Guidetti, D. and Lorenzi, A.
Osaka J. Math.
44 (2007), 579–613

A MIXED TYPE IDENTIFICATION PROBLEM RELATED TO A PHASE-FIELD MODEL WITH MEMORY

DAVIDE GUIDETTI and ALFREDO LORENZI

(Received January 23, 2006, revised October 11, 2006)

Abstract

In this paper we consider an integro-differential system consisting of a parabolic and a hyperbolic equation related to phase transition models. The first equation is integro-differential and of hyperbolic type. It describes the evolution of the temperature and also accounts for memory effects through a memory kernel k via the *Gurtin-Pipkin* heat flux law. The latter equation, governing the evolution of the order parameter, is semilinear, parabolic and of the fourth order (in space). We prove a local in time existence result and a global uniqueness result for the identification problem consisting in recovering the memory kernel k appearing in the first equation.

Introduction

In order to introduce our mathematical problem, let us consider a smooth bounded container $\Omega \subset \mathbf{R}^d$, $1 \leq d \leq 3$, occupied by the substance undergoing the phase transition. Name θ and χ the basic state variables of the process, corresponding to the *relative* temperature and to the order parameter, respectively. Then, the energy balance equation, describing the evolution of θ, can be written, under the *Gurtin-Pipkin* heat flux law (cf. [11]), in the form

$$D_t(\theta + l\chi) - \Delta(k * \theta) = f, \tag{0.1}$$

where $*$ stands for the standard time convolution product for functions with their supports in $\mathbf{R}_+$, $k\colon [0, T] \to \mathbf{R}$, $k(0) > 0$, is the so-called heat conductivity relaxation kernel, f is a heat source also incorporating an additional term depending on the past history of θ up to $t = 0$, which is assumed to be given, and l is a positive constant accounting for the latent heat.

Then, this first equation—of hyperbolic type after differentiation—, ruling the evolution of θ, is coupled with the *kinetic equation* for the phase variable, which is of *parabolic* type. We will consider the *conserved* case

$$D_t\chi - \Delta w = 0, \quad w = -\Delta\chi + \beta(\chi) - l\theta. \tag{0.2}$$

2000 Mathematics Subject Classification. 35R30, 45K05, 35M10.

We stress that system (0.1), (0.2)—of mixed hyperbolic-parabolic type—was studied in the paper [4].

We recall that β in (0.2) is usually assumed to be the sum of a general, possibly multivalued, *maximal monotone graph*, representing the derivative of the convex part of a *double well* free energy potential, and of the derivative of a smooth function accounting for its non-convex part. However, in the present paper to solve our problem we will need to consider only sufficiently regular functions β.

Moreover, relations (0.1) and (0.2) are assumed to be complemented by homogeneous Neumann boundary conditions both for θ, χ, and for the auxiliary unknown w, generally called *chemical potential*, and by the Cauchy conditions for θ and χ:

$$\theta(0,x)=\theta_0(x),\quad \chi(0,x)=\chi_0(x),\quad x\in\Omega, \tag{0.3}$$

$$k*\frac{\partial\theta}{\partial\nu}=\frac{\partial\chi}{\partial\nu}=\frac{\partial}{\partial\nu}[-\Delta\chi+\beta(\chi)-l\theta]=0,\quad \text{on}\quad (0,\tau)\times\partial\Omega, \tag{0.4}$$

where $\partial/\partial\nu$ indicates the normal derivative on $\partial\Omega$.

Actually, on account of the boundary condition on w, it is straightforward to check that the average of χ is constant in time.

The main task of this paper concerns the identification of the kernel k entering equation (0.1) under the following additional information involving the temperature θ:

$$\Phi[\theta(t,\ \cdot\ )]:=\int_\Omega\varphi(x)\theta(t,x)\,dx=g(t),\quad t\in[0,\tau]. \tag{0.5}$$

We stress that, to derive a fixed point equation for k, we need to differentiate (in time) equations (0.1) and (0.2) and working on the resulting expression. Further, to perform rigorously such a procedure, however, we need to deal with smooth solutions to problem (0.1)–(0.5). Consequently, we must first obtain preliminary regularity results for the solutions to hyperbolic and parabolic initial and boundary value problems. We point out that these regularity and continuous dependence results for such direct problems, beyond acting as a basis for the subsequent analysis of the inverse problem, might have some independent interest.

In the latter part of our paper, we will proceed to solving our identification problem for k.

Let us note that identifying *memory kernels* in systems of partial integro-differential equations (PIDE) related to transition models is a quite new problem, mainly when they are of *mixed type*. A pioneering paper on this subject is [5] where the author uses analytic semigroup theory to study (locally in time) both the direct and the inverse problems for a *non-conserved* (parabolic) system of PIDE's (cf. [3, 2, 7]), which couples (0.1)—provided with an additional term $-k_0\Delta\theta$, $k_0>0$—, with a second-order *parabolic* equation for the phase variable.

As far as the *conserved model* considered in this paper is concerned, let us quote the papers [6] and [14], where the authors (using semigroup techniques) study the *local* (*in time*) identification of a kernel for system (0.1), (0.2), when the first equation is provided with an additional term $-k_0\Delta\theta$.

We give now the plan of our paper.

In Section 1 we introduce some notation and recall some basic results used throughout the paper.

In Section 2 we present some higher-order regularity results concerning the abstract hyperbolic problem

$$\begin{cases} u''(t) + Au(t) = f(t), & t \in [0, T], \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases} \tag{0.6}$$

where A is a self-adjoint lower bounded operator in the Hilbert space H. In the study of inverse problems, one often needs to work in spaces of quite regular functions. Section 2 is devoted to proving the regularity results given in Proposition 2.9 and Corollary 2.11, which are both used in the final Section 5. In Remark 2.12 we apply our abstract results to the particular case appearing in the inverse problem.

In Section 3 we study the higher-order regularity of the solution to an abstract parabolic system in an L^p-setting with values in a Hilbert space. Of course, the results obtained are applied in Section 5.

In Section 4 we present a careful study of the nonlinear term in our problem. Here also we have to work in spaces of quite regular functions.

Finally, Section 5 is devoted to solving *locally in time* our identification problem (0.1)–(0.5) as well as to proving a *global in time* uniqueness result.

1. Notation

We introduce some notations we shall use in the sequel.

We shall indicate with $\mathbf{N}$ the set of strictly positive integers, while $\mathbf{N}_0$ will stand for $\mathbf{N} \cup \{0\}$.

$\mathbf{R}^+$ will denote the set of strictly positive real numbers.

If $p \in [1, +\infty[$, we shall indicate with p' the dual index $p/(p-1)$, with $p' = \infty$ if $p = 1$.

The notation $W^{m,p}(0, T; X)$, where X is a Banach space, $T \in \mathbf{R}^+$, $m \in \mathbf{N}$, $p \in [1, +\infty[$, will denote the space of all measurable functions from $(0, T)$ into X whose distributional derivatives up to the order m belong to $L^p(0, T; X)$. Such a space is normed by

$$\| f \|_{W^{m,p}(0,T;X)} := \sum_{k=0}^{m-1} \| f^{(k)}(0)\|_X + \| f^{(m)}\|_{L^p(0,T;X)}. \tag{1.1}$$

We note that this norm is equivalent to the usual one

$$\| f \|_{W^{m,p}(0,T;X)} := \left(\sum_{k=0}^{m-1} \| f^{(k)} \|^p_{L^p(0,T;X)} \right)^{1/p}, \tag{1.2}$$

but the constants yielding the equivalence may burst to $+\infty$ as $T \to 0+$.

Since in some fundamental estimates in this paper occur terms such as $\| f^{(k)}(0) \|_X$ and we are going to prove an existence result *in the little* with respect to time, it will be more appropriate to make use of definition (1.1).

Finally, we norm in a similar way $C^m([0, T]; X)$, $m \in \mathbf{N}\backslash\{0\}$:

$$\| f \|_{C^m([0,T];X)} := \sum_{k=0}^{m-1} \| f^{(k)}(0) \|_X + \| f^{(m)} \|_{C([0,T];X)}. \tag{1.3}$$

It is easily seen that, if $f \in C^1([0, T]; X)$ and $f(0) = 0$, then

$$\| f \|_{C([0,T];X)} \le T \| f \|_{C^1([0,T];X)} \tag{1.4}$$

and, if $f \in C([0, T]; X)$,

$$\| f \|_{L^p(0,T;X)} \le T^{1/p} \| f \|_{C([0,T];X)}. \tag{1.5}$$

If Ω is an open subset in $\mathbf{R}^n$, $s > 0$, $p, q \in [1, +\infty]$, we shall indicate with $B^s_{p,q}(\Omega)$ the corresponding Besov space (see, for example, [9], [19]).

If $\mathcal{X}(\Omega)$ is a space of functions of domain Ω, with Ω open and $\partial\Omega$ smooth, we shall put, whenever the expression has a meaning,

$$\mathcal{X}_B(\Omega) := \left\{ f \in \mathcal{X}(\Omega) \colon \frac{\partial f}{\partial \nu} \equiv 0 \right\}, \tag{1.6}$$

$$\mathcal{X}_{BB}(\Omega) := \left\{ f \in \mathcal{X}(\Omega) \colon \frac{\partial f}{\partial \nu} \equiv \frac{\partial \Delta f}{\partial \nu} \equiv 0 \right\}. \tag{1.7}$$

If (X_0, X_1) is a pair of compatible Banach spaces, $\theta \in (0, 1)$ and $p \in [1, \infty]$, we shall indicate with $(X_0, X_1)_{\theta,p}$ the real interpolation space (see [15]).

We shall indicate with $C(A, B, \ldots)$ positive constants depending on $A, B, \ldots$. In a sequence of estimates, we shall often write $C_1(A, B, \ldots)$, $C_2(A, B, \ldots)$, etc.

2. Hyperbolic problems

We introduce the following basic assumption:

(A1) V and H are Hilbert spaces, $V \subseteq H$ and V is dense in H.

We indicate with $[\cdot, \cdot]$ the scalar product in H, with $(\cdot, \cdot)_V$ the scalar product in V, with $\| \cdot \|_V$ and $\| \cdot \|_H$ the norms in V and H respectively, and with V' the space of antilinear bounded functionals in V. We assume that

(A2) $\|v\|_H \leq \|v\|_V, \ \forall v \in V$
in such a way that V is continuously embedded into H.

Using Riesz theorem, one can easily verify that the natural norm $\|\cdot\|_{V'}$ in V' can be obtained from a scalar product, so that V' is by itself a Hilbert space. If $h \in H$, we identify it with the element $[h, \cdot\,]$ of V'. From (A2) it follows immediately that

$$\|h\|_{V'} \leq \|h\|_H \quad \forall h \in H. \tag{2.1}$$

If $\varphi \in V'$ and $v \in V$, we set

$$[\varphi, v] := \varphi(v). \tag{2.2}$$

(2.2) is consistent with the previous identification of H with a subspace of V'.

Let now a be a sesquilinear, Hermitian continuous functional of domain $V \times V$, such that, for certain $\nu > 0$ and $\mu \geq 0$, the estimate
(A3) $a(v, v) \geq \nu\|v\|_V^2 - \mu\|v\|_H^2 \ \forall v \in V$
holds (observe that, as a is hermitian, $a(v, v) \in \mathbf{R}$). We indicate with $\mathcal{A}$ the linear operator from V to V' such that

$$[\mathcal{A}v, w] = a(v, w), \quad \forall v, w \in V. \tag{2.3}$$

Finally, we introduce the following operator A in H:

$$\begin{cases} D(A) = \{v \in V : \mathcal{A}v \in H\}, \\ Av = \mathcal{A}v, \quad v \in D(A). \end{cases} \tag{2.4}$$

It is convenient to consider also the following operator S in H:

$$S := A + \mu. \tag{2.5}$$

Observe that we should obtain S instead of A if we replaced a with $a + \mu[\,\cdot\,, \,\cdot\,]$. The following properties of S are well known (see, for example, [18], Section 2.2):

Theorem 2.1. *Assume that the assumptions* (A1)–(A3) *hold. Then, if A and S are the operators defined in* (2.4) *and* (2.5), *we have*:
(I) *$D(A)$ is dense in H;*
(II) *S is a positive self-adjoint operator in H;*
(III) *$V = D(S^{1/2})$, with equivalent norms.*

Lemma 2.2. *The operator $S^{1/2}$ can be extended by a linear isomorphism from H to V'.*

Proof. If $h \in H$ and $v \in V$, define

$$[Th, v] := [h, S^{1/2}v]. \tag{2.6}$$

If $\{E(\lambda) \mid \lambda \in \mathbf{R}\}$ is the spectral resolution of S, we have $S^{1/2} = \int_\nu^{+\infty} \lambda^{1/2}\, dE(\lambda)$, so that $S^{1/2}$ is an isomorphism from V to H with inverse $S^{-1/2} = \int_\nu^{+\infty} \lambda^{-1/2}\, dE(\lambda)$. So it is clear that the operator T defined in (2.6) is an extension of $S^{1/2}$ to an element of $\mathcal{L}(H, V')$, as

$$|[h, S^{1/2}v]| \leq \|h\|_H \|S^{1/2}v\|_H \leq \|S^{1/2}\|_{\mathcal{L}(V,H)} \|h\|_H \|v\|_V.$$

Moreover, if $\varphi \in V'$, the equation $Th = \varphi$ has a unique solution $h \in H$: h is the element of H satisfying

$$[h, k] = [\varphi, S^{-1/2}k], \quad \forall k \in H. \qquad \square$$

REMARK 2.3. In the following, we shall indicate with $S^{1/2}$ the operator T introduced in Lemma 2.2.

Proposition 2.4. *Set $X := V \times H$. Define the following operator G in X:*

$$\begin{cases} D(G) := D(A) \times V, \\ G(u, v) = (v, -Au), \quad (u, v) \in D(G). \end{cases} \tag{2.7}$$

Then G is the infinitesimal generator of a semigroup in X.

Proof. With the method developed in the proof of Theorem 2.1, Chapter 5 in [13], if $(u_0, v_0) \in D(A) \times V$, the problem

$$\begin{cases} u''(t) + Au(t) = 0, & t \geq 0, \\ u(0) = u_0, \quad u'(0) = v_0 \end{cases} \tag{2.8}$$

has a unique solution u in $C^2([0, +\infty[; H) \cap C^1([0, +\infty[; V) \cap C([0, +\infty[; D(A))$. If $t \geq 0$, we define

$$S(t)(u_0, v_0) := (u(t), u'(t)). \tag{2.9}$$

From Theorem 8.2 in [12] we have also that for every $T \geq 0$ there exists $C(T) > 0$ such that, if $0 \leq t \leq T$, the estimate

$$\|u(t)\|_V + \|u'(t)\|_H \leq C(T)(\|u_0\|_V + \|v_0\|_H) \tag{2.10}$$

holds. As $D(A) \times V$ is dense in X, we conclude from (2.10) that for any $t \in [0, +\infty[$ $S(t)$ is extensible to an element of $\mathcal{L}(X)$, which we continue to indicate with $S(t)$. From the uniqueness of the solution, it is not difficult to verify that $\{S(t) : t \geq 0\}$ is a strongly continuous semigroup of linear bounded operators in X.

Now we show that its infinitesimal generator is G. We indicate for a moment this generator with G'. Then it is clear that G' is an extension of G, as, if $(u_0, v_0) \in D(G)$, then $S(\cdot)(u_0, v_0) \in C^1([0, +\infty[; X)$ and $S'(0)(u_0, v_0) = G(u_0, v_0)$. On the other hand, from $]-\infty, -\mu] \subseteq \rho(A)$ it follows easily $[\sqrt{\mu}, +\infty[\subseteq \rho(G)$. So, if we pick $\lambda \in \mathbf{R}$ sufficiently large in such a way that $\lambda \in \rho(G) \cap \rho(G')$, we obtain easily that $(\lambda - G)^{-1} = (\lambda - G')^{-1}$, implying immediately $G = G'$. $\square$

Lemma 2.5. *Let G be the infinitesimal generator of a strongly continuous semigroup $\{S(t)\colon t \geq 0\}$ in the Banach space X and let $F \in W^{1,1}(0, T; X)$. Then the function*

$$u(t) := \int_0^t S(t-s)F(s)\, ds, \quad t \in [0, T],$$

belongs to $\in C^1([0, T]; X) \cap C([0, T]; D(G))$ and $u'(t) = Gu(t) + F(t)$ for any $t \in [0, T]$.

