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ON ULTRA WAVE FRONT SETS AND FOURIER INTEGRAL OPERATORS OF INFINITE ORDER

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Introduction. The fundamental solution of the Cauchy problem for a hyperbolic operator is given in the form of Fourier integral operator. As shown in [16] or [20] when the problem is not C^∞ well-posed, the symbol of the fundamental solution has exponential growth, that is, it is estimated not only from above but also from below by

$$(0.1) \quad C \exp [c\xi^{1/\kappa}], \quad c > 0.$$

The constant κ in (0.1) corresponds to the constant in the necessary and sufficient condition for the well-posedness in Gevrey classes given by Ivrii [5].

In the present paper we define $UWF^{(\mu)}(u)$ (ultra wave front sets) for u that belongs to the space of ultradistributions $\mathcal{S}\{\kappa\}'$ by

$$(0.2) \quad (x_0, \xi_0) \in UWF^{(\mu)}(u) \Leftrightarrow \\
 \forall \varepsilon > 0 \exists C; |(\chi u)^\wedge(\xi)| \leq C \exp [\varepsilon \langle \xi \rangle^{1/\mu}],$$

where $\chi \in \mathcal{S}\{\kappa\} \cap C_0^\infty$ and ξ belongs to a conic neighborhood of ξ_0 (see Definition 2.1). Then by using $UWF^{(\mu)}(u)$ we can state the propagation of very high singularities for the solution of not C^∞ well-posed Cauchy problem (see Theorems 3.1 and 3.2). Here, by a very high singularity of u , we mean that its local Fourier transform has an estimate like (0.1).

UWF are first defined by Wakabayashi [22] by the name "generalized wave front sets". But, his definition contains both UWF and Gevrey wave front sets and they are denoted by $WF^{(\kappa)}$ and $WF_{(\kappa)}$ respectively (see Definition 1.3.2 in [22]). He also tried to get non-trivial inner estimates for UWF , but got only a lemma ("not really satisfactory" in his words) and he gave two examples with respect to operators with constant coefficients.

In section 1 we define pseudo-differential operators and Fourier integral operators whose symbols have exponential growth and show that these operators act on the space of ultradistributions $\mathcal{S}\{\kappa\}'$. In section 2 we define the UWF of $u \in \mathcal{S}\{\kappa\}'$ and give the propagation theorem of UWF for Fourier integral operators of infinite order (Theorem 2.2). In section 3 we give exactly the

UWF of the solution of the Cauchy problem for hyperbolic operators with variable multiplicities.

1. Ultradistributions and Fourier integral operators of infinite order.

Let κ satisfy $\kappa > 1$. For positive constants h and ε we define a class $\mathcal{S}\{\kappa; h, \varepsilon\}$ of ultra differentiable functions by a set of functions $u(x)$ satisfying

$$(1.1) \quad |\partial_x^\alpha u(x)| \leq Ch^{-|\alpha|} \alpha!^\kappa \exp(-\varepsilon \langle x \rangle^{1/\kappa})$$

for a positive constant C . For $u \in \mathcal{S}\{\kappa; h, \varepsilon\}$ we define a norm $\|u; \mathcal{S}\{\kappa; h, \varepsilon\}\|$ by

$$\|u; \mathcal{S}\{\kappa; h, \varepsilon\}\| = \inf \{C \text{ of (1.1)}\}.$$

Then, $\mathcal{S}\{\kappa; h, \varepsilon\}$ is a Banach space.

DEFINITION 1.1. We define a class $\mathcal{S}\{\kappa\}$ by

$$\mathcal{S}\{\kappa\} = \text{ind} \lim_{h \rightarrow 0, \varepsilon \rightarrow 0} \mathcal{S}\{\kappa; h, \varepsilon\}$$

and denote by $\mathcal{S}\{\kappa\}'$ the dual space of $\mathcal{S}\{\kappa\}$.

Lemma 1.2. *The Fourier transform $F[u] \equiv \hat{u}(\xi)$ maps $\mathcal{S}\{\kappa\}$ to $\mathcal{S}\{\kappa\}'$ and hence the Fourier transform is also well-defined on $\mathcal{S}\{\kappa\}'$.*

Proof is omitted.

