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ON ASYMPTOTICS OF EIGENVALUES FOR A CERTAIN 1-DIMENSIONAL RANDOM SCHRÖDINGER OPERATOR

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Abstract

The purpose of this paper is to study the limit distribution of individual eigenvalue of 1-dimensional Schrödinger operators with random potentials derived from the derivatives of compound Poisson processes possessing purely positive jumps or purely negative jumps. The central limit theorem for "middle eigenvalue" is also investigated.

1. Introduction

Consider a one-dimensional Schrödinger operator

$$L = -\frac{d^2}{dx^2} + \frac{dQ_x}{dx}$$

on an interval $[0, a]$, where Q_x is a one dimensional compound Poisson process. Imposing a suitable boundary condition, the operator L has countably many eigenvalues denoted by

$$\{\lambda_1(a) < \lambda_2(a) < \cdots < \lambda_k(a) < \cdots\}.$$

In this paper, we investigate limit behavior of individual eigenvalue. This problem has been considered by [3], [5]. In [5], McKean studied L with Gaussian white noise potential and showed

$$aN(\lambda_1(a)) \xrightarrow{\text{weakly}} e^{-x}\,dx$$

as $a \to \infty$, where N is the integrated density of states studied by [2]

$$N(\lambda)^{-1} = \sqrt{2\pi}\int_0^\infty e^{-(x^3/6+2\lambda x)}\frac{dx}{\sqrt{x}}.$$

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His result shows, whatever boundary condition we impose at the boundary of $[0, a]$, the limit distribution remains unchanged. However our Theorem 3.9 asserts that if Q has only positive jumps, then the limit distribution depends on the boundary condition. If the process has purely negative jumps, then the situation approaches to the Gaussian white noise case. Although we have to assume the exponential distribution for random variables describing jumps for a technical reason, we could prove Theorem 4.6. We remark they expressed in [1] the distribution of $\lambda_1(a)$ by using the drift formula.

In [3], Grenkova et al. tried to show the joint distribution of $\{\lambda_k(a)\}$ has a limit distribution after a suitable normalization. They assumed that the magnitude of jumps of the compound Poisson process obeys an exponential distribution with parameter μ. What they pointed out was the independence on μ of the limit distribution of individual $\lambda_k(a)$. However the reality is contrary, and their method does not seem to work to obtain a joint limit distribution. It would be interesting if we could obtain some results in this respect.

The above problems are related to some properties of the spectrum of L defined on $\mathbb{R}$ in infinitesimal neighborhood of the bottom. When we look the n-th eigenvalue, assuming n increases according to the expansion of $[0, a]$, the situation changes drastically and we can obtain a central limit theorem, namely Theorem 5.1.

A beautiful introduction to this and related field is given by Minami [6]. We are grateful to professor Minami for his kind suggestion when we were preparing this paper.

2. Eigenvalues and zeros of eigenfunctions

Let $\{Q(x);\ x \in [0, \infty)\}$ be a function which is of bounded variation on each finite interval of $[0, \infty)$. Then we can define a selfadjoint operator L formally given by

$$L = -\frac{d^2}{dx^2} + \frac{dQ(x)}{dx},$$

on each finite interval $[0, a]$ if we impose a boundary condition

$$\xi(0)\cos\theta + \xi'(0)\sin\theta = 0, \quad \xi(a) = 0.$$

The boundary condition at $x = 0$ is general, however at $x = a$ we assume the Dirichlet condition for technical reason. For any $\lambda \in \mathbb{C}$, let ξ_λ be a solution of the following equation

$$L\xi = \lambda\xi \quad \text{and} \quad \xi(0) = -\sin\theta, \quad \xi'(0) = \cos\theta. \tag{2.1}$$

$\xi_\lambda(x)$ is an analytic function of λ. The eigenvalues of the operator L coincide with

$$\{\lambda \in \mathbb{R} : \xi_\lambda(a) = 0\}$$

and we denote them by $\{\lambda_1(a) < \lambda_2(a) < \cdots < \lambda_k(a) < \cdots\}$. Each $\lambda_k(a)$ is a decreasing function of a. Put $\eta_\lambda(x) = \xi'_\lambda(x)$ and $\zeta(x) = (\xi_\lambda(x), \eta_\lambda(x))$. Then $\zeta(x)$ satisfies

$$\begin{cases} d\xi(x) = \eta(x)\,dx, \\ d\eta(x) = -\lambda\xi(x)\,dx + \xi(x)\,dQ(x). \end{cases} \tag{2.2}$$

Let G be a continuous map from $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{R} \cup \{\infty\}$ defined by

$$G(\zeta) = -\frac{\eta}{\xi} \quad \text{for} \quad \zeta = (\xi, \eta).$$

Setting $Z(x) = G(\zeta(x))$, we see that $Z(x)$ satisfies

$$\begin{cases} dZ(x) = (\lambda + Z(x)^2)\,dx - dQ(x), \\ Z(0) = z = \cot\theta, \end{cases} \tag{2.3}$$

as far as $\xi(x)$ does not vanish. Since $\xi_\lambda(x)$ is a solution of $L\xi = \lambda\xi$, it is known that the set of zeros of $\xi_\lambda(x)$ has no accumulating points. Let $\tau_k(\lambda)$ be the k-th zero from the left end point 0 of $\xi_\lambda(x)$:

$$\begin{cases} \tau_1(\lambda) = \inf\{x > 0;\ Z(x) = \infty\}, \\ \tau_k(\lambda) = \inf\{x > \tau_{k-1};\ Z(x) = \infty\}, \quad k = 2, 3, \ldots. \end{cases}$$

The following lemma can be proved easily (see [4]).

Lemma 2.1. *If $\lambda > \lim_{a\to\infty} \lambda_k(a)$, then $\tau_k(\lambda) < \infty$ and we have*
1. *For any fixed θ, $\tau_k(\lambda)$ is a decreasing continuous function of λ;*
2. *$Z(x)$ is continuous at $\tau_n(\lambda)$ and*

$$Z(\tau_k(\lambda) - 0) = \infty, \quad Z(\tau_k(\lambda) + 0) = -\infty.$$

Owing to the Sturm oscillation theorem, we can replace the study of $\lambda_k(a)$ with that of $\tau_k(\lambda)$ as follows.

