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## A STRUCTURE THEOREM OF COMPACT COMPLEX PARALLELIZABLE PSEUDO-KÄHLER SOLVMANIFOLDS

Dedicated to Professor Yusuke Sakane on his 60th birthday

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### Abstract

In this paper, we prove that the Mostow fibration of a compact complex parallelizable pseudo-Kähler solvmanifold is a complex torus bundle over a complex torus.

### Introduction

A complex manifold  $X^n$  of complex dimension  $n$  is called *complex parallelizable* if there exist  $n$  holomorphic vector fields which are linearly independent at each point. Wang [13] proved that a compact complex parallelizable manifold is of the form  $G/\Gamma$ , where  $G$  is a complex Lie group and  $\Gamma$  is a discrete subgroup of  $G$ . Wang also proved that if a compact complex parallelizable manifold  $X$  admits a Kähler structure, then  $X$  is a complex torus. On the other hand, Matsushima [10] proved that a compact homogeneous Kähler manifold is biholomorphic to a product of a homogeneous rational manifold and a complex torus. By a homogeneous Kähler manifold we mean a Kähler manifold on which the group of holomorphic isometric transformations acts transitively. Borel-Remmert [2] generalized the result of Matsushima to compact Kähler manifolds on which the group of holomorphic transformations acts transitively. Dorfmeister-Guan [3] proved that a compact homogeneous pseudo-Kähler manifold is also biholomorphic to a product of a homogeneous rational manifold and a complex torus. As for compact pseudo-Kähler manifolds on which the group of holomorphic transformations acts transitively, there exist non-toral compact complex parallelizable pseudo-Kähler solvmanifolds. In particular, we see that a compact non-homogeneous pseudo-Kähler manifold is not biholomorphic to a product of a homogeneous rational manifold and a complex torus in general (cf. [17]). It is therefore important to study compact complex parallelizable pseudo-Kähler solvmanifolds. In this paper we prove the following structure theorem, which is our main theorem:

**Theorem 1.6** *Let  $X = G/\Gamma$  be a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure. Then the Mostow fibration of  $X$  is a complex torus bundle over a complex torus.*

We also investigate the Dolbeault cohomology groups of a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure.

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**1. Proof of main theorem**

In this section we prove our main theorem.

**DEFINITION 1.1.** Let  $G$  be a Lie group. A discrete subgroup  $\Gamma$  of  $G$  is called a *lattice* if  $G/\Gamma$  has a finite invariant measure.

If  $G$  is a solvable Lie group, then a discrete subgroup  $\Gamma$  of  $G$  is a lattice if and only if  $\Gamma$  is a discrete co-compact subgroup of  $G$ .

Let  $\mathcal{O}_X = \mathcal{O}$  be the sheaf of holomorphic functions on a complex manifold  $X$ . We denote the Hodge number of  $X$  by  $h^{p,q}(X)$ , i.e.,  $h^{p,q}(X) = \dim H_{\bar{\partial}}^{p,q}(X)$ . Let  $G$  be a connected complex Lie group,  $\Gamma$  a lattice of  $G$ ,  $N$  the maximal connected normal nilpotent subgroup. Let  $G = S \cdot R$  be a Levi decomposition, where  $S$  is a semi-simple part, and  $R$  is the radical. We denote derived Lie subgroups of  $G, N$  and  $R$  by  $G', N'$  and  $R'$  respectively. Winkelmann has proven

**Theorem 1.2** ([14]). *Let  $G, \Gamma, N, S, R, G', N'$  and  $R'$  be as above. Let  $A = [S, R] \cdot N'$ . Furthermore let  $W$  denote the maximal linear subspace of the Lie algebra  $\text{Lie}(R'A/A)$  of  $R'A/A$  such that  $\text{Ad}(\gamma)|_W$ , where  $\text{Ad}$  is the adjoint representation of  $G$ , is a semisimple linear endomorphism with only real eigenvalues for each  $\gamma \in \Gamma$ . Then*

$$\dim H^1(G/\Gamma, \mathcal{O}) \leq \dim G/G' + \dim H^1(G/R\Gamma, \mathcal{O}) + \dim W.$$

**DEFINITION 1.3.** Let  $X$  be a complex manifold. A real  $(1, 1)$ -form  $\omega$  of  $X$  is called a *pseudo-Kähler structure* if  $\omega$  is a non-degenerate closed form.

