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Osaka J. Math.
13 (1976), 361-366

SOME INEQUALITIES FOR t-DESIGNS

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(Received March 12, 1975)

1. Introduction

D.K. Ray-Chaudhuri and R.M. Wilson [3] proved $b\geqq\binom{v}{s}$ for $2s$-designs with $v\geqq k+s$, generalizing Fischer's inequality $b\geqq v$ for 2-designs, and Petrenjuk's inequality $b\geqq\binom{v}{2}$ for 4-designs. In this note we introduce a notion of rank s tactical decompositions of $2s$-designs, and generalize some of well known results for 2-designs.

DEFINITION. *A rank s tactical decomposition* of a $2s$-design $(X, \mathcal{B})$ is a partition of the set $X^{(s)}$ of all s-element subsets of X into s-point classes $X_1, X_2, \cdots, X_m$, together with a partition of $\mathcal{B}$ into block classes $\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_{m'}$, such that the number of elements of X_i contained in a block B of $\mathcal{B}_j$ depends only in i and j, (and does not depend on the choice of B in $\mathcal{B}_j$) and the number of blocks in $\mathcal{B}_h$ containing an element $\{p_1, p_2 \cdots p_s\}$ *of* X_g depends only on h and g.

Our first result is:

Theorem 1. *Let a $2s$-(v, k, λ) design $(X, \mathcal{B})$ with $v\geqq k+s$ admit a rank s tactical decomposition with m s-point classes and m' block classes. Then $m\leqq m'$.*

The case $s=1$ in the above was proved by W.M. Kantor (Theorem 4.1 [2]). Our proof, which will be given in section 2, seems to be more elementary.

If G is a group of automorphisms of a $2s$-design $(X, \mathcal{B})$, then the orbits of G on $X^{(s)}$, together with the orbits of G on $\mathcal{B}$, form a rank s tactical decomposition of $(X, \mathcal{B})$. Therefore, by Theorem 1, we have

Corollary 2. *A group of automorphisms of a $2s$-(v, k, λ) design $(X, \mathcal{B})$ with $v\geqq k+s$ has at least as many orbits on $\mathcal{B}$ as on $X^{(s)}$. In particular a block transitive automorphism group of $(X, \mathcal{B})$ is s-homogeneous on points.*

The following is a slight extension of a theorem of D.K. Ray-Chaudhuri and R.M. Wilson (Theorem 2 [3]).

Theorem 3. *Let a $2s$-$(v, \mathcal{B}, \lambda)$ design $(X, \mathcal{B})$ with $v\geqq k+s$ admit a rank*

s tactical decomposition with m s-point classes and m' block classes. Then b $(=|\mathscr{B}|)\geqq\binom{v}{s}+m'-m$.

Extending the notion of parallelsms of 2-designs we also introduce the following.

Definition. *A rank s parallelism* of a $2s$-design $(X, \mathscr{B})$ is an equivalence relation on $\mathscr{B}$ with the property that each element of $X^{(s)}$ lies in a unique member of each equivalent class. Equivalently a rank s parallelisms of a $2s$-design $(X, \mathscr{B})$ is a partition of $\mathscr{B}$ into "rank s parallel classes", each of which is a partition of $X^{(s)}$.

It is easy to see that a rank s parallelism of a 2s-design is a rank s tactical decomposition with one s-point class and λ_s block classes, each of which consists of $\binom{v}{s}\Big/\binom{k}{s}$ blocks. Here, as usual, λ_s denotes the number of blocks containing given s points. Thus, by Theorem 3 (or by Theorem 2 of [3]), we have

Corollary 4. *Let a $2s$-(v, k, λ) design with $v\geqq k+s$ have a rank s parallelism. Then $b\geqq\binom{v}{s}+\lambda_s-1$.*

Corollary 4 is a generalization of Bose's inequality $b\geqq v+r-1$ for 2-designs with a parallelism [1]. The author does not know whether there exist $2s$-designs, $s\geqq 2$, with a rank s parallelism. But the following is true.

Theorem 5. *If $s\geqq 2$, there exist no $2s$-(v, k, λ) designs, $v\geqq k+s$, with a rank s parallelism having the smallest rumber $b=\binom{v}{s}+\lambda_s-1$ of blocks.*

In the case $s=1$, as is well known, there exist infinitely many 2-designs with a parallelism having $v+\lambda_1-1$ blocks.

2. Proof of Theorem 1

Let N be the $\binom{v}{s}\times b$-matrix whose rows are numbered by elements of $X^{(s)}$ and columns by elements of $\mathscr{B}$, and whose $(\{i_1, i_2, \cdots, i_s\}, B)$ entry is 1 or 0 according as $\{i_1, i_2, \cdots, i_s\}\subset B$ or not. Then N has rank $\binom{v}{s}$ by (the proof of) Theorem 1 [3]. So our Theorem 1 is an immediate consequence of the following.