Proposition 2.2. *For each a and λ, $\lambda_k(a) > \lambda$ if and only if $\tau_k(\lambda) > a$.*

3. Limit theorem of eigenvalue: positive jump case

From now on we assume $\{Q(x)\}_{x\geq 0}$ be a compound Poisson process whose derivative is formally expressed as

$$Q'(x) = \sum_{j=1}^{\infty} q_j \delta(x - x_j),$$

where $\{q_j\}_{j\geq 1}$are *i.i.d.* random variables and $\{x_j\}_{j\geq 1}$ are random variables such that

$$\begin{cases} 0 = x_0 < x_1 < x_2 < \cdots < x_j < \cdots \quad \text{and} \\ \{x_{j+1} - x_j\}_{j\geq 0} \quad \text{are } \textit{i.i.d.} \text{ with} \quad P(x_{j+1} - x_j > x) = e^{-mx}. \end{cases} \tag{3.1}$$

We assume also that two families of random variables are independent. $\{Q(x)\}_{x\geq 0}$ becomes a Lévy process with a Lévy measure

$$n(du) = mF(du),$$

where F is the distribution of q_j. From the equation (2.3), the generator A of the strong Markov process $Z(x)$ has the form

$$\begin{cases} Af(z) = (z^2 + \lambda) f'(z) + \displaystyle\int_0^\infty \{ f(z-u) - f(z)\} n(du), \quad \text{for} \quad z \neq \infty \\ Af(\infty) = -\displaystyle\lim_{z\to\infty} z\{ f(z) - f(\infty)\}. \end{cases}$$

In this section we assume the measure $n(du)$ has non-negative support, i.e.

$$\operatorname{supp} n(du) \subseteq [0, +\infty) \quad (\Leftrightarrow q_j \geq 0 \text{ a.s.}). \tag{3.2}$$

In order to get the asymptotics of τ_k, we employ the method used by Kotani in ([4]) when he obtained the asymptotics of $\mathbb{E}_{-\infty}(\tau_1)$. His method is illustrated as follows. Assume

$$\int_{u>1} (\log u) n(du) < \infty.$$

For any $z \in \mathbb{R} \cup \{\infty\}$ and $\lambda > 0$, $\mathbb{E}_z(\tau_1)$ is finite and $f(z) = \mathbb{E}_z(\tau_1)$ is the unique solution of

$$\begin{cases} (z^2 + \lambda) f'(z) + \displaystyle\int_0^\infty [f(z-u) - f(z)] n(du) = -1, \\ f(+\infty, \lambda) = 0, \quad | f(-\infty, \lambda)| < \infty. \end{cases} \tag{3.3}$$

To investigate the equation (3.3) we apply Fourier transformation to the both sides and obtain

$$\varphi''(s) = \left(\lambda - \frac{\psi(s)}{is}\right)\varphi(s), \quad \varphi(0) = 1, \quad \varphi(\pm\infty) = 0, \tag{3.4}$$

where

$$\begin{cases} \varphi(s) = \displaystyle\int_{\mathbb{R}} e^{-isz} T(z)\, dz \quad \text{with} \quad T(-z) = \frac{-f'(z)}{E_{-\infty}\tau_1}, \\ \psi(s) = \displaystyle\int_0^\infty (e^{isu} - 1) n(du). \end{cases}$$

The integrated density of states $N(\lambda)$ is equal to $(\mathbb{E}_{-\infty}\tau_1)^{-1}$ and can be identified with

$$N(\lambda) = -\frac{1}{\pi}\,\mathrm{Re}\,\varphi'(0+),$$

To study the asymptotic behavior of $N(\lambda)$ near $\lambda = 0$, it is more convenient to make analytic continuation of (3.4) up to the positive imaginary axis and we reach an equation

$$\begin{cases} f_-''(x,\lambda) = \{-\lambda + U(x)\}\, f_-(x,\lambda), \\ f_-(x,\lambda) \sim \exp\left\{-i\displaystyle\int_{x_0}^{x} \sqrt{\lambda - U(y)}\, dy\right\} \quad \text{as} \quad x \to +\infty, \end{cases}$$

where x_0 is any positive number such that $\lambda - U(x_0) \neq 0$ and

$$U(x) = \frac{1}{x}\int_0^\infty (1 - e^{-ux})n(du).$$

With this f_- we have

$$\mathbb{E}_{-\infty}\tau_1(\lambda) = \frac{\pi\,|\,f_-(0,\lambda)|^2}{\sqrt{\lambda}}, \tag{3.5}$$

and after non-trivial calculation we can see that $\mathbb{E}_{-\infty}\tau_1(\lambda)$ $(= N(\lambda)^{-1})$ has the asymptotics

$$\mathbb{E}_{-\infty}\tau_1(\lambda) \sim \pi\,|\,f_0(0)|^2 \exp\left(\frac{n\pi}{\sqrt{\lambda}}\right) \quad \text{as} \quad \lambda \downarrow 0, \tag{3.6}$$

where $n = \int_0^\infty n(du)$ and f_0 is a unique solution of the equation

$$\begin{cases} f_0''(x) = U(x)\, f_0(x), \\ f_0(x) \sim U(x)^{-1/4} \exp\left\{-\displaystyle\int_0^x \sqrt{U(y)}\, dy\right\} \quad \text{as} \quad x \to +\infty. \end{cases} \tag{3.7}$$

Now we turn to the study of the distribution of τ_1. Set

$$N(\lambda) = (\mathbb{E}_{-\infty}(\tau_1))^{-1}.$$

What we should investigate is the normalized random variable $N(\lambda)\tau_1$. Let us denote by ρ_λ its Laplace transform:

$$\rho_\lambda(z) = \mathbb{E}_z e^{-\alpha N(\lambda)\tau_1} \quad \text{for} \quad \alpha \geq 0.$$

Then $\rho_\lambda(z)$ can be interpreted by the following lemma. The proof is analogous to that of Kotani ([4]) and is omitted.

Lemma 3.1. *$\rho_\lambda(z)$ is a unique solution of the equation*

$$\begin{cases} (z^2+\lambda) f'(z) + \int_0^\infty \{ f(z-u) - f(z)\} n(du) = \alpha N(\lambda) f(z), \\ f(+\infty) = 1, \quad | f(-\infty)| < 1. \end{cases} \tag{3.8}$$

Let us introduce a function $T_\lambda(z)$ by

$$T_\lambda(-z) = \rho'_\lambda(z)[1-\rho_\lambda(-\infty)]^{-1}.$$

Then $T_\lambda(z)\,dz$ becomes a probability measure and satisfies

$$(z^2+\lambda)T_\lambda(z) - \int_0^\infty n(du) \int_z^{z+u} T_\lambda(y)\,dy = \frac{\rho_\lambda(-z)\alpha N(\lambda)}{1-\rho_\lambda(-\infty)}. \tag{3.9}$$

Set

$$\varphi_\lambda(s) = \int_{-\infty}^\infty e^{-isz} T_\lambda(z)\,dz.$$

(3.9) implies

$$\begin{cases} \varphi''_\lambda(s) = \{\lambda + \alpha N(\lambda) - V(s)\}\varphi_\lambda(s), \\ \varphi_\lambda(\pm\infty) = 0, \quad \varphi_\lambda(0) = 1, \end{cases} \tag{3.10}$$

where

$$V(s) = \frac{1}{is}\int_0^\infty (e^{ius}-1)n(du), \quad (U(s) = V(is)).$$

One can show an equation

$$\begin{cases} f''(x) = \{\lambda - \alpha N(\lambda) - U(x)\} f(x), \\ f(x) \sim \exp\left\{ -i \int_{x_0}^x \sqrt{\lambda - \alpha N(\lambda) - U(y)}\,dy \right\} \quad \text{as} \quad x \to +\infty, \end{cases} \tag{3.11}$$

has a unique solution, which is denoted by $f_\lambda(x)$. The lemma below can be proved similarly as Theorem 3.2 and Theorem 4.7 in [4].

