Kobayashi, K.
Osaka J. Math.
6 (1969), 409-418

ON THE TRANSVERSAL IMMERSION

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(Received November 14, 1968)

(Revised April 4, 1969)

Zeeman and Armstrong [1], [2], [4] introduced the notion of transversality for the intersection of two manifolds or two polyhedra and proved some results. This notion can be considered as a refinement of general position. In this paper we extend this to the self-intersection of an piecewise linear immersion between manifolds.

The main result of this paper (Theorem) says that any locally flat immersion of a closed manifold into a manifold without boundary can be approximated by a transversal immersion.

It will be assumed, without further mention, that all manifolds have a piecewise linear structure and that all maps are piecewise linear immersions of a closed (i.e. compact and without boundary) manifold into a manifold without boundary.

Definitions and Theorem

Let M be a closed m-manifold and Q be a q-manifold without boundary. Let $f:M\to Q$ be an immersion. Let $\hat{A}$ be the barycenter of a simplex A. We denote E^q the q-dimensional euclidean space.

DEFINITION 1. Let J, K be triangulations of M, Q such that $f:J\to K$ is simplicial. If $(Lk(fx, K), f(Lk(x, J)))$ is unknotted sphere pair for any $x\in M$, we say that f is a *locally flat immersion.* This definition is independent of the triangulation of M, Q. Let $S_f=\{x\in M \mid f^{-1}f(x)\neq\{x\}\}$. Then $S_f=\bar{S}_f$ where $\bar{S}_f$ is the closure of S_f since f is an immersion. And let $S_f(r)=\{x_1\in M \mid f^{-1}f(x_1)=\{x_1,\cdots, x_n\}, n\geq r\}$. Hence $S_f=S_f(2)$.

DEFINITION 2. Let $f^{-1}f(x_1)=\{x_1,\cdots, x_n\}$ for some $x_1\in S_f$. If the following diagram commutes for any $i\,(1\leq i\leq n)$ except j, we call f *transversal to fM at x_j* where φ_j, φ_i, ψ are homeomorphism onto some neighborhoods of x_j, x_i, $f(x_1)$ respectively.

$$\begin{array}{ccccccc} D^{2m-q}\times D^{q-m}, 0\times 0 & \xrightarrow{1\times 1\times 0} & D^{2m-q}\times D^{q-m}\times D^{q-m}, 0\times 0\times 0 & \xleftarrow{1\times 0\times 1} & D^{2m-q}\times D^{q-m}, 0\times 0 \\ \varphi_j\downarrow & & \downarrow\psi & & \downarrow\varphi_i \\ M, x_j & \xrightarrow{\quad f \quad} & Q, f(x_1) & \xleftarrow{\quad f \quad} & M, x_i \end{array}$$

And we simply call *f transversal to fM at* $f(x_1)$ if for any $j(1 \leq j \leq n)$ f is transversal to fM at x_j.

Furthermore we define *f transversal immersion* if f is transversal to fM at any $f(x)$, $x \in S_f$.

DEFINITION 3. An immersion f of M into Q is *in general postition itself* if $\dim S_f(r) \leq rm-(r-1)q$ for all r.

Theorem. *If f is a locally flat immersion of* M^m *into* Q^q, *then we can homotope f into a locally flat transversal immersion g by an arbitrarily small homotopy. Furthermore if f is in general position itself, then g is also in general position itself.*

REMARK 1. The above theorem still holds for an immersion f such that f is locally flat at any point of $St(S_f, M)$ although g is a transversal immersion such that g is locally flat at any point of $St(S_g, M)$.

REMARK 2. Conversely if f is a transversal immersion of M into Q, f is locally flat at any point of S_f. For it is obvious by the left half or the right half of the above diagram.

REMARK 3. When $m=q$, theorem is obvious. So we shall prove it for $m<q$.

