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On the Homogeneous Linear Partial Differential Equation of the First Order

By Takashi KASUGA

§ 1. Introduction

In this paper, we shall treat the following partial differential equation

$$\frac{\partial z}{\partial x} + \sum_{\mu=1}^{n} f_{\mu}(x, y_1, \ldots, y_n) \frac{\partial z}{\partial y_{\mu}} = 0 \quad (n \geq 1)$$

without the usual condition of the total differentiability on the solution $z(x, y_1, \ldots, y_n)$.

For early contributions by R. Baire and P. Montel to this problem in the special case $n = 1$, cf. Baire [1], Montel [6]. Our method is entirely different from theirs and gives more general results even for the case $n = 1$, cf. Kasuga [4]. Also notwithstanding Baire's statement in his paper, it seems to us that their methods cannot be generalized to the case $n > 1$ immediately.

We have not yet succeeded in treating the more general non-homogeneous partial differential equation

$$\frac{\partial z}{\partial x} + \sum_{\mu=1}^{n} f_{\mu}(x, y_1, \ldots, y_n, z) \frac{\partial z}{\partial y_{\mu}} = g(x, y_1, \ldots, y_n, z)$$

in a similar way, except for the case $n = 1$. For this case, cf. Kasuga [5].

1. In this paper, we shall use for points in $R^n$, $R^{n+1}$ or $R^{n+2}$ and for their functions, abbreviations such as:

$$y = (y_1, \ldots, y_n), \quad (x; y) = (x, y_1, \ldots, y_n),$$
$$\eta = (\eta_1, \ldots, \eta_n), \quad (\xi; \eta) = (\xi, \eta_1, \ldots, \eta_n),$$
$$(x, \xi; \eta) = (x, \xi, \eta_1, \ldots, \eta_n), \quad z(x; y) = z(x, y_1, \ldots, y_n),$$

and if $\varphi_{\lambda}(x, \xi; \eta) = \varphi_{\lambda}(x, \xi, \eta_1, \ldots, \eta_n) \quad \lambda = 1, \ldots, n$ are $n$ functions of $(x, \xi; \eta)$,

1) Cf. Baire [1], p. 120.
Also we use the following notations:
For sets of points $A, B$ in $\mathbb{R}^m$ ($m=1, 2, \ldots, n+1$),

$\bar{A} =$ closure in $\mathbb{R}^m$ of $A$, $A^o =$ interior in $\mathbb{R}^m$ of $A$,

$A^b =$ boundary in $\mathbb{R}^m$ of $A$, $A \cap B =$ intersection of $A$ and $B$,

$A[x] =$ the set of the points $(y_1, \ldots, y_n)$ in $\mathbb{R}^n$ such that for a fixed $x (x, y_1, \ldots, y_n) \in A$, if $A \subset \mathbb{R}^{n+1}$.

For two points $y' = (y'_1, \ldots, y'_n)$, $y'' = (y''_1, \ldots, y''_n)$ in $\mathbb{R}^n$,

$$\| y' - y'' \| = \sum_{\mu=1}^n |y'_\mu - y''_\mu|, \quad y' + y'' = (y'_1 + y''_1, \ldots, y'_n + y''_n).$$

In this paper, the so-called degenerated intervals are also included, when we use the word "interval" (open, closed, or half-open). Thus the interval $a < x < a$ or the interval $a \leq x \leq a$ will mean degenerated interval which is empty or is composed of only one point respectively. Similarly for the interval $a < x < a$ or $a < x \leq a$.

2. In the following, we shall denote by $G$ a fixed open set in $\mathbb{R}^{n+1}$, by $f_\lambda(x; y) \lambda = 1, \ldots, n$ fixed continuous functions defined on $G$ which have continuous $\partial f_\lambda / \partial y_\mu \lambda, \mu = 1, \ldots, n$.

Under the above conditions, we shall consider the partial differential equation

$$\frac{\partial z}{\partial x} + \sum_{\mu=1}^n f_\mu(x; y) \frac{\partial z}{\partial y_\mu} = 0. \quad (2.1)$$

With (2.1), we shall associate the simultaneous ordinary differential equations

$$\frac{dy_\lambda}{dx} = f_\lambda(x; y) \quad \lambda = 1, \ldots, n. \quad (2.2)$$

The continuous curves representing the solutions of (2.2) which are prolonged as far as possible on both sides in an open subset $D$ of $G$, will be called characteristic curves of (2.1) in $D$. Through any point $(\xi; \eta)$ in $D$, there passes one and only one characteristic curve in $D^2$. We represent it by

$$y_\lambda = \varphi_\lambda(x, \xi, \eta, \ldots, \eta_n|D) = \varphi_\lambda(x, \xi; \eta|D) \quad \lambda = 1, \ldots, n.$$
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\[ \alpha(\xi; \eta|D), \beta(\xi; \eta|D) \] may be \(-\infty, +\infty\) respectively. Sometimes we abbreviate it as \(C(\xi; \eta|D)\). If an interval (open, closed or half-open) is contained in the interval \(\alpha(\xi; \eta|D) < x < \beta(\xi; \eta|D)\), then we say that \(C(\xi; \eta|D)\) is defined for that interval. Also when a property holds for the portion of \(C(\xi; \eta|D)\) which corresponds to the values of \(x\) belonging to an interval, we say that \(C(\xi; \eta|D)\) has the property for this interval.

We shall use the following properties of \(C(\xi; \eta|D)\) and \(\phi_\lambda(x, \xi; \eta|D)\) often without special reference.

As we can easily see from the definition of \(C(\xi; \eta|D)\), \((\xi; \eta) \in C(\xi; \eta|D)\) and if \((x'; y') \in C(\xi; \eta|D)\), then \(C(\xi; \eta|D) = C(x'; y'|D)\) and so \((\xi; \eta) \in C(x'; y'|D)\).

In terms of the functions \(\phi_\lambda\); this means:

\[ \eta = \phi(\xi, \xi; \eta|D), \]

and if \(y' = \phi(x', \xi; \eta|D)\), then

\[ \alpha(\xi; \eta|D) = \alpha(x'; y'|D) = \alpha, \quad \beta(\xi; \eta|D) = \beta(x'; y'|D) = \beta \]

and

\[ \phi(x, \xi; \eta|D) = \phi(x, x'; y'|D) \quad \text{for} \quad \alpha < x < \beta, \]

especially

\[ \eta = \phi(\xi, x'; y'|D). \]

Also \(C(\xi; \eta|D_1) \supset C(\xi; \eta|D_2)\), if \(D_1 \supset D_2\) and \((\xi; \eta) \in D_1\).

We denote by \(D^*\) the set of the points \((x, \xi; \eta)\) in \(R^{n+2}\) such that \((\xi; \eta) \in D\) and \(\alpha(\xi; \eta|D) < x < \beta(\xi; \eta|D)\). \(D^*\) is the domain of definition of the functions \(\phi_\lambda(x, \xi; \eta|D)\). \(D^*\) is open in \(R^{n+2}\). The functions \(\phi_\lambda(x, \xi; \eta|D)\) are continuous and have continuous partial derivatives with respect to all their arguments on \(D^*\).

A continuous function \(z(x; y)\) defined on \(G\) will be called a quasi-solution of (2.1) on \(G\), if it has \(\partial z/\partial x, \partial z/\partial y_\lambda \lambda = 1, \ldots, n\), except at most at the points of an enumerable set in \(G\) and satisfies (2.1) almost everywhere in \(G\). Here \(\partial z/\partial x, \partial z/\partial y\) need not necessarily be continuous.

On the other hand, a continuous function \(z(x; y)\) defined on \(G\) will be called a solution of (2.1) on \(G\) in the ordinary sense, if it is totally differentiable and satisfies (2.1) everywhere in \(G\).

3. We shall prove the following three theorems in §3, §4.

3) Cf. Kamke [3], §17, Nr. 84, Satz 4.
4) Cf. Kamke [3], §17, Nr. 84, Satz 4 and §18, Nr. 87, Satz 1,
Theorem 1. A quasi-solution \( z(x; y) \) of (2.1) on \( G \) is constant on any characteristic curve of (2.1) in \( G \).

Theorem 2. If for a fixed number \( \xi^{(0)} \), the family of all the characteristic curves \( C(\xi^{(0)}; \eta | G) \) such that \( \eta \in G[\xi^{(0)}] \), covers \( G \) and \( \psi(\eta) = \psi(z, \cdots, \eta_n) \) is a totally differentiable function defined on \( G[\xi^{(0)}] \), then there is one and only one quasi-solution \( z(x; y) \) of (2.1) on \( G \) such that \( z(\xi^{(0)}; \eta) = \psi(\eta) \) on \( G[\xi^{(0)}] \) and this quasi-solution \( z(x; y) \) is also a solution of (2.1) on \( G \) in the ordinary sense.

Theorem 3. If \( n = 1 \), any quasi-solution of (2.1) on \( G \) is also a solution of (2.1) on \( G \) in the ordinary sense.

Remark 1. For the case \( n = 1 \), the proof of Theorem 1 can be partly simplified, cf. Kasuga [4].

Remark 2. In Theorem 1, the condition on \( z(x; y) \) that it has \( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \lambda = 1, \cdots, n \) except at most at the points of an enumerable set in \( G \), cannot be replaced by the condition that it has \( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \lambda = 1, \cdots, n \) almost everywhere in \( G \), as the following example shows it.

Example. \( G : 0 < x < 1, \ 0 < y < 1 \),
the differential equation is
\[
\frac{\partial z}{\partial x} = 0
\]
and a function \( z(x, y) \) is defined by
\[
z(x, y) = \psi(x) \quad \text{on} \quad G
\]
where \( \psi(x) \) is a continuous singular function not constant on the interval \( 0 \leq x \leq 1 \) as given in Saks [8] p. 101.

Then \( z(x, y) \) is continuous on \( G \), has \( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \) almost everywhere in \( G \) and satisfies the differential equation almost everywhere in \( G \). But \( z(x, y) \) is not constant on any characteristic curve \( y = \text{constant} \).

§ 2. Some Lemmas

In this §, the notations are the same as in § 1 and we assume that \( z(x; y) \) is a quasi-solution of (2.1) on \( G \).

4. Set \( K \) and Some Lemmas. We denote by \( K \) the set of the points \( (\xi; \eta) \) of \( G \) such that \( z(x; y) \) is constant on the portion of the characteristic curve \( C(\xi; \eta | G) \) contained in a neighbourhood of \( (\xi; \eta) \).
Lemma 1. If a characteristic curve $C(\xi; \eta | G)$ is defined for an interval $I$ (open, closed, or half-open) and is contained in $K$ for the open interval $I^0$, the interior of $I$, then $z(x; y)$ is constant on the portion of $C(\xi; \eta | G)$ for the interval $I$.

Proof. By the definition of $K$, we easily see that $z(x; y)$ is constant on the portion of $C(\xi; \eta | G)$ for the open interval $I^0$. Then Lemma 1 follows from the continuity of $z(x; y)$ and $\varphi_\lambda(x, \xi; \eta | G) \lambda = 1, \ldots, n$.

Lemma 2. Denote by $D$ an open subset of $G$, and denote by $D_0$ the set of the points $(\xi; \eta)$ of $D$ such that $z(x; y)$ is constant on the characteristic curve $C(\xi; \eta | D)$. Then $D_0$ is closed in $D$.