Proof. It is well known (see, for example, [16], Corollary 2.5) that, if $F \in C^1([0, T]; X)$, then $u \in C^1([0, T]; X) \cap C([0, T]; D(A))$. Moreover, for any $t \in [0, T]$,

$$u'(t) = S(t)F(0) + \int_0^t S(t-s)F'(s)\, ds, \tag{2.11}$$

$$Au(t) = u'(t) - F(t). \tag{2.12}$$

Fix a sequence $(F_k)_{k\in\mathbf{N}}$ in $C^1([0, T]; X)$, converging to F in $W^{1,1}(0, T; X)$ and indicate with u_k the function obtained replacing F with F_k. Then, from (2.11)–(2.12), we deduce that $(u_k(t))_{k\in\mathbf{N}}$, $(u_k'(t))_{k\in\mathbf{N}}$, $(Au_k(t))_{k\in\mathbf{N}}$ converge, uniformly in $[0, T]$, respectively to $u(t)$, $S(t)F(0) + \int_0^t S(t-s)F'(s)\, ds$ and $S(t)F(0) + \int_0^t S(t-s)F'(s)\, ds - F(t)$. From this the conclusion easily follows. $\square$

Corollary 2.6. *Consider the problem*

$$\begin{cases} u''(t) + Au(t) = f(t), & t \in [0, T], \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases} \tag{2.13}$$

under the assumption (A1)–(A3), *where A is the operator defined in* (2.4). *If $f \in W^{1,1}(0, T; H)$, $u_0 \in D(A)$, $u_1 \in V$, then there exists a unique solution u belonging to $C^2([0, T]; H) \cap C^1([0, T]; V) \cap C([0, T]; D(A))$.*

Moreover, if $f \in W^{1,p}(0, T; V')$, with $1 \leq p \leq +\infty$, then u belongs also to $W^{3,p}(0, T; V')$; if $f \in C^1([0, T]; V')$, then u belongs also to $C^3([0, T]; V')$.

Finally, for any $T_0 > 0$ there exists $C(T_0) > 0$ such that, if $0 < T \leq T_0$,

$$\begin{aligned} &\|u\|_{C^2([0,T];H)} + \|u\|_{C^1([0,T];V)} + \|u\|_{C([0,T];D(A))} \\ &\quad \leq C(T_0)(\|u_0\|_{D(A)} + \|u_1\|_V + \|f\|_{W^{1,1}(0,T;H)}), \end{aligned} \tag{2.14}$$

$$\|u\|_{W^{3,p}(0,T;V')} \leq C(T_0)(\|u_0\|_{D(A)} + \|u_1\|_V + \|f\|_{W^{1,1}(0,T;H)} + \|f\|_{W^{1,p}(0,T;V')}). \tag{2.15}$$

Proof. Putting $v := u'$, the problem (2.13) is equivalent to the system

$$\begin{cases} u'(t) = v(t), & 0 \le t \le T, \\ v'(t) = -Au(t) + f(t), & 0 \le t \le T, \\ u(0) = u_0, \quad v(0) = u_1. \end{cases} \tag{2.16}$$

As the function $t \to (0, f(t))$ belongs to $W^{1,1}(0, T; V \times H)$ and $(u_0, u_1) \in D(G)$, G being defined in (2.7), by Lemma 2.5, problem (2.16) has a unique solution $U = (u, v)$ belonging to $C^1([0, T]; V \times H) \cap C([0, T]; D(A) \times V)$. This implies $u \in C^1([0, T]; V) \cap C([0, T]; D(A))$. As $v \in C^1([0, T]; H)$ and $v = u'$, we conclude that $u \in C^2([0, T]; H)$.

Estimate (2.14) can be obtained by the following argument: extend f to an element $\tilde{f} \in W^{1,1}(0, T_0; H)$ setting $\tilde{f}(t) = f(T)$ if $T < t \le T_0$. Next, consider the problem (2.13) in $[0, T_0]$ replacing f with $\tilde{f}$. If $\tilde{u}$ is its solution, evidently u is the restriction of $\tilde{u}$ to $[0, T]$. So we have

$$\begin{aligned} &\|u\|_{C^2([0,T];H)} + \|u\|_{C^1([0,T];V)} + \|u\|_{C([0,T];D(A))} \\ &\le \|\tilde{u}\|_{C^2([0,T_0];H)} + \|\tilde{u}\|_{C^1([0,T_0];V)} + \|\tilde{u}\|_{C([0,T_0];D(A))} \\ &\le C(T_0)(\|u_0\|_{D(A)} + \|u_1\|_V + \|\tilde{f}\|_{W^{1,1}(0,T_0;H)}) \\ &= C(T_0)(\|u_0\|_{D(A)} + \|u_1\|_V + \|f\|_{W^{1,1}(0,T;H)}). \end{aligned}$$

Assume now that $f \in W^{1,1}(0, T; H) \cap W^{1,p}(0, T; V')$. As $u \in C^1([0, T]; V)$, $Au = \mathcal{A}u \in C^1([0, T]; V')$. As $u'' = -Au + f$, we obtain immediately the conclusion $u \in W^{3,p}(0, T; V')$ and the estimate (2.15). The same argument works in case $f \in C^1([0, T]; V')$. $\square$

If $t \ge 0$, $u_0 \in V$, $u_1 \in H$, set

$$S(t)(u_0, u_1) = (S_{11}(t)u_0 + S_{12}(t)u_1, S_{21}(t)u_0 + S_{22}(t)u_1). \tag{2.17}$$

S_{11}, S_{12}, S_{21}, S_{22} are strongly continuous with values in $\mathcal{L}(V)$, $\mathcal{L}(H, V)$, $\mathcal{L}(V, H)$, $\mathcal{L}(H)$, respectively.

Moreover, for any $u_0 \in D(A)$, for any $u_1 \in V$,

$$S_{11}(t)u_0 + S_{12}(t)u_1 = u_0 + \int_0^t [S_{21}(s)u_0 + S_{22}(s)u_1]\, ds. \tag{2.18}$$

By continuity, (2.18) can be extended to $u_0 \in V$, $u_1 \in H$. So $S'_{11}(t)u_0 = S_{21}(t)u_0$ for any $u_0 \in V$, $S'_{12}(t)u_1 = S_{22}(t)u_1$ for any $u_1 \in H$.

Finally, $S_{11}(0) = I_V$, $S_{22}(0) = I_H$, $S_{12}(0)u_1 = 0$ for any $u_1 \in H$, $S_{21}(0)u_0 = 0$ for any $u_0 \in V$.

The solution u of the problem (2.13) under the assumptions of Corollary 2.6 can be represented in the form

$$u(t) = S_{11}(t)u_0 + S_{12}(t)u_1 + \int_0^t S_{12}(t-s) f(s)\, ds. \tag{2.19}$$

We consider the function u given by (2.19) under conditions on u_0, u_1, f which are less restrictive if compared with those of Corollary 2.6.

Proposition 2.7. *Consider the function u defined in* (2.19) *assuming that $f \in L^1(0, T; H)$, $u_0 \in V$, $u_1 \in H$. Then u belongs to $C^1([0, T]; H) \cap C([0, T]; V)$.*

Moreover, if $f \in L^p(0, T; V')$ $(1 \le p \le +\infty)$, then u belongs also to $W^{2,p}(0, T; V')$, if $f \in C([0, T]; V')$, $u \in C^2([0, T]; V')$.

Finally, for any $T_0 > 0$ there exists $C(T_0) > 0$ such that, if $0 < T \le T_0$,

$$\|u\|_{C^1([0,T];H)} + \|u\|_{C([0,T];V)} \le C(T_0)(\|u_0\|_V + \|u_1\|_H + \|f\|_{L^1(0,T;H)}), \tag{2.20}$$

$$\|u\|_{W^{2,p}(0,T;V')} \le C(T_0)(\|u_0\|_V + \|u_1\|_H + \|f\|_{L^1(0,T;H)} + \|f\|_{L^p(0,T;V')}). \tag{2.21}$$

Proof. As S_{11} and S_{12} are strongly continuous with values in $\mathcal{L}(V)$ and $\mathcal{L}(H, V)$ respectively, $u \in C([0, T]; V)$. Moreover, from $S'_{11} = S_{21}$, $S'_{12} = S_{22}$ and $S_{12}(0) = 0$, we obtain that $u \in C^1([0, T]; H)$ and

$$u'(t) = S_{21}(t)u_0 + S_{22}(t)u_1 + \int_0^t S_{22}(t-s) f(s)\, ds \quad \forall t \in [0, T].$$

Consider now a sequence $(u_{0,k})_{k\in\mathbf{N}}$ in $D(A)$, converging to u_0 in V, a sequence $(u_{1,k})_{k\in\mathbf{N}}$ in V, converging to u_1 in H and a sequence $(f_k)_{k\in\mathbf{N}}$ in $W^{1,1}(0, T; H)$, converging to f in $L^1(0, T; H)$. Call u_k the solution of (2.13) obtained replacing u_0 with u_{0k}, u_1 with $u_{1,k}$, f with f_k. Then $u_k \in C^2([0, T]; H) \cap C^1([0, T]; V)$ and the sequence $(u_k)_{k\in\mathbf{N}}$ converges to u in $C^1([0, T]; H) \cap C([0, T]; V)$. As $u''_k = -\mathcal{A}u_k + f_k$, the sequence $(u''_k)_{k\in\mathbf{N}}$ converges to $-\mathcal{A}u + f$ in $L^1(0, T; V')$. So $u'' = -\mathcal{A}u + f$, where u'' is the second derivative of u in the sense of distributions with values in V'. As $\mathcal{A}u \in C([0, T]; V')$, if $f \in L^p(0, T; V')$, then $u \in W^{2,p}(0, T; V')$. In the same way, it follows that, if $f \in C([0, T]; V')$, then $u \in C^2([0, T]; V')$.

Estimates (2.20) and (2.21) can be obtained with the arguments used to prove estimate (2.14), extending f to $]T, T_0]$ by 0. □

DEFINITION 2.8. Let $\alpha \in [-1/2, 0[$. We set

$$D(S^\alpha) := \{\varphi \in V' : \exists v \in D(S^{1/2+\alpha}) : \varphi = S^{1/2} v\},$$

where $S^{1/2}$ is defined in H (see Lemma 2.2 and Remark 2.3). v is obviously unique and coincides with $S^{-1/2}\varphi$, where $S^{-1/2}$ is assumed to be extended to V'. If $\varphi \in D(S^\alpha)$, we set

$$\|\varphi\|_{D(S^\alpha)} := \|S^{-1/2}\varphi\|_{D(S^{1/2+\alpha})}. \tag{2.22}$$

Corollary 2.6 and Proposition 2.7 admit the following extensions, which can be easily obtained using the fractional powers S^α of S:

Proposition 2.9. *Consider the problem* (2.13)*, under the assumption* (A1)–(A3)*, where A is the operator defined in* (2.4)*. Let $\alpha \in [0, +\infty[$. Assume that $f \in W^{1,1}(0, T; D(S^\alpha))$, $u_0 \in D(S^{1+\alpha})$, $u_1 \in D(S^{1/2+\alpha})$. Then the solution u belongs to $C^2([0, T]; D(S^\alpha)) \cap C^1([0, T]; D(S^{1/2+\alpha})) \cap C([0, T]; D(S^{1+\alpha}))$. Moreover, if $f \in W^{1,p}(0, T; D(S^{\alpha-1/2}))$ $(1 \le p \le +\infty)$, then u belongs also to $W^{3,p}(0, T; D(S^{\alpha-1/2}))$; if $f \in C^1([0, T]; D(S^{\alpha-1/2}))$, then u belongs also to $C^3([0, T]; D(S^{\alpha-1/2}))$.*

Finally, for any $T_0 > 0$ there exists $C(T_0) > 0$ such that, if $0 < T \le T_0$,

$$\begin{aligned}&\|u\|_{C^2([0,T];D(S^\alpha))} + \|u\|_{C^1([0,T];D(S^{1/2+\alpha}))} + \|u\|_{C([0,T];D(S^{1+\alpha}))}\\ &\quad \le C(T_0)(\|u_0\|_{D(S^{1+\alpha})} + \|u_1\|_{D(S^{1/2+\alpha})} + \|f\|_{W^{1,1}(0,T;D(S^\alpha))}),\end{aligned} \tag{2.23}$$

$$\begin{aligned}&\|u\|_{W^{3,p}(0,T;D(S^{\alpha-1/2}))}\\ &\le C(T_0)(\|u_0\|_{D(S^{1+\alpha})} + \|u_1\|_{D(S^{1/2+\alpha})} + \|f\|_{W^{1,1}(0,T;D(S^\alpha))} + \|f\|_{W^{1,p}(0,T;D(S^{\alpha-1/2}))}).\end{aligned} \tag{2.24}$$

Proposition 2.10. *Let $\alpha \in [0, +\infty[$. Consider the function u defined in* (2.19) *under the assumptions $f \in L^1(0, T; D(S^\alpha))$, $u_0 \in D(S^{\alpha+1/2})$, $u_1 \in D(S^\alpha)$. Then u belongs to $C^1([0, T]; D(S^\alpha)) \cap C([0, T]; D(S^{\alpha+1/2}))$. Moreover, if $f \in L^p(0, T; D(S^{\alpha-1/2}))$, $(1 \le p \le +\infty)$ then u belongs also to $W^{2,p}(0, T; D(S^{\alpha-1/2}))$; if $f \in C([0, T]; D(S^{\alpha-1/2}))$, then u belongs also to $C^2([0, T]; D(S^{\alpha-1/2}))$. Finally, for any $T_0 > 0$ there exists $C(T_0) > 0$ such that, if $0 < T \le T_0$,*

$$\begin{aligned}&\|u\|_{C^1([0,T];D(S^\alpha))} + \|u\|_{C([0,T];D(S^{\alpha+1/2}))}\\ &\quad \le C(T_0)(\|u_0\|_{D(S^{\alpha+1/2})} + \|u_1\|_{D(S^\alpha)} + \|f\|_{L^1(0,T;D(S^\alpha))}),\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}&\|u\|_{W^{2,p}(0,T;D(S^{\alpha-1/2}))}\\ &\le C(T_0)(\|u_0\|_{D(S^{\alpha+1/2})} + \|u_1\|_{D(S^\alpha)} + \|f\|_{L^1(0,T;D(S^\alpha))} + \|f\|_{L^p(0,T;D(S^{\alpha-1/2}))}).\end{aligned} \tag{2.26}$$

In Section 5 we shall need the following further regularity result:

Corollary 2.11. *Consider problem* (2.13)*, with $f \in W^{1,p}(0, T; V') \cap C([0, T]; V) \cap L^p(0, T; D(S^{3/2}))$, $u_0 = u_1 = 0$. Then the function u defined in* (2.19) *belongs to*

$W^{3,p}(0, T; V') \cap C^2([0, T]; V) \cap C^1([0, T]; D(S^{3/2}))$. *Moreover, if* $T \leq T_0$, *there exist* $C(T_0) > 0$, *such that*

$$\begin{aligned}
&\|u\|_{W^{3,p}(0,T;V')} + \|u\|_{C^2(0,T;V)} + \|u\|_{C^1(0,T;D(S^{3/2}))} \\
&\leq C(T_0)(\|f\|_{W^{1,p}(0,T;V')} + \|f\|_{C([0,T];V)} + \|f\|_{L^p(0,T;D(S^{3/2}))}).
\end{aligned}$$