Lemma 3.2. *Under the condition $\int_{u>1} (\log u) n(du) < \infty$, the followings are valid*

$$\begin{cases} \varphi_\lambda(s) = \dfrac{f_\lambda(-is)}{f_\lambda(0)}, \\ \operatorname{Im} \dfrac{f'_\lambda(0)}{f_\lambda(0)} \sim -\dfrac{1}{| f_0(0)|^2} \exp\left(-\dfrac{n\pi}{\sqrt{\lambda - \alpha N(\lambda)}} \right) \quad \textit{as} \quad \lambda \downarrow 0. \end{cases} \tag{3.12}$$

Now we can connect $\rho_\lambda(-\infty)$ with $f_\lambda(0)$.

Lemma 3.3. *For each $\lambda > 0$ it holds that*

$$\frac{\rho_\lambda(-\infty)}{1-\rho_\lambda(-\infty)} = \frac{-1}{\pi\alpha N(\lambda)} \operatorname{Im} \frac{f_\lambda'(0)}{f_\lambda(0)}.$$

Proof. From (3.9), letting $z \to +\infty$, we see

$$\begin{aligned} \frac{\alpha N(\lambda)\rho_\lambda(-\infty)}{1-\rho_\lambda(-\infty)} &= \lim_{z\to+\infty}(z^2+\lambda)T_\lambda(z) \\ &= \lim_{z\to+\infty}(z^2+\lambda)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isz}\varphi_\lambda(s)\,ds. \end{aligned} \tag{3.13}$$

The identity $\overline{\varphi_\lambda(s)} = \varphi_\lambda(-s)$ implies $\int_{-\infty}^{\infty} e^{isz}\varphi_\lambda(s)\,ds = 2\operatorname{Re}\int_0^\infty e^{isz}\varphi_\lambda(s)\,ds$. On the other hand, the equation (3.4) shows that φ_λ is a holomorphic function in $\mathbb{C}_+$ with exponential type at most $\sqrt{\lambda}$ and is bounded on $i\mathbb{R}_+$ for fixed $\lambda > 0$. This together with the boundedness of φ_λ on $\mathbb{R}_+$ implies that $\varphi_\lambda(z)$ is bounded on the first rectangle of the plane, which guarantees an identity

$$\int_0^\infty e^{isz}\varphi_\lambda(s)\,ds = i\int_0^\infty e^{-sz}\varphi_\lambda(is)\,ds \quad \text{for} \quad z > 0. \tag{3.14}$$

Hence, by (3.12) we obtain

$$\begin{aligned} &\frac{\alpha N(\lambda)\rho_\lambda(-\infty)}{1-\rho_\lambda(-\infty)} \\ &= -\frac{1}{\pi}\lim_{z\to+\infty}(z^2+\lambda)\int_0^\infty e^{-sz}\operatorname{Im}\left(\frac{f_\lambda(s)}{f_\lambda(0)}\right)ds \\ &= -\frac{1}{\pi}\lim_{z\to+\infty}(z^2+\lambda)\frac{1}{z}\int_0^\infty e^{-t}\frac{t}{z}\operatorname{Im}\left(\frac{f_\lambda(t/z)-f_\lambda(0)}{f_\lambda(0)}\times\frac{z}{t}\right)dt \\ &= -\frac{1}{\pi}\operatorname{Im}\frac{f_\lambda'(0)}{f_\lambda(0)}. \end{aligned}$$

□

Then

Lemma 3.4. *As $\lambda \to 0$*

$$\rho_\lambda(-\infty) \to \frac{1}{1+\alpha}.$$

Proof. Let $f_0(x)$ be the unique solution of (3.7). Then from (3.6) and (3.12) it follows that as $\lambda \to 0$

$$\frac{\rho_\lambda(-\infty)}{1-\rho_\lambda(-\infty)} = -\frac{1}{\pi\alpha N(\lambda)}\operatorname{Im}\frac{f_\lambda'(0)}{f_\lambda(0)} \to \frac{1}{\alpha}.$$

□

Now we proceed to the study of $\rho_\lambda(z)$. By the definition of $T_\lambda(z)$ we have

$$\begin{aligned}\rho_\lambda(z) &= 1-(1-\rho_\lambda(-\infty))\int_{-\infty}^{-z} T_\lambda(y)\,dy \\ &= \rho_\lambda(-\infty)+(1-\rho_\lambda(-\infty))F_\lambda(-z),\end{aligned} \tag{3.15}$$

where $F_\lambda(z)=\int_z^\infty T_\lambda(y)\,dy$. Therefore, to get the asymptotics of $\rho_\lambda(z)$, it is sufficient to find the behavior of $F_\lambda(z)$ as $\lambda\to 0$. First we have

Lemma 3.5. *For all fixed $z\in[0,\infty)$, it holds that as $\lambda\to 0$*

$$\lim_{\lambda\to 0} F_\lambda(z)=0.$$

Proof. Rewrite the equation (3.9) in terms of $F_\lambda(z)$

$$\begin{aligned}-F_\lambda'(z) &= \frac{\alpha\rho_\lambda(-\infty)N(\lambda)}{(z^2+\lambda)[1-\rho_\lambda(-\infty)]}+\frac{\alpha F_\lambda(z)N(\lambda)}{z^2+\lambda} \\ &\quad -\frac{\int_0^\infty[F_\lambda(z+u)-F_\lambda(z)]n(du)}{z^2+\lambda}.\end{aligned}$$

Or equivalently

$$[e^{g(z)}F_\lambda(z)]'+c(\lambda)[e^{g(z)}]'=e^{g(z)}\frac{\int_0^\infty F_\lambda(z+u)n(du)}{z^2+\lambda}, \tag{3.16}$$

where

$$g(z)=\int_{-\infty}^{z}\frac{(n+\alpha N(\lambda))}{y^2+\lambda}\,dy,\quad c(\lambda)=\frac{\alpha\rho_\lambda(-\infty)N(\lambda)}{[1-\rho_\lambda(-\infty)](n+\alpha N(\lambda))}.$$

Integrating both sides of (3.16) from z to $+\infty$, we have

$$\begin{aligned}F_\lambda(z) &= c(\lambda)[e^{g(+\infty)-g(z)}-1]-\int_z^\infty\frac{e^{g(y)-g(z)}}{y^2+\lambda}\,dy\int_0^\infty n(du)F_\lambda(y+u) \\ &\le c(\lambda)[e^{g(+\infty)-g(z)}-1].\end{aligned}$$

Moreover, note that as $\lambda\to 0$, we have

$$c(\lambda)\sim\frac{N(\lambda)}{n}\sim \text{const}\times\exp\left(-\frac{n\pi}{\sqrt{\lambda}}\right)$$

and as $\lambda\to 0$

$$g(+\infty)-g(z)=\frac{(n+\alpha N(\lambda))}{\sqrt{\lambda}}\int_{z/\sqrt{\lambda}}^\infty\frac{dx}{x^2+1}\sim\begin{cases}\dfrac{n}{z} & \text{if}\quad z>0,\\[2mm] \dfrac{n\pi}{2\sqrt{\lambda}} & \text{i}\quad z=0.\end{cases}$$

Consequently we see that $0 \le F_\lambda(z) \to 0$ as $\lambda \to 0$. □

Now, we consider $\lim_{\lambda\to 0} F_\lambda(z)$ for $z \in [-\infty, 0)$. The equation (3.17) below formally comes from (3.16) by integrating both sides and then letting $\lambda \to 0$.