Proof. If $C(\xi^{(0)}; \eta^{(0)} | D)$ where $(\xi^{(0)}; \eta^{(0)}) \in D$, is defined for a closed interval $\alpha_0 \leq \eta \leq \beta_0$, then $C(\xi; \eta | D)$ where $(\xi; \eta)$ is sufficiently close to $(\xi^{(0)}; \eta^{(0)})$, is also defined for the interval $\alpha_0 \leq \eta \leq \beta_0$ and

$$\varphi_\lambda(x, \xi; \eta | D) \to \varphi_\lambda(x, \xi^{(0)}; \eta^{(0)} | D) \quad \lambda = 1, \ldots, n$$

uniformly in the interval $\alpha_0 \leq \eta \leq \beta_0$ as $(\xi; \eta) \to (\xi^{(0)}; \eta^{(0)})$. From this and by the continuity of $z(x; y)$, we easily see that $D_0$ is closed in $D$, q.e.d.

Lemma 3. Let $D$ be an open subset of $G$. If

$$|f_\lambda(x; y) - f_\lambda(x; y)| \leq M \|y - y\| \quad \lambda = 1, \ldots, n \quad (4.1)$$

for every pair of points $(x; y), (x; y) \in D$ with the same $x$ coordinate and if $C(\xi; \eta | D)$ and $C(\xi; \eta | D)$ where $(\xi; \eta), (\xi; \eta) \in D$, are both defined for an interval $\alpha_0 \leq \eta \leq \beta_0$ containing $\xi (\alpha_0 \leq \xi \leq \beta_0)$, then

$$\|\varphi(x, \xi; \eta | D) - \varphi(x, \xi; \eta | D)\| = \sum_{\mu=1}^{n} |\varphi_\mu(x, \xi; \eta | D) - \varphi_\mu(x, \xi; \eta | D)|$$

$$\leq \|\eta - \eta\| \exp(nM|x - \xi|) \quad (4.2)$$

and

$$|\varphi_\lambda(x, \xi; \eta | D) - \varphi_\lambda(x, \xi; \eta | D)| \leq |\eta - \eta| + \frac{1}{n} \|\eta - \eta\| \times \{\exp(nM|x - \xi|) - 1\} \quad \lambda = 1, \ldots, n \quad (4.3)$$

for $\alpha_0 \leq \eta \leq \beta_0$.

Proof. We abbreviate $\varphi_\lambda(x, \xi; \eta | D)$ and $\varphi_\lambda(x, \xi; \eta | D)$ as $\tilde{\varphi}_\lambda(x)$ and $\varphi_\lambda(x)$ respectively.

By (4.1) and (2.2), we have

---
5) Cf. Kamke [3], §17, Nr. 84, Satz 4,
\[ |\phi'(x) - \varphi'(x)| \leq M \| \phi(x) - \varphi(x) \| \]  
\[ \lambda = 1, \ldots, n \quad (4.4) \]

for \( \alpha \leq x \leq \beta \), so that

\[ \sum_{\mu=1}^{n} |\phi'(x) - \varphi'(x)| \leq nM \sum_{\mu=1}^{n} |\phi(x) - \varphi(x)| \]

for \( \alpha \leq x \leq \beta \). Hence by a theorem on differential inequalities\(^6\), taking account of \( \alpha \leq \xi \leq \beta \) and \( \eta = \phi(\xi) \), \( \eta = \varphi(\xi) \) (\( \lambda = 1, \ldots, n \)), we obtain

\[ \| \phi(x) - \varphi(x) \| \leq \| \eta - \eta \| \exp(nM|x-\xi|) \quad (4.5) \]

for \( \alpha \leq x \leq \beta \). Thus (4.2) is proved.

By (4.4), (4.5), we get

\[ |\phi'(x) - \varphi'(x)| \leq M \| \eta - \eta \| \exp(nM|x-\xi|) \]  
\[ \lambda = 1, \ldots, n \]

for \( \alpha \leq x \leq \beta \). Hence, again taking account of \( \alpha \leq \xi \leq \beta \) and \( \eta = \phi(\xi) \), \( \eta = \varphi(\xi) \) (\( \lambda = 1, \ldots, n \)), we have

\[ |\phi'(x) - \varphi'(x)| \leq |\eta - \eta| + M \| \eta - \eta \| \int_{\xi}^{x} \exp(nM|x-\xi|) \, dx \]

\[ = |\eta - \eta| + \frac{1}{n} \| \eta - \eta \| \{ \exp(nM|x-\xi|) - 1 \} \]  
\[ \lambda = 1, \ldots, n \]

for \( \alpha \leq x \leq \beta \). Thus (4.3) is also proved.

\[ \frac{\| \eta - \eta \|}{\| \eta - \eta \|} \exp(nM|x-\xi|) \]

\[ (a, b) \in F \]

\[ 6) \text{Cf. Kamke [3], §17, Nr. 85, Hilfssatz 3 and Satz 5,} \]

\[ \exists \eta \]
and such that

\[
\begin{align*}
|z(x+h; y) - z(x; y)| & \leq |h| N \\
|z(x, y, \ldots, y_{\lambda-1}, y_{\lambda}+k_{\lambda}, y_{\lambda+1}, \ldots, y_n) - z(x; y)| & \leq |k_{\lambda}| N \\
\lambda = 1, \ldots, n
\end{align*}
\]  

whenever \((x; y) \in F \cdot Q\) and \((x+h; y+k) \in Q\), where \(k = (k_1, \ldots, k_n)\).

Proof. We denote by \(H\) the at most enumerable set consisting of the points of \(G\) at which \(z(x; y)\) is not derivable with respect to \(x\) and with respect to \(y_{\lambda} \lambda = 1, \ldots, n\) simultaneously.

If a point \((\xi^{(0)}, \eta^{(0)})\) of \(G\) has an open neighbourhood \(V\) such that every point of \(V\) belongs to \(K\) except at most \((\xi^{(0)}; \eta^{(0)})\) itself, then by Lemma 1 \(z(x; y)\) is constant on \(C(\xi^{(0)}; \eta^{(0)})\) where \(\eta\) is any point of \(V[\xi^{(0)}]\) except \(\eta^{(0)}\) and so by Lemma 2, \(z(x; y)\) is also constant on \(C(\xi^{(0)}; \eta^{(0)}; V)\), that is, \((\xi^{(0)}; \eta^{(0)}) \in K\). Hence the set \(F\) which is closed in the open set \(G\), has no isolated point.

Therefore \(F\) is a \(G_\delta\) set in \(R^{n+1}\) without isolated point and so every point of \(F\) is a condensation point of \(F\) and so by Baire's theorem

\[\begin{array}{c}
F = F - H
\end{array}\]  

Also the non-empty \(F - H\) is a \(G_\delta\) set in \(R^{n+1}\) since \(F\) is a \(G_\delta\) set in \(R^{n+1}\) and \(H\) is at most enumerable. Hence \(F - H\) is of the second category in itself by Baire's theorem.

On the other hand, if we denote by \(F_m\) for each positive integer \(m\), the set of the points \((x; y)\) of \(G\) such that

\[
\begin{align*}
|z(x+h; y) - z(x; y)| & \leq |h| m \\
|z(x, y, \ldots, y_{\lambda-1}, y_{\lambda}+k_{\lambda}, y_{\lambda+1}, \ldots, y_n) - z(x; y)| & \leq |k_{\lambda}| m \\
\lambda = 1, \ldots, n
\end{align*}
\]  

\[\begin{array}{c}
(5.3)
\end{array}\]  


whenever $|h|, |k_\lambda| \leq 1/m$ and $(x+h ; y) \in G$, $(x, y_1, \ldots, y_{\lambda-1}, y_\lambda + k_\lambda, y_{\lambda+1}, \ldots, y_n) \in G$, $\lambda=1, \ldots, n$, then the union of the sets $F_m$ covers $F-H$ by the definition of $H$ and each of the set $F_m$ is closed in $G$ by the continuity of $z(x ; y)$.

Therefore there must exist a positive integer $N$ and a $(n+1)$-dimensional open cube $Q$: $|x-a| < L$, $|y_\lambda - b_\lambda| < L$, $\lambda=1, \ldots, n$ ($L > 0$) such that $(a ; b) \in F-H \subset F$ and

$$(F-H) \cdot Q \subset F_N. \quad (5.4)$$

Also we can take $L$ sufficiently small so that

$$0 < L < 1/(2N) \quad (5.5)$$

$$Q \subset G \quad (5.6)$$

since $G$ is open in $\mathbb{R}^{n+1}$.

By (5.4), (5.6) and by observing that $Q$ is open in $\mathbb{R}^{n+1}$ and $F_N$ is closed in $G$, we have

$$F-H \cdot Q = (F-H) \cdot Q \subset F_N \cdot Q \subset F_N \cdot G = F_N$$

so that by (5.2).

$$F_N \supset F \cdot Q.$$ 

Hence by (5.5), (5.6) and by the definition of $F_N$, the inequalities (5.3) for $m=N$ hold whenever $(x ; y) \in F \cdot Q$ and $(x+h ; y) \in Q$, $(x, y_1, \ldots, y_{\lambda-1}, y_\lambda + k_\lambda, y_{\lambda+1}, \ldots, y_n) \in Q$, $\lambda=1, \ldots, n$. This completes the proof of Proposition 1.

In the following, $Q$, $L$, $(a ; b)$ and $N$ have the same meanings as in Proposition 1.

6. Domains $Q_1, \Omega_1, G_1$ and Set $\hat{F}$. $f_\lambda$ and $\partial f_\lambda / \partial y_\mu \lambda, \mu=1, \ldots, n$ are defined and continuous on $Q \subset G$. Hence there is a positive number $M_0$ such that

$$|f_\lambda|, |\partial f_\lambda / \partial y_\mu| < M_0 \lambda, \mu = 1, \ldots, n \text{ on } Q. \quad (6.1)$$

Then we can easily prove

$$|f_\lambda(x ; y) - f_\lambda(x ; y)| \leq M_0 \|y-y\| \quad (6.2)$$

for any pair of points $(x ; y)$, $(x ; y) \in Q$ with the same $x$ coordinate.

We take a positive number $L_1$ such that

$$\exp (2n^2 M_0 L_1) < 2 \quad (6.3)$$

$$L_1 (M_0 + 1) \leq L. \quad (6.4)$$

We denote by $\Omega_1$ the $n$-dimensional open cube: $|y_\lambda - b_\lambda| < L_1$, $\lambda=1,
... and by $Q_i$ the $(n+1)$-dimensional open parallelepiped: $|x-a|<L_1$, $|y_\lambda-b_\lambda|<L \lambda=1, \ldots, n$. By (6.4) $L_1< L$ and so

$$Q_i \subset Q.$$ 

By (6.4), $\eta_\lambda+L_1M_\lambda \leq b_\lambda + L$, $\eta_\lambda - L_1M_\lambda \geq b_\lambda - L \lambda=1, \ldots, n$ whenever $\eta \in \Omega$. Hence the characteristic curves $C(a; \eta|Q_1)$ where $\eta \in \Omega$, are defined just for the interval $|x-a|<L_1$ since $|f_\lambda|<M_\lambda \lambda=1, \ldots, n$ on $Q_i (\subset Q)$ by (6.1).

We denote by $G_i$ the portion of $Q_i$ covered by the family of all the characteristic curves $C(a; \eta|Q_i)$ where $\eta \in \Omega$. Then by the properties of $C(\xi; \eta|Q_1)$ and $\varphi_\lambda(x, \xi; \eta|Q_i)$ as stated in §1.2, observing that $\Omega$ is open in $R^n$, we easily prove that $G_i$ is open in $R^{n+1}$ and the characteristic curves $C(\xi; \eta|G_i)$, where $(\xi; \eta) \in G_i$, are defined just for the interval $|x-a|<L_1$.

