

Title	On the transversal immersion
Author(s)	Kobayashi, Kazuaki
Citation	Osaka Journal of Mathematics. 1969, 6(2), p. 409-418
Version Type	VoR
URL	https://doi.org/10.18910/3589
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

ON THE TRANSVERSAL IMMERSION

KAZUAKI KOBAYASHI

(Received November 14, 1968)

(Revised April 4, 1969)

Zeeman and Armstrong [1], [2], [4] introduced the notion of transversality for the intersection of two manifolds or two polyhedra and proved some results. This notion can be considered as a refinement of general position. In this paper we extend this to the self-intersection of an piecewise linear immersion between manifolds.

The main result of this paper (Theorem) says that any locally flat immersion of a closed manifold into a manifold without boundary can be approximated by a transversal immersion.

It will be assumed, without further mention, that all manifolds have a piecewise linear structure and that all maps are piecewise linear immersions of a closed (i.e. compact and without boundary) manifold into a manifold without boundary.

Definitions and Theorem

Let M be a closed m -manifold and Q be a q -manifold without boundary. Let $f: M \rightarrow Q$ be an immersion. Let \hat{A} be the barycenter of a simplex A . We denote E^q the q -dimensional euclidean space.

DEFINITION 1. Let J, K be triangulations of M, Q such that $f: J \rightarrow K$ is simplicial. If $(Lk(fx, K), f(Lk(x, J)))$ is unknotted sphere pair for any $x \in M$, we say that f is a *locally flat immersion*. This definition is independent of the triangulation of M, Q . Let $S_f = \{x \in M \mid f^{-1}f(x) \neq \{x\}\}$. Then $S_f = \bar{S}_f$ where \bar{S}_f is the closure of S_f since f is an immersion. And let $S_f(r) = \{x_1 \in M \mid f^{-1}f(x_1) = \{x_1, \dots, x_n\}, n \geq r\}$. Hence $S_f = S_f(2)$.

DEFINITION 2. Let $f^{-1}f(x_1) = \{x_1, \dots, x_n\}$ for some $x_1 \in S_f$. If the following diagram commutes for any $i (1 \leq i \leq n)$ except j , we call f *transversal to fM at x_j* where $\varphi_j, \varphi_i, \psi$ are homeomorphism onto some neighborhoods of $x_j, x_i, f(x_1)$ respectively.

$$\begin{array}{ccccc}
 D^{2m-q} \times D^{q-m}, 0 \times 0 & \xrightarrow{1 \times 1 \times 0} & D^{2m-q} \times D^{q-m} \times D^{q-m}, 0 \times 0 \times 0 & \xleftarrow{1 \times 0 \times 1} & D^{2m-q} \times D^{q-m}, 0 \times 0 \\
 \varphi_j \downarrow & & \downarrow \psi & & \downarrow \varphi_i \\
 M, x_j & \xrightarrow{f} & Q, f(x_1) & \xleftarrow{f} & M, x_i
 \end{array}$$

And we simply call f transversal to fM at $f(x_1)$ if for any $j(1 \leq j \leq n)$ f is transversal to fM at x_j .

Furthermore we define f transversal immersion if f is transversal to fM at any $f(x)$, $x \in S_f$.

DEFINITION 3. An immersion f of M into Q is in general position itself if $\dim S_f(r) \leq rm - (r-1)q$ for all r .

Theorem. *If f is a locally flat immersion of M^m into Q^q , then we can homotope f into a locally flat transversal immersion g by an arbitrarily small homotopy. Furthermore if f is in general position itself, then g is also in general position itself.*

REMARK 1. The above theorem still holds for an immersion f such that f is locally flat at any point of $St(S_f, M)$ although g is a transversal immersion such that g is locally flat at any point of $St(S_g, M)$.

REMARK 2. Conversely if f is a transversal immersion of M into Q , f is locally flat at any point of S_f . For it is obvious by the left half or the right half of the above diagram.

REMARK 3. When $m=q$, theorem is obvious. So we shall prove it for $m < q$.

