

Title	Short proofs of Hiramine' results on character values
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Citation	Osaka Journal of Mathematics. 36(4) P.1059-P.1062
Issue Date	1999
Text Version	publisher
URL	<a href="https://doi.org/10.18910/3596">https://doi.org/10.18910/3596</a>
DOI	10.18910/3596
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## SHORT PROOFS OF HIRAMINE' RESULTS ON CHARACTER VALUES

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(Received January 08, 1998)

### 1. Introduction

Let  $G$  and  $H$  be finite groups of order  $n$ . A mapping  $f$  from  $G$  into  $H$  is called a planar function of degree  $n$  if, for each element  $\tau \in H$  and  $u \in G^* = G - \{1\}$ , there exists exactly one  $x \in G$  such that  $f(ux)f(x)^{-1} = \tau$ . In [2] Hiramine has shown that if both  $G$  and  $H$  are abelian groups of order  $3p$  with  $p(\geq 5)$  a prime, then there exists no planar function from  $G$  into  $H$ . To prove this he has established two results on character values. Their proofs are slightly complicated. In this note we shall give short proofs.

In section 2 we shall present Proposition 2 which is useful for the proof of Result 2. In section 3 we shall state Hiramine' results and give short proofs.

We follow the notation and terminology of [2].

### 2. Planar Functions and Equations in Group Algebras

Let  $G$  and  $H$  be finite groups of order  $n$ . Throughout this article elements of  $G$  will be denoted by small Roman letters and elements of  $H$  by small Greek letters. Let  $f$  be a mapping from  $G$  into  $H$  and  $S_\alpha = \{x \in G | f(x) = \alpha\}$ ,  $\alpha \in H$ . If  $S_\alpha \neq \emptyset$ , we set  $\hat{S}_\alpha = \sum_{x \in S_\alpha} x \in C[G]$  and  $\hat{S}_\alpha^{-1} = \sum_{x \in S_\alpha} x^{-1} \in C[G]$ , otherwise  $\hat{S}_\alpha = \hat{S}_\alpha^{-1} = 0$ , where  $C[G]$  is the group algebra of  $G$  over the complex number field  $C$ . Let  $G_0 = G \times H$  be the direct product of groups  $G, H$ .

To prove the results we need two propositions.

The following is Proposition 2.1 [2].

**Proposition 1.** *The following are equivalent.*

- (i) *The function  $f$  is planar.*
- (ii) *In the group algebra  $C[G]$  of  $G$ ,*

$$\sum_{\alpha \in H} \hat{S}_{\tau\alpha} \hat{S}_\alpha^{-1} = \sum_{\alpha \in H} \hat{S}_{\alpha\tau}^{-1} \hat{S}_\alpha = \begin{cases} \hat{G} + n - 1 & \text{if } \tau = 1, \\ \hat{G} - 1 & \text{otherwise.} \end{cases}$$

REMARK 1. If  $\tau \neq 1$ , then it follows from the equation in (ii) of the proposition above that in the group algebra  $C[G_0]$  of  $G_0$ ,

$$\sum_{\alpha \in H} \hat{S}_{\tau\alpha} \tau \alpha \hat{S}_{\alpha}^{-1} \alpha^{-1} = (\hat{G} - 1)\tau.$$

We prove the following

**Proposition 2.** *We have in  $C[G_0]$ ,*

$$\left(\sum_{\alpha \in H} \hat{S}_{\alpha} \alpha\right) \left(\sum_{\beta \in H} \hat{S}_{\beta}^{-1} \beta^{-1}\right) = \hat{G} + n - 1 + \sum_{\tau \in H, \tau \neq 1} (\hat{G} - 1)\tau.$$

**Proof of Proposition 2.**

$$\begin{aligned} \left(\sum_{\alpha \in H} \hat{S}_{\alpha} \alpha\right) \left(\sum_{\beta \in H} \hat{S}_{\beta}^{-1} \beta^{-1}\right) &= \sum_{\tau \in H} \left(\sum_{\beta \in H} \hat{S}_{\tau\beta} \tau \beta \hat{S}_{\beta}^{-1} \beta^{-1}\right) \\ &= \hat{G} + n - 1 + \sum_{\tau \in H, \tau \neq 1} (\hat{G} - 1)\tau, \quad \text{by Remark 1.} \end{aligned}$$

We complete the proof of Proposition 2. □

### 3. Proofs of Hiramine' Results

We start with the following well-known facts about character theory. These facts play important parts in the proofs of his results.