Proof. By Proposition 2.10, with $\alpha = 3/2$, $u \in C^1([0, T]; D(S^{3/2}))$, so that $Au \in C^1([0, T]; V)$. Again by Proposition 2.10, with $\alpha = 0$, $u \in C^2([0, T]; V')$. As $u'' = -Au + f \in C([0, T]; V)$, $u \in C^2([0, T]; V)$. As $-Au + f \in W^{1,p}(0, T; V')$, $u \in W^{3,p}(0, T; V')$. Moreover, by Proposition 2.10,

$$\|u\|_{C^1([0,T];D(S^{3/2}))} \leq C_1(T_0)\|f\|_{L^1(0,T;D(S^{3/2}))} \leq C_1(T_0)T^{1/p'}\|f\|_{L^p(0,T;D(S^{3/2}))}, \tag{2.27}$$

so that

$$\begin{aligned}
\|u\|_{C^2([0,T];V)} &= \|D_t^2 u\|_{C([0,T];V)} \leq \|Au\|_{C([0,T];V)} + \|f\|_{C([0,T];V)} \\
&\leq T\|Au\|_{C^1([0,T];V)} + \|f\|_{C([0,T];V)} \\
&\leq CT\|u\|_{C^1([0,T];D(S^{3/2}))} + \|f\|_{C([0,T];V)} \\
&\leq T^{2-1/p}C_1(T_0)\|f\|_{L^p(0,T;D(S^{3/2}))} + \|f\|_{C([0,T];V)}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\|u\|_{W^{3,p}(0,T;V')} &= \|D_t^2 u\|_{W^{1,p}(0,T;V')} \leq \|Au\|_{W^{1,p}(0,T;V')} + \|f\|_{W^{1,p}(0,T;V')} \\
&\leq C\|Au\|_{W^{1,p}(0,T;V)} + \|f\|_{W^{1,p}(0,T;V')} \\
&\leq CT^{1/p}\|Au\|_{C^1([0,T];V)} + \|f\|_{W^{1,p}(0,T;V')} \\
&\leq C_2(T_0)(\|f\|_{L^p(0,T;D(S^{3/2}))} + \|f\|_{W^{1,p}(0,T;V')}).
\end{aligned}$$ □

REMARK 2.12. Operator S is self adjoint and positive. So, for any $t \in \mathbf{R}^+$, S^{it} is an isometry in H. It follows from Theorem 1.15.3 in [19] that, for any $\alpha \in]0, 1[$, $D(S^\alpha)$ coincides with the complex interpolation space $[H, D(S)]_\alpha$, with equivalent norms.

We apply the previous result to a specific case useful in the sequel. For this purpose we introduce the following condition:

(B1) Ω is an open bounded subset of $\mathbf{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbf{R}^n$ of class C^4.

Under assumption (B1), we set

$$V := H^1(\Omega), \quad H := L^2(\Omega). \tag{2.28}$$

If v and w are elements of V, we set

$$a(v, w) := \int_{\Omega} \nabla v(x) \cdot \overline{\nabla w(x)}\, dx. \tag{2.29}$$

Observe that, for any $v \in V$,

$$a(v, v) = \|v\|_V^2 - \|v\|_H^2,$$

so that (A3) is satisfied. It is well known that, under the assumption (B1),

$$D(A) = D(S) = \left\{ v \in H^2(\Omega) \colon \frac{\partial v}{\partial \nu} = 0 \quad \text{in} \quad \partial\Omega \right\} \tag{2.30}$$

and

$$Au = -\Delta u \quad \forall u \in D(A). \tag{2.31}$$

Owing to Remark 2.12 and Theorem 4.1 in [17], we have, for any $\alpha \in]0, 1[$, with $\alpha \neq 3/4$,

$$D(S^\alpha) = \begin{cases} H^{2\alpha}(\Omega) & \text{if } 0 < \alpha < \dfrac{3}{4}, \\ H_B^{2\alpha}(\Omega) & \text{if } \dfrac{3}{4} < \alpha < 1, \end{cases} \tag{2.32}$$

where $H_B^{2\alpha}(\Omega)$ denotes the space defined by (1.6) with $\mathcal{X} = H^{2\alpha}(\Omega)$.

Owing to the regularity of $\partial\Omega$ and known results of regularity for solutions of elliptic boundary value problems (see, for example, [9] in case $p = q = 2$), we have also, for any $\alpha \in]0, 1[$, with $\alpha \neq 3/4$,

$$D(S^{1+\alpha}) = \begin{cases} H_B^{2(1+\alpha)}(\Omega) & \text{if } 0 < \alpha < \dfrac{3}{4}, \\ H_{BB}^{2(1+\alpha)}(\Omega) & \text{if } \dfrac{3}{4} < \alpha < 1, \end{cases} \tag{2.33}$$

where, if $\alpha > 3/4$, $H_{BB}^{2(1+\alpha)}(\Omega)$ denotes the space defined by (1.7) with $\mathcal{X} = H^{2(1+\alpha)}(\Omega)$.

3. Parabolic problems

We start with the following result, due to De Simon (see [8]):

Theorem 3.1. *Let B be the infinitesimal generator of an analytic semigroup*

$(e^{tB})_{t\geq 0}$ in the Hilbert space X. Let $T \in \mathbf{R}^+$, $p \in]1,+\infty[$, $f \in L^p(0,T;X)$. Set, for $t \in [0,T]$,

$$v(t) := \int_0^t e^{(t-s)B} f(s)\,ds.$$

Then $v \in W^{1,p}(0,T;X) \cap L^p(0,T;D(B))$.

Corollary 3.2. *Let B be the infinitesimal generator of an analytic semigroup $(e^{tB})_{t\geq 0}$ in the Hilbert space X. Let $T \in \mathbf{R}^+$, $p \in]1,+\infty[$, $f \in L^1(0,T;X)$, $v_0 \in X$. Then the Cauchy problem*

$$\begin{cases} v'(t) = Bv(t) + f(t), & t \in [0,T], \\ v(0) = v_0 \end{cases} \tag{3.1}$$

has a solution v in $W^{1,p}(0,T;X) \cap L^p(0,T;D(B))$ if and only if $f \in L^p(0,T;X)$ and $v_0 \in (X, D(B))_{1/p',p}$. In this case the solution with such regularity is unique and can be represented in the form

$$v(t) := e^{tB} v_0 + \int_0^t e^{(t-s)B} f(s)\,ds. \tag{3.2}$$

Finally, for any $T_0 \in \mathbf{R}^+$, there exists $C(T_0) > 0$, such that, if $0 < T \leq T_0$, $f \in L^p(0,T;X)$ and $v_0 \in (X, D(B))_{1/p',p}$,

$$\|v\|_{W^{1,p}(0,T;X)} + \|v\|_{L^p(0,T;D(B))} \leq C(T_0)[\|v_0\|_{(X,D(B))_{1/p',p}} + \|f\|_{L^p(0,T;X)}]. \tag{3.3}$$

Proof. The condition $f \in L^p(0,T;X)$ is clearly necessary to get a solution with the required regularity. Moreover, $\{v(0) \mid v \in W^{1,p}(0,T;X) \cap L^p(0,T;D(B))\} = (X, D(B))_{1/p',p}$ (see [15], Proposition 1.2.10).

On the other hand, assume that $f \in L^p(0,T;X)$ and $v_0 \in (X, D(B))_{1/p',p}$. Let v be the function defined in (3.2). Then, by Theorem 3.1, $t \to \int_0^t e^{(t-s)B} f(s)\,ds \in W^{1,p}(0,T;X) \cap L^p(0,T;D(B))$. Moreover, by Proposition 2.2.2 in [15], $t \to e^{tB}v_0$ belongs to $L^p(0,T;D(B))$. As its derivative is $Be^{tB}v_0$, we conclude that $v \in W^{1,p}(0,T;X) \cap L^p(0,T;D(B))$.

If $f \in C^\alpha([0,T];X)$, for some $\alpha \in \mathbf{R}^+$, and $v_0 \in D(B)$, it is well known that $v \in C^1([0,T];X) \cap C([0,T];D(B))$. Moreover, $v'(t) = Bv(t) + f(t)$ for all $t \in [0,T]$. Approximating f by a sequence $(f_k)_{k\in\mathbf{N}}$ with values in $C^1([0,T];X)$, converging to f in $L^p(0,T;X)$, and v_0 with a sequence $(v_k)_{k\in\mathbf{N}}$ with values in $D(B)$, converging to v_0 in $(X, D(B))_{1/p',p}$ (which exists, owing to Proposition 1.2.12 in [15]), we conclude that

$$v'(t) - Bv(t) = f(t)$$

almost everywhere in $]0,T[$ and in the sense of vector valued distributions.

To show the uniqueness, let $v \in W^{1,p}(0,T;X) \cap L^p(0,T;D(B))$ be such that $v(0) = 0$ and $v'(t) = Bv(t)$ almost everywhere in $]0, T[$. Set, for $t \in [0, T]$,

$$V(t) := \int_0^t v(s)\, ds.$$

Then $V \in C^1([0, T]; X) \cap C([0, T]; D(B))$. Moreover, for all $t \in [0, T]$,

$$V'(t) = v(t) = \int_0^t Bv(s)\, ds = BV(t).$$

Owing to well known properties of semigroups, we conclude that $V \equiv 0$, and so $v = V' \equiv 0$.

To prove estimate (3.3), one can argue as in the proof of the estimate (2.14), extending f to the element $\tilde{f}$ of $L^p(0, T_0; X)$ such that $\tilde{f}(t) = 0$ if $t \in]T, T_0]$. □

Lemma 3.3. *Let B be the infinitesimal generator of an analytic semigroup $(e^{tB})_{t\geq 0}$ in the Hilbert space X. Let $T \in \mathbf{R}^+$, $p \in]1, +\infty[$, $f \in L^1(0, T; X)$, $v_0 \in X$. Then*:

(I) *the Cauchy problem* (3.1) *has a solution v in $W^{2,p}(0, T; X) \cap W^{1,p}(0, T; D(B))$ if and only if $f \in W^{1,p}(0, T; X)$, $v_0 \in D(B)$ and $v_1 := Bv_0 + f(0) \in (X, D(B))_{1/p',p}$*;

(II) *the Cauchy problem* (3.1) *has a solution v in $W^{3,p}(0, T; X) \cap W^{2,p}(0, T; D(B))$ if and only if $f \in W^{2,p}(0, T; X)$, $v_0, v_1 \in D(B)$ and $v_2 := Bv_1 + f'(0) \in (X, D(B))_{1/p',p}$.*

Moreover, for any $T_0 \in \mathbf{R}^+$, there exists $C(T_0) > 0$, such that, in case (I) *with $0 < T \leq T_0$, $f \in W^{1,p}(0, T; X)$, $v_0 \in D(B)$, $v_1 \in (X, D(B))_{1/p',p}$,*

$$\begin{aligned} &\|v\|_{W^{2,p}(0,T;X)} + \|v\|_{W^{1,p}(0,T;D(B))} \\ &\quad \leq C(T_0)[\|v_0\|_{D(B)} + \|v_1\|_{(X,D(B))_{1/p',p}} + \|f\|_{W^{1,p}(0,T;X)}], \end{aligned} \tag{3.4}$$

in case (II), *with $0 < T \leq T_0$, $f \in W^{2,p}(0, T; X)$, $v_0, v_1 \in D(B)$, $v_2 \in (X, D(B))_{1/p',p}$,*

$$\begin{aligned} &\|v\|_{W^{3,p}(0,T;X)} + \|v\|_{W^{2,p}(0,T;D(B))} \\ &\quad \leq C(T_0)[\|v_0\|_{D(B)} + \|v_1\|_{D(B)} + \|v_2\|_{(X,D(B))_{1/p',p}} + \|f\|_{W^{2,p}(0,T;X)}]. \end{aligned} \tag{3.5}$$

Proof. As $W^{1,p}(0,T;D(B)) \subseteq C([0,T];D(B))$, if $v \in W^{2,p}(0,T;X) \cap W^{1,p}(0,T;D(B))$ and solves (3.1), then $v_0 \in D(B)$ and $f \in W^{1,p}(0, T; X)$. Moreover, $w := v'$ belongs to $W^{1,p}(0, T; X) \cap L^p(0, T; D(B))$ and solves the problem

$$\begin{cases} w'(t) = Bw(t) + f'(t), & t \in [0, T], \\ w(0) = v_1. \end{cases} \tag{3.6}$$

So, necessarily $v_1 \in (X, D(B))_{1/p',p}$.

On the other hand, if $w \in W^{1,p}(0, T; X) \cap L^p(0, T; D(B))$ and solves (3.6), setting $v(t) := v_0 + 1 * w$, we get $v \in W^{2,p}(0, T; X) \cap W^{1,p}(0, T; D(B))$. Moreover, for any $t \in]0, T[$,

$$\begin{aligned} v'(t) = w(t) &= v_1 + \int_0^t (Bw(s) + f'(s))\, ds \\ &= v_1 + Bv(t) - Bv_0 + f(t) - f(0) = Bv(t) + f(t). \end{aligned}$$

Thus (I) is proved.

(II) can be shown with similar arguments.

The final estimates can be proved applying (3.3) to the derivatives of v. □

We consider now operator $B := -A^2$, with A defined in (2.30)–(2.31), under assumption $(B1)$, in the space $X = H = L^2(\Omega)$. So, from well known results concerning the regularity of solutions of elliptic problems, its domain is $H^4_{BB}(\Omega)$. It is well known that B is the infinitesimal generator of an analytic semigroup in H. Next, we set $V := H^1(\Omega)$ and consider the following operator $\mathcal{B}$ in V' of domain $H^3_B(\Omega)$:

$$(\mathcal{B}v, w) = \int_\Omega \nabla(\Delta v)(x) \cdot \overline{\nabla w(x)}\, dx, \quad v \in H^3_B(\Omega),\ w \in V. \tag{3.7}$$

Owing to the isomorphism $S^{1/2}$, even $\mathcal{B}$ is the infinitesimal generator of an analytic semigroup in V'. The following result holds:

Proposition 3.4. *Consider the Cauchy problem*

$$\begin{cases} v'(t) = Bv(t) + f(t), & t \in [0, T], \\ v(0) = v_0. \end{cases} \tag{3.8}$$

Let $p \in]1, +\infty[$. Then the following conditions are necessary and sufficient in order that there exist a unique solution $v \in W^{3,p}(0, T; V') \cap W^{2,p}(0, T; H^3_B(\Omega)) \cap W^{1,p}(0, T; H^4_{BB}(\Omega))$:

(I) $f \in W^{2,p}(0, T; V') \cap W^{1,p}(0, T; L^2(\Omega))$;

(II) $v_0 \in H^4_{BB}(\Omega))$;

(III) $v_1 := Bv_0 + f(0) \in (H^3_B(\Omega), H^4_{BB}(\Omega))_{1/p',p}$;

(IV) $v_2 := \mathcal{B}v_1 + f'(0) \in (V', H^3_B(\Omega))_{1/p',p}$.

