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A STRUCTURE THEOREM OF COMPACT COMPLEX PARALLELIZABLE PSEUDO-KÄHLER SOLVMANIFOLDS

Dedicated to Professor Yusuke Sakane on his 60th birthday

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Abstract

In this paper, we prove that the Mostow fibration of a compact complex parallelizable pseudo-Kähler solvmanifold is a complex torus bundle over a complex torus.

Introduction

A complex manifold X^n of complex dimension n is called *complex parallelizable* if there exist n holomorphic vector fields which are linearly independent at each point. Wang [13] proved that a compact complex parallelizable manifold is of the form G/Γ , where G is a complex Lie group and Γ is a discrete subgroup of G . Wang also proved that if a compact complex parallelizable manifold X admits a Kähler structure, then X is a complex torus. On the other hand, Matsushima [10] proved that a compact homogeneous Kähler manifold is biholomorphic to a product of a homogeneous rational manifold and a complex torus. By a homogeneous Kähler manifold we mean a Kähler manifold on which the group of holomorphic isometric transformations acts transitively. Borel-Remmert [2] generalized the result of Matsushima to compact Kähler manifolds on which the group of holomorphic transformations acts transitively. Dorfmeister-Guan [3] proved that a compact homogeneous pseudo-Kähler manifold is also biholomorphic to a product of a homogeneous rational manifold and a complex torus. As for compact pseudo-Kähler manifolds on which the group of holomorphic transformations acts transitively, there exist non-toral compact complex parallelizable pseudo-Kähler solvmanifolds. In particular, we see that a compact non-homogeneous pseudo-Kähler manifold is not biholomorphic to a product of a homogeneous rational manifold and a complex torus in general (cf. [17]). It is therefore important to study compact complex parallelizable pseudo-Kähler solvmanifolds. In this paper we prove the following structure theorem, which is our main theorem:

Theorem 1.6 *Let $X = G/\Gamma$ be a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure. Then the Mostow fibration of X is a complex torus bundle over a complex torus.*

We also investigate the Dolbeault cohomology groups of a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure.

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1. Proof of main theorem

In this section we prove our main theorem.

DEFINITION 1.1. Let G be a Lie group. A discrete subgroup Γ of G is called a *lattice* if G/Γ has a finite invariant measure.

If G is a solvable Lie group, then a discrete subgroup Γ of G is a lattice if and only if Γ is a discrete co-compact subgroup of G .

Let $\mathcal{O}_X = \mathcal{O}$ be the sheaf of holomorphic functions on a complex manifold X . We denote the Hodge number of X by $h^{p,q}(X)$, i.e., $h^{p,q}(X) = \dim H_{\bar{\partial}}^{p,q}(X)$. Let G be a connected complex Lie group, Γ a lattice of G , N the maximal connected normal nilpotent subgroup. Let $G = S \cdot R$ be a Levi decomposition, where S is a semi-simple part, and R is the radical. We denote derived Lie subgroups of G, N and R by G', N' and R' respectively. Winkelmann has proven

Theorem 1.2 ([14]). *Let $G, \Gamma, N, S, R, G', N'$ and R' be as above. Let $A = [S, R] \cdot N'$. Furthermore let W denote the maximal linear subspace of the Lie algebra $\text{Lie}(R'A/A)$ of $R'A/A$ such that $\text{Ad}(\gamma)|_W$, where Ad is the adjoint representation of G , is a semisimple linear endomorphism with only real eigenvalues for each $\gamma \in \Gamma$. Then*

$$\dim H^1(G/\Gamma, \mathcal{O}) \leq \dim G/G' + \dim H^1(G/R\Gamma, \mathcal{O}) + \dim W.$$

DEFINITION 1.3. Let X be a complex manifold. A real $(1, 1)$ -form ω of X is called a *pseudo-Kähler structure* if ω is a non-degenerate closed form.

