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On the Homogeneous Linear Partial Differential Equation of the First Order

By Takashi Kasuga

§ 1. Introduction

In this paper, we shall treat the following partial differential equation

$$\frac{\partial z}{\partial x} + \sum_{\mu=1}^{n} f_{\mu}(x, y_1, \ldots, y_n) \frac{\partial z}{\partial y_\mu} = 0 \quad (n \geq 1)$$

without the usual condition of the total differentiability on the solution $z(x, y_1, \ldots, y_n)$.

For early contributions by R. Baire and P. Montel to this problem in the special case $n = 1$, cf. Baire [1], Montel [6]. Our method is entirely different from theirs and gives more general results even for the case $n = 1$, cf. Kasuga [4]. Also notwithstanding Baire's statement in his paper, it seems to us that their methods cannot be generalized to the case $n > 1$ immediately.

We have not yet succeeded in treating the more general non-homogeneous partial differential equation

$$\frac{\partial z}{\partial x} + \sum_{\mu=1}^{n} f_{\mu}(x, y_1, \ldots, y_n, z) \frac{\partial z}{\partial y_\mu} = g(x, y_1, \ldots, y_n, z)$$

in a similar way, except for the case $n = 1$. For this case, cf. Kasuga [5].

1. In this paper, we shall use for points in $\mathbb{R}^n$, $\mathbb{R}^{n+1}$ or $\mathbb{R}^{n+2}$ and for their functions, abbreviations such as:

$$y = (y_1, \ldots, y_n), \quad (x; y) = (x, y_1, \ldots, y_n),$$

$$\eta = (\eta_1, \ldots, \eta_n), \quad (\xi; \eta) = (\xi, \eta_1, \ldots, \eta_n),$$

$$(x, \xi; \eta) = (x, \xi, \eta_1, \ldots, \eta_n), \quad z(x; y) = z(x, y_1, \ldots, y_n),$$

and if $\varphi_\lambda(x, \xi; \eta) = \varphi_\lambda(x, \xi, \eta_1, \ldots, \eta_n) \quad \lambda = 1, \ldots, n$ are $n$ functions of $(x, \xi; \eta)$,

1) Cf. Baire [1], p. 120.
\[ \varphi(x, \xi; \eta) = (\varphi_1(x, \xi; \eta), \ldots, \varphi_n(x, \xi; \eta)), \]
\[ z(x; \varphi(x, \xi; \eta)) = z(x, \varphi_1(x, \xi; \eta), \ldots, \varphi_n(x, \xi; \eta)). \]

Also we use the following notations:

For sets of points \( A, B \) in \( \mathbb{R}^m \) \( (m = 1, 2, \ldots, n+1) \),

- \( \bar{A} \) = closure in \( \mathbb{R}^m \) of \( A \), \( A^\circ \) = interior in \( \mathbb{R}^m \) of \( A \),
- \( A^b \) = boundary in \( \mathbb{R}^m \) of \( A \), \( A \cap B \) = intersection of \( A \) and \( B \),
- \( A[x] \) = the set of the points \( (y_1, \ldots, y_n) \) in \( \mathbb{R}^n \) such that for a fixed \( x \) \( (x, y_1, \ldots, y_n) \in A \), if \( A \subseteq \mathbb{R}^{n+1} \).

For two points \( y' = (y'_1, \ldots, y'_n), y'' = (y''_1, \ldots, y''_n) \) in \( \mathbb{R}^n \),
\[ \|y' - y''\| = \sum_{k=1}^{n} |y'_k - y''_k|, \quad y' + y'' = (y'_1 + y''_1, \ldots, y'_n + y''_n). \]

In this paper, the so-called degenerated intervals are also included, when we use the word “interval” (open, closed, or half-open). Thus the interval \( a < x < a \) or the interval \( a \leq x \leq a \) will mean degenerated interval which is empty or is composed of only one point respectively. Similarly for the interval \( a \leq x < a \) or \( a < x \leq a \).

2. In the following, we shall denote by \( G \) a fixed open set in \( \mathbb{R}^{n+1} \), by \( f_\lambda(x; y) \) \( \lambda = 1, \ldots, n \) \( n \) fixed continuous functions defined on \( G \) which have continuous \( \partial f_\lambda / \partial y_\mu \) \( \lambda, \mu = 1, \ldots, n \).

Under the above conditions, we shall consider the partial differential equation
\[ \frac{\partial z}{\partial x} + \sum_{\mu=1}^{n} f_\mu(x; y) \frac{\partial z}{\partial y_\mu} = 0. \tag{2.1} \]

With (2.1), we shall associate the simultaneous ordinary differential equations
\[ \frac{dy_\lambda}{dx} = f_\lambda(x; y) \quad \lambda = 1, \ldots, n. \tag{2.2} \]

The continuous curves representing the solutions of (2.2) which are prolonged as far as possible on both sides in an open subset \( D \) of \( G \), will be called characteristic curves of (2.1) in \( D \). Through any point \( (\xi; \eta) \) in \( D \), there passes one and only one characteristic curve in \( D^2 \). We represent it by
\[ y_\lambda = \varphi_\lambda(x, \xi, \eta_1, \ldots, \eta_n | D) = \varphi_\lambda(x, \xi; \eta | D) \quad \lambda = 1, \ldots, n. \]
\[ \alpha(\xi; \eta | D) \ll x \ll \beta(\xi; \eta | D). \]

2) Cf. Kamke [3], §16, Nr. 79, Satz 4.
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\[ \alpha(\xi; \eta|D), \beta(\xi; \eta|D) \text{ may be } -\infty, +\infty \text{ respectively.} \]

Sometimes we abbreviate it as \( C(\xi; \eta|D) \). If an interval (open, closed or half-open) is contained in the interval \( \alpha(\xi; \eta|D) < x < \beta(\xi; \eta|D) \), then we say that \( C(\xi; \eta|D) \) is defined for that interval. Also when a property holds for the portion of \( C(\xi; \eta|D) \) which corresponds to the values of \( x \) belonging to an interval, we say that \( C(\xi; \eta|D) \) has the property for this interval.

We shall use the following properties of \( C(\xi; \eta|D) \) and \( \varphi_\lambda(x, \xi; \eta|D) \) often without special reference.

As we can easily see from the definition of \( C(\xi; \eta|D) \), \((\xi; \eta) \in C(\xi; \eta|D) \) and if \((x'; y') \in C(\xi; \eta|D)\), then \( C(\xi; \eta|D) = C(x'; y'|D) \) and so \((\xi; \eta) \in C(x'; y'|D)\).

In terms of the functions \( \varphi_\lambda \), this means:

\[ \eta = \varphi(\xi, \xi; \eta|D), \]

and if \( y' = \varphi(x', \xi; \eta|D), \) then

\[ \alpha(\xi; \eta|D) = \alpha(x'; y'|D) = \alpha \), \ \beta(\xi; \eta|D) = \beta(x'; y'|D) = \beta \]

and

\[ \varphi(x, \xi; \eta|D) = \varphi(x, x'; y'|D) \text{ for } \alpha < x < \beta, \]

especially

\[ \eta = \varphi(\xi, x'; y'|D). \]

Also \( C(\xi; \eta|D_1) \supset C(\xi; \eta|D_2) \), if \( D_1 \supset D_2 \) and \( (\xi; \eta) \in D_2 \).

We denote by \( D^* \) the set of the points \((x, \xi; \eta)\) in \( \mathbb{R}^{n+2} \) such that \( (\xi; \eta) \in D \) and \( \alpha(\xi; \eta|D) < x < \beta(\xi; \eta|D) \). \( D^* \) is the domain of definition of the functions \( \varphi_\lambda(x, \xi; \eta|D) \). \( D^* \) is open in \( \mathbb{R}^{n+2} \).

The functions \( \varphi_\lambda(x, \xi; \eta|D) \) are continuous and have continuous partial derivatives with respect to all their arguments on \( D^* \).

A continuous function \( z(x; y) \) defined on \( G \) will be called a quasi-solution of (2.1) on \( G \), if it has \( \partial z/\partial x, \partial z/\partial y_\lambda, \lambda = 1, \cdots, n \), except at most at the points of an enumerable set in \( G \) and satisfies (2.1) almost everywhere in \( G \). Here \( \partial z/\partial x, \partial z/\partial y \) need not necessarily be continuous.

On the other hand, a continuous function \( z(x; y) \) defined on \( G \) will be called a solution of (2.1) on \( G \) in the ordinary sense, if it is totally differentiable and satisfies (2.1) everywhere in \( G \).

3. We shall prove the following three theorems in § 3, § 4.

3) Cf. Kamke [3], § 17, Nr. 84, Satz 4.

4) Cf. Kamke [3], § 17, Nr. 84, Satz 4 and § 18, Nr. 87, Satz 1.
Theorem 1. A quasi-solution \( z(x;y) \) of (2.1) on \( G \) is constant on any characteristic curve of (2.1) in \( G \).

Theorem 2. If for a fixed number \( \xi^{(0)} \), the family of all the characteristic curves \( C(\xi^{(0)};\eta|G) \) such that \( \eta \in G[\xi^{(0)}] \), covers \( G \) and \( \psi(\eta) = \psi(\eta_1, \ldots, \eta_n) \) is a totally differentiable function defined on \( G[\xi^{(0)}] \), then there is one and only one quasi-solution \( z(x;y) \) of (2.1) on \( G \) such that \( z(\xi^{(0)};\eta) = \psi(\eta) \) on \( G[\xi^{(0)}] \) and this quasi-solution \( z(x;y) \) is also a solution of (2.1) on \( G \) in the ordinary sense.

Theorem 3. If \( n = 1 \), any quasi-solution of (2.1) on \( G \) is also a solution of (2.1) on \( G \) in the ordinary sense.

Remark 1. For the case \( n=1 \), the proof of Theorem 1 can be partly simplified, cf. Kasuga [4].

Remark 2. In Theorem 1, the condition on \( z(x;y) \) that it has \( \partial z/\partial x, \partial z/\partial y, \lambda = 1, \ldots, n \) except at most at the points of an enumerable set in \( G \), cannot be replaced by the condition that it has \( \partial z/\partial x, \partial z/\partial y, \lambda = 1, \ldots, n \) almost everywhere in \( G \), as the following example shows it.

Example. \( G: 0 < x < 1 \), \( 0 < y < 1 \),
the differential equation is
\[
\frac{\partial z}{\partial x} = 0
\]
and a function \( z(x,y) \) is defined by
\[
z(x,y) = \psi(x) \quad \text{on } G
\]
where \( \psi(x) \) is a continuous singular function not constant on the interval \( 0 \leq x \leq 1 \) as given in Saks [8] p. 101.

Then \( z(x,y) \) is continuous on \( G \), has \( \partial z/\partial x, \partial z/\partial y \) almost everywhere in \( G \) and satisfies the differential equation almost everywhere in \( G \). But \( z(x,y) \) is not constant on any characteristic curve \( y = \text{constant} \).

\section*{§ 2. Some Lemmas}

In this §, the notations are the same as in § 1 and we assume that \( z(x;y) \) is a quasi-solution of (2.1) on \( G \).

4. Set \( K \) and Some Lemmas. We denote by \( K \) the set of the points \( (\xi;\eta) \) of \( G \) such that \( z(x;y) \) is constant on the portion of the characteristic curve \( C(\xi;\eta|G) \) contained in a neighbourhood of \( (\xi;\eta) \).
Lemma 1. If a characteristic curve $C(\xi; \eta|G)$ is defined for an interval $I$ (open, closed, or half-open) and is contained in $K$ for the open interval $I^0$, the interior of $I$, then $z(x;y)$ is constant on the portion of $C(\xi; \eta|G)$ for the interval $I$.