Moreover, for any $T_0 \in \mathbf{R}^+$ there exists $C(T_0) > 0$, such that, if $0 < T \le T_0$ and the conditions (I)–(IV) *are satisfied,*

$$\begin{aligned} &\|v\|_{W^{3,p}(0,T;V')} + \|v\|_{W^{2,p}(0,T;H^3(\Omega))} + \|v\|_{W^{1,p}(0,T;H^4(\Omega))} \\ &\le C(T_0)\big[\|f\|_{W^{2,p}(0,T;V')} + \|f\|_{W^{1,p}(0,T;L^2(\Omega))} + \|v_0\|_{H^4(\Omega)} \\ &\qquad + \|v_1\|_{(H^3_B(\Omega), H^4_{BB}(\Omega))_{1/p',p}} + \|v_2\|_{(V', H^3_B(\Omega))_{1/p',p}}\big]. \end{aligned} \tag{3.9}$$

Proof. Clearly the only possible solution is

$$v(t) = e^{tB}v_0 + \int_0^t e^{(t-s)B} f(s)\, ds = e^{t\mathcal{B}}v_0 + \int_0^t e^{(t-s)\mathcal{B}} f(s)\, ds. \tag{3.10}$$

Owing to Lemma 3.3, necessary and sufficient conditions for (3.8) to have a solution with the desired regularity are:
(a) $f \in W^{2,p}(0, T; V') \cap W^{1,p}(0, T; L^2(\Omega))$;
(b) $v_0 \in H^4_{BB}(\Omega))$;
(c) $v_1 \in (L^2(\Omega), H^4_{BB}(\Omega))_{1/p',p} \cap H^3_B(\Omega)$;
(d) $v_2 \in (V', H^3_B(\Omega))_{1/p',p}$.

So (I)–(II) and (IV) are necessary. As $v' \in W^{1,p}(0, T; H^3_B(\Omega)) \cap L^p(0, T; H^4_{BB}(\Omega))$ and $v'(0) = v_1$, even (III) is necessary, again by [15], Proposition 1.2.10.

On the other hand, (I)–(IV) imply (a)–(d).

(3.9) follows from (3.4) and (3.5). □

Proposition 3.5. *Let* $p \geq 2$. *Then*

$$W^{3,p}(0, T; V') \cap W^{2,p}(0, T; H^3(\Omega)) \subseteq C^2([0, T]; H^1(\Omega)).$$

In particular, if assumptions (I)–(IV) *of* Proposition 3.4 *are fulfilled, the solution* v *of problem* (3.8) *belongs to* $C^2([0, T]; H^1(\Omega))$. *Moreover, for any* $T_0 \in \mathbf{R}^+$, *there exists* $C(T_0) \in \mathbf{R}^+$, *such that, if* $T \in]0, T_0]$,

$$\begin{aligned} &\|v\|_{C^2([0,T];H^1(\Omega))} \\ &\leq C(T_0)\big[\|f\|_{W^{2,p}(0,T;V')} + \|v_0\|_{H^3(\Omega)} + \|v_1\|_{H^3(\Omega)} + \|v_2\|_{(V',H^3_B(\Omega))_{1-1/p,p}}\big]. \end{aligned} \tag{3.11}$$

Proof. Let $v \in W^{3,p}(0, T; V') \cap W^{2,p}(0, T; H^3(\Omega))$. Then we have that $D_t^2 v \in W^{1,p}(0, T; V') \cap L^p(0, T; H^3(\Omega)) \subseteq W^{1,2}(0, T; V') \cap L^2(0, T; H^3(\Omega))$. So, by Theorem 3.1 in Chapter 1 of [12], $D_t^2 v$ is continuous with values in the complex interpolation space $[V', H^3(\Omega)]_{1/2}$, coinciding with $H^1(\Omega)$. As $v(0)$ and $D_t v(0)$ are elements of $H^3(\Omega)$, $v \in C^2([0, T]; H^1(\Omega))$.

Let now v be the solution of (3.8), with $T \in]0, T_0]$. If the conditions (I)–(IV) are satisfied and $p \geq 2$, from the first part of the statement, we have $v \in C^2([0, T]; H^1(\Omega))$. Then, employing again the argument used to show estimate (2.14), we obtain estimate (3.11). □

REMARK 3.6. It is well known (see, for example, [10], Theorem 3.5), that

$$(H^3_B(\Omega), H^4_{BB}(\Omega))_{1/p',p} = \begin{cases} B^{4-1/p}_{2,p,B}(\Omega) & \text{if } 1 < p < 2, \\ B^{4-1/p}_{2,p,BB}(\Omega) & \text{if } 2 < p < +\infty. \end{cases} \tag{3.12}$$

A more involved characterization is known in case $p = 2$ (see [10]).

Proposition 3.7. *Under the assumption* (B1), *we have that*

$$(V', H^3_B(\Omega))_{1/p',p} = \begin{cases} S^{1/2}\big(B^{4(1/p')}_{2,p}(\Omega)\big) & \text{if } 1 < p < \dfrac{8}{5}, \\ B^{3-4/p}_{2,p}(\Omega) & \text{if } \dfrac{4}{3} < p < \dfrac{8}{3}, \\ B^{3-4/p}_{2,p,B}(\Omega) & \text{if } \dfrac{8}{3} < p < +\infty. \end{cases} \tag{3.13}$$

Proof. Owing to Theorem 3.5 in [10], we have

$$(L^2(\Omega), H^4_{BB}(\Omega))_{1/p',p} = \begin{cases} B^{4/p'}_{2,p}(\Omega) & \text{if } \dfrac{4}{p'} < \dfrac{3}{2}, \\ B^{4/p'}_{2,p,B}(\Omega) & \text{if } \dfrac{3}{2} < \dfrac{4}{p'} < \dfrac{5}{2}, \\ B^{4/p'}_{2,p,BB}(\Omega) & \text{if } \dfrac{4}{p'} > \dfrac{5}{2}. \end{cases} \tag{3.14}$$

As $H^4_{BB}(\Omega) = D(S^2)$ and $H^3_B(\Omega) = D(S^{3/2})$ (see (2.33)), $S^{1/2}$ is an isomorphism between $H^4_{BB}(\Omega)$ and $H^3_B(\Omega)$. So, by Lemma 2.2,

$$(V', H^3_B(\Omega))_{1/p',p} = S^{1/2}((L^2(\Omega), H^4_{BB}(\Omega))_{1/p',p}) = S^{1/2}\big(B^{4/p'}_{2,p}(\Omega)\big)$$

if $1 < p < 8/5$.

Moreover, as

$$(L^2(\Omega), H^4_{BB}(\Omega))_{1/4,1} \subseteq H^1(\Omega) \subseteq (L^2(\Omega), H^4_{BB}(\Omega))_{1/4,\infty}$$

(by Theorem 3.5 and Proposition 1.6 in [10]), using again the isomorphism $S^{1/2}$, we can say that

$$(V', H^3_B(\Omega))_{1/4,1} \subseteq L^2(\Omega) \subseteq (V', H^3_B(\Omega))_{1/4,\infty}.$$

So, by the reiteration theorem, if $p > 4/3$ (so that $1/p' > 1/4$), we have

$$(V', H^3_B(\Omega))_{1/p',p} = (L^2(\Omega), H^3_B(\Omega))_{1-4/3p,p}.$$

So we can get the result applying again Theorem 3.5 in [10]. □

4. An auxiliary nonlinear operator

In this section we study the nonlinear operator $V \to S(\chi_0 + 1 * V)V$, where S: $\mathbf{R} \to \mathbf{R}$ is appropriately smooth, V is a suitably regular function of domain $[0, T] \times \Omega$ and

$$(1 * V)(t, x) := \int_0^t V(s, x)\, ds, \quad t \in [0, T]. \tag{4.1}$$

We introduce the following conditions:
(C1) (B1) holds;
(C2) $n \leq 7$.

Lemma 4.1. *Under the conditions* (C1)–(C2), $H^4(\Omega)$ *is continuously embedded in* $C(\overline{\Omega})$ *and is a space of pointwise multipliers for* $H^s(\Omega)$, *for any* $s \in [0, 4]$.

Proof. The first statement follows from Sobolev embedding theorem.

The embedding of $H^4(\Omega)$ into $C(\overline{\Omega})$ implies also that $H^4(\Omega)$ is a space of pointwise multipliers for $L^2(\Omega)$. By Theorem 5.23 in [1], $H^4(\Omega)$ is also a Banach algebra. So the second statement follows by complex interpolation (see [19], Theorem 2 in Section 4.3.1). □

Lemma 4.2. *Assume that* (C1)–(C2) *hold. Let* $j \in \mathbf{Z}$, $0 \leq j \leq 4$, *and* $S \in C^j(\mathbf{R})$. *Then, for any* $u \in H^4(\Omega)$, $S \circ u$ *is a pointwise multiplier for* $H^j(\Omega)$. *Moreover, if* $\|u\|_{H^4(\Omega)} \leq R$ *and* $v \in H^j(\Omega)$,

$$\|S(u)v\|_{H^j(\Omega)} \leq C(R)\|v\|_{H^j(\Omega)}. \tag{4.2}$$

Finally, if $S \in C^{j+1}(\mathbf{R})$, u_0, u_1 *belong to* $H^4(\Omega)$, $\max\{\|u_0\|_{H^4(\Omega)}, \|u_1\|_{H^4(\Omega)}\} \leq R$ *and* v *belongs to* $H^j(\Omega)$

$$\|S(u_0)v - S(u_1)v\|_{H^j(\Omega)} \leq C(R)\|u_0 - u_1\|_{H^4(\Omega)}\|v\|_{H^j(\Omega)}. \tag{4.3}$$

Proof. First of all, if $u \in H^4(\Omega)$ and $v \in H^j(\Omega)$, then $S(u)v \in H^j(\Omega)$. In fact, it is easily seen that, if $\beta \in \mathbf{N}_0^n$ and $|\beta| \leq j$, $\partial^\beta(S(u)v)$ is a linear combination of terms of the form $S^{(k)}(u)\partial^{\beta_1}u \cdot \dots \cdot \partial^{\beta_k}u\partial^\gamma v$, with $k \leq j$, $|\beta_1| + \dots + |\beta_k| + |\gamma| \leq j$ (here we use the convention that there are no derivatives of u if $k = 0$). Now, $S^{(k)}(u) \in L^\infty(\Omega)$, while $\partial^{\beta_1}u \cdot \dots \cdot \partial^{\beta_k}u\partial^\gamma v \in L^2(\Omega)$. To verify this fact, observe that, if $|\beta| < 4 - n/2$, $\partial^\beta u \in L^\infty(\Omega)$, if $|\gamma| < j - n/2$, $\partial^\gamma v \in L^\infty(\Omega)$. So it suffices to show that the two following products are in $L^2(\Omega)$:
(I) products of the form $\partial^{\beta_1}u \cdot \dots \cdot \partial^{\beta_k}u$ with $1 \leq k \leq 4$, $|\beta_i| \geq 4 - n/2$ for each $i \in \{1, \dots, k\}$, $\sum_{i=1}^k |\beta_i| \leq j$;
(II) products of the form $\partial^{\beta_1}u \cdot \dots \cdot \partial^{\beta_k}u\partial^\gamma v$ with $1 \leq k \leq 4$, $|\beta_i| \geq 4 - n/2$ for each $i \in \{1, \dots, k\}$, $|\gamma| \geq j - n/2$, $\sum_{i=1}^k |\beta_i| + |\gamma| \leq j$.

We consider the case (I). By the Sobolev embedding theorem, we have, if $|\beta_i| > 4 - n/2$, $\partial^{\beta_i}u \in L^{2n/(n-2(4-|\beta_i|))}(\Omega)$, while, if $|\beta_i| = 4 - n/2$, $\partial^{\beta_i}u \in \bigcap_{1 \leq p < +\infty} L^p(\Omega)$. So, by Hölder's inequality, we have to show that $\sum_{i=1}^k (n - 2(4 - |\beta_i|))/(2n) \leq 1/2$

and $\sum_{i=1}^{k}(n-2(4-|\beta_i|))/(2n) < 1/2$, when $|\beta_i| = 4 - n/2$ for some i. Recalling that $n-8<0$, we get

$$\sum_{i=1}^{k}\left(n-\frac{2(4-|\beta_i|)}{2n}\right) = \frac{k(n-8)+2\sum_{i=1}^{k}|\beta_i|}{2n} \leq \frac{n-8+2\sum_{i=1}^{k}|\beta_i|}{2n} \leq \frac{1}{2}.$$

We observe, moreover, that we have equality only in case $k=1$ and $|\beta_1| = 4$. This can occur only when $j=4$. So (I) is completely treated.

We consider (II). Arguing as in the first case, we have to show that $\sum_{i=1}^{k}(n-2(4-|\beta_i|))/(2n) + (n-2(j-|\gamma|))/(2n) \leq 1/2$ and $\sum_{i=1}^{k}(n-2(4-|\beta_i|))/(2n) + (n-2(j-|\gamma|))/(2n) < 1/2$, when $|\beta_i| = 4-n/2$ for some i, or $|\gamma| = j - n/2$. We have:

$$\sum_{i=1}^{k}\left(n-\frac{2(4-|\beta_i|)}{2n}\right)+\left(n-\frac{2(j-|\gamma|)}{2n}\right) = \frac{k(n-8)+2\left(\sum_{i=1}^{k}|\beta_i|+|\gamma|\right)+n-2j}{2n} \leq \frac{n-8+n}{2n} < \frac{1}{2}.$$

Estimate (4.2) follows from the previous considerations and Sobolev embedding theorem, implying, for $k \leq j$,

$$\|S^{(k)}\circ u\|_{L^\infty(\Omega)} \leq \sup_{|\xi|\leq\|u\|_{L^\infty(\Omega)}} |S^{(k)}(\xi)| \leq C(R).$$

Finally, estimate (4.3) is a consequence of the fact that, if $S \in C^{j+1}(\mathbf{R})$, $k \leq j$ and $\max\{\|u_0\|_{H^4(\Omega)}, \|u_1\|_{H^4(\Omega)}\} \leq R$, then $\max\{\|u_0\|_{L^\infty(\Omega)}, \|u_1\|_{L^\infty(\Omega)}\} \leq C_1(R)$, so that

$$\|S^{(k)}\circ u_0 - S^{(k)}\circ u_1\|_{L^\infty(\Omega)} \leq C_2(R)\|u_0-u_1\|_{L^\infty(\Omega)}. \qquad \square$$

In the future we shall need the following simple lemma, the proof of which can be obtained by using Hölder and Young's inequalities.

Lemma 4.3. *Let X be a Banach space, $\tau \in \mathbf{R}^+$, $p \in]1,+\infty[$, $z \in W^{1,p}(0,\tau;X)$, with $z(0)=0$. Then*

$$\|z\|_{L^\infty(0,\tau;X)} \leq \tau^{1/p'}\|z\|_{W^{1,p}(0,\tau;X)}, \tag{4.4}$$

and

$$\|z\|_{L^p(0,\tau;X)} \leq \tau\|z\|_{W^{1,p}(0,\tau;X)}. \tag{4.5}$$

Lemma 4.4. *Assume that* (C1)–(C2) *are satisfied,* $S \in C^4(\mathbf{R})$, $\chi_0 \in H^4(\Omega)$. *Let* $R \in \mathbf{R}^+$, $0 < \tau \leq T$, V_1 *and* V_2 *be elements of* $W^{1,p}(0, \tau; H^4(\Omega)) \cap W^{2,p}(0, \tau; L^2(\Omega))$, $V_1(0) = V_2(0)$, $\max_{j\in\{1,2\}} \|V_j\|_{W^{1,p}(0,\tau;H^4(\Omega))} \leq R$, $\chi_0 \in W^{4,p}(\Omega)$. *Then*

$$\begin{aligned}&\|S(\chi_0 + 1 * V_1)V_1 - S(\chi_0 + 1 * V_2)V_2\|_{W^{1,p}(0,\tau;H^2(\Omega))}\\&\leq C(R, T)\tau^{1/(2p)}(\|V_1 - V_2\|_{W^{1,p}(0,\tau;H^4(\Omega))} + \|V_1 - V_2\|_{W^{2,p}(0,\tau;L^2(\Omega))}).\end{aligned}$$

Proof. We begin with some useful estimates.