In the case of a compact complex parallelizable manifold, we have shown the following in the paper [17]:

Theorem 1.4. *Let $X^n = G/\Gamma$ be a compact complex parallelizable manifold which admits a pseudo-Kähler structure. Then*

$$h^{p,q}(X) \geq \binom{n}{p} \cdot \binom{n}{q}.$$

Corollary 1.5. *Let $(G/\Gamma, \omega)$ be an n -dimensional compact complex parallelizable pseudo-Kähler manifold such that Γ is a lattice of G . If $h^{0,1}(G/\Gamma) = \dim H^0(G/\Gamma, d\mathcal{O})$, then G/Γ is a complex torus.*

Proof. Let \mathfrak{g} be the Lie algebra of G , I the complex structure of \mathfrak{g} , and $\mathfrak{g}^+ = \{X \in \mathfrak{g}^{\mathbb{C}} \mid IX = \sqrt{-1}X\}$. We identify \mathfrak{g}^+ with the set of all right invariant holomorphic vector fields of G . Let $H^q(\mathfrak{g}^+)$ be the q th Lie algebra cohomology group of \mathfrak{g}^+ . Since $H^0(G/\Gamma, d\mathcal{O}) \cong H^1(\mathfrak{g}^+)$ and $h^{0,1}(G/\Gamma) \geq n$, we see that $\dim H^1(\mathfrak{g}^+) = n$. Hence \mathfrak{g}^+ is abelian. □

Let G/Γ be a compact complex parallelizable solvmanifold, i.e., G is a simply connected complex solvable Lie group and Γ is a lattice of G . Mostow proved that $\Gamma_N = N \cap \Gamma$ is a lattice of the maximal normal nilpotent Lie subgroup N of G . A fibration $N/\Gamma_N \rightarrow G/\Gamma \rightarrow G/N\Gamma$ is called the *Mostow fibration* of G/Γ .

Theorem 1.6. *Let $X^n = G/\Gamma$ be a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure. Then the Mostow fibration of G/Γ is a complex torus bundle over a complex torus.*

Proof. We use the notation of Theorem 1.2. Since G is solvable, we see that $G = R, S = \{e\}$ and $A = N'$. By Theorems 1.2 and 1.4, we see

$$n \leq \dim H_{\mathfrak{g}}^{0,1}(G/\Gamma) \leq \dim G/G' + \dim W \leq \dim G - \dim G' + \dim G'/N',$$

which implies $\dim N' = 0$. □

By the proof of our main theorem, we have

Corollary 1.7. *If a compact complex parallelizable solvmanifold X^n admits a pseudo-Kähler structure, then $h^{0,1}(X) = n$.*

Corollary 1.8. *If a compact complex parallelizable solvmanifold G/Γ admits a pseudo-Kähler structure, then N must be abelian and in particular the Lie algebra \mathfrak{g} must satisfy $\mathcal{D}^{(2)}\mathfrak{g} = 0$.*

Proof. Since $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$, we have our corollary. □

REMARKS 1.9. (i) There exists a complex solvable Lie group G which has lattices Γ_1, Γ_2 such that G/Γ_1 has a pseudo-Kähler structure, while G/Γ_2 has no pseudo-Kähler structures (see [17]).

(ii) It is well known that a simply connected complex solvable Lie group G is bi-holomorphic to \mathbb{C}^n . Moreover if its Lie algebra \mathfrak{g} has a Chevalley decomposition, then there exists a good system of coordinates (z_1, \dots, z_n) of G which satisfies the following:

(a) The Lie group G is isomorphic to $(\mathbb{C}^n, *)$ as a complex Lie group, where the multiplication $*$ of \mathbb{C}^n is given by

$$(z_1, \dots, z_n) * (y_1, \dots, y_n) = (z_1 + y_1, \dots, z_r + y_r, F_{r+1r+1}(y)z_{r+1} + y_{r+1} + F_{r+1}(z, y), \dots, F_{nn}(y)z_n + y_n + F_n(z, y))$$

for $z = (z_1, \dots, z_n), y = (y_1, \dots, y_n) \in \mathbb{C}^n$, where $r = \dim \mathfrak{g} - \dim[\mathfrak{g}, \mathfrak{g}]$, $F_{ii}(z) = \exp(-\sum_{j=1}^k C_{ji}^i z_j)$, where $k = \dim G/N\Gamma$ and C_{ji}^i are constant, and $F_\lambda(z, y) = F_\lambda(z_1, \dots, z_{\lambda-1}, y_1, \dots, y_{\lambda-1})$ is a holomorphic function with respect to $(z_1, \dots, z_{\lambda-1}, y_1, \dots, y_{\lambda-1})$ for each λ .

(b) Let Γ be a lattice of G . Using the above system of coordinates (z_1, \dots, z_n) of G , we see that any element of $H_{\mathfrak{g}}^{0,1}(G/\Gamma)$ has a representative of the following form:

$$\psi = \sum_{\lambda=1}^k c_\lambda d\bar{z}_\lambda + \sum_{\lambda=k+1}^{k+r(N/\Gamma_N)} f_\lambda(z_1, \dots, z_k) d\bar{z}_\lambda,$$

where $r(N/\Gamma_N) = \dim H_{\mathfrak{g}}^{0,1}(N/\Gamma_N)$, c_λ are constant and $f_\lambda(z_1, \dots, z_k)$ are holomorphic in z_1, \dots, z_k .

We say that a complex solvable Lie algebra \mathfrak{g} has a *Chevalley decomposition* if \mathfrak{g} has a decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ as a vector space, where \mathfrak{a} is a commutative subalgebra and \mathfrak{n} is the maximal nilpotent ideal. For further details see [11].

Using the above system of coordinates, we give another proof of our main theorem for the case where the Lie algebra \mathfrak{g} of G has a Chevalley decomposition (see Section 3).

2. The structure of the sheaf $R^1\pi_*\mathcal{O}_{G/\Gamma}$

For a holomorphic map $f: X \rightarrow Y$ between complex spaces, there exists a Leray spectral sequence for the sheaf \mathcal{O} . The respective lower term sequence yields the following:

$$0 \rightarrow H^1(Y, R^0 f_* \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^0(Y, R^1 f_* \mathcal{O}_X) \rightarrow H^2(Y, R^0 f_* \mathcal{O}_X),$$

where $R^q f_* \mathcal{O}_X$ is the higher direct image sheaf. If f is connected and proper, then $R^0 f_* \mathcal{O}_X = \mathcal{O}_Y$.

In this section we prove the following:

Proposition 2.1. *Let $X^n = G/\Gamma$ be a compact complex parallelizable pseudo-Kähler solvmanifold and $\pi: G/\Gamma \rightarrow G/N\Gamma$ the Mostow fibration. Then $R^1\pi_*\mathcal{O}_X$ is the sheaf of sections of a trivial holomorphic vector bundle.*

To prove this proposition we follow a part of the proof of Theorem 1.2 due to Winkelmann ([14]).

Let G be a connected complex Lie group and Γ a lattice of G . Let V be a complex vector space, $\rho: G \rightarrow GL(V)$ an antiholomorphic representation and V_1 the set of all $v \in V$ which are invariant under $\rho(G')$, where G' is the derived Lie subgroup of G . We denote by V_0 the subspace spanned by all vectors $v \in V_1$ such that v is an eigenvector with a real eigenvalue for every $\rho(\gamma)$ ($\gamma \in \Gamma$).

Let E, E_0 be flat vector bundles over $X = G/\Gamma$ which are induced by $\rho|_\Gamma$ on V, V_0 respectively, i.e., $E = G \times V/\sim$, where $(g, v) \sim (g', v')$ if and only if $(g', v') = (g\gamma^{-1}, \rho(\gamma)v)$. Note that E, E_0 are holomorphic vector bundles.