Proof. By the definition of $K$, we easily see that $z(x;y)$ is constant on the portion of $C(\xi; \eta|G)$ for the open interval $I^0$. Then Lemma 1 follows from the continuity of $z(x;y)$ and $\varphi_\lambda(x, \xi; \eta|G)$ $\lambda = 1, \ldots, n$.

Lemma 2. Denote by $D$ an open subset of $G$, and denote by $D_0$ the set of the points $(\xi; \eta)$ of $D$ such that $z(x;y)$ is constant on the characteristic curve $C(\xi; \eta|D)$. Then $D_0$ is closed in $D$.

Proof. If $C(\xi^{(0)}; \eta^{(0)}|D)$ where $(\xi^{(0)}; \eta^{(0)}) \in D$, is defined for a closed interval $\alpha_0 \leq x \leq \beta_0$, then $C(\xi; \eta|D)$ where $(\xi; \eta)$ is sufficiently close to $(\xi^{(0)}; \eta^{(0)})$, is also defined for the interval $\alpha_0 \leq x \leq \beta_0$ and

$$\varphi_\lambda(x, \xi; \eta|D) \rightarrow \varphi_\lambda(x, \xi^{(0)}; \eta^{(0)}|D) \quad \lambda = 1, \ldots, n$$

uniformly in the interval $\alpha_0 \leq x \leq \beta_0$ as $(\xi; \eta) \rightarrow (\xi^{(0)}; \eta^{(0)})$. From this and by the continuity of $z(x;y)$, we easily see that $D_0$ is closed in $D$, q.e.d.

Lemma 3. Let $D$ be an open subset of $G$. If

$$|f_\lambda(x;y) - f_\lambda(x;y)| \leq M \|y - y\| \quad \lambda = 1, \ldots, n \quad (4.1)$$

for every pair of points $(x;y), (x;y) \in D$ with the same $x$ coordinate and if $C(\xi; \eta|D)$ and $C(\xi; \eta|D)$ where $(\xi; \eta), (\xi; \eta) \in D$, are both defined for an interval $\alpha_0 \leq x \leq \beta_0$ containing $\xi$ ($\alpha_0 \leq \xi \leq \beta_0$), then

$$\|\varphi(x, \xi; \eta|D) - \varphi(x, \xi; \eta|D)\| = \sum_{\lambda=1}^{n} |\varphi_\lambda(x, \xi; \eta|D) - \varphi_\lambda(x, \xi; \eta|D)|$$

$$\leq \|\eta - \eta\| \exp (nM|x - \xi|) \quad (4.2)$$

and

$$|\varphi_\lambda(x, \xi; \eta|D) - \varphi_\lambda(x, \xi; \eta|D)| \leq |\eta - \eta_\lambda| + \frac{1}{n} \|\eta - \eta\|$$

$$\times \{\exp (nM|x - \xi|) - 1\} \quad \lambda = 1, \ldots, n \quad (4.3)$$

for $\alpha_0 \leq x \leq \beta_0$.

Proof. We abbreviate $\varphi_\lambda(x, \xi; \eta|D)$ and $\varphi_\lambda(x, \xi; \eta|D)$ as $\varphi_\lambda(x)$ and $\varphi_\lambda(x)$ respectively.

By (4.1) and (2.2), we have

5) Cf. Kamke [3], §17, Nr. 84, Satz 4.
\[ |\bar{\varphi}_\lambda'(x) - \varphi_\lambda'(x)| \leq M \| \bar{\varphi}(x) - \varphi(x) \| \quad \lambda = 1, \ldots, n \]  
(4.4)

for \( \alpha_0 \leq x \leq \beta_0 \), so that

\[ \sum_{\mu=1}^{n} |\bar{\varphi}_\mu'(x) - \varphi_\mu'(x)| \leq nM \sum_{\mu=1}^{n} |\bar{\varphi}_\mu(x) - \varphi_\mu(x)| \]

for \( \alpha_0 \leq x \leq \beta_0 \). Hence by a theorem on differential inequalities\(^6\), taking account of \( \alpha_0 \leq \xi \leq \beta_0 \) and \( \eta_\lambda = \bar{\varphi}_\lambda(\xi) \), \( \eta_\lambda = \varphi_\lambda(\xi) \) \( \lambda = 1, \ldots, n \), we obtain

\[ \| \bar{\varphi}(x) - \varphi(x) \| \leq \| \eta - \eta \| \exp(nM|x-\xi|) \]  
(4.5)

for \( \alpha_0 \leq x \leq \beta_0 \). Thus (4.2) is proved.

By (4.4), (4.5), we get

\[ |\bar{\varphi}_\lambda'(x) - \varphi_\lambda'(x)| \leq M \| \eta - \eta \| \exp(nM|x-\xi|) \quad \lambda = 1, \ldots, n \]

for \( \alpha_0 \leq x \leq \beta_0 \). Hence, again taking account of \( \alpha_0 \leq \xi \leq \beta_0 \) and \( \eta_\lambda = \bar{\varphi}_\lambda(\xi) \), \( \eta_\lambda = \varphi_\lambda(\xi) \) \( \lambda = 1, \ldots, n \), we have

\[ |\bar{\varphi}_\lambda(x) - \varphi_\lambda(x)| \leq |\eta_\lambda - \eta_\lambda| + M \| \eta - \eta \| \int_{\xi}^{x} \exp(nM|x-\xi|) \, dx \]

\[ = |\eta_\lambda - \eta_\lambda| + \frac{1}{n} \| \eta - \eta \| \{\exp(nM|x-\xi|) - 1\} \quad \lambda = 1, \ldots, n \]

for \( \alpha_0 \leq x \leq \beta_0 \). Thus (4.3) is also proved.

\section*{§ 3. Proof of Theorem 1.}

In this §, the notations are the same as in §1 and §2 and we assume that \( z(x; y) \) is a quasi-solution of (2.1) on \( G \).

5. Set \( F \) and Domain \( Q \). We denote by \( F \) the set \( G - K \cdot G \). Evidently \( F \) is closed in \( G \) and \( K \supset G - F \).

If \( F \) is empty, that is \( G = K \), we can conclude by Lemma 1 that \( z(x; y) \) is constant on any characteristic curve in \( G \) and Theorem 1 is established.

Therefore we suppose in the following that \( F \neq 0 \) and we want to show that such supposition leads to a contradiction.

**Proposition 1.** There is a positive number \( N \) and a \((n+1)\)-dimensional open cube \( Q: |x-a| < L, |y_\lambda - b_\lambda| < L \quad \lambda = 1, \ldots, n \) \((L > 0)\) such that

\[ \bar{Q} \subset G \]

\((a; b) \in F\)

\(^6\) Cf. Kamke [3], §17, Nr. 85, Hilfssatz 3 and Satz 5,
and such that

\[
\begin{align*}
|z(x + h; y) - z(x; y)| & \leq |h| N \\
|z(x, y_1, \ldots, y_{\lambda - 1}, y_\lambda + k_\lambda, y_{\lambda + 1}, \ldots, y_n) - z(x; y)| & \leq |k_\lambda| N
\end{align*}
\]

whenever \((x; y) \in F \cdot Q\) and \((x + h; y + k) \in Q\), where \(k = (k_1, \ldots, k_n)\).

Proof. We denote by \(H\) the at most enumerable set consisting of the points of \(G\) at which \(z(x; y)\) is not derivable with respect to \(x\) and with respect to \(y_\lambda\), \(\lambda = 1, \ldots, n\) simultaneously.

If a point \((\xi^{(0)}; \eta^{(0)})\) of \(G\) has an open neighborhood \(V\) such that every point of \(V\) belongs to \(K\) except at most \((\xi^{(0)}; \eta^{(0)})\) itself, then by Lemma 1 \(z(x; y)\) is constant on \(C(\xi^{(0)}; \eta | V)\) where \(\eta\) is any point of \(V[\xi^{(0)}]\) except \(\eta^{(0)}\) and so by Lemma 2, \(z(x; y)\) is also constant on \(C(\xi^{(0)}; \eta^{(0)} | V)\), that is, \((\xi^{(0)}; \eta^{(0)}) \in K\). Hence the set \(F\) which is closed in the open set \(G\), has no isolated point.

Therefore \(F\) is a \(G_\delta\) set in \(R^{n+1}\) without isolated point and so every point of \(F\) is a condensation point of \(F\). Thus since \(F\) is not empty by the supposition and \(H\) is at most enumerable, \(F - H\) is not empty and

\[
\overline{F - H} \supset F
\]

Also the non-empty \(F - H\) is a \(G_\delta\) set in \(R^{n+1}\) since \(F\) is a \(G_\delta\) set in \(R^{n+1}\) and \(H\) is at most enumerable. Hence \(F - H\) is of the second category in itself by Baire’s theorem.

On the other hand, if we denote by \(F_m\) for each positive integer \(m\), the set of the points \((x; y)\) of \(G\) such that

\[
\begin{align*}
|z(x + h; y) - z(x; y)| & \leq |h| m \\
|z(x, y_1, \ldots, y_{\lambda - 1}, y_\lambda + k_\lambda, y_{\lambda + 1}, \ldots, y_n) - z(x; y)| & \leq |k_\lambda| m
\end{align*}
\]

\(\lambda = 1, \ldots, n\)

whenever $|h|, |k_\lambda| \leq 1/m$ and $(x + h; y) \in G$, $(x, y_1, \ldots, y_{\lambda-1}, y_\lambda + k_\lambda, y_{\lambda+1}, \ldots, y_n) \in G \ \lambda = 1, \ldots, n$, then the union of the sets $F_m$ covers $F - H$ by the definition of $H$ and each of the set $F_m$ is closed in $G$ by the continuity of $z(x; y)$.

Therefore there must exist a positive integer $N$ and an $(n+1)$-dimensional open cube $Q$: $|x - a| < L, |y_\lambda - b_\lambda| < L \ \lambda = 1, \ldots, n \ (L > 0)$ such that $(a; b) \in F - H \subseteq F$ and

$$\quad (F - H) \cdot Q \subseteq F_N . \quad (5.4)$$

Also we can take $L$ sufficiently small so that

$$0 < L < 1/(2N) \quad (5.5)$$
$$\bar{Q} \subseteq G \quad (5.6)$$

since $G$ is open in $R^{n+1}$.

By $(5.4), (5.6)$ and by observing that $Q$ is open in $R^{n+1}$ and $F_N$ is closed in $G$, we have

$$F - H \cdot Q = (F - H) \cdot Q \subseteq \bar{F}_N \cdot Q \subseteq \bar{F}_N \cdot G = F_N$$

so that by $(5.2)$.

$$F_N \supseteq F \cdot Q .$$

Hence by $(5.5), (5.6)$ and by the definition of $F_N$, the inequalities $(5.3)$ for $m = N$ hold whenever $(x; y) \in F \cdot Q$ and $(x + h; y) \in Q$, $(x, y_1, \ldots, y_{\lambda-1}, y_\lambda + k_\lambda, y_{\lambda+1}, \ldots, y_n) \in Q \ \lambda = 1, \ldots, n$. This completes the proof of Proposition 1.

In the following, $Q, L, (a; b)$ and $N$ have the same meanings as in Proposition 1.

