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THE FIRST EIGENVALUE OF *P*-MANIFOLDS

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1. Introduction

Let (M, g) be a compact Riemannian manifold, Δ the Laplacian of (M, g) and $\mathrm{Spec}(M, g) := \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$ the spectrum of Δ of (M, g).

It is an important problem in geometry to find lower bounds for the eigenvalues of Δ of (M, g) in terms of the given geometric data and characterize those Riemannian manifolds (M, g) for which these lower bounds are attained. Lichnerowicz proved in [8] that *if (M, g) is a complete Riemannian manifold of dimension $n \geq 2$ with Ricci curvature $Ric_M \geq l$, where l is a positive constant, then the first eigenvalue λ_1 satisfies the inequality $\lambda_1 \geq n/(n-1)l$.* Later Obata proved in [9] that *equality is attained only for the round sphere of radius $\sqrt{(n-1)/l}$.* Antonio Ros studied this problem for P-manifolds. Let us recall that *a manifold (M, g) is called a P-manifold, if all the geodesics of (M, g) are periodic. It is well known that these geodesics admit a minimum common period. By normalising the metric we may assume that the period is 2π and call the manifold (M, g) a $P_{2\pi}$-manifold* (See [2] for a detailed study of P-manifolds). Antonio Ros proved in [12] that *if (M, g) is a $P_{2\pi}$-manifold of dimension $n \geq 2$ with Ricci curvature $\mathrm{Ric}_M \geq l$, then the first eigenvalue λ_1 satisfies the inequality $\lambda_1 \geq (1/3)(2l + n + 2)$ and equality is attained iff for any first eigenfunction f we have that $f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$ for $u \in UM$.* He further remarked that in view of Obata's theorem, this should happen only for a small class of manifolds.

In this paper we substantiate his claim by proving

Theorem 1. *Let (M, g) be a $P_{2\pi}$-manifold of dimension $n \geq 2$ with Ricci curvature $\mathrm{Ric}_M \geq l$ and $\lambda_1 = (1/3)(2l + n + 2)$. Then*

1. (a) $\lambda_1 = (k(m+1))/2 = \lambda_1(\overline{M})$ *and* $l = \mathrm{Ric}_{\overline{M}}$ *where $\overline{M}$ is a simply connected compact rank-1 symmetric space (CROSS) of dimension $n = km$ with sectional curvature $1/4 \leq K_{\overline{M}} \leq 1$ and $k = 1, 2, 4, 8$ or n is the degree of the generator of $H^*(M, \mathbb{Q}) = H^*(\overline{M}, \mathbb{Q})$ and $H^*(\widetilde{M}, \mathbb{Z}_2) = H^*(\overline{M}, \mathbb{Z}_2)$ where $\widetilde{M}$ is the universal cover of M.*
   (b) *If $k \geq 4$ then M is simply connected and the integral cohomology ring of M is same as that of $\overline{M}$.*

(c) *If* $k = 2$ *then either* M *is simply connected or* M *is non-orientable and it has a two sheeted simply connected cover* $\widetilde{M}$. *Moreover* $H^*(\widetilde{M}, \mathbb{Z}) = H^*(\overline{M}, \mathbb{Z})$.

2. *If* $k = 1$ *then* $(\widetilde{M}, \widetilde{g})$ *is isometric to* S^n *with constant sectional curvature* $1/4$.
3. *If* $k = n$ *then* (M, g) *is isometric to* S^n *with constant sectional curvature* 1 (*Lichnerowicz-Obata theorem*).
4. *If* $k = 2$, 4 *or* 8 *and if there is a first eigenfunction* f *without saddle points then the universal cover* $(\widetilde{M}, \widetilde{g})$ *of* (M, g) *is isometric to* $\overline{M}$ *of dimension* km.

REMARKS.

1. In 1c) it should be noted that, if $(1/2)\dim M$ is even then M is forced to be simply connected (See Lemma 2.12 and Proposition 2.16).
2. In CROSSes there are first eigenfunctions admitting saddle points. For instance on $\mathbb{CP}^n$, consider the function defined by

$$f([z_0, z_1, \ldots, z_n]) = \frac{a_0 \mid z_0 \mid^2 + a_1 \mid z_1 \mid^2 + \ldots + a_n \mid z_n \mid^2}{\mid z_0 \mid^2 + \mid z_1 \mid^2 + \ldots + \mid z_n \mid^2}$$

in homogeneous co-ordiantes. This function has as many critical values as there are distinct a_i's; if there are p distinct a_i's and each a_i occurs m_i times then the number of eigenvalues of hessian of the function f on each critical submanifold is p and the multiplicity of the i-th eigenvalue is $2m_i$. In this example, we get a first eigenfunction without saddle points, only if these a_i's take exactly two values as i runs from 0 to n.
In fact a generic first eigenfunction is a Morse function.

The main step in the proof of Theorem 1 is the following

Theorem 2. *Let* (M, g) *be a* $P_{2\pi}$*-manifold of dimension* $n \geq 2$ *and* λ *be an eigenvalue of* Δ *with an eigenfunction* f *such that* $f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$ *for* $u \in UM$. *Then* $\lambda = (k(m+1))/2 = \lambda_1(\overline{M})$ *where* $\overline{M}$ *is as in Theorem* 1.

REMARK. That the behaviour of f is strikingly similar to that in the model CROSSes is also borne out by the auxillary results proved in this paper.

We refer to [2] and [6] for definitions, basic tools and results used in this paper.

2. Preliminaries

In this section we study the topology of critical sets of the function f of the form $f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$ for $u \in UM$ on a $P_{2\pi}$-manifold (M, g).

DEFINITION. Let (M, g) be a complete Riemannian manifold. A subset $B \subseteq M$

is called *totally a-convex* if for any pair of points a_1, $a_2 \in B$ and any geodesic $\gamma : [0,r] \to M$ with $\gamma(0) = a_1$, $\gamma(r) = a_2$ and $r < a$, we have $\gamma([0,r]) \subseteq B$ (See [7]).

Theorem 3. *Let (M,g) be a $P_{2\pi}$-manifold and $f \in C^\infty(M)$ be such that $f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$ for $u \in UM$. Then*

1. *For each critical value α of the function f, the set $D_\alpha := \{x \in M : f(x) = \alpha \text{ and } \nabla f(x) = 0\}$ is a totally 2π-convex, totally geodesic submanifold of (M,g) without boundary.*
2. *$d(D_\alpha, D_\beta) = \pi$ for $\alpha \neq \beta$.*
3. *The function f has only finitely many critical values.*

2.1. Proof of Theorem 3

Let $x \in M$. Then $f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$ for every $u \in U_x M$, the unit sphere in $T_x M$. If x is a critical point of the function f, then, since $\nabla f(x) = 0$, we have that

$$\begin{aligned} B_u &= \left.\frac{d}{dt}\right|_{t=0} f(\gamma_u(t)) \\ &= \langle \nabla f(x), \gamma_u'(0)\rangle \\ &= 0 \end{aligned}$$

Therefore if x is a critical point of the function f, then $f(\gamma_u(t)) = A_u \cos t + C_u$ for every $u \in U_x M$.

To prove Theorem 3(1) we will first prove some lemmas.

Lemma 2.1. *Let $x \in D_\alpha$, u a unit vector at x and γ_u the corresponding geodesic. If J_v is a normal Jacobi field along γ_u such that $J_v(0) = 0$ and $J_v'(0) = v$, then $(J_v f)(\gamma_u(t)) = -2\langle \nabla^2 f(u), v\rangle(\cos t - 1)$.*

Proof. Without loss of generality we may assume that v is a unit vector orthogonal to u. Let $u_\theta := \cos\theta u + \sin\theta v$ in $U_x M$. Then $f(\gamma_{u_\theta}(t)) = A_{u_\theta}(\cos t - 1) + \alpha$, where

$$\begin{aligned} A_{u_\theta} &= -\langle \nabla^2 f(u_\theta), u_\theta\rangle \\ &= -\cos^2\theta A_u - \sin^2\theta A_v - 2\sin\theta\cos\theta\langle \nabla^2 f(u), v\rangle \end{aligned}$$

Hence

$$\begin{aligned} (J_v f)(\gamma_u(t)) &= \left.\frac{\partial}{\partial\theta}\right|_{\theta=0} f(\gamma_{u_\theta}(t)) \\ &= \left.\frac{\partial}{\partial\theta}\right|_{\theta=0} [A_{u_\theta}(\cos t - 1) + \alpha] \end{aligned}$$

$$= -2\langle \nabla^2 f(u), v\rangle(\cos t - 1)$$ □

Corollary 2.2. *If u is an eigenvector of $\nabla^2 f$ in Lemma 2.1, then ∇f is tangential to γ_u for all t.*

Proof. As v is orthogonal to u in Lemma 2.1, if $-\nabla^2 f(u) = \mu u$, then $(J_v f)(\gamma_u(t)) = 0$ for all t.