Corollary 1. *If g is a transversal immersion and* S_g *consists only on double points, then* S_g *is a closed locally flat* $(2m-q)$*-submanifold of M.*

If the codimension of M and Q is greater than 3, any immersion is locally flat [4]. Hence we obtain the following Cor. 2.

Corollary 2. *If* $m+3 \leq q$, *any piecewise linear immersion of* M^m *into* Q^q *is arbitrarily approximated by an transversal immersion. The approximation is made to be homotopic.*

Corollary 3. *If f is any piecewise linear immersion of* M^2 *into* Q^4, *then f is approximated by an transversal immersion. The approximation can be chosen so near and to be homotopic.*

DEFINITION 4. Let f be an immersion of M into Q and J, K be simplicial subdivisions of M, Q respectively. We call f *in general position with respect to* J *and* K *at* x if for any simplexes $\Delta^i \in J$ and $\Delta^k \in K$ such that $x \in \mathring{\Delta}^i$ and $f(x) \in \mathring{\Delta}^k$,

$$\dim (f\Delta^i \cap \Delta^k) \leq i+k-q .$$

For any $x \in M$ let A be a simplex of K such that $f(x) \in \mathring{A}$. Choose a vertex v of A, let $L=Lk(A, K)$, and $s: AL \to vL$ the simplicial map defined as the join of $A \to v$ to the identity on L.

DEFINITION 5. (Armstrong, Zeeman [1], [2]) Let the map f be an immersion of M into Q and be in general position with respect to J and K where $|J|=M$, $|K|=Q$. The map f is *transimplicial to K at the point $x\in M$* if there exists a neighborhood N of v in vL, and a commutative diagram

$$\begin{array}{ccccc} N\times D^{m+a-q} & \xrightarrow{1\times k} & N\times D^a & \xrightarrow{p} & N \\ \downarrow\alpha & & \downarrow\beta & & \downarrow\subset \\ St(x,J)\cap f^{-1}AL & \xrightarrow{f} & AL & \xrightarrow{s} & vL \end{array}$$

where a is the dimension of A, k is a proper embedding $D^{m+a-q}\to D^a$, p is a projection and α, β are embeddings onto neighborhoods of x, $f(x)$ respectively.

REMARK 4. The definition is independent of the choice of v.

REMARK 5. If f is transimplicial to K at x and $x\in\mathring{\Delta}^i$ $(i<m)$, then $m+a>q$. For since f is in general position and $f(x)\in\mathring{A}$,

$$\dim(f\Delta^i\cap A)\leq i+a-q<m+a-q\,.$$

Hence $f\Delta^i\cap A=\phi$ if $m+a-q\leq 0$. This contradicts $x\in\mathring{\Delta}^i$ and $f(x)\in\mathring{A}$.

If f is transimplicial to K at x and $x\in\mathring{\Delta}^m$, then $m+a\geq q$. For $\dim(f\Delta^m\cap A)\leq m+a-q$. Hence $f\Delta^m\cap A=\phi$ if $m+a<q$. This is contradiction.

REMARK 6. Let K' be a subdivision of K. If f is transimplicial to K' at $x\in M$ (hence f is in general position with respect to K'), then f is also transimplicial to K at x (see [2]).

DEFINITION 6. Let K be a combinatorial manifold of dimension q. Then K is called a *Brouwer manifold* if

(i) For each $v\in K$ there is a linear embedding
$$St(v,K)\to E^q$$
(ii) For each $v\in K$ there is a linear embedding
$$St(v,K),\ St(v,K)\to E^q_+,\ E^{q-1}.$$

REMARK 7. Not all combinatorial manifolds are Brouwer, [3].

REMARK 8. Any subdivision of a Brouwer is Brouwer.

Lemma 0 (Zeeman [2]). *Any combinatorial manifold has a Brouwer subdivision.*

Lemmata

When $m=q$, theorem is obvious. So we assume $m<q$ throughout this section.

