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A STRUCTURE THEOREM OF COMPACT COMPLEX PARALLELIZABLE PSEUDO-KÄHLER SOLVMANIFOLDS

Dedicated to Professor Yusuke Sakane on his 60th birthday

TAKUMI YAMADA

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Abstract

In this paper, we prove that the Mostow fibration of a compact complex parallelizable pseudo-Kähler solvmanifold is a complex torus bundle over a complex torus.

Introduction

A complex manifold $X^n$ of complex dimension $n$ is called complex parallelizable if there exist $n$ holomorphic vector fields which are linearly independent at each point. Wang [13] proved that a compact complex parallelizable manifold is of the form $G/\Gamma$, where $G$ is a complex Lie group and $\Gamma$ is a discrete subgroup of $G$. Wang also proved that if a compact complex parallelizable manifold $X$ admits a Kähler structure, then $X$ is a complex torus. On the other hand, Matsushima [10] proved that a compact homogeneous Kähler manifold is biholomorphic to a product of a homogeneous rational manifold and a complex torus. By a homogeneous Kähler manifold we mean a Kähler manifold on which the group of holomorphic isometric transformations acts transitively. Borel-Remmert [2] generalized the result of Matsushima to compact Kähler manifolds on which the group of holomorphic transformations acts transitively. Dorfmeister-Guan [3] proved that a compact homogeneous pseudo-Kähler manifold is also biholomorphic to a product of a homogeneous rational manifold and a complex torus. As for compact pseudo-Kähler manifolds on which the group of holomorphic transformations acts transitively, there exist non-toral compact complex parallelizable pseudo-Kähler solvmanifolds. In particular, we see that a compact non-homogeneous pseudo-Kähler manifold is not biholomorphic to a product of a homogeneous rational manifold and a complex torus in general (cf. [17]). It is therefore important to study compact complex parallelizable pseudo-Kähler solvmanifolds. In this paper we prove the following structure theorem, which is our main theorem:

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Theorem 1.6 Let $X = G/\Gamma$ be a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure. Then the Mostow fibration of $X$ is a complex torus bundle over a complex torus.

We also investigate the Dolbeault cohomology groups of a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure.

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1. Proof of main theorem

In this section we prove our main theorem.

Definition 1.1. Let $G$ be a Lie group. A discrete subgroup $\Gamma$ of $G$ is called a lattice if $G/\Gamma$ has a finite invariant measure.

If $G$ is a solvable Lie group, then a discrete subgroup $\Gamma$ of $G$ is a lattice if and only if $\Gamma$ is a discrete co-compact subgroup of $G$.

Let $O_X = O$ be the sheaf of holomorphic functions on a complex manifold $X$. We denote the Hodge number of $X$ by $h^{p,q}(X)$, i.e., $h^{p,q}(X) = \dim H^{p,q}_\bar{}(X)$. Let $G$ be a connected complex Lie group, $\Gamma$ a lattice of $G$, $N$ the maximal connected normal nilpotent subgroup. Let $G = S \cdot R$ be a Levi decomposition, where $S$ is a semi-simple part, and $R$ is the radical. We denote derived Lie subgroups of $G$, $N$ and $R$ by $G'$, $N'$ and $R'$ respectively. Winkelmann has proven

Theorem 1.2 ([14]). Let $G$, $\Gamma$, $N$, $S$, $R$, $G'$, $N'$ and $R'$ be as above. Let $A = [S, R] \cdot N'$. Furthermore let $W$ denote the maximal linear subspace of the Lie algebra $\text{Lie}(R'/A/A)$ of $R'/A/A$ such that $\text{Ad}(\gamma)|_W$, where $\text{Ad}$ is the adjoint representation of $G$, is a semisimple linear endomorphism with only real eigenvalues for each $\gamma \in \Gamma$. Then

$$\dim H^1(G/\Gamma, O) \leq \dim G/G' + \dim H^1(G/R\Gamma, O) + \dim W.$$ 

Definition 1.3. Let $X$ be a complex manifold. A real $(1,1)$-form $\omega$ of $X$ is called a pseudo-Kähler structure if $\omega$ is a non-degenerate closed form.