Proposition 2.2. *The flat vector bundle E_0 is a holomorphically trivial vector bundle and $\Gamma(X, E) = \Gamma(X, E_0) \cong V_0$.*

Proof. See [14], Propositions 7.9.1 and 7.9.2. □

Let $\pi: X \rightarrow B$ be a holomorphic fiber bundle with an n -dimensional complex torus $T_{\mathbb{C}}^n$ as typical fiber. Let $V = \Omega^1(T_{\mathbb{C}}^n)$ denote the vector space of holomorphic 1-forms on $T_{\mathbb{C}}^n$. Note that $T_{\mathbb{C}}^n$ is a compact Kähler manifold. Let $\mathfrak{U} = \{U_i\}$ be a trivializing open cover of B such that X is given by transition functions $\phi_{ij}: U_i \cap U_j \rightarrow \text{Aut}(T_{\mathbb{C}}^n)$, where $\text{Aut}(T_{\mathbb{C}}^n)$ is the automorphism group of $T_{\mathbb{C}}^n$. We denote by $\text{Aut}^0(T_{\mathbb{C}}^n)$ the identity component of $\text{Aut}(T_{\mathbb{C}}^n)$.

Lemma 2.3. *Under the above assumptions $R^1\pi_*\mathcal{O}_X$ is a locally free coherent sheaf of B isomorphic to the sheaf of sections of the flat vector bundle E given by transition functions $\varphi_{ij} = \overline{\zeta \circ \phi_{ij}}: U_i \cap U_j \rightarrow GL(V)$, where $1 \rightarrow \text{Aut}^0(T_{\mathbb{C}}^n) \rightarrow \text{Aut}(T_{\mathbb{C}}^n) \xrightarrow{\zeta} GL(V)$ is exact.*

Proof. See [14], CLAIM 8.4.5. □

We apply this lemma to a complex parallelizable manifold G/Γ . Let K be a normal abelian complex Lie subgroup of G and \mathfrak{k} its Lie algebra. Assume that $K/K \cap \Gamma$ is compact. Denote the natural projection map $X = G/\Gamma \rightarrow B = G/K\Gamma$ by π .

Proposition 2.4. *The sheaf $R^1\pi_*\mathcal{O}_X$ is the sheaf of sections of the flat vector bundle E of rank $\dim K$ over B induced by the representation $\rho: \Gamma \rightarrow GL(\mathfrak{k}^*)$ given by $\gamma \mapsto \text{Ad}^*(\gamma)$.*

Proof. See [14], Proposition 8.4.6. □

Moreover, if G/K is abelian, we have

Proposition 2.5 ([14]). *If G/K is abelian, then*

$$\dim H^1(G/\Gamma, \mathcal{O}) \leq \dim G/K + \dim U,$$

where U denotes the maximal linear subspace of \mathfrak{k} such that $\text{Ad}(\gamma)|_U$ is a semisimple linear endomorphism with only real eigenvalues for every γ in Γ .

Proof. Let us consider the lower term of the Leray spectral sequence for $\pi: G/\Gamma \rightarrow G/K\Gamma$. Then we have the following:

$$0 \rightarrow H^1(G/K\Gamma, \mathcal{O}) \rightarrow H^1(G/\Gamma, \mathcal{O}) \rightarrow H^0(G/K\Gamma, R^1\pi_*\mathcal{O}).$$

Since $G/K\Gamma \cong (G/K)/(K\Gamma/K)$ and G/K is abelian, by Propositions 2.2 and 2.4, we see

$$\begin{aligned} \dim H^1(G/\Gamma, \mathcal{O}) &\leq \dim H^1(G/K\Gamma, \mathcal{O}) + \dim U \\ &= \dim G/K + \dim U. \end{aligned}$$