In the case of a compact complex parallelizable manifold, we have shown the following in the paper [17]:

**Theorem 1.4.** *Let  $X^n = G/\Gamma$  be a compact complex parallelizable manifold which admits a pseudo-Kähler structure. Then*

$$h^{p,q}(X) \geq \binom{n}{p} \cdot \binom{n}{q}.$$

**Corollary 1.5.** *Let  $(G/\Gamma, \omega)$  be an  $n$ -dimensional compact complex parallelizable pseudo-Kähler manifold such that  $\Gamma$  is a lattice of  $G$ . If  $h^{0,1}(G/\Gamma) = \dim H^0(G/\Gamma, d\mathcal{O})$ , then  $G/\Gamma$  is a complex torus.*

Proof. Let  $\mathfrak{g}$  be the Lie algebra of  $G$ ,  $I$  the complex structure of  $\mathfrak{g}$ , and  $\mathfrak{g}^+ = \{X \in \mathfrak{g}^{\mathbb{C}} \mid IX = \sqrt{-1}X\}$ . We identify  $\mathfrak{g}^+$  with the set of all right invariant holomorphic vector fields of  $G$ . Let  $H^q(\mathfrak{g}^+)$  be the  $q$ th Lie algebra cohomology group of  $\mathfrak{g}^+$ . Since  $H^0(G/\Gamma, d\mathcal{O}) \cong H^1(\mathfrak{g}^+)$  and  $h^{0,1}(G/\Gamma) \geq n$ , we see that  $\dim H^1(\mathfrak{g}^+) = n$ . Hence  $\mathfrak{g}^+$  is abelian. □

Let  $G/\Gamma$  be a compact complex parallelizable solvmanifold, i.e.,  $G$  is a simply connected complex solvable Lie group and  $\Gamma$  is a lattice of  $G$ . Mostow proved that  $\Gamma_N = N \cap \Gamma$  is a lattice of the maximal normal nilpotent Lie subgroup  $N$  of  $G$ . A fibration  $N/\Gamma_N \rightarrow G/\Gamma \rightarrow G/N\Gamma$  is called the *Mostow fibration* of  $G/\Gamma$ .

**Theorem 1.6.** *Let  $X^n = G/\Gamma$  be a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure. Then the Mostow fibration of  $G/\Gamma$  is a complex torus bundle over a complex torus.*

Proof. We use the notation of Theorem 1.2. Since  $G$  is solvable, we see that  $G = R, S = \{e\}$  and  $A = N'$ . By Theorems 1.2 and 1.4, we see

$$n \leq \dim H_3^{0,1}(G/\Gamma) \leq \dim G/G' + \dim W \leq \dim G - \dim G' + \dim G'/N',$$

which implies  $\dim N' = 0$ . □

By the proof of our main theorem, we have

**Corollary 1.7.** *If a compact complex parallelizable solvmanifold  $X^n$  admits a pseudo-Kähler structure, then  $h^{0,1}(X) = n$ .*

**Corollary 1.8.** *If a compact complex parallelizable solvmanifold  $G/\Gamma$  admits a pseudo-Kähler structure, then  $N$  must be abelian and in particular the Lie algebra  $\mathfrak{g}$  must satisfy  $\mathcal{D}^{(2)}\mathfrak{g} = 0$ .*

Proof. Since  $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$ , we have our corollary. □

REMARKS 1.9. (i) There exists a complex solvable Lie group  $G$  which has lattices  $\Gamma_1, \Gamma_2$  such that  $G/\Gamma_1$  has a pseudo-Kähler structure, while  $G/\Gamma_2$  has no pseudo-Kähler structures (see [17]).

(ii) It is well known that a simply connected complex solvable Lie group  $G$  is bi-holomorphic to  $\mathbb{C}^n$ . Moreover if its Lie algebra  $\mathfrak{g}$  has a Chevalley decomposition, then there exists a good system of coordinates  $(z_1, \dots, z_n)$  of  $G$  which satisfies the following:

(a) The Lie group  $G$  is isomorphic to  $(\mathbb{C}^n, *)$  as a complex Lie group, where the multiplication  $*$  of  $\mathbb{C}^n$  is given by

$$(z_1, \dots, z_n) * (y_1, \dots, y_n) = (z_1 + y_1, \dots, z_r + y_r, F_{r+1r+1}(y)z_{r+1} + y_{r+1} + F_{r+1}(z, y), \dots, F_{nn}(y)z_n + y_n + F_n(z, y))$$

for  $z = (z_1, \dots, z_n), y = (y_1, \dots, y_n) \in \mathbb{C}^n$ , where  $r = \dim \mathfrak{g} - \dim[\mathfrak{g}, \mathfrak{g}]$ ,  $F_{ii}(z) = \exp(-\sum_{j=1}^k C_{ji}^i z_j)$ , where  $k = \dim G/N\Gamma$  and  $C_{ji}^i$  are constant, and  $F_\lambda(z, y) = F_\lambda(z_1, \dots, z_{\lambda-1}, y_1, \dots, y_{\lambda-1})$  is a holomorphic function with respect to  $(z_1, \dots, z_{\lambda-1}, y_1, \dots, y_{\lambda-1})$  for each  $\lambda$ .

(b) Let  $\Gamma$  be a lattice of  $G$ . Using the above system of coordinates  $(z_1, \dots, z_n)$  of  $G$ , we see that any element of  $H_{\mathfrak{g}}^{0,1}(G/\Gamma)$  has a representative of the following form:

$$\psi = \sum_{\lambda=1}^k c_\lambda d\bar{z}_\lambda + \sum_{\lambda=k+1}^{k+r(N/\Gamma_N)} f_\lambda(z_1, \dots, z_k) d\bar{z}_\lambda,$$

where  $r(N/\Gamma_N) = \dim H_{\mathfrak{g}}^{0,1}(N/\Gamma_N)$ ,  $c_\lambda$  are constant and  $f_\lambda(z_1, \dots, z_k)$  are holomorphic in  $z_1, \dots, z_k$ .

We say that a complex solvable Lie algebra  $\mathfrak{g}$  has a *Chevalley decomposition* if  $\mathfrak{g}$  has a decomposition  $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$  as a vector space, where  $\mathfrak{a}$  is a commutative subalgebra and  $\mathfrak{n}$  is the maximal nilpotent ideal. For further details see [11].

Using the above system of coordinates, we give another proof of our main theorem for the case where the Lie algebra  $\mathfrak{g}$  of  $G$  has a Chevalley decomposition (see Section 3).

### 2. The structure of the sheaf $R^1\pi_*\mathcal{O}_{G/\Gamma}$

For a holomorphic map  $f: X \rightarrow Y$  between complex spaces, there exists a Leray spectral sequence for the sheaf  $\mathcal{O}$ . The respective lower term sequence yields the following:

$$0 \rightarrow H^1(Y, R^0 f_* \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^0(Y, R^1 f_* \mathcal{O}_X) \rightarrow H^2(Y, R^0 f_* \mathcal{O}_X),$$

where  $R^q f_* \mathcal{O}_X$  is the higher direct image sheaf. If  $f$  is connected and proper, then  $R^0 f_* \mathcal{O}_X = \mathcal{O}_Y$ .

In this section we prove the following:

**Proposition 2.1.** *Let  $X^n = G/\Gamma$  be a compact complex parallelizable pseudo-Kähler solvmanifold and  $\pi: G/\Gamma \rightarrow G/N\Gamma$  the Mostow fibration. Then  $R^1\pi_*\mathcal{O}_X$  is the sheaf of sections of a trivial holomorphic vector bundle.*

To prove this proposition we follow a part of the proof of Theorem 1.2 due to Winkelmann ([14]).

Let  $G$  be a connected complex Lie group and  $\Gamma$  a lattice of  $G$ . Let  $V$  be a complex vector space,  $\rho: G \rightarrow GL(V)$  an antiholomorphic representation and  $V_1$  the set of all  $v \in V$  which are invariant under  $\rho(G')$ , where  $G'$  is the derived Lie subgroup of  $G$ . We denote by  $V_0$  the subspace spanned by all vectors  $v \in V_1$  such that  $v$  is an eigenvector with a real eigenvalue for every  $\rho(\gamma)$  ( $\gamma \in \Gamma$ ).

Let  $E, E_0$  be flat vector bundles over  $X = G/\Gamma$  which are induced by  $\rho|_\Gamma$  on  $V, V_0$  respectively, i.e.,  $E = G \times V/\sim$ , where  $(g, v) \sim (g', v')$  if and only if  $(g', v') = (g\gamma^{-1}, \rho(\gamma)v)$ . Note that  $E, E_0$  are holomorphic vector bundles.