Now let $f:M\to Q$ be an immersion and J, K be subdivision of M, Q such that f is simplicial with respect to J, K and such that K is Brouwer subdivision. And we may suppose f is in general position itself by [4, Chap. 6]. Let $x_1\in S_f$, and $f^{-1}f(x_1)=\{x_1,\cdots, x_n\}$.

We suppose $x_i\in B_i^b$, $f(x_1)\in\mathring{A}$ where B_i $(i=1, 2,\cdots, n)$ and A are simplexes of J and K respectively.

Let $X_i=St(\hat{B}_i, J')$, $W=St(\hat{A}, K')$.

We construct approximating maps $g_i^b(i=1, 2,\cdots, n-1)$ of f inductively as follows. First take v_1 in $\mathring{W}$ such that $v_1\notin f(M)$ and v_1 is in general position with respect to the vertices of W. Then we define g_1^b as follows

$$g_1^b\,|\,M-\mathring{St}(\hat{B}_1, J')=f\,|\,M-\mathring{St}(\hat{B}_1, J')$$
$$g_1^b(\hat{B}_1)=v_1$$

and for any $y\in St(\hat{B}_1, J')$ if $y=(1-\lambda)d+\lambda\hat{B}_1$ $(d\in\dot{St}(\hat{B}_1, J'))$, $g_1(y)=(1-\lambda)f(d)+\lambda v_1$.

Obviously g_1^b is not simplicial with respect to J' and K'. We take subdivisions J_1, K_1 of J', K' such that they are subdividing the parts $f^{-1}(\mathring{W})$ and $\mathring{W}$ and such that $g_1^b:J_1\to K_1$ is simplicial.

Next take v_2 in $\dot{St}(\hat{A}, K'_1)$ satisfying $v_2\notin g_1^b(M)$ and v_2 is in general position with respect to the vertices of the subcomplex W_1 of K'_1 covering W. And we define g_2^b satisfying

$$g_2^b\,|\,M-\mathring{St}(\hat{B}_2, J')=g_2^b\,|\,M-\mathring{St}(\hat{B}_2, J')$$
$$g_2^b(\hat{B}_2)=v_2 \qquad \text{and}$$

for any $y\in\dot{St}(\hat{B}_2, J')$ it maps linearly same as g_1^b. We take subdivisions J_2, K_2 of J'_1, K'_1 such that they are subdividing the parts $(g_1^b)^{-1}\mathring{W}_1$ and $\mathring{W}_1$ and such that $g_2^b:J_2\to K_2$ is simplicial. We construct g_i^b $(3\leq i\leq n-1)$ in the same way. Furthermore we similarly construct approximating maps of f for any other point $x\in S_f$ such that $x\in\mathring{B}^b$.

And we again put the maps g_j^b.

Obviously $g_i^b\,|\,X_i$ is in general position with respect to J'_{i-1} and K'_{i-1} $(i=1, 2,\cdots, n-1)$ at any point $x\in\mathring{X}_i$ where $J_0=J$, $K_0=K$ and g_i^b is in general position itself.

Lemma 1. *For all i $(i=1, 2,\cdots, n-1)$ the map g_i^b constructed above is transimplicial to K'_{i-1} at any point of $\mathring{X}_i$ where $K_0=K$.*

Proof. First we show g_1^b transimplicial to K' at any point of $\mathring{X}_1$.