Hence we have our proposition. □

Proof of Proposition 2.1. In Section 1, we have seen that if a compact complex parallelizable solvmanifold $X^n = G/\Gamma$ admits a pseudo-Kähler structure, then the maximal normal nilpotent Lie subgroup N of G is abelian and $h^{0,1} = n$. Thus let us consider the Mostow fibration $\pi: G/\Gamma \rightarrow G/N\Gamma$. Since G/N is abelian and G/Γ admits a pseudo-Kähler structure, we have $W = \mathfrak{n}$ by Proposition 2.5, where W is the maximal linear subspace of \mathfrak{n} such that $\text{Ad}(\gamma)|_W$ is a semisimple endomorphism with only real eigenvalues for every $\gamma \in \Gamma$. By Propositions 2.2 and 2.4, this means that the flat vector bundle E induced by the representation $\rho|_\Gamma: \Gamma \rightarrow GL(\mathfrak{n}^*)$ given by $\gamma \mapsto \text{Ad}^*(\gamma)$ is trivial as a holomorphic vector bundle. □

3. Dolbeault cohomology of compact complex parallelizable pseudo-Kähler solvmanifolds

In this section we consider the Dolbeault cohomology groups of compact complex parallelizable pseudo-Kähler solvmanifolds.

Let G be a complex Lie group and \mathfrak{g} its Lie algebra. Let I denote the complex structure of \mathfrak{g} , and \mathfrak{g}^+ (resp. \mathfrak{g}^-) denote the vector space of the $+\sqrt{-1}$ (resp. $-\sqrt{-1}$) eigenvectors of the complex structure I respectively. Then we have $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$. In this section, we identify \mathfrak{g}^+ with the set of all right invariant holomorphic vector fields of G . Recall that $H_{\bar{\partial}}^{p,q}(G/\Gamma) \cong H_{\bar{\partial}}^{0,q}(G/\Gamma) \otimes \wedge^p(\mathfrak{g}^+)^*$ for a compact complex parallelizable manifold G/Γ . Sakane [12] has proved that if G is a complex nilpotent Lie group, then $H_{\bar{\partial}}^{p,q}(G/\Gamma) \cong H^q(\mathfrak{g}^-) \otimes \wedge^p(\mathfrak{g}^+)^*$, where $H^q(\mathfrak{g}^-)$ is the q th Lie algebra cohomology group of \mathfrak{g}^- .

Let $F \rightarrow X \xrightarrow{\pi} B$ be a holomorphic fiber bundle such that X, B, F are connected and F is compact. Then $\bigcup_{b \in B} H_{\bar{\partial}}^{p,q}(F_b)$ is the total space of a differentiable vector bundle over B . This bundle is denoted by $\mathbf{H}^{p,q}(F)$ and $\mathbf{H}_{\bar{\partial}}(F)$ is the direct sum of $\mathbf{H}^{p,q}(F)$. If every connected component of the structure group of $\pi: X \rightarrow B$ acts trivially on $\mathbf{H}_{\bar{\partial}}(F)$, then the vector bundle is a holomorphic vector bundle. Thus if the fiber F is a compact Kähler manifold, then $\mathbf{H}_{\bar{\partial}}(F)$ is a holomorphic vector bundle.

Theorem 3.1 ([8]). *Let $\xi = (X, B, F, \pi)$ be a holomorphic fiber bundle, where X, B, F are connected and F is compact. Assume that every connected component of the structure group of ξ acts trivially on $\mathbf{H}_{\bar{\partial}}(F)$, i.e., $\mathbf{H}_{\bar{\partial}}(F)$ is a holomorphic vector bundle. Then there exists a spectral sequence (E_r, d_r) , $(r \geq 0)$, with the following properties:*

- (i) E_r is 4-graded, by the fiber-degree, the base-degree and the type. Let ${}^{p,q}E_r^{s,t}$ be the subspace of elements of E_r of type (p, q) , fiber-degree s and base-degree t . We have ${}^{p,q}E_r^{s,t} = 0$ if $p + q \neq s + t$ or if one of p, q, s, t is negative. The differential d_r maps ${}^{p,q}E_r^{s,t}$ into ${}^{p,q+1}E_r^{s+r,t-r+1}$.
- (ii) If $p + q = s + t$, then we have

$${}^{p,q}E_2^{s,t} \cong \sum_{i \geq 0} H_{\bar{\partial}}^{i,s-i}(B, \mathbf{H}^{p-i,q-s+i}(F)).$$

- (iii) The spectral sequence converges to $H_{\bar{\partial}}(X) = \bigoplus_{p,q} H_{\bar{\partial}}^{p,q}(X)$.