Corollary 1. *If g is a transversal immersion and S_g consists only on double points, then S_g is a closed locally flat $(2m-q)$ -submanifold of M .*

If the codimension of M and Q is greater than 3, any immersion is locally flat [4]. Hence we obtain the following Cor. 2.

Corollary 2. *If $m+3 \leq q$, any piecewise linear immersion of M^m into Q^q is arbitrarily approximated by an transversal immersion. The approximation is made to be homotopic.*

Corollary 3. *If f is any piecewise linear immersion of M^2 into Q^4 , then f is approximated by an transversal immersion. The approximation can be chosen so near and to be homotopic.*

DEFINITION 4. Let f be an immersion of M into Q and J, K be simplicial subdivisions of M, Q respectively. We call f in general position with respect to J and K at x if for any simplexes $\Delta^i \in J$ and $\Delta^k \in K$ such that $x \in \Delta^i$ and $f(x) \in \Delta^k$,

$$\dim (f\Delta^i \cap \Delta^k) \leq i+k-q.$$

For any $x \in M$ let A be a simplex of K such that $f(x) \in \mathring{A}$. Choose a vertex v of A , let $L=Lk(A, K)$, and $s:AL \rightarrow vL$ the simplicial map defined as the join of $A \rightarrow v$ to the identity on L .

DEFINITION 5. (Armstrong, Zeeman [1], [2]) Let the map f be an immersion of M into Q and be in general position with respect to J and K where $|J|=M$, $|K|=Q$. The map f is *transimplicial to K at the point $x \in M$* if there exists a neighborhood N of v in vL , and a commutative diagram

$$\begin{array}{ccccc}
 N \times D^{m+a-q} & \xrightarrow{1 \times k} & N \times D^a & \xrightarrow{p} & N \\
 \alpha \downarrow & & \beta \downarrow & & \downarrow \subset \\
 St(x, J) \cap f^{-1}AL & \xrightarrow{f} & AL & \xrightarrow{s} & vL
 \end{array}$$

where a is the dimension of A , k is a proper embedding $D^{m+a-q} \rightarrow D^a$, p is a projection and α, β are embeddings onto neighborhoods of $x, f(x)$ respectively.

REMARK 4. The definition is independent of the choice of v .

REMARK 5. If f is transimplicial to K at x and $x \in \Delta^i$ ($i < m$), then $m+a > q$. For since f is in general position and $f(x) \in \mathring{A}$,

$$\dim(f\Delta^i \cap A) \leq i+a-q < m+a-q.$$

Hence $f\Delta^i \cap A = \emptyset$ if $m+a-q \leq 0$. This contradicts $x \in \Delta^i$ and $f(x) \in \mathring{A}$.

If f is transimplicial to K at x and $x \in \Delta^m$, then $m+a \geq q$. For $\dim(f\Delta^m \cap A) \leq m+a-q$. Hence $f\Delta^m \cap A = \emptyset$ if $m+a < q$. This is contradiction.

REMARK 6. Let K' be a subdivision of K . If f is transimplicial to K' at $x \in M$ (hence f is in general position with respect to K'), then f is also transimplicial to K at x (see [2]).

DEFINITION 6. Let K be a combinatorial manifold of dimension q . Then K is called a *Brouwer manifold* if

(i) For each $v \in K$ there is a linear embedding

$$St(v, K) \rightarrow E^q$$

(ii) For each $v \in K$ there is a linear embedding

$$St(v, K), St(v, K) \rightarrow E_+^q, E^{q-1}.$$

REMARK 7. Not all combinatorial manifolds are Brouwer, [3].

REMARK 8. Any subdivision of a Brouwer is Brouwer.

Lemma 0 (Zeeman [2]). *Any combinatorial manifold has a Brouwer subdivision.*

Lemmata

When $m=q$, theorem is obvious. So we assume $m < q$ throughout this section.

