (completeness). Bitwise OR for the set {1,2} results 0111, for the set {1,3} results 1011 and for the set {2,3} results 1100 (nonredundancy). The set {1,3,4} is redundant, because bitwise OR for the sets {1,3,4} and {1,3} are equal (to 1011).

7.3. The lexicographic enumeration of bases and classes of bases

Let d and n denote the numbers of maximal sets and functions or classes of functions respectively. Then we are given n vectors with length d, indexed by $1,\ldots,n$.

To perform an exhaustive enumeration of classes of bases we should enumerate every r-tuple of vectors $a_1, \ldots, a_r$ for each $r = 2, \ldots, d$ (for $r = 1$ it is trivial) and check the completeness (2.1) and redundancy (2.2) conditions for them (rank r base criteria). However this direct method does not work, because of too many r-tuples to be generated. Suppose we are enumerating r vectors $a_1, \ldots, a_r$ for checking the base criteria. Instead of enumerating whole r vectors and checking criteria for them, we will inspect i-tuple of vectors $a_1, \ldots, a_i$ incrementary for $i = 1, \ldots, r$, and at each i-th stage we will certify (by examining simple conditions) that this i-tuple can or cannot be included in a rank r base (addable set). This idea of incremental check can be conveniently implemented in the lexicographic enumeration of subsets.

The lexicographic algorithm enumerates classes of bases and addable sets for every rank at the same time. Moreover the maximal ranks of bases and addable sets are automatically given as a result.

Suppose we are enumerating taken r elements out of n object stored in an array consecutively, i.e. $a(1), \ldots, a(n)$. The selected indexes are to be stored in an array j as $j_1, \ldots, j_r$, $1 \leq j_i \leq n$ for each i, $1 \leq i \leq r$.

Suppose we are examining taken r-subset $a(j_1), \ldots, a(j_r)$, where selected indexes are stored in an array j as $j_1, \ldots, j_r$, $1 \leq j_1 < \ldots < j_r \leq n$ and $a(i)$ denotes a_i. There are three possible cases after the examination: redundant, base and addable set (i.e. nonbase-nonredundant). The enumeration of subsets in lexicographic order can be controlled in the following manner.

If a r-tuple is either redundant or base then it is unnecessary to "extend" it to r+1-tuple, since adding a new vector to them will result in "redundancy"; in the former case the r-tuple is already redundant and in the latter it is already "complete". Hence in these cases we can bypass the lexicographic enumeration of subsets to an appropriate point. The next subset is $j_1, j_2, \ldots, j_r - 1, j_r + 1$ if $j_r \neq n$; otherwise it is the next subset in lexicographic order and the bypass effects nothing. Thus only the remaining addable case can be extended.

As an example we consider the same set M as before. The class 6 (null class) is omitted. In this case $n = 5$ and $d = 4$. The notions "extend", "reduce", "cut", "redundant", "base" and "addable" we denote simply by "e","r","c","n","b","a" respectively.

1-a,e; 1,2-a,e; 1,2,3-b,c;

1,2,4-n,c;

1,2,5-n,c,r;

1,3-a,e; 1,3,4-n,c;

1,3,5-n,c,r;

1,4-n,c;

1,5-n,c,r;

2-a,e; 2,3-a,e; 2,3,4-a,e; 2,3,4,5-b,c,r;

2,3,5-a,r;

2,4-a,e; 2,4,5-a,r;

2,5-a,r;

3-a,e; 3,4-a,e; 3,4,5-a,r;

3,5-a,r;

4-a,e; 4,5-a,r;

5-a.

We can write our algorithm as follows. Let b_r be the number of (classes of) bases of rank r.

```
begin
  read n, d, a(i), i := 1, n; r := 1; j_1 := 1;
  repeat
      if a(j_1), ..., a(j_r) is addable
        then if j_r < n
                then extend
                else reduce
        else begin
                if a(j_1), ..., a(j_r) is a base then b_r := b_r + 1;
                cut;
             end
  until j_1 = n;
  print out b_i, 1 <= i <= d
end.
```

In the algorithm "extend", "reduce" and "cut" are defined as before. Note that the last set n are not checked in the algorithm. It can be easily done before printing results.

7.4. Redundancy checks

We describe a technique (called bitwise pivotality checks) to reduce the computation in redundancy checks.

Suppose we are checking redundancy of $a_1, \ldots, a_r$ (for simplicity we write a_i for $a(j_i)$). For every redundancy check we know that $a_1, \ldots, a_{r-1}$ are included in the tuple which we examined just before (only a_r is a newly added vector). Thus we can assume that we already have $R_k = a_1 \vee \ldots \vee a_k$ for $1 \leq k \leq r-1$ in an array R (for a convenience we add R_0 and assume $R_0 = 0$).

The redundancy condition for the r-tuple can be formulated in the following way (we use a variable B to reduce the number of bitwise OR operations).

For $r \geq 2$.

$R_r = R_{r-1} \vee a_r$ and $R_{r-1} \neq R_r$, (7.1)

$B = B \vee a_{k+1}$ (initial B=0) and $R_{k-1} \vee B \neq R_r$ for $k = r-1, \ldots, 1$ (7.2)

For $r = 1$.

a_1 is addable if it is neither null vector nor unit vector

(if a_1 is a unit vector then it is a base)

The program checks (7.1) and (7.2) for $k = r, \ldots, 1$; $k \geq 2$ in this order, and whenever a condition is not satisfied the check ends immediately with redundancy result.

For a rank r redundancy check we need at most r comparisons and at most $2r$-1 bitwise OR operations.

If the number of components d in vectors a_i is less than the number of bits (usually 16 or 32) of given computer then it is possible to represent a vector a_i by an integer number $c_1 + 2 \cdot c_2 + \ldots + 2^{d-1} \cdot c_d$, where $c_1 c_2 \ldots c_d$ are the components of the vector a_i in the redundancy check we can treat these vectors as integer numbers because OR operation between integer numbers is defined as a machine instruction OR between corresponding components of their binary notations. Otherwise bitwise OR can be realized with (characteristic) vectors as an array of d elements. However, in this case there are another technique called counter redundancy check which is proved faster as well.

In the check of redundancy we use two auxiliary sequences $s_i (1 \leq i \leq d)$ and p_i $(1 \leq i \leq r)$. s_i is the number of units in the i-th position in the vectors $p(j_1), \ldots, p(j_{r-1})$. The sequence $p_1, \ldots, p_r$ has the following property: p_i-th position of each vector is equal to 1 only for $p(j_i)$ (it is equal to 0 for the vectors $p(j_t)$, $1 \leq t \leq r, t \neq i$).

The presented lexicographic algorithm can be supplemented also with this technique. Note that algorithm with bitwise redundancy check using machine command is proved as about twice faster (when n is about 500 and d is about 15) than one with counter redundancy check.

Applying this algorithm classes of bases for several subsets of P_k are determined (cf. [MiS87a]). P_3 has exactly 18 maximal sets [Jab58] and 406 classes of functions [Miy71,Sto84a]. We present the numbers of classes of bases of P_3 of each rank in the following table:

rank	1	2	3	4	5	6	Σ
bases	1	8,265	794,256	4,612,601	810,474	141,124	6,239,721

The lexicographic enumeration algorithm with this bitwise redundancy check requires about 16 minutes computer time (the computer FACOM M380 is used). The total number of examined tuples is N=194759642 for the classes of functions sorted according first to the number of units in the vector and then sorted lexicographically within the same group. Bearing in mind the total number of subsets 2^{406} we can calculate efficiency of cut technique in this case. The program generates in the average 4.41-tuple and consume in the average 2.17 bitwise OR operations to recognize whether it is a base, addable or redundant (bitwise redundancy check is used). Note that computer time depends on the order of characteristic vectors.

7.5. Application of the base enumeration algorithm

Kabulov [Kab82] considered the following problem: Given a complete set F of functions from P_k together with the Boolean matrix displaying the relation "$\in$" between the members of F and maximal sets in P_k (i.e. with characteristic vectors of functions in F), determine all bases composed from functions of the set F. He described a method, using Boolean expressions, to solve this problem.

We can apply the same algorithm described in Section 3 to this problem, because each function is represented by their class of functions. The output in this case are exactly bases instead of classes of bases. Note that in the considered application several function may have the same characteristic vector. However, they compose different bases.

Our algorithm can be used to calculate the number of (classes of) bases composed from vectors $m+1, \ldots, n$ at the same time (for a given $m \leq n$), because in the lexicographic order we examine first all subsets containing vector 1, then all subsets containing vector $2, \ldots$.

In [KuO66,PeS68,Wer42] procedures for determining the number of bases of P_2 consisting of n-ary functions are described and computational results for n=2 and n=3 are obtained. There exist no formulae for numbers of n-ary functions in some classes of functions of P_2, because the number of n-ary monotone functions in P_2 is not known. We present another approach to this problem. It is divided into several subproblems.

1) determination of classes of functions for considered set (not limited to P_2),

2) determination of the number of n-ary functions in each class,

3) determination of all classes of bases,

4) determination of numbers of bases containing n-ary functions (or functions with at most n variables).

The methods presented in [KuO66,PeS68,Wer42] use only step 4) for P_2. Our method can be applied for solving 3) assuming that 1) is already solved. Also, our algorithm can be applied for solving 4) assuming that 2) is solved by applying another procedure. Note that 2) can be done without solving 1) because for each function f we can determine corresponding class of functions. It is sufficient to check inclusion of f in each maximal set of considered closed set; such procedure can be easily written using description of maximal sets [Ros77]. In this manner we can determine classes of functions containing n-ary functions. We can apply our algorithm to count bases. We obtain the number of bases containing n-ary functions in a class of bases by multiplying the numbers of n-ary functions in the classes of functions which compose the base, whenever a class of bases is found. During this procedure we can also enumerate classes of bases consisting of classes of n-ary functions.

Following this description we determined the number of bases of Boolean functions composed from n-ary functions for $n \leq 4$. Obtained data are presented in the following table. For $n = 2$ this result is derived by Wernick [Wer42] and for $n = 3$ by Kudielka and Oliva [KuO66]. Note that the set P_2 of Boolean functions contains 5 maximal sets [Pos21], 15 classes of functions [Jab52,INN63,Krn65] and 42 classes of bases [INN63,Krn65].

n	2	3	4
bases	32	6,664	275,790,502

7.6. Minimal covering problem

Minimal covering problem is one of famous combinatorial problems and there exist a list of solutions for this problem (cf. [Rot69,YoM85]). We will give a solution using the lexicographic enumeration of subsets.

The minimal covering problem is the problem of minimizing the objective function $x_1 + \ldots + x_n$, subject to constraints

$$(x_1, \ldots, x_n)A \geq (1, \ldots, 1) \qquad (7.3)$$

where $A = [a_{ij}]$ is an $n \times d$ coefficient matrix with $a_{ij} = 0$ or 1, and each variable x_j is 0 or 1 for each j.

We will introduce some new notions in order to give a new solution for the problem and to show connection between minimal covering problem and base enumeration.

A vector $(x_1, \ldots, x_n)$ satisfying (7.3) is called *complete* for A. We call a vector $(x_1, \ldots, x_n)$ *nonredundant* in A if

$$(x_1, \ldots, x_n)A > (y_1, \ldots, y_n)A$$

is valid for each vector $(y_1, \ldots, y_n)$ for which $y_i \leq x_i$ for each $i, 1 \leq i \leq n$ and $y_1 + \ldots + y_n < x_1 + \ldots + x_n$ is satisfied.

A vector $(x_1, \ldots, x_n)$ is called *base* in A if it is complete and nonredundant in A. Nonredundant noncomplete vectors we call simply *addable*. The *rank* of a base (addable set) $(x_1, \ldots, x_n)$ is the sum $x_1 + \ldots + x_n$. Thus minimal covering problem is problem of finding a base in A with minimal rank.

There is another definition of minimal covering problem [Kar72]: For a given collection C of subsets of a finite set and positive integer $r \leq |C|$ decide whether C contains a cover for S of size r or less, i.e. a subset $C' \subseteq C$ with $|C'| \leq r$ such that every element of S belongs to at least one member of C'. This problem is exactly to find a base with rank r or less, if we represent a subset by n bits characteristic vector. Karp [Kar72] proved that this problem is NP-complete.

The notions of addable sets, bases and rank have almost the same meaning in both base enumeration and minimal covering problem. Minimal covering problem corresponds directly to finding a base with minimal rank. Thus we can modify our algorithm so that once we find a base with rank r then no subsets of rank $\geq r$ will be considered further.

In the presented branch and bound algorithm $a(i)$ denotes the i-th row of matrix A $(1 \leq i \leq n)$, i.e. $a(i) = (a_{i_1}, \ldots, a_{i_n})$. We suppose that minimal rank of bases (solution of our problem) is between 2 and n-1 to make our algorithm shorter. It is easy to improve our algorithm to deal with these cases. Also some techniques for eliminating some rows or columns (cf.[Rot69]) can be applied before running the algorithm.

```
begin
  read n, d, a(i),  i := 1,n; minrank := d; r := 1; j1 := 1; T := {1};
  repeat
     if a(j1), ..., a(jr) is addable in A
        then if jr < n and r < minrank − 1
                then extend
                else cut
        else begin
                if a(j1), ..., a(jr) is a base in A then
                   begin
                   minrank = r;
                   T := {j1, ..., jr};
                   end;
                cut
             end
  until j1 = n or minrank = 2;
  printout minrank, T
end.
```

The two procedures "extend" and "cut" are defined as before. Note that T corresponds to a solution $(x_1, \ldots, x_n)$ of minimal covering problem so that $x_j = 1$ if and only if $j \in T$.

7.7. Knapsack problem

An input for the knapsack problem are integer numbers $a_1, \ldots, a_n, C$. The problem is to find a subset T of $\{1, \ldots, n\}$ to maximize $\Sigma_{i \in T} a_i$ subject to the requirement that $\Sigma_{i \in T} a_i \leq C$. A more general formulation of the knapsack problem has more applications than this. Namely the input consists of C and two sequences $a_1, \ldots, a_n$ and $p_1, \ldots, p_n$. The problem is to maximize $\Sigma_{i \in T} p_i$ subject to the restraint $\Sigma_{i \in T} a_i \leq C$ where T, as before, is a subset of the indexes.

We give a solution for more general knapsack problem based on the lexicographic order of subsets. Elements i that are a_i greater than C should be eliminated. In the presented algorithm $a(j_i)$ denotes a_{j_i}.

```
begin
   read n, d, a_i, p_i, i = 1, n;
   r := 1; j_1 := 1; maxsum := p_1; T := {1};
   repeat
         S := a(j_1) + ... + a(j_r);
         if S <= C
            then begin
                     P := p(j_1) + ... + p(j_r);
                     if P > maxsum then begin
                                          maxsum := P;
                                          T := {j_1, ..., j_r}
                                        end;
                     if j_r < n then extend else reduce
                  end
            else cut;
   until j_1 = n;
   printout maxsum, T
end.
```

In the algorithm "extend", "reduce" and "cut" are defined as before. The set $\{n\}$ should be examined before printing.

7.8. Concluding remarks

In this chapter we modified backtrack procedures for lexicographic enumeration of subsets and applied the procedure to the base enumeration, knapsack and minimal covering problems. Several variational uses of base enumeration algorithm are presented. The presented "cut" techniques use special properties of bases and addable sets, owing to which, for instance, base enumeration were possible for about n=600 (for the case n=605, d=15 it took about 8 hours using bitwise redundancy check by FACOM 380 computer with 16 MIPS).

Karp [Kar72] proved that the problem of determining of a covering set with rank $\leq r$ for given r is NP-complete. Our algorithms are directly related to the problem. Thus any algorithm for solving these problems takes exponential time according to numbers of rows and columns n and d. There exist a number of algorithms for exact and approximate solution of knapsack and minimal covering problems (see, for example, [Baa78,Rot69,YoM85]).

Chapter 8

Classification of P_{k2}

The set of functions of P_{k2} (mapping the set $\{0,1,...,k-1\}^n$ into $\{0,1\}$, $n = 1,2,\ldots$) is divided into equivalence classes so that two functions are in the same class if their membership in the maximal subclones of P_{k2} coincides. This also leads to a natural classification of the set of bases (i.e. nonredundant complete subsets) of P_{k2}. We determine all nonempty classes of functions of P_{k2} and show that the number of them is $13A_k - 11A_{k-1}$, where A_k is the number of equivalence relations on the set of k elements. The maximal number of elements in a base of P_{k2} is proved to be $k+2$. Computational results for the numbers of classes of bases are also presented for $k = 3$ and 4.

8.1. Introduction

The algebra P_{k2} of all functions whose domain is a Cartesian power of E_k and whose range is E_2 was considered in [Bur73,HaF84,Lau75,Lau82b,Sas84]. Every n-ary function of P_{k2} may be interpreted as an n-ary predicate, or, equivalently, $f^{-1}(1)$ is an n-ary relation on E_k. We mention some applications. Functions of $P_{3,2}$ permit the description of a decision (the values 0, 1) with abstention from voting (the value 2). Special functions of $P_{3,2}$ are of interest in the theory of noncorrect algorithms [Zur78,BDHL79]. In [EFR74] it is mentioned that functions of P_{k2} may be used to describe logical and arithmetical branchings in programs where the arithmetical constants are arguments and the two logical constants form the range. In [Sas84] a minimum sum-of-products expression for the functions of P_{k2} is used to get a minimum PLA (programmable logic array) with decoders (actually, $k = 4$ for PLA with two-bit decoders).

In this chapter we determine classes of functions for the set P_{k2}. The maximal number of elements in a base of P_{k2} is also determined to $k+2$.

8.2. Definitions and notations

In this chapter we are interested in the set

$$P_{k2} = \bigcup_{n \geq 1} \{f : E_k^n \to E_2\}.$$

We recall the following theorem.

Theorem 8.2.1. [Pos21] *P_2 has exactly the following 5 P_2-maximal sets:*

$$T_0 = Pol(0),\ T_1 = Pol(1),\ S = Pol\begin{pmatrix} 01 \\ 10 \end{pmatrix},$$
$$L = Pol(\{(a,b,c,d)^T \in E_2^4 \mid a+b = c+d\ (mod\ 2)\}),\ M = Pol\begin{pmatrix} 010 \\ 011 \end{pmatrix}.$$

Here T_i consists of Boolean functions f such that $f(i,\ldots,i) = i$ $(i = 0,1)$, S is the set of selfdual Boolean functions (satisfying $f(\bar{x}_1,\ldots,\bar{x}_n) = \overline{f(x_1,\ldots,x_n)}$), L is the set of linear Boolean functions and M is the set of monotone (or isotone) Boolean functions.

The 15 nonempty classes of functions of P_2 are shown in the Table 8.1. We remark that the classes 10100, 01100 and 00000 consist only of functions {constant 0 function}, {constant 1 function} and $\{x_i$ (function depending only one variable)$\}$, respectively. We also remark that the set of classes {01100, 10100, 00110, 00001} is a class of basis with a maximum rank 4; for example, a base $\{0, 1, xy, x+y+z\}$ belong to this class of basis.

In this chapter H is the set P_{k2} of all $f : E_k^n \to E_2$ $(n = 1,2,\ldots)$. It is clear that P_{k2} is closed. Let $pr : P_{k2} \to P_2$ be defined by setting $pr\ f :=\ g$ where $g(\boldsymbol{a}) = f(\boldsymbol{a})$ for all $\boldsymbol{a} \in E_2^n$ (the restriction of f to E_2).

We denote the intersection of sets $X_1,\ldots,X_n$ by $X_1\ldots X_n$. For $X \subseteq P_{k2}$ put $\overline{X} = P_{k2} \setminus X$ and for $x \in E_k$, $x^j = (x\ldots x)$ (j times). For $X \subseteq P_2$, the inverse image of X is $X' = pr^{-1}(X) = \{f \in P_{k2} \mid pr\ f\ \in X\}$. For $i,t \in E_k$, put $Z_{it} = P_{k2}Pol\begin{pmatrix} 01i \\ 01t \end{pmatrix}$. Note that $Z_{it} = Z_{ti}$.

Theorem 8.2.2. [Bur73,Lau75,Lau82b,Lau84b] *The set P_{k2} has the following* $5+(1/2)\cdot(k-2)(k+1)$ *maximal sets:*

$$T_0',\ T_1',\ S',\ L',\ M' \text{ and } Z_{it}\ (k > i > t \geq 0,\ i > 1).$$

8.3. Classification of P_{k2}

We denote the characteristic vector of a function f of P_{k2} by

$$c_1c_2c_3c_4c_5c_{02}\dots c_{0(k-1)}c_{12}\dots c_{1(k-1)}\dots c_{(k-2)(k-1)}$$

with respect to the order of the P_{k2}-maximal sets in Theorem 8.2.2. Note that the values of c_1, c_2, c_3, c_4, c_5 coincide with the corresponding characteristic vector for $pr\ f \in P_2$. For each n-ary $f \in P_{k2}$ define a relation Q_f on the set E_k by setting $(i,t) \in Q_f$ if $f(\boldsymbol{a}) = f(\boldsymbol{b})$ whenever $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in \begin{pmatrix} 0\,1\,i \\ 0\,1\,t \end{pmatrix}^n$. Clearly the binary relation Q_f on E_k is reflexive and symmetric. Now we prove several lemmas needed for the description of the equivalence classes ($\equiv$) on P_{k2}.