Since for almost all t, $J_v(t)$ can be made any vector normal to $\gamma_u'(t)$, $\nabla f(\gamma_u(t))$ can have no component normal to $\gamma_u'(t)$. □

REMARK. This Corollary shows that, for $x \in D_\alpha$, if μ is an eigenvalue of $-\nabla^2 f(x)$ and E_μ is the corresponding eigensubspace, then for every $u \in S_\mu$, the unit sphere in E_μ, the geodesics γ_u's are integral curves of $-\nabla f/||\nabla f||$. As a consequence, it follows from Proposition 1 of [11] that ∇f is an eigenvector of $\nabla^2 f$ along such geodesics.

Corollary 2.3. *In Corollary 2.2 above, $\gamma_u(\pi)$ is necessarily a critical point of the function f and $\gamma_u'(\pi)$ is an eigenvector of $-\nabla^2 f$ at $\gamma_u(\pi)$.*

Proof. If $-\nabla^2 f(u) = \mu u$, then, since ∇f is tangential to γ_u, we see that $\nabla f(\gamma_u(t)) = \langle \nabla f(\gamma_u(t)), \gamma_u'(t)\rangle \gamma_u'(t)$. Therefore

$$\begin{aligned} \nabla f(\gamma_u(t)) &= \frac{\partial}{\partial t} f(\gamma_u(t))\gamma_u'(t) \\ &= \frac{\partial}{\partial t}[\mu(\cos t - 1) + \alpha]\gamma_u'(t) \\ &= -\mu \sin t \gamma_u'(t) \end{aligned}$$

Hence $\nabla f(\gamma_u(\pi)) = 0$.

Also $\lim_{t\to\pi}(\nabla f(\gamma_u(t)))/(t-\pi) = \nabla^2 f(\gamma_u'(\pi))$ by L'Hopitâl's Rule. At the same time

$$\begin{aligned} \lim_{t\to\pi} \frac{\nabla f(\gamma_u(t))}{t-\pi} &= \lim_{t\to\pi} \frac{-\mu \sin t \gamma_u'(t)}{t-\pi} \\ &= \mu \gamma_u'(\pi) \end{aligned}$$

Hence $\nabla^2 f(\gamma_u'(\pi)) = \mu \gamma_u'(\pi)$. □

We will now come to the

Proof of Theorem 3(1). Let x, $y \in D_\alpha$ and γ_u be a geodesic joining x and y such that $\gamma_u(0) = x$ and $\gamma_u(r) = y$ for some $r \in \mathbb{R}^+$. Since $f(x) = f(y) = \alpha$ and

$f(\gamma_u(t)) = A_u \cos t + C_u$, we have that $A_u + C_u = A_u \cos r + C_u$. Hence $A_u = 0$ if $r < 2\pi$. This shows that $f(\gamma_u(t)) = \alpha$ for all $t \in [0, r]$.

We will now show that $\gamma_u([0,r]) \subseteq D_\alpha$.

Let $v \perp u$ in U_xM. Then we know from Lemma 2.1 that $(J_v f)(\gamma_u(t)) = -2\langle \nabla^2 f(u), v\rangle(\cos t - 1)$. Since $\gamma_u(r) = y$, for $0 < r < 2\pi$, is a critical point of the function f, we see that $(J_v f)(\gamma_u(r)) = 0$. This proves that $\langle \nabla^2 f(u), v\rangle = 0$ for all $v \perp u$. Hence u is an eigenvector of $-\nabla^2 f$ with eigenvalue μ (say). Then $f(\gamma_u(t)) = \mu(\cos t - 1) + \alpha$. However $f(\gamma_u(t)) = \alpha$. Hence $\mu = 0$. Now by Corollary 2.2 and the proof of Corollary 2.3, we know that $\nabla f(\gamma_u(t)) = -\mu \sin t \gamma_u'(t) = 0$. This shows that $\gamma_u(t)$ is a critical point of f for all t. Therefore $\gamma_u(t) \subseteq D_\alpha$ for all t. Hence D_α is totally 2π-convex. We know from theory of convex sets that D_α is a topological manifold with boundary ∂D_α (possibly empty) and $\mathrm{Int}(D_\alpha)$, the interior of D_α, is non-empty, smooth and totally geodesic. Here $\mathrm{Int}(D_\alpha)$ is not the topological interior as a subset of M but the interior of the manifold D_α (See [6]).

It remains to show that $\partial D_\alpha = \emptyset$

Now let $p \in \partial D_\alpha$ and $q \in \mathrm{Int}(D_\alpha)$. Then the geodesic segment joining p and q has complementary segment of length less than 2π (as all geodesics are periodic of common period 2π). Hence whole of geodesic is actually contained inside D_α and hence there are no boundary points. □

Proof of Theorem 3(2). Let α and β be two critical values of the function f such that $\alpha \neq \beta$. Let $x \in D_\alpha$ and $y \in D_\beta$ with $d(x,y) = t_0$ for some $t_0 \in \mathbb{R}^+$ and γ_u be a geodesic segment such that $\gamma_u(0) = x$ and $\gamma_u(t_0) = y$. Then $f(\gamma_u(t)) = A_u \cos t + C_u$ and

$$\begin{aligned} -A_u \sin t_0 &= \left.\frac{d}{dt}\right|_{t=t_0} f(\gamma_u(t)) \\ &= \langle \nabla f(y), \gamma_u'(t_0)\rangle \\ &= 0 \end{aligned}$$

This can happen only if $t_0 = \pi$. This proves that $d(D_\alpha, D_\beta) = \pi$ for $\alpha \neq \beta$. □

Proof of Theorem 3(3). It is obvious as the critical submanifolds are constant distance apart. □

2.2. In this subsection we will find out the eigenvalues of $\nabla^2 f$ on various D_α's and determine the topology of these D_α's.

Since the function f has only finitely many critical values, we denote these critical values by $\max(f) = \alpha_1, \alpha_2, \cdots, \alpha_p = \min(f)$ and we denote by D_i the critical submanifold $\{x \in M : f(x) = \alpha_i \text{ and } \nabla f(x) = 0\}$.

Let $x_0 \in D_{\max} = \{x \in M : f(x) = \max(f)\}$. Then $-\nabla^2 f(x_0)$ is positive semi-definite for each $x \in D_{\max}$. Therefore we can write the distinct eigenvalues of

$-\nabla^2 f(x_0)$ as $\mu_p > \mu_{p-1} > \cdots > \mu_2 > \mu_1 = 0$ for some $p \in \{1, 2, \cdots, n\}$. p and μ_i's may apriori depend on x_0.

For each i, we denote by E_{μ_i}, the μ_i-eigensubspace of $-\nabla^2 f(x_0)$, by S_{μ_i} the unit sphere in E_{μ_i} and by $S_{\mu_i}(0, r)$ the sphere of radius r centred at origin in E_{μ_i}. Let $u \in S_{\mu_i}$. Then $\max(f) = A_u + C_u$ and $\mu_i = -\nabla^2 f(u, u) = A_u$. Therefore A_u and hence $C_u = \max(f) - A_u$ are constants on S_{μ_i}. Now we define $S(\mu_i, r) := \exp_x(S_{\mu_i}(0, r))$, the exponential image of the sphere $S_{\mu_i}(0, r)$ of radius r. Since $u \in S_{\mu_i}$, it follows from Corollary 2.2 that ∇f is tangential to γ_u for all t and hence $\nabla f(\gamma_u(t)) = -\mu_i \sin t \partial_t$ where ∂_t is the radial vector field $\partial/\partial t$. From this we conclude that $\nabla f(y) = 0$ for $y \in D_i(x_0) := S_{\mu_i}(0, \pi)$.

We will now show that $D_i(x_0) = D_i := \{y \in M : f(y) = \max(f) - 2\mu_i$ and $\nabla f(y) = 0\}$.

It follows from Corollary 2.3 that $D_i(x_0) \subseteq D_i$. To show that $D_i \subseteq D_i(x_0)$ we start with a Lemma which is a sort of converse to Lemma 2.1.