Lemma 3.6. *The following equation has a unique solution f on $(0, \infty)$ satisfying $0 \le f(z) \le 1$.*

$$f(z) = e^{-n/z} + e^{-n/z} \int_z^\infty \frac{e^{n/y}}{y^2}\, dy \int_0^y f(y-u)n(du). \tag{3.17}$$

Proof. First we show (3.17) has a solution $0 \le f(z) \le 1$. Introduce an integral operator

$$K f(z) = e^{-n/z} \int_z^\infty \frac{e^{n/y}}{y^2}\, dy \int_0^y f(y-u)n(du).$$

Define a sequence of functions $\{f_m(z)\}_{m\ge 0}$ by

$$f_0(z) = e^{-n/z}, \quad f_{m+1}(z) = f_0(z) + K f_m(z) \quad \text{for} \quad m \ge 0.$$

We show for all $m \ge 0$

$$0 < e^{-n/z} = f_0(z) \le f_1(z) \le \cdots \le f_m(z) \le 1.$$

Since the operator preserves the positivity, the above increasing property is trivial. On the other hand,

$$\begin{aligned} K1(z) &= e^{-n/z} \int_z^\infty \frac{e^{n/y}}{y^2}\, dy \int_0^y n(du) \\ &\le n e^{-n/z} \int_x^\infty \frac{e^{n/y}}{y^2}\, dy = 1 - e^{-n/z} \end{aligned}$$

is valid. Hence $f_m(z) \le 1$ holds for any $m \ge 0$. Therefore there exists

$$\lim_{m\to+\infty} f_m(z) = f^*(z)$$

and $e^{n/z} \le f^*(z) \le 1$. Clearly it satisfies (3.17) on $(0, \infty)$. The uniqueness of solution is easy to prove. □

Lemma 3.7. *For all fixed $z \in (-\infty, 0)$, as $\lambda \to 0$, it holds that*

$$\lim_{\lambda\to 0} F_\lambda(z) = f(-z),$$

where $f(z)$ is the unique solution of (3.17).

Proof. Integrating both sides of (3.16) from $-\infty$ to z, we have

$$F_\lambda(z) = e^{-g(z)} - c(\lambda)[1 - e^{-g(z)}] + \int_{-\infty}^{z} \frac{e^{g(y)-g(z)}}{y^2+\lambda}\, dy \int_0^{-y} F_\lambda(y+u)n(du)$$
$$+ \int_{-\infty}^{z} \frac{e^{g(y)-g(z)}}{y^2+\lambda}\, dy \int_{-y}^{+\infty} F_\lambda(y+u)n(du).$$

Moreover, the fact $g(z) = -n/z + o(1)$ as $\lambda \to 0$ and Lemma 3.5 imply

$$F_\lambda(z) = e^{n/z} + e^{n/z} \int_{-\infty}^{z} \frac{e^{-n/y}}{y^2}\, dy \int_0^{-y} F_\lambda(y+u)n(du) + o(1).$$

Hence, from Lemma 3.6 it follows that $\lim_{\lambda\to 0} F_\lambda(z) = f(-z)$. □

Corollary 3.8. *For* $z \in [-\infty, \infty)$, *as* $\lambda \to 0$

$$N(\lambda)\tau_k(\lambda) \xrightarrow{weakly} \begin{cases} \dfrac{1}{(k-1)!} x^{k-1} e^{-x}\, dx & \textit{if} \quad z \in [-\infty, 0], \\ \dfrac{1-f(z)}{(k-1)!} x^{k-1} e^{-x}\, dx + \dfrac{f(z)}{(k-2)!} x^{k-2} e^{-x}\, dx & \textit{if} \quad z \in (0, \infty), \end{cases}$$

where the last term should be understood as $f(z)\delta_0(dx)$ *if* $k = 1$.

Proof. Combining (3.15) and Lemmas 3.4, 3.5, 3.7, we can show the result as follows. For $z \in [-\infty, 0]$

$$\lim_{\lambda\to 0} \mathbb{E}_z e^{-\alpha N(\lambda)\tau_1} = \frac{1}{1+\alpha}$$

is valid, hence the distribution of $N(\lambda)\tau_1$ converges to the exponential distribution with parameter 1. For $z > 0$, similarly, from (3.15)

$$\lim_{\lambda\to 0} \mathbb{E}_z e^{-\alpha N(\lambda)\tau_1} = \frac{1}{1+\alpha} + \frac{\alpha}{1+\alpha} f(z)$$
$$= f(z) + \frac{1-f(z)}{1+\alpha}$$

we see $N(\lambda)\tau_1$ converges to $f(z)\delta_0(dx) + (1-f(z))e^{-x}\, dx$. Hence, we have the proof for $k = 1$. For $k \geq 2$, we know that under $\mathbb{P}_z$, the random variables $\{\tau_{k+1} - \tau_k\}_{k\geq 1}$ are identically distributed with distribution $\mathbb{P}_{-\infty}(\tau_1(\lambda) < x)$. Therefore the conclusion is clear. □

Now the asymptotic of $\lambda_k(a)$ can be obtained without difficulty. Recall we are treating a random Schrödinger operator

$$L = -\frac{d^2}{dx^2} + \frac{dQ(x)}{dx},$$

on a finite interval $[0, a]$ imposing the boundary conditions

$$\xi(0)\cos\theta + \xi'(0)\sin\theta = 0, \quad \xi(a) = 0,$$

and we are setting

$$z = \cot\theta.$$

Theorem 3.9. *Assume*

$$\int_1^\infty (\log u) n(du) < \infty.$$

For $k \geq 1$, *as* $a \to +\infty$,

$$aN(\lambda_k(a)) \xrightarrow{weakly} \begin{cases} \dfrac{1}{(k-1)!} x^{k-1} e^{-x}\, dx & \text{if} \quad z \in [-\infty, 0], \\ \dfrac{1 - f(z)}{(k-1)!} x^{k-1} e^{-x}\, dx + \dfrac{f(z)}{(k-2)!} x^{k-2} e^{-x}\, dx & \text{if} \quad z \in (0, \infty), \end{cases}$$

where $f(z)$ *is introduced in* (3.17), *and the last term should be understood as* $f(z)\delta_0(dx)$ *if* $k = 1$.