Lemma 8.3.1. *Let $f \in P_{k2}$. Then $f \in Z_{it}$ if and only if $(i,t) \in Q_f$.*

Proof. ($\Rightarrow$) Let $f \in Z_{it} Pol \begin{pmatrix} 0\,1\,i \\ 0\,1\,t \end{pmatrix}$ and $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in \begin{pmatrix} 0\,1\,i \\ 0\,1\,t \end{pmatrix}^n$. As $f \in Pol \begin{pmatrix} 0\,1\,i \\ 0\,1\,t \end{pmatrix}$ we have $\begin{pmatrix} f(\boldsymbol{a}) \\ f(\boldsymbol{b}) \end{pmatrix} \in \begin{pmatrix} 0\,1\,i \\ 0\,1\,t \end{pmatrix}$. However $f(\boldsymbol{a}) \neq i$ as $f \in P_{k2}$ and $i \geq 2$, hence we have $f(\boldsymbol{a}) = f(\boldsymbol{b})$. Therefore $(i,t) \in Q_f$. ($\Leftarrow$) Let $f \notin Z_{i,t}$. It follows that there are vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in \begin{pmatrix} 0\,1\,i \\ 0\,1\,t \end{pmatrix}^n$ and $\begin{pmatrix} f(\boldsymbol{a}) \\ f(\boldsymbol{b}) \end{pmatrix} \notin \begin{pmatrix} 0\,1\,i \\ 0\,1\,t \end{pmatrix}$. This implies $f(\boldsymbol{a}) \neq f(\boldsymbol{b})$, because $f(\boldsymbol{a})$ and $f(\boldsymbol{b})$ take only values 0 or 1. Hence we conclude $(i,t) \notin Q_f$. □

Lemma 8.3.2. *Let $f \in P_{k2}$. Then $(0,1) \in Q_f$ if and only if the function $pr\ f$ is constant.*

Proof. ($\Leftarrow$) Let $pr\ f$ be constant and let $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in \begin{pmatrix} 0\,1\,0 \\ 0\,1\,1 \end{pmatrix}^n$. Then $f(\boldsymbol{a}) = f(\boldsymbol{b})$. Therefore $(0,1) \in Q_f$. ($\Rightarrow$) Suppose $pr\ f$ not constant. Then there is a vector $\boldsymbol{a} \in E_2^n$ such that $f(\mathbf{o}) \neq f(\boldsymbol{a})$. Since $\begin{pmatrix} \mathbf{o} \\ \boldsymbol{a} \end{pmatrix} \in \begin{pmatrix} 0\,1\,0 \\ 0\,1\,1 \end{pmatrix}^n$, we conclude $(0,1) \notin Q_f$. □

Lemma 8.3.3. *The relation Q_f is an equivalence relation.*

Proof. As mentioned before the reflexivity and symmetry follow from the definition. For transitivity let $(i,t) \in Q_f, (t,j) \in Q_f$ and $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in \begin{pmatrix} 0\,1\,i \\ 0\,1\,j \end{pmatrix}^n$. Put $c_i = a_i$ if

$a_i = b_i$ and $c_i = t$ otherwise. Then $\boldsymbol{c} = (c_1, \ldots, c_n)$ satisfies $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{c} \end{pmatrix} \in \begin{pmatrix} 0\ 1\ i \\ 0\ 1\ t \end{pmatrix}^n$ and $\begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{b} \end{pmatrix} \in \begin{pmatrix} 0\ 1\ t \\ 0\ 1\ j \end{pmatrix}^n$. Thus $f(\boldsymbol{a}) = f(\boldsymbol{c})$ and $f(\boldsymbol{c}) = f(\boldsymbol{b})$ shows $f(\boldsymbol{a}) = f(\boldsymbol{b})$. □

Lemma 7.3.4. *Let $f, g \in P_{k2}$ and $\chi_f = (c_1, \ldots, c_{(k-2)(k-1)})$, $\chi_g = (c'_1, \ldots, c'_{(k-2)(k-1)})$. Then $Q_f = Q_g$ if and only if*

(i) $c_{it} = c'_{it}$ for all $k > i > t \geq 0$, $i > 1$ and (ii) $pr\ f$ constant $\Leftrightarrow$ $pr\ g$ constant.

Note that (ii) is equivalent to $(c_1, \ldots, c_5)$ and $(c'_1, \ldots, c'_5) = (0, 1, 1, 0, 0)$ or $(1, 0, 1, 0, 0)$,

Proof. Assume (i) and (ii). By Lemma 7.3.2 we have $(0, 1) \in Q_f Q_g$, since $pr\ f$ and $pr\ g$ are constant. Consider $k > i > t \geq 0$, $i > 1$ and $(i, t) \in Q_f$. By Lemma 7.3.1 $f \in Z_{it}$ and $c_{it} = c'_{it} = 0$ and so $g \in Z_{it}$. According to Lemma 7.3.1 we conclude $(i, t) \in Q_g$. Together $Q_f \subseteq Q_g$. By symmetry $Q_g \subseteq Q_f$ and so $Q_f = Q_g$. Conversely, assume $Q_f = Q_g = Q$. From Lemma 7.3.1 we have $c_{it} = c'_{it}$ for all $k > i > t \geq 0, i > 1$. Next, $(0, 1) \in Q$ if and only if $pr\ f$ is constant and $pr\ g$ is constant from Lemma 7.3.2. □

Note that the map $f \rightarrow Q_f$ is not injective, i.e. several classes of functions can correspond to the same equivalence relation Q. Next theorem determines these classes of functions and gives their number. We show that the map $f \rightarrow Q_f$ maps P_{k2} onto the set of equivalences on E_k.

Theorem 7.3.1. *Let Q be an equivalence relation on the set E_k. Let $n \geq \max(2, k)$ and let g be an n-ary Boolean function such that g is constant exactly if $(0, 1) \in Q$. Then there exists $f \in P_{k2}$ such that $pr\ f = g$ and $Q = Q_f$.*

Proof. For $l = 2, \ldots, k-1$ put $A_l := \{0, 1, l\}^n \setminus E_2^n$. Let $C_1, \ldots, C_r$ be the equivalence classes of Q and let i_j denote the least element of C_j $(j = 1, \ldots, r)$. Let $1 \leq l \leq r$ and $(i_j, l) \in Q$. To $\boldsymbol{x} = (x_1, \ldots, x_n) \in A_l$ assign $\boldsymbol{x}' = (x'_1, \ldots, x'_n)$ defined by $x'_s = i_j$ if $x_s = l$ and $x'_s = x_s$ otherwise (i.e. if $x_s \in E_2$), $1 \leq s \leq n$. We have two cases:

1). Let $(0, 1) \notin Q$. We may assume that $i_1 = 0$ and $i_2 = 1$. By assumption g is non-constant. For simplicity assume that $g(0^n) = 0$ (if not, replace g by $\bar{g}$). By an appropriate exchange of variables we may obtain $g(1^a 0^{n-a}) = 1$ for some a $(1 \leq a \leq n)$. Define an n-ary $f \in P_{k2}$ as follows:

a) For $\boldsymbol{x} \in E_2^n$ put $f(\boldsymbol{x}) := g(\boldsymbol{x})$.

b) For $2 < p \leq r$ put

$$f(i_p 0^{n-1}) = \ldots = f(i_p^{p-1} 0^{n-p+1}) = 0, f(i_p^p 0^{n-p}) = \ldots = f(i_p^n) = 1,$$

$f(i_p 1^{a-1} 0^{n-a}) := 0$ (where a is defined above) and $f(\boldsymbol{x}) := 1$ elsewhere on A_{i_p}.

c) For $1 \leq p \leq r$, $(i_p, l) \in Q$ and $\boldsymbol{x} \in A_l$ put $f(\boldsymbol{x}) := f(\boldsymbol{x}')$ and finally

d) put $f(\boldsymbol{x}) := 1$ otherwise.

The part c) assures that $Q \subseteq Q_f$. For $2 < p < q \leq r$, we have

$$f(i_p^q 0^{n-q}) = 1 \neq 0 = f(i_q^q 0^{n-q}),$$

hence $(i_p, i_q) \notin Q_f$. Let $2 < p \leq r$. We show that $(0, i_p) \notin Q_f$. Indeed $f(0^n) = g(0^n) = 0$ while $f(i_p^p 0^{n-p}) = 1$ (here we need $r \leq n$ which follows from $r \leq k \leq n$). Similarly from $f(1^a 0^{n-a}) = g(1^a 0^{n-a}) = 1 \neq 0 = f(i_p 1^{a-1} 0^{n-a})$ we get $(1, i_p) \notin Q_f$.
Finally $(0,1) \notin Q_f$ as $f(0^n) = g(0^n) = 0 \neq 1 = g(1^a 0^{n-a}) = f(1^a 0^{n-a})$. Together with c) this shows that $Q_f \subseteq Q$ and $Q_f = Q$.

2). The case $(0,1) \in Q$ is similar but simpler (note that g is constant by assumption).
□

Actually the characteristic vectors for all nonempty classes of functions of P_{k2} can be determined by using Theorem 8.3.1. This is shown simply by an example.

Example 8.3.1. The following table presents the 15 equivalence relations on E_4 and the components $c_{20}, c_{21}, c_{30}, c_{31}$ and c_{32} of the corresponding characteristic vector. These classes are divided into two groups. The one includes $\{0,1\}$ in an equivalence class (the first 5 cases) and the other not. Exactly in the first group we have $c_3 c_4 c_5 = 100$ and $c_1 c_2 \in \{01, 10\}$. Note that within each of these two groups no $\{c_{it}\}$ part of the vector appears twice. The complete list of classes of $P_{4,2}$ is shown in Table 8.1. □

Equivalence classes on E_4	c_{20}	c_{21}	c_{30}	c_{31}	c_{32}
{0,1}, {2}, {3}	1	1	1	1	1
{0,1}, {2,3}	1	1	1	1	0
{0,1,3}, {2}	1	1	0	0	1
{0,1,2}, {3}	0	0	1	1	1
{0,1,2,3}	0	0	0	0	0
{0}, {1}, {2}, {3}	1	1	1	1	1
{0}, {1}, {2,3}	1	1	1	1	0
{0}, {1,3}, {2}	1	1	1	0	1
{0,3}, {1}, {2}	1	1	0	1	1
{0}, {1,2}, {3}	1	0	1	1	1
{0}, {1,2,3}	1	0	1	0	0
{0,3}, {1,2}	1	0	0	1	1
{0,2}, {1}, {3}	0	1	1	1	1
{0,2}, {1,3}	0	1	1	0	1
{0,2,3}, {1}	0	1	0	1	0

The number of equivalence relations on an k-element set is $A_k = \sum_{r=1}^{k} A(k,r)$, where $A(k,r) = (1/r!)\sum_{i=0}^{r}(-1)^i \begin{pmatrix} r \\ i \end{pmatrix} (r-i)^k$ are the well-known Stirling numbers of the second kind [Liu68].

Theorem 7.3.2. *The number of classes of functions of* P_{k2} *is* $13A_k - 11A_{k-1}$.

Proof. In respective case of $(0,1) \in Q$ and $(0,1) \notin Q$ our characteristic vector induced by Q is uniquely determined up to $\{c_{i,t}\}$ part. There are A_{k-1} of equivalence classes Q of the first type because in this case the number of equivalence relations Q on E_k satisfying $(0,1) \in Q$ is A_{k-1}. Accordingly the number of equivalence relation of the second type is $A_k - A_{k-1}$. □

In the following table we give the numbers A_k and the numbers $\mu(k2)$ of P_{k2}-maximal sets and $\gamma(k2)$ of classes of functions of P_{k2} for $1 \leq k \leq 10$.

k	1	2	3	4	5	6	7	8	9	10
$\mu(k2)$	-	5	7	10	14	19	25	32	40	49
A_k	1	2	5	15	52	203	877	4,140	21,147	115,975
$\gamma(k2)$	-	15	43	140	511	2,067	9,168	44,173	229,371	1,275,058

Theorem 7.3.3.

$$T_0'T_1'Z_{i0}Z_{i1} = S'Z_{i0}Z_{i1} = \phi.$$

Proof. Let $g \in Z_{i0}Z_{i1}$. Then $(0,i),(i,0) \in Q_g$ and so $(0,1) \in Q_g$ by Lemmas 7.3.1 and 7.3.3. Together with Lemma 7.3.2 this proves *pr* g constant; however, then $g \notin T_0'T_1' \cup S'$. □

Corollary 7.3.1. *The intersection of all P_{k2}-maximal sets is empty.*

The numbers of classes of bases and pivotal incomplete sets for the sets $P_{3,2}$ and $P_{4,2}$ are shown in the following table. They were obtained by one of the algorithms described in [StM87].

Rank	1	2	3	4	5	6	Σ
bases $P_{3,2}$	1	160	804	272	8	-	1245
pivotals $P_{3,2}$	42	440	435	38	-	-	955
bases $P_{4,2}$	1	1,572	42,822	56,228	6,284	64	106971
pivotals $P_{4,2}$	139	6,336	30,660	10,798	314	-	48,247

7.4. Maximal rank of a base of P_{k2}

We are going to determine the maximal rank of a base of P_{k2}. First we show two combinatorial lemmas. Let $i,t \in E_k$ and $i \neq t$. The set $\{i,t\}$ we call *a pair set.*

Lemma 7.4.1. *For every $k' \geq k$ (> 2) different pair-sets $\{i,t\}$ such that $0 \leq i,t \leq k-1$ and $i \neq t$ there exists a circular sequence $\{i_1,i_2\},\{i_2,i_3\},\ldots,\{i_{s-1},i_s\}$, $\{i_s,i_1\}$ $(0 \leq i_p,i_q \leq k-1$ and $i_p \neq i_q$ for $p \neq q$, $1 \leq p,q \leq s)$ consisting of $s \geq 3$ different pair-sets.*

Proof. The assertion of Lemma can be interpreted as a lemma from graph theory by mapping elements $0,\ldots,k-1$ onto vertices and k' pair sets $\{i,t\}$ as only edges of the graph. It is well-known that each graph with n vertices and at least n edges has a circuit. □

Lemma 7.4.2. *If for a given set T of $k-1$ different pair-sets $\{i,t\}$ $(0 \leq i,t \leq k-1, i \neq t, \{i,t\} \neq \{0,1\})$ there exists no circular sequence (with the definition from Lemma 7.4.1), then there is a sequence which leads from 0 to 1 through at least two pair sets, i.e. there is a sequence $\{0,i_2\},\{i_2,i_3\},\ldots,\{i_{s-1},i_s\},\{i_s,1\}$, where $s \geq 2, \{i_p,i_{p+1}\} \in T$ for $1 \leq p \leq s$ and $i_1 = 0, i_{s+1} = 1$.*

Proof. It is well-known that a graph with k vertices and $k-1$ edges and without circuit is a tree. Thus every two vertices are connected, especially 0 and 1. □

Let $F \subseteq P_{k2}$ be pivotal. To $f \in F$ assign $\Gamma_f := \{\{i,j\} : Z_{ij}$ pivot of $f\}$ (recall that Z_{ij} is a pivot of f if $f \notin Z_{ij}$ while $F \setminus \{f\} \subseteq Z_{ij}$). Put $G_F := (E_k, T)$ where $T = \bigcup\{\Gamma_f : f \in F\}$. We call G_F *pivot graph* for F.

Lemma 7.4.3. *The pivot graph G_F is acyclic.*

Proof. Let $\{i_1,i_2\},\{i_2,i_3\},\ldots,\{i_l,i_{l+1}\} \in T$ where $i_{l+1} = i_1$. Here $\{i_1,i_2\} \in \Gamma_f$ for some $f \in F$ i.e. $f \notin Z_{i_1 i_2}$ while $f \in Z_{i_j i_{j+1}}$ for $j = 2,\ldots,l$ (by the pivot condition). Now by Lemma 7.3.1 we have $(i_j, i_{j+1}) \in Q_f$ for $j = 2,\ldots,l$. In view of Lemma 7.3.3 the relation Q_f is transitive and so $(i_1,i_2) \in Q_f$ and again by Lemma 7.3.1 we get $f \in Z_{i_1 i_2}$, a contradiction. □

Lemma 7.4.4. *The maximal rank of a base of P_{k2} is at most $k+2$.*

Proof. Let F be a base of P_{k2} and G the subset of F such that *pr* G is a base in P_2. Let $Y = \{T_0', T_1', S', L', M'\}$. Assume $|F \setminus G| \geq k-1$ and $H \subseteq F \setminus G$, $|H| = k-1$. The functions from H cannot have a pivot (in P_{k2}) from Y. (If $f \in H$ has a pivot $P \in Y$, then $G \subseteq P$ in contradiction to *pr* G basis of P_2). Consider the graph G_H. By Lemma 7.4.3 it is acyclic and so has at most $k-1$ edges. However, $|\Gamma_h| \geq 1$ for each $h \in H$ and so G_H has exactly $k-1$ edges. It follows that G_H is a tree. In particular, there is a unique path $\{i_1,i_2\},\{i_2,i_3\},\ldots,\{i_{s-1},i_s\}$ in G_H with $i_1 = 0$ and $i_s = 1$. The set G contains a function f such that $f \notin M'$. Clearly f is nonconstant on E_2 and hence we have $(0,1) \notin Q_f$. Therefore, there exists $1 \leq j \leq s-1$ such that $\{i_j, i_{j+1}\} \notin Q_f$ (otherwise we have $(0,1) \in Q_f$ because Q_f is a transitive relation). We have $f \notin Z_{i_j i_{j+1}}$ from Lemma 7.3.1. However, $Z_{i_j i_{j+1}}$ is a pivot of some $h \in H$ and so $f \in Z_{i_j i_{j+1}}$, a contradiction. Thus we conclude that H contains at most $k-2$ functions, But, G contains at most four functions [Jab52,INN63,Krn65,LoW65]. Therefore, F contains at most $k+2$ functions. □

Theorem 7.4.1. *The maximal rank of a base of P_{k2} is $k+2$.*

Proof. Let Q_i $(1 \leq i \leq k-1)$ be the equivalence relations with the two equivalence classes: $\{1,\ldots,i\},\{i+1,\ldots,k-1,0\}$. A base of rank $k+2$ is the set $\{f_1,\ldots,f_{k+2}\}$, defined by

$$\begin{aligned}
Q_{f_i} &= Q_i\,(1 \le i \le k-1), Q_{f_k} = Q_1, Q_{f_{k+1}} = Q_{f_{k+2}} := E_k^2;\\
f_1 &\in T_0'T_1'L'S'\overline{M}',\\
f_i &\in T_0'T_1'L'S'M'\,(2 \le i \le k-1),\\
f_k &\in T_0'T_1'\overline{L}'\overline{S}'M',\\
f_{k+1}(0,\ldots,0) &= 0,\\
f_{k+2}(0,\ldots,0) &= 1.
\end{aligned}$$

We note that $pr\ f_i$ $(2 \le i \le k-1)$ depends only of one variable and $pr\ f_i(0) = 0$, $pr\ f_i(1) = 1$ from $f_i \in T_0'T_1'L'S'M'$. Thus, for example, we can take f_i as unary functions. Then the requirement $f_i \in Q_i$ determines f_i completely, since $Z_{j,0} = 1$ and $Z_{j,1} = 0$ lead $f_i(j) = 1$ for $2 \le j \le i$ and $Z_{j,0} = 0$ and $Z_{j,1} = 1$ lead $f_i(j) = 0$ for $i+1 \le j \le k-1$. It is easy to see that the functions $\{f_1,\ldots,f_{k+2}\}$ actually cover all Z_{it} as well as T_0', T_1', L', S', M'. The pivots of f_1, f_k, f_{k+1} and f_{k+2} are c_5,c_3 and c_4,c_2 and c_1 respectively. The pivots of f_i is $Z_{i+1(\text{mod } k),i}$ $(2 \le i \le k-1)$. □

Example 7.4.1. Let $k = 3$. Put $Q_1 = \{\{1\},\{2,0\}\}, Q_2 = \{\{1,2\},\{0\}\}$. The following is the characteristic vectors of a base $\{f_1,\ldots,f_5\}$ constructed as in the theorem with rank $k+2 = 5$.

	c_1	c_2	c_3	c_4	c_5	$c_{2,0}$	$c_{2,1}$
f_1	0	0	0	0	1	0	1
f_2	0	0	0	0	0	1	0
f_3	0	0	1	1	0	0	1
f_4	0	1	0	1	0	0	0
f_5	1	0	0	1	0	0	0

□

7.5. Concluding remarks

The composition of functions in P_{k2} is closely related to the composition in P_2. Indeed, in a composition of P_{k2}-functions, only the elements in the first layer work as P_{k2} functions; those in the remaining layers work merely as P_2 functions. The proof given in Lemma

7.4.4 indicates that a base needs at most $k-2$ elements from P_{k2} and at most 4 elements from P_2 for the first layer and for the remaining layers, respectively.

The completeness theory of logical functions leads to the classification problems of closed sets by their maximal sets. These has been done for P_2, P_3 and for some other sets [MiS87a], but very little is done in general [Sto86c,Sto85b]. In this chapter we have determined classes of functions of P_{k2} and their exact number. Although the numbers of maximal sets and classes of functions of P_{k2} grow rapidly as $O(k^2)$ and $O(k!)$ respectively, maximal rank of bases of P_{k2} has been proved to be $k+2$. There remains an open problem about the maximal rank of P_k.