Now let $f:M \rightarrow Q$ be an immersion and J, K be subdivision of M, Q such that f is simplicial with respect to J, K and such that K is Brouwer subdivision. And we may suppose f is in general position itself by [4, Chap. 6]. Let $x_1 \in S_f$, and $f^{-1}f(x_1) = \{x_1, \dots, x_n\}$.

We suppose $x_i \in B_i^b, f(x_i) \in \hat{A}$ where B_i ($i=1, 2, \dots, n$) and A are simplexes of J and K respectively.

Let $X_i = St(\hat{B}_i, J'), W = St(\hat{A}, K')$.

We construct approximating maps g_i^b ($i=1, 2, \dots, n-1$) of f inductively as follows. First take v_1 in \hat{W} such that $v_1 \notin f(M)$ and v_1 is in general position with respect to the vertices of W . Then we define g_1^b as follows

$$\begin{aligned} g_1^b | M - \hat{St}(\hat{B}_1, J') &= f | M - \hat{St}(\hat{B}_1, J') \\ g_1^b(\hat{B}_1) &= v_1 \end{aligned}$$

and for any $y \in St(\hat{B}_1, J')$ if $y = (1-\lambda)d + \lambda\hat{B}_1$ ($d \in \hat{St}(\hat{B}_1, J')$), $g_1(y) = (1-\lambda)f(d) + \lambda v_1$.

Obviously g_1^b is not simplicial with respect to J' and K' . We take subdivisions J_1, K_1 of J', K' such that they are subdividing the parts $f^{-1}(\hat{W})$ and \hat{W} and such that $g_1^b: J_1 \rightarrow K_1$ is simplicial.

Next take v_2 in $\hat{St}(\hat{A}, K')$ satisfying $v_2 \notin g_1^b(M)$ and v_2 is in general position with respect to the vertices of the subcomplex W_1 of K'_1 covering W . And we define g_2^b satisfying

$$\begin{aligned} g_2^b | M - \hat{St}(\hat{B}_2, J') &= g_1^b | M - \hat{St}(\hat{B}_2, J') \\ g_2^b(\hat{B}_2) &= v_2 \quad \text{and} \end{aligned}$$

for any $y \in \hat{St}(\hat{B}_2, J')$ it maps linearly same as g_1^b . We take subdivisions J_2, K_2 of J'_1, K'_1 such that they are subdividing the parts $(g_1^b)^{-1}\hat{W}_1$ and \hat{W}_1 and such that $g_2^b: J_2 \rightarrow K_2$ is simplicial. We construct g_i^b ($3 \leq i \leq n-1$) in the same way. Furthermore we similarly construct approximating maps of f for any other point $x \in S_f$ such that $x \in \hat{B}^b$.

And we again put the maps g_i^b .

Obviously $g_i^b | X_i$ is in general position with respect to J'_{i-1} and K'_{i-1} ($i=1, 2, \dots, n-1$) at any point $x \in \hat{X}_i$ where $J_0 = J, K_0 = K$ and g_i^b is in general position itself.

Lemma 1. *For all i ($i=1, 2, \dots, n-1$) the map g_i^b constructed above is transimplicial to K'_{i-1} at any point of \hat{X}_i where $K_0 = K$.*

Proof. First we show g_1^b transimplicial to K' at any point of \hat{X}_1 .

For any $y \in \hat{St}(\hat{B}_1, J')$ let C, D be simplexes of J', K' respectively such that $y \in \hat{C}, g_1^b y \in \hat{D}$. Since K is Brouwer triangulation, we may suppose $St(A, K)$ embedded linearly in E^q . If D is a principal simplex of K', g_1^b is obviously trans-