In the case of a compact complex parallelizable manifold, we have shown the following in the paper [17]:
Theorem 1.4. Let $X^n = G/\Gamma$ be a compact complex parallelizable manifold which admits a pseudo-Kähler structure. Then

$$h^{p,q}(X) \geq \binom{n}{p} \binom{n}{q}.$$  

Corollary 1.5. Let $(G/\Gamma, \omega)$ be an $n$-dimensional compact complex parallelizable pseudo-Kähler manifold such that $\Gamma$ is a lattice of $G$. If $h^{0,1}(G/\Gamma) = \dim H^0(G/\Gamma, d\mathcal{O})$, then $G/\Gamma$ is a complex torus.

Proof. Let $g$ be the Lie algebra of $G$, $I$ the complex structure of $g$, and $g^+ = \{X \in g^C \mid IX = \sqrt{-1}X\}$. We identify $g^+$ with the set of all right invariant holomorphic vector fields of $G$. Let $H^q(g^+)$ be the $q$th Lie algebra cohomology group of $g^+$. Since $H^0(G/\Gamma, d\mathcal{O}) \cong H^1(g^+)$ and $h^{0,1}(G/\Gamma) \geq n$, we see that $\dim H^1(g^+) = n$. Hence $g^+$ is abelian.

Let $G/\Gamma$ be a compact complex parallelizable solvmanifold, i.e., $G$ is a simply connected complex solvable Lie group and $\Gamma$ is a lattice of $G$. Mostow proved that $\Gamma_N = N \cap \Gamma$ is a lattice of the maximal normal nilpotent Lie subgroup $N$ of $G$. A fibration $N/\Gamma_N \to G/\Gamma \to G/N\Gamma$ is called the Mostow fibration of $G/\Gamma$.

Theorem 1.6. Let $X^n = G/\Gamma$ be a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure. Then the Mostow fibration of $G/\Gamma$ is a complex torus bundle over a complex torus.

Proof. We use the notation of Theorem 1.2. Since $G$ is solvable, we see that $G = R$, $S = \{e\}$ and $A = N'$. By Theorems 1.2 and 1.4, we see

$$n \leq \dim H^{0,1}_\beta(G/\Gamma) \leq \dim G/G' + \dim W \leq \dim G - \dim G' + \dim G'/N',$$

which implies $\dim N' = 0$. 

By the proof of our main theorem, we have

Corollary 1.7. If a compact complex parallelizable solvmanifold $X^n$ admits a pseudo-Kähler structure, then $h^{0,1}(X) = n$.

Corollary 1.8. If a compact complex parallelizable solvmanifold $G/\Gamma$ admits a pseudo-Kähler structure, then $N$ must be abelian and in particular the Lie algebra $g$ must satisfy $D^{(2)}g = 0$.

Proof. Since $n \supset [g, g]$, we have our corollary.
REMARKS 1.9. (i) There exists a complex solvable Lie group $G$ which has lattices $\Gamma_1, \Gamma_2$ such that $G/\Gamma_1$ has a pseudo-Kähler structure, while $G/\Gamma_2$ has no pseudo-Kähler structures (see [17]).

(ii) It is well known that a simply connected complex solvable Lie group $G$ is biholomorphic to $\mathbb{C}^n$. Moreover if its Lie algebra $\mathfrak{g}$ has a Chevalley decomposition, then there exists a good system of coordinates $(z_1, \ldots, z_n)$ of $G$ which satisfies the following:

(a) The Lie group $G$ is isomorphic to $(\mathbb{C}^n, \ast)$ as a complex Lie group, where the multiplication $\ast$ of $\mathbb{C}^n$ is given by

$$(z_1, \ldots, z_n) \ast (y_1, \ldots, y_n) = (z_1 + y_1, \ldots, z_r + y_r, F_{r+1} + 1(z)z_{r+1} + y_{r+1} + F_{r+1}(z, y), \ldots, F_{nn}(y)z_n + y_n + F_n(z, y))$$

for $z = (z_1, \ldots, z_n), y = (y_1, \ldots, y_n) \in \mathbb{C}^n$, where $r = \dim \mathfrak{g} - \dim [\mathfrak{g}, \mathfrak{g}]$. $F_{ii}(z) = \exp(-\sum_{j=1}^{k} C_{ji} z_j)$, where $k = \dim G/N \Gamma$ and $C_{ji}$ are constant, and $F_{\lambda}(z, y) = F_{\lambda}(z_1, \ldots, z_{\lambda-1}, y_1, \ldots, y_{\lambda-1})$ is a holomorphic function with respect to $(z_1, \ldots, z_{\lambda-1}, y_1, \ldots, y_{\lambda-1})$ for each $\lambda$.

(b) Let $\Gamma$ be a lattice of $G$. Using the above system of coordinates $(z_1, \ldots, z_n)$ of $G$, we see that any element of $H^{0,1}_\beta(G/\Gamma)$ has a representative of the following form:

$$\psi = \sum_{k=1}^{k} c_{\lambda} d\bar{z}_{\lambda} + \sum_{\lambda=k+1}^{k+r(N/\Gamma_N)} f_{\lambda}(z_1, \ldots, z_k) d\bar{z}_{\lambda},$$

where $r(N/\Gamma_N) = \dim H^{0,1}_\beta(N/\Gamma_N)$, $c_{\lambda}$ are constant and $f_{\lambda}(z_1, \ldots, z_k)$ are holomorphic in $z_1, \ldots, z_k$.

We say that a complex solvable Lie algebra $\mathfrak{g}$ has a Chevalley decomposition if $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ as a vector space, where $\mathfrak{a}$ is a commutative subalgebra and $\mathfrak{n}$ is the maximal nilpotent ideal. For further details see [11].

Using the above system of coordinates, we give another proof of our main theorem for the case where the Lie algebra $\mathfrak{g}$ of $G$ has a Chevalley decomposition (see Section 3).

2. The structure of the sheaf $R^1\pi_*\mathcal{O}_{G/\Gamma}$

For a holomorphic map $f : X \rightarrow Y$ between complex spaces, there exists a Leray spectral sequence for the sheaf $\mathcal{O}$. The respective lower term sequence yields the following:

$$0 \rightarrow H^1(Y, R^0 f_* \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^0(Y, R^1 f_* \mathcal{O}_X) \rightarrow H^2(Y, R^1 f_* \mathcal{O}_X),$$

where $R^q f_* \mathcal{O}_X$ is the higher direct image sheaf. If $f$ is connected and proper, then $R^0 f_* \mathcal{O}_X = \mathcal{O}_Y$. 
In this section we prove the following:

**Proposition 2.1.** Let $X^n = G/\Gamma$ be a compact complex parallelizable pseudo-Kähler solvmanifold and $\pi : G/\Gamma \to G/N\Gamma$ the Mostow fibration. Then $R^1\pi_*\mathcal{O}_X$ is the sheaf of sections of a trivial holomorphic vector bundle.

To prove this proposition we follow a part of the proof of Theorem 1.2 due to Winkelmann ([14]).

Let $G$ be a connected complex Lie group and $\Gamma$ a lattice of $G$. Let $V$ be a complex vector space, $\rho : G \to GL(V)$ an antiholomorphic representation and $V_1$ the set of all $v \in V$ which are invariant under $\rho(G')$, where $G'$ is the derived Lie subgroup of $G$. We denote by $V_0$ the subspace spanned by all vectors $v \in V_1$ such that $v$ is an eigenvector with a real eigenvalue for every $\rho(\gamma)$ ($\gamma \in \Gamma$).