Table 8.1:

Classes of functions of $P_2 = P_{2,2}$
(with respect to the coordinates T_0', T_1', S', L', M' [Jab52,INN63,Krn65]

11111	11011	11001	10111	10101	10100	01111	01101
01100	00111	00110	00011	00010	00001	00000	

Classes of functions of $P_{3,2}$
(with respect to the coordinates $T_0', T_1', S', L', M', Z_{2,0}, Z_{2,1}$)

1111111	1101101	1011110	1010011	0110111	0011111	0011001	0001010	0000011
1111110	1100111	1011101	1010000	0110110	0011110	0001111	0001001	0000010
1111101	1100110	1010111	0111111	0110101	0011101	0001110	0000111	0000001
1101111	1100101	1010110	0111110	0110011	0011011	0001101	0000110	
1101110	1011111	1010101	0111101	0110000	0011010	0001011	0000101	

Classes of functions of $P_{4,2}$
(with respect to the coordinates $T_0', T_1', S', L', M', Z_{2,0}, Z_{2,1}, Z_{3,0}, Z_{3,1}, Z_{3,2}$)

1111111111	1100111111	1010111111	0111110100	0011111111	0001111111	0000111111
1111111110	1100111110	1010111110	0111110011	0011111110	0001111110	0000111110
1111111101	1100111101	1010111101	0111101111	0011111101	0001111101	0000111101
1111111011	1100111011	1010111011	0111101101	0011111011	0001111011	0000111011
1111110111	1100110111	1010110111	0111101010	0011110111	0001110111	0000110111
1111110100	1100110100	1010110100	0110111111	0011110100	0001110100	0000110100
1111110011	1100110011	1010110011	0110111110	0011110011	0001110011	0000110011
1111101111	1100101111	1010101111	0110111101	0011101111	0001101111	0000101111
1111101101	1100101101	1010101101	0110111011	0011101101	0001101101	0000101101
1111101010	1100101010	1010101010	0110110111	0011101010	0001101010	0000101010
1101111111	1011111111	1010011111	0110110100	0011011111	0001011111	0000011111
1101111110	1011111110	1010011110	0110110011	0011011110	0001011110	0000011110
1101111101	1011111101	1010011001	0110101111	0011011101	0001011101	0000011101
1101111011	1011111011	1010000111	0110101101	0011011011	0001011011	0000011011
1101110111	1011110111	1010000000	0110101010	0011010111	0001010111	0000010111
1101110100	1011110100	0111111111	0110011111	0011010100	0001010100	0000010100
1101110011	1011110011	0111111110	0110011110	0011010011	0001010011	0000010011
1101101111	1011101111	0111111101	0110011001	0011001111	0001001111	0000001111
1101101101	1011101101	0111111011	0110000111	0011001101	0001001101	0000001101
1101101010	1011101010	0111110111	0110000000	0011001010	0001001010	0000001010

Chapter 9

Classifications of Maximal Sets of P_{k2}

In the previous chapter the set of functions of P_{k2} mapping the set $\{0, 1, \ldots, k-1\}^n$ into $\{0, 1\}$ has been classified. It is shown that the number of P_{k2}-classes is $13A_k - 11A_{k-1}$, where A_k is the number of equivalence relations on the set of k elements. The maximal number of elements in a base of P_{k2} has been also proved to be $k+2$.

In this chapter we consider maximal sets of P_{k2}. We determine classes of functions for all P_{k2}-maximal sets T_0', T_1', S', L' and Z_{it} $(0 \leq t < i \leq k-1, i \geq 2)$ except M'. We also give maximal number of elements in a base (maximal rank f a base) for each of these sets (for S' we prove its upper bound to be $2k$).

We also classify the symmetric functions of P_{k2} and its maximal sets. In the last section we give numerical data for the respective numbers of classes of functions and classes of symmetric functions of Z_{it}, T_0', S' and L' for $2 \leq k \leq 10$. We also give numerical data for bases, pivotals, S-bases, S-pivotals for each of the P_{k2}-maximal sets Z_{it}, T_0', L' and S' for k up to 4.

9.1. Classification of Z_{it}

All the maximal sets of the P_{k2}-maximal set Z_{it} are given by the following theorem. Recall that $Z_{it}' := P_{k2}Pol\begin{pmatrix} 01i \\ 01t \end{pmatrix}$.

Theorem 9.1.1. [Lau84b] *Maximal sets of the set* Z_{it} $(0 \leq t < i \leq k-1,\ 2 \leq i)$ *are*

$Z'_{jl} := Z_{it}Z_{jl},\ 0 \le l < j \le k-1,\ 2 \le j,\ l \ne t \ or\ i \ne j,\ \{0,1\} \ne \{t,l\} \ for\ i = j;$
$R_j := Pol\begin{pmatrix} 01ij \\ 01tj \end{pmatrix},\ 2 \le j \le k-1,\ j \ne i \ for\ t \in E_2,$
$Z_{it}\ pr^{-1}B,\ B \in \{T_0, T_1, L, S, M\}.$

As we will see below all the above $\{R_j\}$ and $Z'_{jl} = Z_{it}Z_{jl}$ are not necessarily distinct.

Note 9.1.1. $R_i = Z_{it}$ for $t \in E_2$. This is easily seen from that $f(\boldsymbol{a}) = f(\boldsymbol{b})$ for $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in \begin{pmatrix} 01ii \\ 01ti \end{pmatrix}$ and for $f \in Z_{it}, t \in E_2$.

Next lemma shows that R_i coincides with R_t in Z_{it}.

Lemma 9.1.1. *$R_i = R_t$ in Z_{it} for $0 \le t < i \le k-1,\ 2 \le i$.*

Proof. Let $f \in R_i$ and $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in \begin{pmatrix} 01it \\ 01tt \end{pmatrix}$. If there is no j such that $a_j = b_j = t$ then obviously $f(\boldsymbol{a}) = f(\boldsymbol{b})$. Otherwise let $\boldsymbol{c}$ be $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{pmatrix} \in \begin{pmatrix} 01it \\ 01tt \\ 01ii \end{pmatrix}$. Then we have $f(\boldsymbol{a}) = f(\boldsymbol{c})$ since $f \in R_i$ and $f(\boldsymbol{c}) = f(\boldsymbol{b})$ since $f \in Z_{it}$. Hence $f(\boldsymbol{a}) = f(\boldsymbol{b})$. □

Thus we have to skip R_i in counting maximal sets of Z_{it} all the time (not only in the case of $t \in E_2$). In the proof of the next theorem we show that several sets among above $\{Z'_{jl}\}$ also coincide. Thus the numbers of the maximal sets of Z_{it} reported in [Lau84b] as $k(k+1)/2+1$ for $t \ge 2$ and $k(k+1)/2-1$ for $t = 0$ or 1, are not correct.

Theorem 9.1.2. *The number of the maximal sets of Z_{it} is $k(k-1)/2+2$ $(k \ge 3)$.*

Proof. It follows from relational product that $Z_{it}Z_{ts} \subseteq Z_{is}$ $(i,t,s \in E_k)$. Therefore, we conclude

$$Z'_{is} = Z_{it}Z_{is} = Z_{it}Z_{st} = Z'_{st}. \tag{9.1}$$

Thus, several maximal sets of the type $\{Z_{jl}\}$ coincide in Z_{it}. For $t \ge 2$ (9.1) is meaningful for $s \in E_k, s \ne t, s \ne i$ $(k-2$ values). For $t = 0$ or $t = 1$ (9.1) is meaningful for $s \in E_k, s \ne 0, s \ne 1, s \ne i$ $(k-3$ values). Hence, the number of maximal sets in Z_{it} is $k(k+1)/2+1-(k-2)-1 = k(k-1)/2+2$ for $t \ge 2$ (from Lemma 9.1.1 together) and $k(k+1)/2-1-(k-3) = k(k-1)/2+2$ for $t = 0$ or $t = 1$. □

Theorem 9.1.3. *The number of classes of functions of Z_{it} is $2^{k-3}(13A_{k-1} - 11A_{k-2})$.*

Proof. Consider an equivalence relation Q_f defined as in the case of P_{k2}:

$$(j,l) \in Q \Leftrightarrow f \in Z_{jl}.$$

Then $(i,t) \in Q$ always holds, because $f \in Z_{it}$. The number of such relations Q is A_{k-1}. Similarly as in the case of P_{k2} (Theorem 8.3.2) we can prove that there are $13(A_{k-1} - A_{k-2}) + 2A_{k-2}$ classes of functions of Z_{it}, according to the maximal sets $\{Z_{it}Z_{jl}\}$ and $\{Z_{it}pr^{-1}B \mid B$ maximal set in $P_2\}$. Now consider R_j $(2 \le j \le k-1, j \ne i)$. We show that a representative exists in each of both cases of R_j and $\overline{R}_j$ for each j and for each such class. For $f \in R_j$ we put $f(x_1,\dots,x_n) = f(y_1,\dots,y_n)$, where each of $x_1,\dots,x_n \in \{0,1,i,j\}$, $y_i = x_i$ for $x_i \in \{0,1,j\}$ and $y_i = t$ for $x_i = i$. For $f \notin R_j$ $(n \ge 2)$ we put

$$f(j,i,0,\dots,0) \ne f(j,t,0,\dots,0). \tag{9.2}$$

All the considered conditions of type (9.2) are independent, because only values $f(j,0,\dots,0)$ (for $t=0$) can be fixed with respect to other maximal sets from $\{R_j\}$. Therefore, there are 2^{k-3} possibilities with respect to the sets $\{R_j\}$. Hence the number of classes of functions of Z_{it} is $2^{k-3}(13A_{k-1} - 11A_{k-2})$. □

Example 9.1.1. Classes of $Z_{2,0}$ and $Z_{2,1}$ in $P_{3,2}$ are isomorphic to those of P_2. □

Example 9.1.2. We consider classes of $Z_{3,0}$ in $P_{4,2}$. The maximal sets are intersections $Z_{3,0}pr^{-1}B$ (B is one of the five maximal sets of E_2), $Z_{3,0}Z_{it}$, $2 \le i \le 3$, $0 \le t \le 2$, $i \ne 3$ for $t = 0$ (i.e. $Z_{2,0}$ and $Z_{2,1}$; $Z_{3,2}$ is omitted since $Z_{3,0}Z_{2,0} = Z_{3,0}Z_{3,2}$ as indicated in Theorem 9.1.2), and $R_2 = Pol\begin{pmatrix} 0132 \\ 0102 \end{pmatrix}$. Note that $R_3 = Pol\begin{pmatrix} 0133 \\ 0103 \end{pmatrix} = Pol\begin{pmatrix} 013 \\ 010 \end{pmatrix}$. We show all the equivalence relations on E_4 which include $\{0,3\}$.

Equivalence class	$c_{2,0}c_{2,1}$
{0,1,3}, {2}	1 1
{0,1,2,3}	0 0
{0,3}, {1}, {2}	1 1
{0,3}, {1,2}	1 0
{0,2,3}, {1}	0 1

It is easy to check that $c_{3,2}$ coincides with $c_{2,0}$. This confirms that $Z_{3,2}$ coincides with $Z_{2,0}$ in $Z_{3,0}$. To demonstrate our construction of a representative for each of the above classes, let our example equivalence relation Q on E_4 be {0,3},{1},{2}. We proceed analogously as the steps of Theorem 8.3.1 for $f(x_1, x_2)$ for a given $g(x_1, x_2) \in P_2$. 1) $pr\ f = g$ and $g(x_1, x_2)$ is an arbitrary nonconstant function on E_2 since $\{0,1\} \notin Q$. 2) Only $\{0,3\} \in Q$. So $f(x_1, x_2) = f(y_1, y_2)$, where $x_1, x_2 \in \{0,1,3\}, y_j = x_j$ for $x_j \in E_2$ and $y_j = 0$ for $x_j = 3, 1 \leq j \leq 2$. 4) $\{2,0\} \notin Q$ and $\{2,1\} \notin Q$. We put $f(0,2) \neq f(0,0)$ and $f(1,2) \neq f(1,1)$. As for R_2 we construct two cases. Case of $f \in R_2$. $f(x_1, x_2) = f(y_1, y_2)$ where $x_1, x_2 \in \{0,1,2,3\}, y_j = x_j$ for $x_j \in \{0,1,2\}$ and $y_j = 0$ for $x_j = 3$. Case $f \notin R_2$. We put $f(2,3) \neq f(2,0)$. Thus we can see that our construction for f in Theorem 9.1.3 is compatible with that in Theorem 8.3.1. □

Maximal rank of a base of Z_{it}

As we have seen in the previous chapter, an equivalence relation on E_K induced by $f \in P_{k2}$ by setting $(i,t) in Q_f \Leftrightarrow f \in Z_{it}$ restricts the number of functions in a base. This can be summarized in the following lemma.

Lemma 9.1.2. *The number of pivots from the sets Z_{it} in any pivotal set of any closed set containing some of sets Z_{it} as its maximal sets is $\leq k-1$.*

Proof. Suppose a pivotal set contains at least k functions which give pivots from the sets $\{Z_{it}\}$. Then from Lemma 8.4.1 follows that there is a circular sequence. Then from Lemma 8.4.3 follows that circular sequence cannot be in a set of pivots for a set of pivotal functions. A contradiction. □

Theorem 9.1.4. *Maximal rank of a base of Z_{it} is $2k-2$.*

Proof. According to the maximal sets Z_{jl} and T_0', T_1', L', S', M' there exists a base with a maximal rank $k+1$, because $(i,t) \in Q$ for every Q and we consider the equivalence relation on the set with in fact $k-1$ elements (the proof is similar to that of P_{k2}). The sets R_j can give $k-3$ new functions for a base. Hence maximal rank is $k+1+k-3 = 2k-2$.

We are to give an example of a base with the maximal rank. Let $i = 3, t = 2$ for simplicity (examples for $t \in E_2$ and any i can be constructed similarly). A base $\{f_j | 1 \leq j \leq 2k-1, j \neq k+3\}$ and corresponding relations Q_j for f_j are defined as follows:

$$
\begin{array}{lll}
Q_1 & := & \{1\}, \{2,3,\ldots,k-1,0\}, \\
Q_{j-1} & := & \{1,2,\ldots,j\}, \{j+1,\ldots,k-1,0\},\ 3 \leq j \leq k-1, \\
Q_{k-1} = Q_r & := & Q_1\ (k+2 \leq r \leq 2k-2,\ r \neq k+3), \\
Q_k = Q_{k+1} & := & E_k^2;
\end{array}
$$

$$
\begin{array}{lll}
f_1 & \in & T_0'T_1'L'S'\overline{M}', \\
f_{k-1} & \in & T_0'T_1'\overline{L}'\overline{S}'M', \\
f_k(0,\ldots,0)=0 & \in & T_0\overline{T}_1L\overline{S}'M, \\
f_{k+1}(0,\ldots,0)=1 & \in & \overline{T}_0T_1L\overline{S}'M, \\
f_i & \in & R_j\ (1\le i\le k+1,\ 2\le j\le k-1,\ j\ne 3), \\
f_r & \in & T_0'T_1'L'S'M'\ (2\le r\le k-2, k+2\le r\le 2k-1, r\ne k+3), \\
f_r & \in & \overline{R}_{r-k}R_s,\ s\ne r-k, 3 \text{ for } k+2\le r\le 2k-1,\ r\ne k+3.
\end{array}
$$

We note that $Z_{l,3}=Z_{l,2},\ 0\le l\le k-1,\ l\ne 2,\ l\ne 3$. Pivots for f_1, f_{k-1}, f_k and f_{k+1} are c_5, c_3, c_2 and c_1, respectively, and pivots for f_j are $Z_{j+2(\text{mod } k),j+1}$ for $2\le j\le k-2$. Finally, pivots for f_r is R_{r-k} for $k+2\le r\le 2k-1,\ r\ne k+3$. We remark that functions $f_r\in T_0'T_1'L'S'M'$ $(k+2\le r\le 2k-1, r\ne k+3)$ cannot be unary (unary functions lead to $f\in R_j$). □

Example 9.1.3. We give an example of the above base for $Z_{3,2}$ in $P_{4,2}$ $(k=4, i=3, t=2)$. Note that $Z_{3,0}=Z_{2,0}, Z_{3,1}=Z_{2,1}$ and $R_3=R_2$ in $Z_{3,2}$ (in $P_{4,2}$).

	T_0T_1LSM	$Z_{2,0}Z_{3,1}$	R_2
f_1	0 0 0 0 1	0 1	0
f_2	0 0 0 0 0	1 0	0
f_3	0 0 1 1 0	0 1	0
f_4	0 1 0 1 0	0 0	0
f_5	1 0 0 1 0	0 0	0
f_6	0 0 0 0 0	0 1	1

$Q_1=\{\{1\},\{2,3,0\}\};\ Q_2=\{\{1,2,3\},\{0\}\}$. □

9.2. Classification of the maximal set T_0': the functions preserving 0

Theorem 9.2.1. [Lau84b] *The maximal set T_0' of P_{k2} has $4+(k+1)(k-2)/2$ maximal sets:*

$$
\begin{array}{lll}
T_{0,1} & := & T_0'T_1', \\
L_0 & := & T_0'L', \\
M_0 & := & T_0'M', \\
N_0 & := & T_0'Pol\begin{pmatrix}001\\010\end{pmatrix}, \\
T_0'Z_{it} & & \textit{for } 1\le t\le k-2,\ 2\le i\le k-1,\ t<i, \\
T_{0i} & := & P_{k2}Pol\,(0i),\ 2\le i\le k-1.
\end{array}
$$

Note 9.2.1. The first four sets are the intersections with the maximal sets of T_0 in P_2. We note that respective cases of $i = 1$ and $t = 0$ are not included in the above list. It is easy to see that $T_{0i} = Pol(0i) = \{f \mid f(\{0,i\})^n = 0 \text{ for } n = 1,2,\ldots\}$ for $2 \le i \le k-1$ (we write simply $Pol(0i)$ for $P_{k2}Pol(0i)$). For $i = 1$ this does not hold, because we have $T_{01} = Pol(01) = P_2' = P_{k2}$. Putting $t = 0$ for $T_0'Z_{it}$, we have $T_0'Z_{i,0} \subseteq T_0Z_{i,0} = Pol(0)Pol\begin{pmatrix} 01i \\ 010 \end{pmatrix} = T_{0i},\ 2 \le i \le k-1$.

Since the sets $\{Z_{i,0}\}$ do not appear as maximal sets, our equivalence relation induced by $f \in T_0$ is on the $k-1$ elements of $\{1,\ldots,k-1\}$ (i.e. 0 is excluded). We give several lemmas for the classification.

Lemma 9.2.1. [Sto85] *There are exactly 10 classes of functions of* T_0*:*

$$1111,\ 1110,\ 1011,\ 1000,\ 0111,\ 0110,\ 0101,\ 0100,\ 0011,\ 0000,$$

where the coordinates are in the order of T_1, L, M *and* $Pol\begin{pmatrix} 001 \\ 010 \end{pmatrix}$.

The maximal rank of a base of T_0 is 3. The set $\{(0100), (0011), (1000)\}$ is an example of base. The class 1000 consists only of the constant function 0. The set of T_0'-functions corresponding to this class is called *0-class* in the classification below (functions constant 0 on $\{0,1\}^n$). The next lemma includes an assertion on this 0-class as the case $i = 1$.

Lemma 9.2.2. $f \in Z_{ij}$ *and* $f(\{0,1\}^n) = 0 \Rightarrow f(\{0,j\}^n) = 0$ *for* $1 \le i,j \le k-1$.

Proof. Suppose $f \in Z_{ij}$ and $f(\boldsymbol{a}) = 0$ for $\boldsymbol{a} \in (0i)$. Then $f(\boldsymbol{b}) = 0$ for $\boldsymbol{b} \in (0j)$ because $f \in Z_{ij}$ and $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in Z_{ij}$. □

Corollary 9.2.1. $Z_{ij}T_{0i} \subseteq T_{0j}$ *for* $2 \le i \le k-1$.

Theorem 9.2.2. *The number of classes of functions of* T_0' *is* $10\sum_{r=1}^{k-1} A(k-1,r)2^{r-1}$.

Proof. As we have seen in the previous chapter the equivalence relation Q_f on the sets $\{1,2,\ldots,k-1\}$ induced by a function $f \in T_0$ determines the characteristic vector of f for $\{Z_{ij}\}$ by the rule $(i,j) \in Q_f \Leftrightarrow f \in Z_{ij}$. Let Q_f divide $\{1,\ldots,k-1\}$ into r classes. Let one of these classes be $\{i_1,\ldots,i_p\}$. For these numbers we have $Z_{i_s,i_t} = 0$ $(1 \le s,t \le p)$. If 1 is included in the set $\{i_1,\ldots,i_p\}$ $(p > 1)$, we have $f \in Z_{m,1}$ for any such $m := i_s > 1$, i.e. $f \in Pol(0)Pol\begin{pmatrix} 01m \\ 011 \end{pmatrix}$. Further, assume that f is from 0-class (i.e. $f(\boldsymbol{a}) = 0$ for any $\boldsymbol{a} \in (01)$), then from $f \in Pol\begin{pmatrix} 01m \\ 011 \end{pmatrix}$ we have $f(\{0,m\}^n) = 0$, i.e. $f \in T_{0m}$; assume otherwise, then from $f \in Pol(0)Pol\begin{pmatrix} 01m \\ 011 \end{pmatrix}$ we conclude $f \notin T_{0m}$. Thus we distinguish two cases.

Case 1. $1 \notin \{i_1, \ldots, i_p\}$. From Lemma 9.2.2 only the two possibilities exist for $T_{0i_m}, m = 1, \ldots, p$, namely $f \in T_{0i_1} \ldots T_{0i_p}$ or $f \in \overline{T}_{0i_1} \ldots \overline{T}_{0i_p}$.

Case 2. $1 \in \{i_1, \ldots, i_p\}$. There exists exactly one possible case depending on the values of f on $\{0,1\}^n$:

$$\begin{aligned} f &\in T_{0i_1} \ldots T_{0i_p} \text{ for } f \text{ from 0-class } (f(\{0,1\}^n) = 0) \text{ or} \\ f &\in \overline{T}_{0i_1} \ldots \overline{T}_{0i_p} \text{ for } f \text{ not from 0-class.} \end{aligned}$$

So, if Q_f divide $\{1, \ldots, k-1\}$ into r classes (one of them includes 1 as its member), there are 2^{r-1} classes of functions with respect to the sets $\{Z_{it}\}$ and $\{T_{0i}\}$. Further, 10 classes will be derived for each of these vectors if we add first 4 coordinates. We show that these classes are actually nonempty by giving a representative for each class.

Let g be a function with $n \geq 2$ variables in P_2 such that g is a function of the corresponding class with respect to T_0-maximal sets. Let Q be an equivalence relation induced by Z_{ij}. Conditions for f are as follows:

1) $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ for $x_1, \ldots, x_n \in E_2$.

2) $(i,t) \in Q \Leftrightarrow f(x_1, \ldots, x_{j-1}, i, x_{j+1}, \ldots, x_n) = f(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)$ for each $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \in E_k$ and $1 \leq t < i \leq k-1$.