Lemma 2.4. *Let γ_u be a geodesic such that $\gamma_u(0)$ and $\gamma_u(\pi)$ are critical points of the function f. Then both $\gamma_u'(0)$ and $\gamma_u'(\pi)$ are eigenvectors of $\nabla^2 f$.*

Proof. Let J_v be the Jacobi field along γ_u such that $J_v(0) = 0$ and $J_v'(0) = v$ for $v \perp u$. We know from Lemma 2.1 that $(J_v f)(\gamma_u(t)) = -2\langle \nabla^2 f(u), v\rangle(\cos t - 1)$. Since $\gamma_u(\pi)$ is a critical point of the function f, at $t = \pi$, $(J_v f)(\gamma_u(\pi)) = 0$. This forces $\langle \nabla^2 f(u), v\rangle = 0$. i.e., $\nabla^2 f$ has u as an eigenvector.

Similarly arguing from the other side we see that $\gamma_u'(\pi)$ is also an eigenvector of $\nabla^2 f$ at $\gamma_u(\pi)$. □

Corollary 2.5. *If $y \in D_\alpha$ and $\beta \neq \alpha$ is another critical value of the function f, then for each $z \in D_\beta$ and each geodesic γ joining y and z, $\gamma'(0)$ is in the same eigenspace of $-\nabla^2 f$ at y. Moreover, the eigenvalue is independent of the points y and z.*

Proof. If $u = \gamma'(0)$ then $A_u = -\langle \nabla^2 f(u), u\rangle$, and $f(\gamma(t)) = A_u(\cos t - 1) + \alpha$ and so $\beta = -2A_u + \alpha$. Therefore $-A_u = (\beta - \alpha)/2$.

Since it follows from Lemma 2.4 that u is necessarily an eigenvector of $-\nabla^2 f$, the eigenvalue is $(\alpha - \beta)/2$ which is independent of y and z. □

This proves that $D_i \subseteq D_i(x_0)$ and hence $D_i = D_i(x_0)$.

As a consequence of the Corollary 2.5 above we prove the following

Lemma 2.6. *The spectrum of $-\nabla^2 f$ is constant along $D_{\max}$.*

Proof. Let $x_0 \in D_{\max}$. Then for each eigenvalue μ_i, we have the submanifold

$D_i(x_0) = \exp_{x_0}(S_{\mu_i}(0,\pi))$. Also it follows from Lemma 2.4 that, for every $x \in D_{\max}$, the set of unit vectors $\{u \in U_xM : \gamma_u(0) = x \text{ and } \gamma_u(\pi) \in D_i(x_0)\}$ is the unit sphere of the eigenspace of $\nabla^2 f$ with eigenvalue μ_i. This implies that

1. The number of distinct eigenvalues of $-\nabla^2 f$ on $D_{\max}$ and hence on all the critical submanifolds is constant.
2. Each eigenvalue μ_i is constant on $D_{\max}$. □

REMARK. This Lemma 2.6 verifies that each critical submanifold D_α is non-degenerate in the sense of R. Bott.

Now, since μ_i are the only eigenvalues of $-\nabla^2 f$ on $D_{\max}$ it follows from Corollary 2.5 and Lemma 2.6 above that any critical submanifold D_α coincides with one of the D_i's. Hence the only critical values of the function f are $\max(f) - 2\mu_i$ where μ_i's are the eigenvalues of $-\nabla^2 f$ on $D_{\max}$ and the eigenvalues of $-\nabla^2 f$ on D_i are $\{\mu_{ij} := \mu_j - \mu_i, 1 \le j \le p\}$. Thus we have proved the following

Corollary 2.7.

1. *For each critical value $\alpha \neq \max(f)$, the critical submanifold D_α coincicdes with D_i for some i where $2 \le i \le p$.*
2. *The only critical values of the function f are $\max(f) - 2\mu_i$ where μ_i's are the eigenvalues of $-\nabla^2 f$ on $D_{\max}$ for $1 \le i \le p$. Moreover the eigenvalues of $-\nabla^2 f$ on D_i are $\{\mu_{ij} := \mu_j - \mu_i, 1 \le j \le p\}$.*

We will now prove the following

Lemma 2.8. *Let $x \in D_\alpha$, $u \in S_\mu(x)$ and $v \in S_{\mu'}(x)$ where $\mu \neq \mu'$ and $S_\mu(x)$ be the unit sphere in the eigenspace of $\nabla^2 f(x)$ with eigenvalue μ. Let J_v, as before, denote the Jacobi field along γ_u such that $J_v(0) = 0$ and $J_v'(0) = v$. Then $\langle \nabla^2 f(J_v(\pi)), J_v(\pi)\rangle = -4(\mu' - \mu)$.*

Moreover, if $v' \in S_{\mu''}(x)$ such that $\mu'' \neq \mu'$ and v' is orthogonal to u, then $\langle \nabla^2 f(J_v(\pi)), J_v'(\pi)\rangle = 0$.

Proof. By Corollary 2.1 and Corollary 2.3, $\gamma_u(\pi)$ is a critical point of the function f. Hence $\nabla^2 f$ at $\gamma_u(\pi)$ can be identified with the matrix of second partial derivative at this point. Therefore $\langle \nabla^2 f(J_v(\pi)), J_v(\pi)\rangle = -2(\partial^2/\partial\theta^2)\,|_{\theta=0}\, A_{u_\theta}$ where $u_\theta = \cos\theta u + \sin\theta v$ and $A_{u_\theta} = -\langle \nabla^2 f(u_\theta), u_\theta\rangle$. In our situation

$$\begin{aligned} -A_{u_\theta} &= \langle \nabla^2 f(u_\theta), u_\theta\rangle \\ &= \cos^2\theta\mu + \sin^2\theta\mu' \end{aligned}$$

Hence $-(\partial^2/\partial\theta^2)\,|_{\theta=0}\, A_{u_\theta} = -2(\mu' - \mu)$ and $\langle \nabla^2 f(J_v(\pi)), J_v(\pi)\rangle = -4(\mu' - \mu)$.

Similarly considering the two parameter variation defined by $u_{\theta,\phi} := \cos\theta u +$

$\sin\theta(\cos\phi v + \sin\phi v')$ we have that

$$\begin{aligned} A_{u_{\theta,\phi}} &= -\langle \nabla^2 f(u_{\theta,\phi}), u_{\theta,\phi}\rangle \\ &= -(\mu\cos^2\theta + \sin^2\theta(\cos^2\phi\mu' + \sin^2\phi\mu'')) \end{aligned}$$

and

$$\begin{aligned} \langle \nabla^2 f(J_v(\pi)), J_{v'}(\pi)\rangle &= -2\left.\frac{\partial}{\partial\theta}\frac{\partial}{\partial\phi}\right|_{\theta=\phi=0} A_{u_{\theta,\phi}} \\ &= 0 \end{aligned}$$

□

Corollary 2.9. *Let $x \in D_\alpha$, $0 \neq v \perp u$, $u \in S_\mu(x)$ and J_v be the Jacobi field along γ_u such that $J_v(0) = 0$ and $J_v'(0) = v$. If $J_v(\pi) = 0$, then $v \in S_\mu(x)$.*

Proof. Let $v = \sum_\nu v_\nu$ be the decomposition into eigenvectors. Then $J_v(\pi) = \sum_\nu J_{v_\nu}(\pi) = 0$. In particular $\langle \nabla^2 f(J_v(\pi)), J_{v_\nu}(\pi)\rangle = 0$ for each eigenvalue v_ν. By the Lemma 2.8 above this gives $4(\mu - \nu) \parallel v_\nu \parallel^2 = 0$. Therefore $v_\nu = 0$ whenever $\mu \neq \nu$. □

Corollary 2.10. *For $x \in D_\alpha$ and for any non-zero eigenvalue μ of $-\nabla^2 f(x)$, the map $\exp_x : S_\mu(0,\pi) \to D_\mu(x) = D_{\alpha-2\mu}$ is a fibration with $(k-1)$-dimensional fibres and hence the multiplicity of μ is divisible by k where $k-1$ is the index of geodesics γ of length 2π in (M, g).*

Proof. For each $u \in S_\mu(x)$, the geodesic γ_u has index $k-1$ on $[0, 2\pi)$ and its segments $[0,\pi]$ and $[\pi, 2\pi]$ are both minimizing. Hence all the conjugate points to $\gamma_u(0)$ are concentrated at π. By the Corollary 2.9 above the Jacobi fields must come from $v \in S_\mu(x)$. This proves the first part of the Corollary.