**Proposition 2.2.** *The flat vector bundle  $E_0$  is a holomorphically trivial vector bundle and  $\Gamma(X, E) = \Gamma(X, E_0) \cong V_0$ .*

Proof. See [14], Propositions 7.9.1 and 7.9.2. □

Let  $\pi: X \rightarrow B$  be a holomorphic fiber bundle with an  $n$ -dimensional complex torus  $T_{\mathbb{C}}^n$  as typical fiber. Let  $V = \Omega^1(T_{\mathbb{C}}^n)$  denote the vector space of holomorphic 1-forms on  $T_{\mathbb{C}}^n$ . Note that  $T_{\mathbb{C}}^n$  is a compact Kähler manifold. Let  $\mathfrak{U} = \{U_i\}$  be a trivializing open cover of  $B$  such that  $X$  is given by transition functions  $\phi_{ij}: U_i \cap U_j \rightarrow \text{Aut}(T_{\mathbb{C}}^n)$ , where  $\text{Aut}(T_{\mathbb{C}}^n)$  is the automorphism group of  $T_{\mathbb{C}}^n$ . We denote by  $\text{Aut}^0(T_{\mathbb{C}}^n)$  the identity component of  $\text{Aut}(T_{\mathbb{C}}^n)$ .

**Lemma 2.3.** *Under the above assumptions  $R^1\pi_*\mathcal{O}_X$  is a locally free coherent sheaf of  $B$  isomorphic to the sheaf of sections of the flat vector bundle  $E$  given by transition functions  $\varphi_{ij} = \overline{\zeta \circ \phi_{ij}}: U_i \cap U_j \rightarrow GL(V)$ , where  $1 \rightarrow \text{Aut}^0(T_{\mathbb{C}}^n) \rightarrow \text{Aut}(T_{\mathbb{C}}^n) \xrightarrow{\zeta} GL(V)$  is exact.*

Proof. See [14], CLAIM 8.4.5. □

We apply this lemma to a complex parallelizable manifold  $G/\Gamma$ . Let  $K$  be a normal abelian complex Lie subgroup of  $G$  and  $\mathfrak{k}$  its Lie algebra. Assume that  $K/K \cap \Gamma$  is compact. Denote the natural projection map  $X = G/\Gamma \rightarrow B = G/K\Gamma$  by  $\pi$ .

**Proposition 2.4.** *The sheaf  $R^1\pi_*\mathcal{O}_X$  is the sheaf of sections of the flat vector bundle  $E$  of rank  $\dim K$  over  $B$  induced by the representation  $\rho: \Gamma \rightarrow GL(\mathfrak{k}^*)$  given by  $\gamma \mapsto \text{Ad}^*(\gamma)$ .*

Proof. See [14], Proposition 8.4.6. □

Moreover, if  $G/K$  is abelian, we have

**Proposition 2.5** ([14]). *If  $G/K$  is abelian, then*

$$\dim H^1(G/\Gamma, \mathcal{O}) \leq \dim G/K + \dim U,$$

where  $U$  denotes the maximal linear subspace of  $\mathfrak{k}$  such that  $\text{Ad}(\gamma)|_U$  is a semisimple linear endomorphism with only real eigenvalues for every  $\gamma$  in  $\Gamma$ .

Proof. Let us consider the lower term of the Leray spectral sequence for  $\pi: G/\Gamma \rightarrow G/K\Gamma$ . Then we have the following:

$$0 \rightarrow H^1(G/K\Gamma, \mathcal{O}) \rightarrow H^1(G/\Gamma, \mathcal{O}) \rightarrow H^0(G/K\Gamma, R^1\pi_*\mathcal{O}).$$

Since  $G/K\Gamma \cong (G/K)/(K\Gamma/K)$  and  $G/K$  is abelian, by Propositions 2.2 and 2.4, we see

$$\begin{aligned} \dim H^1(G/\Gamma, \mathcal{O}) &\leq \dim H^1(G/K\Gamma, \mathcal{O}) + \dim U \\ &= \dim G/K + \dim U. \end{aligned}$$