3) $(i,t) \notin Q$ $(2 \leq i \leq k-1, 1 \leq t < i)$ and $f(i, \ldots, i) = f(t, \ldots, t)$ $\Rightarrow \{f(t, i, \ldots, i), f(i, i, \ldots, i)\} = E_2$.

4) Let $\{i_1, i_2, \ldots, i_l\}$ be a class included in Q_f. In the case $f \in T_{0i_1} \ldots T_{0i_l}$ let $f(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in (0i_j)$ $(1 \leq j \leq l)$. In the case $f \in \overline{T}_{0i_1} \ldots \overline{T}_{0i_l}$ let $f(i_j, \ldots, i_j) = 1$. □

Example 9.2.1. For $k = 3$ all maximal sets of T_0' in $P_{3,2}$ are $T_{01}, L_0, M_0, N_0, Z_{2,1}$ and T_{02}. The vectors for $Z_{2,1}$ are determined by an equivalence relation on {1,2} as follows.

Equivalence class	$Z_{2,1}T_{02}$	$T_{0,1}L_0M_0N_0$
{1}, {2}	1 1	for each of 10 T_0 classes,
	1 0	for each of 10 T_0 classes,
{1,2}	0 1	for each of 9 T_0 classes except 0-class,
	0 0	for 0-class.

We give all its 30 classes (the coordinates are in the order of $T_{01}, L_0, M_0, N_0, Z_{2,1}$ and T_{02}). In the table of characteristic vectors * at the end of the vector denotes the class having no symmetric representative (cf. Section 9.5).

111111	111110	111101	111011	111010	111001
101111	101110	101101	100011	100010	100000
011111	011110	011101	011011	011010	011001
010111	010110	010101	010011	010010	010001
001111	001110	001101	000011*	000010	000001

□

Example 9.2.2. Classes of T_0' in $P_{4,2}$. Equivalence classes are on {1,2,3}. Maximal sets are $Z_{2,1}, Z_{3,1}, Z_{3,2}, T_{02}$ and T_{03}.

Equivalence class	$Z_{2,1}Z_{3,1}Z_{3,2}$	$T_{02}T_{03}$	number of classes
{1},{2},{3}	1 1 1	1 1	for each of 10 T_0 classes,
		1 0	for each of 10 T_0 classes,
		0 1	for each of 10 T_0 classes,
		0 0	for each of 10 T_0 classes,
{1},{2,3}	1 1 0	1 1	for each of 10 T_0 classes,
		0 0	for each of 10 T_0 classes,
{1,2},{3}	0 1 1	1 0	for each of 9 T_0 classes except 0-class,
		1 1	for each of 9 T_0 classes except 0-class,
		0 1	for 0-class,
		0 0	for 0-class,
{1,3},{2}	1 0 1	1 1	for each of 9 T_0 classes except 0-class,
		0 1	for each of 9 T_0 classes except 0-class,
		1 0	for 0-class,
		0 0	for 0-class,
{1,2,3}	0 0 0	1 1	for each of 9 T_0 classes except 0-class,
		0 0	for 0-class.

All 110 classes are listed in Table 9.4. □

We are going to determine the maximal rank of a base of T_0'.

Theorem 9.2.3. *The maximal rank of bases of T_0' is $k+1$.*

Proof. We note that $Z_{i,0} = T_{0i}$ in T_0 for $2 \le i \le k-1$. We can consider sets Z_{ij}, $2 \le i \le k-1$, $0 \le j \le k-2$, $j < i$. Rank of a base for these sets is greater than rank of a base for T_0' (the proof is analogous to that of P_{k2}). Let $P = \{f_1, \dots, f_p\}$ be a base with respect to considered sets, V be a subsets of P which is a base with respect to the sets $Y = \{T_{01}, L_0, M_0, N_0\}$ and $W = Y \setminus V$. The set V contains at most 3 elements from Lemma 9.2.1. The set W contains at most $k-2$ functions (the same proof as in P_{k2}). Thus the rank of a base is less than or equals to $3 + k - 2 = k + 1$. We show an example of a base with the rank $k+1$.

Let Q_i $(1 \le i \le k-1)$ be equivalence relations defined by $\{1, .., i\}, \{i+1, \dots, k-1, 0\}$. Put $Q_k := Q_1$ and $Q_{k+1} := E_k^2$. The base of rank $k+1$ is the set $\{f_1, \dots, f_{k+1}\}$ defined by $Q_{f_i} := Q_i$ and in the following way.

$$
\begin{array}{lll}
f_i & \in & T_{0i} \Leftrightarrow (0,i) \in Q_{f_i} \text{ for } 1 \le i \le k-1, \\
f_1 & \in & T_{01}L\overline{MN}_0, \\
f_k & \in & T_{01}\overline{L}MN_0, \\
f_i & \in & T_{01}LMN_0 \ (2 \le i \le k-1), \\
f_{k+1} & \in & \overline{T}'_{01}LMN_0. \square
\end{array}
$$

9.3. Classification of L': the set of functions from P_{k2} that are linear on $\{0,1\}$

Theorem 9.3.1. [Lau84b] *There are $(k-1)(k-2)/2+4$ maximal sets in $L' := Pr^{-1}(L)$:*

$$
\begin{array}{lll}
L_0' & := & L'T_0', \\
L_1' & := & L'T_1', \\
L_s' & := & L'S', \\
L^{(1)\prime} & := & [a_0 + a_1 x \mid a_0, a_1 \in \{0,1\}]', \\
L_q & := & P_{k2}Pol\{(q,q,q,q), (a,b,c,d) \mid (a,b,c,d) \in E_2^4, \ a+b = c+d \pmod 2\}, \\
 & & 2 \le q \le k-1, \\
Z_{it}' & := & Z_{it}Pr^{-1}L, \ 2 \le t < i \le k-1.
\end{array}
$$

We show lemmas for the classification and determine the number of classes of L'. For simplicity we write Z_{it} for Z_{it}' in this section.

Lemma 9.3.1. [Sto85] *There are 8 classes of functions in L of P_2:*

$$0000, 0001, 0110, 1010, 1100, 0111, 1011, 1101,$$

where the coordinates are L_0, L_1, L_S and $L^{(1)}$.

The maximal rank of a base of L in P_2 is 3. An example of a base with the maximal rank is $\{(0110), (1010), (0001)\}$.

Lemma 9.3.2. *$Z_{it}L_i \subseteq L_t$ and $Z_{ti}L_i \subseteq L_t$, $2 \le i,\ t \le k-1$.*

Proof. For convenience let Z_{it}, L_i and L_t denote the relations instead of the functions preserving these relations. It is easy to see that we can construct L_t by repeated applications of relational product and permutations of rows of the relation from Z_{it} and L_i; we use that the relation $L = \{(a, b, c, d)^T \in E_2^4,\ a + b = c + d \pmod 2\}$ is invariant under permutations of rows. Thus the lemma is proved. □

Theorem 9.3.2. *The number of classes of functions of L' is $8\sum_{r=1}^{k-2} A(k-2, r)2^r$.*

Proof. If i and t are in the same equivalence class induced by $Q = Q_f$, i.e. $f \in Z_{it}$, then there are only two possibilities from Lemma 9.3.2: $f \in L_iL_t$ or $f \in \overline{L}_i\overline{L}_t$. Let Q divide $\{2, \ldots, k-1\}$ into r equivalence classes and $\{i_1, \ldots, i_l\}$ be one such class. For each equivalence class there are only two possibilities by Lemma 9.3.2: $f \in L_{i_1} \cdots L_{i_l}$ or $f \in \overline{L}_{i_1} \cdots \overline{L}_{l_l}$. Hence there are 2^r possible classes corresponding to a Q with respect to the sets $\{L_q\} \cup \{Z_{it}\}$, and for each of this class there are 8 different prefixes corresponding to the maximal sets of L in P_2. We are to give a representative for each possible class of L'.

Let $g(x_1, \ldots, x_n) \in P_2$ be a function of one of the 8 classes with respect to the first 4 maximal sets ($n \ge 3$). Let Q be an equivalence relation on $\{2, \ldots, k-1\}$ defined by $\{Z_{it}\}$. Put $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ for $x_1, \ldots, x_n \in \{0, 1\}$. Further, define $f(x_1, \ldots, x_n)$ in the following way.

If $(i, t) \in Q$ then set

$$f(x_1, \ldots, x_{j-1}, i, x_{j+1}, \ldots, x_n) = f(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)$$

for each i, t, $2 \le i, t \le k-1$, $i \ne t$ and $1 \le j \le n$ and for each $x_m \in E_k$ $(1 \le m \le n)$.

If $(i, t) \notin Q$ $(2 \le i < t \le k-1)$ and $f(i, \ldots, i) = f(t, \ldots, t)$ then set

$$f(t, i, 0, \ldots, 0) \ne f(i, i, 0, \ldots, 0).$$

Let an equivalence class induced by Q be $\{i_1, \ldots, i_l\}$.

If $f \in L_{i_1} \cdots L_{i_l}$ then set

$$f(x_1, \ldots, x_{j-1}, q, x_{j+1}, \ldots, x_n) = f(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n)$$

for each $q \in \{i_1, \ldots, i_l\}$, $1 \leq j \leq n$ and for $x_1, \ldots, x_n \in E_2 \cup \{q\}$.

If $f \in \overline{L}_{i_1} \cdots \overline{L}_{i_l}$ then set

$$\begin{aligned} f(q,0,0,x_4,\ldots,x_n) &= 0, \\ f(q,0,1,x_4,\ldots,x_n) &= 0, \\ f(q,1,0,x_4,\ldots,x_n) &= 0, \\ f(q,1,1,x_4,\ldots,x_n) &= 1, \end{aligned}$$

for each $x_4, \ldots, x_n \in E_2$ and $q \in \{i_1, \ldots, i_l\}$. That is, $f(q, x_2, \ldots, x_n) = 0$ for each $x_2, \ldots, x_n \in E_2$ except $f(q, 1, \ldots, 1) = 1$. Thus the result of the theorem follows. □

Example 9.3.1. Classes of L' in $P_{3,2}$. All its maximal sets are L'_0, L'_1, L'_S, $L^{(1)\prime}$ and L_2, which are the coordinates from left to right.

11011	11010	10111	10110	11001*	11000	10101*	10100
01111	01110	01101*	01100	00011	00010	00001*	00000

The intersection of all maximal sets contains a unary function s_{010} and in this case the intersection is nonempty. □

Example 9.3.2. Classes of L' in $P_{4,2}$. All its maximal sets are L'_0, L'_1, L'_S, $L^{(1)\prime}$, L_2, L_3 and $Z_{3,2}$, which are the coordinates from left to right.

1101111	1101101	1101011	1101001	1101110	1101000
1011111	1011101	1011011	1011001	1011110	1011000
1100111*	1100101*	1100011*	1100001	1100110*	1100000
1010111*	1010101*	1010011*	1010001	1010110*	1010000
0111111	0111101	0111011	0111001	0111110	0111000
0110111*	0110101*	0110011*	0110001	0110110*	0110000
0001111	0001101	0001011	0001001	0001110	0001000
0000111*	0000101*	0000011*	0000001	0000110*	0000000

□

We are going to determine the maximal rank of a base of L'.

Theorem 9.3.3. *Maximal rank of a base of L' is $k+1$.*

Proof. Let P be a base for L', and $A \subseteq P$ be a subset which is a base for the set L'_0, L'_1, L'_S and $L^{(1)\prime}$. The set A contains at most three functions from Lemma 9.3.1.

Let B be a subset of $P \setminus A$ which is a base for $\{Z_{it}\}$. We know that B contains at most $k-3$ functions; if there are $k-2$ functions then a circular sequence results, which contradicts to a base (the discussion is analogous to P_{k2} case). Let $C := P \setminus A \setminus B$. C covers sets $\{L_q\}$. We will show that C and B together contain at most $k-2$ functions. Let $Z_{i_1 i'_1}, Z_{i_2 i'_2}, \ldots, Z_{i_l i'_l}$ $(l \leq k-3)$ be pivots for functions in B. Let $2, 3, \ldots, k-1$ be $k-2$ nodes of a graph constructed in such a way that a pair i and j is connected if and only if Z_{ij} is a pivot for a function in B. Let the graph obtained has s connected components $(s \geq 1)$. As an elementary property of graph, the number l of the pivots is $l = k-2-s$, since there is no isolated point in the graph. Now, let f be a function from C whose pivot in L' is L_{i_1}, i.e. $f \in \overline{L}_{i_1}$. From $f \in Z_{i_i, i_2}\overline{L}_{i_1}$ follows $f \in \overline{L}_{i_2}$. Hence f covers all L_i for each node i in the same connected component containing i_1. In other words, there is at most one pivot in L_i for each of the s connected components of the graph. Thus the number of the pivots in B and C together is at most $s+k-2-s = k-2$.

We show a base with rank $k+1$ in L'. We take maximal 3 functions for A, $k-3$ functions for B and a function for C. These $k-3$ functions for B are defined by the equivalence relations

$$Q_i : \{2, \ldots i\},\ \{i+1, \ldots, k-1\},\ 2 \leq i \leq k-2.$$

One function for C should be $f \in Z_{2,3}Z_{3,4} \ldots Z_{k-2,k-1}$ and $f \notin L_i$ for exactly one i (the construction is similar to P_{k2}). □

9.4. Classification of S'

Theorem 9.4.1. [Lau84b] *There are* $2 + (k-2)(k-1)$ *maximal sets of the set* S':

$$\begin{array}{lll} S'_L & := & S'L', \\ S'_{01} & := & S'T'_0\ (= S'T'_1), \\ S'Z_{it}, & & 0 \leq t < i < k,\ i \geq 2, \\ S^{(it)} & := & Pol\begin{pmatrix} 01i \\ 10t \end{pmatrix},\ 2 \leq t < i < k. \end{array}$$

We need the following property of P_2-maximal set S.

Lemma 9.4.1. [Sto85] *There are 4 classes of functions of* S *in* P_2: 11, 10, 01, 00, *where the coordinates are* S_L *and* S_{01} *in this order. The maximal rank of a base of* S *is 2.*

Lemma 9.4.2. $f \in Z_{ij} \Rightarrow f \notin S^{(ij)}$.

Proof. From $f \in Z_{ij}$ follows $f(i,\ldots,i) = f(j,\ldots,j)$. Then immediately $f \notin S^{(ij)}$. □

Lemma 9.4.3. $f \in Z_{ij} \Rightarrow f \in S^{(it)}S^{(tj)} \cup \overline{S}^{(it)}\overline{S}^{(tj)}$.

Proof. It is sufficient to prove $\overline{S}^{(it)}S^{(tj)} \subseteq \overline{Z}_{i,j}$. From $f \notin S^{(it)}$ follows that there are $\boldsymbol{a}$, $\boldsymbol{b}$ such that $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in \begin{pmatrix} 01i \\ 10t \end{pmatrix}$ and $f(\boldsymbol{a}) = f(\boldsymbol{b})$. Let $\boldsymbol{c}$ be a vector such that $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{pmatrix} \in \begin{pmatrix} 01i \\ 10t \\ 01j \end{pmatrix}$. From $f \in S^{(tj)}$ follows $\{f(\boldsymbol{b}), f(\boldsymbol{c})\} = E_2$. Therefore $\{f(\boldsymbol{a}), f(\boldsymbol{c})\} = E_2$ and from $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{c} \end{pmatrix} \in \begin{pmatrix} 01i \\ 01j \end{pmatrix}$ we conclude $f \notin Z_{ij}$. □

Lemma 9.4.4. $S^{(it)}S^{(tj)} \subseteq Z_{ij}$.

Proof. The relational product of $S^{(it)}$ and $S^{(tj)}$ equals to Z_{ij}. □

Theorem 9.4.2. *The number of classes of functions of S' is*

$$4\sum_{r=1}^{k}(A(k-2,r-2)B_{r-2} + 2(r-1)A(k-2,r-1)B_{r-1} + r(r-1)A(k-2,r)B_r),$$

where $B_r = \sum_{m=0}^{[r/2]} \begin{pmatrix} r \\ 2m \end{pmatrix} (2m)!/(2^m m!)$ is the number of possible choices of several pairs in the set of r elements.

Proof. There are 4 classes with respect to S'_{01} and S'_L. Let Q be an equivalence relation on $\{0,1,\ldots,k-1\}$ and let c_{01}, c_L, c_{it} and $c_{(it)}$ denote components corresponding to S'_{01}, S'_L, $S'Z_{it}$ and $S^{(it)}$, respectively, of a characteristic vector of a function f. Then $(0,1) \notin Q$ (constant functions are not elements of S').

From $(i,j) \in Q$ follow $c_{ij} = 0$ and $c_{(ij)} = 1$ (Lemma 9.4.2). Let $K_1,\ldots,K_r$ be equivalence classes defined by Q $(1 \le r \le k)$. Suppose $(i_1,i_2) \in Q$ and $(j_1,j_2) \in Q$. From Lemma 9.4.3 we conclude $c_{(i_1j_1)} = c_{(i_1j_2)} = c_{(j_1i_2)} = c_{(i_2j_2)}$. Therefore we can consider $c_{(K_iK_j)}$ instead of individual components $c_{(ij)}$. From Lemma 9.4.4 we get $c_{(K_iK_t)} = 0 \Rightarrow c_{(K_tK_j)} = 1$ for $i \ne t$, $t \ne j$, $j \ne i$. So the set of pairs $\{K_i, K_j\}$ from $\{K_1,\ldots,K_r\}$ such that $c_{(K_iK_j)} = 0$ has no member K_i in common between any of two pairs. The number of such possible choices for these pairs are B_r (the numbers B_r are given in Table 9.1 for $1 \le r \le 10$). We have $2 \le t < i \le k-1$ for the maximal sets $S^{(it)}$. So we must omit 0 and 1 from above consideration. There are three cases:

1) $\{0\}$ and $\{1\}$ are two equivalence classes in Q. Then after removing them we consider the equivalence relation Q'' on the set $\{2, \ldots, k-1\}$ with $r-2$ equivalence classes (member $A(k-2, r-2)B_{r-2}$),

2) $\{0\}$ is an equivalence class in Q and 1 is one of the members of a class with ≥ 2 elements. There is $r-1$ possibilities for position of 1 in one of the remaining $r-1$ classes of Q'' and the number of such equivalence relations Q'' is $(r-1)A(k-2, r-1)$ with B_{r-1} possible choices of pair-classes. Similarly we can consider the case interchanging 0 and 1.

3) 0 and 1 are members of two different equivalence classes in Q with ≥ 2 elements (0 and 1 do not enter into the same equivalence class). There still remain r equivalence classes in Q'' and number of positions of 0 and 1 in these classes is $r(r-1)$ (third member $r(r-1)A(k-2, r)B_r$).

We sketch construction of a representative function for each possible class. Let $n \geq k$.

1) $f(x_1, \ldots, x_n) := g(x_1, \ldots, x_n)$ for $x_1, \ldots, x_n \in E_2$, where g is a function on E_2 from one of the 4 possible classes of S.

2) $f \in Z_{it} \Rightarrow f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$, where $\{x_1, \ldots, x_n\} \subseteq \{0, 1, i\}$, $y_j = x_j$ for $x_j \in E_2$ and $y_j = t$ otherwise.

3) $f \in Z_{it}$ or $f \notin Z_{it}$ we can realize as before.

4) $f \in S^{(it)} \Rightarrow f(x_1, \ldots, x_n) \neq f(y_1, \ldots, y_n)$, where $\{x_1, \ldots, x_n\} \subseteq \{0, 1, i\}$, $y_j = x_j + 1 \pmod 2$ for $x_j \in E_2$ and $y_j = t$ otherwise.

5) $f \notin S^{(it)}$. If $(i, t) \in Q$ then $f \notin S^{(it)}$ is satisfied from Lemma 9.4.2.

Now, consider r equivalence classes K_j, $1 \leq j \leq r$, defined by Q. We can divide them into two groups such that if $c_{(K_i, K_j)} = 0$ then K_i and K_j are in different groups. We can define $f(0, i, \ldots, i) = f(1, i, \ldots, i) = 0$ for all numbers i in the first group and $f(0, i, \ldots, i) = f(1, i, \ldots, i) = 1$ for all numbers i in the second group. □

Example 9.4.1. Classes of functions of S' in $P_{3,2}$. In this case we have no $S^{(it)}$ maximal sets. The coordinates are in the order of S'_L, T_{01}, $Z_{2,0}$ and $Z_{2,1}$.

1111	1011	0111	0011
1110	1010	0110	0010
1101	1001	0101	0001

□

Example 9.4.2. Classes of functions of S' in $P_{4,2}$. There are $17 \cdot 4 = 68$ classes of functions of S' in $P_{4,2}$. We show only 17 classes with respect to S'-maximal sets $Z_{2,0}, Z_{2,1}, Z_{3,0}, Z_{3,1}, Z_{3,2}, S^{(3,2)}$, which are determined by an equivalence relation Q on $\{0,1,2,3\}$. Each of these vectors becomes a class of S' by appending each of two-component vectors 11, 10, 01 and 00 (corresponding S'_L and S'_{01}). We also show the corresponding relation Q for each of these classes.