The class $\mathcal{S}\{\kappa\}'$ is a class of ultradistributions (see [2] and [9]), and as we shall prove later (Lemma 1.7) the class $\mathcal{S}\{\kappa\}'$ is characterized by the following: Let $u \in \mathcal{S}\{\kappa\}'$. Then, for any function $\chi(x)$ in $\mathcal{S}\{\kappa\}$ with compact support the Fourier transform $(\chi u)^\wedge(\xi)$ of χu is a measurable function and has an estimate

$$|(\chi u)^\wedge(\xi)| \leq C_\varepsilon \exp[\varepsilon \langle \xi \rangle^{1/\kappa}]$$

for any $\varepsilon > 0$.

Let ρ and δ be real numbers satisfying $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, $\kappa(1-\delta) \geq 1$ and $\kappa\rho \geq 1$.

DEFINITION 1.3 (cf. [6], [12], [17]). i) Let $w(\theta)$ be a positive and non-decreasing function in $[1, \infty)$ or a function of the type θ^m . We say that a symbol $p(x, \xi)$ belongs to a class $S_{\rho, \delta, G(\kappa)}[w]$ if $p(x, \xi)$ satisfies

$$\begin{aligned} |p_{(\beta)}^{(\alpha)}(x, \xi)| &\leq CM^{-|\alpha+\beta|} (\alpha!^\kappa + \alpha!^{\kappa\rho} \langle \xi \rangle^{(1-\rho)|\alpha|}) \\ &\quad \times (\beta!^\kappa + \beta!^{\kappa(1-\delta)} \langle \xi \rangle^{\delta|\beta|}) \langle \xi \rangle^{-|\alpha|} w(\langle \xi \rangle) \\ &\quad \text{for all } x \text{ and } \xi, \end{aligned}$$

where $p_{(\beta)}^{(\alpha)} = \partial_{\xi}^{\alpha} (-i\partial_x)^{\beta} p$. We call the above function $w(\theta)$ an order function.

ii) We say that a symbol $p(x, \xi) (\in S^{-\infty})$ belongs to a class $\mathcal{R}_{G(\kappa)}$ if for any α there exists a constant C_{α} such that

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leq C_{\alpha} M^{-|\beta|} \langle \xi \rangle^{-|\alpha|} \beta!^{\kappa} \exp(-c\langle \xi \rangle^{1/\kappa})$$

hold with a positive constant c independent of α and β . We call a pseudo-differential operator with a symbol in $\mathcal{R}_{G(\kappa)}$ a regularizing operator.

REMARK 1. When $w(\theta) = \theta^m$ for a real m we denote $S_{\rho, \delta, G(\kappa)}[w]$ by $S_{\rho, \delta, G(\kappa)}^m$.

REMARK 2. When $w(\theta) = \exp(C\theta^{\sigma})$ for a $\sigma > 0$, the class $S_{\rho, \delta, G(\kappa)}[w]$ is a symbol class of exponential type, and this corresponds to the class investigated in [23], [14] and [1]. We also remark that the class of symbols in Gevrey classes are investigated in [10], [11], [3] and [19].

EXAMPLE. For $a(x, \xi) \in S_{1,0,G(\kappa)}^m$ the symbol $p(x, \xi) = a(x, \xi) \exp(\langle \xi \rangle^{\sigma})$ belongs to $S_{1,0,G(\kappa)}[\exp(2\theta^{\sigma})]$.

DEFINITION 1.4. Let $0 \leq \tau < 1$. We say that a phase function $\phi(x, \xi)$ belongs to a class $\mathcal{P}_{G(\kappa)}(\tau)$ if $\phi(x, \xi)$ is real-valued and for $J(x, \xi) \equiv \phi(x, \xi) - x \cdot \xi$ the estimates

$$(1.2) \quad \sum_{|\alpha|+|\beta| \leq 2} |J_{(\beta)}^{(\alpha)}(x, \xi)| \langle \xi \rangle^{1-|\alpha|} \leq \tau$$

and

$$(1.3) \quad |J_{(\beta)}^{(\alpha)}(x, \xi)| \leq \tau M^{-(|\alpha|+|\beta|)} (\alpha! \beta!)^{\kappa} \langle \xi \rangle^{1-|\alpha|}$$

hold for a constant M independent of α and β . We also set

$$\mathcal{P}_{G(\kappa)} = \bigcup_{0 \leq \tau < 1} \mathcal{P}_{G(\kappa)}(\tau).$$