Hence we have our proposition. □

Proof of Proposition 2.1. In Section 1, we have seen that if a compact complex parallelizable solvmanifold  $X^n = G/\Gamma$  admits a pseudo-Kähler structure, then the maximal normal nilpotent Lie subgroup  $N$  of  $G$  is abelian and  $h^{0,1} = n$ . Thus let us consider the Mostow fibration  $\pi: G/\Gamma \rightarrow G/N\Gamma$ . Since  $G/N$  is abelian and  $G/\Gamma$  admits a pseudo-Kähler structure, we have  $W = \mathfrak{n}$  by Proposition 2.5, where  $W$  is the maximal linear subspace of  $\mathfrak{n}$  such that  $\text{Ad}(\gamma)|_W$  is a semisimple endomorphism with only real eigenvalues for every  $\gamma \in \Gamma$ . By Propositions 2.2 and 2.4, this means that the flat vector bundle  $E$  induced by the representation  $\rho|_\Gamma: \Gamma \rightarrow GL(\mathfrak{n}^*)$  given by  $\gamma \mapsto \text{Ad}^*(\gamma)$  is trivial as a holomorphic vector bundle. □

### 3. Dolbeault cohomology of compact complex parallelizable pseudo-Kähler solvmanifolds

In this section we consider the Dolbeault cohomology groups of compact complex parallelizable pseudo-Kähler solvmanifolds.

Let  $G$  be a complex Lie group and  $\mathfrak{g}$  its Lie algebra. Let  $I$  denote the complex structure of  $\mathfrak{g}$ , and  $\mathfrak{g}^+$  (resp.  $\mathfrak{g}^-$ ) denote the vector space of the  $+\sqrt{-1}$  (resp.  $-\sqrt{-1}$ ) eigenvectors of the complex structure  $I$  respectively. Then we have  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ . In this section, we identify  $\mathfrak{g}^+$  with the set of all right invariant holomorphic vector fields of  $G$ . Recall that  $H_{\bar{\partial}}^{p,q}(G/\Gamma) \cong H_{\bar{\partial}}^{0,q}(G/\Gamma) \otimes \wedge^p(\mathfrak{g}^+)^*$  for a compact complex parallelizable manifold  $G/\Gamma$ . Sakane [12] has proved that if  $G$  is a complex nilpotent Lie group, then  $H_{\bar{\partial}}^{p,q}(G/\Gamma) \cong H^q(\mathfrak{g}^-) \otimes \wedge^p(\mathfrak{g}^+)^*$ , where  $H^q(\mathfrak{g}^-)$  is the  $q$ th Lie algebra cohomology group of  $\mathfrak{g}^-$ .

Let  $F \rightarrow X \xrightarrow{\pi} B$  be a holomorphic fiber bundle such that  $X, B, F$  are connected and  $F$  is compact. Then  $\bigcup_{b \in B} H_{\bar{\partial}}^{p,q}(F_b)$  is the total space of a differentiable vector bundle over  $B$ . This bundle is denoted by  $\mathbf{H}^{p,q}(F)$  and  $\mathbf{H}_{\bar{\partial}}(F)$  is the direct sum of  $\mathbf{H}^{p,q}(F)$ . If every connected component of the structure group of  $\pi: X \rightarrow B$  acts trivially on  $\mathbf{H}_{\bar{\partial}}(F)$ , then the vector bundle is a holomorphic vector bundle. Thus if the fiber  $F$  is a compact Kähler manifold, then  $\mathbf{H}_{\bar{\partial}}(F)$  is a holomorphic vector bundle.

**Theorem 3.1** ([8]). *Let  $\xi = (X, B, F, \pi)$  be a holomorphic fiber bundle, where  $X, B, F$  are connected and  $F$  is compact. Assume that every connected component of the structure group of  $\xi$  acts trivially on  $\mathbf{H}_{\bar{\partial}}(F)$ , i.e.,  $\mathbf{H}_{\bar{\partial}}(F)$  is a holomorphic vector bundle. Then there exists a spectral sequence  $(E_r, d_r)$ ,  $(r \geq 0)$ , with the following properties:*