Proof. Proposition 2.2 implies

$$\mathbb{P}_z(aN(\lambda_k(0)) < x) = \mathbb{P}_z\left(\lambda_k(a) < N^{-1}\left(\frac{x}{a}\right)\right) = \mathbb{P}_z(N(\lambda)\tau_k(\lambda) < x),$$

where λ is introduction by $N(\lambda) = x/a$ for fixed $x > 0$. Then the rest of the proof is obvious from Corollary 3.8. □

REMARK 3.10. One can replace $N(\lambda)$ with its asymptotic from

$$\pi^{-1} |f_0(0)|^{-2} \exp\left(-\frac{n\pi}{\sqrt{\lambda}}\right)$$

in the statement of Theorem 3.9.

4. Negative jump case

The method used in the positive jump case may not work here, since some difficulties arise in analyzing the equations (3.3) or (3.4). Therefore, in this case we restrict ourselves only in a special case

$$P\{-q_i > x\} = \begin{cases} 1 & \text{if} \quad x \leq 0, \\ \exp(-\mu x) & \text{if} \quad x > 0, \end{cases}$$

where μ is a positive constant. We assume $n = 1$ in (3.1), hence $n(du)$ takes a special form:

$$n(du) = \begin{cases} 0 & \text{if} u > 0, \\ \mu e^{\mu u}\, du & \text{if} \quad u \le 0. \end{cases}$$

Thus, the generator A of the process $Z(x)$ becomes

$$\begin{cases} Af(z) = -f(z) + (z^2+\lambda)f'(z) + \mu \displaystyle\int_{-\infty}^{0} f(z-y)e^{\mu y}\, dy, \quad \text{for} \quad z \ne \infty, \\ Af(\infty) = -\displaystyle\lim_{z\to\infty} z\{f(z) - f(\infty)\}. \end{cases}$$

To get the asymptotics of τ_k, we use the method of moments. Set

$$u(z) = u(z,\alpha) = \mathbb{E}_z e^{-\alpha\tau_1}.$$

$u(z)$ is support to be a unique solution of the equation

$$\begin{cases} -u(z) + (z^2+\lambda)u'(z) + \mu \displaystyle\int_{-\infty}^{0} u(z-y)e^{\mu y}\, dy = \alpha u(z), \\ u(+\infty) = 1, \quad 0 \le u(z) \le 1, \end{cases}$$

which is non-trivial to be shown. For the time being we proceed by assuming the existence. Introduce a new function by

$$v(z) = \mu \int_{-\infty}^{0} u(z-y)e^{\mu y}\, dy = \mu e^{\mu z} \int_{z}^{\infty} u(x)e^{-\mu x}\, dx.$$

We get the boundary value problem for $u(z)$, $v(z)$:

$$\begin{cases} (z^2+\lambda)u'(z) + v(z) - (1+\alpha)u(z) = 0, \\ v'(z) = \mu(v(z) - u(z)), \\ u(+\infty) = 1, \quad v(\infty) = 1. \end{cases}$$

Hence, by excluding $v(z)$ we get an equation for $u(z)$

$$\begin{cases} (z^2+\lambda)u''(z) + (-\mu z^2 + 2z - \mu\lambda - 1 - \alpha)u'(z) + \mu\alpha u = 0, \\ u(\infty) = 1, \quad u'(\infty) = 0. \end{cases} \tag{4.1}$$

This procedure was found in [3]. Introducing

$$u_k(z) = \left.\frac{\partial^k u(z,\alpha)}{\partial \alpha^k}\right|_{\alpha=0},$$

we see that

$$\begin{cases} u_0(z) = 1, \\ u_k(z) = (-1)^k \mathbb{E}_z \tau_1^k, \quad k \geq 1. \end{cases}$$

Differentiating (4.1) with respect to α and setting $\alpha = 0$, we come to a system of equations

$$(z^2 + \lambda)u_k''(z) + (-\mu z^2 + 2z - \mu\lambda - 1)u_k'(z) - k(u_{k-1}'(z) - \mu u_{k-1}(z)) = 0.$$

For simplicity we set

$$\beta = \sqrt{-\lambda}.$$

Lemma 4.1. *For $k \geq 1$ the equation*

$$\begin{cases} (z^2 - \beta^2)u_k'' + (-\mu z^2 + 2z + \mu\beta^2 - 1)u_k' - k(u_{k-1}' - \mu u_{k-1}) = 0, \\ u_k(\infty) = 0, \quad u_k'(\infty) = 0, \quad u_0(z) = 1. \end{cases} \tag{4.2}$$

have unique continuous and bounded solution.

Proof. Since the coefficient of the second derivative has singularity, we separate the equations in three cases below by transforming them into integral forms.

1. $z \in (\beta, \infty)$

$$u_k(z) = k \int_z^\infty \frac{e^{\mu x}(x-\beta)^{1/(2\beta)-1}}{(x+\beta)^{1/(2\beta)+1}}\,dx \int_x^\infty [u_{k-1}'(t) - \mu u_{k-1}(t)]e^{-\mu t}\left(\frac{t+\beta}{t-\beta}\right)^{1/(2\beta)} dt.$$

2. $z \in (-\beta, \beta)$

$$\begin{aligned} u_k(z) = k &\int_z^\beta \frac{e^{\mu x}(\beta - x)^{1/(2\beta)-1}}{(\beta+x)^{1/(2\beta)+1}}\,dx \\ &\times \int_{-\beta}^x [u_{k-1}'(t) - \mu u_{k-1}(t)]e^{-\mu t}\left(\frac{\beta+t}{\beta-t}\right)^{1/(2\beta)} dt + u_k(\beta). \end{aligned}$$

3. $z \in (-\infty, -\beta)$

$$\begin{aligned} u_k(z) = k &\int_z^{-\beta} \frac{e^{\mu x}(\beta - x)^{1/(2\beta)-1}}{(-\beta-x)^{1/(2\beta)+1}}\,dx \\ &\times \int_x^{-\beta} [u_{k-1}'(t) - \mu u_{k-1}(t)]e^{-\mu t}\left(\frac{-\beta-t}{\beta-t}\right)^{1/(2\beta)} dt + u_k(-\beta). \end{aligned}$$

Then it is not difficult to show that u_k $(\pm\beta)$ are finite and these integrals are convergent, which shows the lemma. □

Now to simplify the situation we introduce

$$\tilde{u}_k(z) = u_k(\beta z).$$

Lemma 4.2 ($z \geq 1$ case). *For each $k \geq 1$, there exists a constant c_k such that*

$$|\tilde{u}_k(z)| \leq c_k$$

holds for all $z \geq 1$ and $\beta \geq 1$.