Proposition 1.5. *Let $w(\theta)$ be an order function satisfying*

$$(1.4) \quad w(\theta) \leq \exp[C\theta^{\sigma}]$$

for a constant σ with $0 \leq \sigma < 1/\kappa$. For a phase function $\phi(x, \xi) \in \mathcal{P}_{G(\kappa)}$ and a symbol $p(x, \xi) \in S_{\rho, \delta, G(\kappa)}[w]$ we define a Fourier integral operator P_{ϕ} and a conjugate Fourier integral operator P_{ϕ^*} by

$$P_{\phi} u(x) = \int e^{i\phi(x, \xi)} p(x, \xi) \hat{u}(\xi) d\xi,$$

$$P_{\phi^*} u(x) = \int e^{ix \cdot \xi} \left\{ \int e^{-i\phi(y, \xi)} p(y, \xi) u(y) dy \right\} d\xi,$$

where $d\xi = (2\pi)^{-n} d\xi$. Then, the operators P_{ϕ} and P_{ϕ^*} map $S\{\kappa\}$ to $S\{\kappa\}$ continuously.

Proof. For $u(x) \in \mathcal{S}\{\kappa\}$ we denote

$$f(x) = P_\phi u(x) \equiv \int e^{i\phi(x,\xi)} p(x, \xi) \hat{u}(\xi) d\xi.$$

Define $L = \{1 + |\nabla_\xi \phi(x, \xi)|^2\}^{-1} \{1 - i\nabla_\xi \phi(x, \xi) \cdot \nabla_\xi\}$. Then, we have $Le^{i\phi(x,\xi)} = e^{i\phi(x,\xi)}$ and hence

$$f(x) = \int e^{i\phi(x,\xi)} (L^\dagger)^N \{p(x, \xi) \hat{u}(\xi)\} d\xi.$$

By the induction on N we can prove

$$(1.5) \quad \begin{aligned} |\partial_x^\alpha \partial_x^\beta (L^\dagger)^N \{p(x, \xi) \hat{u}(\xi)\}| &\leq CM_1^{-N} M_2^{-|\alpha+\beta|} (|\alpha| + N)! \kappa \langle x \rangle^{-N} \\ &\quad \times (\beta!^\kappa + \beta!^{\kappa(1-\delta)} \langle \xi \rangle^{\delta|\beta|}) \exp(C_1 \langle \xi \rangle^\sigma - \varepsilon \langle \xi \rangle^{1/\kappa}) \end{aligned}$$

for positive constants C, M_1, M_2, C_1 and ε , since $\hat{u}(\xi)$ belongs to $\mathcal{S}\{\kappa\}$. Assume that x satisfies $C_0 N^\kappa \leq \langle x \rangle \leq C_0(N+1)^\kappa$ for a constant C_0 to be determined later. Then, using (1.5) with $\alpha=0$ and denoting $\phi_\beta(x, \xi) = e^{-i\phi(x,\xi)} \partial_x^\beta e^{i\phi(x,\xi)}$ we have

$$\begin{aligned} |\partial_x^\beta f(x)| &= \left| \sum_{\beta'+\beta''=\beta} \binom{\beta}{\beta'} \int e^{i\phi(x,\xi)} \phi_{\beta'}(x, \xi) D_x^{\beta''} (L^\dagger)^N \{p(x, \xi) \hat{u}(\xi)\} d\xi \right| \\ &\leq C \sum_{\beta'+\beta''=\beta} \binom{\beta}{\beta'} \int M_3^{-|\beta'|} \left\{ \sum_{j=1}^{|\beta'|} (|\beta'| - j)! \kappa \langle \xi \rangle^j \right\} M_1^{-N} M_2^{-|\beta''|} N! \kappa \langle x \rangle^{-N} \\ &\quad \times (\beta''!^\kappa + \beta''!^{\kappa(1-\delta)} \langle \xi \rangle^{\delta|\beta''|}) \exp(C_1 \langle \xi \rangle^\sigma - \varepsilon \langle \xi \rangle^{1/\kappa}) d\xi \\ &\leq CM_4^{|\beta|} \beta!^\kappa M_1^{-N} N! \kappa \langle x \rangle^{-N} \\ &\leq CM_4^{|\beta|} \beta!^\kappa M_1^{-N} N! \kappa (C_0 N^\kappa)^{-N} \exp(\varepsilon_1 C_0^{1/\kappa} (N+1)) \exp(-\varepsilon_1 C_0^{1/\kappa} \langle x \rangle^{1/\kappa}) \end{aligned}$$

for any positive constant ε_1 . Now, take C_0 and ε_1 satisfying

$$C_0 \geq 2M_1^{-1}, \quad \exp(\varepsilon_1 C_0^{1/\kappa}) \leq 2.$$

Then, $f(x)$ satisfies (1.1) with $h=M_4$ and $\varepsilon = \varepsilon_1 C_0^{1/\kappa}$. Consequently, we have proved that P_ϕ maps $\mathcal{S}\{\kappa\}$ to $\mathcal{S}\{\kappa\}$ continuously. In the same way we can prove that P_{ϕ^*} maps $\mathcal{S}\{\kappa\}$ to $\mathcal{S}\{\kappa\}$ continuously. Q.E.D.