Lemma. *Let M be a (real) $n\times b$-matrix with rank n. Assume that M can be decomposed into mm' rectangular submatrices M_{ij}, $1\leqq i\leqq m$, $1\leqq j\leqq m'$, such that M_{ij} is an $n_i\times b_j$-matrix with constant column sum k_{ij}. Then $m\leqq m'$.*

Proof. Let x_i $(1\leqq i\leqq n)$ denote the i-th row vector of M. Set $y_1= x_1+\cdots+x_{n_1}, y_2=x_{n_1+1},+\cdots+x_{n_1+n_2},\cdots, y_m=x_{n_1+n_2\cdots+n_{m-1}+1}+\cdots+x_{n_1+n_2+\cdots+n_m}$. Then y_i is the vector of the form:

$$y_i=(\overbrace{k_{i_1}\cdots k_{i_1}}^{b_1},\ \overbrace{k_{i_2}\cdots k_{i_2}}^{b_2},\cdots,\ \overbrace{k_{im'}\cdots, k_{im'}}^{b_{m'}}),\qquad 1\leqq i\leqq m_i\,.$$

Then since the m vectors y_i are linearly independent, it follows that $m\leqq m'$.

3. Proof of Theorem 3

We make use of an argument of D.K. Ray-Chaudhuri and R.M. Wilson [3]. Let V_s denote the free vector space over the rationals generated by $X^{(s)}$. Claerly V_s is $\binom{v}{s}$ dimensional over rationals. Now for each $A\in\mathcal{B}$, define a vector $\hat{A}\in V_s$ as the sum of all s-subsets of A, *i.e.*

$$\hat{A}=\sum(S\colon S\in X^{(s)},\ S\subseteq A)\,.$$

D.K. Ray-Chaudhuri and R.M. Wilson showed that the vectors $\{\hat{A}\colon A\in\mathcal{B}\}$ span V_s. Put $\hat{X}_j=\sum(S\colon S\in X_j)$. Then, by our assumption

$$\sum\{\hat{A}\colon A\in\mathcal{B}_i\}=\sum_{j=1}^{m}\lambda_{ij}\sum(S\colon S\in X_j)=\sum_{j=1}^{m}\lambda_{ij}\,\hat{X}, \text{ for some } \lambda_{ij},\qquad 1\leqq i\leqq m'$$

So, if we choose one block A_i from each $\mathcal{B}_i$, then

$$\{\hat{A}\colon A\in\mathcal{B}-\{A_1,\cdots,A_{m'}\}\}\cup\{\hat{X}_1\,\hat{X}_2\cdots,\hat{X}_m\}$$

spans V_s. The stated inequality follows.

4. Proof of Theorem 5

Assume by way of contradiction that there exists a $2s$-(v, k, λ) design $(X, \mathcal{B})$, $s\geqq 2$, $v\geqq k+s$ with a rank s parallelism having the smallest number $b=\binom{v}{s}+\lambda_s-1$ of blocks. Then we have

$$\frac{\binom{v}{s}}{\binom{k}{s}}\lambda_s=\binom{v}{s}+\lambda_s-1\,,$$

$$\lambda_s\left\{\frac{\binom{v}{s}}{\binom{k}{s}}-1\right\}=\binom{v}{s}-1 \tag{4.1}$$

Case 1. $s=2r$ is even.

Applying the theorem of D.K. Ray-Chaudhuri and R.M. Wilson (Theorem 1 [3]), to a contracted s-$(v-s, k-s, \lambda)$ design of $(X, \mathcal{B})$, we have

$$\lambda_s \geqq \binom{v-s}{r} \tag{4.2}$$

Then (4.1) and (4.2) yield

$$\binom{v-s}{r}\left\{\frac{\binom{v}{s}}{\binom{k}{s}}-1\right\} \leqq \binom{v}{s}-1\,, \tag{4.3}$$

Now let $\mathcal{B}_1$ be a rank s parallel class of $(X, \mathcal{B})$. Then $(X, \mathcal{B}_1)$ is a s-$(v, k, 1)$ design, and hence, again by the theorem of D.K. Ray-Chaudhuri and R.M. Wilson, we have

$$\frac{\binom{v}{s}}{\binom{k}{s}} \geqq \binom{v}{r} \tag{4.4}$$

Then (4.3) and (4.4) imply

$$\binom{v-s}{r}\left\{\binom{v}{r}-1\right\} \leqq \binom{v}{s}-1$$

$$\left\{\binom{v-s}{r}-1\right\}\binom{v}{r} \leqq \binom{v-s}{r}\left\{\binom{v}{r}-1\right\} \lneqq \binom{v}{s}$$

$$\binom{v-s}{r}-1 \lneqq \frac{\binom{v}{s}}{\binom{v}{r}} = \frac{(v-r)(v-r-1)\cdots(v-2r+1)}{2r\quad(2r-1)\quad\cdots\quad(r+1)} \tag{4.5}$$