Let $E, E_0$ be flat vector bundles over $X = G/\Gamma$ which are induced by $\rho|_\Gamma$ on $V, V_0$ respectively, i.e., $E = G \times V /\sim$, where $(g, v) \sim (g', v')$ if and only if $(g', v') = (g \gamma^{-1}, \rho(\gamma)v)$. Note that $E, E_0$ are holomorphic vector bundles.

**Proposition 2.2.** The flat vector bundle $E_0$ is a holomorphically trivial vector bundle and $\Gamma(X, E) = \Gamma(X, E_0) \cong V_0$.

**Proof.** See [14], Propositions 7.9.1 and 7.9.2.

Let $\pi : X \to B$ be a holomorphic fiber bundle with an $n$-dimensional complex torus $T_C^n$ as typical fiber. Let $V = \Omega^1(T_C^n)$ denote the vector space of holomorphic 1-forms on $T_C^n$. Note that $T_C^n$ is a compact Kähler manifold. Let $\mathcal{U} = \{U_i\}$ be a trivializing open cover of $B$ such that $X$ is given by transition functions $\phi_{ij} : U_i \cap U_j \to Aut(T_C^n)$, where $Aut(T_C^n)$ is the automorphism group of $T_C^n$. We denote by $Aut^0(T_C^n)$ the identity component of $Aut(T_C^n)$.

**Lemma 2.3.** Under the above assumptions $R^1\pi_*\mathcal{O}_X$ is a locally free coherent sheaf of $B$ isomorphic to the sheaf of sections of the flat vector bundle $E$ given by transition functions $\varphi_{ij} = \overline{\xi} \circ \phi_{ij} : U_i \cap U_j \to GL(V)$, where $1 \to Aut^0(T_C^n) \to Aut(T_C^n) \xrightarrow{\xi} GL(V)$ is exact.

**Proof.** See [14], CLAIM 8.4.5.

We apply this lemma to a complex parallelizable manifold $G/\Gamma$. Let $K$ be a normal abelian complex Lie subgroup of $G$ and $\mathfrak{k}$ its Lie algebra. Assume that $K/K \cap \Gamma$ is compact. Denote the natural projection map $X = G/\Gamma \to B = G/K\Gamma$ by $\pi$. 
Proposition 2.4. The sheaf $R^{1}\pi_{*}\mathcal{O}_{X}$ is the sheaf of sections of the flat vector bundle $E$ of rank $\dim K$ over $B$ induced by the representation $\rho: \Gamma \to GL(\mathfrak{k}^{*})$ given by $\gamma \mapsto \text{Ad}^\gamma(\gamma)$.

Proof. See [14], Proposition 8.4.6.

Moreover, if $G/K$ is abelian, we have

Proposition 2.5 ([14]). If $G/K$ is abelian, then

$$\dim H^1(G/\Gamma, \mathcal{O}) \leq \dim G/K + \dim U,$$

where $U$ denotes the maximal linear subspace of $\mathfrak{k}$ such that $\text{Ad}(\gamma)|_{U}$ is a semisimple linear endomorphism with only real eigenvalues for every $\gamma$ in $\Gamma$.

Proof. Let us consider the lower term of the Leray spectral sequence for $\pi: G/\Gamma \to G/K\Gamma$. Then we have the following:

$$0 \to H^1(G/K\Gamma, \mathcal{O}) \to H^1(G/\Gamma, \mathcal{O}) \to H^0(G/K\Gamma, R^1\pi_{*}\mathcal{O}).$$

Since $G/K\Gamma \cong (G/K)/(K\Gamma/K)$ and $G/K$ is abelian, by Propositions 2.2 and 2.4, we see

$$\dim H^1(G/\Gamma, \mathcal{O}) \leq \dim H^1(G/K\Gamma, \mathcal{O}) + \dim U = \dim G/K + \dim U.$$

Hence we have our proposition.