By Corollary 2.3, as u runs over $S_\mu(x)$, the unit vectors $\gamma_u'(\pi)$ exhaust all the eigenvectors of $\nabla^2 f$ with eigenvalue $-\mu$ sitting along $D_{\alpha-2\mu}$. Hence the multiplicity of μ is divisible by k. □

REMARK. Since dim$M$ is divisible by k, even for $\mu = 0$, the multiplicity is divisible by k.

We will now study these fibrations.

Let $x \in D_\alpha$ and μ a non-zero eigenvalue of $-\nabla^2 f(x)$ on D_α. Then we have seen in Corollary 2.10 that $\exp_x : S_\mu(0,\pi) \to D_{\alpha-2\mu}$ is a constant rank map and the rank of $\exp_x$ is $\dim E_\mu - k$. If $k = 1$, then $\exp_x : S_\mu(0,\pi) \to D_{\alpha-2\mu}$ is a covering. If $k \geq 2$, then either

F1. $k-1 = 1$, 3, or 7 in which case the connected components of the fibres are homotopy spheres $\sum^{k-1}$ and $k-1 = 7$ occurs only when $S_\mu = S^{15}$ (see [4]) or

F2. $k-1 \neq 1$, 3 and 7 in which case the fibration has to be trivial.

When F2 holds we have the following

Proposition 2.11. *Let* $x \in D_\alpha$ *and the fibration* $\exp_x : S_\mu(0,\pi) \to D_{\alpha-2\mu}$ *be such that* $k-1 \neq 1$, 3 *and* 7. *Then*

1. *the fibration is trivial for all critical values* α,
2. *the function* f *does not have saddle points and*
3. M *is homeomorphic to* S^n.

Proof. If the fibration $\exp_x : S_\mu(0,\pi) \to D_{\alpha-2\mu}$ is non-trivial for some critical value α and some non-zero eigenvalue μ of $-\nabla^2 f$ on D_α, then from [4] it follows that the connected components of the fibres are homotopy spheres $\sum^{k-1}$, $k-1=1$, 3, or 7. Hence by our assumption the fibration has to be trivial for all critical values α of the function f. This also shows that all critical submanifolds are singleton.

Since the geodesics from D_α to $D_{\min}$ for $\alpha > \min(f)$ must neccessarily be in the direction of negative eigenvalues of $\nabla^2 f$, the local minimum i.e., index$=0$, must necessarily be unique.

Now starting with $D_{\min}$ we attach the discs of radius π from each eigenspace at every level. Since these discs are simply connected and the boundary, being the sphere of dimension greater than or equal to 2 is simply connected, by Van Kampen's Theorem, we get a simply connected space at every stage. Hence M is simply connected. Further from our construction, it is clear that M is also an integral cohomology CROSS and the degree of generator of $H^*(M,\mathbb{Z})$ is k where $k \neq 2$, 4 and 8.

Now it is a result in cohomology theory that in this case $k=n$, the dimension of M (See [2]). Hence there are only two critical submanifolds $D_{\max}$ and $D_{\min}$ and they are singletons.

This proves that the function f does not have saddle points and from our construction it is clear that M is homeomorphic to S^n. □

REMARK. By case (3) of Theorem 1 (to be proved later), we have isometry with S^n.

Now we come to case F1. First we start with the following

Lemma 2.12.

1. *Either all* D_α*'s are simply connected integral cohomology CROSSes, or*
2. *all* D_α*'s are non-orientable and* $\pi_1(D_\alpha) \simeq \mathbb{Z}_2$.

Proof. Let D_α and D_β be two distinct critical submanifolds. Then, we know from corollaries 2.5 and 2.10 that $\exp_x : S_{(\alpha-\beta)/2}(0,\pi) \to D_\beta$ is a fibration for $x \in D_\alpha$ and $\exp_y : S_{(\beta-\alpha)/2}(0,\pi) \to D_\alpha$ is a fibration for $y \in D_\beta$. If the number of connected components in each fibre is r for the fibration $\exp_x : S_{(\alpha-\beta)/2}(0,\pi) \to D_\beta$, then by symmetry we see that for the fibration $\exp_y : S_{(\beta-\alpha)/2}(0,\pi) \to D_\alpha$ also the number of connected components in each fibre is r. Therefore $\#\pi_1(D_\alpha) = \#\pi_1(D_\beta)$ and we have shown that

1. either all D_α's are simply connected, or
2. all D_α's are non-simply connected and they all have fundamental groups of same cardinality.

We will now show that when $\pi_1(D_\alpha)$ is non-trivial all D_α's are non-orientable and $\pi_1(D_\alpha) \simeq \mathbb{Z}_2$.

Since $\exp_x : S_{(\alpha-\beta)/2}(0,\pi) \to D_\beta$ is of constant rank, we have a foliation $\mathcal{F}_{\alpha\beta}$ of $S_{(\alpha-\beta)/2}(0,\pi)$ given by the family of $(k-1)$-planes $\ker(d\exp_x)_u$ for $u \in S_{(\alpha-\beta)/2}(0,\pi)$. For each point $u \in S_{(\alpha-\beta)/2}(0,\pi)$, the leaf through u is the connected component through u in the fibre $\exp_x^{-1}(\exp_x(u))$. Let $\mathcal{L}_{\alpha\beta}$ be the leaf space of this foliation and $\Pi_{\alpha\beta} : S_{(\alpha-\beta)/2}(0,\pi) \to \mathcal{L}_{\alpha\beta}$ the natural projection. Then $S_{(\alpha-\beta)/2}(0,\pi)$ is a $(k-1)$-sphere bundle over $\mathcal{L}_{\alpha\beta}$ and the map $\mathcal{L}_{\alpha\beta} \to D_\beta$ is a covering [2]. Since $\Pi_{\alpha\beta} : S_{(\alpha-\beta)/2}(0,\pi) \to \mathcal{L}_{\alpha\beta}$ is a sphere bundle, it follows that $\mathcal{L}_{\alpha\beta}$ is a simply connected integral cohomlogy CROSS. If we now show that a simply connected integral cohomology CROSSes can have only non-orientable 2-sheeted quotients, then we will be through.

Let Y be a simply connected integral cohomology CROSS. If G is a nontrivial finite group acting fixed point freely on Y, then a simple application of Lefschetz's fixed point Theorem tells us that $G \simeq \mathbb{Z}_2$. Again a simple application of Lefschetz's fixed point Theorem tells us that any $\mathbb{Z}_2$ action on Y has a fixed point if $H^*(Y,\mathbb{Z}) = H^*(\mathbb{C}a\mathbb{P}^2,\mathbb{Z})$. In other cases it follows from [3] that

1. if $H^*(Y,\mathbb{Z}) = H^*(\mathbb{Q}\mathbb{P}^h,\mathbb{Z})$, then any $\mathbb{Z}_2$-action on Y must have a fixed point, and
2. if $H^*(Y,\mathbb{Z}) = H^*(\mathbb{C}\mathbb{P}^h,\mathbb{Z})$ then a fixed point free action of $\mathbb{Z}_2$ is possible only when h is odd and in this case the quotient is not orientable.

Thus we have proved that

1. if $k-1=1$ then exactly one of the following holds true :
   (a) For each α, D_α is a simply connected integral cohomology CROSS and the degree of the generator of $H^*(D_\alpha,\mathbb{Z})$ is 2, or
   (b) For each α, D_α is non-orientable, $\pi_1(D_\alpha) \simeq \mathbb{Z}_2$ and $(1/2)\dim D_\alpha$ is odd.
2. if $k-1 = 3$ or 7, then each D_α is a simply connected integral cohomolgy CROSS and the degree of the generator of $H^*(D_\alpha,\mathbb{Z})$ is k.

□

For each $\alpha \neq \beta$, we denote by $D_\alpha * D_\beta$, the submanifold obtained by attaching the disc bundles of $E_{(\alpha-\beta)/2}$ and $E_{(\beta-\alpha)/2}$ along the boundary set. Then we have the following

Lemma 2.13. *Each D_α is orientable iff $D_\alpha * D_\beta$ is orientable. Further if D_α is not orientable then $\pi_1(D_\alpha)$ is isomorphic to $\pi_1(D_\alpha * D_\beta)$.*

Proof. Let us assume that each D_α is orientable. We saw in the Lemma 2.12 that D_α is orienatble iff D_α is simply connected.