On the other hand, since $v\geqq 6r$, we have

$$\frac{v-s-i}{r-i} \gneqq \frac{v-r-i}{2r-i}\,, \qquad 0\leqq i\leqq r-2 \tag{4.6}$$

and

$$v-3r+1 \geqq \frac{v-2r+1}{r+1}+1\,. \tag{4.7}$$

Then (4.6) and (4.7) yield

$$\binom{v-s}{r}-1=\frac{(v-s)}{r}\cdot\frac{(v-s-1)}{r-1}\cdots\frac{(v-3r+2)}{2}\cdot(v-3r+1)-1$$

$$\gneqq \frac{(v-r)}{2r}\cdot\frac{(v-r-1)}{2r-1}\cdots\frac{(v-2r+2)}{r+2}\left\{\frac{v-2r+1}{r+1}+1\right\}-1$$

$$= \frac{\binom{v}{s}}{\binom{v}{r}} + \frac{(v-r)}{s} \frac{\binom{v}{s-1}}{\binom{v}{r+1}} - 1$$

$$\geqq \frac{\binom{v}{s}}{\binom{v}{r}} .$$

This contradicts (4.5).

Case 2. $s=2r+1$ $(r\geqq 1)$ is odd.

Applying the theorem of D.K. Ray-Chaudhuri and R.M. Wilson to a contracted $2(s-1)$-$(v-2,\ k-2,\ \lambda)$ design of $(X,\ \mathcal{B})$, we have

$$\lambda_2 = \frac{\binom{v-2}{s-2}}{\binom{k-2}{s-2}} \lambda_s \geqq \binom{v-2}{s-1} \tag{4.8}$$

Then (4.1) and (4.8) yield

$$\frac{\binom{k-2}{s-2}(v-s)}{s-1} \left\{ \frac{\binom{v}{s}}{\binom{k}{s}} - 1 \right\} \leqq \binom{v}{s} - 1$$

$$\frac{s(v-s)}{k(k-1)} \left\{ \binom{v}{s} - \binom{k}{s} \right\} \leqq \binom{v}{s} - 1 \tag{4.9}$$

On the other hand applying Fischer's inequality to a contracted 2-$(v-s+2, k-s+2, 1)$ design of a s-$(v, k, 1)$ design $(X, \mathcal{B}_1)$, where $\mathcal{B}_1$ is a rank s parallel class of $(X,\ \mathcal{B})$, we have

$$(k-s+2)\,(k-s+1) \leqq v-s+1 \,. \tag{4.10}$$

We shall now show that

$$\binom{v}{s} - \binom{k}{s} \geqq \frac{4}{5} \left\{ \binom{v}{s} - 1 \right\} \tag{4.11}$$

Deny (4.11). Then

$$\frac{1}{5} \binom{v}{s} \lneqq \binom{k}{s}$$

$$v(v-1) \cdots (v-s+1) \lneqq k(k-1) \cdots (k-s+1)\, 5 \tag{4.12}$$

Then (4.10) and (4.12) give

$$v(v-1)\cdots(v-s+2) \lneqq k(k-1)\cdots(k-s+3)\,5$$
$$v(v-1)\cdots(v-s+3) \lneqq k(k-1)\cdots(k-s+3)$$
$$v \lneqq k, \text{ a contradiction.}$$

Now by (4.9) and (4.11) we obtain

$$s(v-s) \leqq \frac{5}{4}\, k(k-1)$$

Combining this with (4.10) gives

$$(k-s+2)(k-s+1) \leqq \frac{5k(k-1)}{4s}+1 \tag{4.13}$$

Then since $k \geqq 2s$ (4.13) implies

$$\left(\frac{1}{2}\,k+2\right)\left(\frac{1}{2}\,k+1\right) \geqq \frac{5k(k-1)}{4s}+1 \tag{4.14}$$

If $s \leqq 5$ then (4.14) gives

$$\left(\frac{1}{2}\,k+2\right)\left(\frac{1}{2}\,k+1\right) \leqq \frac{k(k-1)}{4}+1$$

$$\frac{3}{2}\,k+2- \leqq -\frac{1}{4}\,k+1\,, \text{ a contradiction.}$$

So we must have $s=3$. But then (4.13) gives

$$(k-1)\,(k-2) \leqq \frac{5}{12}\,k(k-1)+1$$

$$7k^2-31k+12 \leqq 0$$

$$k \leqq 4, \text{ a contradiction.}$$

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