implicial to K' at y . Hence we may suppose $\dim D < q$. $g_1^b C$ is the linear join in E^q of v_1 to some simplex of \dot{W} by the construction of g_1^b . And since v_1 is in general position with respect to the vertices of W , $g_1^b C$ and D span E^q . $g_1^b C \cap D$ is a $(c+d-q)$ -linear convex cell. Let $(g_1^b)^{-1}(g_1^b C \cap D) \cap C = E$ and F a $(q-d)$ -cell through y that is perpendicular to E in C . Let C^* be a simplex of J' having C as a face and $E^* = (g_1^b)^{-1}(g_1^b C^* \cap D) \cap C^*$. Then although E^* is not necessarily perpendicular to F in C^* , it has the property that any $(q-d)$ -cell parallel to F in C^* , and sufficiently close to F , meets it in exactly one point. Therefore for some sufficiently small neighborhood U of y in $St(\hat{B}_1, J')$ we can define a map $\rho_1: U \rightarrow (g_1^b)^{-1}D$ by projecting $U \cap C^*$ parallel to F onto the corresponding E^* . Now return to E^q . Since we defined F perpendicular to E in C , we know that the linear subspace $[g_1^b F]$ and $[D]$, spanned by $g_1^b F$ and D , are complementary in E^q . Hence we define the map

$$\rho_2: E^q \rightarrow [D] \text{ parallel to } [g_1^b F].$$

Then $\rho_2 g_1^b = g_1^b \rho_1$ on U .

Choose a vertex v of D and let $L = lk(D, K')$, $s: DL \rightarrow vL$ and define $\alpha = s g_1^b \times \rho_1: U \rightarrow vL \times (g_1^b)^{-1}D$

$$\beta = s \times \rho_2: St(D, K') \rightarrow vL \times [D].$$

We can check that α and β are both piecewise linear embeddings onto neighborhoods of (v, y) , $(v, g_1^b y)$ respectively. Choose ball neighborhoods N of v in vL ,

$$D^{m+d-q} \text{ of } y \text{ in } (g_1^b)^{-1}D \quad \text{and}$$

$$D^d \text{ of } g_1^b y \text{ in } D$$

such that

$$N \times D^{m+d-q} \text{ image of } \alpha$$

$$N \times D^d \text{ image of } \beta, \quad \text{and}$$

$$g_1^b D^{m+d-q} \subset D^d$$

Hence the following diagram commutes

$$\begin{array}{ccccc} N \times D^{m+d-q} & \xrightarrow{1 \times (g_1^b | D^{m+d-q})} & N \times D^d & \xrightarrow{\hat{p}} & N \\ \alpha^{-1} \downarrow & & \beta^{-1} \downarrow & & \downarrow \subset \\ g^{-1}DL \cap St(x_1, J') & \xrightarrow{g_1^b} & DL & \xrightarrow{s} & vL \end{array}$$

This complete the proof for g_1^b .

Next we show g_2^b transimplicial to K'_1 at any point of \dot{X}_2 . The main part of the proof is equally for g_1^b and we shall not repeat all, but give the difference part of g_2^b . Since g_2^b is simplicial with respect to J' and $v_2 * \dot{W}$ on $St(\hat{A}_2, J')$, if C_2 is the simplex of J' such that $y_2 \in \dot{C}_2$ where y_2 is any point of \dot{X}_2 , $g_2^b C_2$ is

the linear join of v_2 and some simplex of \dot{W} . Let D_2 be the simplex of K'_1 such that $g_2^b y_2 \in \dot{D}_2$. Since v_2 is in general position with respect to the vertices of W_1 , $g_2^b C_2$ and D_2 span E^q . Hence $(g_2^b)^{-1}(g_2^b C_2 \cap D_2) \cap C_2$ is the $(c+d-q)$ -linear convex cell E_2 . We define F_2 the $(q-d)$ -cell through y_2 that is perpendicular to E_2 . Let C_2^* be a simplex of J' having C_2 as a face. In the same way as g_1^b , g_2^b is transimplicial to K'_1 at any point of \dot{X}_2 . We can prove the lemma similarly for g_i^b ($3 \leq i \leq n-1$).