- (i)  $E_r$  is 4-graded, by the fiber-degree, the base-degree and the type. Let  ${}^{p,q}E_r^{s,t}$  be the subspace of elements of  $E_r$  of type  $(p, q)$ , fiber-degree  $s$  and base-degree  $t$ . We have  ${}^{p,q}E_r^{s,t} = 0$  if  $p + q \neq s + t$  or if one of  $p, q, s, t$  is negative. The differential  $d_r$  maps  ${}^{p,q}E_r^{s,t}$  into  ${}^{p,q+1}E_r^{s+r,t-r+1}$ .
- (ii) If  $p + q = s + t$ , then we have

$${}^{p,q}E_2^{s,t} \cong \sum_{i \geq 0} H_{\bar{\partial}}^{i,s-i}(B, \mathbf{H}^{p-i,q-s+i}(F)).$$

- (iii) The spectral sequence converges to  $H_{\bar{\partial}}(X) = \bigoplus_{p,q} H_{\bar{\partial}}^{p,q}(X)$ .

We put  ${}^{p,q}E_r = \sum_{s,t \geq 0} {}^{p,q}E_r^{s,t}$ . We call the above spectral sequence the *Borel's spectral sequence*.

**REMARK 3.2.** If  $F$  is a complex torus, then the vector bundle  $\mathbf{H}_{\bar{\partial}}^{0,1}(F) \rightarrow B$  is isomorphic to the holomorphic vector bundle  $E$  considered in Section 2 (see Lemma 2.3).

By Theorem 1.4 and the Borel's spectral sequence, we have

**Proposition 3.3.** *Let  $X = G/\Gamma$  be a compact complex parallelizable manifold which admits a pseudo-Kähler structure and  $F \rightarrow X \xrightarrow{\pi} B$  a holomorphic fiber bun-*



dle such that  $F, B$  are complex tori. If  $\mathbf{H}_{\bar{\partial}}(F) \rightarrow B$  is trivial as a holomorphic vector bundle, then

$$h^{p,q}(X) = \binom{n}{p} \cdot \binom{n}{q}.$$

Proof. By our assumption we see that the Borel’s spectral sequence satisfies

$${}^{p,q}E_2^{s,t} \cong \sum_{i \geq 0} H_{\bar{\partial}}^{i,s-i}(B) \otimes H_{\bar{\partial}}^{p-i,q-s+i}(F).$$

Thus by the relation  $\dim {}^{p,q}E_{\infty} \leq \dim {}^{p,q}E_2$ , we see that  $h^{p,q}(X) \leq \binom{n}{p} \cdot \binom{n}{q}$ . Thus we have our proposition by Theorem 1.4. □

If  $\mathbf{H}^{0,1}(T_{\mathbb{C}}^n) \rightarrow B$ , where  $T_{\mathbb{C}}^n$  is an  $n$ -dimensional complex torus, admits global holomorphic sections  $\sigma_1, \dots, \sigma_n$  which are linearly independent at each point, then  $\mathbf{H}^{0,q}(T_{\mathbb{C}}^n) \rightarrow B$  is trivial as a holomorphic vector bundle. Indeed, consider  $\sigma_J = \sigma_{j_1} \wedge \dots \wedge \sigma_{j_q}$  (Note that  $h^{0,q}(T_{\mathbb{C}}^n) = \binom{n}{q}$ ). Thus by Proposition 2.1 and Lemma 2.3 we have

**Corollary 3.4.** *If a compact complex parallelizable solvmanifold  $X^n = G/\Gamma$  admits a pseudo-Kähler structure, then*

$$h^{p,q}(X) = \binom{n}{p} \cdot \binom{n}{q}.$$

Proof. By our assumption, we see that  $\mathbf{H}_{\bar{\partial}}^{0,q}(N/\Gamma_N) \rightarrow G/N\Gamma$  is trivial as a holomorphic vector bundle. □

Let  $(X^n, \omega)$  be a compact pseudo-Kähler manifold. We say that  $(X^n, \omega)$  has the *hard Lefschetz property with respect to the Dolbeault cohomology* if for any  $p+q \leq n$ , the homomorphism

$$L^{n-p-q} : H_{\bar{\partial}}^{p,q}(X) \rightarrow H_{\bar{\partial}}^{n-q,n-p}(X), \quad L^{n-p-q}([\alpha]) = [\alpha \wedge \omega^{n-p-q}]$$

is an isomorphism.