6. Domains $Q_1, \Omega, G_1$ and Set $\bar{\Omega}$. $f_\lambda$ and $\partial f_\lambda / \partial y_\mu \ \lambda, \mu = 1, \ldots, n$ are defined and continuous on $\bar{Q} \subseteq G$. Hence there is a positive number $M_0$ such that

$$|f_\lambda|, |\partial f_\lambda / \partial y_\mu| < M_0 \ \lambda, \mu = 1, \ldots, n \ \text{on} \ Q . \quad (6.1)$$

Then we can easily prove

$$|f_\lambda(x; y) - f_\lambda(x; y)| \leq M_0 \|y - y\| \quad (6.2)$$

for any pair of points $(x; y), (x; y) \in Q$ with the same $x$ coordinate. We take a positive number $L_1$ such that

$$\exp(2nM_0L_1) < 2$$
$$L_1(M_0 + 1) \leq L . \quad (6.3)$$

We denote by $\Omega_1$ the $n$-dimensional open cube: $|y_\lambda - b_\lambda| < L_1 \ \lambda = 1,$
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..., n and by \( Q \) the \((n+1)\)-dimensional open parallelepiped: \(|x-a|<L_1\), \(|y_\lambda-b_\lambda|<L\) \( \lambda=1, \ldots, n \). By (6.4) \( L_1<L \) and so

\[ Q_1 \subset Q. \]

By (6.4), \( \eta_\lambda+L_2M_\lambda \leq b_\lambda+L, \eta_\lambda-L_1M_\lambda \geq b_\lambda-L \) \( \lambda=1, \ldots, n \) whenever \( \eta \in \Omega_1 \). Hence the characteristic curves \( C(a; \eta|Q) \) where \( \eta \in \Omega_1 \), are defined just for the interval \(|x-a|<L_1\) since \(|f_\lambda|<M_\lambda \) \( \lambda=1, \ldots, n \) on \( Q_1 (\subset Q) \) by (6.1).

We denote by \( G_1 \) the portion of \( Q_1 \) covered by the family of all the characteristic curves \( C(a; \eta|Q) \) where \( \eta \in \Omega_1 \). Then by the properties of \( C(\xi; \eta|Q) \) and \( \varphi_\lambda(x, \xi; \eta|Q) \) as stated in §1.2, observing that \( \Omega_1 \) is open in \( \mathbb{R}^n \), we easily prove that \( G_1 \) is open in \( \mathbb{R}^{n+1} \) and the characteristic curves \( C(\xi; \eta|G_1) \) where \( (\xi; \eta) \in G_1 \), are defined just for the interval \(|x-a|<L_1\).

We denote by \( \widetilde{F} \) the set of the points \( \eta \) of \( \Omega_1 \) such that \( C(a; \eta|G_1) \) has at least one point in common with \( F \). \( \widetilde{F} \) is not empty since \( (a; b) \in F \) and \( b=(b_1, \ldots, b_n) \in \Omega_1 \). Now we prove

**Proposition 2.** \( \widetilde{F}^0 \), the interior of \( \widetilde{F} \) in \( \mathbb{R}^n \), is not empty.

**Proof.** Suppose, if possible, that \( \widetilde{F}^0=0 \).

If \( \eta \in \Omega_1, \widetilde{F} \), then \( C(a; \eta|G_1) \) is contained in \( K \) by the definition of \( \widetilde{F} \) and so by Lemma 1 \( \varepsilon(x; y) \) is constant on \( C(a; \eta|G_1) \). Hence by
Lemma 2, \( z(x; y) \) is constant on any \( C(a; \eta | G_1) \) such that \( \eta \in \Omega_1 - \bar{F} \cdot \Omega_1 \).

But \( \Omega_1 - \bar{F} \cdot \Omega_1 = \Omega_1 \), since \( \bar{F}^0 = 0 \) by supposition and \( \Omega_1 \) is open in \( R^n \).

Therefore \( z(x; y) \) is constant on any \( C(a; \eta | G_1) \) such that \( \eta \in \Omega_1 \) and so by the definition of \( K, G_1 \subset K \) since \( G_1 \) is covered by the family of all the characteristic curves \( C(a; \eta | G_1) \) where \( \eta \in \Omega_1 \). Hence, observing that \( G_1 \) is open in \( R^{n+1} \), we have

\[
F = G \cdot G - K \subset G \cdot G - G_1 \subset G - G_1.
\]

But this is a contradiction since \( (a; b) \in F \cdot G_1 \). Thus Proposition 2 is proved.

7. Domains \( \Omega_2, G_z \). By Proposition 2, \( \bar{F}^0 \) is not empty. Hence we take a point \( b^{(1)} = (b_1^{(1)}, \ldots, b_n^{(1)}) \in \bar{F}^0 \).

Then we can construct a domain \( \Omega_2 \) in \( R^n \) defined by

\[
\eta : \| \eta - b^{(1)} \| < L_2 \quad (L_2 > 0)
\]  (7.1)

such that

\[
\bar{\Omega}_2 \subset \bar{F}. \quad (7.2)
\]

Evidently

\[
\bar{\Omega}_2 \subset \Omega_1 \quad (7.3)
\]

since \( \bar{F} \subset \Omega_1 \) by the definition of \( \bar{F} \).

We denote by \( G_z \) the portion of \( Q_1 \) covered by the family of all the characteristic curves \( C(a; \eta | Q_1) \) where \( \eta \in \Omega_2 \). In the same way as in the case of \( G_1 \), we easily prove that \( G_z \) is open in \( R^{n+1} \) and so \( G_1[x] \)
is open in $R^n$ for any $x$ in the interval $|x-a|<L_1$. Also $C(\xi; \eta | G_2)$ where $(\xi; \eta) \in G_2$, is defined just for the interval $|x-a|<L_1$. Evidently by (7.3) and the definition of $G_1$, $G_2$,

$$G_2 \subset G_1 \subset Q_1 \subset Q.$$ 

**Proposition 3.** $C(\xi; \eta | G_2)$ where $(\xi; \eta) \in G_2$, has at least one point in common with $F \cap G_2$.

Proof. If $(\xi; \eta) \in G_2$ and $\eta^{(0)} = \varphi(a, \xi; \eta | G_2)$, then $\eta^{(0)} \in \Omega_2$ and $C(\xi; \eta | G_2) = C(a; \eta^{(0)} | G_1)$ by the definition of $G_2$. Then by (7.2) and the definition of $\tilde{F}$, Proposition 3 follows.

**Proposition 4.** If $|\xi-a|<L_1$, then

$$G_1[\xi] \supset (G_2[\xi])^c, \text{ the boundary in } R^n \text{ of } G_2[\xi].$$

(7.4)

Further if $|\xi-a|<L_1$ and $\eta \in (G_2[\xi])^c$, then

$$\varphi(a, \xi; \eta | G_1) \in \Omega_2^c, \text{ the boundary in } R^n \text{ of } \Omega_2.$$

Proof. We consider the continuous mapping $\mathcal{A}_\xi$ of $\Omega_1$ onto $G_1[\xi]$ defined by

$$\eta^{(0)} \rightarrow \varphi(\xi, a; \eta^{(0)} | G_1).$$

That $\mathcal{A}_\xi$ maps $\Omega_1$ onto $G_1[\xi]$ follows from the definition of $G_1$.

By the properties of $C(\xi; \eta | G_1)$ and $\varphi_\xi(x, \xi; \eta | G_1)$ as stated in §1.2, we easily see that $\mathcal{A}_\xi$ is one to one and bicontinuous and $\mathcal{A}_\xi^{-1}$ is represented by

$$\eta \rightarrow \varphi(a, \xi; \eta | G_1)$$

(7.5)

We have

$$\mathcal{A}_\xi(\Omega_2) = G_1[\xi]$$

(7.6)

by (7.3) and the definition of $G_2$. Hence again taking account of (7.3), by the continuity of $\mathcal{A}_\xi$ we have $\mathcal{A}_\xi(\Omega_2) \subset G_1[\xi]$.

On the other hand, since $\Omega_2$ is closed and bounded in $R^n$, its continuous image $\mathcal{A}_\xi(\Omega_2)$ is closed in $R^n$ and so, taking account of (7.6), we have $\mathcal{A}_\xi(\Omega_2) \supset G_1[\xi]$.

Therefore $\mathcal{A}_\xi(\Omega_2) = G_1[\xi]$. Hence by (7.6), (7.3), and $\mathcal{A}_\xi(\Omega_1) = G_1[\xi]$, observing that $\Omega_2, G_1[\xi]$ are both open in $R^n$, we get $\mathcal{A}_\xi(\Omega_2) = (G_2[\xi])^c \subset G_1[\xi]$.

From this, taking account of the representation (7.5) of $\mathcal{A}_\xi^{-1}$, Proposition 4 follows.
8. Classes \( S_\lambda, \ S^{(\lambda)} \) and Operations \( T_\lambda, \ T, \ T^{(\lambda)} \). We take two points \((x'; y'), \ (x'; y')\) of \(R^{n+1}\) with the same \(x\) coordinate such that \((x'; y') \in F \cdot G_2, \ (x'; y') \in G_2\) and further \((x', y'_1, \ldots, y'_{\lambda-1}, y'_{\lambda}, y'_{\lambda+1}, \ldots, y'_n) \in G_2\). In the following, we denote the class of all such ordered pairs \((x'; y'), \ (x'; y')\) of points of \(R^{n+1}\) by \(S_\lambda \ (\lambda = 1, \ldots, n)\). If we put \(y' = (y'_1, \ldots, y'_{\lambda-1}, y'_{\lambda}, y'_{\lambda+1}, \ldots, y'_n)\), then \((x'; y') \in G_2\).

Now there is the nearest \(x\) to \(x'\) in the interval \(|x-a| < L_1\) such that either \((x, \varphi(x, x'; y'|G_2)) \in F \cdot G_2\) or \((x, \varphi(x, x'; y'|G_2)) \in F \cdot G_2\), since by Proposition 3 each of the continuous curves \(C(x'; y'|G_2)\) and \(C(x'; y'|G_2)\) which are just defined for the interval \(|x-a| < L_1\) and are contained in \(G_2\), has at least one point in common with \(F \cdot G_2\) which is closed in \(G_2\). We denote such \(x\) by \(x''\). If incidently two such \(x\) exist, then we take as \(x''\) the one on the right side of \(x'\).

Now we distinguish two cases;

i) If \((x'', \varphi(x'', x'; y'|G_2)) \in F \cdot G_2\), then we put
\[ y'' = \varphi(x'', x'; y'|G_2) \quad \text{and} \quad y'' = \varphi(x'', x'; y'|G_2). \]

ii) If \((x'', \varphi(x'', x'; y'|G_2)) \notin F \cdot G_2\) and so by the definition of \(x''\), \((x'', \varphi(x'', x'; y'|G_2)) \in F \cdot G_2\), then we put
\[ y'' = \varphi(x'', x'; y'|G_2) \quad \text{and} \quad y'' = \varphi(x'', x'; y'|G_2). \]

In any case, \((x'''; y'') \in F \cdot G_2\) and \((x'''; y'') \in G_2\).