First of all,

$$\|V_j\|_{L^\infty(0,\tau;H^4(\Omega))} \leq C(R, T). \tag{4.6}$$

In fact, employing (4.4)

$$\begin{aligned}\|V_j\|_{L^\infty(0,\tau;H^4(\Omega))} &\leq \|V_j(0)\|_{H^4(\Omega)} + \|1 * D_t V_j\|_{L^\infty(0,\tau;H^4(\Omega))}\\&\leq \|V_j(0)\|_{H^4(\Omega)} + \tau^{1/p'}\|1 * D_t V_j\|_{W^{1,p}(0,\tau;H^4(\Omega))}\\&= \|V_j(0)\|_{H^4(\Omega)} + \tau^{1/p'}\|D_t V_j\|_{L^p(0,\tau;H^4(\Omega))} \leq (1 \vee T^{1/p'})R.\end{aligned}$$

Next,

$$\|\chi_0 + 1 * V_j\|_{L^\infty(0,\tau;H^4(\Omega))} \leq C(R, T). \tag{4.7}$$

In fact,

$$\begin{aligned}\|\chi_0 + 1 * V_j\|_{L^\infty(0,\tau;H^4(\Omega))} &\leq \|\chi_0\|_{H^4(\Omega)} + \|1 * V_j\|_{L^\infty(0,\tau;H^4(\Omega))}\\&\leq \|\chi_0\|_{H^4(\Omega)} + \tau\|V_j\|_{L^\infty(0,\tau;H^4(\Omega))} \leq R + T\|V_j\|_{L^\infty(0,\tau;H^4(\Omega))}.\end{aligned}$$

We show that

$$\|1 * (V_1 - V_2)\|_{L^\infty(0,\tau;H^4(\Omega))} \leq \tau^{1+1/p'}\|V_1 - V_2\|_{W^{1,p}(0,\tau;H^4(\Omega))}. \tag{4.8}$$

In fact, using (4.4), we have

$$\begin{aligned}\|1 * (V_1 - V_2)\|_{L^\infty(0,\tau;H^4(\Omega))} &\leq \tau\|V_1 - V_2\|_{L^\infty(0,\tau;H^4(\Omega))}\\&\leq \tau^{1+1/p'}\|V_1 - V_2\|_{W^{1,p}(0,\tau;H^4(\Omega))}.\end{aligned}$$

We show that

$$\begin{aligned}&\|D_t(V_1 - V_2)\|_{L^p(0,\tau;H^2(\Omega))}\\&\leq C(R, T)\tau^{1/(2p)}(\|V_1 - V_2\|_{W^{2,p}(0,\tau;L^2(\Omega))} + \|V_1 - V_2\|_{W^{1,p}(0,\tau;H^4(\Omega))}).\end{aligned} \tag{4.9}$$

In fact, as $H^2(\Omega) \in J_{1/2}(L^2(\Omega), H^4(\Omega))$, by Hölder's inequality,

$$\begin{aligned}
&\left(\int_0^\tau \|D_t(V_1 - V_2)(t)\|^p_{H^2(\Omega)}\, dt\right)^{1/p} \\
&\leq C\left(\int_0^\tau \|D_t(V_1 - V_2)(t)\|^{p/2}_{L^2(\Omega)} \|D_t(V_1 - V_2)(t)\|^{p/2}_{H^4(\Omega)}\, dt\right)^{1/p} \\
&\leq C\|D_t(V_1 - V_2)\|^{1/2}_{L^p(0,\tau;L^2(\Omega))} \|D_t(V_1 - V_2)\|^{1/2}_{L^p(0,\tau;H^4(\Omega))}.
\end{aligned}$$

Moreover, by Minkowski's inequality,

$$\begin{aligned}
&\|D_t(V_1 - V_2)\|_{L^p(0,\tau;L^2(\Omega))} \\
&\leq \tau^{1/p}\|D_t(V_1 - V_2)(0)\|_{L^2(\Omega)} + \|1 * D_t^2(V_1 - V_2)\|_{L^p(0,\tau;L^2(\Omega))} \\
&\leq \tau^{1/p}\|D_t(V_1 - V_2)(0)\|_{L^2(\Omega)} + \tau\|D_t^2(V_1 - V_2)\|_{L^p(0,\tau;L^2(\Omega))} \\
&\leq \max\{1, T^{1/p'}\}\tau^{1/p}\|V_1 - V_2\|_{W^{2,p}(0,\tau;L^2(\Omega))},
\end{aligned}$$

so that

$$\begin{aligned}
&\|D_t(V_1 - V_2)\|_{L^p(0,\tau;H^2(\Omega))} \\
&\leq C(T)\tau^{1/(2p)}\|V_1 - V_2\|^{1/2}_{W^{2,p}(0,\tau;L^2(\Omega))}\|D_t(V_1 - V_2)\|^{1/2}_{L^p(0,\tau;H^4(\Omega))} \\
&\leq C(T)\tau^{1/(2p)}(\|V_1 - V_2\|_{W^{2,p}(0,\tau;L^2(\Omega))} + \|V_1 - V_2\|_{W^{1,p}(0,\tau;H^4(\Omega))}).
\end{aligned}$$

Now we prove the lemma.

As $V_1(0) = V_2(0)$, we have

$$\begin{aligned}
&\|S(\chi_0 + 1 * V_1)V_1 - S(\chi_0 + 1 * V_2)V_2\|_{W^{1,p}(0,\tau;H^2(\Omega))} \\
&= \|S'(\chi_0 + 1 * V_1)V_1^2 + S(\chi_0 + 1 * V_1)D_t V_1 \\
&\quad - S'(\chi_0 + 1 * V_2)V_2^2 - S(\chi_0 + 1 * V_2)D_t V_2\|_{L^p(0,\tau;H^2(\Omega))}.
\end{aligned}$$

For almost every $t \in [0, \tau]$, employing Lemmata 4.1, 4.2, 4.3, (4.7), (4.8), we get

$$\begin{aligned}
&\|S'(\chi_0 + 1 * V_1(t))V_1(t)^2 + S(\chi_0 + 1 * V_1(t))D_t V_1(t) \\
&\quad - S'(\chi_0 + 1 * V_2(t))V_2(t)^2 - S(\chi_0 + 1 * V_2(t))D_t V_2(t)\|_{H^2(\Omega)} \\
&\leq \|V_1(t)[S'(\chi_0 + 1 * V_1(t)) - S'(\chi_0 + 1 * V_2(t))]V_1(t)\|_{H^2(\Omega)} \\
&\quad + \|(V_1(t) + V_2(t))S'(\chi_0 + 1 * V_2(t))(V_1(t) - V_2(t))\|_{H^2(\Omega)} \\
&\quad + \|[S(\chi_0 + 1 * V_1(t)) - S(\chi_0 + 1 * V_2(t))]D_t V_1(t)\|_{H^2(\Omega)} \\
&\quad + \|S(\chi_0 + 1 * V_2(t))[D_t V_1(t) - D_t V_2(t)]\|_{H^2(\Omega)} \\
&\leq C(R, T)[\|1 * (V_1 - V_2)(t)\|_{H^4(\Omega)}(1 + \|D_t V_1(t)\|_{H^2(\Omega)}) \\
&\qquad + \|V_1(t) - V_2(t)\|_{H^2(\Omega)} + \|D_t(V_1 - V_2)(t)\|_{H^2(\Omega)}]
\end{aligned}$$

$$\leq C(R,T)[\tau^{1+1/p'}\|V_1 - V_2\|_{W^{1,p}(0,\tau;H^4(\Omega))}(1 + \|D_t V_1(t)\|_{H^2(\Omega)}) + \|V_1(t) - V_2(t)\|_{H^2(\Omega)} + \|D_t(V_1 - V_2)(t)\|_{H^2(\Omega)}].$$

We conclude, applying (4.5) and (4.9), that

$$\begin{aligned}
&\|S(\chi_0 + 1 * V_1)V_1 - S(\chi_0 + 1 * V_2)V_2\|_{W^{1,p}(0,\tau;H^2(\Omega))} \\
&\leq C(R,T)[\tau^{1+1/p'}\|V_1 - V_2\|_{W^{1,p}(0,\tau;H^4(\Omega))}(\tau^{1/p} + \|D_t V_1\|_{L^p(0,\tau;H^2(\Omega))}) \\
&\qquad + \|V_1 - V_2\|_{L^p(0,\tau;H^2(\Omega))} + \|D_t(V_1 - V_2)\|_{L^p(0,\tau;H^2(\Omega))}] \\
&\leq C_1(R,T)\tau^{1/(2p)}(\|V_1 - V_2\|_{W^{2,p}(0,\tau;L^2(\Omega))} + \|V_1 - V_2\|_{W^{1,p}(0,\tau;H^4(\Omega))}). \qquad \square
\end{aligned}$$

In the following, we shall use without comment the elementary fact that, if $0 < \tau \leq T$ and $0 < \alpha \leq \beta$,

$$\tau^\beta \leq C\tau^\alpha,$$

with $C \in \mathbf{R}^+$, depending only on $\beta - \alpha$ and T.

Lemma 4.5. *Assume that* (C1)–(C2) *are satisfied and* $S \in C^4(\mathbf{R})$. *Let* $R \in \mathbf{R}^+$, $0 < \tau \leq T$, V_1 *and* V_2 *be elements of* $W^{1,p}(0,\tau;H^4(\Omega)) \cap W^{2,p}(0,\tau;H^3_B(\Omega)) \cap W^{3,p}(0,\tau;V')$, *such that* $V_1(0) = V_2(0)$, $D_t V_1(0) = D_t V_2(0)$, $\max_{j\in\{1,2\}}(\|V_j\|_{W^{2,p}(0,\tau;H^3(\Omega))} + \|V_j\|_{W^{1,p}(0,\tau;H^4(\Omega))}) \leq R$. *Then*

$$\begin{aligned}
&\|S(\chi_0 + 1 * V_1)V_1 - S(\chi_0 + 1 * V_2)V_2\|_{W^{2,p}(0,\tau;H^1(\Omega))} \\
&\leq C(R)\tau^{(1/p')\wedge 1/(2p)}(\|V_1 - V_2\|_{W^{1,p}(0,\tau;H^4(\Omega))} \\
&\qquad + \|V_1 - V_2\|_{W^{2,p}(0,\tau;H^3(\Omega))} + \|V_1 - V_2\|_{W^{3,p}(0,\tau;V')}).
\end{aligned}$$

Proof. We start with a couple of useful estimates. First of all,

$$\begin{aligned}
&\|D_t(V_1 - V_2)\|_{L^p(0,\tau;H^1(\Omega))} \\
&\leq \tau\|V_1 - V_2\|_{W^{2,p}(0,\tau;H^3(\Omega))}.
\end{aligned} \tag{4.10}$$

In fact, as $D_t(V_1 - V_2)(0) = 0$,

$$\begin{aligned}
\|D_t(V_1 - V_2)\|_{L^p(0,\tau;H^1(\Omega))} &= \|1 * D_t^2(V_1 - V_2)\|_{L^p(0,\tau;H^1(\Omega))} \\
&\leq \tau\|D_t^2(V_1 - V_2)\|_{L^p(0,\tau;H^1(\Omega))} \leq \tau\|V_1 - V_2\|_{W^{2,p}(0,\tau;H^3(\Omega))}.
\end{aligned}$$

Next,

$$\|D_t^2(V_1 - V_2)\|_{L^p(0,\tau;H^1(\Omega))} \leq C\tau^{1/(2p)}(\|V_1 - V_2\|_{W^{3,p}(0,\tau;V')} + \|V_1 - V_2\|_{W^{2,p}(0,\tau;H^3(\Omega))}). \tag{4.11}$$

This estimate can be obtained with the same method used to prove (4.9), using the fact that $H^1(\Omega) \in J^{1/2}(V', H^3_B(\Omega))$.

We pass to prove the stated result. As $V_1(0) = V_2(0)$ and $D_t V_1(0) = D_t V_2(0)$, we have

$$\begin{aligned}
&\|S(\chi_0 + 1 * V_1)V_1 - S(\chi_0 + 1 * V_2)V_2\|_{W^{2,p}(0,\tau;H^1(\Omega))} \\
&= \|D_t^2[S(\chi_0 + 1 * V_1)V_1 - S(\chi_0 + 1 * V_2)V_2]\|_{L^p(0,\tau;H^1(\Omega))} \\
&= \|S''(\chi_0 + 1 * V_1)V_1^3 - S''(\chi_0 + 1 * V_2)V_2^3 \\
&\quad + 3[S'(\chi_0 + 1 * V_1)V_1 D_t V_1 - S'(\chi_0 + 1 * V_2)V_2 D_t V_2] \\
&\quad + S(\chi_0 + 1 * V_1)D_t^2 V_1 - S(\chi_0 + 1 * V_2)D_t^2 V_2\|_{L^p(0,\tau;H^1(\Omega))}.
\end{aligned}$$

For almost every t in $[0, \tau]$ we have, employing Lemma 4.1, Lemma 4.2, (4.6), (4.8), (4.9), (4.4),

$$\begin{aligned}
&\|S''(\chi_0 + 1 * V_1(t))V_1(t)^3 - S''(\chi_0 + 1 * V_2(t))V_2(t)^3 \\
&\quad + 3[S'(\chi_0 + 1 * V_1(t))V_1(t)D_t V_1(t) - S'(\chi_0 + 1 * V_2(t))V_2(t)D_t V_2(t)] \\
&\quad + S(\chi_0 + 1 * V_1(t))D_t^2 V_1(t) - S(\chi_0 + 1 * V_2(t))D_t^2 V_2(t)\|_{H^1(\Omega)} \\
&\leq \|V_1(t)^2[S''(\chi_0 + 1 * V_1(t)) - S''(\chi_0 + 1 * V_2(t))]V_1(t)\|_{H^1(\Omega)} \\
&\quad + \|(V_1(t)^2 + V_1(t)V_2(t) + V_2(t)^2)S''(\chi_0 + 1 * V_2(t))(V_1(t) - V_2(t))\|_{H^1(\Omega)} \\
&\quad + 3\|V_1(t)[S'(\chi_0 + 1 * V_1(t)) - S'(\chi_0 + 1 * V_2(t))]D_t V_1(t)\|_{H^1(\Omega)} \\
&\quad + 3\|(V_1(t) - V_2(t))S'(\chi_0 + 1 * V_2(t))D_t V_1(t)\|_{H^1(\Omega)} \\
&\quad + 3\|V_2(t)S'(\chi_0 + 1 * V_2(t))D_t(V_1 - V_2)(t)\|_{H^1(\Omega)} \\
&\quad + \|[S(\chi_0 + 1 * V_1(t)) - S(\chi_0 + 1 * V_2(t))]D_t^2 V_1(t)\|_{H^1(\Omega)} \\
&\quad + \|S(\chi_0 + 1 * V_2(t))D_t^2(V_1 - V_2)(t)\|_{H^1(\Omega)} \\
&\leq C(R, T)[\|1 * (V_1 - V_2)(t)\|_{H^4(\Omega)}(1 + \|D_t V_1(t)\|_{H^1(\Omega)} + \|D_t^2 V_1(t)\|_{H^1(\Omega)}) \\
&\qquad + \|V_1(t) - V_2(t)\|_{H^1(\Omega)} + \|V_1(t) - V_2(t)\|_{H^4(\Omega)}\|D_t V_1(t)\|_{H^1(\Omega)} \\
&\qquad + \|D_t(V_1 - V_2)(t)\|_{H^1(\Omega)} + \|D_t^2(V_1 - V_2)(t)\|_{H^1(\Omega)}] \\
&\leq C(R, T)\{\|V_1 - V_2\|_{W^{1,p}(0,\tau;H^4(\Omega))}[\tau^{1+1/p'}(1 + \|D_t V_1(t)\|_{H^1(\Omega)} + \|D_t^2 V_1(t)\|_{H^1(\Omega)}) \\
&\qquad + \tau^{1/p'}(1 + \|D_t V^1(t)\|_{H^1(\Omega)})] \\
&\qquad + \|D_t(V_1 - V_2)(t)\|_{H^1(\Omega)} + \|D_t^2(V_1 - V_2)(t)\|_{H^1(\Omega)}\}.
\end{aligned}$$

So, using (4.10)–(4.11), we get

$$\begin{aligned}
&\|S(\chi_0 + 1 * V_1)V_1 - S(\chi_0 + 1 * V_2)V_2\|_{W^{2,p}(0,\tau;H^1(\Omega))} \\
&\leq C_1(R, T)\{\tau^{1/p'}\|V_1 - V_2\|_{W^{1,p}(0,\tau;H^4(\Omega))} + \|D_t(V_1 - V_2)(t)\|_{L^p(0,\tau;H^1(\Omega))} \\
&\qquad + \|D_t^2(V_1 - V_2)(t)\|_{L^p(0,\tau;H^1(\Omega))}\}
\end{aligned}$$

$$\leq C_1(R, T)\{\tau^{1/p'}\|V_1 - V_2\|_{W^{1,p}(0,\tau;H^4(\Omega))} + \tau\|V_1 - V_2\|_{W^{2,p}(0,\tau;H^3(\Omega))}$$
$$+ \tau^{1/(2p)}(\|V_1 - V_2\|_{W^{3,p}(0,\tau;V')} + \|V_1 - V_2\|_{W^{2,p}(0,\tau;H^3(\Omega))})\},$$

which implies the conclusion. □

5. The inverse problem

In this section we will be concerned with the identification of the kernel k entering in the boundary value problem (0.1)–(0.4).