Proof of Proposition 2.1. In Section 1, we have seen that if a compact complex parallelizable solvmanifold $X^n = G/\Gamma$ admits a pseudo-Kähler structure, then the maximal normal nilpotent Lie subgroup $N$ of $G$ is abelian and $h^{0,1} = n$. Thus let us consider the Mostow fibration $\pi: G/\Gamma \to G/N\Gamma$. Since $G/N$ is abelian and $G/\Gamma$ admits a pseudo-Kähler structure, we have $W = n$ by Proposition 2.5, where $W$ is the maximal linear subspace of $\mathfrak{n}$ such that $\text{Ad}(\gamma)|_{W}$ is a semisimple endomorphism with only real eigenvalues for every $\gamma \in \Gamma$. By Propositions 2.2 and 2.4, this means that the flat vector bundle $E$ induced by the representation $\rho|_{\gamma}: \Gamma \to GL(n^{*})$ given by $\gamma \mapsto \text{Ad}^\gamma(\gamma)$ is trivial as a holomorphic vector bundle.

3. Dolbeault cohomology of compact complex parallelizable pseudo-Kähler solvmanifolds

In this section we consider the Dolbeault cohomology groups of compact complex parallelizable pseudo-Kähler solvmanifolds.
Let $G$ be a complex Lie group and $\mathfrak{g}$ its Lie algebra. Let $I$ denote the complex structure of $\mathfrak{g}$, and $\mathfrak{g}^+$ (resp. $\mathfrak{g}^-$) denote the vector space of the $+\sqrt{-1}$ (resp. $-\sqrt{-1}$) eigenvectors of the complex structure $I$ respectively. Then we have $\mathfrak{g}^C = \mathfrak{g}^+ \oplus \mathfrak{g}^-$. In this section, we identify $\mathfrak{g}^+$ with the set of all right invariant holomorphic vector fields of $G$. Recall that $H^q_{\partial\bar{\partial}}(G/\Gamma) \cong H^q_{\partial\bar{\partial}}(G/\Gamma) \otimes \wedge^p(\mathfrak{g}^+)^s$ for a compact complex parallelizable manifold $G/\Gamma$. Sakane [12] has proved that if $G$ is a complex nilpotent Lie group, then $H^p_{\partial\bar{\partial}}(G/\Gamma) \cong H^p(\mathfrak{g}^-) \otimes \wedge^p(\mathfrak{g}^+)^s$, where $H^p(\mathfrak{g}^-)$ is the $q$th Lie algebra cohomology group of $\mathfrak{g}^-.$

Let $F \rightarrow X \rightarrow B$ be a holomorphic fiber bundle such that $X, B, F$ are connected and $F$ is compact. Then $\bigcup_{b \in B} H^p_{\partial\bar{\partial}}(F_b)$ is the total space of a differentiable vector bundle over $B$. This bundle is denoted by $H^p_{\partial\bar{\partial}}(F)$ and $H^0_{\partial\bar{\partial}}(F)$ is the direct sum of $H^p_{\partial\bar{\partial}}(F)$. If every connected component of the structure group of $\pi : X \rightarrow B$ acts trivially on $H^0(F)$, then the vector bundle is a holomorphic vector bundle. Thus if the fiber $F$ is a compact Kähler manifold, then $H^0(F)$ is a holomorphic vector bundle.

**Theorem 3.1** ([8]). Let $\xi = (X, B, F, \pi)$ be a holomorphic fiber bundle, where $X, B, F$ are connected and $F$ is compact. Assume that every connected component of the structure group of $\xi$ acts trivially on $H^0(F)$, i.e., $H^0(F)$ is a holomorphic vector bundle. Then there exists a spectral sequence $(E_r, d_r)$, $(r \geq 0)$, with the following properties:

(i) $E_r$ is 4-graded, by the fiber-degree, the base-degree and the type. Let $^{p,q,E_r^{s,t}}$ be the subspace of elements of $E_r$ of type $(p, q)$, fiber-degree $s$ and base-degree $t$. We have $^{p,q,E_r^{s,t}} = 0$ if $p + q \neq s + t$ or if one of $p, q, s, t$ is negative. The differential $d_r$ maps $^{p,q,E_r^{s,t}}$ into $^{p,q+1,E_r^{s+r,t-r+1}}$.