For any $y\in\dot{St}(\hat{B}_1, J')$ let C, D be simplexes of J', K' respectively such that $y\in\mathring{C}$, $g_1^b y\in\mathring{D}$. Since K is Brouwer triangulation, we may suppose $St(A, K)$ embedded linearly in E^q. If D is a principal simplex of K', g_1^b is obviously trans-

implicial to K' at y. Hence we may suppose dim $D<q$. $g_1^b C$ is the linear join in E^q of v_1 to some simplex of $\mathring{W}$ by the construction of g_1^b. And since v_1 is in general position with respect to the vertices of W, $g_1^b C$ and D span E^q. $g_1^b C \cap D$ is a $(c+d-q)$-linear convex cell. Let $(g_1^b)^{-1}(g_1^b C \cap D) \cap C = E$ and F a $(q-d)$-cell through y that is perpendicular to E in C. Let C^* be a simplex of J' having C as a face and $E^* = (g_1^b)^{-1}(g_1^b C^* \cap D) \cap C^*$. Then although E^* is not necessarily perpendicular to F in C^*, it has the property that any $(q-d)$-cell parallel to F in C^*, and sufficiently close to F, meets it in exactly one point. Therefore for some sufficiently small neighborhood U of y in $St(\hat{B}_1, J')$ we can define a map $\rho_1 : U \to (g_1^b)^{-1} D$ by projecting $U \cap C^*$ parallel to F onto the corresponding E^*. Now return to E^q. Since we defined F perpendicular to E in C, we know that the linear subspace $[g_1^b F]$ and $[D]$, spanned by $g_1^b F$ and D, are complementary in E^q. Hence we define the map

$$\rho_2 : E^q \to [D] \text{ parallel to } [g_1^b F] .$$

Then $\rho_2 g_1^b = g_1^b \rho_1$ on U.

Choose a vertex v of D and let $L = lk(D, K')$, $s : DL \to vL$ and define
$\alpha = s g_1^b \times \rho_1 : U \to vL \times (g_1^b)^{-1} D$

$$\beta = s \times \rho_2 : St(D, K') \to vL \times [D] .$$

We can check that α and β are both piecewise linear embeddings onto neighborhoods of (v, y), $(v, g_1^b y)$ respectively. Choose ball neighborhoods N of v in vL,

D^{m+d-q} of y in $(g_1^b)^{-1} D$ and
D^d of $g_1^b y$ in D
such that $N \times D^{m+d-q}$ image of α
$N \times D^d$ image of β, and
$g_1^b D^{m+d-q} \subset D^d$

Hence the following diagram commutes

$$\begin{array}{ccccc} N \times D^{m+d-q} & \xrightarrow{1 \times (g_1^b \mid D^{m+d-q})} & N \times D^d & \xrightarrow{p} & N \\ \downarrow \alpha^{-1} & & \downarrow \beta^{-1} & & \downarrow \subset \\ g^{-1} DL \cap St(x_1, J') & \xrightarrow{g_1^b} & DL & \xrightarrow{s} & vL \end{array}$$

This complete the proof for g_1^b.

Next we show g_2^b transimplicial to K_1' at any point of $\mathring{X}_2$. The main part of the proof is equally for g_1^b and we shall not repeat all, but give the difference part of g_2^b. Since g_2^b is simplicial with respect to J' and $v_2 * \mathring{W}$ on $St(\hat{A}_2, J')$, if C_2 is the simplex of J' such that $y_2 \in \mathring{C}_2$ where y_2 is any point of $\mathring{X}_2$, $g_2^b C_2$ is

the linear join of v_2 and some simplex of $\dot{W}$. Let D_2 be the simplex of K_1' such that $g_2^b y_2 \in \dot{D}_2$. Since v_2 is in general position with respect to the vertices of W_1, $g_2^b C_2$ and D_2 span E^q. Hence $(g_2^b)^{-1}(g_2^b C_2 \cap D_2) \cap C_2$ is the $(c+d-q)$-linear convex cell E_2. We define F_2 the $(q-d)$-cell through y_2 that is perpendicular to E_2. Let C_2^* be a simplex of J' having C_2 as a face. In the same way as g_1^b, g_2^b is transimplicial to K_1' at any point of $\mathring{X}_2$. We can prove the lemma similarly for g_i^b $(3 \leq i \leq n-1)$.