Lemma 2. *If f is a locally flat immersion, g_i^b ($i=1, 2, \dots, n-1$) are also a locally flat immersion.*

Proof. By the construction of g_i^b , it is sufficiently to show that g_1^b is locally flat. Since g_1^b is different from f only on $\dot{St}(\dot{B}_1, J')$ and since f is locally flat, $(Lk(f\dot{B}_1, K'), f(Lk(\dot{B}_1, J')))$ is the unknotted $(q-1, m-1)$ -sphere pair. Hence

$$\begin{aligned} & (St(g_1^b B, K_1), g_1^b(St(B_1, J))) \\ & = v_{1*}(Lk(fB_1, K), f(Lk(B_1, J))) \end{aligned}$$

is an unknotted (q, m) -ball pair.

Therefore g_1^b is locally flat at $x \in \dot{St}(\dot{B}_1, J')$. Obviously g_1^b is locally flat at the point $x \in M - St(\dot{B}_1, J')$. Let u be a vertex in $Lk(\dot{B}_1, J')$. Since f is locally flat, $(Lk(f(u), K'), f(Lk(u, J')))$ is an unknotted sphere pair.

Case 1. If u is a vertex of J in $Lk(\dot{B}_1, J')$,

$$(Lk(f(u), K'), f(Lk(u, J'))) \cong \partial(\square, f\triangleright)$$

where \triangleright is the dual cell of u in J and where \square is the dual cell of $f(u)$ in K . And

$\partial(\square, f\triangleright) \cap St(f\dot{B}_1, K') = (\widetilde{\square}, f(\widetilde{\triangleright}))$ where $\widetilde{\triangleright} = St(\dot{B}_1, \partial\triangleright)$ and where $\widetilde{\square} = St(f\dot{B}_1, \partial\square)$. Since f is locally flat, $(\widetilde{\square}, f(\widetilde{\triangleright}))$ is the unknotted $(q-1, m-1)$ -ball pair. Hence $\partial(\square, f\triangleright) - \text{Int}(\widetilde{\square}, f(\widetilde{\triangleright}))$ is the unknotted ball pair (D, C) by [5. Cor. 8] and

$$(Lk(g_1^b(u), K'), g_1^b(Lk(u, J'))) \cong (D, C) \cup v_{1*}\partial(D, C)$$

is the unknotted sphere pair. Therefore g_1^b is locally flat at u .

Case 2. If u is a vertex of J' not J in $Lk(\dot{B}_1, J')$,

$$\begin{aligned} & (Lk(f(u), K'), f(Lk(u, J'))) \cap St(f\dot{B}_1, K') \\ & = (St(f\dot{B}_1, Lk(f(u)), f(St(\dot{B}_1, Lk(u, J')))). \end{aligned}$$

And since f is locally flat, it is the unknotted $(q-1, m-1)$ -ball pair (E, F) . Hence $(Lk(f(u), K'), f(Lk(u, J')) - \text{Int}(E, F))$ is the unknotted $(q-1, m-1)$ -ball pair (D, C) . Therefore g_1^b is locally flat at u as Case 1.

Lemma 3. *If a locally flat immersion $f: M \rightarrow Q$ is transimplicial to K at*

$x_j \in S_f$ where $f^{-1}f(x_j) = \{x_1, \dots, x_j, \dots, x_n\}$ and $|K| = Q$ and if $f|M - \hat{St}(x_j, J)$ is simplicial with respect to J and K where $|J| = M$, then f is transversal to $f(M)$ at x_j .

Proof. Since f is a locally flat immersion, g_i^b is also by Lemma 2. Let A be the simplex of K satisfying $f(x_j) \in \hat{\Delta}$ and $L = Lk(A, K)$. Choose a vertex v of A , let $s: AL \rightarrow vL$ the simplicial map as defined before. Since f is transimplicial to K at x_j , $f|St(x_j, J)$ is in general position with respect to K and there exists a neighborhood N of v in vL such that the following diagram commutes (see Remark 5),

$$(0) \quad \begin{array}{ccccc} N \times D^{m+a-q} & \xrightarrow{1 \times k} & N \times D^a & \xrightarrow{p} & N \\ \alpha \downarrow & & \beta \downarrow & & \downarrow \subset \\ f^{-1}AL \cap St(x_j, J) & \xrightarrow{f} & AL & \xrightarrow{s} & vL \end{array}$$