Now, $D_\alpha * D_\beta$ is obtained by attaching the disc bundles of $E_{(\alpha-\beta)/2}$ and $E_{(\beta-\alpha)/2}$ along the boundary set. These disc bundles are simply connected and the boundary set being S^{rk-1} bundles over D_α's with $r \geq 1$ and $k \geq 2$, is connected. Hence by Van Kampen's Theorem $D_\alpha * D_\beta$ is simply connected. This proves that if each D_α is orientable then $D_\alpha * D_\beta$ is orientable.

Let us now assume that each D_α is non-orientable and we will show that $D_\alpha * D_\beta$ is non-orientable and $\pi_1(D_\alpha)$ is isomorphic to $\pi_1(D_\alpha * D_\beta)$.

For each critical value α, we denote by $\widetilde{D_\alpha}$, the simply connected two sheeted cover of D_α. Then by the arguments above, it follows that $\widetilde{D_\alpha} * \widetilde{D_\beta}$ (constructed in an obvious way) is a simply connected integral cohomology CROSS covering $D_\alpha * D_\beta$. This proves that $D_\alpha * D_\beta$ is non-orientable and $\pi_1(D_\alpha * D_\beta) \simeq \mathbb{Z}_2$.

From the inclusion $i : D_\alpha \to D_\alpha * D_\beta$, we have the natural map $i_* : \pi_1(D_\alpha) \to \pi_1(D_\alpha * D_\beta)$. We will be through if this map is non-trivial.

Let γ be a non-trivial geodesic loop in D_α. Let $\widetilde{\gamma}$ be the lift of γ in $\widetilde{D_\alpha}$. Now, if $i_*(\gamma)$ is trivial in $\pi_1(D_\alpha * D_\beta)$, then its lift $\widetilde{i_*(\gamma)}$ is a closed geodesic loop in $\widetilde{D_\alpha} * \widetilde{D_\beta}$ which is contained in $\widetilde{D_\alpha}$. But $\widetilde{\gamma} = \widetilde{i_*(\gamma)}$. This implies that $\widetilde{\gamma}$ is a closed geodesic loop in $\widetilde{D_\alpha}$. Therefore γ must be homotopically trivial, a contradiction. Hence $i_*(\gamma)$ is non-trivial in $\pi_1(D_\alpha * D_\beta)$ and this proves that $\pi_1(D_\alpha)$ is isomorphic to $\pi_1(D_\alpha * D_\beta)$. □

Next we prove the following

Lemma 2.14. *For each α, the normal bundle $N_M(D_\alpha)$ of D_α is orientable along D_α.*

Proof. If D_α is orientable then it is simply connected and hence the normal bundle $N_M(D_\alpha)$ of D_α is orientable along D_α.

We will now assume that D_α is not orientable. It suffices to show that for each critical value $\beta \neq \alpha$, the subbundle $E_{(\alpha-\beta)/2}$ of the normal bundle $N_M(D_\alpha)$ is orientable along D_α.

For a vector bundle E over D_α, we denote by $\Lambda^{\text{top}}(E)$, the top exterior line bundle of E over D_α.

We know that

$$\begin{aligned}\Lambda^{\text{top}}(T(D_\alpha * D_\beta)\mid_{D_\alpha}) &= \Lambda^{\text{top}}(TD_\alpha \oplus E_{\frac{\alpha-\beta}{2}}) \\ &= \Lambda^{\text{top}}(TD_\alpha) \otimes \Lambda^{\text{top}}(E_{\frac{\alpha-\beta}{2}})\end{aligned}$$

Hence from the properties of the Stiefel-Whitney classes, it follows that $w_1(\Lambda^{\text{top}}(T(D_\alpha * D_\beta)\mid_{D_\alpha})) = w_1(\Lambda^{\text{top}}(TD_\alpha)) + w_1(\Lambda^{\text{top}}(E_{(\alpha-\beta)/2}))$ in $H^1(D_\alpha, \mathbb{Z}_2)$; here $w_1(*)$ denotes the first Stiefel-Whitney class.

Since $i_* : \pi_1(D_\alpha) \to \pi_1(D_\alpha * D_\beta)$ is an isomorphism, the natural map $i^* : H^1(D_\alpha * D_\beta, \mathbb{Z}_2) \to H^1(D_\alpha, \mathbb{Z}_2)$ is also an isomorphism. Under this isomorphism $w_1(\Lambda^{\text{top}}T(D_\alpha * D_\beta)) \mapsto w_1(\Lambda^{\text{top}}TD_\alpha) + w_1(\Lambda^{\text{top}}E_{(\alpha-\beta)/2})$. Since $D_\alpha * D_\beta$ is non-orientable, $w_1(T(D_\alpha * D_\beta))$ is the unique non-zero element in $H^1(D_\alpha * D_\beta, \mathbb{Z}_2)$ and hence its image $w_1(\Lambda^{\text{top}}TD_\alpha) + w_1(\Lambda^{\text{top}}E_{(\alpha-\beta)/2})$ is the non-zero element in $H^1(D_\alpha, \mathbb{Z}_2)$. This implies that $w_1(\Lambda^{\text{top}}E_{(\alpha-\beta)/2}) = 0$ in $H^1(D_\alpha, \mathbb{Z}_2)$ and hence the normal bundle $N_M(D_\alpha)$ of D_α is orientable along D_α. □

Now we are in a position to prove the following

Proposition 2.15. *The following statements are equivalent*

1. *M is orientable.*
2. *D_α's are orientable.*
3. *D_α's are simply connected.*
4. *M is simply connected.*

Proof. The proof of the claims that $4 \Rightarrow 1 \Rightarrow 2 \Rightarrow 3$ is obvious.

We will now come to the proof of $3 \Rightarrow 4$.

We again remark here that the local minimum i.e., index$=0$ is unique (See proposition 2.11). Hence starting with $D_{\min}$ which is simply connected, we attach disc bundles at every level along the boundary set. These disc bundles are simply connected and the boundary set being the S^{rk-1} bundle, for $r \geq 1$ and $k \geq 2$, over D_α is connected. Hence by Van Kampen's Theorem, we get a simply connected space at every stage. This implies that M is simply connected. □

Similar statement can also be made when M is not orientable. We state this as

Proposition 2.16. *The following statements are equivalent*

1. *M is not orientable.*
2. *M is not simply connected and $\pi_1(M)$ is isomorphic to $\mathbb{Z}_2$.*
3. *D_α's are not simply connected and $\pi_1(D_\alpha)$is isomorphic to $\mathbb{Z}_2$.*
4. *D_α's are not orientable.*

Proof. If M is not orientable, then we take the orientable two sheeted cover

$(\widetilde{M}, \widetilde{g})$ of (M, g). Then $(\widetilde{M}, \widetilde{g})$ is also a $P_{2\pi}$-manifold. For otherwise the common index of geodesics of length 4π in $(\widetilde{M}, \widetilde{g})$ will be $2k+n-1 > n+1$, a contradiction. Now the rest of the proof goes through by appealing to proposition 2.15. $\square$

3. Proof of Theorem 2

Let λ be an eigenvalue of Δ with an eigenfunction f such that $f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$ for $u \in UM$. We know from Theorem 3 that the function has only finitely many critical values say $\{\alpha_i : 1 \leq i \leq p\}$. Let $D_{\max} = D_1, D_2, \cdots, D_p = D_{\min}$ be the critical submanifolds of the function f with critical values α_i.

Let $\mu_p > \mu_{p-1} > \cdots > \mu_2 > \mu_1 = 0$ be the eigenvalues of $-\nabla^2 f$ on $D_{\max}$. We saw in Corollary 2.10 that for each $x \in D_{\max}$, the map $\exp_x|_{S_{\mu_j}(0,\pi)}$: $S_{\mu_j}(0,\pi) \to D_j$ is a fibration with fibres of dimension $k-1$. Therefore we can write $\dim E_{\mu_j} = kr_j$ for some non-negative integer $r_j \in \{1, 2, \cdots, n\}$. Hence $\dim D_j = k(r_j - 1)$.

We also know from Corollary 2.7 that the eigenvalues of $-\nabla^2 f$ on D_i are $\{\mu_{ij} : \mu_j - \mu_i, 1 \leq j \leq p\}$ and from Corollary 2.10 that $\exp|_{S_{\mu_{ij}}(0,\pi)}$: $S_{\mu_{ij}}(0,\pi) \to D_j$ is a fibration for $j \neq i$. In particular $\exp : S_{-\mu_i}(0,\pi) \to D_{\max}$ is a fibration. Hence $\dim E_{\mu_{ij}} = \dim E_{\mu_j} = kr_j$ and $\dim E_{-\mu_i} = \dim D_{\max} + k = k(r_1 + 1)$.