$Z_{2,0}$	$Z_{2,1}$	$Z_{3,0}$	$Z_{3,1}$	$Z_{3,2}$	$S^{(3,2)}$	Q
1	1	1	1	1	1	{0}, {1}, {2}, {3}
1	1	1	1	1	0	
1	1	1	1	0	1	{0}, {1}, {2,3}
1	1	1	0	1	1	{0}, {1,3}, {2}
1	1	1	0	1	0	
1	1	0	1	1	1	{0,3}, {1}, {2}
1	1	0	1	1	0	
1	0	1	1	1	1	{0}, {1,2}, {3}
1	0	1	1	1	0	
1	0	1	0	0	1	{0}, {1,2,3}
1	0	0	1	1	1	{0,3}, {1,2}
1	0	0	1	1	0	
0	1	1	1	1	1	{0,2}, {1}, {3}
0	1	1	1	1	0	
0	1	1	0	1	1	{0,2}, {1,3}
0	1	1	0	1	0	
0	1	0	1	0	1	{0,2,3}, {1}

For a relation Q, for example, if $(2,3) \in Q$ is satisfied then $c_{(3,2)} = 1$ is uniquely possible for $S^{(3,2)}$. Otherwise both 0 and 1 are possible for $c_{(3,2)}$, because in this case $(2,3) \notin Q$ and we can choose pairs from classes $\{2\}, \{3\} \in E_4 \setminus E_2$ in two ways: take one pair $\{\{2\},\{3\}\}$ (value 0) or take no pair (value 1). □

Maximal rank of a base of S'

Lemma 9.4.5. *Let $Q_1, \ldots, Q_r$ be equivalence relations on E_k each of which consists exactly of 2 equivalence classes and satisfying the property that for each i $(1 \le i \le r)$ there exist two elements $j, l \in E_k$ such that $(j,l) \in Q_i$ and $(j,l) \notin Q_s$ $(1 \le s \le r, s \ne i)$. Then $r \le k$.*

Proof. Let us call such (j,l) as indicated in Lemma *pivot* induced by Q_i (recall that $(j,l) \in Q_i$ implies $f \notin S^{(jl)}$ for any f induced by Q_i). Let U_i and V_i denote two classes on E_k defined by Q_i and assume $0 \in U_i$. Suppose $r > k$ and consider k relations $Q_1, \dots, Q_k$. There is a circular sequence in the set of pair sets $\{(j,l)\}$, where (j,l) is a pivot induced by Q_i. We denote this sequence by $(0,1),(1,2),\dots,(m-1,m),(m,0)$ and assume $(j,j+1) \in Q_{j+1}$, $0 \le j \le m-1$; $(m,0) \in Q_{m+1}$ (because of isomorphism we can do this). Consider Q_{k+1}. Then $0,2,4,\dots,m-1 \in U_{k+1}$ and $1,3,\dots,m \in V_{k+1}$. So, if m is even then we have a contradiction. Thus m is odd. Consider Q_2. $0 \in U_2$, $1 \in V_2$, $2 \in V_2$ (because (1,2) is a pivot of Q_2), $4 \in V_2$ (because (3,4) is a pivot of Q_4). Thus for $i \ge 3$ odd number belongs to U_2 and even number to V_2. So $m-1 \in V_2$ and $m \in U_2$. Since $(m,0)$ is a pivot of Q_{m+1} we get $0 \in V_2$. Again this is a contradiction. □

Theorem 9.4.3. *Maximal rank of a base of S' is $\le 2k$.*

Proof. We know that rank of a base for S'_L and S'_{01} is ≤ 2 (Lemma 9.4.1), and for $\{Z_{ij}\}$ is $\le k-1$ (Lemma 9.1.2). Assume that a base for $\{Z_{ij}\}$ has rank $k-1$. Then there is a sequence $\{0,i_1\}$, $\{i_1,i_2\},\dots,$ $\{i_{k-2},1\}\}$ of pivots $Z_{i_l,i_{l+1}}$ for f_l, $l = 0,\dots,k-2$; $i_0 = 0, i_{k-1} = 1$ from Lemma 8.4.2. But from $(0,1) \notin Q_f$ for every Q_f there exists l $(0 \le l \le k-2)$ such that $(i_l, i_{l+1}) \notin Q_f$, i.e. $f \notin Z_{i_l i_{l+1}}$. Thus $Z_{i_l i_{l+1}}$ is not a pivot of f_l for $\{S_L, S_{01}, Z_{ij},\ 0 \le j,i \le k-1, i \ge 2\}$. (similar proof as in P_{k2}). Thus S'_L, S'_{01} and $\{Z_{ij}\}$ have maximal rank k (for $k=3$ there exists no $S^{(j,l)}$ maximal set and hence maximal rank of a base is $k=3$, which can also be seen from the computational data in Table 9.3).

Consider sets $S^{(lj)}$. Let $f_1,\dots,f_r$ be functions which have a pivot from $S^{(lj)}$ in S'. We prove $r \le k$. Let $Q_1,\dots,Q_r$ be equivalence relations for $f_1,\dots,f_r$. The condition $c_{(lj)} = 0$ can be satisfied for $l \in K_s$ and $j \in K_t$, where K_s and K_t $(\subseteq E_k,\ s \ne t)$ are different equivalence classes of a relation Q_i. The set of pairs of such different equivalence classes $\{\{K_{s_1}, K_{t_1}\},\dots,\{K_{s_r}, K_{t_r}\}\}$ are mutually disjoint, as it has been proved in the proof of Theorem 9.4.2. $f_1,\dots,f_r$ will again be pivots for the same sets $\{S^{(lj)}\}$ if we replace every $c_{(lj)} = 1$ by $c_{(lj)} = 0$ for any function except when $c_{(lj)}$ is a pivot. Some "1" among $c_{(lj)}$ will became 0 by this replacement. This corresponds that we consider new equivalence relations $Q''_1,\dots,Q''_r$ such that Q''_i consist exactly of two equivalence classes on E_k. Let $f''_1,\dots,f''_r$ be new functions taken out from these new classes. Since the replacement of the values does not effect the pivotality, new pivot of f_i'' coincides with that of f_i. If $S^{(lj)}$ is a pivot of f_i'' then $c_{(lj)} = 1$ for f_i''. Since Q_i'' has only two equivalence relations and since $c_{(lj)} = 0$ is satisfied only for $(l,j) \notin Q_i''$, we have $(l,j) \in Q_i''$. From the property of pivot, $c_{(lj)} = 0$ is satisfied for other functions f_s'', hence $(l,j) \notin Q_s''$ for $s \neq i$. From Lemma 9.4.5 we conclude $r \leq k$. □

9.5. Classifications of Symmetric Functions of P_{k2}

In this section we determine classes of functions for the set of symmetric functions in P_{k2} and its all maximal sets except M'. The problem in the classification of symmetric functions of P_{k2} is mainly related to the fact that there exists only one representative $f(x) := x$ (identity function) in the T_0T_1LSM-class ("identity class") of symmetric functions of P_2. Since we used n-ary functions of P_2 for $n > 2$ (cf. Theorem 8.3.1) in the construction of representatives of the classes of functions of P_{k2}, we need a separate consideration for the set of symmetric P_{k2}-functions corresponding to this identity class.

First we recall some notions about symmetric functions. A function $f(x_1, \ldots, x_n)$ is said to be symmetric if $f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ holds for all $x_1, \ldots, x_n \in E_k$ and every permutation π of $\{1, \ldots, n\}$.

S-base (S-pivotal set) is a base (pivotal set) consisting solely of symmetric functions. Hence a class of S-base (S-pivotal) is a set of classes of functions each of which contains a symmetric function. Thus we need to determine classes of functions for the set of symmetric functions. We use the following fact (this is a corollary of Theorem 3.3.1).

Lemma 9.5.1. [Tos72] *Each of 15 classes of functions of P_2 contain a symmetric function. Unary function $f(x) := x$ is a unique symmetric function of the class T_0T_1LSM. The other 14 classes contain symmetric functions of n variables for any given n ($n > 1$).*

9.5.1. Classification of symmetric functions of P_{k2}

Theorem 9.5.1. *The number of classes of symmetric functions in P_{k2} is*

$$12A_k - 10A_{k-1} + 2^{k-2}.$$

Proof. As we have seen in Theorem 8.3.2, classes with respect to $\{Z_{it}\}$ are determined by an equivalence relation on E_k (numbers of the classes are A_{k-1} if $(0,1) \in Q$ and $A_k - A_{k-1}$

if $(0,1) \notin Q$). Further, there are two P_2-classes (corresponding to constant functions) for each former class and 13 P_2-classes for each latter class. However, among these 13 classes, the class T_0T_1SLM contains only a unary symmetric function ($f(x) := x$). We can show that there is a symmetric representative in each P_{k2}-class corresponding to the other 14 classes, because they contain n-ary symmetric functions for any $n > 1$. Thus consider the set of symmetric functions in P_{k2} defined by $\{pr^{-1}(f(x) := x)\}$. It is easy to see that $f \in Z_{it} \Leftrightarrow (i,t) \in Q_f \Leftrightarrow f(i) = f(t)$. Thus in this case an equivalence relation Q is defined exactly by two equivalence classes: $I_0 := \{i \mid f(i) = 0\}$ and $I_1 := \{i \mid f(i) = 1\}$. The number of such equivalence classes is 2^{k-2}, because $f(0) = 0$ and $f(1) = 1$. The assertion of the theorem follows from this and Theorem 8.3.2. □

Note 9.5.1. The number of classes of functions of P_{k2} which contain no symmetric function is $A_k - A_{k-1} - 2^{k-2}$.

9.5.2. Symmetric functions of Z_{it}

Theorem 9.5.2. *The number of classes of symmetric functions of Z_{it} is*

$$2^{k-3}(12A_{k-1} - 10A_{k-2}) + 2^{k-3}.$$

Proof. Again, consider symmetric functions of P_{k2} corresponding to unary function $f(x) := x$ of the set T_0T_1LSM. Let $I_0 = \{j \mid f(j) = 0\}$ and $I_1 = \{j \mid f(j) = 1\}$. Obviously $0 \in I_0$ and $1 \in I_1$. From $f \in Z_{it}$, either $i,t \in I_0$ or $i,t \in I_1$ is satisfied, because $f(i) = f(t)$. Further, $f \in Z_{jl}$ if either $j,l \in I_0$ or $j,l \in I_1$. It follows immediately that $f \in R_j$ for every j. The assertion of the theorem follows obviously from these considerations and Theorem 9.1.3. □

Note 9.5.2. The number of classes of functions of Z_{it} which contain no symmetric function is $2^{k-3}(A_{k-1} - A_{k-2}) - 2^{k-3}$.

9.5.3. Symmetric functions of T_0' in P_{k2}

Theorem 9.5.3. *The number of classes of symmetric functions in T_0' is*

$$9\sum_{r=1}^{k-1} A(k-1,r)2^{r-1} + 2^{k-2}.$$

Proof. Classes of T_0 which contain symmetric functions with $n \geq 2$ variables can correspond to classes of symmetric functions of T_0 in P_{k2} by the same construction as in Theorem 9.2.2. However, in case of the class with only one variable we need another construction. Only the class 0000 contains no symmetric function with $n \geq 2$ variables; the identity function $f(x) := x \in P_2$ is a unique symmetric function in this class. Again let $I_0 = \{i \mid f(i) = 0\}$ and $I_1 = \{i \mid f(i) = 1\}$ ($I_0 \cup I_1 = E_k$). The induced equivalence relation Q_f divides $E_k \setminus \{0\}$ into exactly two classes and in this case there are 2^{k-2} such Q_f ($0 \in I_0$ and $1 \in I_1$). The assertion of the theorem follows from this and Theorem 9.2.2. □

Note 9.5.3. The number of classes of functions of T_0' which contain no symmetric function is $\sum_{r=1}^{k-1} A(k-1,r)2^{r-1} - 2^{k-2} = \sum_{r=1}^{k-2} A(k-1,r)2^{r-1}$.

9.5.4. Symmetric functions of L'

Theorem 9.5.4. *The number of classes of symmetric functions of L' is*

$$4\sum_{r=1}^{k-2} A(k-2,r)(2^r+1).$$

Proof. The following classes of L in P_2: 0001, 0111, 1011 and 1101 contain symmetric functions with $n \geq 3$ variables, hence $4\sum_{r=1}^{k-2} A(k-2,r)2^r$ classes contain symmetric functions. The other classes 0000, 0110, 1010 and 1100 contains only symmetric functions $\{0, 1, x, x+1\}$ of only one variables. Hence f must have only one variable because $f = g$ on $\{0,1\}$ for $g \in P_2$. In this case $f \in L_2L_3 \ldots L_{k-1}$. Hence the number of symmetric class in this case is $4\sum_{r=1}^{k-2} A(k-2,r)$. The assertion of the theorem follows from this and Theorem 9.3.2. □

Note 9.5.4. The number of classes of functions of L' which contain no symmetric function is $4\sum_{r=1}^{k-2} A(k-2,r)(2^r-1)$.

9.5.5. Symmetric functions of S'

All classes of S' contain a symmetric function, because all classes of functions with respect to S_L' and S_{01}' contain symmetric functions with n variables for any $n \geq 1$ [Sto85]. Hence the classes of functions and the classes of symmetric functions coincide in this case (the number of them is given in Theorem 9.4.2).

8.6. Concluding remarks

Classifications are done for a few general cases of closed sets of P_k [Sto86c] (also cf. [MSLR87]). In [MiS87b] classes of functions of P_{k2} and their exact number is determined. In this chapter we have determined classes of functions and classes of symmetric functions for maximal sets of P_{k2} (all except M'). We have seen that although the numbers of maximal sets and the numbers of classes of functions for both P_{k2} and its maximal sets grow rapidly as $O(k^2)$ and $O(k!)$, respectively, maximal ranks of a base for both P_{k2} and its maximal sets have been proved to be $O(k)$.

In the following Table 9.2 we give the numbers $\mu(X)$ of X-maximal sets, $\gamma(X)$ of classes of functions of X and $\sigma(X)$ of classes of functions of X containing symmetric functions, where X denote P_k, P_{k2} and some maximal sets of P_{k2} for $1 \leq k \leq 10$. We note that these numbers of the maximal sets of P_k are given in [Ros73,Ros77], the number of classes of functions of P_2 in [INN63,Krn65], the number of classes of functions of P_3 in [Miy71,Sto84a], the numbers of the classes of symmetric functions of P_2 and P_3 in [Tos72] and [Sto85], respectively.

The numbers A_k and B_k needed for the computation of these data are given in Fig. 9.1.

The numbers of classes of bases, pivotal incomplete sets, S-bases and S-pivotal incomplete sets for the sets $P_{3,2}$ and $P_{4,2}$ and for some their maximal sets are shown in the following Table 9.3. One of the algorithms described in [StM86a] is used. The symbol * in the table denotes that S-bases (S-pivotals) and bases (pivotals) coincide on the set marked by it.

In the last Table 9.4 we give the characteristic vectors of the classes of maximal sets $Z_{3,0}$ and T'_0 both in the set $P_{4,2}$.

Table 9.1: A_k and B_k ($0 \leq k \leq 10$).

k	0	1	2	3	4	5	6	7	8	9	10
A_k	-	1	2	5	15	52	203	877	4,140	21,147	115,975
B_k	1	1	2	4	10	26	76	232	764	2,620	9,496

Table 9.2: Numbers of maximal sets, classes and classes of symmetric functions.

k	1	2	3	4	5	6	7	8	9	10
$\mu(P_k)$	-	5	18	82	643	7,848,984				
$\gamma(P_k)$	-	15	406	?	?	?	?	?	?	?
$\sigma(P_k)$	-	15	394	?	?	?	?	?	?	?
$\mu(P_{k2})$	-	5	7	10	14	19	25	32	40	49
$\gamma(P_{k2})$	-	15	43	140	511	2,067	9,168	44,173	229,371	1,275,058
$\sigma(P_{k2})$	-	15	42	134	482	1,932	8,526	40,974	212,492	1,180,486
$\mu(Z_{it})$	-	-	5	8	12	17	23	30	38	47
$\gamma(Z_{it})$	-	-	15	86	560	4,088	33,072	293,376	2,827,072	29,359,488
$\sigma(Z_{it})$	-	-	15	82	524	3,800	30,672	271,840	2,618,304	27,182,720
$\mu(S')$	-	2	4	8	14	22	32	44	58	74
$\gamma(S')$	-	4	12	68	388	2,492	17,676	136,500	1,138,916	10,203,420
$\sigma(S')$	-	4	12	68	388	2,492	17,676	136,500	1,138,916	10,203,420
$\mu(T_0')$	-	4	6	9	13	18	24	31	39	48
$\gamma(T_0')$	-	10	30	110	480	2,270	12,150	71,070	449,590	3,050,910
$\sigma(T_0')$	-	10	29	103	440	2,059	10,967	64,027	404,759	2,746,075
$\mu(L')$	-	4	5	7	10	14	19	25	32	40
$\gamma(L')$	-	8	16	48	176	752	3,632	19,440	113,712	719,344
$\sigma(L')$	-	8	12	32	108	436	2,024	10,532	60,364	376,232
$\mu(M')$	-	4	7	13	22	34	49	67	88	112
$\gamma(M')$	-	?	?	?	?	?	?	?	?	?
$\sigma(M')$	-	?	?	?	?	?	?	?	?	?

Table 9.3: Numbers of bases, pivotals, S-bases and S-pivotals.

rank	1	2	3	4	5	6	Σ
bases $P^*_{2,2}$	1	17	22	2	-	-	42
pivotals $P^*_{2,2}$	13	31	7	-	-	-	51
bases $P_{3,2}$	1	160	804	272	8	-	1,245
S-bases $P_{3,2}$	1	158	770	228	4	-	1,161
pivotals $P_{3,2}$	42	440	435	38	-	-	955
S-pivotals $P_{3,2}$	41	416	374	24	-	-	855
bases $P_{4,2}$	1	1,572	42,822	56,228	6,284	64	106,971
S-bases $P_{4,2}$	1	1,533	39,501	42,652	3,132	16	86,835
pivotals $P_{4,2}$	139	6,336	30,660	10,798	314	-	48,247
S-pivotals $P_{4,2}$	133	5,721	24,293	6,202	126	-	36,475
$P_{3,2}$							
bases $Z^*_{2,0}$	1	17	22	2	-	-	42
pivotals $Z^*_{2,0}$	13	31	7	-	-	-	51
bases T_0	1	98	217	30	-	-	346
S-bases T_0	1	96	198	18	-	-	313
pivotals T_0	29	174	73	-	-	-	276
S-pivotals T_0	28	158	53	-	-	-	239
bases L'	-	27	45	3	-	-	75
S-bases L'	-	15	12	-	-	-	27
pivotals L'	15	46	9	-	-	-	70
S-pivotals L'	11	21	3	-	-	-	35
bases S'^*	1	20	4	-	-	-	25
pivotals S'^*	11	13	-	-	-	-	24
$P_{4,2}$							
bases $Z_{3,0}$	1	522	8,506	9,314	932	8	19,283
S-bases $Z_{3,0}$	1	509	7,733	6,508	280	-	15,031
pivotals $Z_{3,0}$	85	2,181	6,780	1,938	40	-	11,024
S-pivotals $Z_{3,0}$	81	1,963	5,171	874	4	-	8,093
bases T_0	1	1,174	19,253	16,013	952	-	37,398
S-bases T_0	1	1,127	16,436	8,656	392	-	26,610
pivotals T_0	109	3,600	10,802	1,916	-	-	16,427
S-pivotals T_0	102	3,061	7,219	967	-	-	11,349
bases L'	-	171	1,845	912	33	-	2,961
S-bases L'	-	75	393	96	-	-	564
pivotals L'	47	648	938	96	-	-	1,729
S-pivotals L'	31	243	198	3	-	-	475
bases S'^*	1	639	3,430	400	2	-	4,472
pivotals S'^*	67	1,140	762	10	-	-	1,979
$P_{5,2}$							
bases S'^*	1	19,246	1,083,933	1,102,264	47,832	118	2,253,394
pivotals S'^*	387	49,740	371,903	71,650	519	-	494,199

Table 9.4:
(* at the end of the vector denotes that the class has no symmetric representative.)

Classes of functions of $Z_{3,0}$ in $P_{4,2}$
(coordinates are $T_0', T_1', S', L', M', Z_{2,0}, Z_{2,1}, R_2$).

11111111	11011011	10111101	10100111	01101111	00111111	00110011	00010101	00000111*
11111110	11011010	10111100	10100110	01101110	00111110	00110010	00010100	00000110*
11111101	11001111	10111011	10100001	01101101	00111101	00011111	00010011	00000101*
11111100	11001110	10111010	10100000	01101100	00111100	00011110	00010010	00000100
11111011	11001101	10101111	01111111	01101011	00111011	00011101	00001111	00000011*
11111010	11001100	10101110	01111110	01101010	00111010	00011100	00001110	00000010
11011111	11001011	10101101	01111101	01100111	00110111	00011011	00001101	
11011110	11001010	10101100	01111100	01100110	00110110	00011010	00001100	
11011101	10111111	10101011	01111011	01100001	00110101	00010111	00001011	
11011100	10111110	10101010	01111010	01100000	00110100	00010110	00001010	

Classes of functions of T_0' in $P_{4,2}$
(coordinates are $T_{01}, L', M', N_0, Z_{2,1}, Z_{3,1}, Z_{3,2}, T_{03}$ and T_{04}).

111111111	111111110	111111101	111111100	111111011	111111000
111011111	111011110	111011101	111011100	111011011	111011000
101111111	101111110	101111101	101111100	101111011	101111000
100011111	100011110	100011101	100011100	100011011	100011000
011111111	011111110	011111101	011111100	011111011	011111000
011011111	011011110	011011101	011011100	011011011	011011000
010111111	010111110	010111101	010111100	010111011	010111000
010011111	010011110	010011101	010011100	010011011	010011000
001111111	001111110	001111101	001111100	001111011	001111000
000011111*	000011110*	000011101*	000011100*	000011011*	000011000

111101110	111101111	111110111	111110101	111100011
111001110	111001111	111010111	111010101	111000011
101101110	101101111	101110111	101110101	101100011
100001101	100001100	100010110	100010100	100000000
011101110	011101111	011110111	011110101	011100011
011001110	011001111	011010111	011010101	011000011
010101110	010101111	010110111	010110101	010100011
010001110	010001111	010010111	010010101	010000011
001101110	001101111	001110111	001110101	001100011
000001110	000001111*	000010111*	000010101	000000011

Chapter 10

Concluding discussions, an overview and some open problems

The number of P_k-maximal sets was approximated in [ZKJ69,ZKJ71] and the exact formula for it was determined in [Ros73]:

k	2	3	4	5	6	7
maximal sets	5	18	82	643	15,182	7,848,984

Classification of P_k is barely possible for $k = 4$.