We put ${}^{p,q}E_r = \sum_{s,t \geq 0} {}^{p,q}E_r^{s,t}$. We call the above spectral sequence the *Borel's spectral sequence*.

REMARK 3.2. If F is a complex torus, then the vector bundle $\mathbf{H}_{\bar{\partial}}^{0,1}(F) \rightarrow B$ is isomorphic to the holomorphic vector bundle E considered in Section 2 (see Lemma 2.3).

By Theorem 1.4 and the Borel's spectral sequence, we have

Proposition 3.3. *Let $X = G/\Gamma$ be a compact complex parallelizable manifold which admits a pseudo-Kähler structure and $F \rightarrow X \xrightarrow{\pi} B$ a holomorphic fiber bun-*

dle such that F, B are complex tori. If $\mathbf{H}_{\bar{\partial}}(F) \rightarrow B$ is trivial as a holomorphic vector bundle, then

$$h^{p,q}(X) = \binom{n}{p} \cdot \binom{n}{q}.$$

Proof. By our assumption we see that the Borel’s spectral sequence satisfies

$${}^{p,q}E_2^{s,t} \cong \sum_{i \geq 0} H_{\bar{\partial}}^{i,s-i}(B) \otimes H_{\bar{\partial}}^{p-i,q-s+i}(F).$$

Thus by the relation $\dim {}^{p,q}E_{\infty} \leq \dim {}^{p,q}E_2$, we see that $h^{p,q}(X) \leq \binom{n}{p} \cdot \binom{n}{q}$. Thus we have our proposition by Theorem 1.4. □

If $\mathbf{H}^{0,1}(T_{\mathbb{C}}^n) \rightarrow B$, where $T_{\mathbb{C}}^n$ is an n -dimensional complex torus, admits global holomorphic sections $\sigma_1, \dots, \sigma_n$ which are linearly independent at each point, then $\mathbf{H}^{0,q}(T_{\mathbb{C}}^n) \rightarrow B$ is trivial as a holomorphic vector bundle. Indeed, consider $\sigma_J = \sigma_{j_1} \wedge \dots \wedge \sigma_{j_q}$ (Note that $h^{0,q}(T_{\mathbb{C}}^n) = \binom{n}{q}$). Thus by Proposition 2.1 and Lemma 2.3 we have

Corollary 3.4. *If a compact complex parallelizable solvmanifold $X^n = G/\Gamma$ admits a pseudo-Kähler structure, then*

$$h^{p,q}(X) = \binom{n}{p} \cdot \binom{n}{q}.$$

Proof. By our assumption, we see that $\mathbf{H}_{\bar{\partial}}^{0,q}(N/\Gamma_N) \rightarrow G/N\Gamma$ is trivial as a holomorphic vector bundle. □

Let (X^n, ω) be a compact pseudo-Kähler manifold. We say that (X^n, ω) has the *hard Lefschetz property with respect to the Dolbeault cohomology* if for any $p+q \leq n$, the homomorphism

$$L^{n-p-q} : H_{\bar{\partial}}^{p,q}(X) \rightarrow H_{\bar{\partial}}^{n-q,n-p}(X), \quad L^{n-p-q}([\alpha]) = [\alpha \wedge \omega^{n-p-q}]$$

is an isomorphism.