We denote by \(T_\lambda \ (\lambda = 1, \ldots, n)\) the above operation which assigns to every \((ordered)\) pair \({(x', y')\} \in S_\lambda, \ (ordered)\) pair \({(x', y')\} \in S_\lambda\) with the same \(x\) coordinate such that \((x'; y'') \in F \cdot G_2\) and \((x'''; y'') \in G_2\). Also we write \(T_\mu\) again on \({(x'''; y')\} \in S_\mu\), we can apply \(T_\mu\) again on \({(x'''; y')\} \in S_\mu\).}

**Proposition 5.** If \((x'; y'), \ (x'; y')\) \(\in S_\lambda\) and if we put \((x'''; y''), \ (x'''; y''), \ z' = z(x'; y'), \ z' = z(x'; y'), \ z'' = z''(x'''; y''), \ \) and \(z'' = z''(x'''; y'')\), then
\[
|z' - z'| \leq |z'' - z''| + N \|y' - y'\| \tag{1.1}
\]

**Proof.** We put \(y' = (y'_1, \ldots, y'_{\lambda-1}, y'_{\lambda}, y'_{\lambda+1}, \ldots, y'_n)\). Then \((x'; y') \in F \cdot G_2 \subset F \cdot Q\) and \((x'; y') \in G_2 \subset Q\) since \({(x'; y'), \ (x'; y')} \in S_\lambda, \ (x'; y') \in S_\lambda\). Hence by Proposition 1 (5.1), if we put \(z' = z(x'; y')\)
\[
|z' - z' \leq N \|y' - y'\| \leq N \|y' - y'\|. \tag{2.1}
\]
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\[
\begin{cases}
    y'' = \varphi(x', x'; y'|G_2) \\
    \bar{y}'' = \varphi(x'', x'; y'|G_2)
\end{cases}
\text{or}
\begin{cases}
    y'' = \varphi(x'', x'; y'|G_2) \\
    \bar{y}'' = \varphi(x', x'; y'|G_2)
\end{cases}  \tag{8.3}
\]

On the other hand, each of \( C(x'; y'|G_2) \) and \( C(x'; y'|G_2) \) has no point in common with \( F \) for the interval \( x'<x<x'' \) or \( x''<x<x' \) by the definition of \( T_\lambda \) so that they are contained in \( K \) for the interval \( x''<x<x' \) or \( x'<x<x'' \). Therefore by Lemma 1 and (8.3)

\[
\begin{cases}
    z'' = z(x''; y'') = z(x'; y') = \bar{z}' \\
    \bar{z}'' = z(x''; y'') = z(x'; y') = \bar{z}'
\end{cases}
\text{or}
\begin{cases}
    z'' = z(x''; y'') = z(x'; y') = \bar{z}' \\
    \bar{z}'' = z(x''; y'') = z(x'; y') = \bar{z}'
\end{cases}
\]

so that

\[ |\bar{z}'' - z''| = |\bar{z}' - z'|. \tag{8.4} \]

By (8.4), (8.2) we have

\[ |\bar{z}' - z'| \leq |\bar{z}' - \bar{z}''| + |\bar{z}'' - z'| \leq |\bar{z}'' - z''| + N \|y' - y''\|, \]

q.e.d.

We denote by \( T_\mu \cdot T_\lambda \) the operation which assigns to a pair \( \{(x'; y'), (x'; y')\} \) of points of \( R^{n+1} \), the pair \( T_\mu \{T_\lambda \{(x'; y'), (x'; y')\}\} \) of points of \( R^{n+1} \) if

\[ \{(x'; y'), (x'; y')\} \in S_\lambda \quad \text{and} \quad T_\lambda \{(x'; y'), (x'; y')\} \in S_\mu; \]

and similarly for products of any number of operations \( T_\lambda (\lambda = 1, \cdots, n) \).

We put \( T = T_n \cdot T_{n-1} \cdots T_2 \cdot T_1 \) and \( T^m = T \cdot T \cdots T \) (\( T^0 = \text{identity operator} \)) for any non-negative integer \( m \) and \( T^{(l)} = T_\nu \cdot T_{\nu-1} \cdots T_2 \cdot T_1 \cdot T^m \) for any non-negative integer \( l \) if \( l = mn + \nu, \ 0 \leq \nu \leq n-1 \) and \( m, \nu = \text{non-negative integer} \) if \( \nu = 0, T^{(0)} = T^m \).

We denote by \( S^{(l)} \) (\( l > 0 \)) the class of all the pairs \( \{(x'; y'), (x'; y')\} \) of points of \( R^{n+1} \) on which we can apply the operation \( T^{(l)} \) (\( l > 0 \)) and by \( S^{(0)} \) the class of all the pairs \( \{(x'; y'), (x'; y')\} \) such that \( (x'; y') \in F \cdot G_2 \) and \( (x'; y') \in G_2 \). We regard \( S^{(0)} \) as the domain of definition of the identity operator \( T^{(0)} = T^0 \).

In the following, we put

\[ M_i = \exp \left( 2n\lambda M_i L_i \right) - 1 \tag{8.5} \]
\[ M_2 = \exp \left( 2nM_2 L_i \right) \tag{8.6} \]
then by (6.3)

\[ 1 > M_i > 0. \]  

(8.7)

Also

\[ M_2 > 1. \]  

(8.8)

**Proposition 6.** If \( \{(x^{(0)}; y^{(0)}), (x^{(0)}; y^{(0)})\} \in S^{(l)} \) where \( l = mn + \nu \), \( n-1 \geq \nu \geq 0 \) and \( m, \nu \) = non-negative integer, and if we put \( T^{(l)} \{(x^{(0)}; y^{(0)}), (x^{(0)}; y^{(0)})\} = \{(x^{(l)}; y^{(l)}), (x^{(l)}; y^{(l)})\} \), then

\[ \| \tilde{y}^{(l)} - y^{(l)} \| \leq (M_i + 1)M_1^n \| y^{(0)} - y^{(0)} \| \]  

(8.9)

Proof. In the following lines, we shall prove by induction on \( l \) more precise results,

\[ \| \tilde{y}^{(l)} - y^{(l)} \| \leq M_2^\nu M_1^n \| y^{(0)} - y^{(0)} \| \]  

(8.10)

and

\[ |\tilde{y}_i^{(l)} - y_i| \leq \frac{1}{n} (M_2^\nu - M_2^{\nu-1})M_1^n \| y^{(0)} - y^{(0)} \| \]  

(8.11)

whenever

\[ \{(x^{(0)}; y^{(0)}), (x^{(0)}; y^{(0)})\} \in S^{(l)} \quad (l = mn + \nu). \]

(8.9) follows from (8.10) since \( M_2 < M_i + 1 \) by \( n-1 \geq \nu \geq 0 \) and (8.5), (8.6).

For \( l = 0 \), \( T^{(0)} = T^{(0)} = \text{identity operator} \) and \( m = \nu = 0 \). Hence here (8.10) and (8.11) are trivial.

Now we assume that (8.10), (8.11) are true for \( l = l' = mn' + \nu' \) and let \( \{(x^{(0)}; y^{(0)}), (x^{(0)}; y^{(0)})\} \in S^{(l'+1)} \subseteq S^{(l)} \). Then \( \{(x^{(0)}; y^{(0)}), (x^{(0)}; y^{(0)})\} \in S^{(l'+1)} \) since \( T^{(l'+1)} = T_{l'+1} \cdot T^{(0)}. \) Also \( \{(x^{(0)}; y^{(0)}), (x^{(0)}; y^{(0)}), (x^{(0)}; y^{(0)})\} = T^{(l'+1)} \{(x^{(0)}; y^{(0)}), (x^{(0)}; y^{(0)}), (x^{(0)}; y^{(0)})\} \).

Then by the definition of \( T^{(l'+1)} \), (6.2), Lemma 3 (4.2) and (8.6), taking account of \( |x^{(l'+1)} - x^{(l')}| < 2L_1 \), we have

\[ \| \tilde{y}^{(l'+1)} - y^{(l'+1)} \| \leq \left( \sum_{\mu=1}^{n} |y_{i,\mu}^{(l') - y_{i,\mu}^{(l')}}| + \sum_{\mu=n+1}^{n} |y_{i,\mu}^{(l') - y_{i,\mu}^{(l')}}| \right) \times \exp (nM_0 |x^{(l'+1)} - x^{(l')}|) \leq \| \tilde{y}^{(l')} - y^{(l')} \| \exp (2nM_0 L_1) = M_2 \| y^{(0)} - y^{(0)} \| \]  

and so by (8.10) for \( l = l' \),

\[ \| \tilde{y}^{(l'+1)} - y^{(l'+1)} \| \leq M_2^{l'+1}M_1^n \| y^{(0)} - y^{(0)} \| . \]  

(8.12)

Also by the definition of \( T^{(l'+1)} \), (6.2), Lemma 3 (4.3) and (8.6), we have
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\[ |y(x^{(l+1)}) - y(x^{(l+1)})| \leq |\sum_{\mu=1}^{n} y(x^{(l)} - y(x^{(l)})| + \frac{1}{n} (\sum_{\mu=1}^{n} |\sum_{\nu=\nu_{\mu}+1}^{n} y(x^{(l)}) - y(x^{(l)})|) \{\exp(nM_{\nu} |x^{(l+1)} - x^{(l)})| - 1\} \leq |y(x^{(l)}) - y(x^{(l)})|
\]

and so by (8.10) and (8.11) for \( l = l' \),

\[ |y(x^{(l+1)}) - y(x^{(l+1)})| \leq \frac{1}{n} (M_{2}^{l} - M_{2}^{l-1})M_{l}^{w} \|y^{(0)} - y^{(0)}\| + \frac{1}{n} (M_{2}^{l} - M_{2}^{l-1})M_{l}^{w} \|y^{(0)} - y^{(0)}\|
\]

for \( \nu \geq \lambda \geq 1 \).

Again by the definition of \( T_{\nu+1} \), (6.2) and Lemma 3 (4.3), we have

\[ |\sum_{\mu=1}^{n} y(x^{(l)}) - y(x^{(l)})| \leq \frac{1}{n} (\sum_{\mu=1}^{n} |\sum_{\nu=\nu_{\mu}+1}^{n} y(x^{(l)}) - y(x^{(l)})|) \{\exp(nM_{\nu} |x^{(l+1)} - x^{(l)})| - 1\}
\]

and so by (8.10) for \( l = l' \),

\[ |y(x^{(l+1)}) - y(x^{(l+1)})| \leq \frac{1}{n} (M_{2}^{l} - M_{2}^{l-1})M_{l}^{w} \|y^{(0)} - y^{(0)}\| . \]

If \( n - 2 \geq \nu \geq 0 \), then (8.12), (8.13), (8.14) prove (8.10), (8.11) for \( l = l' + 1 \) since \( l' + 1 = n \mu + \nu + 1 \), \( n - 1 \geq \nu + 1 \geq 1 \) in this case.

If \( \nu = n - 1 \), then \( l' + 1 = n (\mu + 1) \). In this case, by (8.13), (8.14), we obtain

\[ \|y^{(l+1)} - y^{(l+1)}\| \leq \frac{1}{n} \{\sum_{\nu=\nu_{\mu}+1}^{n} (M_{\nu}^{\mu}) \|y^{(0)} - y^{(0)}\| \leq (M_{\nu}^{\mu}) \|y^{(0)} - y^{(0)}\|
\]

since \( M_{\nu}^{\mu} \geq 1 \) for \( n \geq \mu \geq 1 \) and \( M_{\nu} = M_{\nu}^{\mu} - 1 \) by (8.8), (8.5), (8.6). This proves (8.10) for \( l = l' + 1 \) in the case \( \nu = n - 1 \). In this case (8.11) for \( l = l' + 1 \) is trivial, since then there is no \( \lambda \) for which it should be established.

Thus (8.10), (8.11) are proved for any non-negative integer \( l \) and so the proof of Proposition 6 is completed.

9. Further on the operation \( T^{(l)} \).

In the following we put
By (8.7), (8.8), $M_3$ is positive.