where $k: D^{m+a-q} \rightarrow D^a$ is a proper embedding. Furthermore as f is locally flat, the pair $(N \times D^a, N \times k(D^{m+a-q}))$ is unknotted (q, m) -ball pair. Hence we can write the diagram (0) as follows,

$$\begin{array}{ccc} N \times D^{m+a-q}, v \times 0 & \xrightarrow{1 \times 1 \times 0} & N \times D^{m+a-q} \times D^{q-m}, v \times 0 \times 0 \\ \alpha \downarrow & & \downarrow \beta \\ f^{-1}AL \cap St(x_j, J), x_j & \xrightarrow{f} & AL, f(x_j) \end{array}$$

On the other hand $f|M - St(x_j, J)$ is simplicial, there exist simplexes B_i of J such that $\hat{B}_i \ni x_i$ and $fB_i = A$ ($i=1, 2, \dots, j, \dots, n$). Let $L_1 = f(Lk(B_i, J))$ and N_1 be a neighborhood of v in vL_1 , then $N_1 = N \cap vL_1$. And since f is locally flat, (N, N_1) is an unknotted $(q-a, m-a)$ -ball pair. Thus there exists an unknotting homeomorphism

$$h: D^{q-m} \times D^{m-a}, 0 \times D^{m-a}, 0 \times 0 \rightarrow N, N_1, v.$$

Hence the following diagram commutes.

$$\begin{array}{ccccc} D^{a-m} \times D^{m-a} \times & \xrightarrow{1 \times 1 \times 1 \times 0} & D^{q-m} \times D^{m-a} \times D^{m+a-q} & \xleftarrow{0 \times 1 \times 1 \times 1} & D^{m-a} \times D^{m+a-q} \\ D^{m+a-q}, 0 \times 0 \times 0 & & \times D^{q-m}, 0 \times 0 \times 0 \times 0 & & \times D^{q-m}, 0 \times 0 \times 0 \\ h \times 1 \downarrow & & \downarrow h \times 1 \times 1 & & (h|D^{m-a}) \times 1 \times 1 \downarrow \\ N \times D^{m+a-q}, v \times 0 & \xrightarrow{1 \times 1 \times 0} & N \times D^{m+a-q} \times D^{q-m}, v \times 0 \times 0 & \xleftarrow{\subset \times 1 \times 1} & N_1 \times D^{m+a-q} \times D^{q-m}, v \times 0 \times 0 \\ \alpha \downarrow & & \beta \downarrow & & f^{-1} \alpha \downarrow \\ f^{-1}AL \cap St(x_j, J), x_j & \xrightarrow{f} & AL, f(x_j) & \xleftarrow{f} & f^{-1}AL \cap St(x_i, J), x_i. \end{array}$$

From the top and the bottom of the diagram, f is transversal to $f(M)$ at x_j .

Lemma 4. *If $y_k \in Sg_k^b - Sg_{k-1}^b$ and $(g_k^b)^{-1}(g_k^b)(y_k) = \{\dots y_k \dots y_m\}$ where $y_i \in \dot{S}t(\dot{B}_i, J')$ and $\dot{B}_i \ni x_i$, then g_k^b is transversal to $g_i^b M$ at y_i ($1 \leq i \leq k$).*