There are several classification results for subsets in P_k [Sto86c,Sto85,MiS87b,MSL87]. A function is *linear* if there are $a_0, \dots, a_n \in E_k$ so that, under a certain abelian structure on E_k,

$$f(x_1, \dots, x_n) = a_0 + a_1 x_1 + \dots + a_n x_n$$

holds for all $x_1, \dots, x_n \in E_k$. The set of linear functions has been investigated (cf. [BaD78,BaD80,Lau84b]). It is P_k-maximal if and only if k is a prime power [Jab58]. Let L denote the set of linear functions of P_k and $T_m = \{f \mid f(m, \dots, m) = m\}$ the set of functions preserving m $(0 \le m \le k-1)$.

Theorem 10.1. [BaD78,BaD80] *There are exactly $p+2$ maximal sets of L in prime-valued logic P_p:*

$$\begin{array}{rl} L_m = & LT_m,\ 0 \le m \le p-1, \\ L_S = & LS = \{a_0 + a_1x_1 + \dots + a_nx_n \mid a_1 + \dots + a_n = 1\} \\ & \text{(the set of linear selfdual functions)}, \\ L^{(1)} = & \{a_0 + a_1 x_i \mid a_0, a_1 \in E_p,\ i > 0\} \\ & \text{(the set of essentially unary linear functions)}. \end{array}$$

There are exactly $2p+4$ classes of functions of the set L [Sto86c]. Their characteristic vectors listed with respect to the above order of maximal sets are:

$$\begin{array}{ll} 1: & 0^{p+2} \text{ (i.e. } p+2 \text{ zeros)} \\ 2: & 0^{p+1}1 \\ 3 \leq r \leq p+3: & 1^{r-3}01^{p+3-r}0 \\ p+4 \leq r \leq 2p+4: & 1^{r-p-4}01^{2p+5-r}. \end{array}$$

Let $f(x_1,\ldots,x_n) = a_0+a_1x_1+\ldots+a_nx_n$ be a linear function in P_p. The function x is in class 1, and the function $a_1x_1+\ldots+a_nx_n$ is in the class 2 for $n \geq 2$ and $a_1+\ldots+a_n = 1$. The functions a_0+x are in class $p+3$ for $a_0 \neq 0$, and the functions $a_0+a_1x_1+\ldots+a_nx_n$ for $a_0 \neq 0$ and $a_1+\ldots+a_n = 1$, $n \geq 2$ are in class $2p+4$. The constant function $f = i$ belongs to class $i+3$ $(0 \leq i \leq p-1)$. Let $a_1+\ldots+a_n \neq 1$ and let a be the number determined uniquely by $a(1-a_1-\ldots-a_n) = a_0$, i.e. $a_0+a_1a+\ldots+a_na = a$ $(a \in E_p)$. Then the function $f(x_1,\ldots,x_n) = a_0+a_1x_1+\ldots+a_nx_n$ belongs to class $p+4+a$, because it preserves $\{a\}$.

No Sheffer function for L exists. However, each $f \in L\backslash L^{(1)}$ is c-Sheffer as $0 \notin T_m$ $(m \geq 1), 1 \notin T_0, 0 \notin S$. The number of such n-ary functions is $p^{n+1}-np(p-1)-p$ $(n \geq 2)$. As $n \to \infty$ the proportion of c-Sheffer n-ary linear functions (among n-ary linear functions) goes rapidly to 1.

Bases of rank 2 are composed of any two functions of classes i and j, where i and j satisfy the condition

a) $p+4 \leq i < j \leq 2p+4$, or
b) $3 \leq i \leq p+3 < j \leq 2p+4$ and $j \neq i+p+1$.

Bases of rank 3 contain a function of class 2 and two functions, each from classes i and j, where $3 \leq i < j \leq p+3$. Thus L contains exactly $4\binom{p+1}{2}$ aggregates; $3\binom{p+1}{2}$ of rank 2 and $\binom{p+1}{2}$ of rank 3. The maximal rank of a base of L is 3.

The H-maximal sets for the above $p+2$ L-maximal sets H (p prime) are determined in [BaD78] and their classification is in [Sto86c].

Let $S = Pol\begin{pmatrix} 0 & 1 & \ldots & k-2 & k-1 \\ 1 & 2 & \ldots & k-1 & 0 \end{pmatrix}$. It is easy to verify that S is a set of selfdual functions in k-valued logic (i.e. f such that $f(x_1+1,\ldots,x_n+1) = f(x_1,\ldots,x_n)+1$). Note that there are other types of selfdual functions (cf. [Ros70]).

Theorem 10.2. [Sze82] *There are exactly two S-maximal sets in prime-valued logic P_p:*

$$S_L = SL \text{ and } S_0 = ST_0.$$

A linear function $a_0 + a_1x_1 + \ldots + a_nx_n$ is selfdual if $a_1 + \ldots + a_n = 1$. When this holds, the function $a_1x_1 + \ldots + a_nx_n$ belongs to the set S_LS_0 (class 00) and the functions $a_0 + a_1x_1 + \ldots + a_nx_n$ for $a_0 \neq 0$ belong to the set $S_L\overline{S}_0$ (class 01).

The number of n-ary Sheffer functions in S is $(p-1)p^{n-1}(p^{p^{n-1}-1} - 1)$. Note that the notions of c-Shefferness and Shefferness coincide, because no constant function belongs to S. There are exactly two aggregates for S; each for ranks 1 and 2.

An overview and some open problems

We give some subsets of P_k whose maximal sets are known. Perhaps the most interesting P_k-maximal set is the set L of linear functions for k not prime. Let $k = p_1^{\alpha_1} \ldots p_m^{\alpha_m}$, $\alpha_1, \ldots, \alpha_m \geq 1, p_1, \ldots, p_m$: prime numbers ($p_i \neq p_j$ for $i \neq j$). All the maximal sets of L are described as follows [Lau84b]:

1) $2^m - 1$ maximal sets
$T_d := L_d \cup \bigcup_{n\geq 1}\{f \in L \mid \exists b, a_0, \ldots, a_n : \; b|d \;\wedge\; b \neq 1 \;\wedge f(\boldsymbol{x}) = a_0 + b\sum_{i=1}^{n} a_ix_i\}$,
where $\boldsymbol{x} = (x_1, \ldots, x_n)$ and
$L_d := \bigcup_{n\geq 1}\{f \in L \mid \exists a_0, \ldots, a_n, j \;:\; f(\boldsymbol{x}) = a_0 + a_jx_j + d\sum_{i=1, i\neq j}^{n} a_ix_i\}$,
$d = p_{i_1} \ldots p_{i_t}$, $\{p_{i_1}, \ldots, p_{i_t}\} \subseteq \{p_1, \ldots, p_m\}$, $1 \leq t \leq m$.

2) m maximal sets of type
$L_{*,p_i} := \bigcup_{n\geq 1}\{f \in L \mid \exists a_0, \ldots, a_n \in E_k :$
$f(\boldsymbol{x}) = a_0 + \sum_{i=1}^{n} a_ix_i \wedge a_1 + \ldots + a_n = 1 \pmod{p_i}\}$, $1 \leq i \leq m$.

3) $p_1 + \ldots + p_m$ maximal sets
$L \cap Pol(j, p_i + j, 2p_i + j, \ldots, k - p_i + j)$ for all j
satisfying $0 \leq j \leq p_i - 1$, $1 \leq i \leq m$.

The special case of $k = p^m$ or $k = 2 \cdot p$ ($m > 1$, $p > 2, p$: prime) is also investigated in [Lau84b] and [Schr87].

Another interesting maximal set is the set of special selfdual functions S for k not a prime number [Lau84b] (for the case k prime number we have the simple result as described above [Sto85b]). All $2\prod_{i=1}^{m}(\alpha_i + 1) - 3$ maximal sets of S are described as

$$\{S \cap Pol\;\gamma_r,\; S \cap Pol\;\rho_t \;\mid\; r \in T\setminus\{1\},\; t \in T\setminus\{1, k\}\},$$

where $T := \{x \mid k \equiv 0 \pmod{x}\}$, $\gamma_r := \{x \in E_k \mid x \equiv 0 \pmod{r}\}$ and $\rho_t := \{(x,y) \in E_k^2 \mid y - x \equiv 0 \pmod{t}\}$.

Some cases of selfdual functions are also described in [Mar79].

Compositions of partial k-valued functions are investigated in [Fre66,Lou84,Rom80]. Define $P_{k,l} := \bigcup_{n\geq 1}\{f | f : E_k^n \to E_l\}$, $l > k$, with the operation of composition defined by:

$$f \circ g = \begin{cases} f * g & \text{if } W(g) \subseteq \{0,\ldots,k-1\}, \\ f & \text{otherwise,} \end{cases}$$

where $W(g)$ denotes the range of g. $P_{k,l}$ is a generalization of the partial k-valued logic. $P_{2,l}$ has exactly the following 8 maximal sets [Lau77,Fre66]:

$$\{f \in P_{2,l} \mid |W(f)| \leq l-1\},\ Pol^*(0),\ Pol^*(1),$$
$$Pol^*\begin{pmatrix} 0 \\ 1 \end{pmatrix},\ Pol^*\begin{pmatrix} 01 \\ 10 \end{pmatrix},\ Pol^*\begin{pmatrix} 001 \\ 011 \end{pmatrix},$$
$$Pol^*\begin{pmatrix} 000111 \\ 001101 \\ 010011 \\ 011001 \end{pmatrix},\ Pol^*\begin{pmatrix} 00011011 \\ 00110101 \\ 01001101 \\ 01100011 \end{pmatrix},$$

where

$$\begin{aligned} Pol^*\rho \ := \ & \{f \in P_{2,l} \mid \\ & (\boldsymbol{a}_1,\ldots,\boldsymbol{a}_h)^T \in \rho^n \Rightarrow (f(\boldsymbol{a}_1),\ldots,f(\boldsymbol{a}_h))^T \in \rho \cup (\{0,\ldots,l-1\}^h \setminus \{0,1\}^h)^T\}. \end{aligned}$$

The set $P_{2,l}$ is being classified [LMS87]. Besides, it is known that $P_{3,l}$, $l > 3$, has exactly 58 maximal sets ([Lau77], a slightly different number of the maximal sets is reported in [Rom80]).

Define $P_3(2) := \bigcup_{n\geq 1}\{f(x_1,\ldots,x_n) \in P_3 \mid |W(f)| \leq 2\}$. $P_3(2)$ has exactly the following 13 maximal sets [Fre66,Lau77]:

$$\begin{aligned} \bigcup_{n\geq 1}\{f \ \mid \ & \exists f_0,\ldots,f_n \in P_3^1(2)(\text{the set of unary functions}) : \\ & f(x_1,\ldots,x_n) = f_0(f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n) \bmod 2)\} \end{aligned}$$

and the classes $P_3(2) \cap Pol\rho$ where $\rho \in$

$$\{\begin{pmatrix} 012001 \\ 012122 \end{pmatrix}, \begin{pmatrix} 012112 \\ 012200 \end{pmatrix}, \begin{pmatrix} 012220 \\ 012011 \end{pmatrix}, \begin{pmatrix} 01201 \\ 01210 \end{pmatrix},$$
$$\begin{pmatrix} 01202 \\ 01220 \end{pmatrix}, \begin{pmatrix} 01212 \\ 01221 \end{pmatrix}, (0\ 1), (0\ 2), (1\ 2), \begin{pmatrix} 0120102 \\ 0121020 \end{pmatrix},$$

$$\begin{pmatrix} 0120121 \\ 0121012 \end{pmatrix}, \begin{pmatrix} 0120212 \\ 0122021 \end{pmatrix}\}.$$

Define $P_{\{0,1\},\{a,3\}} := \bigcup_{n,m\geq 1}\{f^{n,m} \mid f^{n,m} : \{0,1\}^n \times \{a,3\}^m \to \{0,1\}\}$, $a \in \{0,2\}$, and with a similar generalization of superposition. $P_{\{0,1\},\{a,3\}}$ has exactly 10 maximal sets for $a = 0$ and 21 maximal ones for $a = 2$ ([BBK73] also cf. [Lau80]). The maximal sets for $Pol\ (0)$ are also known [Lau82a].

Finding maximal sets for other subsets of P_k and under various modifications of composition are open problems. Among them we find part of automata theory [Das81,Kud60], where some maximal sets are given. Uniform delay composition with unit-delay for P_3 was solved in [Noz70], and with positive-integer-delays for P_3 in [Hik78] (30 and 49 maximal sets). Composition with delay was also treated in the general case in [MRR83,RoH83].

The enumeration of Sheffer functions as well as c-Sheffer may be considered in many of the above cases (cf. [Ros77]). For example, the number of n-ary 3-valued c-Sheffer functions is known only for n=2 [Muz75].

Maximal rank of a base is an open problem in many cases. The problem is mentioned early in [Jab58,But60], especially for P_k. It is known that some closed subsets of P_k, $k \geq 3$ have an infinite base or no base [JaM59]. It is also known that for $k \geq 8$ some P_k-maximal sets have no finite basis [Mik86,Tar86].

Classification and basis enumeration can be used to calculate the number of n-ary bases [StM86a,Wer42,KuO66,PeS68,Ber80,Ber83]. In many cases, this has not yet been done. The corresponding classifications and basis enumerations for symmetric functions are surveyed in [StM86b]. The classification of P_3 may be shortened if one uses relational calculation extensively as we had done for the maximal set T_0 in Section 6.5.

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Appendix 1. Classes of P_3.

wt	#no	TLS	$M_1M_2M_0$	$U_2U_0U_1$	$B_0B_1B_2$	$T_0T_1T_2$	$T_{01}T_{12}T_{20}$	*no	representative
18	#1	111	111	111	111	111	111	*406	$f8.14$ (Sheffer)
17	#2	111	111	111	111	111	110	*405	σ_0-similar of $f8.13$
17	#3	111	111	111	111	111	101	*404	σ_1-similar of $f8.13$
17	#4	111	111	111	111	111	011	*403	$f8.13$
17	#5	111	111	111	111	110	111	*397	σ_1-similar of $f8.12$
17	#6	111	111	111	111	101	111	*402	σ_2-similar of $f8.12$
17	#7	111	111	111	111	011	111	*392	$f8.12$
17	#8	111	111	110	111	111	111	*322	σ_2-similar of $f6.31$
17	#9	111	111	101	111	111	111	*236	$f6.31$
17	#10	111	111	011	111	111	111	*279	σ_1-similar of $f6.31$
17	#11	110	111	111	111	111	111	*86	$f4.4$
16	#12	111	111	111	111	110	110	*396	σ_3-similar of $f8.10$
16	#13	111	111	111	111	110	101	*394	σ_1-similar of $f8.10$
16	#14	111	111	111	111	110	011	*395	σ_1-similar of $f8.11$
16	#15	111	111	111	111	101	110	*400	σ_2-similar of $f8.11$
16	#16	111	111	111	111	101	101	*401	σ_4-similar of $f8.10$
16	#17	111	111	111	111	101	011	*399	σ_2-similar of $f8.10$
16	#18	111	111	111	111	100	111	*375	$f8.8$
16	#19	111	111	111	111	011	110	*391	σ_0-similar of $f8.10$
16	#20	111	111	111	111	011	101	*390	$f8.11$
16	#21	111	111	111	111	011	011	*389	$f8.10$
16	#22	111	111	111	111	010	111	*387	σ_2-similar of $f8.8$
16	#23	111	111	111	111	001	111	*381	σ_1-similar of $f8.8$
16	#24	111	111	111	110	110	111	*348	σ_1-similar of $f7.9$
16	#25	111	111	111	101	101	111	*361	σ_2-similar of $f7.9$
16	#26	111	111	111	011	011	111	*335	$f7.9$
16	#27	111	111	110	111	111	110	*311	σ_2-similar of $f6.23$
16	#28	111	111	110	111	111	101	*319	σ_4-similar of $f6.28$
16	#29	111	111	110	111	111	011	*316	σ_2-similar of $f6.28$
16	#30	111	111	110	111	101	111	*321	σ_2-similar of $f6.30$
16	#31	111	111	101	111	111	110	*233	σ_0-similar of $f6.28$
16	#32	111	111	101	111	111	101	*225	$f6.23$
16	#33	111	111	101	111	111	011	*230	$f6.28$
16	#34	111	111	101	111	011	111	*235	$f6.30$
16	#35	111	111	011	111	111	110	*276	σ_3-similar of $f6.28$
16	#36	111	111	011	111	111	101	*273	σ_1-similar of $f6.28$
16	#37	111	111	011	111	111	011	*268	σ_1-similar of $f6.23$
16	#38	111	111	011	111	110	111	*278	σ_1-similar of $f6.30$
16	#39	101	111	111	111	110	111	*83	σ_1-similar of $f4.1$
16	#40	101	111	111	111	101	111	*82	σ_2-similar of $f4.1$
16	#41	101	111	111	111	011	111	*81	$f4.1 = x + 2y$
16	#42	100	111	111	111	111	111	*84	$f4.2 = 2x + 2y + 1$
15	#43	111	111	111	111	110	100	*393	σ_1-similar of $f8.9$
15	#44	111	111	111	111	101	001	*398	σ_2-similar of $f8.9$
15	#45	111	111	111	111	100	110	*374	σ_0-similar of $f8.6$
15	#46	111	111	111	111	100	101	*373	$f8.7$
15	#47	111	111	111	111	100	011	*372	$f8.6$
15	#48	111	111	111	111	011	010	*388	$f8.9$
15	#49	111	111	111	111	010	110	*385	σ_2-similar of $f8.7$
15	#50	111	111	111	111	010	101	*386	σ_4-similar of $f8.6$

wt	#no	TLS	$M_1M_2M_0$	$U_2U_0U_1$	$B_0B_1B_2$	$T_0T_1T_2$	$T_{01}T_{12}T_{20}$	*no	representative
15	#51	111	111	111	111	010	011	*384	σ_2-similar of $f8.6$
15	#52	111	111	111	111	001	110	*380	σ_3-similar of $f8.6$
15	#53	111	111	111	111	001	101	*378	σ_1-similar of $f8.6$
15	#54	111	111	111	111	001	011	*379	σ_1-similar of $f8.7$
15	#55	111	111	111	111	000	111	*369	$f8.4$
15	#56	111	111	111	110	110	110	*346	σ_3-similar of $f7.7$
15	#57	111	111	111	110	110	101	*345	σ_1-similar of $f7.7$
15	#58	111	111	111	110	110	011	*347	σ_1-similar of $f7.8$
15	#59	111	111	111	101	101	110	*360	σ_2-similar of $f7.8$
15	#60	111	111	111	101	101	101	*359	σ_4-similar of $f7.7$
15	#61	111	111	111	101	101	011	*358	σ_2-similar of $f7.7$
15	#62	111	111	111	011	011	110	*333	σ_0-similar of $f7.7$
15	#63	111	111	111	011	011	101	*334	$f7.8$
15	#64	111	111	111	011	011	011	*332	$f7.7$
15	#65	111	111	110	111	110	110	*306	σ_4-similar of $f6.18$
15	#66	111	111	110	111	101	110	*310	σ_2-similar of $f6.22$
15	#67	111	111	110	111	101	101	*318	σ_4-similar of $f6.27$
15	#68	111	111	110	111	101	011	*315	σ_2-similar of $f6.27$
15	#69	111	111	110	111	011	110	*299	σ_2-similar of $f6.18$
15	#70	111	111	110	101	101	111	*320	σ_2-similar of $f6.29$
15	#71	111	111	101	111	110	101	*220	σ_0-similar of $f6.18$
15	#72	111	111	101	111	101	101	*213	$f6.18$
15	#73	111	111	101	111	011	110	*232	σ_0-similar of $f6.27$
15	#74	111	111	101	111	011	101	*224	$f6.22$
15	#75	111	111	101	111	011	011	*229	$f6.27$
15	#76	111	111	101	011	011	111	*234	$f6.29$
15	#77	111	111	011	111	110	110	*275	σ_3-similar of $f6.27$
15	#78	111	111	011	111	110	101	*272	σ_1-similar of $f6.27$
15	#79	111	111	011	111	110	011	*267	σ_1-similar of $f6.22$
15	#80	111	111	011	111	101	011	*256	σ_1-similar of $f6.18$
15	#81	111	111	011	111	011	011	*263	σ_3-similar of $f6.18$
15	#82	111	111	011	110	110	111	*277	σ_1-similar of $f6.29$
15	#83	000	111	111	111	111	111	*2	$x+1, x+2$
14	#84	111	111	111	111	100	100	*371	σ_0-similar of $f8.5$
14	#85	111	111	111	111	100	001	*370	$f8.5$
14	#86	111	111	111	111	010	100	*383	σ_4-similar of $f8.5$
14	#87	111	111	111	111	010	010	*382	σ_2-similar of $f8.5$
14	#88	111	111	111	111	001	010	*377	σ_3-similar of $f8.5$
14	#89	111	111	111	111	001	001	*376	σ_1-similar of $f8.5$
14	#90	111	111	111	111	000	110	*368	σ_0-similar of $f8.3$
14	#91	111	111	111	111	000	101	*367	σ_1-similar of $f8.3$
14	#92	111	111	111	111	000	011	*365	$f8.3$
14	#93	111	111	111	110	110	100	*344	σ_1-similar of $f7.6$
14	#94	111	111	111	110	100	101	*340	σ_1-similar of $f7.5$
14	#95	111	111	111	110	010	110	*343	σ_3-similar of $f7.5$
14	#96	111	111	111	101	101	001	*357	σ_2-similar of $f7.6$
14	#97	111	111	111	101	100	101	*356	σ_4-similar of $f7.5$
14	#98	111	111	111	101	001	011	*353	σ_2-similar of $f7.5$
14	#99	111	111	111	011	011	010	*331	$f7.6$
14	#100	111	111	111	011	010	110	*330	σ_0-similar of $f7.5$