Now we will compute Δf along D_i's.

Since f is an eigenfunction of Δ with eigenvalue λ, for each $x \in D_{\max}$

$$\begin{aligned} \lambda \max(f) &= \Delta f(x) \\ &= Tr(-\nabla^2 f(x)) \\ &= k \sum_{i=1}^{p} r_i \mu_i \end{aligned}$$

and for each $y \in D_j$

$$\lambda \alpha_j = \Delta f(y)$$

But we know that $\alpha_j = \max(f) - 2\mu_j$. Therefore

$$\begin{aligned} \lambda(\max(f) - 2\mu_j) &= k(r_1 + 1)(\mu_1 - \mu_j) + k \sum_{i \geq 2} r_i(\mu_i - \mu_j) \\ &= -k\mu_j + k \sum_{i=1}^{p} r_i(\mu_i - \mu_j) \\ &= -k\mu_j + k \sum_i r_i \mu_i - k\mu_j \sum_i r_i \\ &= -k(1 + \sum_i r_i)\mu_j + \lambda \max(f) \end{aligned}$$

This proves that

$$\lambda = \frac{k(m+1)}{2}$$

where $m = \sum_i r_i$.

We know from Bott-Samelson Theorem for P-manifolds that $H^*(M,\mathbb{Q})$ has exactly one generator (See [1], [2]). From Lemma 2.12 and the discussion towards the end of its proof, it follows that the degree of the generator is k. Therefore $\lambda = k(m+1)/2 = \lambda_1(\overline{M})$ where $\overline{M}$ is a CROSS of dimension km with sectional curvarture $1/4 \leq K_{\overline{M}} \leq 1$ and $H^*(M,\mathbb{Q}) = H^*(\overline{M},\mathbb{Q})$. □

4. Proof of Theorem 1

By hypothesis $\mathrm{Ric}_M \geq l$ and $\lambda_1 = (1/3)(2l+n+2)$. Hence for any first eigenfunction f we have that $f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$ for $u \in UM$ (See [12]).

Proof of 1a. It follows from Theorem 2 that $\lambda_1 = (k(m+1))/2$. Since λ_1 is also equal to $(1/3)(2l+n+2)$, we get that $l = (k(m-1))/4+(k-1) = \mathrm{Ric}_{\overline{M}}$. Again from the proof of Theorem 2 it follows that $H^*(M,\mathbb{Q}) = H^*(\overline{M},\mathbb{Q})$ and also that $H^*(\widetilde{M},\mathbb{Z}_2) = H^*(\overline{M},\mathbb{Z}_2)$. □

Proof of 1b. Since $k \geq 4$, it follows from Lemma 2.12 that all D_α's are simply connected and from proposition 2.15 shows that M is simply connected.

Since each D_α is a simply connected integral cohomology CROSS and we are attaching only rk-dimensional cells at each level along D_α's, we see that M is also an integral cohomolgy CROSS and the degree of the generator of $H^*(M,\mathbb{Z})$ is k.

□

REMARK. If the integral cohomology ring of M is same as that of the cohomology projective plane then the function can have at most three critical submanifolds $D_{\max}$ and $D_{\min}$ and one saddle. If there are three critical submanifolds then all of them are points; if there are only two critical submanifolds $D_{\max}$ and $D_{\min}$ again one of them is a point.

Proof of 1c. Since $k=2$, it follows from Lemma 2.12 and propositons 2.15 and 2.16 that, M is either simply connected or it has orientable 2-sheeted simply connected cover. That the integral cohomology ring of $\widetilde{M}$ is same as that of $\overline{M}$ follows from the proof of Theorem 1b). This completes the proof of Theorem 1c).

□

Proof of 2. Let $(\widetilde{M},\widetilde{g})$ be the universal cover of (M,g) and $\Pi : \widetilde{M} \to M$ the covering map. Since $k=1$, we have that $\mathrm{Ric}_M = (n-1)/4$. Therefore $\mathrm{Ric}_{\widetilde{M}} =$

$\mathrm{Ric}_M = (n-1)/4$. Now by Bonnet-Myers' Theorem it follows that $\mathrm{diam}(\widetilde{M}, \widetilde{g}) \leq 2\pi$. We will now show that $\mathrm{diam}(\widetilde{M}, \widetilde{g}) \geq 2\pi$. Then it will follow from the rigidity of Bonnet-Myers' Theorem [6] or Cheng's maximal diameter Theorem [5] that $(\widetilde{M}, \widetilde{g})$ is isometric to S^n with constant sectional curvature $1/4$.

Since (M, g) is a $P_{2\pi}$-manifold it follows that $(\widetilde{M}, \widetilde{g})$ is a $P_{4\pi}$ manifold (See [2]). If γ is a geodesic between two critical submanifolds then the index of $\gamma|_{[0,2\pi)} = 0$. Since the index of geodesics of length 2π in (M, g) is constant, we see that all such geodesics must have index 0. Hence $\gamma(2\pi)$ is conjugate $\gamma(0)$ with full multiplicity $n-1$, for any geodesic γ in (M, g). This implies that, in $(\widetilde{M}, \widetilde{g})$ also, we must have that $\widetilde{\gamma}(2\pi)$ is conjugate to $\widetilde{\gamma}(0)$ with full multiplicity $n-1$ for any geodesic $\widetilde{\gamma}$ and no more conjugate points in between. This proves that every point in $(\widetilde{M}, \widetilde{g})$ has conjugate locus at constant distance 2π. Therefore for every $x \in \widetilde{M}$ and $u \in T_x\widetilde{M}$ a unit vector $d(\exp_x)_{tu} : T_x\widetilde{M} \to T_{\widetilde{\gamma}_u(t)}\widetilde{M}$ is non-singular for $0 \leq t < 2\pi$ and $d(\exp_x)_{2\pi u}(v)$ for all $v \perp u$. This implies that $\exp_x$ is a local diffeomorphism on the open ball $B(0, 2\pi)$ of radius 2π centred at origin in $T_x\widetilde{M}$ and $\exp_x(S(0, 2\pi))$ is singleton. Hence $\exp_x : D(0, 2\pi)/S(0, 2\pi) \to \widetilde{M}$ is a covering. Here $D(0, 2\pi)$ is the disc of radius 2π and $S(0, 2\pi)$ is the sphere of radius 2π both centred at origin in $T_x\widetilde{M}$. This implies that $\widetilde{M}$ is diffeomorphic to S^n. Since $\exp_x : D(0, 2\pi)/S(0, 2\pi) \to \widetilde{M}$ is a diffeomorphism the cut points to x can not occur before 2π. This implies that $\mathrm{diam}(\widetilde{M}, \widetilde{g}) \geq 2\pi$. Hence $(\widetilde{M}, \widetilde{g})$ is isometric to S^n with constant sectional curvature. □

REMARKS.

1. If $\dim M$ is even then (M, g) is isometric to $\mathbb{RP}^n$ with constant sectional curvature $1/4$. If $\dim M$ is odd only even order lens spaces can occur. i.e., $\pi_1(M)$ is of even order. In this case $\pi_1(M)$ acts linearly on S^n, leaving invariant, at least as many great spheres as the number of critical levels of the function f.
2. We can in fact show that any $P_{2\pi}$-metric g on $\mathbb{RP}^n$ is standard. We give a proof below.
   Let $(\mathbb{RP}^n, g)$ be a $P_{2\pi}$-manifold. Then its universal cover $(S^n, \widetilde{g})$ is a $P_{4\pi}$-manifold. We also know that the index of geodesics of length 2π in $\mathbb{RP}^n$ is constant and the same is true about the geodesics of length 4π. In $(\mathbb{RP}^n, g)$ for any geodesic γ, the point $\gamma(2\pi)$ is conjugate to $\gamma(0)$ with full multiplicity $n-1$. Hence $\widetilde{\gamma}$, the lift of γ, will have $\widetilde{\gamma}(2\pi)$ conjugate to $\widetilde{\gamma}(0)$ with full multiplicity $n-1$ and hence no more conjugate points can occur in between. Hence for all geodesics $\widetilde{\gamma}$ in S^n, the point $\widetilde{\gamma}(2\pi)$ is conjugate to $\widetilde{\gamma}(0)$ with full multiplicity $n-1$. This implies that for any point $x \in S^n$, the cojugate locus occurs at constant distance 2π. From the proof above we can deduce that the injectivity radius at any point is a constant equal to 2π. This means that $(S^n, \widetilde{g})$ is a Blaschke manfold. Now from Blaschke conjecture for spheres [2], it follows

that $(S^n, \widetilde{g})$ is isometric to S^n with constant sectional curvature $1/4$. Hence $(\mathbb{RP}^n, g)$ is ismoetric to the standard $\mathbb{RP}^n$ with constant sectional curvature $1/4$.