**Proposition 7.** If $\{(x^{(0)}; y^{(0)}), (x^{(0)}; y^{(0)})\} \in S^{(I)}$ for a non-negative integer $l$ and if we put
\[
\{\{x^{(p)}; y^{(p)}\}, (x^{(p)}; y^{(p)})\} = T^{(p)}\{\{x^{(0)}; y^{(0)}\}, (x^{(0)}; y^{(0)})\},
\]
\[
\eta^{(p)} = \varphi(a, x^{(p)}; y^{(p)}|G_3), \quad \bar{\eta}^{(p)} = \varphi(a, x^{(p)}; y^{(p)}|G_3)
\]
for $p = 0, 1, \ldots, l$,
then
\[
\|\bar{\eta}^{(p)} - \eta^{(p)}\| \leq M_3 \|\bar{\eta}^{(0)} - \eta^{(0)}\|.
\]
(9.2)

**Proof.** By the definition of $T^{(p)}$,
\[
y^{(p)} = \varphi(x^{(p)}, x^{(p-1)}; y^{(p-1)}|G_3) \quad \text{or} \quad \bar{y}^{(p)} = \varphi(x^{(p)}, x^{(p-1)}; y^{(p-1)}|G_3)
\]
for $p = 1, \ldots, l$. Hence
\[
\eta^{(p)} = \bar{\eta}^{(p-1)} \quad \text{or} \quad \bar{\eta}^{(p)} = \bar{\eta}^{(p-1)} \quad p = 1, \ldots, l,
\]
so that
\[
\|\bar{\eta}^{(p)} - \bar{\eta}^{(p-1)}\| = \|\bar{\eta}^{(p)} - \eta^{(p)}\| \quad \text{or} \quad \|\bar{\eta}^{(p)} - \bar{\eta}^{(p-1)}\| = 0
\]
for $p = 1, \ldots, l$.
Therefore
\[
\|\bar{\eta}^{(p)} - \eta^{(p)}\| \leq \sum_{p=1}^{l} \|\bar{\eta}^{(p)} - \bar{\eta}^{(p-1)}\| + \|\bar{\eta}^{(0)} - \eta^{(0)}\| \leq \sum_{p=0}^{l} \|\bar{\eta}^{(p)} - \eta^{(p)}\|.
\]
(9.3)

By (6.2) and Lemma 3 (4.2), we obtain
\[
\|\bar{\eta}^{(p)} - \eta^{(p)}\| \leq \|\bar{y}^{(p)} - y^{(p)}\| \exp(nM_3|a - x^{(p)}|)
\]
\[
\leq \|\bar{y}^{(p)} - y^{(p)}\| \exp(nM_3L_1) = \sqrt{M_3} \|\bar{y}^{(p)} - y^{(p)}\|
\]
for $p = 0, \ldots, l$,
(9.4)
since $|a - x^{(p)}| < L_1$ and $\sqrt{M_3} = \exp(nM_3L_1)$ by (8.6). Similarly by (6.2) and Lemma 3 (4.2), we have
\[
\|\bar{y}^{(0)} - y^{(0)}\| \leq \|\bar{\eta}^{(0)} - \eta^{(0)}\| \exp(nM_3|x^{(0)} - a|) \leq \sqrt{M_3} \|\bar{\eta}^{(0)} - \eta^{(0)}\|.
\]
(9.5)
On the other hand, by Proposition 6, if $p = qu + s$, $n - 1 \geq s \geq 0$
$q, s = \text{non-negative integer}$, then
\[
\|\bar{y}^{(p)} - y^{(p)}\| \leq (M_1 + 1)M_l \|\bar{y}^{(0)} - y^{(0)}\|
\]
for $p = 0, \ldots, l$.
(9.6)
By (9.4), (9.5), (9.6), we obtain
\[
\|\bar{\eta}^{(p)} - \eta^{(p)}\| \leq M_3(M_1 + 1)M_l \|\bar{\eta}^{(0)} - \eta^{(0)}\|
\]
for $p = 0, \ldots, l$.
(9.7)
By (9.3), (9.7), taking account of (8.7) and (9.1), if \( l = nm + \nu \), \( n - 1 \geq \nu \geq 0 \) and \( m, \nu \) = non-negative integer, then we get

\[
\| \tilde{\eta}^{(I)} - \eta^{(I)} \| \leq \sum_{p=0}^{nm-1} \| \tilde{\eta}^{(p)} - \eta^{(p)} \| + \sum_{p=nm}^{nm+\nu} \| \tilde{\eta}^{(p)} - \eta^{(p)} \|
\]
\[
\leq nM_4(M_4+1)(\sum_{q=0}^{m-1} M_q^i) \| \tilde{\eta}^{(0)} - \eta^{(0)} \| + (\nu + 1)M_4(M_4+1)M_q^i \| \tilde{\eta}^{(0)} - \eta^{(0)} \|
\]
\[
\leq nM_4(M_4+1)(\sum_{q=0}^{m-1} M_q^i) \| \tilde{\eta}^{(0)} - \eta^{(0)} \| = nM_4(1+M_4)(1-M_i)^{-1} \| \tilde{\eta}^{(0)} - \eta^{(0)} \|
\]
\[
= M_i \| \tilde{\eta}^{(0)} - \eta^{(0)} \| , \quad \text{q.e.d.}
\]

In the following we put

\[
M_i = nN(1+M_4)(1-M_i)^{-1} \quad (9.8)
\]

By (8.7), \( M_i \) is positive.

**Proposition 8.** Let the premises and the notations be the same as in the proposition 7 and further let

\[
\tilde{z}^{(p)} = z(x^{(p)}; y^{(p)}) \quad \text{and} \quad \tilde{z}^{(p)} = z(x^{(p)}; y^{(p)}) \quad \text{for} \quad p = 0, \ldots, l,
\]

then

\[
| \tilde{z}^{(0)} - z^{(0)} | \leq M_i \| \tilde{y}^{(0)} - y^{(0)} \| + | \tilde{z}^{(I)} - z^{(I)} | . \quad (9.9)
\]

**Proof.** We use the same notations as in the proof of Proposition 7. If \( l = 0 \), (9.9) is obvious by \( M_i > 0 \). Hence we suppose \( l > 0 \) in the following lines.

By Proposition 5,

\[
| \tilde{z}^{(p)} - z^{(p)} | - | \tilde{z}^{(p+1)} - z^{(p+1)} | \leq N \| \tilde{y}^{(p)} - y^{(p)} \| \quad \text{for} \quad p = 0, \ldots, l-1 , \quad (9.10)
\]

Adding the inequalities (9.10) for \( p = 0, \ldots, l-1 \) side by side, we obtain

\[
| \tilde{z}^{(0)} - z^{(0)} | - | \tilde{z}^{(I)} - z^{(I)} | \leq N \sum_{p=0}^{l-1} \| \tilde{y}^{(p)} - y^{(p)} \| . \quad (9.11)
\]

On the other hand, by Proposition 6,

\[
\| \tilde{y}^{(p)} - y^{(p)} \| \leq (M_i + 1)M_i \| \tilde{y}^{(0)} - y^{(0)} \| \quad \text{for} \quad p = 0, \ldots, l ,
\]

where \( q \) is determined for \( p \) as in Proposition 7 (9.6). Hence \( m, \nu \) being determined for \( l \) as in the definition of \( T^{(l)} \),

\[
\sum_{p=0}^{l-1} \| \tilde{y}^{(p)} - y^{(p)} \| \leq n(M_i + 1)(\sum_{q=0}^{m-1} M_q^i) \| \tilde{y}^{(0)} - y^{(0)} \|
\]
\[
+ \nu(M_i + 1)M_q^i \| \tilde{y}^{(0)} - y^{(0)} \| \leq n(M_i + 1)(\sum_{q=0}^{m-1} M_q^i) \| \tilde{y}^{(0)} - y^{(0)} \|
\]
\[
= n(1+M_4)(1-M_i)^{-1} \| \tilde{y}^{(0)} - y^{(0)} \| , \quad (9.12)
\]
since \( n-1 \geq \nu \geq 0 \) and \( 0 < M_i < 1 \) by (8.7).

By (9.11), (9.12), taking account of (9.8), we obtain finally
\[
|\tilde{z}^{(0)} - \tilde{x}^{(0)}| - |\tilde{z}^{(1)} - \tilde{x}^{(1)}| \leq n N (1 + M_i) (1 - M_i)^{-1} \| y^{(0)} - y^{(0)} \|
\leq M_i \| y^{(0)} - y^{(0)} \|
\]
q.e.d.

10. Domain \( \Omega_3, G_3 \). We put
\[
M_5 = 1 + 2 (M_1 + 1) M_2 + 2 M_3.
\]
(10.1)

By (8.7), (8.8), (9.1), \( M_5 \geq 1 \). Hence if we take a positive number \( L_3 \) such that
\[
L_3 M_5 < L_2,
\]
(10.2)
then
\[
L_3 < L_2.
\]
(10.3)

Now we take a domain \( \Omega_3 \) in \( R^n \) defined by
\[
\eta : \| \eta - b^{(1)} \| < L_3.
\]
(10.4)

Then by (7.1), (10.3), (10.4), (7.3),
\[
\Omega_3 \subset \Omega_2 \subset \Omega_1
\]
(10.5)

We denote by \( G_3 \) the portion of \( Q \) covered by the family of all the characteristic curves \( C(a; \eta | Q) \) where \( \eta \in \Omega_3 \). In the same way as in the cases of \( G_1, G_2 \), we easily prove that \( G_3 \) is open in \( R^{n+1} \), and \( C(\xi; \eta | G_3) \) where \( (\xi; \eta) \in G_3 \), is defined just for the interval \( |x - a| < L_3 \). By (10.5) and the definitions of \( G_1, G_2, G_3 \),
\[
G_3 \subset G_2 \subset G_1 \subset Q_1.
\]

**Proposition 9.** If \( (x^{(0)}; y^{(0)}) \in F \cdot G_3 \) and \( (x^{(0)}; y^{(0)}) \in G_3 \), then \( \{(x^{(0)}; y^{(0)}), (x^{(0)}; \tilde{y}^{(0)})\} \in S^{(1)} \) for any non-negative integer \( l \).

Proof. We use the same notations as in the Proposition 7 and 8.
We prove Proposition 9 by induction on \( l \). For \( l = 0 \), Proposition 9 is obvious by the definition of \( S^{(0)} \) and \( G_3 \subset G_2 \). We assume that
Proposition 9 is true for $l = l'$. Then if $(x^{(0)}; y^{(0)}) \in F \cdot G_3$ and $(x^{(0)}; \tilde{y}^{(0)}) \in G_3$, the pair $\{ (x^{(v)}; y^{(v)}), (x^{(0)}; \tilde{y}^{(0)}) \} = T^{(v)} \{ (x^{(0)}; y^{(0)}), (x^{(0)}; \tilde{y}^{(0)}) \}$ is uniquely determined and $(x^{(v)}; y^{(v)}) \in F \cdot G_2$ and $(x^{(v)}; \tilde{y}^{(0)}) \in G_2$ by the definition of $T^{(v)}$.

Let $l' = m'n + \nu'$, $n - 1 \geq \nu' \geq 0$ and $m', \nu' = \text{integer}$. Then we put $\tilde{y}^{(v')} = (y_{1}^{(v')}, \ldots, y_{p+1}^{(v')}, y_{p+2}^{(v')}, \ldots, y_{n}^{(v')})$. If $\tilde{y}^{(v')} \in G_2[x^{(v')}]$, that is, $(x^{(v')}; \tilde{y}^{(v')}) \in G_2$, then $\{(x^{(v')}; y^{(v')}), (x^{(v')}; \tilde{y}^{(v')})\} \in S_{v'+1} \text{ so that } \{(x^{(0)}; y^{(0)}), (x^{(0)}; \tilde{y}^{(0)})\} \in S^{(v'+1)}$ and the proof of Proposition 9 is completed.