We denote by $\tilde{F}$ the set of the points $\eta$ of $\Omega$ such that $C(a; \eta|G_i)$ has at least one point in common with $F$. $\tilde{F}$ is not empty since $(a; b) \in F$ and $b=(b_1, \ldots, b_n) \in \Omega$. Now we prove

**Proposition 2.** $\tilde{F}^0$, the interior of $\tilde{F}$ in $R^n$, is not empty.

Proof. Suppose, if possible, that $\tilde{F}^0=0$.

If $\eta \in \Omega - \tilde{F}$, then $C(a; \eta|G_i)$ is contained in $K$ by the definition of $\tilde{F}$ and so by Lemma 1 $\varepsilon(x; y)$ is constant on $C(a; \eta|G_i)$. Hence by
Lemma 2, $z(x;y)$ is constant on any $C(a;\eta|G_1)$ such that $\eta \in \Omega_2 - \overline{F} \cdot \Omega_1$. 
But $\Omega_1 - \overline{F} \cdot \Omega_1 = \Omega_1$, since $\overline{F_0} = 0$ by supposition and $\Omega_1$ is open in $R^n$.

Therefore $z(x;y)$ is constant on any $C(a;\eta|G_1)$ such that $\eta \in \Omega_1$ and so by the definition of $K$, $G_1 \subseteq K$ since $G_1$ is covered by the family of all the characteristic curves $C(a;\eta|G_1)$ where $\eta \in \Omega_1$. Hence, observing that $G_1$ is open in $R^{n+1}$, we have

$$F = G \cdot G - K \subseteq G \cdot G - G_1 \subseteq G - G_1.$$ 

But this is a contradiction since $(a;b) \in F \cdot G_1$. Thus Proposition 2 is proved.

7. Domains $\Omega_2, G_2$. By Proposition 2, $\overline{F_0}$ is not empty. Hence we take a point $b^{(1)} = (b_1^{(1)}, \ldots, b_n^{(1)}) \in \overline{F_0}$.

Then we can construct a domain $\Omega_2$ in $R^n$ defined by

$$\eta: \|\eta - b^{(1)}\| < L_2 \quad (L_2 > 0)$$

such that

$$\tilde{\Omega}_2 \subset \overline{F}. \quad (7.2)$$

Evidently

$$\tilde{\Omega}_2 \subset \Omega_1 \quad (7.3)$$

since $\overline{F} \subseteq \Omega_1$ by the definition of $\overline{F}$.

We denote by $G_2$ the portion of $Q_1$ covered by the family of all the characteristic curves $C(a;\eta|Q_1)$ where $\eta \in \Omega_2$. In the same way as in the case of $G_1$, we easily prove that $G_2$ is open in $R^{n+1}$ and so $G_2[\tilde{x}]$
is open in $R^n$ for any $x$ in the interval $|x-a|<L_1$. Also $C(\xi; \eta|G_2)$ where $(\xi; \eta)\in G_2$, is defined just for the interval $|x-a|<L_1$. Evidently by (7.3) and the definition of $G_1$, $G_2$,

$$G_2 \subset G_1 \subset Q_1 \subset Q.$$ 

**Proposition 3.** $C(\xi; \eta|G_2)$ where $(\xi; \eta)\in G_2$, has at least one point in common with $F \cdot G_2$.

Proof. If $(\xi; \eta)\in G_2$ and $\eta^{(0)} = \varphi(a, \xi; \eta|G_2)$, then $\eta^{(0)} \in \Omega_2$ and $C(\xi; \eta|G_2) = C(a; \eta^{(0)}|G_1)$ by the definition of $G_2$. Then by (7.2) and the definition of $F'$, Proposition 3 follows.

**Proposition 4.** If $|\xi-a|<L_1$, then

$$G_1[\xi] \supset (G_2[\xi])^b,$$ 

the boundary in $R^n$ of $G_2[\xi]$. \hspace{1cm} (7.4)

Further if $|\xi-a|<L_1$ and $\eta \in (G_2[\xi])^b$, then

$$\varphi(a, \xi; \eta|G_1) \in \Omega_2^b,$$ 

the boundary in $R^n$ of $\Omega_2$.

Proof. We consider the continuous mapping $\mathcal{A}_\xi$ of $\Omega_1$ onto $G_1[\xi]$ defined by

$$\eta^{(0)} \mapsto \varphi(\xi, a; \eta^{(0)}|G_1).$$

That $\mathcal{A}_\xi$ maps $\Omega_1$ onto $G_1[\xi]$ follows from the definition of $G_1$.

By the properties of $C(\xi; \eta|G_1)$ and $\varphi(x, \xi; \eta|G_1)$ as stated in §1.2, we easily see that $\mathcal{A}_\xi$ is one to one and bicontinuous and $\mathcal{A}_\xi^{-1}$ is represented by

$$\eta \mapsto \varphi(a, \xi; \eta|G_1) \hspace{1cm} (7.5)$$

We have

$$\mathcal{A}_\xi(\Omega_2) = G_2[\xi] \hspace{1cm} (7.6)$$

by (7.3) and the definition of $G_2$. Hence again taking account of (7.3), by the continuity of $\mathcal{A}_\xi$ we have $\mathcal{A}_\xi(\Omega_2) \subset G_2[\xi]$.

On the other hand, since $\Omega_2$ is closed and bounded in $R^n$, its continuous image $\mathcal{A}_\xi(\Omega_2)$ is closed in $R^n$ and so, taking account of (7.6), we have $\mathcal{A}_\xi(\Omega_2) \supset G_2[\xi]$.

Therefore $\mathcal{A}_\xi(\Omega_2) = G_2[\xi]$. Hence by (7.6), (7.3), and $\mathcal{A}_\xi(\Omega_1) = G_1[\xi]$, observing that $\Omega_2$, $G_1[\xi]$ are both open in $R^n$, we get $\mathcal{A}_\xi(\Omega_2) = (G_2[\xi])^b \subset G_1[\xi]$.

From this, taking account of the representation (7.5) of $\mathcal{A}_\xi^{-1}$, Proposition 4 follows.
8. Classes $S_\lambda$, $S^{(\ell)}$ and Operations $T_\lambda$, $T$, $T^{(\ell)}$. We take two points $(x'; y')$, $(x''; y'')$ of $\mathbb{R}^{n+1}$ with the same $x$ coordinate such that $(x'; y') \in F \cdot G_2$, $(x''; y'') \in G_2$ and further $(x', y', \cdots, y_{\lambda-1}', y_{\lambda}', y_{\lambda+1}', \cdots, y_\ell') \in G_2$. In the following, we denote the class of all such ordered pairs \{(x'; y'), (x''; y'')\} of points of $\mathbb{R}^{n+1}$ by $S_\lambda$ ($\lambda = 1, \cdots, n$). If we put $y' = (y', \cdots, y_{\lambda-1}', y_{\lambda}', y_{\lambda+1}', \cdots, y_\ell')$, then $(x''; y'') \in G_2$.

Now there is the nearest $x$ to $x'$ in the interval $|x - a| < L$, such that either $(x, \varphi(x, x'; y'|G_2)) \in F \cdot G_2$ or $(x, \varphi(x, x'; y'|G_2)) \in F \cdot G_2$, since by Proposition 3 each of the continuous curves $C(x'; y'|G_2)$ and $C(x''; y'|G_2)$ which are just defined for the interval $|x - a| < L$, and are contained in $G_2$, has at least one point in common with $F \cdot G_2$ which is closed in $G_2$. We denote such $x$ by $x''$. If incidently two such $x$ exist, then we take as $x''$ the one on the right side of $x'$.

Now we distinguish two cases;

i) If $(x'', \varphi(x'', x'; y'|G_2)) \in F \cdot G_2$, then we put $y'' = \varphi(x'', x'; y'|G_2)$ and $y'' = \varphi(x'', x'; y'|G_2)$.

ii) If $(x'', \varphi(x'', x'; y'|G_2)) \notin F \cdot G_2$ and so by the definition of $x''$, $(x'', \varphi(x'', x'; y'|G_2)) \in F \cdot G_2$, then we put $y'' = \varphi(x'', x'; y'|G_2)$ and $y'' = \varphi(x'', x'; y'|G_2)$.

In any case, $(x''; y'') \in F \cdot G_2$ and $(x''; y'') \in G_2$.

We denote by $T_\lambda$ ($\lambda = 1, \cdots, n$) the above operation which assigns to every (ordered) pair \{(x'; y'), (x' y')\} of points of $\mathbb{R}^{n+1}$ belonging to the class $S_\lambda$, an (ordered) pair \{(x''; y''), (x'' y'')\} of points of $\mathbb{R}^{n+1}$ with the same $x$ coordinate such that $(x''; y'') \in F \cdot G_2$ and $(x''; y'') \in G_2$. Also we write $T_\lambda, \{(x'; y'), (x' y')\} = \{(x''; y''), (x'' y'')\}$.

If \{(x''; y''), (x'' y'')\} $\in S_\lambda$, we can apply $T_\lambda$ again on \{(x''; y''), (x'' y'')\}.

**Proposition 5.** If \{(x'; y'), (x' y')\} $\in S_\lambda$ and if we put \{(x''; y''), (x'' y'')\}, $z' = z(x'; y')$, $z' = z(x', y')$, $z'' = z(x''; y'')$, and $z'' = z(x''; y'')$, then

$$|z' - z'| \leq |z'' - z'| + N \|y' - y'\| \quad (8.1)$$

Proof. We put $y' = (y', \cdots, y_{\lambda-1}', y_{\lambda}', y_{\lambda+1}', \cdots, y_\ell')$. Then $(x'; y') \in F \cdot G_2 \subset F \cdot Q$ and $(x'; y') \in G_2 \subset Q$ since \{(x'; y'), (x' y')\} $\in S_\lambda$. Hence by Proposition 1 (5.1), if we put $z' = z(x'; y')$

$$|z' - z'| \leq N |y' - y'| \leq N \|y' - y'\|. \quad (8.2)$$

By the definition of $T_\lambda$, 

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\[
\begin{align*}
\{ y'' &= \varphi(x', x'; y'|G_z) \quad \text{or} \quad y'' = \varphi(x', x'; y'|G_z) \\
y'' &= \varphi(x', x'; y'|G_z) \quad \text{or} \quad y'' = \varphi(x', x'; y'|G_z)
\end{align*}
\] (8.3)

On the other hand, each of \( C(x'; y'|G_z) \) and \( C(x'; y'|G_z) \) has no point in common with \( F \) for the interval \( x' < x < x'' \) or \( x'' < x < x' \) by the definition of \( T_\lambda \) so that they are contained in \( K \) for the interval \( x'' < x < x' \) or \( x' < x < x'' \). Therefore by Lemma 1 and (8.3)

\[
\begin{align*}
\{ z'' &= z(x'' ; y'') = z(x' ; y') = \bar{z}' \\
\bar{z}'' &= z(x'' ; y'') = z(x' ; y') = \bar{z}'
\end{align*}
\]

or

\[
\begin{align*}
\{ z'' &= z(x'' ; y'') = z(x' ; y') = \bar{z}' \\
\bar{z}'' &= z(x'' ; y'') = z(x' ; y') = \bar{z}'
\end{align*}
\]

so that

\[
|\bar{z}'' - z''| = |\bar{z}' - z'|. \tag{8.4}
\]

By (8.4), (8.2) we have

\[
|\bar{z}' - z'| \leq |\bar{z}' - \bar{z}'| + |\bar{z}' - z'| \leq |\bar{z}'' - z''| + N \| \bar{y}' - y' \|,
\]

q.e.d.