Proof of 3. Since $\lambda_1 = n$, this case is nothing but Obata's Theorem. □

4.1. Proof of Theorem 1(4)

First we assume that M is simply connected and that $\max(f)$ and $\min(f)$ are the only critical values of the function f. Hence $D_{\max}$ and $D_{\min}$ are the only critical submanifolds of the function f in (M, g). Therefore $-\nabla^2 f$ has only two eigenvalues on $D_{\max}$. By normalizing the function f, we may assume that these two eigenvalues are 1 and 0. Hence we can write $f(\gamma_u(t)) = \cos t + C$ for $u \in UD^{\perp}_{\max}$, the unit normal bundle of $D_{\max}$, and the tubular hypersurfaces around $D_{\max}$ are level sets of the function f.

Now we get bounds for $\nabla^2 f(u, u)$ for every $u \in UM$.

Let $S(t)$ be the tubular hypersurface of radius t around $D_{\max}$. Then $f(x) = \cos t + C$ for $x \in S(t)$ and $f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$ for $u \in U_x M$. Then $\gamma_u(0) \in S(t)$ and $\gamma_u(\pi) \in S(t_1)$ for some t_1 such that $0 \leq t_1 \leq \pi$. Since $A_u + C_u = \cos t + C$ and $-A_u + C_u = \cos t_1 + C$, we have that $A_u = (1/2)(\cos t - \cos t_1)$. Therefore

$$\begin{aligned} -\nabla^2 f(u,u) &= A_u \\ &= \frac{1}{2}(\cos t - \cos t_1) \end{aligned}$$

and we get that

$$\frac{1-\cos t}{2} \geq \nabla^2 f(u,u) \geq -\frac{1+\cos t}{2}$$

Having got these bounds for $\nabla^2 f$, we define two eigensubbundles of $\nabla^2 f$

$$\begin{aligned} E_{\frac{1-\cos t}{2}} &:= \{E \in T_x M : x \in S(t) \text{ and } \nabla^2 f(E) = \frac{1-\cos t}{2} E\} \\ E_{-\frac{1+\cos t}{2}} &:= \{E \in T_x M : x \in S(t) \text{ and } \nabla^2 f(E) = -\frac{1+\cos t}{2} E\} \end{aligned}$$

Then we have the following

Lemma 4.1.

1. *The eigensubbundles $E_{(1-\cos t)/2}$ and $E_{-(1+\cos t)/2}$ of $\nabla^2 f$ are parallel along the trajectories of ∇f. More over* $\dim E_{(1-\cos t)/2} + \dim E_{-(1+\cos t)/2} = k(m-1)$.

2. *$E_{(1-\cos t)/2}$ and $E_{-(1+\cos t)/2}$ are eigensubbundles of $R(.,\nabla f)\nabla f$ with eigenvalue $(1/4)||\nabla f||^2$.*

Proof. Let $x \in D_{\max}$ and γ be a geodesic starting at x such that $\gamma'(0) \in UD_{\max}{}^{\perp}$. Let J be a Jacobi field along γ describing the variation of the geodesic γ such that $J(0) \in TD_{\max}$ and $J(\pi) = 0$. We normalise J such that $||J'(\pi)|| = 1$. Then, since J is a Jacobi field, $[J, \gamma'(t)] = 0$ along the geodesic γ. Further, since $\gamma'(t) = -\nabla f/||\nabla f||$, we note that $J(||\nabla f||) = 0$. Hence

$$\begin{aligned} -\langle J', J\rangle &= \frac{1}{||\nabla f||}\langle \nabla_J \nabla f, J\rangle \\ &\leq \frac{||J||^2}{||\nabla f||}\frac{1-\cos t}{2} \\ \frac{\langle J', J\rangle}{||J||^2} &\geq -\frac{1}{2}\frac{\sin\frac{t}{2}}{\cos\frac{t}{2}} \end{aligned}$$

The function $||J||^2/\cos^2(t/2)$ is smooth and non-vanishing on $\mathbb{R}$. Hence we can take the positive square root $||J||/|\cos(t/2)|$ of $||J||^2/\cos^2(t/2)$ which is again smooth. The function $\cos(t/2)$ is positive on $(-\pi, \pi)$. Therefore from the last step of the above equation it follows that

$$\frac{d}{dt}\log\left(\frac{||J||}{\cos\frac{t}{2}}\right) \geq 0$$

on $(-\pi, \pi)$. Now since (M, g) is a $P_{2\pi}$-manifold, we have that $J(t) = J(t + 2\pi)$. Hence $||J||/\cos(t/2)\ |_{t=-\pi} = ||J||/\cos(t/2)\ |_{t=\pi} = 2$. This proves that $||J||/\cos(t/2) = 2$ for $t \in [-\pi, \pi]$ and equality must hold everywhere in the above inequalities. This proves that J is an eigenvectorfield of $\nabla^2 f$ with eigenvalue $(1-\cos t)/2$. Since $||J|| = 2\cos(t/2)$, we can write $J(t) = 2\cos(t/2)E(t)$ where $E(t) \in E_{(1-\cos t)/2}$ is a unit vector field along γ. Since J is a Jacobi field along γ

$$\begin{aligned} J' &= \nabla_J \gamma' \\ &= \frac{1}{||\nabla f||}\nabla_J \nabla f \\ &= \frac{1-\cos t}{2}\frac{1}{||\nabla f||}J \\ &= \frac{1-\cos t}{2}\frac{1}{||\nabla f||}2\cos\frac{t}{2}E. \end{aligned}$$

On the other hand $J' = -\sin(t/2)E + \cos(t/2)E'$. This shows that E' is along the direction of the vector field E. Since E is a unit vector field along γ, $E' \perp E$. Therefore $E' = 0$ along γ. Thus we have shown that any Jacobi field J along γ

with $J(0) \in TD_{\max}$ and $J(\pi) = 0$ is of the form $J(t) = 2\cos(t/2)E(t)$, where $E(t) \in E_{(1-\cos t)/2}$ and $E(t)$ is parallel along γ. On the other hand it follows from Lemma 2.8 that every element of $E_{(1-\cos t)/2}$ can be expressed as a Jacobi field $J(t)$ described above. This proves that $E_{(1-\cos t)/2}$ is parallel along the trajectories of ∇f.

Now by a similar argument we can show that the eigensubbundle $E_{-(1+\cos t)/2}$ is also parallel along the trajectories of ∇f by using the inequality that $\nabla^2 f(u,u) \leq -(1+\cos t)/2$. (For a proof see also [11]).

Now we set out to prove the second part of Lemma 4.1. Let $E \in E_{(1-\cos t)/2}$ be a unit vector at $t = 0$ and J be a Jacobi field describing the variation of a normal geodesic γ starting $D_{\max}$, such that $J(0) = 2E$. Then from what we have seen above $J(t) = 2\cos(t/2)E(t)$; $E(t)$ parallel along γ. Therefore

$$\begin{aligned} R(J,\gamma')\gamma' &= -J'' \\ &= \frac{1}{4}J \end{aligned}$$

and this proves that $E_{(1-\cos t)/2}$ is eigensubbundle of $R(.,\nabla f)\nabla f$ with eigenvalue $(1/4)||\nabla f||^2$ along the trajectories of ∇f. The same arguments will prove that $E_{-(1+\cos t)/2}$ is also an eigensubbundle of $R(.,\nabla f)\nabla f$ with eigenvalue $(1/4)||\nabla f||^2$.