From Proposition 1.5 the following definition is well-defined

DEFINITION 1.6. Let $w(\theta)$ be an order function satisfying (1.4), that is, it satisfies

$$w(\theta) \leq \exp(C\theta^\sigma)$$

for a constant σ with $0 \leq \sigma < 1/\kappa$. Then for $\phi(x, \xi) \in \mathcal{P}_{G(\kappa)}$ and $p(x, \xi) \in S_{\rho,\delta,G(\kappa)}[w]$, the following operators

$$\begin{aligned}
 P_\phi &: \mathcal{S}\{\kappa\}' \rightarrow \mathcal{S}\{\kappa\}', \\
 P_{\phi^*} &: \mathcal{S}\{\kappa\}' \rightarrow \mathcal{S}\{\kappa\}'
 \end{aligned}$$

are defined by the principle of duality.

EXAMPLE. For $a(x, \xi) \in S_{1,0,G(\kappa)}^m$ ($\kappa < 2$) we consider a symbol $p(x, \xi) = a(x, \xi) \exp(c\langle \xi \rangle^{1/2})$ with $c > 0$. Then, it belongs to $S_{1,0,G(\kappa)}[\exp(2c\theta^{1/2})]$ and for $1 < \kappa < 2$ the following maps are well-defined:

$$\begin{aligned}
 P_\phi &: \mathcal{S}\{\kappa\}' \rightarrow \mathcal{S}\{\kappa\}', \\
 P_{\phi^*} &: \mathcal{S}\{\kappa\}' \rightarrow \mathcal{S}\{\kappa\}',
 \end{aligned}$$

where ϕ is a phase function in $\mathcal{P}_{G(\kappa)}$.

Lemma 1.7. For $u \in \mathcal{S}\{\kappa\}'$ and $\chi \in \mathcal{S}\{\kappa\} \cap C_0^\infty$ the Fourier transform $(\chi u)^\wedge(\xi)$ of χu is a measurable function and has an estimate

$$|(\chi u)^\wedge(\xi)| \leq C_\varepsilon \exp(\varepsilon \langle \xi \rangle^{1/\kappa})$$

for any $\varepsilon > 0$.

Proof. We may assume that $u \in \mathcal{S}\{\kappa\}'$ has a compact support and prove that, for any fixed ε , $\exp(-\varepsilon \langle \xi \rangle^{1/\kappa}) \hat{u}$ is a functional on L^1 and has the following estimate

$$(1.6) \quad |\langle \exp(-\varepsilon \langle \xi \rangle^{1/\kappa}) \hat{u}, \psi \rangle| \leq C \|\psi\|_{L^1}$$

for $\psi \in L^1$. Then, we find that $\exp(-\varepsilon \langle \xi \rangle^{1/\kappa}) \hat{u}$ belongs to L^∞ and we have an estimate

$$|\hat{u}(\xi)| \leq C_\varepsilon \exp(\varepsilon \langle \xi \rangle^{1/\kappa})$$

for any ε . Denote by $\tilde{\psi}(x)$ the inverse Fourier transform of $\psi(\xi)$ and take a function $\chi(x)$ in $\mathcal{S}\{\kappa; h, 1\} \cap C_0^\infty(\mathbb{R}^n)$ with $h = \varepsilon^\kappa \kappa^{-\kappa}/2$ such that $\chi(x) = 1$ on the support of u . Then, we have for $\psi \in \mathcal{S}\{\kappa; h, 1\}$

$$\begin{aligned}
 \langle \exp(-\varepsilon \langle \xi \rangle^{1/\kappa}) \hat{u}, \psi \rangle &= \langle \hat{u}, \exp(-\varepsilon \langle \xi \rangle^{1/\kappa}) \psi \rangle \\
 &= \langle u, \exp(-\varepsilon \langle D \rangle^{1/\kappa}) \tilde{\psi} \rangle \\
 &= \langle u, \chi(x) \exp(-\varepsilon \langle D \rangle^{1/\kappa}) \tilde{\psi} \rangle.
 \end{aligned}$$