Lemma 2. *If f is a locally flat immersion, g_i^b $(i=1, 2, \cdots, n-1)$ are also a locally flat immersion.*

Proof. By the construction of g_i^b, it is sufficiently to show that g_1^b is locally flat. Since g_1^b is different from f only on $\mathring{St}(\hat{B}_1, J')$ and since f is locally flat, $(Lk(f\hat{B}_1, K'), f(Lk(\hat{B}_1, J'))$ is the unknotted $(q-1, m-1)$-sphere pair. Hence

$$(St(g_1^b B, K_1), g_1^b(St(B_1, J))) = v_1 * (Lk(fB_1, K), f(Lk(B_1, J)))$$

is an unknotted (q, m)-ball pair.

Therefore g_1^b is locally flat at $x \in \mathring{St}(\hat{B}_1, J')$. Obviously g_1^b is locally flat at the point $x \in M - St(\hat{B}_1, J')$. Let u be a vertex in $Lk(\hat{B}_1, J')$. Since f is locally flat, $(Lk(f(u), K'), f(Lk(u, J'))$ is an unknotted sphere pair.

Case 1. If u is a vertex of J in $Lk(\hat{B}_1, J')$,

$$(Lk(f(u), K'), f(Lk(u, J')) \cong \partial(\square, f\triangledown)$$

where $\triangledown$ is the dual cell of u in J and where $\square$ is the dual cell of $f(u)$ in K. And $\partial(\square, f\triangledown) \cap St(f(\hat{B}_1), K') = (\widetilde{\square}, f(\widetilde{\triangledown}))$ where $\widetilde{\triangledown} = St(\hat{B}_1, \partial\triangledown)$ and where $\widetilde{\square} = St(f(\hat{B}_1), \partial\square)$. Since f is locally flat, $(\widetilde{\square}, f(\widetilde{\triangledown}))$ is the unknotted $(q-1, m-1)$-ball pair. Hence $\partial(\square, f\triangledown)$-Int$(\widetilde{\square}, f(\widetilde{\triangledown}))$ is the unknotted ball pair (D, C) by [5. Cor. 8] and

$$(Lk(g_1^b(u), K'), g_1^b(Lk(u, J')) \cong (D, C) \cup v_1 * \partial(D, C)$$

is the unknotted sphere pair. Therefore g_1^b is locally flat at u.

Case 2. If u is a vertex of J' not J in $Lk(\hat{B}_1, J')$,

$$(Lk(f(u), K'), f(Lk(u, J'))) \cap St(f\hat{B}_1, K') = (St(f\hat{B}_1), Lk(f(u)), f(St(\hat{B}_1, Lk(u, J')))) .$$

And since f is locally flat, it is the unknotted $(q-1, m-1)$-ball pair (E, F). Hence $(Lk(f(u), K'), f(Lk(u, J')))$-Int$(E, F)$ is the unknotted $(q-1, m-1)$-ball pair (D, C). Therefore g_1^b is locally flat at u as Case 1.

Lemma 3. *If a locally flat immersion $f: M \to Q$ is transimplicial to K at*

$x_j \in S_f$ where $f^{-1}f(x_j)=\{x_1,\cdots, x_j,\cdots, x_n\}$ and $|K|=Q$ and if $f|M-\mathring{St}(x_j, J)$ is simplicial with respect to J and K where $|J|=M$, then f is transversal to $f(M)$ at x_j.

Proof. Since f is a locally flat immersion, g_i^b is also by Lemma 2. Let A be the simplex of K satisfying $f(x_j)\in \mathring{A}$ and $L=Lk(A, K)$. Choose a vertex v of A, let $s:AL\to vL$ the simplicial map as defined before. Since f is transimplicial to K at x_j, $f|St(x_j, J)$ is in general position with respect to K and there exists a neighborhood N of v in vL such that the following diagram commutes (see Remark 5),