We start by introducing some notations, which will be useful in the following. As in the previous sections, we indicate with V' the dual space of $H^1(\Omega)$.

Let $s \in \mathbf{N} \cup \{0\}, p \in [2, +\infty[$. We set:

$$\begin{aligned} X^{s+3,p}(\tau) := W^{s+3,p}(0, \tau; V') \cap C^{s+2}([0, \tau]; H^1(\Omega)) \cap C^{s+1}([0, \tau]; H^2_B(\Omega)) \\ \cap C^s([0, \tau]; H^3_B(\Omega)), \end{aligned} \tag{5.1}$$

$$Y^{s+3,p}(\tau) := W^{s+3,p}(0, \tau; V') \cap W^{s+2,p}(0, \tau; H^3_B(\Omega)) \cap W^{s+1,p}(0, \tau; H^4_{BB}(\Omega)), \tag{5.2}$$

$$Z^{s+3,p}(\tau) := X^{s+3,p}(\tau) \times Y^{s+3,p}(\tau) \times W^{s+1,p}(0, \tau) \tag{5.3}$$

endowed with their natural norms $\|\cdot\|_{X^{s+3,p}(\tau)}$, $\|\cdot\|_{Y^{s+3,p}(\tau)}$, $\|\cdot\|_{Z^{s+3,p}(\tau)}$ defined by

$$\|w\|_{X^{s+3,p}(\tau)} := \|w\|_{W^{s+3,p}(0,\tau;V')} + \sum_{j=1}^{3} \|w\|_{C^{s+3-j}([0,\tau];H^j(\Omega))}, \tag{5.4}$$

$$\begin{aligned} \|v\|_{Y^{s+3,p}(\tau)} := \|v\|_{W^{s+3,p}(0,\tau;V')} + \|v\|_{W^{s+2,p}(0,\tau;H^3(\Omega))} \\ + \|v\|_{W^{s+1,p}(0,\tau;H^4(\Omega))} + \|v\|_{C^{s+2}([0,\tau];H^1(\Omega))}, \end{aligned} \tag{5.5}$$

$$\|(u, v, h)\|_{Z^{s+3,p}(\tau)} := \|u\|_{X^{s+3,p}(\tau)} + \|v\|_{Y^{s+3,p}(\tau)} + \|h\|_{W^{s+1,p}(0,\tau)}. \tag{5.6}$$

The norm (5.5) is motivated by the fact that, owing to Proposition 3.5, in the case $p \geq 2$, $Y^{s+3,p}(\tau) \subseteq C^{s+2}([0, \tau]; H^1(\Omega))$.

Explicitly we will search for a solution $(\theta, \chi, k) \in Z^{4,p}(\tau)$ to the following problem, where $\tau \in (0, T]$:

$$\begin{cases} D_t(\theta + l\chi)(t, x) - (k * \Delta_x\theta)(t, x) = f(t, x), & (t, x) \in [0, \tau] \times \Omega, \\ D_t\chi(t, x) - \Delta_x[-\Delta_x\chi + \beta(\chi) - l\theta](t, x) = 0, & (t, x) \in [0, \tau] \times \Omega, \\ \theta(0, x) = \theta_0(x), \quad \chi(0, x) = \chi_0(x), & x \in \Omega, \\ \dfrac{\partial\theta}{\partial\nu}(t, x) = \dfrac{\partial\chi}{\partial\nu}(t, x) = \dfrac{\partial\Delta_x\chi}{\partial\nu}(t, x) = 0, & (t, x) \in [0, \tau] \times \partial\Omega, \\ \displaystyle\int_\Omega \varphi(x)\theta(t, x)\, dx = g(t), & t \in [0, \tau]. \end{cases} \tag{5.7}$$

As far as the data are concerned, we shall make the following assumptions:

(D1) Ω is an open bounded subset of $\mathbf{R}^n$, $n \le 7$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbf{R}^n$ of class C^4;
(D2) $\beta \in C^5(\mathbf{R})$, $\varphi \in H^1(\Omega)$;
(D3) $f \in W^{3,p}(0, T; H^1(\Omega))$, $g \in W^{4,p}(0, T)$, $T \in \mathbf{R}^+$, $p \in [2, +\infty[$;
(D4) $\theta_0 \in H^3_B(\Omega)$, $\chi_0 \in H^4_{BB}(\Omega)$;
(D5) $v_0 := -\Delta_x^2 \chi_0 + \Delta_x[\beta(\chi_0) - l\theta_0] \in H^4_{BB}(\Omega)$;
(D6) $u_0 := f(0, \cdot\,) - lv_0 \in H^3_B(\Omega)$;
(D7) $v_1 := -\Delta_x^2 v_0 + \Delta_x[\beta'(\chi_0)v_0 - lu_0] \in (H^3_B(\Omega), H^4_{BB}(\Omega))_{1-1/p,p}$;
(D8) $\int_\Omega \varphi(x)\Delta_x\theta_0(x)\,dx \neq 0$;
(D9) $k_0 := \left\{g''(0) + \int_\Omega \varphi(x)[lv_1(x) - D_t f(0, x)]\,dx\right\}\left\{\int_\Omega \varphi(x)\Delta_x\theta_0(x)\,dx\right\}^{-1} \in \mathbf{R}^+$;
(D10) $u_1 := k_0\Delta_x\theta_0 + D_t f(0, \cdot\,) - lv_1 \in H^2_B(\Omega)$;
(D11) $\int_\Omega \varphi(x)\theta_0(x)\,dx = g(0)$, $\int_\Omega \varphi(x)u_0(x)\,dx = g'(0)$, $\int_\Omega \varphi(x)u_1(x)\,dx = g''(0)$;
(D12) $v_2 := \mathcal{B}v_1 + \Delta_x[\beta''(\chi_0)v_0^2 + \beta'(\chi_0)v_1] - l\Delta_x u_1 \in (V', H^3_B(\Omega))_{1/p',p}$, $p' = p/(p-1)$.

REMARK 5.1. Observe that the more complex boundary conditions (0.4) can be replaced with the simpler ones in (5.7) under the assumption $k(0) > 0$ (cf. (D9) and (5.19). Indeed, since $(\theta, \chi, k) \in Z^{4,p}(\tau)$, the following relations hold:

$$k(0)\frac{\partial\theta}{\partial\nu} + k' * \frac{\partial\theta}{\partial\nu} = 0, \text{ on } (0, \tau) \times \partial\Omega, \iff \frac{\partial\theta}{\partial\nu} = 0, \text{ on } (0, \tau) \times \partial\Omega, \tag{5.8}$$

$$\frac{\partial\beta(\chi)}{\partial\nu} = \beta'(\chi)\frac{\partial\chi}{\partial\nu} = 0, \quad \text{on } (0, \tau) \times \partial\Omega. \tag{5.9}$$

Consequently, from (5.7), (5.8), (5.9) we easily deduce the simpler boundary conditions in (5.7).

Assume now that a solution $(\theta, \chi, k) \in Z^{4,p}(\tau)$ does exist. Then, differentiating with respect to t the first equation in (5.7), we obtain, for $(t, x) \in [0, \tau] \times \Omega$,

$$D_t^2\theta(t, x) + lD_t^2\chi(t, x) = k(0)\Delta_x\theta(t, x) + (k' * \Delta_x\theta)(t, x) + D_t f(t, x). \tag{5.10}$$

Differentiating again (5.10) and the second equation in (5.7) and setting

$$u := D_t\theta, \quad v := D_t\chi, \quad h := D_t k, \tag{5.11}$$

we get

(5.12)
$$\begin{aligned} &D_t^2 u(t, x) + lD_t^2 v(t, x) \\ &\quad = k(0)\Delta_x u(t, x) + h(t)\Delta_x\theta_0(x) + (h * \Delta_x u)(t, x) + D_t^2 f(t, x), \quad (t, x) \in [0, \tau] \times \Omega, \end{aligned}$$

$$D_t v(t, x) - \Delta_x[-\Delta_x v + \beta'(\chi)v - lu](t, x) = 0, \quad (t, x) \in [0, \tau] \times \Omega. \tag{5.13}$$

Moreover,

$$
\begin{gathered}
v(0,\ \cdot\ ) = D_t\chi(0,\ \cdot\ ) = v_0,\quad u(0,\ \cdot\ ) = f(0,\ \cdot\ ) - lD_t\chi(0,\ \cdot\ ) = u_0, \\
D_t^2\chi(0,\ \cdot\ ) = D_t v(0,\ \cdot\ ) = v_1.
\end{gathered} \tag{5.14}
$$

If $w \in V'$, we set

$$
\Phi[w] := [w, \overline{\varphi}], \tag{5.15}
$$

where $\overline{\varphi}$ denotes the function conjugate to φ. Therefore the last condition in (5.7) can be written in the form

$$
\Phi[\theta(t,\ \cdot\ )] = g(t), \quad t \in [0, \tau]. \tag{5.16}
$$

Applying φ to (5.10), we obtain

$$
\begin{aligned}
&g''(t) + l\Phi[D_t^2\chi(t,)] \\
&\quad = k(0)\Phi[\Delta_x\theta(t,\ \cdot\ )] + \Phi[(h * \Delta_x\theta)(t,\ \cdot\ )] + \Phi[D_t f(t,\ \cdot\ )], \quad t \in [0, \tau],
\end{aligned} \tag{5.17}
$$

implying

$$
g''(0) + l\Phi[v_1] = k(0)\Phi[\Delta_x\theta_0] + \Phi[D_t f(0,\ \cdot\ )]. \tag{5.18}
$$

It follows from (D8) and (D9) that

$$
k(0) = \frac{g''(0) + l\Phi[v_1] - \Phi[D_t f(0,\ \cdot\ )]}{\Phi[\Delta_x\theta_0]} = k_0. \tag{5.19}
$$

Next,

$$
\begin{aligned}
D_t u(0,\ \cdot\ ) = D_t^2\theta(0,\ \cdot\ ) &= k(0)\Delta_x\theta_0 + D_t f(0,\ \cdot\ ) - lD_t^2\chi(0,\ \cdot\ ) \\
&= k_0\Delta_x\theta_0 + D_t f(0,\ \cdot\ ) - lv_1 = u_1.
\end{aligned} \tag{5.20}
$$

Finally,

$$
\frac{\partial u}{\partial \nu}(t, x) = \frac{\partial v}{\partial \nu}(t, x) = \frac{\partial \Delta_x v}{\partial \nu}(t, x) = 0, \quad (t, x) \in [0, \tau] \times \partial\Omega. \tag{5.21}
$$

From these considerations, if $(\theta, \chi, k) \in Z^{4,p}(\tau)$ solves (5.7), then triplet $(u, v, h) \in Z^{3,p}(\tau)$. Moreover, if A is the operator defined in (2.30)–(2.31), $B = -A^2$ and $S := \beta'$, then $(u, v, h) \in Z^{3,p}(\tau)$ solves the problem

(5.22)
$$\begin{cases} D_t^2 u(t, \cdot) + l D_t^2 v(t, \cdot) = -k_0 A u(t,) - h(t) A\theta_0 \\ \qquad - (h * Au)(t, \cdot) + D_t^2 f(t, \cdot), & t \in [0, \tau], \\ D_t v(t, \cdot) - Bv(t, \cdot) + A[S((\chi_0 + 1 * v)(t, \cdot))v(t, \cdot)] - lAu(t, \cdot) \\ \quad = 0, & t \in [0, \tau], \\ u(0, \cdot) = u_0, \quad D_t u(0, \cdot) = u_1, \quad v(0, \cdot) = v_0, \\ g^{(3)}(t) + l\Phi[D_t^2 v(t, \cdot)] = -k_0 \Phi[Au(t, \cdot)] - h(t)\Phi[A\theta_0] \\ \qquad - \Phi[(h * Au)(t, \cdot)] + \Phi[D_t^2 f(t, \cdot)], & t \in [0, \tau]. \end{cases}$$

Lemma 5.2. *Let assumptions* (D1)–(D12) *be satisfied. Let* $(u, v, h) \in Z^{3,p}(\tau)$ *solve system* (5.22) *and set* $\theta := \theta_0 + 1 * u$, $\chi := \chi_0 + 1 * v$, $k := k_0 + 1 * h$. *Then* $(\theta, \chi, k) \in Z^{4,p}(\tau)$ *and solves problem* (5.7).

Proof. Since $\partial u/\partial \nu = 0$ and $\partial v/\partial \nu = \partial \Delta v/\partial \nu = 0$ on $\partial\Omega$, we deduce that also θ and χ satisfy such boundary conditions according to (D4). This implies that $(\theta, \chi, k) \in Z^{4,p}(\tau)$ according to definitions (5.1)–(5.3). Moreover, it is easy to check that the initial conditions $\theta(0, \cdot) = \theta_0$, $\chi(0, \cdot) = \chi_0$ in (5.7) are satisfied.

Since the second equation in (5.22) can be rewritten in the form

$$D_t\{D_t \chi(t, x) - \Delta_x[-\Delta_x \chi + \beta(\chi) - l\theta](t, x)\} \equiv 0,$$

by virtue of (D5), the second equation in (5.7) is satisfied.

Next, we have

$$D_t^2(k * \Delta_x \theta) = D_t(k_0 \Delta_x \theta + h * \Delta_x \theta) = k_0 \Delta_x u + h \Delta_x \theta_0 + h * \Delta_x u. \tag{5.23}$$

Hence, the first equation in (5.22) can be rewritten in the form

$$D_t^2\{D_t(\theta + l\chi) - (k * \Delta_x \theta)(t, x)\} \equiv D_t^2 f.$$

Owing to (D6), (D7) and (D10), we get

$$\begin{aligned} D_t\{D_t(\theta + l\chi) - (k * \Delta_x \theta)\}(0, \cdot) &= D_t u(0, \cdot) + l D_t v(0, \cdot) - k_0 \Delta_x \theta_0 \\ &= u_1 + l v_1 - k_0 \Delta_x \theta_0 = D_t f(0, \cdot), \end{aligned} \tag{5.24}$$

i.e. even the first equation in (5.7) is satisfied.