Here, we have used Proposition 1.5 for well-definedness of the third and fourth members of the above equation. Hence, by the definition and the fact that $u \in \mathcal{S}\{\kappa\}'$ we have

$$(1.7) \quad |\langle \exp(-\varepsilon \langle \xi \rangle^{1/\kappa}) \hat{u}, \psi \rangle| \leq C \|\chi(x) \exp(-\varepsilon \langle D \rangle^{1/\kappa}) \tilde{\psi}; \mathcal{S}\{\kappa; h, 1\}\|.$$

Write

$$\chi(x) \exp(-\varepsilon \langle D \rangle^{1/\kappa}) \tilde{\psi}(x) = \int e^{ix \cdot \xi} \chi(x) \exp(-\varepsilon \langle \xi \rangle^{1/\kappa}) \psi(\xi) d\xi.$$

Then, from $h = \varepsilon^\kappa \kappa^{-\kappa} / 2$, we have

$$|\partial_x^\alpha (\chi(x) \exp(-\varepsilon \langle D \rangle^{1/\kappa}) \tilde{\psi})| \leq Ch^{-|\alpha|} \alpha!^\kappa \exp(-\langle x \rangle^{1/\kappa}) \|\psi\|_{L^1}$$

and hence

$$\|\chi(x) \exp(-\varepsilon \langle D \rangle^{1/\kappa}) \tilde{\psi}; \mathcal{S}\{\kappa; h, 1\}\| \leq C \|\psi\|_{L^1}.$$

This and (1.7) yields (1.6) for $\psi \in \mathcal{S}\{\kappa; h, 1\}$. Finally, using the limiting process we have (1.6) for any $\psi \in L^1(\mathbb{R}^n)$. Q.E.D.

From Lemma 1.7 we get the following Lemma 1.8, which states that the pseudo-differential operator with a symbol in $\mathcal{R}_{G(\kappa)}$ is a regularizing operator.

Lemma 1.8. *For $u \in \mathcal{S}\{\kappa\}'$ with compact support and $r(x, \xi) \in \mathcal{R}_{G(\kappa)}$ we have*

$$r(X, D_x)u \in \mathcal{B}\{\kappa\}.$$

Here, $f(x) \in \mathcal{B}\{\kappa\}$ means that there exists a constant C such that

$$|\partial_x^\alpha f(x)| \leq CM^{-|\alpha|} \alpha!^\kappa \quad \text{for any } x.$$

In the following section we also need

Lemma 1.9. *Let $r(x, \xi)$ satisfies*

$$(1.8) \quad |r_{(\beta)}^{(\alpha)}(x, \xi)| \leq CM^{-|\alpha+\beta|} \alpha!^\kappa \times (\beta!^\kappa + \beta!^{\kappa(1-\delta)} \langle \xi \rangle^{\delta|\beta|}) \exp(-c_0 \langle x \rangle^{1/\kappa} - c_0 \langle \xi \rangle^{1/\kappa})$$

for a positive constant c_0 . Then, for $u \in \mathcal{S}\{\kappa\}'$, $r(X, D_x)u$ is well-defined and belongs to $\mathcal{B}\{\kappa\}$.

We can prove the lemma as Proposition 1.5 and Lemma 1.7. The details are omitted.

2. Ultra wave front set

DEFINITION 2.1. Let κ and μ satisfy $\kappa \leq \mu$. For $u \in \mathcal{S}\{\kappa\}'$ we define a *UWF* (ultra wave front set) of u as follows: We say that a point (x_0, ξ_0) in $T^*\mathbb{R}^n \setminus \{0\}$ does not belong to $UWF^{(\mu)}(u)$ if there exist a function $\chi(x)$ in $\mathcal{S}\{\kappa\} \cap C_0^\infty$ with $\chi(x_0) \neq 0$, a conic neighborhood Γ of ξ_0 , and for any positive constant ε there exists a constant C such that

$$(2.1) \quad |(\chi u)^\wedge(\xi)| \leq C \exp[\varepsilon \langle \xi \rangle^{1/\mu}] \quad \text{for } \xi \in \Gamma.$$

REMARK 1. As stated in Introduction this definition is the same as that of Wakabayashi. (See Definition 1.3.2 in [22]).