(ii) If $p + q = s + t$, then we have

$$^{p,q,E_r^{s,t}} \cong \sum_{i \geq 0} H^i_{\partial\bar{\partial}}(B, H^{p-i,q-s+i}(F)).$$

(iii) The spectral sequence converges to $H^0(X) = \bigoplus_{p,q} H^0_{\partial\bar{\partial}}(X)$.

We put $^{p,q,E_r} = \sum_{s,t \geq 0}^{p,q,E_r^{s,t}}$. We call the above spectral sequence the Borel’s spectral sequence.

**Remark 3.2.** If $F$ is a complex torus, then the vector bundle $H^0_{\partial\bar{\partial}}(F) \rightarrow B$ is isomorphic to the holomorphic vector bundle $E$ considered in Section 2 (see Lemma 2.3).

By Theorem 1.4 and the Borel’s spectral sequence, we have

**Proposition 3.3.** Let $X = G/\Gamma$ be a compact complex parallelizable manifold which admits a pseudo-Kähler structure and $F \rightarrow X \rightarrow B$ a holomorphic fiber bun-
dle such that $F, B$ are complex tori. If $H_\delta(F) \to B$ is trivial as a holomorphic vector bundle, then

$$h^{p,q}(X) = \binom{n}{p} \cdot \binom{n}{q}.$$ 

Proof. By our assumption we see that the Borel's spectral sequence satisfies

$$E^{s,t}_2 \cong \sum_{i \geq 0} H^{i,s-i}_\delta(B) \otimes H^{p-i,q-i}_\delta(F).$$

Thus by the relation $\dim h^{p,q} E_\infty \leq \dim h^{p,q} E_2$, we see that $h^{p,q}(X) \leq \binom{n}{p} \cdot \binom{n}{q}$. Thus we have our proposition by Theorem 1.4.

If $H^0_\delta(T^n_C) \to B$, where $T^n_C$ is an $n$-dimensional complex torus, admits global holomorphic sections $\sigma_1, \ldots, \sigma_n$ which are linearly independent at each point, then $H^0_\delta(T^n_C) \to B$ is trivial as a holomorphic vector bundle. Indeed, consider $\sigma_j = \sigma_{j_1} \land \cdots \land \sigma_{j_k}$ (Note that $H^0_\delta(T^n_C) = \binom{n}{q}$). Thus by Proposition 2.1 and Lemma 2.3 we have

**Corollary 3.4.** If a compact complex parallelizable solvmanifold $X^n = G/\Gamma$ admits a pseudo-Kähler structure, then

$$h^{p,q}(X) = \binom{n}{p} \cdot \binom{n}{q}.$$ 

Proof. By our assumption, we see that $H^0_\delta(N/\Gamma_N) \to G/N\Gamma$ is trivial as a holomorphic vector bundle.

Let $(X^n, \omega)$ be a compact pseudo-Kähler manifold. We say that $(X^n, \omega)$ has the **hard Lefschetz property with respect to the Dolbeault cohomology** if for any $p+q \leq n$, the homomorphism

$$L^{n-p-q} : H^{p,q}_\delta(X) \to H^{n-q,n-p}_\delta(X), \quad L^{n-p-q}([\alpha]) = [\alpha \land \omega^{n-p-q}]$$

is an isomorphism.

**Corollary 3.5.** Let $(G/\Gamma, \omega)$ be an $n$-dimensional compact complex parallelizable pseudo-Kähler solvmanifold. Then $(G/\Gamma, \omega)$ has the hard Lefschetz property with respect to the Dolbeault cohomology.