Finally, applying Φ to the first equation in (5.22) and comparing the result with the final equation, we get, for all $t \in [0, T]$,

$$\Phi[D_t^3 \theta(t, \cdot)] = \Phi[D_t^2 u(t, \cdot)] = g^{(3)}(t).$$

Thus the final equation in (5.7) follows from (D11). $\square$

REMARK 5.2. If (u, v, h) is a solution of the system, from the last equation in (5.22) we get

$$h(0) = h_0 := \frac{\Phi[-lv_2 - k_0 Au_0 + D_t^2 f(0, \cdot)] - g^{(3)}(0)}{\Phi[A\theta_0]}. \tag{5.25}$$

Lemma 5.3. *Let X be a Banach space and let $\tau \in \mathbf{R}^+$, $p \in [1, +\infty[$, $h \in W^{1,p}(0, \tau)$, $z \in C([0, \tau]; X)$. Then $h * z \in W^{1,p}(0, \tau; X)$ and*

$$\|h * z\|_{W^{1,p}(0,\tau;X)} \le (\tau^{1/p} \vee \tau)\|h\|_{W^{1,p}(0,\tau)}\|z\|_{C([0,\tau];X)}. \tag{5.26}$$

Proof. As $(h * z)(0) = 0$ and

$$\|h' * z\|_{L^p(0,\tau;X)} \le \|h'\|_{L^p(0,\tau)}\|z\|_{L^1(0,\tau;X)},$$

we get

$$\begin{aligned} \|h * z\|_{W^{1,p}(0,\tau;X)} &= \|(h * z)'\|_{L^p(0,\tau;X)} = \|h(0)z + h' * z\|_{L^p(0,\tau;X)} \\ &\le |h(0)|\|z\|_{L^p(0,\tau;X)} + \|h'\|_{L^p(0,\tau)}\|z\|_{L^1(0,\tau;X)} \\ &\le (\tau^{1/p}|h(0)| + \tau\|h'\|_{L^p(0,\tau)})\|z\|_{C([0,\tau];X)} \\ &\le (\tau^{1/p} \vee \tau)\|h\|_{W^{1,p}(0,\tau)}\|z\|_{C([0,\tau];X)}. \end{aligned} \tag{5.27}$$

□

Lemma 5.4. *Let assumptions* (D1)–(D12) *be satisfied and let U be the solution to the Cauchy problem*

$$\begin{cases} D_t^2 U(t, \cdot) + k_0 AU(t, \cdot) = 0, & t \in [0, T], \\ U(0, \cdot) = u_0, \quad D_t U(0, \cdot) = u_1. \end{cases} \tag{5.28}$$

Then $U \in \bigcap_{j=0}^3 C^{3-j}([0, T]; H^j(\Omega))$.

Proof. It follows from Proposition 2.9, with $\alpha = 1/2$, $p = +\infty$. □

Lemma 5.5. *Let assumption* (D1) *be satisfied. Let $v_0 \in H^4_{BB}(\Omega)$, $v_1 \in (H^3_B(\Omega), H^4_{BB}(\Omega))_{1/p',p}$, $v_2 \in (V', H^3_B(\Omega))_{1/p',p}$. Then there exists $V \in Y^{3,p}(T)$ (cf.* (5.2)) *such that $V(0) = v_0$, $D_t V(0) = v_1$, $D_t^2 V(0) = v_2$.*

Proof. As $v_2 - \mathcal{B}v_1 \in (V', L^2(\Omega))_{1-1/p,p}$, by virtue of Proposition 1.2.10 in [15] there exists $g \in W^{1,p}(0, T; V') \cap L^p(0, T; L^2(\Omega))$, such that $g(0) = v_2 - \mathcal{B}v_1$. Then we set

$$f := v_1 - Bv_0 + 1 * g$$

and consider the solution V to problem (3.8). It is easily seen that all conditions (I)–(IV) in Proposition 3.4 are fulfilled. So $V \in Y^{3,p}(T)$ and it is easily seen that $D_t V(0) = v_1$, $D_t^2 V(0) = v_2$. □

Now we are able to state and prove the first of the main results of our paper.

Theorem 5.6. *Let assumptions* (D1)–(D12) *be fulfilled. Then there exists* $\tau \in]0, T]$ *such that the problem* (5.7) *has a solution* $(\theta, \chi, k) \in Z^{4,p}(\tau)$.

Proof. We define

$$\tilde{Z}^{3,p}(\tau) := \{(u, v, h) \in X^{3,p}(\tau) \times Y^{3,p}(\tau) \times W^{1,p}(0, \tau): \\ u(0) = u_0,\ D_t u(0) = u_1,\ v(0) = v_0,\ D_t v(0) = v_1,\ D_t^2 v(0) = v_2,\ h(0) = h_0\}. \tag{5.29}$$

Observe first that, owing to Lemmata 5.4 and 5.5, $\tilde{Z}^{3,p}(\tau)$ is a nonempty closed subset of $Z^{3,p}(\tau)$.

Then, for any fixed $(U, V, H) \in \tilde{Z}^{3,p}(\tau)$ we consider the problem

$$\begin{cases} D_t^2 u(t, \cdot) + k_0 A u(t, \cdot) = D_t^2 f(t, \cdot) - H(t) A\theta_0 \\ \qquad - (H * AU)(t, \cdot) - l D_t^2 V(t, \cdot), & t \in [0, \tau], \\ D_t v(t, \cdot) - B v(t, \cdot) = l A U(t, \cdot) - A[S((\chi_0 + 1 * V)(t, \cdot))V(t, \cdot)], \\ u(0, \cdot) = u_0, \quad D_t u(0, \cdot) = u_1, \quad v(0, \cdot) = v_0, \\ h(t)\Phi[A\theta_0] = -g^{(3)}(t) + \Phi[D_t^2 f(t, \cdot)] - k_0 \Phi[AU(t, \cdot)] \\ \qquad - \Phi[(H * AU)(t, \cdot)] - l\Phi[D_t^2 V(t, \cdot)], & t \in [0, \tau]. \end{cases} \tag{5.30}$$

It is easily seen that, for any fixed $(U, V, H) \in \tilde{Z}^{3,p}(\tau)$, problem (5.30) has a unique solution (u, v, h) in $\tilde{Z}^{3,p}(\tau)$. In fact, owing to (D3), (D4) and Lemma 5.3, $D_t^2 f(t, \cdot) - H(\cdot)A\theta_0 - H * AU \in W^{1,p}(0, \tau; H^1(\Omega))$. So, by virtue of Proposition 2.9, with $\alpha = 1/2$, the problem

$$\begin{cases} D_t^2 z(t, \cdot) + k_0 A z(t, \cdot) = D_t^2 f(t, \cdot) - H(t) A\theta_0 \\ \qquad - (H * AU)(t, \cdot), & t \in [0, \tau], \\ z(0, \cdot) = u_0, \quad D_t z(0, \cdot) = u_1 \end{cases} \tag{5.31}$$

has a unique solution $z \in C^2([0, \tau]; H^1(\Omega)) \cap C^1([0, \tau]; H_B^2(\Omega)) \cap C([0, \tau]; H_B^3(\Omega)) \cap W^{3,p}(0, \tau; L^2(\Omega))$. Moreover, by Proposition 3.5 and Corollary 2.11, the problem

$$\begin{cases} D_t^2 \eta(t, \cdot) + k_0 A\eta(t, \cdot) = -l D_t^2 V(t, \cdot), \quad t \in [0, \tau], \\ \eta(0, \cdot) = 0, \quad D_t \eta(0, \cdot) = 0 \end{cases} \tag{5.32}$$

has a unique solution η belonging to $W^{3,p}(0,\tau;V') \cap C^2([0,\tau];H^1(\Omega)) \cap C^1([0,\tau];H_B^3(\Omega))$. Moreover, $lAU \in C^2([0,\tau];V') \cap C^1([0,\tau];L^2(\Omega))$, so that $lAU - A[S((\chi_0+1*V))V(t,\cdot)] \in$

$W^{2,p}(0,\tau;V')\cap W^{1,p}(0,\tau;L^2(\Omega))$ due to Lemmata 4.4 and 4.5. So, owing to assumptions (D5), (D7) and (D12), Proposition 3.4 allows to conclude that the second equation in (5.30) has a unique solution $v\in Y^{3,p}(\tau)$. Finally, as $\varphi\in H^1(\Omega)$ and $AU\in C^1([0,\tau];V')$, $-g^{(3)}+\Phi[D_t^2 f]-k_0\Phi[AU]-\Phi[H*AU]-l\Phi[D_t^2V]\in W^{1,p}(0,\tau)$, owing to (D3) and Lemma 5.3. So problem (5.30) has a unique solution in $Z^{3,p}(\tau)$. It is not difficult to check that it belongs, in fact, to $\tilde Z^{3,p}(\tau)$. We denote this solution by

$$\mathcal{S}(U,V,H)=(\mathcal{S}_1(U,V,H),\mathcal{S}_2(U,V),\mathcal{S}_3(U,V,H)),$$

and stress that $\mathcal{S}_2$ is independent of H. We observe that $\mathcal{S}$ is a (nonlinear) operator from $\tilde Z^{3,p}(\tau)$ into itself. Clearly, to solve our identification problem (5.22) we have to look for a fixed point of $\mathcal{S}$.

Let $R\in\mathbf{R}^+$ and $(U_j,V_j,H_j)\in\tilde Z^{3,p}(\tau)$, $j\in\{1,2\}$ and $0<\tau\le T$, be such that

$$\max_{j\in\{1,2\}}\|(U_j,V_j,H_j)\|_{Z^{3,p}(\tau)}\le R. \tag{5.33}$$

Taking Proposition 2.9, Corollary 2.11 and Lemma 5.3 into account, we get

$$\begin{aligned}
&\|\mathcal{S}_1(U_1,V_1,H_1)-\mathcal{S}_1(U_2,V_2,H_2)\|_{X^{3,p}(\tau)}\\
&\le C(\|(H_1-H_2)A\theta_0\|_{W^{1,p}(0,\tau;H^1(\Omega))}+\|H_1*A(U_1-U_2)\|_{W^{1,p}(0,\tau;H^1(\Omega))}\\
&\qquad+\|(H_1-H_2)*AU_2\|_{W^{1,p}(0,\tau;H^1(\Omega))}+\|D_t^2(V_1-V_2)\|_{L^p(0,\tau;H^3(\Omega))}\\
&\qquad+\|D_t^2(V_1-V_2)\|_{W^{1,p}(0,\tau;V')}+\|D_t^2(V_1-V_2)\|_{C([0,\tau];H^1(\Omega))})\\
&\le C_1(R)\tau^{1/p}(\|U_1-U_2\|_{X^{3,p}(\tau)}+\|H_1-H_2\|_{W^{1,p}(0,\tau)})\\
&\quad+C_1(\|V_1-V_2\|_{Y^{3,p}(\tau)}+\|H_1-H_2\|_{W^{1,p}(0,\tau)}).
\end{aligned}\tag{5.34}$$

Next, from Propositions 3.4 and 3.5, Lemmata 4.4 and 4.5 we obtain

$$\begin{aligned}
&\|\mathcal{S}_2(U_1,V_1)-\mathcal{S}_2(U_2,V_2)\|_{Y^{3,p}(\tau)}\\
&\le C(\|U_1-U_2\|_{W^{2,p}(0,\tau;H^1(\Omega))}+\|U_1-U_2\|_{W^{1,p}(0,\tau;H^2(\Omega))}\\
&\qquad+\|S(\chi_0+1*V_1)V_1-S(\chi_0+1*V_2)V_2\|_{W^{2,p}(0,\tau;H^1(\Omega))}\\
&\qquad+\|S(\chi_0+1*V_1)V_1-S(\chi_0+1*V_2)V_2\|_{W^{1,p}(0,\tau;H^2(\Omega))})\\
&\le C_2(\|D_t^2(U_1-U_2)\|_{L^p(0,\tau;H^1(\Omega))}+\|D_t(U_1-U_2)\|_{L^p(0,\tau;H^2(\Omega))})\\
&\quad+C_2(R)\tau^{1/(2p)}\|V_1-V_2\|_{Y^{3,p}(\tau)}\\
&\le C_2\tau^{1/p}\|U_1-U_2\|_{X^{3,p}(\tau)}+C_2(R)\tau^{1/(2p)}\|V_1-V_2\|_{Y^{3,p}(\tau)}.
\end{aligned}\tag{5.35}$$

Finally, as $\varphi \in H^1(\Omega)$, using again Lemma 5.3, we get

$$
\begin{aligned}
&\|\mathcal{S}_3(U_1, V_1, H_1) - \mathcal{S}_3(U_2, V_2, H_2)\|_{W^{1,p}(0,\tau)} \\
&\le C_3(\|A(U_1 - U_2)\|_{W^{1,p}(0,\tau;V')} + \|H_1 * AU_1 - H_2 * AU_2\|_{W^{1,p}(0,\tau;V')} \\
&\qquad + \|V_1 - V_2\|_{W^{3,p}(0,\tau;V')}) \\
&\le C_4(\|U_1 - U_2\|_{W^{1,p}(0,\tau;H^1(\Omega))} + \|H_1 * A(U_1 - U_2)\|_{W^{1,p}(0,\tau;V')} \\
&\qquad + \|(H_1 - H_2) * AU_2\|_{W^{1,p}(0,\tau;V')} + \|V_1 - V_2\|_{Y^{3,p}(\tau)}) \\
&\le C_3(R)\tau^{1+1/p}(\|U_1 - U_2\|_{X^{3,p}(\tau)} + \|H_1 - H_2\|_{W^{1,p}(0,\tau)}) + C_4\|V_1 - V_2\|_{Y^{3,p}(\tau)}.
\end{aligned}
\tag{5.36}
$$

Now we set

$$\mathcal{N}_1(U, V, H) = \mathcal{S}_1(U, \mathcal{S}_2(U, V), S_3(U, \mathcal{S}_2(U, V), H)), \tag{5.37}$$

$$\mathcal{N}_2(U, V) = \mathcal{S}_2(U, V), \tag{5.38}$$

$$\mathcal{N}_3(U, V, H) = \mathcal{S}_3(U, \mathcal{S}_2(U, V), H), \tag{5.39}$$

$$\mathcal{N}(U, V, H) = (\mathcal{N}_1(U, V, H), \mathcal{N}_2(U, V), \mathcal{N}_3(U, V, H)). \tag{5.40}$$

It is straightforward to check that a fixed point of $\mathcal{N}$ in $\tilde{Z}^{3,p}(\tau)$ is also a fixed point of $\mathcal{S}$. From (5.34), (5.35), (5.36), we easily deduce that, for any $R \in \mathbf{R}^+$ there exists $C(R) > 0$ such that

$$
\begin{aligned}
&\|\mathcal{N}(U_1, V_1, H_1) - \mathcal{N}(U_2, V_2, H_2)\|_{Z^{3,p}(\tau)} \\
&\le C(R)\tau^{1/(2p)}\|(U_1 - U_2, V_1 - V_2, H_1 - H_2)\|_{Z^{3,p}(\tau)}
\end{aligned}
\tag{5.41}
$$

for all $\tau \in (0, T]$ and all $(U_1, V_1, H_1), (U_2, V_2, H_2) \in \tilde{Z}^{3,p}(\tau)$ with

$$\max_{j\in\{1,2\}} \|(U_j, V_j, H_j)\|_{Z^{3,p}(\tau)} \le R.$$

Fix now (U^*, V^*, H^*) in $\tilde{Z}^{3,p}(T)$ and set $(\tilde{U}, \tilde{V}, \tilde{H}) := \mathcal{N}(U^*, V^*, H^*)$. Let $\rho \in \mathbf{R}^+$, $\tau \in (0, T]$. Then, if $(U_j, V_j, H_j) \in \tilde{Z}^{3,p}(\tau)$, $j \in \{1, 2\}$, and

$$\max_{j\in\{1,2\}} \|(U_j, V_j, H_j) - (\tilde{U}, \tilde{V}, \tilde{H})\|_{Z^{3,p}(\tau)} \le \rho, \tag{5.42}$$

we obtain

$$
\begin{aligned}
&\|\mathcal{N}(U_1, V_1, H_1) - \mathcal{N}(U_2, V_2, H_2)\|_{Z^{3,p}(\tau)} \\
&\le C(R_1(\rho))\tau^{1/(2p)}\|(U_1 - U_2, V_1 - V_2, H_1 - H_2)\|_{Z^{3,p}(\tau)}
\end{aligned}
\tag{5.43}
$$

with

$$R_1(\rho) := \|(\tilde{U}, \tilde{V}, \tilde{H})\|_{Z^{3,p}(T)} + \rho. \tag{5.44}$$

Moreover, if

$$R_2(\rho) := \max\{R_1(\rho), \|(U^*, V^*, H^*)\|_{Z^{3,p}(T)}\}, \tag{5.45}$$

we have

$$\begin{aligned} &\|\mathcal{N}(U_1, V_1, H_1) - (\tilde{U}, \tilde{V}, \tilde{H})\|_{Z^{3,p}(\tau)} \\ &\leq C(R_2(\rho))\tau^{1/(2p)}\|(U_1, V_1, H_1) - (U^*, V^*, H^*)\|_{Z^{3,p}(\tau)} \\ &\leq C(R_2(\rho))\tau^{1/(2p)}(\rho + \|(\tilde{U}, \tilde{V}, \tilde{H}) - (U^*, V^*, H^*)\|_{Z^{3,p}(T)}). \end{aligned} \tag{5.46}$$

Choose now $\tau \in (0, \tau(\rho))$, where $\tau(\rho)$ satisfies the inequalities

$$C(R_1(\rho))\tau(\rho)^{1/(2p)} \leq \frac{1}{2}, \tag{5.47}$$

$$C(R_2(\rho))\tau(\rho)^{1/(2p)}(\rho + \|(\tilde{U}, \tilde{V}, \tilde{H}) - (U^*, V^*, H^*)\|_{Z^{3,p}(T)}) \leq \rho. \tag{5.48}$$

Then, from the contraction mapping theorem we deduce that $\mathcal{N}$ has a unique fixed point in $\{(U, V, H) \in \tilde{Z}^{3,p}(\tau) \colon \|(U, V, H) - (\tilde{U}, \tilde{V}, \tilde{H})\|_{Z^{3,p}(\tau)} \leq \rho\}$. □

We conclude our paper with a result of global uniqueness.