REMARK 2. Let $u \in \mathcal{S}\{\kappa\}'$ and let $\kappa \leq \mu$. Then, $(x_0, \xi) \notin UWF^{(\kappa)}(u)$ for all ξ is equivalent to that $\chi u \in \mathcal{S}\{\mu\}'$ for some $\chi \in \mathcal{S}\{\kappa\}$ with $\chi(x_0) \neq 0$. (See Lemma 1.3.3 of [22]). Especially, from Lemma 1.7 we have $UWF^{(\kappa)}(u) = \emptyset$ for $u \in \mathcal{S}\{\kappa\}$.

Theorem 2.2. *Let $\kappa < \mu$ and let $\phi(x, \xi) \in \mathcal{P}_{G(\kappa)}$ and $p(x, \xi) \in S_{\rho, \delta, G(\kappa)}[\exp(c\theta^\sigma)]$ for some σ with $\sigma < 1/\mu$. Assume that $\phi(x, \xi)$ is positively homogeneous for large $|\xi|$. Then, for $u \in \mathcal{S}\{\kappa\}'$ and $(y_0, \eta_0) \in T^*R^n \setminus \{0\}$ with $|\eta_0| \gg 1$, $(y_0, \eta_0) \notin UWF^{(\mu)}(u)$ yields*

$$(2.2) \quad (x_0, \xi_0) \notin UWF^{(\mu)}(P_\phi u),$$

where

$$(2.3) \quad \xi_0 = \nabla_x \phi(x_0, \eta_0), \quad y_0 = \nabla_\xi \phi(x_0, \eta_0).$$

This theorem corresponds to the theorem for the propagation of Gevrey wave front sets investigated in Theorem 4 in [18].

Proof. Assume $(y_0, \eta_0) \notin UWF^{(\mu)}(u)$. Then, from the definition we can take a neighborhood V_2 of y_0 and a conic neighborhood Γ_2 of η_0 such that for any ε and $\chi \in \mathcal{S}\{\kappa\}$ with $\text{supp } \chi \subset V_2$ an inequality

$$(2.4) \quad |(\chi u)^\wedge(\eta)| \leq C_\varepsilon \exp[\varepsilon \langle \eta \rangle^{1/\mu}] \quad \text{for } \eta \in \Gamma_2$$

holds. Next, using (2.3) we take neighborhoods V_1 and V_2' of x_0 and y_0 , and conic neighborhoods Γ_1 and Γ_2' of ξ_0 and η_0 satisfying

$$V_2' \subset V_2, \quad \Gamma_2' \cap S_\eta^{n-1} \subset \Gamma_2 \cap S_\eta^{n-1}$$

and

$$(2.5) \quad \begin{cases} \text{i) } \nabla_\xi \phi(x, \eta) \in V_2' & \text{for } x \in V_1, \eta \in \Gamma_2', \\ \text{ii) } \nabla_x \phi^{-1}(x, \xi) \in \Gamma_2' & \text{for } x \in V_1, \xi \in \Gamma_1, \end{cases}$$

where $\eta = \nabla_x \phi^{-1}(x, \xi)$ is the inverse function of $\xi = \nabla_x \phi(x, \eta)$. Let $\chi_1(x)$ and $\chi_2(x)$ be functions in $\mathcal{S}\{\kappa\}$ and $\psi_1(\xi)$ and $\psi_2(\xi)$ be symbols in $S_{1,0,G(\kappa)}^0$ satisfying

$$(2.6) \quad \text{supp } \chi_1 \subset V_1,$$

$$(2.7) \quad \text{supp } \chi_2 \subset V_2, \quad \chi_2(y) = 1 \quad \text{for } y \in V_2',$$

$$(2.8) \quad \text{supp } \psi_1 \subset \Gamma_1, \quad \psi_1(\xi) = 1 \quad \text{for } \xi \in \Gamma_1^0$$

with some conic neighborhood Γ_1^0 of ξ_0 , and

$$(2.9) \quad \text{supp } \psi_2 \subset \Gamma_2, \quad \psi_2(\eta) = 1 \quad \text{for } \eta \in \Gamma_2'.$$

Now, write $\chi_1(x)P_\phi u$ as

$$(2.10) \quad \begin{aligned} \chi_1 P_\phi u &= \chi_1 P_\phi \psi_2(D)\chi_2 u + \chi_1 P_\phi \psi_2(D)(1-\chi_2)u + \chi_1 P_\phi(1-\psi_2(D))u \\ &\equiv f_1(x) + f_2(x) + f_3(x). \end{aligned}$$