wt	#no	TLS	$M_1M_2M_0$	$U_2U_0U_1$	$B_0B_1B_2$	$T_0T_1T_2$	$T_{01}T_{12}T_{20}$	*no	representative
14	#101	111	111	111	011	001	011	*327	$f7.5$
14	#102	111	111	110	111	110	100	*308	σ_4-similar of $f6.20$
14	#103	111	111	110	111	101	001	*313	σ_2-similar of $f6.25$
14	#104	111	111	110	111	100	110	*302	σ_4-similar of $f6.14$
14	#105	111	111	110	111	011	010	*301	σ_2-similar of $f6.20$
14	#106	111	111	110	111	010	110	*292	σ_2-similar of $f6.13$
14	#107	111	111	110	111	001	110	*295	σ_2-similar of $f6.14$
14	#108	111	111	110	101	101	110	*309	σ_2-similar of $f6.21$
14	#109	111	111	110	101	101	101	*317	σ_4-similar of $f6.26$
14	#110	111	111	110	101	101	011	*314	σ_2-similar of $f6.26$
14	#111	111	111	101	111	110	100	*222	σ_0-similar of $f6.20$
14	#112	111	111	101	111	101	001	*215	$f6.20$
14	#113	111	111	101	111	100	101	*206	$f6.13$
14	#114	111	111	101	111	011	010	*227	$f6.25$
14	#115	111	111	101	111	010	101	*216	σ_0-similar of $f6.14$
14	#116	111	111	101	111	001	101	*209	$f6.14$
14	#117	111	111	101	011	011	110	*231	σ_0-similar of $f6.26$
14	#118	111	111	101	011	011	101	*223	$f6.21$
14	#119	111	111	101	011	011	011	*228	$f6.26$
14	#120	111	111	011	111	110	100	*270	σ_1-similar of $f6.25$
14	#121	111	111	011	111	101	001	*258	σ_1-similar of $f6.20$
14	#122	111	111	011	111	100	011	*252	σ_1-similar of $f6.14$
14	#123	111	111	011	111	011	010	*265	σ_3-similar of $f6.20$
14	#124	111	111	011	111	010	011	*259	σ_3-similar of $f6.14$
14	#125	111	111	011	111	001	011	*249	σ_1-similar of $f6.13$
14	#126	111	111	011	110	110	110	*274	σ_3-similar of $f6.26$
14	#127	111	111	011	110	110	101	*271	σ_1-similar of $f6.26$
14	#128	111	111	011	110	110	011	*266	σ_1-similar of $f6.21$
14	#129	111	110	111	111	100	110	*165	σ_3-similar of $f5.8$
14	#130	111	110	111	111	100	011	*159	σ_2-similar of $f5.8$
14	#131	111	101	111	111	001	110	*131	σ_0-similar of $f5.8$
14	#132	111	101	111	111	001	101	*137	σ_4-similar of $f5.8$
14	#133	111	011	111	111	010	101	*109	σ_1-similar of $f5.8$
14	#134	111	011	111	111	010	011	*103	$f5.8$
14	#135	110	111	111	111	000	111	*88	$f4.6$
13	#136	111	111	111	111	000	100	*366	σ_0-similar of $f8.2$
13	#137	111	111	111	111	000	010	*364	σ_2-similar of $f8.2$
13	#138	111	111	111	111	000	001	*363	$f8.2$
13	#139	111	111	111	110	100	100	*338	σ_1-similar of $f7.3$
13	#140	111	111	111	110	100	001	*339	σ_1-similar of $f7.4$
13	#141	111	111	111	110	010	100	*341	σ_3-similar of $f7.3$
13	#142	111	111	111	110	010	010	*342	σ_3-similar of $f7.4$
13	#143	111	111	111	101	100	100	*355	σ_4-similar of $f7.4$
13	#144	111	111	111	101	100	001	*354	σ_4-similar of $f7.3$
13	#145	111	111	111	101	001	010	*352	σ_2-similar of $f7.4$
13	#146	111	111	111	101	001	001	*351	σ_2-similar of $f7.3$
13	#147	111	111	111	011	010	100	*329	σ_0-similar of $f7.4$
13	#148	111	111	111	011	010	010	*328	σ_0-similar of $f7.3$
13	#149	111	111	111	011	001	010	*325	$f7.3$
13	#150	111	111	111	011	001	001	*326	$f7.4$

wt	#no	TLS	$M_1M_2M_0$	$U_2U_0U_1$	$B_0B_1B_2$	$T_0T_1T_2$	$T_{01}T_{12}T_{20}$	*no	representative
13	#151	111	111	110	111	100	100	*305	σ_4-similar of $f6.17$
13	#152	111	111	110	111	010	100	*294	σ_4-similar of $f6.12$
13	#153	111	111	110	111	010	010	*291	σ_2-similar of $f6.12$
13	#154	111	111	110	111	001	010	*298	σ_2-similar of $f6.17$
13	#155	111	111	110	111	000	110	*280	σ_2-similar of $f6.4$
13	#156	111	111	110	110	110	100	*307	σ_4-similar of $f6.19$
13	#157	111	111	110	101	101	001	*312	σ_2-similar of $f6.24$
13	#158	111	111	110	011	011	010	*300	σ_2-similar of $f6.19$
13	#159	111	111	101	111	100	100	*208	σ_0-similar of $f6.12$
13	#160	111	111	101	111	100	001	*205	$f6.12$
13	#161	111	111	101	111	010	100	*219	σ_0-similar of $f6.17$
13	#162	111	111	101	111	001	001	*212	$f6.17$
13	#163	111	111	101	111	000	101	*194	$f6.4$
13	#164	111	111	101	110	110	100	*221	σ_0-similar of $f6.19$
13	#165	111	111	101	101	101	001	*214	$f6.19$
13	#166	111	111	101	011	011	010	*226	$f6.24$
13	#167	111	111	011	111	100	001	*255	σ_1-similar of $f6.17$
13	#168	111	111	011	111	010	010	*262	σ_3-similar of $f6.17$
13	#169	111	111	011	111	001	010	*251	σ_3-similar of $f6.12$
13	#170	111	111	011	111	001	001	*248	σ_1-similar of $f6.12$
13	#171	111	111	011	111	000	011	*237	σ_1-similar of $f6.4$
13	#172	111	111	011	110	110	100	*269	σ_1-similar of $f6.24$
13	#173	111	111	011	101	101	001	*257	σ_1-similar of $f6.19$
13	#174	111	111	011	011	011	010	*264	σ_3-similar of $f6.19$
13	#175	111	110	111	111	100	100	*163	σ_3-similar of $f5.6$
13	#176	111	110	111	111	100	001	*157	σ_2-similar of $f5.6$
13	#177	111	110	110	111	100	110	*164	σ_3-similar of $f5.7$
13	#178	111	110	011	111	100	011	*158	σ_2-similar of $f5.7$
13	#179	111	101	111	111	001	010	*129	σ_0-similar of $f5.6$
13	#180	111	101	111	111	001	001	*135	σ_4-similar of $f5.6$
13	#181	111	101	110	111	001	110	*130	σ_0-similar of $f5.7$
13	#182	111	101	101	111	001	101	*136	σ_4-similar of $f5.7$
13	#183	111	011	111	111	010	100	*107	σ_1-similar of $f5.6$
13	#184	111	011	111	111	010	010	*101	$f5.6$
13	#185	111	011	101	111	010	101	*108	σ_1-similar of $f5.7$
13	#186	111	011	011	111	010	011	*102	$f5.7$
13	#187	100	111	111	111	000	111	*85	$f4.3 = 2x + 2y$
13	#188	011	111	110	010	111	110	*79	σ_0-similar of $f3.13$
13	#189	011	111	101	100	111	101	*55	σ_1-similar of $f3.13$
13	#190	011	111	011	001	111	011	*31	$f3.13$
12	#191	111	111	111	111	000	000	*362	$f8.1$
12	#192	111	111	111	110	000	100	*337	σ_1-similar of $f7.2$
12	#193	111	111	111	101	000	001	*350	σ_2-similar of $f7.2$
12	#194	111	111	111	011	000	010	*324	$f7.2$
12	#195	111	111	110	111	000	100	*282	σ_2-similar of $f6.6$
12	#196	111	111	110	111	000	010	*284	σ_4-similar of $f6.6$
12	#197	111	111	110	110	100	100	*304	σ_4-similar of $f6.16$
12	#198	111	111	110	110	010	100	*293	σ_4-similar of $f6.11$
12	#199	111	111	110	101	100	100	*303	σ_4-similar of $f6.15$
12	#200	111	111	110	101	001	010	*296	σ_2-similar of $f6.15$

wt	#no	TLS	$M_1M_2M_0$	$U_2U_0U_1$	$B_0B_1B_2$	$T_0T_1T_2$	$T_{01}T_{12}T_{20}$	*no	representative
12	#201	111	111	110	011	010	010	*290	σ_2-similar of $f6.11$
12	#202	111	111	110	011	001	010	*297	σ_2-similar of $f6.16$
12	#203	111	111	101	111	000	100	*196	$f6.6$
12	#204	111	111	101	111	000	001	*198	σ_0-similar of $f6.6$
12	#205	111	111	101	110	100	100	*207	σ_0-similar of $f6.11$
12	#206	111	111	101	110	010	100	*218	σ_0-similar of $f6.16$
12	#207	111	111	101	101	100	001	*204	$f6.11$
12	#208	111	111	101	101	001	001	*211	$f6.16$
12	#209	111	111	101	011	010	100	*217	σ_0-similar of $f6.15$
12	#210	111	111	101	011	001	001	*210	$f6.15$
12	#211	111	111	100	110	110	100	*189	σ_1-similar of $f6.3$
12	#212	111	111	011	111	000	010	*239	σ_1-similar of $f6.6$
12	#213	111	111	011	111	000	001	*241	σ_3-similar of $f6.6$
12	#214	111	111	011	110	100	001	*253	σ_1-similar of $f6.15$
12	#215	111	111	011	110	010	010	*260	σ_3-similar of $f6.15$
12	#216	111	111	011	101	100	001	*254	σ_1-similar of $f6.16$
12	#217	111	111	011	101	001	001	*247	σ_1-similar of $f6.11$
12	#218	111	111	011	011	010	010	*261	σ_3-similar of $f6.16$
12	#219	111	111	011	011	001	010	*250	σ_3-similar of $f6.11$
12	#220	111	111	010	011	011	010	*185	$f6.3$
12	#221	111	111	001	101	101	001	*193	σ_2-similar of $f6.3$
12	#222	111	110	111	111	000	010	*172	σ_2-similar of $f5.13$
12	#223	111	110	111	110	100	100	*161	σ_3-similar of $f5.4$
12	#224	111	110	111	101	100	001	*155	σ_2-similar of $f5.4$
12	#225	111	110	110	111	100	100	*162	σ_3-similar of $f5.5$
12	#226	111	110	011	111	100	001	*156	σ_2-similar of $f5.5$
12	#227	111	101	111	111	000	100	*144	σ_0-similar of $f5.13$
12	#228	111	101	111	101	001	001	*133	σ_4-similar of $f5.4$
12	#229	111	101	111	011	001	010	*127	σ_0-similar of $f5.4$
12	#230	111	101	110	111	001	010	*128	σ_0-similar of $f5.5$
12	#231	111	101	101	111	001	001	*134	σ_4-similar of $f5.5$
12	#232	111	011	111	111	000	001	*116	$f5.13$
12	#233	111	011	111	110	010	100	*105	σ_1-similar of $f5.4$
12	#234	111	011	111	011	010	010	*99	$f5.4$
12	#235	111	011	101	111	010	100	*106	σ_1-similar of $f5.5$
12	#236	111	011	011	111	010	010	*100	$f5.5$
12	#237	011	111	110	010	110	110	*64	σ_3-similar of $f3.4$
12	#238	011	111	110	010	011	110	*60	σ_0-similar of $f3.4$
12	#239	011	111	101	100	110	101	*36	σ_1-similar of $f3.4$
12	#240	011	111	101	100	101	101	*40	σ_4-similar of $f3.4$
12	#241	011	111	100	100	111	101	*56	σ_4-similar of $f3.12$
12	#242	011	111	100	010	111	110	*78	$j_0 = \sigma_0$-similar of $f3.12$
12	#243	011	111	011	001	101	011	*16	σ_2-similar of $f3.4$
12	#244	011	111	011	001	011	011	*12	$f3.4$
12	#245	011	111	010	010	111	110	*80	σ_3-similar of $f3.12$
12	#246	011	111	010	001	111	011	*32	σ_2-similar of $f3.12$
12	#247	011	111	001	100	111	101	*54	σ_1-similar of $f3.12$
12	#248	011	111	001	001	111	011	*30	$f3.12 = s_{100}$
12	#249	001	111	110	101	101	110	*5	$2x + 2 = \sigma_2$-, σ_4-sim. $2x$
12	#250	001	111	101	011	011	101	*3	$2x$

wt	#no	TLS	$M_1M_2M_0$	$U_2U_0U_1$	$B_0B_1B_2$	$T_0T_1T_2$	$T_{01}T_{12}T_{20}$	*no	representative
12	#251	001	111	011	110	110	011	*4	$2x+1=\sigma_1$-, σ_3-sim. $2x$
11	#252	111	111	111	110	000	000	*336	σ_1-similar of $f7.1$
11	#253	111	111	111	101	000	000	*349	σ_2-similar of $f7.1$
11	#254	111	111	111	011	000	000	*323	$f7.1$
11	#255	111	111	110	111	000	000	*289	σ_2-similar of $f6.10$
11	#256	111	111	110	110	000	100	*281	σ_2-similar of $f6.5$
11	#257	111	111	110	011	000	010	*283	σ_4-similar of $f6.5$
11	#258	111	111	101	111	000	000	*203	$f6.10$
11	#259	111	111	101	110	000	100	*195	$f6.5$
11	#260	111	111	101	101	000	001	*197	σ_0-similar of $f6.5$
11	#261	111	111	100	110	100	100	*187	σ_1-similar of $f6.2$
11	#262	111	111	100	110	010	100	*188	σ_3-similar of $f6.2$
11	#263	111	111	011	111	000	000	*246	σ_1-similar of $f6.10$
11	#264	111	111	011	101	000	001	*240	σ_3-similar of $f6.5$
11	#265	111	111	011	011	000	010	*238	σ_1-similar of $f6.5$
11	#266	111	111	010	011	010	010	*184	σ_0-similar of $f6.2$
11	#267	111	111	010	011	001	010	*183	$f6.2$
11	#268	111	111	001	101	100	001	*192	σ_4-similar of $f6.2$
11	#269	111	111	001	101	001	001	*191	σ_2-similar of $f6.2$
11	#270	111	110	111	111	000	000	*181	σ_2-similar of $f5.19$
11	#271	111	110	111	011	000	010	*169	σ_2-similar of $f5.11$
11	#272	111	110	110	111	000	010	*171	σ_3-similar of $f5.12$
11	#273	111	110	110	110	100	100	*160	σ_3-similar of $f5.3$
11	#274	111	110	011	111	000	010	*170	σ_2-similar of $f5.12$
11	#275	111	110	011	101	100	001	*154	σ_2-similar of $f5.3$
11	#276	111	101	111	111	000	000	*153	σ_0-similar of $f5.19$
11	#277	111	101	111	110	000	100	*141	σ_0-similar of $f5.11$
11	#278	111	101	110	111	000	100	*142	σ_0-similar of $f5.12$
11	#279	111	101	110	011	001	010	*126	σ_0-similar of $f5.3$
11	#280	111	101	101	111	000	100	*143	σ_4-similar of $f5.12$
11	#281	111	101	101	101	001	001	*132	σ_4-similar of $f5.3$
11	#282	111	011	111	111	000	000	*125	$f5.19$
11	#283	111	011	111	101	000	001	*113	$f5.11$
11	#284	111	011	101	111	000	001	*115	σ_1-similar of $f5.12$
11	#285	111	011	101	110	010	100	*104	σ_1-similar of $f5.3$
11	#286	111	011	011	111	000	001	*114	$f5.12$
11	#287	111	011	011	011	010	010	*98	$f5.3$
11	#288	110	111	111	111	000	000	*87	$f4.5$
11	#289	011	111	110	010	110	100	*63	σ_3-similar of $f3.3$
11	#290	011	111	110	010	011	010	*59	σ_0-similar of $f3.3$
11	#291	011	111	110	010	010	110	*72	σ_0-similar of $f3.11$
11	#292	011	111	101	100	110	100	*35	σ_1-similar of $f3.3$
11	#293	011	111	101	100	101	001	*39	σ_4-similar of $f3.3$
11	#294	011	111	101	100	100	101	*48	σ_1-similar of $f3.11$
11	#295	011	111	100	100	101	101	*38	σ_4-similar of $f3.2$
11	#296	011	111	100	010	011	110	*58	σ_0-similar of $f3.2$
11	#297	011	111	011	001	101	001	*15	σ_2-similar of $f3.3$
11	#298	011	111	011	001	011	010	*11	$f3.3$
11	#299	011	111	011	001	001	011	*24	$f3.11$
11	#300	011	111	010	010	110	110	*62	σ_3-similar of $f3.2$

wt	#no	TLS	$M_1M_2M_0$	$U_2U_0U_1$	$B_0B_1B_2$	$T_0T_1T_2$	$T_{01}T_{12}T_{20}$	*no	representative
11	#301	011	111	010	001	101	011	*14	σ_2-similar of $f3.2$
11	#302	011	111	001	100	110	101	*34	σ_1-similar of $f3.2$
11	#303	011	111	001	001	011	011	*10	$f3.2$
11	#304	011	110	110	010	110	110	*66	σ_3-similar of $f3.5$
11	#305	011	110	011	001	101	011	*18	σ_2-similar of $f3.5$
11	#306	011	101	110	010	011	110	*65	$j_1 = \sigma_0$-similar of $f3.5$
11	#307	011	101	101	100	101	101	*42	σ_4-similar of $f3.5$
11	#308	011	011	101	100	110	101	*41	σ_1-similar of $f3.5$
11	#309	011	011	011	001	011	011	*17	$f3.5 = s_{001}$
10	#310	111	111	110	110	000	000	*288	σ_4-similar of $f6.9$
10	#311	111	111	110	101	000	000	*285	σ_2-similar of $f6.7$
10	#312	111	111	110	011	000	000	*287	σ_2-similar of $f6.9$
10	#313	111	111	101	110	000	000	*202	σ_0-similar of $f6.9$
10	#314	111	111	101	101	000	000	*201	$f6.9$
10	#315	111	111	101	011	000	000	*199	$f6.7$
10	#316	111	111	100	110	000	100	*186	σ_1-similar of $f6.1$
10	#317	111	111	011	110	000	000	*242	σ_1-similar of $f6.7$
10	#318	111	111	011	101	000	000	*244	σ_1-similar of $f6.9$
10	#319	111	111	011	011	000	000	*245	σ_3-similar of $f6.9$
10	#320	111	111	010	011	000	010	*182	$f6.1$
10	#321	111	111	001	101	000	001	*190	σ_2-similar of $f6.1$
10	#322	111	110	111	110	000	000	*180	σ_3-similar of $f5.18$
10	#323	111	110	111	101	000	000	*179	σ_2-similar of $f5.18$
10	#324	111	110	110	111	000	000	*178	σ_3-similar of $f5.17$
10	#325	111	110	110	011	000	010	*168	σ_3-similar of $f5.10$
10	#326	111	110	011	111	000	000	*176	σ_2-similar of $f5.17$
10	#327	111	110	011	011	000	010	*167	σ_2-similar of $f5.10$
10	#328	111	101	111	101	000	000	*152	σ_4-similar of $f5.18$
10	#329	111	101	111	011	000	000	*151	σ_0-similar of $f5.18$
10	#330	111	101	110	111	000	000	*148	σ_0-similar of $f5.17$
10	#331	111	101	110	110	000	100	*139	σ_0-similar of $f5.10$
10	#332	111	101	101	111	000	000	*150	σ_4-similar of $f5.17$
10	#333	111	101	101	110	000	100	*140	σ_4-similar of $f5.10$
10	#334	111	011	111	110	000	000	*124	σ_1-similar of $f5.18$
10	#335	111	011	111	011	000	000	*123	$f5.18$
10	#336	111	011	101	111	000	000	*122	σ_1-similar of $f5.17$
10	#337	111	011	101	101	000	001	*112	σ_1-similar of $f5.10$
10	#338	111	011	011	111	000	000	*120	$f5.17$
10	#339	111	011	011	101	000	001	*111	$f5.10$
10	#340	011	111	110	010	010	100	*75	σ_3-similar of $f3.8$
10	#341	011	111	110	010	010	010	*69	σ_0-similar of $f3.8$
10	#342	011	111	101	100	100	100	*45	σ_1-similar of $f3.8$
10	#343	011	111	101	100	100	001	*51	σ_4-similar of $f3.8$
10	#344	011	111	100	100	110	100	*33	σ_1-similar of $f3.1$
10	#345	011	111	100	010	110	100	*61	σ_3-similar of $f3.1$
10	#346	011	111	011	001	001	010	*21	$f3.8$
10	#347	011	111	011	001	001	001	*27	σ_2-similar of $f3.8$
10	#348	011	111	010	010	011	010	*57	σ_0-similar of $f3.1$
10	#349	011	111	010	001	011	010	*9	$f3.1$
10	#350	011	111	001	100	101	001	*37	σ_4-similar of $f3.1$