Proof. We shall prove the lemma by induction on k . Since g_k^b is transimplicial to K'_{k-1} at y_k by Lemma 1, g_k^b is transversal to $g_k^b M$ at y_k by Lemma 3. We suppose g_k^b transversal to $g_j^b M$ at y_j ($1 \leq j \leq k$). Next we shall show g_k^b transversal to $g_l^b M$ at y_l where l is the largest number satisfying $l' < l$. Since $g_k^b | \dot{S}t(\dot{B}_{l'}, J') = g_{l'}^b | St(\dot{B}_{l'}, J')$ and g_k^b is transimplicial to K'_i at $y_{l'}$ ($i < l'$), $g_k^b | \dot{S}t(\dot{B}_{l'}, J')$ is transversal to $g_k^b(\dot{S}t(\dot{B}_i, J'))$ at $y_{l'}$ ($i < l$) by Lemma 3. And g_k^b is transversal to $g_j^b(\dot{S}t(\dot{B}_j, J'))$ ($1 \leq j \leq k$) at $y_{l'}$ by inductive hypothesis. For $j < k$, since g_k^b does not effect on $\dot{S}t(\dot{B}_j, J')$, $g_k^b(\dot{S}t(\dot{B}_j, J'))$ is embedded as a subcomplex of K'_i for any i . Hence g_k^b is transversal to $g_j^b(St(\dot{B}_j, J'))$ ($j < k$) at $y_{l'}$ because g_k^b is transimplicial to K'_i ($i < l'$) at $y_{l'}$. Therefore g_k^b is transversal to $g_l^b M$ at $y_{l'}$.

We denote the last map like g_{n-1}^b for all other point $x \in S_f$ such that $x \in B^b$ as g^b .

Proof of Theorem.

We order k -simplexes of J containing the point of S_f such $\Delta_{i,j}^k$, that if $f(\Delta_{i,j}^k) = f(\Delta_{m,n}^k)$, $i=m$ and that the set of k -simplexes of $f^{-1}f(\Delta_{i,j}^k)$ is properly numbered for the second suffix j . We number point contained $|\Delta_{i,j}^k| \cap S_f$ as $x_{i,j}^k$.

We perform the shift $g_{i,j}^b$ for every $x_{i,j}^b \in S_f$ such that $x_{i,j}^b \in B_{i,j}^b$ and in order to decreasing dimension of b . Since J is a finite dimensional finite complex, the time of shifts is finite.

First perform on m -shift $g_{i,j}^m$ for $x_{i,j}^m \in S_f$ ($i=1, 2, \dots, l; j=1, 2, \dots, q$) where l is the number of m -simplexes such that $|\Delta_{i,j}^m| \cap S_f \neq \phi$, $|\Delta_{s,t}^m| \cap S_f \neq \phi$ and $f(\Delta_{i,j}^m) \cap f(\Delta_{s,t}^m) = \phi$ and where $f^{-1}f(\Delta_{i,j}^m) = \{\Delta_{i,i}^m, \dots, \Delta_{i,q}^m\}$. We denote $g_{i,q}^m = g^m$ then $g^m | J - J^{m-1}$ is non-singular. Next perform on $(m-1)$ -shift $g_{i,j}^{m-1}$ for $x_{i,j}^{m-1} \in S_f$ ($i=1, 2, \dots, p; j=1, 2, \dots, r$) where p and r are same as above. We denote $g_{p,r}^{m-1} = g^{m-1}$. Then $x_{i,j}^{m-1} \in S_{g^{m-1}}$ for $i=1, 2, \dots, p; j=1, 2, \dots, r$. For any point $y \in S_{g^{m-1}} - S_f$ such that $y \in \dot{S}t(\dot{B}_{i,j}^{m-1}, J')$, g^{m-1} is transversal to $g^{m-1} M$ at $g^{m-1}(y)$ by Lemma 4.

Furthermore since for any k -simplex $\Delta^k \in J'$ whose interior is contained in $\dot{S}t(\dot{B}^{m-1}, J') \cap \dot{S}t(\dot{B}^m, J')$, $\Delta^k = \dot{B}^{m-1} * \bar{\Delta}^{k-1}$ where $\bar{\Delta}^{k-1}$ is the opposite face of \dot{B}^{m-1} in Δ^k and since $g^{m-1} = g^m$ at $M - \bigcup_{B \in J'} \dot{S}t(\dot{B}^{m-1}, J')$, $g^{m-1}(\Delta^k) = v * g^m(\bar{\Delta}^{k-1})$ where v is a point of $\dot{S}t(f(\dot{B}^{m-1}), K')$ satisfying the conditions 1) $v \notin f(M)$ and 2) v is in general position with respect to the vertices of the complex covered by $St(f(\dot{B}^{m-1}), K')$. Hence it is obviously that if g^m is transversal to $g^m M$ at $St(\dot{B}^m, J') \cap St(\dot{B}^m, J')$, g^{m-1} is also transversal to $g^{m-1} M$ at there. Therefore

g^{m-1} is transversal to $g^{m-1}M$ at every point of $|J-J^{m-2}|$. In this way any singular point of S_f become non-singular point one time, and if $y \in \dot{St}(\dot{B}^b, J')$ become again a singular point by g^b , then, g^b is transversal to g^bM at y by Lemma 4.