From (2.5) and (2.8)–(2.9) we can show that $\sigma(\psi_1(D)\chi_1 P_\phi(1-\psi_2(D)))$ satisfies (1.8) and hence from Lemma 1.9 we have

$$\psi_1(D)f_3 = \psi_1(D)\chi_1 P_\phi(1-\psi_2(D))u \in \mathcal{B}\{\kappa\}$$

and

$$(2.11) \quad |\hat{f}_3(\xi)| \leq C \quad \text{for } \xi \in \Gamma_1^0.$$

Similarly, from (2.5)–(2.7) we obtain that $\sigma(\chi_1 P_\phi \psi_2(D)(1-\chi_2))$ satisfies (1.8) and hence we get

$$f_2(x) = \chi_1 P_\phi \psi_2(D)(1-\chi_2)u \in \mathcal{B}\{\kappa\}.$$

This yields

$$(2.12) \quad |\hat{f}_2(\xi)| \leq C \quad \text{for all } \xi.$$

Next, we consider $f_1(x)$. Let τ be a constant satisfying (1.2)–(1.3) and write

$$(2.13) \quad \begin{aligned} \hat{f}_1(\xi) &= \iint e^{i(-x \cdot \xi + \phi(x, \eta))} \chi_1(x) p(x, \eta) \psi_2(\eta) (\chi_2 u)^\wedge(\eta) \hat{d}\eta dx \\ &= \iint_{|\xi - \eta| \leq \lambda \langle \eta \rangle} e^{i(-x \cdot \xi + \phi(x, \eta))} \chi_1(x) p(x, \eta) \psi_2(\eta) (\chi_2 u)^\wedge(\eta) \hat{d}\eta dx \\ &\quad + \iint_{|\xi - \eta| \geq \lambda \langle \eta \rangle} e^{i(-x \cdot \xi + \phi(x, \eta))} \chi_1(x) p(x, \eta) \psi_2(\eta) (\chi_2 u)^\wedge(\eta) \hat{d}\eta dx \\ &\equiv I_1 + I_2 \end{aligned}$$

with $\lambda = (1 + \tau)/2$. Since the absolute value of the integrand of I_1 is estimated by

$$\begin{aligned} C \exp [c \langle \eta \rangle^\sigma + \varepsilon \langle \eta \rangle^{M\mu}] &\leq C' \exp [2\varepsilon \langle \eta \rangle^{M\mu}] \\ &\leq C' \exp [2\varepsilon \{2/(1-\tau)\}^{M\mu} \langle \xi \rangle^{M\mu}], \end{aligned}$$

we have

$$(2.14) \quad |I_1| \leq C'' \exp [2\varepsilon \{2/(1-\tau)\}^{M\mu} \langle \xi \rangle^{M\mu}].$$

Let $L = -i | -\xi + \nabla_x \phi(x, \eta) |^{-2} (-\xi + \nabla_x \phi(x, \eta)) \cdot \nabla_x$. Then, we have $L \exp [i(-x \cdot \xi + \phi(x, \eta))] = \exp [i(-x \cdot \xi + \phi(x, \eta))]$. Hence, using the integration by parts and $| -\xi + \nabla_x \phi(x, \eta) | \geq C(\langle \xi \rangle + \langle \eta \rangle)$ on the support of the integrand of I_2 we can obtain

$$(2.15) \quad |I_2| \leq C.$$

Combining (2.10)–(2.15) we obtain

$$|(\mathcal{X}_1 P_\phi u)^\wedge(\xi)| \leq C \exp [2\varepsilon \{2/(1-\tau)\}^{1/\mu} \langle \xi \rangle^{1/\mu}] \quad \text{for } \xi \in \Gamma^0.$$

Since we can take ε arbitrary, we obtain (2.2). Q.E.D.

3. Propagation of ultra wave front sets. The propagation of Gevrey wave front sets are investigated in [8], [13] and [15] for the solutions of not C^∞ well-posed Cauchy problem of hyperbolic operators. In this section, we give the propagation of the *UWF* for the solutions of the following two degenerate hyperbolic operators in $[s, T] \times R_x^1$:

$$L = D_t^2 - t^{2j} D_x^2 + a i t^k D_x$$

and

$$L = D_t^2 - g(x)^{2j} D_x^2 + a i D_x,$$

where $D_t = -i\partial_t$ and $D_x = -i\partial_x$. First, we consider the former degenerate hyperbolic operator

$$(3.1) \quad L = D_t^2 - t^{2j} D_x^2 + a i t^k D_x \quad \text{in } [s, T] \times R_x^1,$$

where a is a real constant. Then, Shinkai [16] proves that the fundamental solution $E(t, s)$ for the Cauchy problem

$$(3.2) \quad Lu(t) = 0, \quad u(s) = 0, \quad \partial_t u(s) = u_0,$$

when $s < 0 < t$, is constructed in the form

$$(3.3) \quad E(t, s) = \sum_{m, n=1}^2 E_{m, n, \phi_{m, n}}(t, s),$$

where $\phi_{m, n}(t, s) \equiv \phi_{m, n}(t, s; \xi)$ are phase functions defined by

$$\phi_{m, n}(t, s; \xi) = x\xi + \{(-1)^m t^{j+1} + (-1)^n s^{j+1}\} \xi / (j+1).$$