We denote by \( T_\mu \cdot T_\lambda \) the operation which assigns to a pair \( \{(x'; y') \}, \{(x'; y') \} \) of points of \( R^{n+1} \), the pair \( T_\mu \{ (x'; y'), (x'; y') \} \) of points of \( R^{n+1} \) if

\[
\{(x'; y'), (x'; y') \} \in S_\lambda \quad \text{and} \quad T_\lambda \{ (x'; y'), (x'; y') \} \in S_\mu;
\]

and similarly for products of any number of operations \( T_\lambda(\lambda = 1, \ldots , n) \).

We put \( T = \overline{T_n \cdots T_2 T_1} \) and \( T^m = \overline{\cdots T_2 T_1} \) for any non-negative integer \( m \) and \( T^{(l)} = \overline{T_{\nu} T_{\nu-1} \cdots T_2 T_1} \) for any non-negative integer \( l \) if \( l = mn + \nu, \ 0 \leq \nu \leq n - 1 \) and \( m, \nu = \) non-negative integer (if \( \nu = 0, \ T^{(l)} = T^m \)).

We denote by \( S^{(l)} (l > 0) \) the class of all the pairs \( \{(x'; y'), (x'; y') \} \) of points of \( R^{n+1} \) on which we can apply the operation \( T^{(l)} (l > 0) \) and by \( S^{(0)} \) the class of all the pairs \( \{(x'; y'), (x'; y') \} \) such that \( (x'; y') \in F \cdot G_z \) and \( (x'; y') \in G_z \). We regard \( S^{(0)} \) as the domain of definition of the identity operator \( T^{(0)} = T^0 \).

In the following, we put

\[
\begin{align*}
M_i &= \exp (2n^2 M_z L_i) - 1 \tag{8.5} \\
M_z &= \exp (2n M_z L_i), \tag{8.6}
\end{align*}
\]
then by (6.3)

\[ 1 > M_\nu > 0. \]  

(8.7)

Also

\[ M_\nu > 1. \]  

(8.8)

**Proposition 6.** If \( \{(x^{(0)}; y^{(0)}), (x^{(0)}; \bar{y}^{(0)})\} \in S^{(l)} \) where \( l = mn + \nu \), \( n - 1 \geq \nu \geq 0 \) and \( m, \nu \) = non-negative integer, and if we put \( T^{(l)} \{ (x^{(0)}; y^{(0)}), (x^{(0)}; \bar{y}^{(0)}) \} = \{(x^{(l)}; y^{(l)}), (x^{(l)}; \bar{y}^{(l)})\} \), then

\[ \| \bar{y}^{(l)} - y^{(l)} \| \leq (M_\nu + 1) M_1^m \| y^{(0)} - \bar{y}^{(0)} \| \]  

(8.9)

Proof. In the following lines, we shall prove by induction on \( l \) more precise results,

\[ \| \bar{y}^{(l)} - y^{(l)} \| \leq M_\nu^m M_1^m \| y^{(0)} - \bar{y}^{(0)} \| \]  

(8.10)

and

\[ |\bar{y}_\lambda^{(l)} - y_\lambda| \leq \frac{1}{n} (M_\nu^2 - M_\nu^{-1}) M_1^m \| y^{(0)} - \bar{y}^{(0)} \| \]  

(8.11)

whenever

\[ \{(x^{(0)}; y^{(0)}), (x^{(0)}; \bar{y}^{(0)})\} \in S^{(l)} \quad (l = mn + \nu). \]

(8.9) follows from (8.10) since \( M_\nu^2 < M_\nu + 1 \) by \( n - 1 \geq \nu \geq 0 \) and (8.5), (8.6).

For \( l = 0 \), \( T^{(0)} = T^{(0)} = \text{identity operator and } m = \nu = 0 \). Hence here (8.10) and (8.11) are trivial.

Now we assume that (8.10), (8.11) are true for \( l = l' = mn + \nu' \) and let \( \{(x^{(0)}; y^{(0)}), (x^{(0)}; \bar{y}^{(0)})\} \in S^{(l'+1)} \subset S^{(l')} \). Then \( \{(x^{(l')}; y^{(l')}), (x^{(l')}; \bar{y}^{(l')})\} \in S^{(l'+1)} \) since \( T^{(l'+1)} = T_{l'+1} \cdot T^{(l')} \). Also \( \{(x^{(l'+1)}; y^{(l'+1)}), (x^{(l'+1)}; \bar{y}^{(l'+1)})\} = T^{(l'+1)} \{(x^{(0)}; y^{(0)}), (x^{(0)}; \bar{y}^{(0)})\} \).

Then by the definition of \( T_{l'+1} \), (6.2), Lemma 3 (4.2) and (8.6), taking account of \( |x^{(l'+1)} - x^{(l')}| < 2L_1 \), we have

\[ \| \bar{y}^{(l'+1)} - y^{(l'+1)} \| \leq (\sum_{\mu=1}^{\nu} |\bar{y}_\mu^{(l') - y_\mu^{(l')}}| + \sum_{\mu=\nu+2}^{n} |y_\mu^{(l') - y_\mu^{(l')}}|) \times \exp (nM_0 |x^{(l'+1)} - x^{(l')}|) \leq \| \bar{y}^{(l')} - y^{(l')} \| \exp (2nM_0 L_1) = M_\nu \| y^{(0)} - \bar{y}^{(0)} \| \]  

and so by (8.10) for \( l = l' \),

\[ \| \bar{y}^{(l'+1)} - y^{(l'+1)} \| \leq M_\nu^{l'+1} M_1^m \| y^{(0)} - \bar{y}^{(0)} \|. \]  

(8.12)

Also by the definition of \( T_{l'+1} \), (6.2), Lemma 3 (4.3) and (8.6), we have
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\[ |y^{(t')}_{\lambda} - y^{(t')}_{\lambda'}| \leq |\tilde{y}^{(t')}_{\lambda} - y^{(t')}_{\lambda'}| + \frac{1}{n} \left( \sum_{\mu=1}^{n} |\tilde{y}^{(t')}_{\mu} - y^{(t')}_{\mu'}| \right) \\
+ \sum_{\mu=\nu+2}^{n} |y^{(t')}_{\mu} - y^{(t')}_{\mu'}| \{\exp(nM_{\mu} |x^{(t')} - x^{(t')}|) - 1\} \leq |\tilde{y}^{(t')}_{\lambda} - y^{(t')}_{\lambda'}| \\
+ \frac{1}{n} (M_{\nu} - 1) \times \|\tilde{y}^{(t')} - y^{(t')}\| \quad \text{for} \quad n \geq \lambda, \lambda = \nu + 1, \]

and so by (8.10) and (8.11) for \( l = l' \),

\[ |\tilde{y}^{(t'+1)}_{\lambda} - y^{(t'+1)}_{\lambda'}| \leq \frac{1}{n} (M_{\nu}^{2} - M_{\nu}^{1}) M_{\nu}^{\nu'} \|\tilde{y}^{(t')} - y^{(t')}\| \\
+ \frac{1}{n} (M_{\nu} - 1) \times M_{\nu}^{\nu'} \|\tilde{y}^{(t')} - y^{(t')}\| = \frac{1}{n} (M_{\nu}^{2} + 1 - M_{\nu}^{1}) M_{\nu}^{\nu'} \|\tilde{y}^{(t')} - y^{(t')}\| \\
\text{for} \quad \nu \geq \lambda \geq 1. \quad (8.13) \]

Again by the definition of \( T_{\nu+1}, (6.2) \) and Lemma 3 (4.3), we have

\[ |\tilde{y}^{(t+1)}_{\nu+1} - y^{(t+1)}_{\nu+1}| \leq \frac{1}{n} (\sum_{\mu=1}^{n} |\tilde{y}^{(t')}_{\mu} - y^{(t')}_{\mu'}|) \\
+ \sum_{\mu=\nu+2}^{n} |y^{(t')}_{\mu} - y^{(t')}_{\mu'}| \{\exp(nM_{\mu} |x^{(t')} - x^{(t')}|) - 1\} \leq \frac{1}{n} (M_{\nu} - 1) \|\tilde{y}^{(t')} - y^{(t')}\|, \]

and so by (8.10) for \( l = l' \),

\[ |\tilde{y}^{(t+1)}_{\nu+1} - y^{(t+1)}_{\nu+1}| \leq \frac{1}{n} (M_{\nu}^{2} + 1 - M_{\nu}^{1}) M_{\nu}^{\nu'} \|\tilde{y}^{(t')} - y^{(t')}\|. \quad (8.14) \]

If \( n - 2 \geq \nu \geq 0 \), then (8.12), (8.13), (8.14) prove (8.10), (8.11) for \( l = l'+1 \) since \( l' + 1 = n\nu' + \nu' + 1, n - 1 \geq \nu' + 1 \geq 1 \) in this case.

If \( \nu = n - 1 \), then \( l' + 1 = n(n' + 1) \). In this case, by (8.13), (8.14), we obtain

\[ \|\tilde{y}^{(t+1)} - y^{(t+1)}\| = \sum_{\mu=1}^{n-1} |\tilde{y}^{(t+1)}_{\mu} - y^{(t+1)}_{\mu'}| + |\tilde{y}^{(t+1)} - y^{(t+1)}_{\nu+1}| \leq \frac{1}{n} \{ \sum_{\mu=1}^{n-1} (M_{\mu}^{2} - M_{\mu}^{1}) \} M_{\nu}^{\nu'} \|\tilde{y}^{(t')} - y^{(t')}\| \leq (M_{\nu} - 1) M_{\nu}^{\nu'} \|\tilde{y}^{(t')} - y^{(t')}\| \]

\[ = M_{\nu}^{\nu'+1} \|\tilde{y}^{(t')} - y^{(t')}\|, \]

since \( M_{\nu}^{\nu'+1} \geq 1 \) for \( n \geq \nu \geq 1 \) and \( M_{\nu} = M_{\nu}^{1} \) by (8.8), (8.5), (8.6). This proves (8.10) for \( l = l'+1 \) in the case \( \nu = n - 1 \). In this case (8.11) for \( l = l'+1 \) is trivial, since then there is no \( \lambda \) for which it should be established.

Thus (8.10), (8.11) are proved for any non-negative integer \( l \) and so the proof of Proposition 6 is completed.

9. Further on the operation \( T^{(\nu)} \).
In the following we put
\[ M_3 = nM_i(1 + M_i)(1 - M_i)^{-1}. \]  

(9.1)

By (8.7), (8.8), \( M_3 \) is positive.

**Proposition 7.** If \( \{(x^{(0)}; \tilde{x}^{(0)}), (x^{(0)}; \tilde{y}^{(0)})\} \in S^{(1)} \) for a non-negative integer \( l \) and if we put

\[
\{(x^{(p)}; \tilde{x}^{(p)}), (x^{(p)}; \tilde{y}^{(p)})\} = T^{(p)}\{(x^{(0)}; \tilde{x}^{(0)}), (x^{(0)}; \tilde{y}^{(0)})\},
\]

\[
\eta^{(p)} = \varphi(a, x^{(p)}; \tilde{y}^{(p)}|G_3), \quad \tilde{\eta}^{(p)} = \varphi(a, x^{(p)}; \tilde{y}^{(p)}|G_3)
\]

for \( p = 0, 1, \ldots, l \),

then

\[
\| \tilde{\eta}^{(l)} - \eta^{(0)} \| \leq M_3 \| \tilde{\eta}^{(0)} - \eta^{(0)} \|.
\]

(9.2)

**Proof.** By the definition of \( T^{(p)} \),

\[
y^{(p)} = \varphi(x^{(p)}, x^{(p-1)}; \tilde{y}^{(p-1)}|G_3) \quad \text{or} \quad \tilde{y}^{(p)} = \varphi(x^{(p)}, x^{(p-1)}; \tilde{y}^{(p-1)}|G_3)
\]

for \( p = 1, \ldots, l \). Hence

\[
\eta^{(p)} = \tilde{\eta}^{(p-1)} \quad \text{or} \quad \tilde{\eta}^{(p)} = \tilde{\eta}^{(p-1)} \quad p = 1, \ldots, l,
\]

so that

\[
\| \tilde{\eta}^{(p)} - \eta^{(p-1)} \| = \| \tilde{\eta}^{(p)} - \eta^{(p-1)} \| \quad \text{or} \quad \| \tilde{\eta}^{(p)} - \tilde{\eta}^{(p-1)} \| = 0
\]

for \( p = 1, 1, \ldots, l \).