Corollary 3.5. *Let $(G/\Gamma, \omega)$ be an n -dimensional compact complex parallelizable pseudo-Kähler solvmanifold. Then $(G/\Gamma, \omega)$ has the hard Lefschetz property with respect to the Dolbeault cohomology.*

Proof. Put $\bar{\tau}_i = i(X_i^+) \omega$, where $\{X_1^+, \dots, X_n^+\}$ is a basis of \mathfrak{g}^+ . We denote the dual basis of \mathfrak{g}^+ by $\{\omega_1^+, \dots, \omega_n^+\}$. Then ω can be written as $\omega = \sum_{i=1}^n \bar{\tau}_i \wedge \omega_i^+$. In particular, we see that $\bar{\tau}_i$ are non-exact $\bar{\partial}$ -closed. We also see that $\alpha = \sum_{JK} a_{JK} \bar{\tau}_J \wedge \omega_K^+$

is non-exact $\bar{\partial}$ -closed, where $a_{JK} \in \mathbb{C}$, $\bar{\tau}_J = \bar{\tau}_{j_1} \wedge \cdots \wedge \bar{\tau}_{j_q}$ for $J = (j_1, \dots, j_q)$ and $\omega_K^+ = \omega_{k_1}^+ \wedge \cdots \wedge \omega_{k_p}^+$ for $K = (k_1, \dots, k_p)$. Thus by Corollary 3.4 for each Dolbeault cohomology class we can choose a representative of the form $\alpha = \sum_{JK} a_{JK} \bar{\tau}_J \wedge \omega_K^+$. Hence $(G/\Gamma, \omega)$ has the hard Lefschetz property. \square

By the proof of Corollary 3.5, we see that if an n -dimensional compact complex parallelizable solvmanifold G/Γ admits a pseudo-Kähler structure, then $H_{\bar{\partial}}(G/\Gamma) = \bigoplus_{p,q} H_{\bar{\partial}}^{p,q}(G/\Gamma)$ is isomorphic to the cohomology ring $H_{\bar{\partial}}(T_{\mathbb{C}}^n)$.

REMARK 3.6. Mathieu’s theorem of the Dolbeault cohomology on a compact pseudo-Kähler manifold (X, ω) also holds (see [18], [19]), i.e., the following two assertions are equivalent: (a) every Dolbeault cohomology class contains a $\bar{\partial}$ -harmonic representative. (b) (X, ω) has the hard Lefschetz property with respect to the Dolbeault cohomology. We define $\partial^* : \Omega^{p,q}(X) \rightarrow \Omega^{p-1,q}(X)$ by $\partial^* = (-1)^{p+q} * \bar{\partial} *$, where $\Omega^{p,q}(X)$ is the set of all differential (p, q) -forms on X . A form α is called a $\bar{\partial}$ -harmonic form if it satisfies $\bar{\partial}\alpha = \partial^*\alpha = 0$, where $*$: $\Omega^{p,q}(X^n) \rightarrow \Omega^{n-q,n-p}(X^n)$ is defined as an analogy of the star operator for a compact Riemannian manifold. In the above case, for each Dolbeault cohomology class, we can choose a $\bar{\partial}$ -harmonic representative of the form $\alpha = \sum_{JK} a_{JK} \bar{\tau}_J \wedge \omega_K^+$.

Let $(G/\Gamma, \omega)$ be an n -dimensional compact complex parallelizable pseudo-Kähler solvmanifold. We now give another proof of our main theorem for the case where the Lie algebra \mathfrak{g} of G has a Chevalley decomposition. We use a system of coordinates of Remarks 1.9 and the notation of the proof of Corollary 3.5. Then the pseudo-Kähler structure ω on G/Γ can be written as follows:

$$\omega = \sum_{i=1}^n \bar{\tau}_i \wedge \omega_i^+.$$

Note that $\bar{\tau}_i, \omega_i^+$ are $\bar{\partial}$ -closed. By Remarks 1.9, $\bar{\tau}_i$ are expressed by

$$\bar{\tau}_i = \psi_i + \bar{\partial}\gamma_i,$$

where $\psi_i = \sum_{\lambda=1}^k c_{\lambda}^i d\bar{z}_{\lambda} + \sum_{\lambda=k+1}^{k+r(F)} f_{\lambda}^i(z) d\bar{z}_{\lambda}$, $F = N/\Gamma_N$, c_{λ}^i are constant and f_{λ}^i are holomorphic. Hence we can write