We suppose therefore, if possible, that $\tilde{y}^{(v')} \notin G_2[x^{(v')}]$. Then there is a point $y^{*} \in (G_2[x^{(v')}])^b$ on the segment of straight line which joins $y^{(v')}$ and $\tilde{y}^{(v')}$ since $y^{(v')} \in G_2[x^{(v')}]$ by $(x^{(v')}; \tilde{y}^{(v')}) \in G_2$.

We can easily see that

$$\| y^{*} - \tilde{y}^{(v')} \| \leq \| y^{(v')} - \tilde{y}^{(v')} \| \leq \| y^{(v')} - \tilde{y}^{(v')} \|,$$

so that by Proposition 6 (8.9)

$$\| y^{*} - \tilde{y}^{(v')} \| \leq (M_1 + 1) M_1^{\nu'} \| y^{(0)} - y^{(0)} \|. \quad (10.6)$$

By Proposition 4, since $y^{*} \in (G_2[x^{(v')}])^b$, $y^{*} \in G_1[x^{(v')}]$, that is, $(x^{(v')}; y^{*}) \in G_1$ and if we put $\eta^{*} = \varphi(a, x^{(v')}; y^{*}|G_1)$, $\eta^{*} \in \Omega_2$. Hence by (7.1)

$$\| \eta^{*} - b^{(1)} \| = L_2. \quad (10.7)$$

By (6.2) and Lemma 3 (4.2), taking account of (8.6) and $|a - x^{(v')}| < L_1$, we have

$$\| \eta^{*} - y^{(v')} \| \leq \| y^{*} - \tilde{y}^{(v')} \| \exp(n M_0 |a - x^{(v')}|)$$

$$\leq \| y^{*} - \tilde{y}^{(v')} \| \exp(n M_0 L_1) = \sqrt{M_2} \| y^{*} - \tilde{y}^{(v')} \|$$

and so by (10.6)

$$\| \eta^{*} - y^{(v')} \| \leq \sqrt{M_2} (M_1 + 1) M_1^{\nu'} \| y^{(0)} - y^{(0)} \|. \quad (10.8)$$

On the other hand, by the definitions of $\eta^{(0)}$, $\eta^{(0)}$, $G_3$ and by $(x^{(0)}; y^{(0)}) \in F \cdot G_2$, $(x^{(0)}; \tilde{y}^{(0)}) \in G_3$, we have

$$\eta^{(0)}, \eta^{(0)} \in \Omega_3,$$

so that by (10.4),

$$\| \eta^{(0)} - b^{(1)} \| < L_3, \quad \| \eta^{(0)} - b^{(1)} \| < L_3. \quad (10.9)$$

Also, as we have seen in Proposition 7 (9.5),

$$\| \tilde{y}^{(0)} - y^{(0)} \| \leq \sqrt{M_2} \| \eta^{(0)} - \eta^{(0)} \|. $$
Hence, by (10.9)
\[ \| \tilde{y}^{(0)} - y^{(0)} \| \leq \sqrt{M_2} \left( \| \tilde{y}^{(0)} - b^{(1)} \| + \| y^{(0)} - b^{(1)} \| \right) < 2L_3 \sqrt{M_2}. \]

From this and (10.8) we get
\[ \| \eta^* - \tilde{y}^{(r)} \| < 2L_3 M_2 (M_1 + 1) M_1^w \]  
(10.10)

Also, by Proposition 7, (9.2) and (10.9), we have
\[ \| \tilde{y}^{(r)} - y^{(0)} \| \leq M_3 \| \tilde{y}^{(0)} - y^{(0)} \| < 2L_3 M_3. \]  
(10.11)

By (10.7), (10.9), (10.10), and (10.11), taking account of (8.7), (10.1), we obtain finally
\[ L_n = \| \eta^* - \tilde{y}^{(r)} \| \leq \| \eta^* - \tilde{y}^{(0)} \| + \| \tilde{y}^{(r)} - y^{(0)} \| + \| y^{(0)} - b^{(1)} \| \]
\[ \leq L_3 \{2M_2 (M_1 + 1) M_1^w + 2M_2 + 1 \} \leq L_3 \{2M_2 (M_2 + 1) + 2M_2 + 1 \} = L_3 M_5. \]

But this contradicts (10.2) and Proposition 9 is completely proved.

**Proposition 10.** Let \( \{(x', y'), (x'' ; y'')\} \) be any pair of points of \( G_3 \) with the same \( x \) coordinate and let
\[ z' = z(x', y'), \quad \tilde{z}' = z(x'' ; y'). \]

Then
\[ |\tilde{z}' - z'| \leq M_3 M_4 \| y' - y'' \| \]  
(10.12)

**Proof.** There is the nearest \( x \) to \( x' \) in the interval \( |x - a| \leq L_1 \) such that either \( (x, \varphi(x, x'; y'(G_3)) \in F \cdot G_3 \) or \( (x, \varphi(x, x'; y''|G_3)) \in F \cdot G_3 \), since by Proposition 3 and \( G_2 \subset G_3 \), each of the continuous curves \( C(x'; y'|G_3) \) and \( C(x'' ; y''|G_3) \) which are defined just for the interval \( |x - a| \leq L_1 \) and are contained in \( G_3 \), has at least one point in common with \( F \cdot G_3 \) which is closed in \( G_3 \). We denote such \( x \) by \( x^{(0)} \). If incidently two such \( x \) exist, we take as \( x^{(0)} \) for example the one on the right side of \( x' \).

Now we distinguish two cases.

i) If \( (x^{(0)}, \varphi(x^{(0)}, x'; y'|G_3)) \in F \cdot G_3 \), then we put
\[ y^{(0)} = \varphi(x^{(0)}, x'; y'|G_3), \quad \tilde{y}^{(0)} = \varphi(x^{(0)}, x'; y'|G_3) \]

ii) If \( (x^{(0)}, \varphi(x^{(0)}, x'; y'|G_3)) \notin F \cdot G_3 \) and so by the definition of \( x^{(0)} \), \( (x^{(0)}, \varphi(x^{(0)}, x'; y'|G_3)) \notin F \cdot G_3 \), then we put
\[ y^{(0)} = \varphi(x^{(0)}, x'; y'|G_3), \quad \tilde{y}^{(0)} = \varphi(x^{(0)}, x'; y'|G_3). \]

In any case \((x^{(0)}; y^{(0)}) \in F \cdot G_3 \) and \((x^{(0)}; y^{(0)}) \in G_3 \).

By the definition of \( x^{(0)} \), each of the characteristic curves \( C(x'; y'|G_3) \)
and $C(x'; y' | G_s)$ has no point in common with $F$ and so is contained in $K$ for the interval $x' < x < x^{(0)}$ or $x^{(0)} < x < x'$. Hence by Lemma 1, if we put \[ z^{(0)} = z(x^{(0)} ; y^{(0)}) \] and \[ z^{(0)} = z(x^{(0)} ; y^{(0)}) \],

\[
\begin{cases}
  z^{(0)} = z' \\
  z^{(0)} = \hat{z}' \\
  z^{(0)} = z'.
\end{cases}
\]

Therefore in any case,

\[
|z' - z'| = |z^{(0)} - z^{(0)}|. \tag{10.13}
\]

By Lemma 3 (4.2) and (6.2), taking account of (8.6), we have

\[
\| z^{(0)} - y^{(0)} \| \leq \| \hat{y}' - y' \| \exp(nM_s |x^{(0)} - x'|)
\leq \| \hat{y}' - y' \| \exp(2nM_s L_s) \leq M_s \| \hat{y}' - y' \| , \tag{10.14}
\]

since $|x^{(0)} - x'| \leq 2L_s$.

Now, by Proposition 9, \{$(x^{(0)} ; y^{(0)})$, $(x^{(0)} ; y^{(0)})$\} $\in S^{(l)}$ for any non-negative integer $l$, since $(x^{(0)} ; y^{(0)}) \in F \cdot G_s$ and $(x^{(0)} ; y^{(0)}) \in G_s$. Hence we put

\[
\{(x^{(l)} ; y^{(l)}), (x^{(l)} ; y^{(l)})\} = T^{(l)}\{(x^{(0)} ; y^{(0)}), (x^{(0)} ; y^{(0)})\},
\]

\[z^{(l)} = z(x^{(l)} ; y^{(l)}), \quad \hat{z}^{(l)} = z(x^{(l)} ; y^{(l)})\]

for any non-negative integer $l$. Then by Proposition 8

\[
|z^{(0)} - z^{(0)}| \leq M_s \| \hat{y}^{(0)} - y^{(0)} \| + |z^{(l)} - z^{(l)}| \tag{10.15}
\]

On the other hand, by Proposition 6, if $l = nm + \nu, n - 1 \geq \nu \geq 0$ and $m, \nu$ integer, then

\[
\| \hat{y}^{(l)} - y^{(l)} \| \leq (M_s + 1)M_s \| \hat{y}^{(0)} - y^{(0)} \| .
\]

Hence observing that $m \to \infty$ as $l \to \infty$ and $0 < M_s < 1$,

\[
\| \hat{y}^{(l)} - y^{(l)} \| \to 0 \quad \text{as} \quad l \to \infty ,
\]

Thus

\[
|z^{(l)} - z^{(l)}| = \| z(x^{(l)} ; y^{(l)}) - z(x^{(l)} ; y^{(l)}) \| \to 0 \quad \text{as} \quad l \to \infty ,
\]

since by the continuity of $z(x ; y)$ on $\tilde{Q} \subset G$, $z(x ; y)$ is uniformly continuous on $\tilde{Q}$ which is closed and bounded in $\mathbb{R}^{n+1}$ and by the definition of $T^{(l)}$, $(x^{(l)} ; y^{(l)})$, $(x^{(l)} ; y^{(l)}) \in G_s \subset Q$ for any non-negative integer $l$.

Therefore letting $l \to \infty$ on the right side of (10.15), we have

\[
|z^{(0)} - z^{(0)}| \leq M_s \| \hat{y}^{(0)} - y^{(0)} \| . \tag{10.16}
\]
By (10.13), (10.14) and (10.16), we obtain finally

$$|\tilde{z}' - z'| \leq M_5 M_4 \| \tilde{y}' - y' \|,$$

q.e.d.

11. Domains $Q'$, $Q''$, $\Omega_4$, $G_4$ and Mapping $\Psi$. Since $G_3$ is open in $R^{n+1}$, and $F \cdot G_3 \neq 0$ by the way of the construction of $G_3$ and Proposition 3, we can take a $(n+1)$-dimensional open parallelepiped $Q'$:

$$|x-a'| < L_4, \ |y_\lambda - b_\lambda'| < L_4(M_5 + 1) \ \lambda = 1, \ldots, n \ (L_4 > 0)$$

such that

$$(a'; b') = (a', b'_1, \ldots, b'_n) \in F \cdot G_3 \ \text{and} \ Q' \subseteq G_3.$$

Evidently $Q' \subseteq G_3 \subseteq Q$.