Therefore

\[
\| \tilde{\eta}^{(l)} - \eta^{(0)} \| \leq \sum_{p=1}^{l} \| \tilde{\eta}^{(p)} - \eta^{(p-1)} \| + \| \tilde{\eta}^{(0)} - \eta^{(0)} \| \leq \sum_{p=0}^{l} \| \tilde{\eta}^{(p)} - \eta^{(p)} \|.
\]

(9.3)

By (6.2) and Lemma 3 (4.2), we obtain

\[
\| \tilde{\eta}^{(p)} - \eta^{(p)} \| \leq \| \tilde{y}^{(p)} - y^{(p)} \| \exp(nM_3|a - x^{(0)}|) \leq \| \tilde{y}^{(p)} - y^{(p)} \| \exp(nM_3L_3) = \sqrt{M_3} \| \tilde{y}^{(p)} - y^{(p)} \|
\]

for \( p = 0, \ldots, l \),

(9.4)

since \( |a - x^{(0)}| < L_4 \), and \( \sqrt{M_3} = \exp(nM_3L_3) \) by (8.6). Similarly by (6.2) and Lemma 3 (4.2), we have

\[
\| \tilde{y}^{(0)} - y^{(0)} \| \leq \| \tilde{\eta}^{(0)} - \eta^{(0)} \| \exp(nM_3|x^{(0)} - a|) \leq \sqrt{M_3} \| \tilde{\eta}^{(0)} - \eta^{(0)} \|.
\]

(9.5)

On the other hand, by Proposition 6, if \( p = qn + s, n - 1 \geq s \geq 0 \) \( q, s = \text{non-negative integer} \), then

\[
\| y^{(p)} - \tilde{y}^{(p)} \| \leq (M_i + 1)M_i \| \tilde{y}^{(p)} - y^{(p)} \| \quad \text{for} \quad p = 0, \ldots, l.
\]

(9.6)

By (9.4), (9.5), (9.6), we obtain

\[
\| \tilde{\eta}^{(p)} - \eta^{(p)} \| \leq M_3(M_i + 1)M_i \| \tilde{\eta}^{(0)} - \eta^{(0)} \| \quad \text{for} \quad p = 0, \ldots, l.
\]

(9.7)
By (9.3), (9.7), taking account of (8.7) and (9.1), if \( l = nm + \nu \), \( n - 1 \geq \nu \geq 0 \) and \( m, \nu \) = non-negative integer, then we get

\[
\| \eta^{(l)} - \eta^{(l)} \| \leq \sum_{p=0}^{nm-1} \| \eta^{(p)} - \eta^{(p)} \| + \sum_{p=nm}^{nm+\nu} \| \eta^{(p)} - \eta^{(p)} \|
\]

\[
\leq nM_4(M_i + 1)\left( \sum_{q=0}^{m-1} M_1 \right) \| \eta^{(0)} - \eta^{(0)} \| \leq (\nu + 1)M_4(M_i + 1)M_1 \| \eta^{(0)} - \eta^{(0)} \|
\]

\[
\leq nM_4(M_i + 1)\left( \sum_{q=0}^{m} M_1 \right) \| \eta^{(0)} - \eta^{(0)} \| = nM_4(1 + M_i)(1 - M_i)^{-1} \| \eta^{(0)} - \eta^{(0)} \|
\]

\[
= M_4 \| \eta^{(0)} - \eta^{(0)} \| , \quad \text{q.e.d.}
\]

In the following we put

\[
M_i = nN(1 + M_i)(1 - M_i)^{-1}
\]

(9.8)

By (8.7), \( M_i \) is positive.

**Proposition 8.** Let the premises and the notations be the same as in the proposition 7 and further let

\[
z^{(p)} = z(x^{(p)}; y^{(p)}) \quad \text{and} \quad \tilde{z}^{(p)} = z(x^{(p)}; \tilde{y}^{(p)}) \quad \text{for} \quad p = 0, \ldots, l,
\]

then

\[
|z^{(0)} - \tilde{z}^{(0)}| \leq M_i \|y^{(0)} - \eta^{(0)}\| + |z^{(l)} - \tilde{z}^{(l)}|.
\]

(9.9)

Proof. We use the same notations as in the proof of Proposition 7.

If \( l = 0 \), (9.9) is obvious by \( M_i > 0 \). Hence we suppose \( l > 0 \) in the following lines.

By Proposition 5,

\[
|\tilde{z}^{(p)} - z^{(p)}| - |\tilde{z}^{(p+1)} - z^{(p+1)}| \leq N \|\tilde{y}^{(p)} - y^{(p)}\| \quad \text{for} \quad p = 0, \ldots, l-1,
\]

(9.10)

Adding the inequalities (9.10) for \( p = 0, \ldots, l-1 \) side by side, we obtain

\[
|z^{(0)} - \tilde{z}^{(0)}| - |z^{(l)} - \tilde{z}^{(l)}| \leq N \sum_{p=0}^{l-1} \|\tilde{y}^{(p)} - y^{(p)}\|.
\]

(9.11)

On the other hand, by Proposition 6,

\[
\|\tilde{y}^{(p)} - y^{(p)}\| \leq (M_i + 1)M_4 \|\tilde{y}^{(0)} - \eta^{(0)}\| \quad \text{for} \quad p = 0, \ldots, l,
\]

where \( q \) is determined for \( p \) as in Proposition 7 (9.6). Hence \( m, \nu \) being determined for \( l \) as in the definition of \( T^{(l)} \),

\[
\sum_{p=0}^{l-1} \|\tilde{y}^{(p)} - y^{(p)}\| \leq n(M_i + 1)(\sum_{q=0}^{m-1} M_i) \|\tilde{y}^{(0)} - \eta^{(0)}\|
\]

\[
+ \nu(M_i + 1)M_1 \|\tilde{y}^{(0)} - \eta^{(0)}\| \leq n(M_i + 1)(\sum_{q=0}^{m} M_i) \|\tilde{y}^{(0)} - \eta^{(0)}\|
\]

\[
= n(1 + M_i)(1 - M_i)^{-1} \|\tilde{y}^{(0)} - \eta^{(0)}\|,
\]

(9.12)
since \( n-1 \geq \nu \geq 0 \) and \( 0 < M_i < 1 \) by (8.7).

By (9.11), (9.12), taking account of (9.8), we obtain finally
\[
|\tilde{z}^{(0)} - \tilde{z}^{(D)}| - |\tilde{z}^{(D)}| \leq nN(1 + M_i)(1 - M_i)^{-1} ||y^{(0)} - y^{(D)}||
\]
\[
\leq M_i ||y^{(0)} - y^{(D)}||
\]
q.e.d.

10. Domain \( \Omega_3, G_5 \). We put
\[
M_5 = 1 + 2(M_i + 1)M_2 + 2M_3.
\]  (10.1)
By (8.7), (8.8), (9.1), \( M_5 > 1 \). Hence if we take a positive number \( L_3 \) such that
\[
L_3M_5 < L_2,
\]  (10.2)
then
\[
L_3 < L_2.
\]  (10.3)

Now we take a domain \( \Omega_3 \) in \( \mathbb{R}^n \) defined by
\[
\eta : ||\eta - b^{(1)}|| < L_3. \quad (10.4)
\]
Then by (7.1), (10.3), (10.4), (7.3),
\[
\Omega_3 \subset \Omega_2 \subset \Omega_1 \quad (10.5)
\]
We denote by \( G_3 \) the portion of \( Q \) covered by the family of all the characteristic curves \( C(a; \eta | Q) \) where \( \eta \in \Omega_3 \). In the same way as in the cases of \( G_1, G_2 \), we easily prove that \( G_3 \) is open in \( \mathbb{R}^{n+1} \), and \( C(\xi; \eta | G_3) \) where \( (\xi; \eta) \in G_3 \), is defined just for the interval \( |x-a| < L_3 \). By (10.5) and the definitions of \( G_1, G_2, G_3 \),
\[
G_3 \subset G_2 \subset G_1 \subset Q_1.
\]

**Proposition 9.** If \((x^{(0)}; y^{(0)}) \in F \cdot G_3 \) and \((x^{(0)}; y^{(0)}) \in G_3 \), then \\{(x^{(0)}; y^{(0)}), (x^{(0)}; y^{(0)})\} \in S^{(l)} for any non-negative integer \( l \).

Proof. We use the same notations as in the Proposition 7 and 8.
We prove Proposition 9 by induction on \( l \). For \( l=0 \), Proposition 9 is obvious by the definition of \( S^{(0)} \) and \( G_3 \subset G_2 \). We assume that
Proposition 9 is true for \( l = l' \). Then if \((x^{(0)}; y^{(0)}) \in F \cdot G_3\) and \((x^{(0)}; y^{(0)}) \in G_3\), the pair \(\{(x^{(0)}; y^{(0)}), (x^{(0)}; y^{(0)})\} = T^{(l') - l} \{(x^{(0)}; y^{(0)}), (x^{(0)}; y^{(0)})\}\) is uniquely determined and \((x^{(l')}; y^{(l')}) \in F \cdot G_2\) and \((x^{(l')}; y^{(l')}) \in G_2\) by the definition of \(T^{(l') - l}\).

Let \( l' = m'n + n', \ n - 1 \geq n' \geq 0 \) and \( m', n' = \text{integer}. \) Then we put \(\tilde{y}^{(l')} = (y_1^{(l')}, \ldots, y_n^{(l')}, y_{n+1}^{(l')}, \ldots, y_{m'n}^{(l')})\). If \(\tilde{y}^{(l')} \in G_2[x^{(l')}], \) that is, \((x^{(l')}; y^{(l')}) \in G_2, \) then \(\{(x^{(l')}; y^{(l')}), (x^{(l')}; y^{(l')})\} \in S_{l'-1} \) so that \(\{(x^{(0)}; y^{(0)}), (x^{(0)}; y^{(0)})\} \in S^{(l') + 1}\) and the proof of Proposition 9 is completed.