$$
\begin{array}{ccccc}
 & & N\times D^{m+a-q} & \xrightarrow{1\times k} & N\times D^a \xrightarrow{p} N \\
(0) & & \alpha\downarrow & & \beta\downarrow \quad \downarrow\subset \\
 & f^{-1}AL\cap St(x_j, J) & \xrightarrow{f} & AL & \xrightarrow{s} vL
\end{array}
$$

where $k:D^{m+a-q}\to D^a$ is a proper embedding. Furthermore as f is locally flat, the pair $(N\times D^a,\ N\times k(D^{m+a-q}))$ is unknotted $(q,\ m)$-ball pair. Hence we can write the diagram (0) as follows,

$$
\begin{array}{ccc}
N\times D^{m+a-q},\ v\times 0 & \xrightarrow{1\times 1\times 0} & N\times D^{m+a-q}\times D^{q-m},\ v\times 0\times 0 \\
\alpha\downarrow & & \downarrow\beta \\
f^{-1}AL\cap St(x_j, J),\ x_j & \xrightarrow{\qquad f\qquad} & AL,\ f(x_j)
\end{array}
$$

On the other hand $f|M-St(x_j, J)$ is simplicial, there exist simplexes B_i of J such that $\mathring{B}_i\ni x_i$ and $fB_i=A$ $(i=1, 2,\cdots, \overset{*}{j},\cdots, n)$. Let $L_1=f(Lk(B_i, J))$ and N_1 be a neighborhood of v in vL_1, then $N_1=N\cap vL_1$. And since f is locally flat, $(N,\ N_1)$ is an unknotted $(q-a,\ m-a)$-ball pair. Thus there exists an unknotting homeomorphism

$$h:D^{q-m}\times D^{m-a},\ 0\times D^{m-a},\ 0\times 0\to N,\ N_1,\ v\ .$$

Hence the following diagram commutes.

$$
\begin{array}{ccccc}
\begin{array}{c} D^{a-m}\times D^{m-a}\times \\ D^{m+a-q},\ 0\times 0\times 0 \end{array} & \xrightarrow{1\times 1\times 1\times 0} & \begin{array}{c} D^{q-m}\times D^{m-a}\times D^{m+a-q} \\ \times D^{q-m},\ 0\times 0\times 0\times 0 \end{array} & \xleftarrow{0\times 1\times 1\times 1} & \begin{array}{c} D^{m-a}\times D^{m+a-q} \\ \times D^{q-m},\ 0\times 0\times 0 \end{array} \\
h\times 1\downarrow & & \downarrow h\times 1\times 1 & & (h|D^{m-a})\times 1\times 1\downarrow \\
N\times D^{m+a-q},\ v\times 0 & \xrightarrow{1\times 1\times 0} & N\times D^{m+a-q}\times D^{q-m},\ v\times 0\times 0 & \xleftarrow{\subset\times 1\times 1} & N_1\times D^{m+a-q}\times D^{q-m}, v\times 0\times 0 \\
\alpha\downarrow & & \beta\downarrow & & f^{-1}\alpha\downarrow \\
f^{-1}AL\ St(x_j, J),\ x_j & \xrightarrow{\qquad f\qquad} & AL,\ f(x_j) & \xleftarrow{\qquad f\qquad} & f^{-1}AL\cap St(x_i, J),\ x_i.
\end{array}
$$

From the top and the bottom of the diagram, f is transversal to $f(M)$ at x_j.

Lemma 4. *If* $y_k \in Sg^b_k - Sg^b_{k-1}$ *and* $(g^b_k)^{-1}(g^b_k)(y_k) = \{\cdots\ y_k\ \cdots\ y_m\}$ *where* $y_i \in \mathring{S}t(\hat{B}_i, J')$ *and* $\hat{B}_i \ni x_i$, *then* g^b_k *is transversal to* $g^b_k M$ *at* y_i $(1 \leq i \leq k)$.