Proof. Put $\bar{\xi}_i = i(X^+_i)\omega$, where $\{X^+_1, \ldots, X^+_n\}$ is a basis of $\mathfrak{g}^\ast$. We denote the dual basis of $\mathfrak{g}^\ast$ by $\{\omega^+_1, \ldots, \omega^+_n\}$. Then $\omega$ can be written as $\omega = \sum_{i=1}^n \bar{\xi}_i \land \omega^+_i$. In particular, we see that $\bar{\xi}_i$ are non-exact $\bar{\delta}$-closed. We also see that $\alpha = \sum_{j,k} a_{jk} \bar{\xi}_j \land \omega^+_k$.
is non-exact $\bar{\partial}$-closed, where $a_{jK} \in \mathbb{C}$, $\bar{\tau}_j = \bar{\tau}_{j_1} \wedge \cdots \wedge \bar{\tau}_{j_q}$ for $J = (j_1, \ldots, j_q)$ and $\omega_{k}^+ = \omega_{k_1}^+ \wedge \cdots \wedge \omega_{k_p}^+$ for $K = (k_1, \ldots, k_p)$. Thus by Corollary 3.4 for each Dolbeault cohomology class we can choose a representative of the form $\alpha = \sum_{jK} a_{jK} \bar{\tau}_j \wedge \omega_{k}^+$. Hence $(G/\Gamma, \omega)$ has the hard Lefschetz property.

By the proof of Corollary 3.5, we see that if an $n$-dimensional compact complex parallelizable solvmanifold $G/\Gamma$ admits a pseudo-Kähler structure, then $H_{\bar{\partial}}(G/\Gamma) = \bigoplus_{p,q} H^{p,q}_{\bar{\partial}}(G/\Gamma)$ is isomorphic to the cohomology ring $H_{\bar{\partial}}(T^n_\mathbb{C})$.

**Remark 3.6.** Mathieu's theorem of the Dolbeault cohomology on a compact pseudo-Kähler manifold $(X, \omega)$ also holds (see [18], [19]), i.e., the following two assertions are equivalent: (a) every Dolbeault cohomology class contains a $\bar{\partial}$-harmonic representative. (b) $(X, \omega)$ has the hard Lefschetz property with respect to the Dolbeault cohomology. We define $\bar{\partial}^*: \Omega^{p,q}(X) \to \Omega^{p-1,q}(X)$ by $\bar{\partial}^* = (-1)^{p+q} \bar{\partial}^*$, where $\Omega^{p,q}(X)$ is the set of all differential $(p,q)$-forms on $X$. A form $\alpha$ is called a $\bar{\partial}$-harmonic form if it satisfies $\bar{\partial} \alpha = \bar{\partial}^* \alpha = 0$, where $*: \Omega^{p,q}(X^n) \to \Omega^{n-q,n-p}(X^n)$ is defined as an analogy of the star operator for a compact Riemannian manifold. In the above case, for each Dolbeault cohomology class, we can choose a $\bar{\partial}$-harmonic representative of the form $\alpha = \sum_{jK} a_{jK} \bar{\tau}_j \wedge \omega_{k}^+$.

Let $(G/\Gamma, \omega)$ be an $n$-dimensional compact complex parallelizable pseudo-Kähler solvmanifold. We now give another proof of our main theorem for the case where the Lie algebra $\mathfrak{g}$ of $G$ has a Chevalley decomposition. We use a system of coordinates of Remarks 1.9 and the notation of the proof of Corollary 3.5. Then the pseudo-Kähler structure $\omega$ on $G/\Gamma$ can be written as follows:

$$\omega = \sum_{i=1}^n \bar{\tau}_i \wedge \omega_i^+.$$ 

Note that $\bar{\tau}_i$, $\omega_i^+$ are $\bar{\partial}$-closed. By Remarks 1.9, $\bar{\tau}_i$ are expressed by

$$\bar{\tau}_i = \psi_i + \bar{\partial} \gamma_i,$$

where $\psi_i = \sum_{\lambda=1}^k c_i^\lambda d\bar{z}_\lambda + \sum_{\lambda=k+1}^{k+r(F)} f_i^\lambda(z) d\bar{z}_\lambda$, $F = N/\Gamma N$, $c_i^\lambda$ are constant and $f_i^\lambda$ are holomorphic. Hence we can write