Proof. The formula for $\tilde{u}_k(z)$ is

$$\tilde{u}_k(z) = k \int_z^\infty \frac{e^{\mu\beta x}(x-1)^{1/(2\beta)-1}}{(x+1)^{1/(2\beta)+1}}\, dx \times \int_x^\infty \left[\frac{\tilde{u}'_{k-1}(t)}{\beta} - \mu\tilde{u}_{k-1}(t)\right] e^{-\mu\beta t}\left(\frac{t+1}{t-1}\right)^{1/(2\beta)} dt.$$

By induction, we will show for each $k \geq 1$, there exist two constants $d_{1,k}$, c_k which depend only on k such that for all $z \in [1, \infty]$

$$|\tilde{u}'_k(z)| \leq \frac{d_{1,k}}{\beta}\frac{(z-1)^{1/(2\beta)-1}}{(z+1)^{1+1/(2\beta)}}\left\{\sum_{m=0}^{k-1}\frac{1}{\beta^m}\left(\log\frac{z+1}{z-1}\right)^m\right\}, \tag{4.3}$$

$$|\tilde{u}_k(z)| \leq c_k$$

holds for all $z, \beta \geq 1$. For $k = 1$, we have

$$\tilde{u}'_1(z) = \mu\frac{e^{\mu\beta z}(z-1)^{1/(2\beta)-1}}{(z+1)^{1/(2\beta)+1}}\int_z^\infty e^{-\mu\beta t}\left(\frac{t+1}{t-1}\right)^{1/(2\beta)} dt.$$

Changing variable $t = z + s/(\mu\beta)$ leads us to

$$\begin{aligned}\tilde{u}'_1(z) &= \frac{1}{\beta}\frac{(z-1)^{1/(2\beta)-1}}{(z+1)^{1/(2\beta)+1}}\int_0^\infty e^{-s}\left(\frac{z+s/(\mu\beta)+1}{z+s/(\mu\beta)-1}\right)^{1/(2\beta)} ds \\ &\leq \frac{d_1}{\beta}\frac{(z-1)^{1/(2\beta)-1}}{(z+1)^{1/(2\beta)+1}}.\end{aligned}$$

Then $\tilde{u}_1(z)$ is bounded and (4.3) is valid for $k = 1$. Now for $k \geq 2$, suppose (4.3) is true for $k-1$. Observing $\log((z+1)/(z-1))$ is decreasing, we have for $z>1$

$$\begin{aligned}|\tilde{u}_k'(z)| &\le k\frac{e^{\mu\beta z}(z-1)^{1/(2\beta)-1}}{(z+1)^{1/(2\beta)+1}}\int_z^\infty \frac{|\tilde{u}_{k-1}'(t)|}{\beta}e^{-\mu\beta t}\left(\frac{t+1}{t-1}\right)^{1/(2\beta)}dt + kc_{k-1}|\tilde{u}_1'(z)|\\ &\le k\frac{(z-1)^{1/(2\beta)-1}}{(z+1)^{1/(2\beta)+1}}\left\{\sum_{m=0}^{k-2}\frac{d_{1,k-1}}{\beta^{m+1}}\left(\log\frac{z+1}{z-1}\right)^m\int_z^\infty\frac{dt}{(t-1)(t+1)}+c_{k-1}\right\}\\ &\le d_{1,k}\frac{(z-1)^{1/(2\beta)-1}}{(z+1)^{1/(2\beta)+1}}\sum_{m=0}^{k-1}\frac{1}{\beta^m}\left(\log\frac{z+1}{z-1}\right)^m,\end{aligned}$$

with $d_{1,k}=k(d_{1,k-1}\vee c_{k-1}d_1)$. Noting for fixed $\varepsilon>0$ and large β

$$\begin{aligned}\int_1^{1+\varepsilon}\frac{(x-1)^{1/(2\beta)-1}}{(x+1)^{1/(2\beta)+1}}\left(\log\frac{x+1}{x-1}\right)^m dx &\sim (2\beta)^{m+1}m!\\ \int_{1+\varepsilon}^\infty\frac{(x-1)^{1/(2\beta)-1}}{(x+1)^{1/(2\beta)+1}}\left(\log\frac{x+1}{x-1}\right)^m dx &\sim \frac{1}{m+1}\left(\log\frac{2+\varepsilon}{\varepsilon}\right)^{m+1},\end{aligned}$$

we see $\tilde{u}_k$ is bounded and (4.3) is true for k as well, which completes the proof. □

Lemma 4.3 ($-1\le z<1$ case). *For any $k\ge 1$, as $\beta\to\infty$, we have*

$$\tilde{u}_k(z)\sim(-1)^k k!\exp(2k\mu\beta).$$

Proof. In this case

$$\begin{aligned}\tilde{u}_k(z) = k\int_z^1 &\frac{e^{\mu\beta x}(1-x)^{1/(2\beta)-1}}{(x+1)^{1/(2\beta)+1}}dx\\ &\times\int_{-1}^x\left[\frac{\tilde{u}_{k-1}'(t)}{\beta}-\mu\tilde{u}_{k-1}(t)\right]e^{-\mu\beta t}\left(\frac{1+t}{1-t}\right)^{1/(2\beta)}dt+\tilde{u}_k(\beta)\end{aligned}$$

is valid. We show by induction that for $\beta\ge 1$

$$\begin{cases}|\tilde{u}_k'(z)|\le d_{2,k}e^{\mu\beta(2k-1+z)}(1-z)^{1/(2\beta)-1}\left\{\displaystyle\sum_{m=0}^{k-1}\frac{|\log(1-z)|^m}{\beta^m}\right\},\\ \tilde{u}_k(z)\sim(-1)^k k!\,e^{2k\mu\beta}\quad\text{as}\quad\beta\to\infty\end{cases}\tag{4.4}$$

holds. For $k = 1$, changing the variables leads us to

$$\begin{aligned}
&\tilde{u}_1(z) \\
&= -\mu \int_z^1 \frac{e^{\mu\beta}(1-x)^{1/(2\beta)-1}}{(1+x)^{1+1/(2\beta)}} dx \int_{-1}^x e^{-\mu\beta t}\left(\frac{1+t}{1-t}\right)^{1/(2\beta)} dt + \tilde{u}_1(1) \qquad (4.5)\\
&\quad - \frac{e^{2\mu\beta}}{\beta}\int_0^{2\mu\beta} e^{-y} dy \int_0^{(y/(\mu\beta))\wedge(1-z)} \frac{s^{1/(2\beta)-1}}{(2-s)^{1+1/(2\beta)}}\left(\frac{y/(\mu\beta)-s}{2+s-y/(\mu\beta)}\right)^{1/(2\beta)} ds + \tilde{u}_1(1)\\
&\sim \frac{e^{2\mu\beta}}{2\beta}\int_0^\infty e^{-x} dx \int_0^{(x/(\mu\beta))\wedge(1-z)} s^{1/(2\beta)-1} ds \sim -e^{2\mu\beta}, \qquad (4.6)
\end{aligned}$$

since we already know from the last lemma that $\tilde{u}_1(1)$ remains bounded. Consequently, from

$$\int_{-1}^z e^{-\mu\beta t}\left(\frac{1+t}{1-t}\right)^{1/(2\beta)} dt \le 2e^{\mu\beta}(z+1)^{1/(2\beta)+1} \tag{4.7}$$