FACT 1. *Let  $G$  be an abelian group and  $\chi$  an arbitrary (linear) character of  $G$ . Then  $\chi$  is a homomorphism from  $G$  into  $C^* = C - \{0\}$ . So we can extend this homomorphism  $\chi$  to an algebra homomorphism  $\bar{\chi}$  from  $C[G]$  into  $C$ .*

FACT 2. *Let  $H_1, H_2$  be finite groups and  $G_1$  the direct product of  $H_1, H_2$ . Then all irreducible characters of  $G_1$  are obtained as follows. Let  $\chi_0, \dots, \chi_{s-1}$  be the irreducible characters of  $H_1$ ,  $\rho_0, \dots, \rho_{t-1}$  the irreducible characters of  $H_2$ . Then  $G_1$  has exactly  $st$  irreducible characters  $\Psi_{ij} (0 \leq i \leq s - 1, 0 \leq j \leq t - 1)$ , satisfying  $\Psi_{ij}(h_1 h_2) = \chi_i(h_1) \rho_j(h_2)$ , where  $h_1 \in H_1, h_2 \in H_2$ .*

Proof. See[1, p.54]. □

REMARK 2. In Fact 2 if both  $\chi_i$  and  $\rho_j$  are linear characters, then  $\Psi_{ij}$  is a homomorphism from  $G_1$  to  $C^*$ . As in Fact 1, we have an algebra homomorphism  $\bar{\Psi}_{ij}$  from  $C[G_1]$  into  $C$  which is an extension of  $\Psi_{ij}$ .

Now we shall start to state Hiramine' results and prove them. In the remainder of this section we assume that  $f$  is a planar function and that  $G$  is an abelian group of order  $n$ . Let  $\chi_0 (= 1_G), \dots, \chi_{n-1}$  be the irreducible (linear) characters of  $G$ , where  $1_G$  denote the principal character of  $G$ . We set

$$d_i^{(\alpha)} = \begin{cases} \sum_{x \in S_\alpha} \chi_i(x) & \text{if } S_\alpha \neq \emptyset, \\ 0 & \text{if } S_\alpha = \emptyset \end{cases}$$

for each  $0 \leq i \leq n - 1$  and for each  $\alpha \in H$ . Now we state one of Hiramine' results[2].

RESULT 1. *The following hold*

(i)  $d_0^{(\alpha)} = |S_\alpha|$  and

$$\sum_{\alpha \in H} d_0^{(\tau\alpha)} d_0^{(\alpha)} = \sum_{\alpha \in H} d_0^{(\alpha\tau)} d_0^{(\alpha)} = \begin{cases} 2n - 1 & \text{if } \tau = 1, \\ n - 1 & \text{otherwise.} \end{cases}$$

(ii) For  $i \neq 0$ ,

$$\sum_{\alpha \in H} d_i^{(\tau\alpha)} \overline{d_i^{(\alpha)}} = \sum_{\alpha \in H} \overline{d_i^{(\alpha\tau)}} d_i^{(\alpha)} = \begin{cases} n - 1 & \text{if } \tau = 1, \\ -1 & \text{otherwise.} \end{cases}$$

(Here  $\bar{d}$  denotes the complex conjugate of  $d \in C$ .)

Proof. Since  $G$  is abelian, from Fact 1 we note that for each  $0 \leq i \leq n - 1$ ,  $\overline{\chi_i}$  is an algebra homomorphism from  $C[G]$  into  $C$ . We shall prove (i). It is immediate that  $d_0^{(\alpha)} = |S_\alpha|$ . If  $\tau = 1$ , then the equation in (ii) of Proposition 1 becomes

$$\sum_{\alpha \in H} \hat{S}_\alpha \hat{S}_\alpha^{-1} = \sum_{\alpha \in H} \hat{S}_\alpha^{-1} \hat{S}_\alpha = \hat{G} + n - 1.$$

We apply  $\overline{\chi_0}$  to this equation. Then

$$\sum_{\alpha \in H} \overline{\chi_0}(\hat{S}_\alpha) \overline{\chi_0}(\hat{S}_\alpha^{-1}) = \sum_{\alpha \in H} \overline{\chi_0}(\hat{S}_\alpha^{-1}) \overline{\chi_0}(\hat{S}_\alpha) = \overline{\chi_0}(\hat{G} + n - 1),$$

which implies the equation for  $\tau = 1$  in (i). Similarly we can prove the equation for  $\tau \neq 1$  in (i). We have proved (i). Next we shall prove (ii). By applying the algebra homomorphism  $\overline{\chi_i}$  ( $i \neq 0$ ) to the equation in (ii) of Proposition 1 we can prove (ii). This completes the proof of Result 1.  $\square$

We state another result on character values in [2].

**RESULT 2.** *With the same notation and assumption as in Result 1, suppose that  $H$  is abelian and let  $\rho_0 (= 1_H), \dots, \rho_{n-1}$  be the irreducible characters of  $H$ . Set  $z_{ij} = \sum_{\alpha \in H} d_i^{(\alpha)} \rho_j(\alpha)$ . Then,*

(i)  $z_{0,0} = n$ , and  $z_{i,0} = 0 (i \neq 0)$

(ii) For  $j \neq 0$ ,  $z_{ij} \overline{z_{ij}} = n$ .

**Proof.** Since  $\chi_i$  and  $\rho_j$  are linear, from Remark 2 we see that  $\overline{\Psi}_{ij}$  is an algebra homomorphism from  $C[G_0]$  into  $C$ . First we shall prove (ii). We apply  $\overline{\Psi}_{ij}$  ( $j \neq 0$ ) to the equation in Proposition 2. Then we get (ii). We have proved (ii). Next we shall prove (i). Similarly by using  $\overline{\Psi}_{i0}$  we can prove (i). This completes the proof of Result 2.  $\square$

**REMARK 3.** Nakagawa[3] has proved Result 2 by using Gaussian sums.

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