We suppose therefore, if possible, that \(y^{(l')} \notin G_2[x^{(l')}]. \) Then there is a point \(y^{*} \in (G_2[x^{(l')}])^0\) on the segment of straight line which joins \(y^{(l')}\) and \(y^{(l')}\) since \(y^{(l')} \in G_2[x^{(l')}\}] \) by \((x^{(l')}; y^{(l')}) \in G_2.\)

We can easily see that

\[ \|y^* - \tilde{y}^{(l')}\| \leq \|\tilde{y}^{(l')} - y^{(l')}\| \leq \|y^{(l')} - \tilde{y}^{(l')}\|, \]

so that by Proposition 6 (8.9)

\[ \|y^* - \tilde{y}^{(l')}\| \leq (M_1 + 1)M_1^{m'} \|\tilde{y}^{(0)} - y^{(0)}\|. \] (10.6)

By Proposition 4, since \(y^* \in (G_2[x^{(l')}])^0, \) \(y^* \in G_1[x^{(l')}], \) that is, \((x^{(l')}; y^*) \in G_1\) and if we put \(\eta^* = \varphi(a, x^{(l')}; y^*|G_1), \eta^* \in \Omega_1.\) Hence by (7.1)

\[ \|\eta^* - b^{(1)}\| = L_2. \] (10.7)

By (6.2) and Lemma 3 (4.2), taking account of (8.6) and \(|a - x^{(l')}| \leq L_1,\) we have

\[ \|\eta^* - \tilde{\eta}^{(l')}\| \leq \|y^* - \tilde{y}^{(l')}\| \exp (nM_0 |a - x^{(l')}|) \leq \|y^* - \tilde{y}^{(l')}\| \exp (nM_0 L_1) = \sqrt{M_2} \|y^* - \tilde{y}^{(l')}\| \]

and so by (10.6)

\[ \|\eta^* - \tilde{\eta}^{(l')}\| \leq \sqrt{M_2} (M_1 + 1)M_1^{m'} \|\tilde{y}^{(0)} - y^{(0)}\|. \] (10.8)

On the other hand, by the definitions of \(\tilde{\eta}^{(0)}, \eta^{(0)}, G_3\) and by \((x^{(0)}; y^{(0)}) \in F \cdot G_2, \) \((x^{(0)}; y^{(0)}) \in G_3,\) we have

\(\tilde{\eta}^{(0)}, \eta^{(0)} \in \Omega_1,\)

so that by (10.4),

\[ \|y^{(0)} - b^{(1)}\| \leq L_3, \quad \|\tilde{\eta}^{(0)} - y^{(0)}\| \leq \sqrt{M_2} \|\tilde{\eta}^{(0)} - \eta^{(0)}\|. \] (10.9)

Also, as we have seen in Proposition 7 (9.5),

\[ \|\tilde{y}^{(0)} - y^{(0)}\| \leq \sqrt{M_2} \|\tilde{y}^{(0)} - \eta^{(0)}\|. \]
Hence, by (10.9)
\[ \| \tilde{y}^{(0)} - y^{(0)} \| \leq \sqrt{M_s} (\| \tilde{b}^{(0)} - b^{(0)} \| + \| \eta^{(0)} - b^{(1)} \| ) < 2L_3 \sqrt{M_s}. \]

From this and (10.8) we get
\[ \| \eta^{(0)} - \tilde{\eta}^{(0)} \| < 2L_3 M_s (M_t + 1) M_t^{\infty} \]
(10.10)

Also, by Proposition 7, (9.2) and (10.9), we have
\[ \| \tilde{\eta}^{(1)} - \eta^{(0)} \| \leq M_s \| \tilde{\eta}^{(0)} - \eta^{(0)} \| < 2L_3 M_s \]
(10.11)

By (10.7), (10.9), (10.10), and (10.11), taking account of (8.7), (10.1), we obtain finally
\[ L_s = \| \eta^{(0)} - b^{(0)} \| \leq \| \eta^{(0)} - \eta^{(0)} \| + \| \eta^{(1)} - \eta^{(0)} \| + \| \eta^{(0)} - b^{(0)} \| \]
\[ < L_3 (2M_s (M_t + 1) M_t^{\infty} + 2M_s + 1) \leq L_3 (2M_s (M_t + 1) + 2M_s + 1) = L_3 M_s. \]

But this contradicts (10.2) and Proposition 9 is completely proved.

**Proposition 10.** Let \( \{ (x', y'), (x'; y') \} \) be any pair of points of \( G_3 \) with the same x coordinate and let

\[ z' = z(x'; y'), \quad \tilde{z}' = z(x'; \tilde{y}'). \]

Then
\[ |\tilde{z}' - z'| \leq M_s M_t \| y' - y' \| \]
(10.12)

**Proof.** There is the nearest \( x \) to \( x' \) in the interval \( |x - a| < L_1 \) such that either \( (x, \varphi(x, x'; y' | G_3)) \in F \cdot G_3 \) or \( (x, \varphi(x, x'; y' | G_3)) \in F \cdot G_3 \), since by Proposition 3 and \( G_3 \subseteq G_2 \), each of the continuous curves \( C(x'; y' | G_3) \) and \( C(x'; \tilde{y}' | G_3) \) which are defined just for the interval \( |x - a| < L_1 \) and are contained in \( G_3 \), has at least one point in common with \( F \cdot G_3 \) which is closed in \( G_3 \). We denote such \( x \) by \( x^{(0)} \). If incidently two such \( x \) exist, we take as \( x^{(0)} \) for example the one on the right side of \( x' \).

Now we distinguish two cases.

i) If \( (x^{(0)}, \varphi(x^{(0)}, x'; y' | G_3)) \in F \cdot G_3 \), then we put

\[ y^{(0)} = \varphi(x^{(0)}, x'; y' | G_3), \quad \tilde{y}^{(0)} = \varphi(x^{(0)}, x'; \tilde{y}' | G_3) \]

ii) If \( (x^{(0)}, \varphi(x^{(0)}, x'; y' | G_3)) \notin F \cdot G_3 \) and so by the definition of \( x^{(0)} \), \( (x^{(0)}, \varphi(x^{(0)}, x'; \tilde{y}' | G_3)) \in F \cdot G_3 \), then we put

\[ y^{(0)} = \varphi(x^{(0)}, x'; \tilde{y}' | G_3), \quad \tilde{y}^{(0)} = \varphi(x^{(0)}, x'; y' | G_3). \]

In any case \( (x^{(0)}; y^{(0)}) \in F \cdot G_3 \) and \( (x^{(0)}; \tilde{y}^{(0)}) \in G_3 \).

By the definition of \( x^{(0)} \), each of the characteristic curves \( C(x'; y' | G_3) \)
and \( C(x'; y' \mid G_s) \) has no point in common with \( F \) and so is contained in \( K \) for the interval \( x' \lessdot x \lessdot x^{(0)} \) or \( x^{(0)} \lessdot x \lessdot x' \). Hence by Lemma 1, if we put \( \tilde{z}^{(0)} = z(x^{(0)}; y^{(0)}) \) and \( \check{z}^{(0)} = z(x^{(0)}; \check{y}^{(0)}) \),

\[
\begin{align*}
\begin{cases}
\tilde{z}^{(0)} &= z' \\
\check{z}^{(0)} &= \check{z}'
\end{cases} \quad \text{or} \quad \begin{cases}
\tilde{z}^{(0)} &= \check{z}' \\
\check{z}^{(0)} &= z'
\end{cases}
\end{align*}
\]

Therefore in any case,

\[ |z' - z| = |\tilde{z}^{(0)} - z^{(0)}|. \tag{10.13} \]

By Lemma 3 (4.2) and (6.2), taking account of (8.6), we have

\[
\| \hat{y}^{(0)} - y^{(0)} \| \leq \| \hat{y}' - y' \| \exp \left( nM_s |x^{(0)} - x'| \right) \]

\[
\leq \| \hat{y}' - y' \| \exp \left( 2nM_sL_s \right) \leq M_s \| \hat{y}' - y' \| , \tag{10.14}
\]

since \( |x^{(0)} - x'| < 2L_s \).

Now, by Proposition 9, \( \{(x^{(0)}; y^{(0)}), (x^{(0)}; \hat{y}^{(0)})\} \in S^{(l)} \) for any non-negative integer \( l \), since \( (x^{(0)}; y^{(0)}) \in F \cdot G_s \) and \( (x^{(0)}; \hat{y}^{(0)}) \in G_s \). Hence we put

\[
\{(x^{(l)}; y^{(l)}), (x^{(l)}; \hat{y}^{(l)})\} = T^{(l)}\{(x^{(0)}; y^{(0)}), (x^{(0)}; \hat{y}^{(0)})\} ,
\]

\[
\tilde{z}^{(l)} = z(x^{(l)}; y^{(l)}), \quad \check{z}^{(l)} = z(x^{(l)}; \hat{y}^{(l)})
\]

for any non-negative integer \( l \). Then by Proposition 8

\[
|\tilde{z}^{(0)} - z^{(0)}| \leq M_s \| \hat{y}^{(0)} - y^{(0)} \| + |\tilde{z}^{(l)} - z^{(l)}| \tag{10.15}
\]

On the other hand, by Proposition 6, if \( l = nm + \nu, \quad n - 1 \geq \nu \geq 0 \) and \( m, \nu = \text{integer} \), then

\[
\| \hat{y}^{(l)} - y^{(l)} \| \leq (M_s + 1)M_1^n \| \hat{y}^{(0)} - y^{(0)} \| .
\]

Hence observing that \( m \to \infty \) as \( l \to \infty \) and \( 0 < M_s < 1 \),

\[
\| \hat{y}^{(l)} - y^{(l)} \| \to 0 \quad \text{as} \quad l \to \infty ,
\]

Thus

\[
|\tilde{z}^{(l)} - z^{(l)}| = |z(x^{(l)}; y^{(l)}) - z(x^{(l)}; \hat{y}^{(l)})| \to 0 \quad \text{as} \quad l \to \infty ,
\]

since by the continuity of \( z(x; y) \) on \( \bar{Q}(\subset G) \), \( z(x; y) \) is uniformly continuous on \( \bar{Q} \) which is closed and bounded in \( R^{n+1} \) and by the definition of \( T^{(l)}, \quad (x^{(l)}; y^{(l)}), \quad (x^{(l)}; \hat{y}^{(l)}) \in G_s \subset \bar{Q} \) for any non-negative integer \( l \).

Therefore letting \( l \to \infty \) on the right side of (10.15), we have

\[
|\tilde{z}^{(0)} - z^{(0)}| \leq M_s \| \hat{y}^{(0)} - y^{(0)} \| . \tag{10.16}
\]
By (10.13), (10.14) and (10.16), we obtain finally
\[ |\bar{z}' - z'| \leq M_4 M_1 \| \bar{y}' - y' \| , \]
q.e.d.

11. Domains $Q'$, $Q''$, $\Omega_4$, $G_4$ and Mapping $\Psi$. Since $G_3$ is open in $R^{n+1}$, and $F \cdot G_3 \equiv 0$ by the way of the construction of $G_3$ and Proposition 3, we can take a $(n+1)$-dimensional open parallelepiped $Q'$:
\[ |x-a'| < L_4, \quad |y_\lambda - b'_\lambda| < L_4(M_0 + 1) \quad \lambda = 1, \ldots, n \quad (L_4 > 0) \]
such that
\[ (a'; b') = (a'_1, b'_1, \ldots, b'_n) \in F \cdot G_3 \quad \text{and} \quad Q' \subset G_3. \]
Evidently $Q' \subset G_3 \subset Q$.

We denote by $\Omega_4$ the $n$-dimensional open cube
\[ \eta : |\eta - b'_\lambda| < L_4 \quad \lambda = 1, \ldots, n. \]
Then if $\eta \in \Omega_4$,
\[ \eta + L_4 M_0 \leq b'_\lambda + (M_0 + 1)L_4, \quad \eta - L_4 M_0 \geq b'_\lambda - (M_0 + 1)L_4 \quad \lambda = 1, \ldots, n. \]
Hence the characteristic curves $C(a'; \eta |Q')$ where $\eta \in \Omega_4$, are defined just for the interval $|x-a'| < L_4$ since $|f_\lambda| < M_0 \quad \lambda = 1, \ldots, n$ on $Q'(< Q)$ by (6.1).
We denote by $G_4$ the portion of $Q'$ covered by the family of all the characteristic curves $C(a'; \eta | Q')$ where $\eta \in \Omega_4$. Evidently $G_4 \subset Q' \subset G \subset Q$.