**Corollary 3.5.** *Let  $(G/\Gamma, \omega)$  be an  $n$ -dimensional compact complex parallelizable pseudo-Kähler solvmanifold. Then  $(G/\Gamma, \omega)$  has the hard Lefschetz property with respect to the Dolbeault cohomology.*

Proof. Put  $\bar{\tau}_i = i(X_i^+) \omega$ , where  $\{X_1^+, \dots, X_n^+\}$  is a basis of  $\mathfrak{g}^+$ . We denote the dual basis of  $\mathfrak{g}^+$  by  $\{\omega_1^+, \dots, \omega_n^+\}$ . Then  $\omega$  can be written as  $\omega = \sum_{i=1}^n \bar{\tau}_i \wedge \omega_i^+$ . In particular, we see that  $\bar{\tau}_i$  are non-exact  $\bar{\partial}$ -closed. We also see that  $\alpha = \sum_{JK} a_{JK} \bar{\tau}_J \wedge \omega_K^+$

is non-exact  $\bar{\partial}$ -closed, where  $a_{JK} \in \mathbb{C}$ ,  $\bar{\tau}_J = \bar{\tau}_{j_1} \wedge \cdots \wedge \bar{\tau}_{j_q}$  for  $J = (j_1, \dots, j_q)$  and  $\omega_K^+ = \omega_{k_1}^+ \wedge \cdots \wedge \omega_{k_p}^+$  for  $K = (k_1, \dots, k_p)$ . Thus by Corollary 3.4 for each Dolbeault cohomology class we can choose a representative of the form  $\alpha = \sum_{JK} a_{JK} \bar{\tau}_J \wedge \omega_K^+$ . Hence  $(G/\Gamma, \omega)$  has the hard Lefschetz property.  $\square$

By the proof of Corollary 3.5, we see that if an  $n$ -dimensional compact complex parallelizable solvmanifold  $G/\Gamma$  admits a pseudo-Kähler structure, then  $H_{\bar{\partial}}(G/\Gamma) = \bigoplus_{p,q} H_{\bar{\partial}}^{p,q}(G/\Gamma)$  is isomorphic to the cohomology ring  $H_{\bar{\partial}}(T_{\mathbb{C}}^n)$ .

REMARK 3.6. Mathieu’s theorem of the Dolbeault cohomology on a compact pseudo-Kähler manifold  $(X, \omega)$  also holds (see [18], [19]), i.e., the following two assertions are equivalent: (a) every Dolbeault cohomology class contains a  $\bar{\partial}$ -harmonic representative. (b)  $(X, \omega)$  has the hard Lefschetz property with respect to the Dolbeault cohomology. We define  $\partial^* : \Omega^{p,q}(X) \rightarrow \Omega^{p-1,q}(X)$  by  $\partial^* = (-1)^{p+q} * \bar{\partial} *$ , where  $\Omega^{p,q}(X)$  is the set of all differential  $(p, q)$ -forms on  $X$ . A form  $\alpha$  is called a  $\bar{\partial}$ -harmonic form if it satisfies  $\bar{\partial}\alpha = \partial^*\alpha = 0$ , where  $*$  :  $\Omega^{p,q}(X^n) \rightarrow \Omega^{n-q,n-p}(X^n)$  is defined as an analogy of the star operator for a compact Riemannian manifold. In the above case, for each Dolbeault cohomology class, we can choose a  $\bar{\partial}$ -harmonic representative of the form  $\alpha = \sum_{JK} a_{JK} \bar{\tau}_J \wedge \omega_K^+$ .

Let  $(G/\Gamma, \omega)$  be an  $n$ -dimensional compact complex parallelizable pseudo-Kähler solvmanifold. We now give another proof of our main theorem for the case where the Lie algebra  $\mathfrak{g}$  of  $G$  has a Chevalley decomposition. We use a system of coordinates of Remarks 1.9 and the notation of the proof of Corollary 3.5. Then the pseudo-Kähler structure  $\omega$  on  $G/\Gamma$  can be written as follows:

$$\omega = \sum_{i=1}^n \bar{\tau}_i \wedge \omega_i^+.$$

Note that  $\bar{\tau}_i, \omega_i^+$  are  $\bar{\partial}$ -closed. By Remarks 1.9,  $\bar{\tau}_i$  are expressed by

$$\bar{\tau}_i = \psi_i + \bar{\partial}\gamma_i,$$

where  $\psi_i = \sum_{\lambda=1}^k c_{\lambda}^i d\bar{z}_{\lambda} + \sum_{\lambda=k+1}^{k+r(F)} f_{\lambda}^i(z) d\bar{z}_{\lambda}$ ,  $F = N/\Gamma_N$ ,  $c_{\lambda}^i$  are constant and  $f_{\lambda}^i$  are holomorphic. Hence we can write