It follows from Lemma 2.8 that both the subbundles are of constant dimension at any point in M and also that $\dim E_{(1-\cos t)/2} = ka$ and $\dim E_{-(1+\cos t)/2} = k(m-a-1)$ where $\dim D_{\max} = ka$ and $\dim D_{\min} = k(m-a-1)$. This proves that $\dim E_{(1-\cos t)/2} + \dim E_{-(1+\cos t)/2} = k(m-1)$. □

Let $E_{-\cos t} := (E_{(1-\cos t)/2} \oplus E_{-(1+\cos t)/2})^{\perp}$ be the orthogonal complement of $E_{(1-\cos t)/2} \oplus E_{-(1+\cos t)/2}$ in TM. Then we have the following

Lemma 4.2. *$E_{-\cos t}$ is an eigensubbundle of*
1. *$\nabla^2 f$ with eigenvalue $-\cos t$*
2. *$R(.,\nabla f)\nabla f$ with eigenvalue $||\nabla f||^2$*

Proof. First we note that $\dim(E_{(1-\cos t)/2} \oplus E_{-(1+\cos t)/2}) = k(m-1)$. Therefore the dimension of $E_{-\cos t}$ is k. Let us choose an orthonormal basis $E_1 = \nabla f/||\nabla f||$, E_2, E_3, $\cdots$, E_k of $E_{-\cos t}$, E_{k+1}, E_{k+2}, $\cdots$, $E_{k(a+1)}$ of $E_{(1-\cos t)/2}$ and $E_{k(a+1)+1}$, E_{ka+2}, $\cdots$, E_{km} of $E_{-(1+\cos t)/2}$. Then

$$\begin{aligned} \sum_{i=2}^{k}\langle R(E_i,\nabla f)\nabla f, E_i\rangle &= \mathrm{Ric}_M(\nabla f,\nabla f) - \sum_{j=k+1}^{km}\langle R(E_j,\nabla f)\nabla f, E_j\rangle \\ &= \left[\frac{k(m-1)}{4} + (k-1)\right]||\nabla f||^2 - \frac{k(m-1)}{4}||\nabla f||^2 \\ &= (k-1)||\nabla f||^2 \end{aligned}$$

Now, for $2 \le i \le k$, we define the vector fields $W_i(t) = \sin t E_i(t)$, where each E_i is a parallel vector field along γ such that $E_i(0) = E_i$. Then Index Lemma shows that

$$0 \le I(W_i, W_i) = \int_0^\pi (\langle W_i', W_i'\rangle - \langle R(W_i, \gamma')\gamma', W_i\rangle)$$

Therefore

$$\begin{aligned} 0 &\le \sum_{i=2}^k I(W_i, W_i) \\ &= \sum_{i=2}^k \int_0^\pi \cos^2 t \langle E_i, E_i\rangle - \sin^2 t K(E_i, \gamma') \\ &= (k-1)\int_0^\pi (\cos^2 t - \sin^2 t) \\ &= 0 \end{aligned}$$

Hence $W_i(t) = \sin t E_i(t)$ are Jacobi fields along γ for $2 \le i \le k$. Now it can be easily verified that $E_{-\cos t}$ is an eigensubbundle of $\nabla^2 f$ with eigenvalue $-\cos t$ and also an eigensubbundle of $R(., \gamma')\gamma'$ with eigenvalue 1. □

An intersting Remark. When $k = 2$, we don't need the condition on Ric_M to show that $E_{-\cos t}$ is an eigensubbundle of $\nabla^2 f$ with eigenvalue $-\cos t$ and also an eigensubbundle of $R(., \gamma')\gamma'$ with eigenvalue 1. We give the proof below.

Let μ_1 and μ_2 be the eigenvalues of $\nabla^2 f|_{E_{-\cos t}}$. Then for $x \in D_{\max}$

$$\begin{aligned} \Delta f(x) &= \frac{k(m+1)}{2} f(x) \\ &= \frac{k(m+1)}{2}(1+C) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{k(m+1)}{2}(1+C) &= Tr(-\nabla^2 f(x)) \\ &= -Tr(\nabla^2 f(x)\,|_{E_{-\frac{1+\cos t}{2}}}) - Tr(\nabla^2 f(x)\,|_{E_{-\cos t}}) \\ &= k(m-a) \end{aligned}$$

Hence $C = m - (2a+1)/m + 1$.

Now let $p \in M$. Then $f(p) = \cos t + C$ for some t and

$$\begin{aligned} \frac{k(m+1)}{2}[\cos t + C] &= Tr(-\nabla^2 f(p)) \\ &= -\mu_1 - \mu_2 - Tr(\nabla^2 f(p)\,|_{E_{-\frac{1+\cos t}{2}}}) \end{aligned}$$

$$
\begin{aligned}
&-Tr(\nabla^2 f(p) \mid_{E_{\frac{1-\cos t}{2}}}) \\
&= \cos t - \mu_2 - ka\left(\frac{1-\cos t}{2}\right) \\
&+k(m-(a+1))\left(\frac{1+\cos t}{2}\right)
\end{aligned}
$$

Hence by substituting the value $m - (2a+1)/m + 1$ for C we get that $\mu_2 = -\cos t$.

An important consequence of Lemma 4.1 is that, for each $x \in D_{\max}$, the map $\exp_x : S(0,\pi) \to D_{\min}$ and for each $y \in D_{\min}$, the map $\exp_y : S(0,\pi) \to D_{\max}$ are great sphere fibrations; here $S(0,\pi)$ denotes the normal sphere of radius π at the corresponding points. Now we state the following

Lemma 4.3. *For every $x \in D_{\max}$, the map*

$$\exp_x : S(0,\pi) \to D_{\min}$$

and for every $x \in D_{\min}$, the map

$$\exp_x : S(0,\pi) \to D_{\max}$$

are congruent to Hopf fibrations.

Proof. See [7] and [11]. □

Proof of Theorem 1(4). Let us fix a $\mathbb{P}^a(k) \subseteq \mathbb{P}^m(k)$. We denote by $TD_{\max}^{\perp}$, the normal bundle of $D_{\max}$ and by $(T\mathbb{P}^a(k))^{\perp}$, the normal bundle of $\mathbb{P}^a(k)$ in $\mathbb{P}^m(k)$. Since the map $\exp_x : S(0,\pi) \to D_{\min}$ is congruent to Hopf fibration for each $x \in D_{\max}$ there is a fibre preserving isometry $I : TD_{\max}^{\perp} \to (T\mathbb{P}^a(k))^{\perp}$. Using this isometry we define a map

$$\Phi : M \setminus D_{\min} \to \mathbb{P}^m(k)$$

as follows: For every $q \in M \setminus D_{\min}$ there is a unique $x \in D_{\max}$ and a unique geodesic segement joining x and q and we define $\Phi(q) := \exp \circ I \circ \exp_x^{-1}(q)$. This map carries the geodesics orthogonal to $D_{\max}$ to geodesics orthogonal to $\mathbb{P}^a(k)$ and matches the tubular hypersurfaces around $D_{\max}$. To complete the proof we only have to show that $d\Phi$ preserves the length of the Jacobi fields along these normal geodesics. This follows from [11]. This finishes the proof when M is simply connected.

We will now come to the case when M is not simply connected.

If M is not simply connected, then from our earlier analysis we conclude that the universal cover $(\widetilde{M},\widetilde{g})$ of (M,g) is isometric to $\mathbb{CP}^{2d-1}$ with its standard metric of sectional curvature $1/4 \leq K_{\mathbb{CP}^{2d-1}} \leq 1$. This completes the proof of Theorem 1(4). □

CONCLUDING REMARKS

1. If $k = 2$ and M is not simply connected then we have seen that (M, g) is a quotient of $\mathbb{CP}^{2d-1}$ by a fixed point free involutive isometry. For the existence of such a map consider

$$\phi : \mathbb{CP}^{2d-1} \to \mathbb{CP}^{2d-1}$$

defined by

$$\phi([z_1, z_2, \ldots, z_{2d}]) = [\overline{z}_2, -\overline{z}_1, \ldots, \overline{z}_{2d}, -\overline{z}_{2d-1}]$$

in homogeneous co-ordinates. Then ϕ is a fixed point free involutive isometry of $\mathbb{CP}^{2d-1}$.
For example, consider the eigenfunction

$$f : \mathbb{CP}^{2d-1} \to \mathbb{CP}^{2d-1}$$

defined by

$$f([z_1, z_2, \ldots, z_{2d}]) = \frac{a_0(|z_1|^2 + |z_2|^2) + a_1(|z_3|^2 + \ldots + |z_{2d}|^2)}{|z_1|^2 + |z_2|^2 + \ldots + |z_{2d}|^2}$$

For $a_0 \neq a_1$, f goes down to $M = \mathbb{CP}^{2d-1}/\mathbb{Z}_2$ to give a first eigenfunction without saddle points.

2. Theorem 1(4) has been used to give an intrinsic proof of Lichnerowicz conjecture on harmonic manifolds by the first author (See [13] for a proof using *Nice imbeddings*). The details will appear in *An Intrinsic Approach to Lichnerowicz Conjecture* [10].

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