In the same way as in the cases of $G_1, G_2$ and $G_3$, we easily prove that $G_4$ is open in $R^{n+1}$ and any characteristic curve $C(\xi; \eta | G_4)$ where $(\xi; \eta) \in G_4$ is defined just for $|x-a'| < L_4$.

We put

$$M_5 = N + nM_0M_1 + M_2M_4. \quad (11.1)$$

**Proposition 11.** Let $(x^{(1)}; y^{(1)})$ and $(x^{(0)}; y^{(0)})$ be any pair of points of $G_4$ and let

$$z^{(1)} = z(x^{(1)}; y^{(1)}), \ z^{(0)} = z(x^{(0)}; y^{(0)}).$$

Then

$$|z^{(1)} - z^{(0)}| \leq M_4(|x^{(1)} - x^{(0)}| + \|y^{(1)} - y^{(0)}\|). \quad (11.2)$$

**Proof.** We denote by $x^{(2)}$:

(Case I) the nearest $x$ to $x^{(1)}$ in the interval $x^{(0)} \leq x \leq x^{(1)}$ or $x^{(1)} \leq x \leq x^{(0)}$ such that $(x; \varphi(x, x^{(1)}; y^{(1)} | G_4)) \in F$, if the portion of $C(x^{(1)}; y^{(1)} | G_4)$ for that interval has some points in common with $F$. Such $x$ exists in this case since the continuous curve $C(x^{(1)}; y^{(1)} | G_4)$ which is defined for the interval $x^{(0)} \leq x \leq x^{(1)}$ or $x^{(1)} \leq x \leq x^{(0)}$, is contained in $G_4$ and $F \cdot G_4$ is closed in $G_4$.

(Case II) the number $x^{(0)}$, if the portion of $C(x^{(1)}; y^{(1)} | G_4)$ for the interval $x^{(0)} \leq x \leq x^{(1)}$ or $x^{(1)} \leq x \leq x^{(0)}$ has no point in common with $F$.

We put $y^{(2)} = \varphi(x^{(2)}; x^{(1)}; y^{(1)} | G_4), \ z^{(2)} = z(x^{(2)}; y^{(2)})$. In Case I, $(x^{(2)}; y^{(2)}) \in F \cdot G_4$ and in Case II, $(x^{(2)}; y^{(2)}) = (x^{(0)}; \varphi(x^{(0)}, x^{(1)}; y^{(1)} | G_4)) \in G_4$.

Then in both Cases, by Lemma 1,

$$z^{(1)} = z^{(2)}, \quad (11.3)$$

since the portion of $C(x^{(1)}; y^{(1)} | G_4)$ for the open interval $x^{(1)} \leq x \leq x^{(2)}$ or $x^{(2)} \leq x \leq x^{(1)}$ has no point in common with $F$ and so is contained in $K$ by the definition of $x^{(2)}$.

Also by (6.1), (2.2), observing that $C(x^{(1)}; y^{(1)} | G_4)$ is contained in $Q$, we have

$$|y^{(2)}_\lambda - y^{(1)}_\lambda| = |\varphi_\lambda(x^{(2)}, x^{(1)}; y^{(1)} | G_4) - \varphi_\lambda(x^{(1)}, x^{(1)}; y^{(1)} | G_4)|$$

$$\leq M_0 |x^{(2)} - x^{(1)}| \quad \lambda = 1, \ldots, n.$$ 

Hence

$$\|y^{(2)} - y^{(1)}\| \leq \sum_{\alpha=1}^n |y^{(2)}_\alpha - y^{(1)}_\alpha| \leq nM_0 |x^{(2)} - x^{(1)}| \leq nM_0 |x^{(1)} - x^{(0)}|,$$

since $x^{(0)} \leq x^{(2)} \leq x^{(1)}$ or $x^{(1)} \leq x^{(2)} \leq x^{(0)}$ by the definition of $x^{(2)}$. 


Therefore we have
\[ \| y^{(2)} - y^{(0)} \| \leq \| y^{(2)} - y^{(1)} \| + \| y^{(1)} - y^{(0)} \| \leq nM_0 |x^{(1)} - x^{(0)}| + \| y^{(1)} - y^{(0)} \|. \] (11.4)

Now
\[ (x^{(0)}; y^{(2)}) \in Q' \subset G_5 \subset Q, \]
since \((x^{(0)}; y^{(0)}) \in G_4 \subset Q' \) and \((x^{(2)}; y^{(2)}) \in G_5 \subset Q' \) in both Cases. We put
\[ z^{(3)} = z(x^{(0)}; y^{(2)}). \]

Then we have
\[ |z^{(3)} - z^{(2)}| = |z(x^{(0)}; y^{(2)}) - z(x^{(2)}; y^{(2)})| \leq N |x^{(2)} - x^{(0)}| \]
in Case I, by Proposition 1 and \((x^{(0)}; y^{(2)}) \in Q, (x^{(2)}; y^{(2)}) \in F \cdot G_4 \subset F \cdot Q, \)
and in Case II, simply as \(x^{(2)} = x^{(0)}\). Hence, by (11.3), we get
\[ |z^{(3)} - z^{(1)}| = |z^{(3)} - z^{(2)}| \leq N |x^{(3)} - x^{(2)}| \leq N |x^{(1)} - x^{(0)}|, \] (11.5)
since \(x^{(0)} \leq x^{(2)} \leq x^{(1)}\) or \(x^{(1)} \leq x^{(2)} \leq x^{(0)}\) by the definition of \(x^{(2)}\).

Also, by Proposition 10 (10.12), since \((x^{(0)}; y^{(2)}) \in G_5, (x^{(0)}; y^{(0)}) \in G_5 \)
in both Cases, we have
\[ |z^{(3)} - z^{(0)}| \leq M_5 M_4 \| y^{(2)} - y^{(0)} \|. \]

Hence by (11.4), we get
\[ |z^{(3)} - z^{(2)}| \leq nM_0 M_2 M_4 |x^{(1)} - x^{(0)}| + M_5 M_4 \| y^{(1)} - y^{(0)} \|. \] (11.6)

By (11.5), (11.6), taking account of (11.1), we obtain finally
\[ |z^{(1)} - z^{(0)}| \leq |z^{(3)} - z^{(1)}| + |z^{(3)} - z^{(0)}| \]
\[ \leq (N + nM_0 M_2 M_4) |x^{(1)} - x^{(0)}| + M_5 M_4 \| y^{(1)} - y^{(0)} \| \]
\[ \leq M_6 (|x^{(1)} - x^{(0)}| + \| y^{(1)} - y^{(0)} \|), \quad \text{q.e.d.} \]

We denote by \(Q''\) the \((n+1)\)-dimensional open cube defined by
\[ (x; \xi) : |x - a'| < L_4, \]
\[ |\xi - b'| < L_4 \quad \lambda = 1, \ldots, n. \]
We put \(X_\lambda(x; \xi) = \varphi_\lambda(x, a'; \xi; G_\lambda)\). \(\lambda = 1, \ldots, n. \) Then \(X_\lambda(x; \xi)\) are defined and continuous on \(Q''\) and have continuous partial derivatives with respect to all their arguments on \(Q''\), by the corresponding properties of \(\varphi_\lambda(x, \xi; \eta; G_\lambda). \)

We denote by \(\mathfrak{Y}\) the continuous

\[ \text{Fig. 7} \]
mapping of $Q''$ onto $G_4$:

$$(x; \eta) \rightarrow (x; \chi(x; \eta))$$

That $\mathfrak{A}$ maps $Q''$ onto $G_4$, follows from the definition of $G_4$.

By the properties of $C(\xi; \eta|G_4)$ and $\varphi_{\lambda}(x, \xi; \eta|G_4)$ as stated in §1.2, we easily see that $\mathfrak{A}$ is one to one and bicontinuous, and $\mathfrak{A}^{-1}$ is represented by

$$(x; y) \rightarrow (x; \gamma(x; y)),$$

if we put $\gamma_{\lambda}(x; y) = \varphi_{\lambda}(a', x; y|G_4)$ for $x, y \in G_4$. Further $\gamma_{\lambda}(x; y)$ have continuous partial derivatives with respect to all their arguments by the corresponding properties of $\varphi_{\lambda}(x, \xi; \eta|G_4)$.

From this, we can easily prove that $\mathfrak{A}^{-1}$ maps any null set in $G_4$ onto a null set in $Q''$.

Thus we have

**Proposition 12.** The mapping $\mathfrak{A}$ of $Q''$ onto $G_4$ is one to one and bicontinuous, and $\mathfrak{A}^{-1}$ maps any null set in $G_4$ onto a null set in $Q''$.

12. Completion of the proof. By Proposition 11, we have

$$\lim \sup_{(x; y) \to (x_0; y_0)} \frac{|z(x; y) - z(x^{(0)}; y^{(0)})|}{|x - x^{(0)}| + \|y - y^{(0)}\|} \leq M,$$  \hspace{1cm} (12.1)

whenever $(x^{(0)}; y^{(0)}) \in G_4$. Hence $z(x; y)$ is totally differentiable almost everywhere in $G_4$, by a theorem of Rademacher on almost everywhere total differentiability\(^9\). Also, by Proposition 12, $\mathfrak{A}^{-1}$ maps any null set in $G_4$ onto a null set in $Q''$. Therefore, if we write $\xi(x; \eta) = z(x; \chi(x; \eta))$, and $y_{\lambda} = \chi_{\lambda}(x; \eta)$ for $(x; \eta) \in Q''$, we obtain

$$\frac{\partial}{\partial x} \xi(x; \eta) = \frac{\partial z}{\partial x}(x; y) + \sum_{\mu=1}^{n} \frac{\partial z}{\partial y_{\mu}}(x; y) \frac{\partial \chi_{\mu}}{\partial x}(x; \eta) \hspace{1cm} (12.2)$$

for almost all $(x; \eta)$ of $Q''$.

Since $\chi_{\lambda}(x; \eta) = \varphi_{\lambda}(a'; x; \eta|G_4)$ for $(x; \eta) \in Q''$, we obtain by (2.2),

$$\frac{\partial}{\partial x} \chi_{\lambda}(x; \eta) = f_{\lambda}(x; \chi(x; \eta)) = f_{\lambda}(x; y) \lambda = 1, \ldots, n \hspace{1cm} (12.3)$$

for $(x; \eta) \in Q''$. Substituting this into (12.2), we get

$$\frac{\partial}{\partial x} \xi(x; \eta) = \frac{\partial z}{\partial x}(x; y) + \sum_{\mu=1}^{n} f_{\mu}(x; y) \frac{\partial z}{\partial y_{\mu}}(x; y) \hspace{1cm} (12.4)$$


\(^{10}\) Cf. Rademacher [7], pp. 341-347. Also Cf. Saks [9], p. 311.
for almost all \((x; \eta)\) of \(Q''\).

Since by assumption, \(z(x; y)\) satisfies (2.1) almost everywhere in \(G(\supset G_4)\) and by Proposition 12, \(\mathfrak{A}^{-1}\) maps any null set in \(G_4\) onto null set in \(Q''\), the right side of (12.4) regarded as a function of \((x; \eta)_0\), vanishes almost everywhere in \(Q''\). Therefore

\[
\frac{\partial}{\partial x} \zeta(x; \eta) = 0 \tag{12.5}
\]

almost everywhere in \(Q''\).