$$\omega = \sum_{i=1}^n \psi_i \wedge \omega_i^+ + \bar{\partial}\theta.$$

Put  $\omega' = \sum_{i=1}^n \psi_i \wedge \omega_i^+$ . Then a volume form  $\omega^n$  is expressed by  $\omega^n = \omega'^n + \bar{\partial}\Omega$ , where  $\Omega \in \Omega^{n,n-1}(X)$ . Since  $\int_{G/\Gamma} \omega^n = \int_{G/\Gamma} \omega'^n + \int_{G/\Gamma} \bar{\partial}\Omega = \int_{G/\Gamma} \omega'^n$ , we see that  $r(F) = \dim H_{\bar{\partial}}^{0,1}(F) = \dim F = n - k$ . Since  $F$  is a compact complex parallelizable

nilmanifold, we see  $n - k = \dim H_{\bar{\partial}}^{0,1}(N/\Gamma_N) = \dim H^1(\mathfrak{n}^-)$  by Sakane's theorem. Thus  $\mathfrak{n}^-$  is abelian, which implies that  $F$  is a complex torus.

**4. Examples**

EXAMPLE 4.1 ([11]). Define a multiplication  $*$  of  $\mathbb{C}^{n+2m}$  by

$$\begin{aligned} & (z_1, \dots, z_n, w_1, w_2, \dots, w_{2m-1}, w_{2m}) * (z'_1, \dots, z'_n, w'_1, w'_2, \dots, w'_{2m-1}, w'_{2m}) \\ &= (z_1 + z'_1, \dots, z_n + z'_n, \dots, e^{-\sum_i a_i^k z_i} w'_{2k-1} + w_{2k-1}, e^{\sum_i a_i^k z_i} w'_{2k} + w_{2k}, \dots), \end{aligned}$$

where  $a_i^k$  are integers. The solvable Lie group  $G = (\mathbb{C}^{n+2m}, *)$  has a lattice  $\Gamma$  such that  $G/\Gamma$  has a pseudo-Kähler structure. Indeed, for a suitable lattice  $\Gamma$  of  $G$ ,

$$\omega = \sqrt{-1} \sum_{k=1}^n dz_k \wedge d\bar{z}_k + \sum_{k=1}^m (dw_{2k-1} \wedge d\bar{w}_{2k} + d\bar{w}_{2k-1} \wedge dw_{2k})$$

is a pseudo-Kähler structure on  $G/\Gamma$  (for details, see [17]). By Corollary 3.4, we see  $h^{p,q}(G/\Gamma) = \binom{n+2m}{p} \cdot \binom{n+2m}{q}$ .

EXAMPLE 4.2 (cf. [4]). Let us consider the following solvable Lie group:

$$G = \left\{ \left( \begin{array}{cccccc} e^{z_1} & 0 & z_2 e^{z_1} & 0 & 0 & 0 & w_1 \\ 0 & e^{-z_1} & 0 & z_2 e^{-z_1} & 0 & 0 & w_2 \\ 0 & 0 & e^{z_1} & 0 & 0 & 0 & w_3 \\ 0 & 0 & 0 & e^{-z_1} & 0 & 0 & w_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & z_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & z_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \mid z_1, z_2, w_1, w_2, w_3, w_4 \in \mathbb{C} \right\}.$$

The Lie algebra  $\mathfrak{g}$  of  $G$  is given by

$$\mathfrak{g} = \text{span}_{\mathbb{C}}\{Z_1, Z_2, W_1, W_2, W_3, W_4\}$$

with

$$\begin{aligned} [Z_1, W_{2k-1}] &= W_{2k-1}, & [Z_1, W_{2k}] &= -W_{2k}, \\ [Z_2, W_3] &= W_1, & [Z_2, W_4] &= W_2 \end{aligned}$$

for  $k = 1, 2$ . The solvable Lie group  $G$  admits a lattice  $\Gamma$  (see [16]). Since the maximal nilpotent ideal  $\mathfrak{n}$  is not abelian, we see that for any lattice  $\Gamma$ ,  $G/\Gamma$  has no pseudo-Kähler structures by Corollary 1.8.

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