wt	#no	TLS	$M_1M_2M_0$	$U_2U_0U_1$	$B_0B_1B_2$	$T_0T_1T_2$	$T_{01}T_{12}T_{20}$	*no	representative
10	#351	011	111	001	001	101	001	*13	σ_2-similar of $f3.1$
9	#352	111	111	110	010	000	000	*286	σ_2-similar of $f6.8$
9	#353	111	111	101	100	000	000	*200	$f6.8$
9	#354	111	111	011	001	000	000	*243	σ_1-similar of $f6.8$
9	#355	111	110	110	110	000	000	*177	σ_3-similar of $f5.16$
9	#356	111	110	011	101	000	000	*175	σ_2-similar of $f5.16$
9	#357	111	110	010	011	000	010	*166	σ_2-similar of $f5.9$
9	#358	111	101	110	011	000	000	*147	σ_0-similar of $f5.16$
9	#359	111	101	101	101	000	000	*149	σ_4-similar of $f5.16$
9	#360	111	101	100	110	000	100	*138	σ_0-similar of $f5.9$
9	#361	111	011	101	110	000	000	*121	σ_1-similar of $f5.16$
9	#362	111	011	011	011	000	000	*119	$f5.16$
9	#363	111	011	001	101	000	001	*110	$f5.9$
9	#364	011	111	100	100	100	100	*43	σ_1-similar of $f3.6$
9	#365	011	111	100	010	010	100	*73	σ_3-similar of $f3.6$
9	#366	011	111	010	010	010	010	*67	σ_0-similar of $f3.6$
9	#367	011	111	010	001	001	010	*19	$f3.6$
9	#368	011	111	001	100	100	001	*49	σ_4-similar of $f3.6$
9	#369	011	111	001	001	001	001	*25	σ_2-similar of $f3.6$
9	#370	011	110	110	010	010	010	*70	σ_0-similar of $f3.9$
9	#371	011	110	101	100	100	100	*47	σ_1-similar of $f3.10$
9	#372	011	110	101	100	100	001	*53	σ_4-similar of $f3.10$
9	#373	011	110	011	001	001	010	*22	$f3.9$
9	#374	011	101	110	010	010	100	*76	σ_3-similar of $f3.9$
9	#375	011	101	101	100	100	100	*46	σ_1-similar of $f3.9$
9	#376	011	101	011	001	001	010	*23	$f3.10$
9	#377	011	101	011	001	001	001	*29	σ_2-similar of $f3.10$
9	#378	011	011	110	010	010	100	*77	σ_3-similar of $f3.10$
9	#379	011	011	110	010	010	010	*71	σ_0-similar of $f3.10$
9	#380	011	011	101	100	100	001	*52	σ_4-similar of $f3.9$
9	#381	011	011	011	001	001	001	*28	σ_2-similar of $f3.9$
8	#382	111	110	101	100	000	000	*174	σ_2-similar of $f5.15$
8	#383	111	110	010	011	000	000	*173	σ_2-similar of $f5.14$
8	#384	111	101	100	110	000	000	*145	σ_0-similar of $f5.14$
8	#385	111	101	011	001	000	000	*146	σ_0-similar of $f5.15$
8	#386	111	100	110	101	000	000	*97	σ_2-similar of $f5.2$
8	#387	111	011	110	010	000	000	*118	$f5.15$
8	#388	111	011	001	101	000	000	*117	$f5.14$
8	#389	111	010	011	110	000	000	*94	σ_1-similar of $f5.2$
8	#390	111	001	101	011	000	000	*91	$f5.2$
7	#391	011	100	100	100	100	100	*44	σ_1-similar of $f3.7$
7	#392	011	100	010	001	001	010	*20	$f3.7 = s_{010}$
7	#393	011	010	010	010	010	010	*68	$j_2 = \sigma_0$-similar of $f3.7$
7	#394	011	010	001	100	100	001	*50	σ_4-similar of $f3.7$
7	#395	011	001	100	010	010	100	*74	σ_3-similar of $f3.7$
7	#396	011	001	001	001	001	001	*26	σ_2-similar of $f3.7$
6	#397	111	100	100	100	000	000	*96	σ_4-similar of $f5.1$
6	#398	111	100	010	001	000	000	*95	σ_2-similar of $f5.1$
6	#399	111	010	010	010	000	000	*93	σ_3-similar of $f5.1$
6	#400	111	010	001	100	000	000	*92	σ_1-similar of $f5.1$

wt	#no	TLS	$M_1M_2M_0$	$U_2U_0U_1$	$B_0B_1B_2$	$T_0T_1T_2$	$T_{01}T_{12}T_{20}$	*no	*representative*
6	#401	111	001	100	010	000	000	*90	σ_0-similar of $f5.1$
6	#402	111	001	001	001	000	000	*89	$f5.1 = \min(x, y)$
4	#403	001	000	000	000	110	100	*8	$2 = \sigma_1$-, σ_3-similar of 0
4	#404	001	000	000	000	101	001	*7	$1 = \sigma_2$-, σ_4-similar of 0
4	#405	001	000	000	000	011	010	*6	0 (constant)
0	#406	000	000	000	000	000	000	*1	x (projection functions)

Appendix 2. Representatives of classes of P_3 ($f3.1$–$f8.14$).

$f3.5$	0	1	2
$f(x)$	0	0	1

$f3.7$	0	1	2
$f(x)$	0	1	0

$f3.12$	0	1	2
$f(x)$	1	0	0

$f \setminus xy$	00	01	02	10	11	12	20	21	22
$f3.1$	0	0	0	1	0	1	0	0	0
$f3.2$	0	1	1	0	0	0	0	0	0
$f3.3$	0	0	0	0	0	1	0	1	0
$f3.4$	0	0	1	0	0	0	1	0	0
$f3.8$	0	1	0	0	1	1	0	0	0
$f3.9$	0	1	0	1	1	0	0	0	0
$f3.10$	0	1	0	0	1	1	0	1	0
$f3.11$	0	0	1	0	1	0	0	0	0
$f3.13$	1	1	0	1	0	0	0	0	0
$f4.1$	0	2	1	1	0	2	2	1	0
$f4.2$	1	0	2	0	2	1	2	1	0
$f4.3$	0	2	1	2	1	0	1	0	2
$f4.4$	1	0	0	1	2	1	2	2	0
$f4.5$	0	0	2	0	1	1	2	1	2
$f4.6$	0	0	1	2	1	1	2	0	2
$f5.1$	0	0	0	0	1	1	0	1	2
$f5.2$	0	0	0	0	1	1	0	2	2
$f5.3$	0	0	0	0	0	1	0	1	2
$f5.4$	0	0	0	0	0	1	0	2	2
$f5.7$	0	0	1	0	0	1	1	1	2
$f5.8$	0	0	1	0	0	1	0	2	2
$f5.9$	0	1	1	1	1	1	1	1	2
$f5.10$	0	0	1	0	1	1	1	1	2
$f5.11$	0	0	1	1	1	1	1	2	2
$f5.12$	0	1	1	0	1	1	2	2	2
$f5.13$	0	0	1	0	1	2	1	2	2
$f5.15$	0	0	0	0	1	2	2	2	2
$f5.16$	0	0	0	0	1	1	2	2	2
$f6.2$	0	0	2	0	1	2	0	0	0
$f6.3$	0	0	0	1	0	1	2	2	0
$f6.4$	0	2	2	2	1	1	1	1	2
$f6.5$	0	2	2	1	1	2	2	2	2
$f6.10$	0	0	0	1	1	2	2	1	2
$f6.11$	1	0	0	1	1	1	2	1	2
$f6.12$	1	0	0	1	1	2	2	1	2
$f6.13$	2	2	1	0	1	2	0	2	2
$f6.14$	0	2	1	2	1	1	1	1	1
$f6.15$	0	0	0	0	1	2	0	2	1

$f \setminus xy$	00	01	02	10	11	12	20	21	22
$f6.16$	0	1	1	1	1	2	1	2	1
$f6.17$	0	1	1	0	1	2	0	2	1
$f6.18$	2	2	1	0	1	2	0	2	1
$f6.19$	1	1	1	0	1	2	0	1	1
$f6.20$	1	0	0	1	1	2	2	1	1
$f6.21$	0	0	0	0	2	2	0	1	1
$f6.22$	0	0	0	2	2	2	1	2	1
$f6.23$	2	1	2	0	2	1	0	2	1
$f6.24$	0	1	2	0	0	0	0	0	0
$f6.25$	0	1	2	1	0	0	2	0	0
$f6.27$	0	1	2	1	0	0	1	0	0
$f6.28$	1	1	2	1	0	0	1	0	0
$f6.29$	0	2	1	0	0	0	0	0	0
$f6.30$	0	2	1	2	0	0	1	0	0
$f6.31$	2	2	1	2	0	0	1	0	0
$f7.2$	0	0	2	0	1	0	0	1	2
$f7.3$	0	1	0	0	1	2	0	0	0
$f7.4$	0	0	0	0	1	2	1	1	1
$f7.5$	0	0	1	1	1	0	0	2	1
$f7.6$	0	0	0	0	0	1	0	2	0
$f7.7$	0	0	0	0	0	2	1	1	1
$f7.8$	0	0	1	0	2	1	0	1	1
$f7.9$	0	2	1	0	0	1	0	2	0
$f8.1$	0	1	0	0	1	2	0	2	2
$f8.2$	0	0	1	1	1	2	0	1	2
$f8.3$	0	0	1	1	1	0	1	2	2
$f8.4$	0	1	1	2	1	0	0	1	2
$f8.5$	1	0	1	0	1	1	0	2	2
$f8.6$	1	0	0	0	1	0	1	2	2
$f8.7$	1	0	1	2	1	1	0	2	2
$f8.8$	2	0	1	0	1	0	1	0	2
$f8.9$	0	0	2	0	0	1	2	2	0
$f8.10$	0	1	1	0	0	2	1	2	0
$f8.11$	0	0	1	0	2	2	0	2	1
$f8.12$	0	2	1	1	0	2	1	1	0
$f8.13$	1	1	0	1	0	2	0	2	0
$f8.14$	1	0	1	0	2	0	1	0	0

Representatives of classes of P_3 (continued) $f3.6$–$f7.1$

$f3.6$	00	01	10	11	12	21	22	20	02
0	0	1	0	1	0	1	0	0	0
1	1	0	1	1	1	0	1	1	1
2	0	1	0	1	0	1	0	0	0

$f5.5$	00	01	10	11	12	21	22	20	02
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1	1
2	0	0	1	1	2	2	2	2	2

$f5.6$	00	01	10	11	12	21	22	20	02
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1	1
2	0	1	1	2	2	2	2	2	2

$f5.14$	00	01	10	11	12	21	22	20	02
0	0	0	0	1	1	1	2	0	0
1	1	1	1	1	1	1	2	1	1
2	2	2	2	2	2	2	2	2	2

$f5.17$	00	01	10	11	12	21	22	20	02
0	0	0	0	0	0	0	0	0	0
1	0	1	0	1	1	1	1	0	1
2	0	1	0	1	2	2	2	2	2

$f5.18$	00	01	10	11	12	21	22	20	02
0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	1	2	2	0	0
2	0	0	2	2	2	2	2	2	2

$f5.19$	00	01	10	11	12	21	22	20	02
0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	1	1	2	1	0
2	2	2	2	2	2	2	2	2	2

$f6.1$	00	01	10	11	12	21	22	20	02
0	0	0	0	0	0	2	2	2	0
1	0	0	1	1	1	2	2	2	0
2	2	2	2	2	2	0	2	0	2

$f6.6$	00	01	10	11	12	21	22	20	02
0	0	2	0	2	1	1	2	0	2
1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2

$f6.7$	00	01	10	11	12	21	22	20	02
0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	2	2	2	0	0
2	0	0	0	1	1	2	2	0	0

$f6.8$	00	01	10	11	12	21	22	20	02
0	0	1	1	1	1	1	2	2	2
1	1	1	1	1	1	2	1	1	1
2	2	1	1	1	2	1	2	2	2

$f6.9$	00	01	10	11	12	21	22	20	02
0	0	0	1	1	1	1	2	2	0
1	1	1	1	1	1	2	1	1	1
2	2	1	1	1	2	1	2	2	2

$f6.26$	00	01	10	11	12	21	22	20	02
0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	2
2	0	2	0	0	0	0	0	0	1

$f7.1$	00	01	10	11	12	21	22	20	02
0	0	0	0	0	1	0	0	0	0
1	0	0	1	1	1	2	1	0	0
2	0	0	0	2	1	1	2	0	0

Appendix 3. List of basic inclusions in P_3.

Lemma 5.1.3.

$$M_1M_0 \subseteq U_2,\ M_2M_1 \subseteq U_0, \text{ and } M_0M_2 \subseteq U_1.$$

Corollary 5.1.1.

$$M_1M_2M_0 \subseteq U_2U_0U_1.$$

Lemma 5.1.4.

$$U_2U_0U_1 \subseteq M_1M_2M_0.$$

Note 5.1.1.

$$\begin{array}{rcll} D(0,1)U_2U_0 & \subseteq & M_1, & (5.4) \\ D(2,0)U_2U_1 & \subseteq & M_1, & (5.5) \\ D(0,1)U_0U_1 & \subseteq & M_0, & (5.6) \\ D(0,1)U_1U_0 & \subseteq & M_2. & (5.7) \end{array}$$

$$D(0,1)U_0U_1 = \{0,1\}\ , D(1,2)U_1U_2 = \{1,2\} \text{ and } D(2,0)U_2U_0 = \{2,0\}.$$

Lemma 5.1.5.

$$M_1M_2 \subseteq B_0,\ M_2M_0 \subseteq B_1 \text{ and } M_0M_1 \subseteq B_2.$$

Corollary 5.1.2.

$$M_0M_1M_2 \subseteq B_0B_1B_2.$$

Lemma 5.1.6.

$$U_2U_0 \subseteq B_1,\ U_0U_1 \subseteq B_2 \text{ and } U_1U_2 \subseteq B_0.$$

Corollary 5.1.3.

$$U_0U_1U_2 \subseteq B_0B_1B_2.$$

Lemma 5.1.7.

$$B_0B_1 \subseteq U_2,\ B_1B_2 \subseteq U_0 \text{ and } B_2B_0 \subseteq U_1.$$

Corollary 5.1.4.

$$B_0B_1B_2 \subseteq U_0U_1U_2.$$

Theorem 5.1.4

$$K = M_0M_1M_2 = B_0B_1B_2 = U_0U_1U_2 = \{0,1,2,\ x_i\ (i = 1,2,\ldots)\}.$$

Lemma 5.1.8.

$$T_{01}T_{12} \subseteq T_1,\ T_{12}T_{20} \subseteq T_2 \text{ and } T_{20}T_{01} \subseteq T_0.$$

Corollary 5.1.5.

$$T_{01}T_{12}T_{20} \subseteq T_0T_1T_2.$$

Lemma 5.1.9.

$$M_1 \cup M_2 \cup M_0 \subseteq T_{01} \cup T_{12} \cup T_{20}.$$

Note 5.1.2.

$$
\begin{aligned}
U_0 \cup U_1 \cup U_2 &\not\subseteq T_{01}T_{12}T_{20}, \\
B_0 \cup B_1 \cup B_2 &\not\subseteq T_{01}T_{12}T_{20}.
\end{aligned}
$$

Lemma 5.1.10.

$$
\begin{array}{lll}
B_0B_1 \subseteq T_{01},\ M_0M_1 \subseteq T_{01} & \text{except constant function} & f = 2, \\
B_1B_2 \subseteq T_{12},\ M_1M_2 \subseteq T_{12} & \text{except constant function} & f = 0, \\
B_2B_0 \subseteq T_{01},\ M_2M_0 \subseteq T_{20} & \text{except constant function} & f = 1.
\end{array}
$$

Corollary 5.1.6 (**Lemma** 5.1.11).

$$
\begin{array}{llll}
U_0 = M_2 & \text{on} & D(0,1)M_1 & (5.11) \\
U_0 = M_1,\ U_1 = M_0 & \text{on} & D(0,1)M_2 & (5.9), (5.12) \\
U_1 = M_2 & \text{on} & D(0,1)M_0 & (5.13) \\
U_1 = M_0 & \text{on} & D(1,2)M_2 & (5.14) \\
U_1 = M_2,\ U_2 = M_1 & \text{on} & D(1,2)M_0 & (5.15) \\
U_2 = M_0 & \text{on} & D(1,2)M_1 & (5.16) \\
U_2 = M_1 & \text{on} & D(2,0)M_0 & (5.17) \\
U_2 = M_0,\ U_0 = M_2 & \text{on} & D(2,0)M_1 & (5.18), (5.18) \\
U_0 = M_1 & \text{on} & D(2,0)M_2. & (5.19)
\end{array}
$$

Let $P^{(1)}_{onto} := \{f | f \in P^{(1)}_3 \text{ and } f \text{ is onto}\}$ and $D'(0,1) := D(0,1) \setminus \{0,1\}$, $D'(1,2) := D(1,2) \setminus \{1,2\}$, $D'(2,0) := D(2,0) \setminus \{2,0\}$. Let $D := P_3 \setminus D(0,1) \cup D(1,2) \cup D(2,0)$.

Lemma 5.3.1.

$$D'(0,1) \subseteq \overline{S}.$$

Corollary 5.3.1.

$$D \subseteq \overline{S}.$$

Corollary 5.3.2.

$$TS = \{x_i, x_i + 1, x_i + 2 \ (i = 1, 2, \ldots)\}.$$

Lemma 5.3.2.

$$D'(0,1) \subseteq \overline{L}.$$

Corollary 5.3.2.

$$D \setminus \{0,1,2\} \subset \overline{L}.$$

Corollary 5.3.4.

$$TL = P^{(1)}_{onto} + \{0,1,2\}.$$

Lemma 5.3.5.

$$D'(0,1)U_0\overline{U}_1 \subseteq \overline{T}_{20}.$$

Lemma 5.4.3.

$$\overline{T}S \subseteq \tilde{M}\tilde{U}\tilde{B}, \text{ where } \tilde{M} = \overline{M}_0\overline{M}_1\overline{M}_2,\ \tilde{U} = \overline{U}_0\overline{U}_1\overline{U}_2, \text{ and } \tilde{B} = \overline{B}_0\overline{B}_1\overline{B}_2.$$

Lemma 5.4.4.

$$S\overline{T}_0\overline{T}_1\overline{T}_2 \subseteq \overline{T}_{01}\overline{T}_{12}\overline{T}_{20}.$$

Lemma 5.4.5.

$$\overline{T}L \subseteq \overline{T}_{01}\overline{T}_{12}\overline{T}_{20}.$$

Let $L_a := \{f | f = c_0 + \sum c_i x_i \text{ and } \sum_{i=1}^{n} = a\}$ and $L_{ab} := \{f | f \in L_a \text{ and } f(\mathbf{o}) = c_0 = b\}$ $(L_a = L_{a0} + L_{a1} + L_{a2})$.

Lemma 5.4.7.

$$LS = L_1.$$

Lemma 5.4.8.

$$\begin{aligned} 1)\ L_{00} + L_{20} &\subseteq T_0\overline{T}_1\overline{T}_2, \\ 2)\ L_{01} + L_{22} &\subseteq \overline{T}_0 T_1\overline{T}_2, \\ 3)\ L_{02} + L_{21} &\subseteq \overline{T}_0\overline{T}_1 T_2. \end{aligned}$$

Lemma 5.4.9.

$$\overline{T}L \subseteq \tilde{M}.$$

Lemma 5.4.10.

$$\overline{T}L \subseteq \tilde{U}.$$

Lemma 5.4.11.

$$\overline{T}L \subseteq \tilde{B}.$$

Lemma 5.5.1.

$$\begin{aligned} 1)\ M_q\overline{T} &\subseteq T_pT_r, \\ 2)\ M_qT_pT_q &\subseteq T_{pq}, \\ 3)\ M_qT_qT_r &\subseteq T_{qr}. \end{aligned}$$

Corollary 5.5.1.

$$M_1M_2\overline{T} \subseteq T_0T_1T_2T_{01}T_{12}T_{20}.$$

Corollary 5.5.2.

$$U_2 = B_1 \text{ and } U_1 = B_2 \text{ in } M_1M_2\overline{T}.$$

Lemma 5.5.8.

$$B_1T_{20}M_1 \subseteq U_2U_0.$$

Lemma 5.5.9.

$$U_1 \subseteq \overline{B}_1,\ U_2 \subseteq \overline{B}_2 \text{ and } U_0 \subseteq \overline{B}_0 \text{ in } M_1\overline{M}_2\overline{M}_0.$$

Lemma 5.5.10.

$$\overline{B}_0B_1\overline{B}_2 = B_2B_0\overline{B}_1 \text{ in } M_1\overline{M}_2\overline{M}_0T_{20}.$$

Lemma 5.5.11.

$$U_1 \subseteq B_2B_0 \text{ in } M_1.$$

Note 5.5.2.

$$B_2B_0 = U_1 \text{ in } M_1.$$

Lemma 5.6.1.

$$U_2U_1 \subseteq T_{01}T_{20}T_0B_0.$$

Lemma 5.6.3.

$$\overline{M}_0U_2U_1 \subseteq \overline{T}_{12}.$$

Lemma 5.6.4.

$$U_2\overline{T}_{12} \subseteq \overline{B}_1.$$

Corollary 5.6.1.

$$U_2U_1\overline{DT}_{12} \subseteq \overline{B}_1\overline{B}_2.$$

Lemma 5.6.6.

$$\begin{aligned} U_rT_{pq}\overline{T}_{pr}\overline{D} &\subseteq \overline{B}_p, \\ U_rT_{pq}\overline{T}_{qr}\overline{D} &\subseteq \overline{B}_q. \end{aligned}$$

Lemma 5.6.7.

$$T_p\overline{T}_{pq} \subseteq \overline{B}_p.$$

Corollary 5.6.3.

$$T_pT_q\overline{T}_r \subseteq \overline{B}_r.$$

Lemma 5.6.8.

$$\overline{T}_pT_{pq}\overline{D} \subseteq \overline{T}_{pr}\overline{B}_p.$$

Lemma 5.6.9.

$$\overline{T}_p\overline{T}_q\overline{T}_rT_{pq} \subseteq \overline{B}_r.$$

Lemma 5.6.10.

$$T_{pq}U_r\overline{D} \subseteq \overline{T}_p\overline{T}_q\overline{B}_p\overline{B}_q.$$

Lemma 5.6.11.

$$\overline{T}_p\overline{D} \subseteq \overline{B}_p.$$

Lemma 5.7.1.

$$B_p\overline{D} \subseteq T_p.$$

Lemma 5.7.2.

$$T_pB_q \subseteq T_{pq}.$$