Proof. We shall prove the lemma by induction on k. Since g^b_k is transimplical to K'_{k-1} at y_k by Lemma 1, g^b_k is transversal to $g^b_k M$ at y_k by Lemma 3. We suppose g^b_k transversal to $g^b_k M$ at y_j $(1 \leq j \leq k)$. Next we shall show g^b_k transversal to $g^b_k M$ at y_l where l' is the largest number satisfying $l' < l$. Since $g^b_k | \mathring{S}t(\hat{B}_l, J') = g^b_{l'} | St(\hat{B}_{l'}, J')$ and g^b_k is transimplicial to K'_i at $y_{l'}$ $(i < l')$, $g^b_k | \mathring{S}t$ $(\hat{B}_{l'}, J')$ is transversal to $g^b_k(\mathring{S}t(\hat{B}_i, J'))$ at $y_{l'}$, $(i < l)$ by Lemma 3. And g^b_k is transversal to $g^b_k(\mathring{S}t(\hat{B}_j, J'))$ $(1 \leq j \leq k)$ at $y_{l'}$ by inductive hypothesis. For $j < k$, since g^b_k does not effect on $\mathring{S}t(\hat{B}_j, J')$, $g^b_k(\mathring{S}t(\hat{B}_j, J'))$ is embedded as a subcomplex of K_i' for any i. Hence g^b_k is transversal to $g^b_k(St(\hat{B}_j, J'))$ $(j < k)$ at $y_{l'}$ because g^b_k is transimplicial to K'_i $(i < l')$ at y_l. Therefore g^b_k is transversal to $g^b_k M$ at $y_{l'}$.

We denote the last map like g^b_{n-1} for all other point $x \in S_f$ such that $x \in B^b$ as g^b.

Proof of Theorem.

We order k-simplexes of J containing the point of S_f such Δ^k_{ij} that if $f(\Delta^k_{ij}) = f(\Delta^k_{mn})$, $i = m$ and that the set of k-simplexes of $f^{-1}f(\Delta^k_{ij})$ is properly numbered for the second suffix j. We number point contained $|\Delta^k_{ij}| \cap S_f$ as x^k_{ij}.

We perform the shift g^b_{ij} for every $x^b_{ij} \in S_f$ such that $x^b_{ij} \in B^b_{ij}$ and in order to decreasing dimension of b. Since J is a finite dimensional finite complex, the time of shifts is finite.

First perform on m-shift g^m_{ij} for $x^m_{ij} \in S_f$ $(i = 1, 2, \cdots, l;\ j = 1, 2, \cdots, q)$ where l is the number of m-simplexes such that $|\Delta^m_{ij}| \cap S_f \neq \phi$, $|\Delta^m_{st}| \cap S_f \neq \phi$ and $f(\mathring{\Delta}^m_{ij}) \cap f(\mathring{\Delta}^m_{st}) = \phi$ and where $f^{-1}f(\Delta^k_{ij}) = \{\Delta^k_{i1}, \cdots, \Delta^k_{iq}\}$. We denote $g^m_{iq} = g^m$ then $g^m | J - J^{m-1}$ is non-singular. Next perform on $(m-1)$-shift g^{m-1}_{ij} for $x^m_{ij} \in S_f$ $(i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, r)$ where p and r are same as above. We denote $g^{m-1}_{pr} = g^{m-1}$. Then $x^{m-1}_{ij} \notin S_{g^{m-1}}$ for $i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, r$. For any point $y \in S_{g^{m-1}} - S_f$ such that $y \in \mathring{S}t(\hat{B}^{m-1}_{ij}, J')$, g^{m-1} is transversal to $g^{m-1}M$ at $g^{m-1}(y)$ by Lemma 4.