Furthermore for any point $y \in \dot{St}(\dot{B}^b, J') \cap \dot{St}(\dot{B}^{b+1}, J')$ g^b is transversal to g^bM at y as above. Hence g^b is transversal to g^bM at every point of $|J-J^{b-1}|$. Therefore, in final, g^0 is transversal to g^0M at all points of M particularly of S_{g^0} . Then g^0 is the required g .

And if f is in general position itself, g is obviously in general position itself by construction. Complete the proof.

Proof of Corollary 1.

By the hypothesis if $x \in S_g$, $g^{-1}g(x) = x \cup y$ and the following diagram commutes

$$\begin{array}{ccccc}
 D^{2m-q} \times D^{q-m}, 0 \times 0 & \xrightarrow{k_x} & D^{2m-q} \times D^{q-m} \times D^{q-m}, 0 \times 0 \times 0 & \xleftarrow{k_y} & D^{2m-q} \times D^{q-m}, 0 \times 0 \\
 \varphi_x \downarrow & & \downarrow \psi & & \downarrow \varphi_y \\
 M, x & \xrightarrow{g} & Q, g(x) & \xleftarrow{g} & M, y
 \end{array}$$

where $k_x = 1 \times 1 \times 0$, $k_y = 1 \times 0 \times 1$ and where $\varphi_x, \varphi_y, \psi$ are homeomorphism onto some neighborhoods of $x, y, g(x)$ respectively.

Since $St(x, S_g) \cong St(k_x(0 \times 0))$, $k_x(D^{2m-q} \times D^{q-m}) \cap k_y(D^{2m-q} \times D^{q-m}) = St(0 \times 0 \times 0, D^{2m-q}) \cong D^{2m-q}$, S_g is a closed $(2m-q)$ -manifold. And in the left side of the above diagram $(St(x, M), St(x, S_g)) \cong (D^{2m-q} \times D^{q-m}, D^{2m-q})$ since $\varphi_x: (D^{2m-q} \times D^{q-m}, D^{2m-q} \times 0, 0 \times 0) \rightarrow (M, S_g, x)$.

Hence S_g is a closed locally flat $(2m-q)$ -submanifold of M .

Proof of Cor. 3.

Let J_0 and K_0 be the subdivisions of M, Q and f be in general position with respect to J_0, K_0 . Let J, K be the subdivisions of J_0, K_0 such that $f: J \rightarrow K$ is simplicial.

Then S_f consists only of the vertices of J and the local knotness rises only on the vertices of J . Hence if $y_k \in S_{g^k} - S_{g^{k-1}}$, $y_k \notin J^0$ where y_k, g_k are same as Lemma 3.

Then g_k is locally flat at y_k and g_k is transversal to g_kM at y_k as the proof of theorem.

References

- [1] M.A. Armstrong: *Transversality for polyhedra*, Ann. of Math. **86** (1967), 172–191.
- [2] ——— and E.C. Zeeman: *Transversality for piecewise linear manifolds*, Topology **6** (1967), 433–466.
- [3] S.S. Cairns: *Triangulated manifolds which are not Brouwer manifolds*, Ann. of Math. **41** (1940), 792–795.
- [4] E.C. Zeeman: *Seminar on Combinatorial Topology* (mimeographed), Inst. Hautes Etudes Sci., Paris, 1963.
- [5] J.F.P. Hudson and E.C. Zeeman: *On regular neighborhoods*, Proc. London Math. Soc. (3) **14** (1964), 719–745.