We denote by $\Omega_4$ the $n$-dimensional open cube

$$\eta: |\eta_\lambda - b_\lambda'| < L_4 \ \lambda = 1, \ldots, n.$$

Then if $\eta \in \Omega_4$,

$$\eta_\lambda + L_4 M_6 \leq b_\lambda' + (M_5 + 1)L_4, \ \eta_\lambda - L_4 M_6 \geq b_\lambda' - (M_5 + 1)L_4 \ \lambda = 1, \ldots, n.$$

Hence the characteristic curves $C(a'; \eta \mid Q')$ where $\eta \in \Omega_4$, are defined just for the interval $|x-a'| < L_4$ since $|f_\lambda| < M_5 \ \lambda = 1, \ldots, n$ on $Q'(\subseteq Q)$ by (6.1).
We denote by \( G_4 \) the portion of \( Q' \) covered by the family of all the characteristic curves \( C(\alpha' ; \eta | Q') \) where \( \eta \in \Omega_4 \). Evidently \( G_4 \subset Q' \subset G_5 \subset Q \).

In the same way as in the cases of \( G_1, G_2 \) and \( G_3 \), we easily prove that \( G_4 \) is open in \( R^{n+1} \) and any characteristic curve \( C(\xi ; \eta | G_4) \) where \( (\xi ; \eta) \in G_4 \) is defined just for \( |x-a'| < L_4 \).

We put

\[
M_5 = N + nM_0M_1 + M_2M_4.
\]

Proposition 11. Let \((x^{(1)} ; y^{(1)})\) and \((x^{(0)} ; y^{(0)})\) be any pair of points of \( G_4 \) and let

\[
z^{(1)} = z(x^{(1)} ; y^{(1)}), \quad z^{(0)} = z(x^{(0)} ; y^{(0)}).
\]

Then

\[
|z^{(1)} - z^{(0)}| \leq M_5(|x^{(1)} - x^{(0)}| + \|y^{(1)} - y^{(0)}\|)
\]

Proof. We denote by \( x^{(2)} \):

(Case I) the nearest \( x \) to \( x^{(1)} \) in the interval \( x^{(0)} < x < x^{(1)} \) or \( x^{(1)} < x < x^{(0)} \) such that \( (x ; \varphi(x, x^{(1)} ; y^{(1)} | G_4)) \in F \), if the portion of \( C(x^{(1)} ; y^{(1)} | G_4) \) for that interval has some points in common with \( F \). Such \( x \) exists in this case since the continuous curve \( C(x^{(1)} ; y^{(1)} | G_4) \) which is defined for the interval \( x^{(0)} < x < x^{(1)} \) or \( x^{(1)} < x < x^{(0)} \), is contained in \( G_4 \) and \( F \cdot G_4 \) is closed in \( G_4 \).

(Case II) the number \( x^{(0)} \), if the portion of \( C(x^{(1)} ; y^{(1)} | G_4) \) for the interval \( x^{(0)} < x < x^{(1)} \) or \( x^{(1)} < x < x^{(0)} \) has no point in common with \( F \).

We put \( y^{(2)} = \varphi(x^{(2)} ; x^{(1)} ; y^{(1)} | G_4), \quad z^{(2)} = z(x^{(2)} ; y^{(2)}). \) In Case I, \((x^{(2)} ; y^{(2)}) \in F \cdot G_4 \) and in Case II, \((x^{(2)} ; y^{(2)}) = (x^{(0)} ; \varphi(x^{(0)} ; x^{(1)} ; y^{(1)} | G_4)) \in G_4 \).

Then in both Cases, by Lemma 1,

\[
z^{(1)} = z^{(2)},
\]

(11.3)
since the portion of \( C(x^{(1)} ; y^{(1)} | G_4) \) for the open interval \( x^{(1)} < x < x^{(2)} \) or \( x^{(2)} < x < x^{(1)} \) has no point in common with \( F \) and so is contained in \( K \) by the definition of \( x^{(2)} \).

Also by (6.1), (2.2), observing that \( C(x^{(1)} ; y^{(1)} | G_4) \) is contained in \( Q \), we have

\[
|y^{(2)} - y^{(1)}| = |\varphi_\lambda(x^{(2)} ; x^{(1)} ; y^{(1)} | G_4) - \varphi_\lambda(x^{(1)} ; x^{(1)} ; y^{(1)} | G_4)|
\]

\[
\leq M_6 |x^{(2)} - x^{(1)}| \quad \lambda = 1, \ldots, n.
\]

Hence

\[
\|y^{(2)} - y^{(1)}\| \leq \sum_{\lambda=1}^{n} |y^{(2)}_{\lambda} - y^{(1)}_{\lambda}| \leq nM_6 |x^{(2)} - x^{(1)}| \leq nM_6 |x^{(1)} - x^{(0)}|,
\]

since \( x^{(0)} < x^{(2)} < x^{(1)} \) or \( x^{(1)} < x^{(2)} < x^{(0)} \) by the definition of \( x^{(2)} \).
Therefore we have
\[ \| y^{(2)} - y^{(6)} \| \leq \| y^{(2)} - y^{(1)} \| + \| y^{(1)} - y^{(6)} \| \]
\[ \leq nM_0 \| x^{(1)} - x^{(6)} \| + \| y^{(1)} - y^{(6)} \|. \]  \hspace{1cm} (11.4)

Now
\[ (x^{(6)}; y^{(2)}) \in Q' \subset G_s \subset Q, \]
since \((x^{(6)}; y^{(2)})\) is \(G_s \subset Q'\) and \((x^{(2)}; y^{(2)})\) is \(G_s \subset Q'\) in both Cases. We put
\[ z^{(3)} = z(x^{(6)}; y^{(2)}). \]

Then we have
\[ |z^{(3)} - z^{(2)}| = |z(x^{(6)}; y^{(2)}) - z(x^{(2)}; y^{(2)})| \leq N|x^{(2)} - x^{(6)}| \]
in Case I, by Proposition 1 and \((x^{(6)}; y^{(2)}) \in Q, (x^{(2)}; y^{(2)}) \in F \cdot G_s \subset F \cdot Q\), and in Case II, simply as \(x^{(2)} = x^{(6)}\). Hence, by (11.3), we get
\[ |z^{(3)} - z^{(1)}| = |z^{(3)} - z^{(2)})| \leq N|x^{(2)} - x^{(6)}| \leq N|x^{(1)} - x^{(6)}|, \] \hspace{1cm} (11.5)
since \(x^{(6)} \leq x^{(2)} \leq x^{(1)}\) or \(x^{(1)} \leq x^{(2)} \leq x^{(6)}\) by the definition of \(x^{(2)}\).

Also, by Proposition 10 (10.12), since \((x^{(6)}; y^{(2)}) \in G_s, (x^{(6)}; y^{(6)}) \in G_s\) in both Cases, we have
\[ |z^{(3)} - z^{(6)}| \leq M_2M_4 \| y^{(2)} - y^{(6)} \|. \]
Hence by (11.4), we get
\[ |z^{(3)} - z^{(2)}| \leq nM_0M_2M_4M_5 \| x^{(1)} - x^{(6)} \| + M_2M_4 \| y^{(1)} - y^{(6)} \|. \] \hspace{1cm} (11.6)

By (11.5), (11.6), taking account of (11.1), we obtain finally
\[ |z^{(1)} - z^{(6)}| \leq |z^{(3)} - z^{(1)}| + |z^{(3)} - z^{(6)}| \]
\[ \leq (N + nM_0M_2M_4) \| x^{(1)} - x^{(6)} \| + M_2M_4 \| y^{(1)} - y^{(6)} \| \]
\[ \leq M_5(|x^{(1)} - x^{(6)}| + \| y^{(1)} - y^{(6)} \|), \] \hspace{1cm} q.e.d.

We denote by Q'' the \((n + 1)\)-dimensional open cube defined by
\[ (x; \eta): \| x - a' \| < L_4, \]
\[ \| \eta - b' \| < L_4 \quad \lambda = 1, \ldots, n. \]
We put \( \chi_{\lambda}(x; \eta) = \phi_{\lambda}(x, a'; \eta | G_\lambda) \), \( \lambda = 1, \ldots, n \). Then \( \chi_{\lambda}(x; \eta) \) are defined and continuous on Q'' and have continuous partial derivatives with respect to all their arguments on Q'', by the corresponding properties of \( \phi_{\lambda}(x, \xi; \eta | G_\lambda) \).

We denote by \( \mathcal{S} \) the continuous
mapping of $Q''$ onto $G_4$:

$$(x; \eta) \rightarrow (x; \chi(x; \eta)).$$

That $\mathcal{A}$ maps $Q''$ onto $G_4$, follows from the definition of $G_4$.

By the properties of $C(\xi; \eta | G_4)$ and $\varphi_\lambda(x, \xi; \eta | G_4)$ as stated in §1.2, we easily see that $\mathcal{A}$ is one to one and bicontinuous, and $\mathcal{A}^{-1}$ is represented by

$$(x; y) \rightarrow (x; \gamma(x; y)),$$

if we put $\gamma_\lambda(x; y) = \varphi_\lambda(a'; x; y | G_4) \lambda = 1, \ldots, n$ for $(x; y) \in G_4$.

Further $\gamma_\lambda(x; y)$ have continuous partial derivatives with respect to all their arguments by the corresponding properties of $\varphi_\lambda(x, \xi; \eta | G_4)$.

From this, we can easily prove that $\mathcal{A}^{-1}$ maps any null set in $G_4$ onto a null set in $Q''$.

Thus we have

**Proposition 12.** The mapping $\mathcal{A}$ of $Q''$ onto $G_4$ is one to one and bicontinuous, and $\mathcal{A}^{-1}$ maps any null set in $G_4$ onto a null set in $Q''$.

12. Completion of the proof. By Proposition 11, we have

$${\lim}_{(x; y) \rightarrow (x^{(0)}; y^{(0)})} \left| \frac{z(x; y) - z(x^{(0)}; y^{(0)})}{|x - x^{(0)}| + \|y - y^{(0)}\|} \right| \leq M_6, \quad (12.1)$$

whenever $(x^{(0)}; y^{(0)}) \in G_4$. Hence $z(x; y)$ is totally differentiable almost everywhere in $G_4$, by a theorem of Rademacher on almost everywhere total differentiability\(^9\). Also, by Proposition 12, $\mathcal{A}^{-1}$ maps any null set in $G_4$ onto a null set in $Q''$. Therefore, if we write $\zeta(x; \eta) = z(x; \chi(x; \eta))$, and $y_\lambda = \chi_\lambda(x; \eta) \lambda = 1, \ldots, n$ for $(x; \eta) \in Q''$, we obtain

$$\frac{\partial}{\partial x} \zeta(x; \eta) = \frac{\partial z}{\partial x}(x; y) + \sum_{\mu=1}^{n} \frac{\partial z}{\partial y_\mu}(x; y) \frac{\partial \chi_\mu}{\partial x}(x; \eta) \quad (12.2)$$

for almost all $(x; \eta)$ of $Q''$.

Since $\chi_\lambda(x; \eta) = \varphi_\lambda(a'; x; \eta | G_4)$ for $(x; \eta) \in Q''$, we obtain by (2.2),

$$\frac{\partial}{\partial x} \chi_\lambda(x; \eta) = f_\lambda(x; \chi(x; \eta)) = f_\lambda(x; y) \quad \lambda = 1, \ldots, n \quad (12.3)$$

for $(x; \eta) \in Q''$. Substituting this into (12.2), we get

$$\frac{\partial}{\partial x} \zeta(x; \eta) = \frac{\partial z}{\partial x}(x; y) + \sum_{\mu=1}^{n} f_\mu(x; y) \frac{\partial z}{\partial y_\mu}(x; y) \quad (12.4)$$

---


for almost all \((x; \eta)\) of \(Q''\).