Theorem 5.7. *Let $\tau \in (0, T]$, (θ, χ, k) and $(\tilde{\theta}, \tilde{\chi}, \tilde{k})$ be solutions of* (5.7) *in $Z^{4,p}(\tau)$. Then $(\theta, \chi, k) = (\tilde{\theta}, \tilde{\chi}, \tilde{k})$.*

Proof. Set

$$h := D_t k, \quad u := D_t\theta, \quad v := D_t\chi, \quad \tilde{h} := D_t\tilde{k}, \quad \tilde{u} := D_t\tilde{\theta}, \quad \tilde{v} := D_t\tilde{\chi}. \tag{5.49}$$

Then (u, v, h) and $(\tilde{u}, \tilde{v}, \tilde{h})$ belong to $Z^{3,p}(\tau)$. Moreover, (u, v, h) and $(\tilde{u}, \tilde{v}, \tilde{h})$ are both solutions of (5.22). We assume, by contradiction, that they do not coincide in $[0, T]$. Consequently, we set

$$\tau_1 := \inf\{t \in [0, \tau] \colon \|u(t, \cdot) - \tilde{u}(t, \cdot)\|_{L^2(\Omega)} + \|v(t, \cdot) - \tilde{v}(t, \cdot)\|_{L^2(\Omega)} + |h(t) - \tilde{h}(t)| > 0\}. \tag{5.50}$$

Of course, $0 \leq \tau_1 < \tau$ and $u(t, \cdot) = \tilde{u}(t, \cdot)$, $v(t, \cdot) = \tilde{v}(t, \cdot)$, $h(t) = \tilde{h}(t)$ for all $t \in [0, \tau_1]$. We set now, for $t \in [0, \tau - \tau_1]$,

$$\eta(t, \cdot) := u(\tau_1 + t, \cdot) - \tilde{u}(\tau_1 + t, \cdot), \tag{5.51}$$

$$\zeta(t, \cdot) := v(\tau_1 + t, \cdot) - \tilde{v}(\tau_1 + t, \cdot), \tag{5.52}$$

$$\iota(t) := h(\tau_1 + t) - \tilde{h}(\tau_1 + t). \tag{5.53}$$

From

$$h * Au - \tilde{h} * A\tilde{u} = h * A(u - \tilde{u}) + (h - \tilde{h}) * A\tilde{u}$$

we get, for all $t \in [0, \tau - \tau_1]$,

$$
\begin{aligned}
[h * A(u - \tilde{u})](\tau_1 + t, \cdot) &= \int_{\tau_1}^{\tau_1+t} h(\tau_1 + t - s) A(u - \tilde{u})(s)\, ds \\
&= \int_0^t h(t - s) A\eta(s)\, ds = (h * A\eta)(t, \cdot).
\end{aligned} \tag{5.54}
$$

Analogously,

$$
\begin{aligned}
[(h - \tilde{h}) * A\tilde{u}](\tau_1 + t, \cdot) &= \int_0^t [h(\tau_1 + t - s) - \tilde{h}(\tau_1 + t - s)] A\tilde{u}(s, \cdot)\, ds \\
&= (\iota * A\tilde{u})(t, \cdot).
\end{aligned} \tag{5.55}
$$

So, the triplet (η, ζ, ι) solves the problem

$$
\begin{cases}
D_t^2 \eta(t, \cdot) + l D_t^2 \zeta(t, \cdot) = -k_0 A\eta(t, \cdot) - \iota(t) A\theta_0 & \\
\qquad\qquad - (h * A\eta)(t, \cdot) - (\iota * A\tilde{u})(t, \cdot), & t \in [0, \tau - \tau_1], \\
D_t \zeta(t, \cdot) - B\zeta(t, \cdot) & \\
\quad + A[S((\chi_1 + 1 * v(\cdot + \tau_1, \cdot))(t, \cdot)) v(t + \tau_1, \cdot) & t \in [0, \tau - \tau_1], \\
\qquad - S((\chi_1 + 1 * \tilde{v}(\cdot + \tau_1, \cdot))(t, \cdot)) \tilde{v}(t + \tau_1, \cdot)] - l A\eta(t, \cdot) = 0, & \\
\eta(0, \cdot) = 0, \quad D_t \eta(0, \cdot) = 0, \quad \zeta(0, \cdot) = 0, & \\
l \Phi[D_t^2 \zeta(t, \cdot)] = -k_0 \Phi[A\eta(t, \cdot)] - \iota(t) \Phi[A\theta_0] & \\
\qquad\qquad - \Phi[(h * A\eta)(t, \cdot) + (\iota * A\tilde{u})(t, \cdot)], & t \in [0, \tau - \tau_1],
\end{cases} \tag{5.56}
$$

with

$$
\chi_1 := \chi_0 + \int_0^{\tau_1} v(t, \cdot)\, dt = \chi_0 + \int_0^{\tau_1} \tilde{v}(t, \cdot)\, dt.
$$

From the last equation in (5.56), using Lemma 5.3, we obtain, for $\sigma \in (0, \tau - \tau_1]$,

$$
\begin{aligned}
\|\iota\|_{W^{1,p}(0,\sigma)} &\le C_1 \big(\|\zeta\|_{W^{3,p}(0,\sigma;V')} + \|\eta\|_{W^{1,p}(0,\sigma;H^1(\Omega))} + \|h * \eta\|_{W^{1,p}(0,\sigma;H^1(\Omega))} \\
&\qquad + \|\iota * \tilde{u}\|_{W^{1,p}(0,\sigma;H^1(\Omega))} \big) \\
&\le C_1 [\|\zeta\|_{W^{3,p}(0,\sigma;V')} + \|\eta\|_{W^{1,p}(0,\sigma;H^1(\Omega))} + (\sigma^{1/p} \vee \sigma) \|h\|_{W^{1,p}(0,\sigma)} \\
&\qquad \times \|\eta\|_{C([0,\sigma];H^1(\Omega))} + (\sigma^{1/p} \vee \sigma) \|\iota\|_{W^{1,p}(0,\sigma)} \|\tilde{u}\|_{C([0,\sigma];H^1(\Omega))}].
\end{aligned} \tag{5.57}
$$

Set now

$$
\rho := \max\{\|h\|_{W^{1,p}(0,\tau)}, \|\tilde{u}\|_{X^{3,p}(\tau)}, \|v\|_{Y^{3,p}(\tau)}, \|\tilde{v}\|_{Y^{3,p}(\tau)}\}, \tag{5.58}
$$

and choose, from now on, σ satisfying

$$
C_1 \rho (\sigma^{1/p} \vee \sigma) \le \frac{1}{2}. \tag{5.59}
$$

Then, since $\eta(0,\ \cdot\ ) = D_t\eta(0,\ \cdot\ ) = 0$, from (5.57) we obtain

$$\begin{aligned} \|\iota\|_{W^{1,p}(0,\sigma)} &\le C_2(\|\zeta\|_{W^{3,p}(0,\sigma;V')} + \|\eta\|_{W^{1,p}(0,\sigma;H^1(\Omega))}) \\ &\le C_3(\|\zeta\|_{W^{3,p}(0,\sigma;V')} + \sigma^{1+1/p}\|\eta\|_{X^{3,p}(\sigma)}), \end{aligned} \tag{5.60}$$

for suitable positive constants C_2 and C_3 independent of $\sigma \le \tau - \tau_1$.

Moreover, from Proposition 2.9, Corollary 2.11, Lemma 5.3, for some constant $C_4 \in \mathbf{R}^+$ independent of $\sigma \le \tau - \tau_1$, we get

$$\begin{aligned} \|\eta\|_{X^{3,p}(\sigma)} &\le C_4(\|h * A\eta\|_{W^{1,p}(0,\sigma;H^1(\Omega))} + \|\iota * A\tilde{u}\|_{W^{1,p}(0,\sigma;H^1(\Omega))} + \|\iota\|_{W^{1,p}(0,\sigma)} \\ &\quad + \|D_t^2\zeta\|_{W^{1,p}(0,\sigma;V')} + \|D_t^2\zeta\|_{C([0,\sigma];H^1(\Omega))} + \|D_t^2\zeta\|_{L^p([0,\sigma];H^3(\Omega))}) \\ &\le C_5[\sigma^{1/p}\rho\|\eta\|_{X^{3,p}(\sigma)} + (\rho+1)\|\iota\|_{W^{1,p}(0,\sigma)} + \|\zeta\|_{Y^{3,p}(\sigma)}]. \end{aligned} \tag{5.61}$$

If $\sigma \le \sigma_1 := (2C_5\rho)^{-p}$ and (5.59) holds, we get

$$\|\eta\|_{X^{3,p}(\sigma)} \le C_6(\rho+1)(\|\zeta\|_{Y^{3,p}(\sigma)} + \sigma^{1+1/p}\|\eta\|_{X^{3,p}(\sigma)}), \tag{5.62}$$

implying

$$\|\eta\|_{X^{3,p}(\sigma)} \le C_7(\rho+1)\|\zeta\|_{Y^{3,p}(\sigma)}, \quad \text{if} \quad C_6(\rho+1)\sigma^{1+1/p} \le \frac{1}{2}. \tag{5.63}$$

Finally, as

$$\begin{aligned} \|A\eta\|_{W^{2,p}(0,\sigma;V')} + \|A\eta\|_{W^{1,p}(0,\sigma;L^2(\Omega))} &\le C_7(\|\eta\|_{W^{2,p}(0,\sigma;H^1(\Omega))} + \|\eta\|_{W^{1,p}(0,\sigma;H^2(\Omega))}) \\ &\le C_8\sigma^{1/p}\|\eta\|_{X^{3,p}(\sigma)}, \end{aligned} \tag{5.64}$$

from Proposition 3.4 and 3.5 and Lemmata 4.4 and 4.5, for $\sigma \le \sigma(\rho)$, we obtain

$$\|\zeta\|_{Y^{3,p}(\sigma)} \le C(\rho)\sigma^{1/p}\|\eta\|_{X^{3,p}(\sigma)}. \tag{5.65}$$

Hence

$$\|\eta\|_{X^{3,p}(\sigma)} \le C_9(\rho+1)\sigma^{1/p}\|\eta\|_{X^{3,p}(\sigma)}, \tag{5.66}$$

implying that η vanishes in some right neighbourhood of 0. From (5.65) and (5.60), we obtain also that ζ and ι vanish in some right neighbourhood of 0. So, $u \equiv \tilde{u}$, $v \equiv \tilde{v}$ and $h \equiv \tilde{h}$ in some right neighbourhood of τ_1, contrarily to the definition of τ_1. Therefore we conclude that $u \equiv \tilde{u}$, $v \equiv \tilde{v}$ and $h \equiv \tilde{h}$, implying $\theta \equiv \tilde{\theta}$, $\chi \equiv \tilde{\chi}$ and $k \equiv \tilde{k}$. □

References

[1] R.A. Adams: Sobolev Spaces, Pure and Applied Mathematics **65**, Plenum Press, 1975.

[2] G. Bonfanti and L. Luterotti: *Regularity and convergence results for a phase-field model with memory*, Math. Methods Appl. Sci. **21** (1998), 1085–1105.

[3] G. Bonfanti and F. Luterotti: *Global solution to a phase-field model with memory and quadratic nonlinearity*, Adv. Math. Sci. Appl. **9** (1999), 523–538.

[4] P. Colli, G. Gilardi, M. Grasselli and G. Schimperna: *The conserved phase-field system with memory*, Adv. Math. Sci. Appl. **11** (2001), 265–291.

[5] F. Colombo: *Direct and inverse problems for a phase-field model with memory*, J. Math. Anal. Appl. **260** (2001), 517–545.

[6] F. Colombo and D. Guidetti: *An inverse problem for a phase-field model in Sobolev spaces*; in Nonlinear Elliptic and Parabolic Problems, Progr. Nonlinear Differential Equations Appl. **64**, 2005, Birkhäuser, 189–210.

[7] F.Colombo and A. Lorenzi: *Identification of time and space dependent relaxation kernels for materials with memory related to cylindrical domains*, I, II, J. Math. Anal. Appl. **213** (1997), 32–62, 63–90.

[8] L. de Simon: *Un'applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine*, Rend. Sem. Mat. Univ. Padova **34** (1964), 205–223.

[9] D. Guidetti: *On elliptic problems in Besov spaces*, Math. Nachr. **152** (1991), 247–275.

[10] D. Guidetti: *On interpolation with boundary conditions*, Math. Z. **207** (1991), 439-460.

[11] M.E. Gurtin and A.C. Pipkin: *A general theory of the heat conduction with finite wave speeds*, Arch. Rational Mech. Anal. **31** (1968), 113–126.

[12] J.L. Lions and E. Magenes: Non-Homogeneous Boundary Value Problems and Applications I, Grundlehren der mathematischen Wissenschaften **181**, Springer-Verlag, 1972.

[13] J.L. Lions and E. Magenes: Non-Homogeneous Boundary Value Problems and Applications II, Grundlehren der mathematischen Wissenschaften **182**, Springer-Verlag, 1972.

[14] A. Lorenzi: *Identification problems for conserved phase-field models with memory*, Math. Methods Appl. Sci. **28** (2005), 1315–1339.

[15] A. Lunardi: Analytic Semigroups and Optimal Regularity in Parabolic Problems, Progress in Nonlinear Differential Equations and Their Applications **16**, Birkhäuser Verlag, Basel, 1995.

[16] A. Pazy: Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences **44**, Springer-Verlag, New York, 1983.

[17] R. Seeley: *Interpolation in L^p with boundary conditions*, Studia Math. **44** (1972), 47–60.

[18] H. Tanabe: Equations of Evolution, Pitman, 1979.

[19] H. Triebel: Interpolation Theory, Function Spaces, Differential Operators, North-Holland Mathematical Library **18**, North-Holland Publishing Co., Amsterdam-New York, 1978.

Davide Guidetti
Dipartimento di Matematica
Piazza di Porta S. Donato 5, Bologna
Italy
e-mail: guidetti@dm.unibo.it

Alfredo Lorenzi
Dipartimento di Matematica F. Enriques
Via C. Saldini 50, Milan
Italy
e-mail: Alfredo.Lorenzi@mat.unimi.it