(4.4) follows for $k = 1$. Suppose the statement is true for $k-1$ ($k \ge 2$). Since

$$\begin{aligned}
&\tilde{u}'_k(z)\\
&= -k\frac{e^{\mu\beta z}(1-z)^{1/(2\beta)-1}}{(z+1)^{1/(2\beta)-1}} \int_{-1}^z \left[\frac{\tilde{u}'_{k-1}(t)}{\beta} - \mu\tilde{u}_{k-1}(t)\right] e^{-\mu\beta t}\left(\frac{1+t}{1-t}\right)^{1/(2\beta)} dt,
\end{aligned}$$

using (4.4) for $k-1$, we see

$$\begin{aligned}
&\int_{-z}^z \frac{|\tilde{u}'_{k-1}(t)|}{\beta} e^{-\mu\beta t}\left(\frac{1+t}{1-t}\right)^{1/(2\beta)} dt\\
&\le d_{2,k-1}\frac{e^{\mu\beta(2k-3)}}{\beta}(z+1)^{1/(2\beta)+1}\left\{\sum_{m=0}^{k-2}\frac{|\log(1-z)|^{m+1}}{(m+1)\beta^m}\right\}.
\end{aligned}$$

Moreover, by (4.7) we have

$$\int_{-1}^z \mu|\tilde{u}_{k-1}(t)|e^{-\mu\beta t}\left(\frac{1+t}{1-t}\right)^{1/(2\beta)} dt \le \text{const} \times e^{\mu\beta(2k-1)}(z+1)^{1/(2\beta)+1}.$$

Therefore

$$|\tilde{u}'_k(z)| \le d_{2,k}e^{\mu\beta(2k-1+z)}(1-z)^{1/(2\beta)-1}\left\{\sum_{m=0}^{k-1}\frac{|\log(1-z)|^m}{\beta^m}\right\}$$

with some constant $d_{2,k}$, which proves (4.4). Noting

$$\int_0^1 (1-x)^{1/(2\beta)-1}(-\log(1-x))^m\, dt = (2\beta)^{m+1}m!,$$

similarly as (4.6) we see

$$\begin{aligned}
&\frac{\tilde{u}_k(z)}{k!}\\
&= (-1)^k e^{2(k-1)\mu\beta}\mu \int_z^1 \frac{e^{\mu\beta x}(1-x)^{1/(2\beta)-1}}{(x+1)^{1/(2\beta)+1}} \int_{-1}^x e^{-\mu\beta t}\left(\frac{1+t}{1-t}\right)^{1/(2\beta)} dt + o(e^{2k\mu\beta})\\
&\sim (-1)^k e^{2k\mu\beta},
\end{aligned}$$

which completes the proof. □

Lemma 4.4 ($-\infty \le z < -1$ case). *As $\beta \to \infty$, we have*

$$\tilde{u}_k(z) \sim (-1)^k k!\, e^{2k\mu\beta}[1 + o(1)].$$

Proof. It is enough to show that for β large,

$$\begin{cases} |\tilde{u}'_k(z)| \le d_{3,k} \times e^{2(k-1)\mu\beta}, \\ \tilde{u}_k(z) - \tilde{u}_k(-1) = o(e^{2k\mu\beta}) \end{cases} \tag{4.8}$$

holds for all $z \in [-\infty, -1]$, where $d_{3,k}$ is a constant depending only on k.

In this case we have

$$\begin{aligned}
\tilde{u}_k(z) = k \int_z^{-1} &\frac{e^{\mu x\beta}(-x+1)^{1/(2\beta)-1}}{(-x-1)^{1/(2\beta)+1}}\, dx\\
&\times \int_x^{-1}\left[\frac{\tilde{u}'_{k-1}(t)}{\beta} - \mu\tilde{u}_{k-1}(t)\right] e^{-\mu\beta t}\left(\frac{-t-1}{-t+1}\right)^{1/(2\beta)} dt + \tilde{u}_k(-1).
\end{aligned}$$

The estimate (4.8) can be shown by induction on k as follows. For $k = 1$,

$$\begin{aligned}
&\tilde{u}_1(z) - \tilde{u}_1(-1)\\
&= -\mu \int_z^{-1} e^{\mu\beta x}\frac{(-x+1)^{1/(2\beta)-1}}{(-x-1)^{1/(2\beta)+1}}\, dx \int_x^{-1} e^{-\mu\beta t}\left(\frac{-t-1}{-t+1}\right)^{1/(2\beta)} dt\\
&= -\frac{1}{\beta}\int_0^{-(1+z)\mu\beta} e^{-s}\, ds \int_z^{-1-s/(\mu\beta)} \frac{(-x+1)^{1/(2\beta)-1}}{(-x-1)^{1/(2\beta)+1}}\left(\frac{-x-s/(\mu\beta)-1}{-x-s/(\mu\beta)+1}\right)^{1/(2\beta)} dx\\
&= O\left(\frac{\log\beta}{\beta}\right).
\end{aligned}$$

Moreover, it is easy to see that

$$\begin{aligned}
&\int_z^{-1} e^{-\mu\beta t}\left(\frac{-t-1}{-t+1}\right)^{1/(2\beta)} dt\\
&\le e^{-\mu\beta z}\frac{(\mu+1)}{\mu\beta}[(-z+1)^{1/(2\beta)+1}1_{[-1-1/\beta,-1]}(z) + 1_{(-\infty,-1-1/\beta)}].
\end{aligned} \tag{4.9}$$

Then we have

$$|\tilde{u}_1'(z)| \le \frac{2(\mu+1)}{\beta}.$$

Therefore we know the statement (4.8) is true for $k = 1$. Now suppose (4.8) is true for $k-1$ ($k \ge 2$). The estimate (4.9) shows

$$\begin{cases} |\tilde{u}_k'(z)| \sim k!\, e^{2(k-1)\mu\beta} |\tilde{u}_1'(z)|, \\ |\tilde{u}_k(z) - \tilde{u}_k(-1)| \sim k!\, e^{2(k-1)\mu\beta} |\tilde{u}_1(z) - \tilde{u}_1(-\beta)|, \end{cases}$$

which concludes the proof. □

The integrated density state $N(\lambda)$ is equal to $(\mathbb{E}_{-\infty}\tau_1(\lambda))^{-1}$. Lemma 4.4 implies.

$$N(\lambda) \sim e^{-2\mu\sqrt{-\lambda}} \quad \text{as} \quad \lambda \downarrow -\infty.$$

Corollary 4.5. *For $z \in [-\infty, \infty)$ and $k \ge 1$, as $\lambda \to -\infty$ the random variables $N(\lambda)\tau_k(\lambda)$ converge weakly to*

$$\frac{1}{(k-1)!} x^{k-1} e^{-x}\, dx.$$

Proof. Since we have for all $z \in [-\infty, \infty)$ and $m \ge 1$, as $\beta \to \infty$

$$\mathbb{E}_z \tau_1^m = m!\, e^{2m\beta\mu}[1 + o(1)],$$

the conclusion holds for $k = 1$. For $k \ge 2$, since under $\mathbb{P}_z$, $\{\tau_{n+1} - \tau_n\}_{n \ge 1}$ are identically distributed and the distribution coincides with that of $\tau_1(\lambda)$ under $\mathbb{P}_{-\infty}$, we have the proof. □

Now the asymptotics of $\lambda_n(a)$ can be obtained similarly as Theorem 3.9.