Furthermore since for any k-simplex $\Delta^k \in J'$ whose interior is contained in $\mathring{S}t(\hat{B}^{m-1}, J') \cap \mathring{S}t(\hat{B}^m, J')$, $\Delta^k = \hat{B}^{m-1} * \tilde{\Delta}^{k-1}$ where $\tilde{\Delta}^{k-1}$ is the opposite face of $\hat{B}^{m-1}$ in Δ^k and since $g^{m-1} = g^m$ at $M - \bigcup_{B \in J} \mathring{S}t(\hat{B}^{m-1}, J')$, $g^{m-1}(\Delta^k) = v * g^m(\tilde{\Delta}^{k-1})$ where v is a point of $\mathring{S}t(f(\hat{B}^{m-1}), K')$ satisfying the conditions 1) $v \notin f(M)$ and 2) v is in general position with respect to the vertices of the complex covered by $St(f(\hat{B}^{m-1}), K')$. Hence it is obviously that if g^m is transversal to $g^m M$ at $St(\hat{B}^m, J') \cap St(\hat{B}^m, J')$, g^{m-1} is also transversal to $g^{m-1}M$ at there. Therefore

g^{m-1} is transversal to $g^{m-1}M$ at every point of $|J-J^{m-2}|$. In this way any singular point of S_f become non-singular point one time, and if $y\in\mathring{St}(\hat{B}^b, J')$ become again a singular point by g^b, then, g^b is transversal to g^bM at y by Lemma 4.

Furthermore for any point $y\in\mathring{St}(\hat{B}^b, J')\cap\mathring{St}(\hat{B}^{b+1}, J')$ g^b is transversal to g^bM at y as above. Hence g^b is transversal to g^bM at every point of $|J-J^{b-1}|$. Therefore, in final, g^0 is transversal to g^0M at all points of M particularly of S_{g^0}. Then g^0 is the required g.

And if f is in general position itself, g is obviously in general position itself by construction. Complete the proof.

Proof of Corollary 1.

By the hypothesis if $x\in S_g$, $g^{-1}g(x)=x\cup y$ and the following diagram commutes

$$\begin{array}{ccccc} D^{2m-q}\times D^{q-m},\ 0\times 0 & \xrightarrow{k_x} & D^{2m-q}\times D^{q-m}\times D^{q-m},\ 0\times 0\times 0 & \xleftarrow{k_y} & D^{2m-q}\times D^{q-m},\ 0\times 0 \\ \downarrow{\scriptstyle \varphi_x} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi_y} \\ M,\ x & \xrightarrow{\quad g\quad} & Q,\ g(x) & \xleftarrow{\quad g\quad} & M,\ y \end{array}$$

where $k_x=1\times 1\times 0$, $k_y=1\times 0\times 1$ and where φ_x, φ_y, ψ are homeomorphism onto some neighborhoods of x, y, $g(x)$ respectively.

Since $St(x,\ S_g)\cong St(k_x(0\times 0),\ k_x(D^{2m-q}\times D^{q-m})\cap k_y(D^{2m-q}\times D^{q-m}))=St(0\times 0\times 0,\ D^{2m-q})\cong D^{2m-q}$, S_g is a closed $(2m-q)$-manifold. And in the left side of the above diagram $(St(x,\ M),\ St(x,\ S_g))\cong(D^{2m-q}\times D^{q-m},\ D^{2m-q})$ since φ_x: $(D^{2m-q}\times D^{q-m},\ D^{2m-q}\times 0,\ 0\times 0)\to(M,\ S_g,\ x)$.

Hence S_g is a closed locally flat $(2m-q)$-submanifold of M.

Proof of Cor. 3.

Let J_0 and K_0 be the subdivisions of M, Q and f be in general position with respect to J_0, K_0. Let J, K be the subdivisions of J_0, K_0 such that $f:J\to K$ is simplicial.

Then S_f consists only of the vertices of J and the local knotness rises only on the vertices of J. Hence if $y_k\in S_{g^k}-S_{g^{k-1}}$, $y_k\notin J^0$ where y_k, g_k are same as Lemma 3.

Then g_k is locally flat at y_k and g_k is transversal to g_kM at y_k as the proof of theorem.

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