On the other hand, if we write \(y_\lambda = \chi_\lambda(x^{(0)}; \eta^{(0)}), \lambda = 1, \ldots, n\) for a point \((x^{(0)}; \eta^{(0)}) \in Q''\), we have \((x^{(0)}; y^{(0)}) \in G_4\) and

\[
\limsup_{x \to x^{(0)}} \frac{|\zeta(x; \eta^{(0)}) - \zeta(x^{(0)}; \eta^{(0)})|}{|x - x^{(0)}|} \leq \limsup_{x \to x^{(0)}} \frac{|\zeta(x; \eta^{(0)}) - \zeta(x^{(0)}; \eta^{(0)})|}{|x - x^{(0)}| + \|x(x; \eta^{(0)}) - x(x^{(0)}; \eta^{(0)})\|} \times \limsup_{x \to x^{(0)}} \frac{|x - x^{(0)}| + \|x(x; \eta^{(0)}) - x(x^{(0)}; \eta^{(0)})\|}{|x - x^{(0)}|}
\]

\[
\leq \left( \limsup_{(x; y) \to (x^{(0)}; y^{(0)})} \frac{|\zeta(x; \eta^{(0)}) - \zeta(x^{(0)}; \eta^{(0)})|}{|x - x^{(0)}| + \|y - y^{(0)}\|} \right) \left( 1 + \sum_{\mu=1}^{n} \left| \frac{\partial X_\mu}{\partial x}(x^{(0)}; \eta^{(0)}) \right| \right).
\]

Hence by (12.1) and (12.3), observing that \(|f_\lambda(x; y)| < M_6\) on \(G_4(\subset Q)\) by (6.1), we obtain

\[
\limsup_{x \to x^{(0)}} \frac{|\zeta(x; \eta^{(0)}) - \zeta(x^{(0)}; \eta^{(0)})|}{|x - x^{(0)}|} \leq M_6 \left( 1 + \sum_{\mu=1}^{n} \left| f_\mu(x^{(0)}; y^{(0)}) \right| \right)
\]

\[
\leq M_6(1 + nM_6) \tag{12.6}
\]

whenever \((x^{(0)}; \eta^{(0)}) \in Q''\).

By Fubini’s theorem, \(\zeta(x; \eta)\) as a function of \(x\), satisfies (12.5) almost everywhere in the interval \(|x - a'| < L_4\), for almost every \(\eta\) in the domain \(\Omega_4\) and by (12.6), \(\zeta(x; \eta)\) as a function of \(x\), is absolutely continuous in the interval \(|x - a'| < L_4\) for any \(\eta\) in the domain \(\Omega_4\).

Therefore by Lebesgue’s theorem, \(\zeta(x; \eta)\) as a function of \(x\), is constant in the interval \(|x - a'| < L_4\) for almost every \(\eta\) in the domain \(\Omega_4\). Hence, by the continuity of \(z(x; y)\) and \(\chi_\lambda(x; \eta)\), accordingly of \(\zeta(x; \eta)\), it follows that \(\zeta(x; \eta)\) as a function of \(x\), is constant in the interval \(|x - a'| < L_4\) for any \(\eta\) in the domain \(\Omega_4\).

From this, by the definition of \(\zeta(x; \eta)\), we easily see that \(z(x; y)\) is constant on any characteristic curve of (2.1) in \(G_4\). Hence, by the definition of \(K\), we have \(G_4 \subset K\) and so observing that \(G_4(\subset G)\) is open in \(R^{n+1}\).
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\[ F = G - K \cdot G \subset G - G_i \cdot G = G - G_i. \]

This is however excluded, since \((a'; b') \in F \cdot G_i\). Thus we arrive at a contradiction and this completes the proof of Theorem 1.

§ 4. Proof of Theorem 2 and Theorem 3

In this §, the notations are the same as in § 1 and § 2.

13. Proof of Theorem 2. By the assumption on \(G\) and Theorem 1, if we put \(\omega_\lambda(x; y) = \varphi_\lambda(\xi^{(c)}; x; y|G)\), \(\lambda = 1, \ldots, n\) for \((x; y) \in G\), then we have \(\omega(x; y) \in G[\xi^{(c)}]\) for \((x; y) \in G\) and

\[ z(x; y) = \psi(\omega(x; y)) \quad \text{on} \quad G \]  

(13.1)

for any quasi-solution \(z(x; y)\) of (2.1) on \(G\) such that \(z(\xi^{(c)}; \eta) = \psi(\eta)\) on \(G[\xi^{(c)}]\). Hence there is at most only one such quasi-solution.

Conversely if we define a function \(z(x; y)\) by the right side of (13.1) on \(G\), then by the total differentiability of \(\psi(\eta)\) on \(G[\xi^{(c)}]\) and of \(\omega_\lambda(x; y)\) on \(G\), \(z(x; y)\) is totally differentiable on \(G\) and

\[ \frac{\partial z}{\partial x} = \sum_{\mu=1}^{n} \frac{\partial \psi}{\partial \eta_\mu} \frac{\partial \omega_\mu}{\partial x}, \quad \frac{\partial z}{\partial y_\lambda} = \sum_{\mu=1}^{n} \frac{\partial \psi}{\partial \eta_\mu} \frac{\partial \omega_\mu}{\partial y_\lambda} \quad \lambda = 1, \ldots, n \]

on \(G\). Hence

\[ \frac{\partial z}{\partial x} + \sum_{\mu=1}^{n} f_\mu(x; y) \frac{\partial z}{\partial y} = \sum_{\lambda=1}^{n} \frac{\partial \psi}{\partial \eta_\lambda} \left( \frac{\partial \omega_\lambda}{\partial x} + \sum_{\mu=1}^{n} \frac{\partial \omega_\lambda}{\partial y_\mu} f_\mu(x; y) \right) \quad \text{on} \quad G. \]  

(13.2)

But for \(\omega_\lambda(x; y) = \varphi_\lambda(\xi^{(c)}; x; y|G)\), we have\(^{11}\)

\[ \frac{\partial \omega_\lambda}{\partial x} + \sum_{\mu=1}^{n} f_\mu(x; y) \frac{\partial \omega_\lambda}{\partial y_\mu} = 0 \quad \lambda = 1, \ldots, n \quad \text{on} \quad G. \]

Therefore by (13.2), for \(z(x; y)\) defined by (13.1)

\[ \frac{\partial z}{\partial x} + \sum_{\mu=1}^{n} f_\mu(x; y) \frac{\partial z}{\partial y_\mu} = 0 \quad \text{on} \quad G. \]

Also for \(z(x; y)\) defined by (13.1), we have

\[ z(\xi^{(c)}; \eta) = \psi(\eta) \quad \text{on} \quad G[\xi^{(c)}], \]

since \(\omega_\lambda(\xi^{(c)}; \eta) = \varphi_\lambda(\xi^{(c)}; \xi^{(c)}; \eta|G) = \eta\).

Thus there is at least one quasi-solution \(z(x; y)\) of (2.1) on \(G\) such that \(z(\xi^{(c)}; \eta) = \psi(\eta)\) on \(G[\xi^{(c)}]\) and this quasi-solution is also a solution

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\(^{11}\) Cf. Kamke [3], §18, Nr. 87, Satz 1.
of (2.1) on \( G \) in the ordinary sense. This completes the proof of Theorem 2.

14. Proof of Theorem 3. For the special case \( n = 1 \), we write (2.1) in the form

\[
\frac{\partial z}{\partial x} + f(x, y) \frac{\partial z}{\partial y} = 0
\]  
(14.1)

and the characteristic curve of (14.1) in \( G \) which passes through the point \((\xi, \eta)\) of \( G \), in the form

\[
y = \varphi(x, \xi, \eta|G) \quad \alpha(\xi, \eta|G) \leq x \leq \beta(\xi, \eta|G).
\]

Let \( z(x, y) \) be any quasi-solution of (14.1) on \( G \) and \((x^{(0)}, y^{(0)})\) be any point of \( G \). Then there is at least one point \((\xi^{(0)}, \eta^{(0)})\) on \( C(x^{(0)}, y^{(0)}|G) \) where \( z(x, y) \) has \( \partial z/\partial y \), since \( z(x, y) \) has \( \partial z/\partial y \) except at most at the points of an enumerable set in \( G \).

If we put \( \omega(x, y) = \varphi(\xi^{(0)}, x, y|G) \) and for \( \eta \in G[\xi^{(0)}], \psi(\eta) = z(\xi^{(0)}, \eta) \), then by the properties of the family of the characteristic curves as stated in §1.2, \( \omega(x, y) \) is defined and \( \omega(x, y) \in G[\xi^{(0)}] \) for \((x, y)\) in some neighbourhood of \((x^{(0)}, y^{(0)})\) and by Theorem 1

\[
z(x, y) = \psi(\omega(x, y))
\]  
(14.2)

in that neighbourhood. Evidently \( \omega(x^{(0)}, y^{(0)}) = \varphi(\xi^{(0)}, x^{(0)}, y^{(0)}|G) = \eta^{(0)} \in G[\xi^{(0)}] \). Also \( \psi(\eta) (= z(\xi^{(0)}, \eta)) \) is differentiable at \( \eta^{(0)} \) since \( z(x, y) \) has \( \partial z/\partial y \) at \((\xi^{(0)}, \eta^{(0)})\).

Since \( \psi(\eta) \) is differentiable at \( \eta^{(0)} = \omega(x^{(0)}, y^{(0)}) \) and \( \omega(x, y) \)

\[
= \varphi(\xi^{(0)}, x, y|G)
\]

is totally differentiable at \((x^{(0)}, y^{(0)})\), by (14.2) \( z(x, y) \) is totally differentiable at \((x^{(0)}, y^{(0)})\) and

\[
\frac{\partial z}{\partial x}(x^{(0)}, y^{(0)}) = \psi'(\eta^{(0)}) \frac{\partial \omega}{\partial x}(x^{(0)}, y^{(0)}),
\]

\[
\frac{\partial z}{\partial y}(x^{(0)}, y^{(0)}) = \psi'(\eta^{(0)}) \frac{\partial \omega}{\partial y}(x^{(0)}, y^{(0)}).
\]

Hence

\[
\frac{\partial z}{\partial x}(x^{(0)}, y^{(0)}) + f(x^{(0)}, y^{(0)}) \frac{\partial z}{\partial y}(x^{(0)}, y^{(0)})
\]

\[
= \psi'(\eta^{(0)}) \left\{ \frac{\partial \omega}{\partial x}(x^{(0)}, y^{(0)}) + f(x^{(0)}, y^{(0)}) \frac{\partial \omega}{\partial y}(x^{(0)}, y^{(0)}) \right\}.
\]  
(14.3)

But for \( \omega(x, y) = \varphi(\xi^{(0)}, x, y) \), we have\(^{12}\)

\(^{12}\) Cf. Kamke [3], §18, Nr. 87, Satz 1.
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\[
\frac{\partial \omega}{\partial x}(x^{(0)}, y^{(0)}) + f(x^{(0)}, y^{(0)}) \frac{\partial \omega}{\partial y}(x^{(0)}, y^{(0)}) = 0.
\]

Hence, by (14.3)

\[
\frac{\partial z}{\partial x}(x^{(0)}, y^{(0)}) + f(x^{(0)}, y^{(0)}) \frac{\partial z}{\partial y}(x^{(0)}, y^{(0)}) = 0.
\]

Therefore \( z(x, y) \) is totally differentiable and satisfies (14.1) at any point \((x^{(0)}, y^{(0)})\) of \( G \), q.e.d.

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References