Theorem 4.6. *For $z \in [-\infty, \infty)$ and $k \ge 1$, as $a \to \infty$ it holds that*

$$aN(\lambda_k(a)) \xrightarrow{weakly} \frac{1}{(k-1)!} x^{k-1} e^{-x}\, dx.$$

5. Central limit theorem for "middle eigenvalue"

In this section we consider a limiting property of $\lambda_n(nc)$ for a fixed constant c. Since we have

$$\mathbb{P}_z(\lambda_n(nc) < \lambda) = \mathbb{P}_z(\tau_n(\lambda) < nc).$$

and under $\mathbb{P}_z$, $\{\tau_{k+1}(\lambda) - \tau_k(\lambda)\}_{k\geq 1}$ are identically distributed with distribution $\mathbb{P}_{-\infty}(\tau_1(\lambda) < a)$, therefore

$$\frac{\tau_n(\lambda)}{n} \to \mathbb{E}_{-\infty}(\tau_1(\lambda)) = N(\lambda)^{-1} \quad \text{a.s.}$$

holds, hence we see, as $n \to \infty$

$$\lambda_n(nc) \to N^{-1}(c^{-1}) \quad \text{or equivalently} \quad cN(\lambda_n(nc)) \to 1$$

in probability. Now we consider the central limit theorem for the difference.

Let Σ be the spectrum of globally defined L and set

$$\lambda_0 = \inf \Sigma.$$

Assume

$$(A) \qquad \mathbb{E}_{-\infty}\tau_1(\lambda) \quad \text{is } C^1\text{-class and} \quad \mathbb{E}_{-\infty}\tau_1^2(\lambda) \quad \text{is continuous on} \quad (\lambda_0, \infty).$$

In most cases including the two cases treated in the above argument, this condition is satisfied. Let us introduce $\sigma_1(\lambda)$ by

$$\mathbb{E}_{-\infty}\tau_1(\epsilon + \lambda) = \mathbb{E}_{-\infty}\tau_1(\lambda) + \epsilon\sigma_1(\lambda) + o(\epsilon) \quad \text{as} \quad \epsilon \to 0.$$

Actually

$$\sigma_1(\lambda) = \frac{d}{d\lambda}\mathbb{E}_{-\infty}\tau_1(\lambda).$$

For simplicity of the notation, set

$$\kappa = N^{-1}(c^{-1}).$$

Define

$$\begin{cases} \sigma^2 = \mathbb{E}_{-\infty}\tau_1(\kappa)^2 - c^2 \quad (= Var\{\tau_1(\kappa)\}), \\ M = -\sigma_1(\kappa) > 0. \end{cases}$$

Denoting by $\Phi(x)$ the standard normal distribution, we have

Theorem 5.1. *Under the assumption* (A), *for any* $c > 0$ *and* $z \in [-\infty, \infty)$ *it holds that*

$$\lim_{n\to+\infty} \mathbb{P}_z(\sqrt{n}(\lambda_n(nc) - N^{-1}(c^{-1})) < \lambda) = \Phi\left(\frac{\lambda}{\sigma}M\right).$$

Proof. Note

$$\begin{aligned}\mathbb{P}_z(\sqrt{n}(\lambda_n(nc) - N^{-1}(c^{-1})) < \lambda) &= \mathbb{P}_z\left(\lambda_n(nc) < \frac{\lambda}{\sqrt{n}} + \kappa\right)\\ &= \mathbb{P}_z\left(\tau_n\left(\frac{\lambda}{\sqrt{n}} + \kappa\right) < nc\right)\\ &= \mathbb{P}_z\left(X_n\frac{\sqrt{n}}{\sigma} < \frac{\lambda}{\sigma}M\right),\end{aligned}$$

where

$$X_n = \frac{\tau_n(\lambda/\sqrt{n} + \kappa)}{n} - c + \frac{\lambda}{\sqrt{n}}M.$$

We compute their characteristic functions. Since $\{\tau_i - \tau_{i-1}\}$ are *i.i.d.*, we see

$$\mathbb{E}_z\left(\exp\left(i\xi X_n\frac{\sqrt{n}}{\sigma}\right)\right) = \mathbb{E}_z\left(\exp\left(i\frac{\xi}{\sigma\sqrt{n}}Y\right)\right)\left\{\mathbb{E}_{-\infty}\left(\exp\left(i\frac{\xi}{\sigma\sqrt{n}}Y\right)\right)\right\}^{n-1},$$

where

$$Y = \tau_1\left(\frac{\lambda}{\sqrt{n}} + \kappa\right) - c + \frac{\lambda}{\sqrt{n}}M.$$

Apparently the first term converges to 1 as $n \to \infty$. To compute the second term, we remark

$$e^{ix} = 1 + ix - \frac{1}{2}x^2\delta(x) \tag{5.1}$$

with a smooth function satisfying

$$|\delta(x)| \leq 1 \quad \text{for all} \quad x \in \mathbb{R} \quad \text{and} \quad \delta(x) \to 1 \quad \text{as} \quad x \to 0. \tag{5.2}$$

Applying (5.1) yields

$$\exp\left(i\frac{\xi}{\sigma\sqrt{n}}Y\right) = 1 + i\frac{\xi}{\sigma\sqrt{n}}Y - \frac{1}{2}\frac{\xi^2}{\sigma^2 n}Y^2\delta\left(\frac{\xi}{\sigma\sqrt{n}}Y\right).$$

Taking its expectation, we have

$$\mathbb{E}_{-\infty}\left(\exp\left(i\frac{\xi}{\sigma\sqrt{n}}Y\right)\right) = 1 + i\frac{\xi}{\sigma\sqrt{n}}I_1 - \frac{1}{2}\frac{\xi^2}{\sigma^2 n}I_2.$$

Each term can be computed as follows:

$$\begin{aligned}
I_1\ (=\mathbb{E}_{-\infty}Y) &= \mathbb{E}_{-\infty}\tau_1\left(\frac{\lambda}{\sqrt{n}}+\kappa\right)-c+\frac{\lambda}{\sqrt{n}}M\\
&= \mathbb{E}_{-\infty}\tau_1(\kappa)+\sigma_1(\kappa)\frac{\lambda}{\sqrt{n}}-c+\frac{\lambda}{\sqrt{n}}M+o\left(\frac{1}{\sqrt{n}}\right)\\
&= o\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

and by (5.2), (A)

$$\begin{aligned}
I_2\ \left(=\mathbb{E}_{-\infty}\left\{Y^2\delta\left(\frac{\xi}{\sigma\sqrt{n}}Y\right)\right\}\right) &= \mathbb{E}_{-\infty}Y^2+\mathbb{E}_{-\infty}Y^2\left(\delta\left(\frac{\xi}{\sigma\sqrt{n}}Y\right)-1\right)\\
&= \mathbb{E}_{-\infty}\tau_1(\kappa)^2-c^2+o(1).
\end{aligned}$$

Consequently it follows that

$$\mathbb{E}_{-\infty}\left(\exp\left(i\frac{\xi}{\sigma\sqrt{n}}Y\right)\right)=1-\frac{\xi^2}{2n}+o\left(\frac{1}{n}\right),$$

which implies the present central limit theorem. □

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