$$\omega = \sum_{i=1}^n \psi_i \wedge \omega_i^+ + \bar{\partial}\theta.$$

Put $\omega' = \sum_{i=1}^n \psi_i \wedge \omega_i^+$. Then a volume form ω^n is expressed by $\omega^n = \omega'^n + \bar{\partial}\Omega$, where $\Omega \in \Omega^{n,n-1}(X)$. Since $\int_{G/\Gamma} \omega^n = \int_{G/\Gamma} \omega'^n + \int_{G/\Gamma} \bar{\partial}\Omega = \int_{G/\Gamma} \omega'^n$, we see that $r(F) = \dim H_{\bar{\partial}}^{0,1}(F) = \dim F = n - k$. Since F is a compact complex parallelizable

nilmanifold, we see $n - k = \dim H_{\bar{\partial}}^{0,1}(N/\Gamma_N) = \dim H^1(\mathfrak{n}^-)$ by Sakane's theorem. Thus \mathfrak{n}^- is abelian, which implies that F is a complex torus.

4. Examples

EXAMPLE 4.1 ([11]). Define a multiplication $*$ of \mathbb{C}^{n+2m} by

$$\begin{aligned} & (z_1, \dots, z_n, w_1, w_2, \dots, w_{2m-1}, w_{2m}) * (z'_1, \dots, z'_n, w'_1, w'_2, \dots, w'_{2m-1}, w'_{2m}) \\ &= (z_1 + z'_1, \dots, z_n + z'_n, \dots, e^{-\sum_i a_i^k z_i} w'_{2k-1} + w_{2k-1}, e^{\sum_i a_i^k z_i} w'_{2k} + w_{2k}, \dots), \end{aligned}$$

where a_i^k are integers. The solvable Lie group $G = (\mathbb{C}^{n+2m}, *)$ has a lattice Γ such that G/Γ has a pseudo-Kähler structure. Indeed, for a suitable lattice Γ of G ,

$$\omega = \sqrt{-1} \sum_{k=1}^n dz_k \wedge d\bar{z}_k + \sum_{k=1}^m (dw_{2k-1} \wedge d\bar{w}_{2k} + d\bar{w}_{2k-1} \wedge dw_{2k})$$

is a pseudo-Kähler structure on G/Γ (for details, see [17]). By Corollary 3.4, we see $h^{p,q}(G/\Gamma) = \binom{n+2m}{p} \cdot \binom{n+2m}{q}$.

EXAMPLE 4.2 (cf. [4]). Let us consider the following solvable Lie group:

$$G = \left\{ \left(\begin{array}{cccccc} e^{z_1} & 0 & z_2 e^{z_1} & 0 & 0 & 0 & w_1 \\ 0 & e^{-z_1} & 0 & z_2 e^{-z_1} & 0 & 0 & w_2 \\ 0 & 0 & e^{z_1} & 0 & 0 & 0 & w_3 \\ 0 & 0 & 0 & e^{-z_1} & 0 & 0 & w_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & z_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & z_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \mid z_1, z_2, w_1, w_2, w_3, w_4 \in \mathbb{C} \right\}.$$

The Lie algebra \mathfrak{g} of G is given by

$$\mathfrak{g} = \text{span}_{\mathbb{C}}\{Z_1, Z_2, W_1, W_2, W_3, W_4\}$$

with

$$\begin{aligned} [Z_1, W_{2k-1}] &= W_{2k-1}, & [Z_1, W_{2k}] &= -W_{2k}, \\ [Z_2, W_3] &= W_1, & [Z_2, W_4] &= W_2 \end{aligned}$$

for $k = 1, 2$. The solvable Lie group G admits a lattice Γ (see [16]). Since the maximal nilpotent ideal \mathfrak{n} is not abelian, we see that for any lattice Γ , G/Γ has no pseudo-Kähler structures by Corollary 1.8.

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