$$\omega = \sum_{i=1}^n \psi_i \wedge \omega_i^+ + \bar{\partial} \theta.$$

Put $\omega' = \sum_{i=1}^n \psi_i \wedge \omega_i^+$. Then a volume form $\omega^n$ is expressed by $\omega^n = \omega^n + \bar{\partial} \Omega$, where $\Omega \in \Omega^{n,n-1}(X)$. Since $\int_{G/\Gamma} \omega^n = \int_{G/\Gamma} \omega^n + \int_{G/\Gamma} \bar{\partial} \Omega = \int_{G/\Gamma} \omega^n$, we see that $r(F) = \dim H^{0,1}_{\bar{\partial}}(F) = \dim F = n - k$. Since $F$ is a compact complex parallelizable
nilmanifold, we see \( n - k = \dim H^{0,1}_\partial (N/\Gamma_N) = \dim H^1(n^-) \) by Sakane's theorem. Thus \( n^- \) is abelian, which implies that \( F \) is a complex torus.

4. Examples

**Example 4.1** ([11]). Define a multiplication \( * \) of \( \mathbb{C}^{n+2m} \) by

\[
(z_1, \ldots, z_n, w_1, w_2, \ldots, w_{2m-1}, w_{2m}) * (z'_1, \ldots, z'_n, w'_1, w'_2, \ldots, w'_{2m-1}, w'_{2m}) = (z_1 + z'_1, \ldots, z_n + z'_n, e^{-\sum a_i z_i} w'_{2k-1} + w_{2k-1}, e^{\sum a_i z_i} w'_{2k} + w_{2k}, \ldots),
\]

where \( a_i \) are integers. The solvable Lie group \( G = (\mathbb{C}^{n+2m}, *) \) has a lattice \( \Gamma \) such that \( G/\Gamma \) has a pseudo-Kähler structure. Indeed, for a suitable lattice \( \Gamma \) of \( G \),

\[
\omega = \sqrt{-1} \sum_{k=1}^n d\bar{z}_k \wedge d\bar{z}_k + \sum_{k=1}^m (dw_{2k-1} \wedge d\bar{w}_{2k} + d\bar{w}_{2k-1} \wedge dw_{2k})
\]

is a pseudo-Kähler structure on \( G/\Gamma \) (for details, see [17]). By Corollary 3.4, we see \( \text{dim} (G/\Gamma) = C^{n+2m} / \text{dim} (n+2m) \).

**Example 4.2** (cf. [4]). Let us consider the following solvable Lie group:

\[
G = \left\langle \begin{pmatrix}
e^{z_1} & 0 & z_2 e^{z_1} & 0 & 0 & 0 & w_1 \\
0 & e^{-z_1} & 0 & z_2 e^{-z_1} & 0 & 0 & w_2 \\
0 & 0 & e^{z_1} & 0 & 0 & 0 & w_3 \\
0 & 0 & 0 & e^{-z_1} & 0 & 0 & w_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : z_1, z_2, w_1, w_2, w_3, w_4 \in \mathbb{C} \right\rangle.
\]

The Lie algebra \( g \) of \( G \) is given by

\[
g = \text{span}_\mathbb{C}\{Z_1, Z_2, W_1, W_2, W_3, W_4\}
\]

with

\[
[Z_1, W_{2k-1}] = W_{2k-1}, \quad [Z_1, W_{2k}] = -W_{2k},
\]

\[
\]

for \( k = 1, 2 \). The solvable Lie group \( G \) admits a lattice \( \Gamma \) (see [16]). Since the maximal nilpotent ideal \( n \) is not abelian, we see that for any lattice \( \Gamma \), \( G/\Gamma \) has no pseudo-Kähler structures by Corollary 1.8.
References


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