Since by assumption, \(z(x; y)\) satisfies (2.1) almost everywhere in \(G \supset G_4\) and by Proposition 12, \(R^{-1}\) maps any null set in \(G_4\) onto null set in \(Q''\), the right side of (12.4) regarded as a function of \((x; \eta)\), vanishes almost everywhere in \(Q''\). Therefore

\[
\frac{\partial}{\partial x} \zeta(x; \eta) = 0 \tag{12.5}
\]

almost everywhere in \(Q''\).

On the other hand, if we write \(y^{(0)} = x^{(0)}; \eta^{(0)}\) \(\lambda = 1, \ldots, n\) for a point \((x^{(0)}; \eta^{(0)}) \in Q''\), we have \((x^{(0)}; y^{(0)}) \in G_4\) and

\[
\limsup_{x \to x^{(0)}} \frac{|\zeta(x; \eta^{(0)}) - \zeta(x^{(0)}; \eta^{(0)})|}{|x - x^{(0)}|} \leq \limsup_{x \to x^{(0)}} \frac{|\zeta(x; \eta^{(0)}) - \zeta(x^{(0)}; \eta^{(0)})|}{|x - x^{(0)}| + \|x(x; \eta^{(0)}) - x(x^{(0)}; \eta^{(0)})\|}
\]

\[
\times \limsup_{x \to x^{(0)}} \frac{|x - x^{(0)}| + \|x(x; \eta^{(0)}) - x(x^{(0)}; \eta^{(0)})\|}{|x - x^{(0)}|}
\]

\[
\leq \left( \limsup_{(x; y) \to (x^{(0)}; y^{(0)})} \frac{|z(x; y) - z(x^{(0)}; y^{(0)})|}{|x - x^{(0)}| + \|y - y^{(0)}\|} \right) \left( 1 + \sum_{\mu=1}^{n} \left| \frac{\partial x_{\mu}}{\partial x} (x^{(0)}; \eta^{(0)}) \right| \right).
\]

Hence by (12.1) and (12.3), observing that \(|f_\lambda(x; y)| \leq M_0\) on \(G_4 \subseteq Q\) by (6.1), we obtain

\[
\limsup_{x \to x^{(0)}} \frac{|\zeta(x; \eta^{(0)}) - \zeta(x^{(0)}; \eta^{(0)})|}{|x - x^{(0)}|} \leq M_0(1 + \sum_{\mu=1}^{n} |f_\mu(x^{(0)}; y^{(0)})|)
\]

\[
\leq M_0(1 + nM_0) \tag{12.6}
\]

whenever \((x^{(0)}; \eta^{(0)}) \in Q''\).

By Fubini's theorem, \(\zeta(x; \eta)\) as a function of \(x\), satisfies (12.5) almost everywhere in the interval \(|x - a'| < L_4\), for almost every \(\eta\) in the domain \(\Omega_4\), and by (12.6), \(\zeta(x; \eta)\) as a function of \(x\), is absolutely continuous in the interval \(|x - a'| < L_4\) for any \(\eta\) in the domain \(\Omega_4\).

Therefore by Lebesgue's theorem, \(\zeta(x; \eta)\) as a function of \(x\), is constant in the interval \(|x - a'| < L_4\) for almost every \(\eta\) in the domain \(\Omega_4\). Hence, by the continuity of \(z(x; y)\) and \(\chi_\lambda(x; \eta)\), accordingly of \(\zeta(x; \eta)\), it follows that \(\zeta(x; \eta)\) as a function of \(x\), is constant in the interval \(|x - a'| < L_4\) for any \(\eta\) in the domain \(\Omega_4\).

From this, by the definition of \(\zeta(x; \eta)\), we easily see that \(z(x; y)\) is constant on any characteristic curve of (2.1) in \(G_4\). Hence, by the definition of \(K\), we have \(G_4 \subseteq K\) and so observing that \(G_4 \subseteq G\) is open in \(R^{n+1}\),
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\[ F = G - K \cdot G \subseteq G - G = -K \cdot G. \]

This is however excluded, since \((a'; b') \in F \cdot G\). Thus we arrive at a contradiction and this completes the proof of Theorem 1.

§ 4. Proof of Theorem 2 and Theorem 3

In this §, the notations are the same as in § 1 and § 2.

13. Proof of Theorem 2. By the assumption on \(G\) and Theorem 1, if we put \(\omega_\lambda(x; y) = \varphi_\lambda(\xi^{(0)}, x; y|G) \ \lambda = 1, \ldots, n\) for \((x; y) \in G\), then we have \(\omega(x; y) \in G[\xi^{(0)}]\) for \((x; y) \in G\) and

\[ z(x; y) = \psi(\omega(x; y)) \quad \text{on} \ G \quad (13.1) \]

for any quasi-solution \(z(x; y)\) of (2.1) on \(G\) such that \(z(\xi^{(0)}; \eta) = \psi(\eta)\) on \(G[\xi^{(0)}]\). Hence there is at most only one such quasi-solution.

Conversely if we define a function \(z(x; y)\) by the right side of (13.1) on \(G\), then by the total differentiability of \(\psi(\eta)\) on \(G[\xi^{(0)}]\) and of \(\omega_\lambda(x; y)\) on \(G\), \(z(x; y)\) is totally differentiable on \(G\) and

\[ \frac{\partial z}{\partial x} = \sum_{\mu=1}^{n} \frac{\partial \varphi}{\partial \eta_\mu} \frac{\partial \omega_\mu}{\partial x}, \quad \frac{\partial z}{\partial y_\lambda} = \sum_{\mu=1}^{n} \frac{\partial \varphi}{\partial \eta_\mu} \frac{\partial \omega_\mu}{\partial y_\lambda} \quad \lambda = 1, \ldots, n \]

on \(G\). Hence

\[ \frac{\partial z}{\partial x} + \sum_{\mu=1}^{n} f_\mu(x; y) \frac{\partial z}{\partial y} = \sum_{\lambda=1}^{n} \frac{\partial \varphi}{\partial \eta_\lambda} \left( \frac{\partial \omega_\lambda}{\partial x} + \sum_{\mu=1}^{n} \frac{\partial \omega_\mu}{\partial y_\mu} f_\mu(x; y) \right) \quad \text{on} \ G. \quad (13.2) \]

But for \(\omega_\lambda(x; y) = \varphi_\lambda(\xi^{(0)}, x; y|G)\), we have

\[ \frac{\partial \omega_\lambda}{\partial x} + \sum_{\mu=1}^{n} f_\mu(x; y) \frac{\partial \omega_\lambda}{\partial y_\mu} = 0 \quad \lambda = 1, \ldots, n \quad \text{on} \ G. \]

Therefore by (13.2), for \(z(x; y)\) defined by (13.1)

\[ \frac{\partial z}{\partial x} + \sum_{\mu=1}^{n} f_\mu(x; y) \frac{\partial z}{\partial y_\mu} = 0 \quad \text{on} \ G. \]

Also for \(z(x; y)\) defined by (13.1), we have

\[ z(\xi^{(0)}; \eta) = \psi(\eta) \quad \text{on} \ G[\xi^{(0)}], \]

since \(\omega_\lambda(\xi^{(0)}; \eta) = \varphi_\lambda(\xi^{(0)}, \xi^{(0)}; \eta|G) = \eta\).

Thus there is at least one quasi-solution \(z(x; y)\) of (2.1) on \(G\) such that \(z(\xi^{(0)}; \eta) = \psi(\eta)\) on \(G[\xi^{(0)}]\) and this quasi-solution is also a solution

11) Cf. Kamke [3], §18, Nr. 87, Satz 1.
of (2.1) on $G$ in the ordinary sense. This completes the proof of Theorem 2.

14. Proof of Theorem 3. For the special case $n = 1$, we write (2.1) in the form

$$\frac{\partial z}{\partial x} + f(x, y)\frac{\partial z}{\partial y} = 0 \quad (14.1)$$

and the characteristic curve of (14.1) in $G$ which passes through the point $(\xi, \eta)$ of $G$, in the form

$$y = \varphi(x, \xi, \eta|G) \quad \alpha(\xi, \eta|G) < x < \beta(\xi, \eta|G).$$

Let $z(x, y)$ be any quasi-solution of (14.1) on $G$ and $(x^{(0)}, y^{(0)})$ be any point of $G$. Then there is at least one point $(\xi^{(0)}, \eta^{(0)})$ on $C(x^{(0)}, y^{(0)}|G)$ where $z(x, y)$ has $\partial z/\partial y$, since $z(x, y)$ has $\partial z/\partial y$ except at most at the points of an enumerable set in $G$.

If we put $\omega(x, y) = \varphi(\xi^{(0)}, x, y|G)$ and for $\eta \in G[\xi^{(0)}]$, $\psi(\eta) = z(\xi^{(0)}, \eta)$, then by the properties of the family of the characteristic curves as stated in §1.2, $\omega(x, y)$ is defined and $\omega(x, y) \in G[\xi^{(0)}]$ for $(x, y)$ in some neighbourhood of $(x^{(0)}, y^{(0)})$ and by Theorem 1

$$z(x, y) = \psi(\omega(x, y)) \quad (14.2)$$

in that neighbourhood. Evidently $\omega(x^{(0)}, y^{(0)}) = \varphi(\xi^{(0)}, x^{(0)}, y^{(0)}|G) = \eta^{(0)} \in G[\xi^{(0)}]$. Also $\psi(\eta) (= z(\xi^{(0)}, \eta))$ is differentiable at $\eta^{(0)}$ since $z(x, y)$ has $\partial z/\partial y$ at $(\xi^{(0)}, \eta^{(0)})$.

Since $\psi(\eta)$ is differentiable at $\eta^{(0)} = \omega(x^{(0)}, y^{(0)})$ and $\omega(x, y) (= \varphi(\xi^{(0)}, x, y|G))$ is totally differentiable at $(x^{(0)}, y^{(0)})$, by (14.2) $z(x, y)$ is totally differentiable at $(x^{(0)}, y^{(0)})$ and

$$\frac{\partial z}{\partial x}(x^{(0)}, y^{(0)}) = \psi'(\eta^{(0)})\frac{\partial \omega}{\partial x}(x^{(0)}, y^{(0)}),$$

$$\frac{\partial z}{\partial y}(x^{(0)}, y^{(0)}) = \psi'(\eta^{(0)})\frac{\partial \omega}{\partial y}(x^{(0)}, y^{(0)}).$$

Hence

$$\frac{\partial z}{\partial x}(x^{(0)}, y^{(0)}) + f(x^{(0)}, y^{(0)})\frac{\partial z}{\partial y}(x^{(0)}, y^{(0)})$$

$$= \psi'(\eta^{(0)})\left\{\frac{\partial \omega}{\partial x}(x^{(0)}, y^{(0)}) + f(x^{(0)}, y^{(0)})\frac{\partial \omega}{\partial y}(x^{(0)}, y^{(0)})\right\}. \quad (14.3)$$

But for $\omega(x, y) = \varphi(\xi^{(0)}, x, y)$, we have

12) Cf. Kamke [3], §18, Nr. 87, Satz 1.
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\[ \frac{\partial \omega}{\partial x}(x^{(0)}, y^{(0)}) + f(x^{(0)}, y^{(0)}) \frac{\partial \omega}{\partial y}(x^{(0)}, y^{(0)}) = 0. \]

Hence, by (14.3)

\[ \frac{\partial z}{\partial x}(x^{(0)}, y^{(0)}) + f(x^{(0)}, y^{(0)}) \frac{\partial z}{\partial y}(x^{(0)}, y^{(0)}) = 0. \]

Therefore \( z(x, y) \) is totally differentiable and satisfies (14.1) at any point \( (x^{(0)}, y^{(0)}) \) of \( G \). q.e.d.

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References