In (3.3) the symbols $e_{m, n}(t, s; \xi)$ of $E_{m, n, \phi_{m, n}}(t, s)$ satisfy

$$(3.4) \quad e_{m, n}(t, s; \xi) = a_{m, n} \exp [C_{m, n} \xi^\sigma] \xi^{-1} (1 + o(1)), \quad \xi \rightarrow +\infty,$$

where

$$\sigma = (j - k - 1) / (2j - k).$$

So, in (3.4), if $\text{Re } C_{m, n} > 0$, then $E_{m, n, \phi_{m, n}}(t, s)$ is a Fourier integral operator of infinite order. Using the fundamental solution in (3.3) we have the following theorem

Theorem 3.1 ([16]). *Assume $k < j - 1$. Let $u(t, x)$ be the solution of (3.2)*

for (3.1) with $u_0(x) = \delta(x)$ (Dirac function). Let $\Gamma_{m,n}$ be the trajectory associated to $\phi_{m,n}$ for $t > 0$. Then we get

$$(3.5) \quad UWF^{(l\sigma)}(u(t)) = \cup_P \Gamma_{m,n},$$

where $P = \{(m, n); \operatorname{Re} C_{m,n} > 0\}$.

REMARK. The result (3.5) shows that if $k < j - 1$, then (3.2) for (3.1) is not C^∞ well-posed and is $\gamma^{(\kappa)}$ -well-posed for $1 < \kappa < (2j - k)/(j - k - 1)$ (for the $\gamma^{(\kappa)}$ -well-posedness see also [5]).

Next, we consider a degenerate hyperbolic operator with respect to the space variable:

$$(3.6) \quad L = D_t^2 - g(x)^{2j} D_x^2 + aiD_x$$

with a positive constant a , where j is an even number and $g(x)$ is an function in $\mathcal{B}\{x\}$ satisfying $g(x) = x$ for $|x| \leq 1$, $g(x) \geq 1$ for $x > 1$ and $g(x) \leq -1$ for $x < -1$. It is well-known that the Cauchy problem (3.2) for (3.6) is not C^∞ well-posed (see [5], [21] and [4]). Assume

$$2j/(2j - 1) \leq \kappa \leq 2j/(j + 1).$$

Let $\phi_\pm(t, s; x, \xi)$ be the phase functions corresponding to the characteristic roots $\pm g(x)^j \xi$ of (3.6). Then, the fundamental solution of the Cauchy problem (3.2) for (3.6) is constructed in the form

$$(3.7) \quad E(t, s) = E_{+, \phi_+}(t, s) + E_{-, \phi_-}(t, s) + (\text{regularizing operator})$$

and the symbols $e_\pm(t, s; x, \xi)$ of the Fourier integral operators $E_{\pm, \phi_\pm}(t, s)$ can be written in the form

$$(3.8) \quad e_\pm(t, s; x, \xi) = \exp [f_\pm(t, s; x, \xi)] e'_\pm(t, s; x, \xi)$$

with symbols $f_\pm(t, s; x, \xi)$ in $S_{1-\delta, \delta, G(\kappa)}^{1/2}$ and elliptic symbols $e'_\pm(t, s; x, \xi)$ in $S_{1-\delta, \delta, G(\kappa)}^0$. Here, $\delta = 1/(2j)$. Moreover, when $s < t$, the symbols $f_\pm(t, s; x, \xi)$ of (3.8) satisfy

$$(3.9) \quad \operatorname{Re} f_+(t, s; x, \xi) \geq C(t-s) \langle \xi \rangle^{1/2} / (|x|^j \langle \xi \rangle^{1/2} + 1),$$

$$(3.10) \quad \operatorname{Re} f_-(t, s; x, \xi) \leq -C(t-s) \langle \xi \rangle^{1/2} / (|x|^j \langle \xi \rangle^{1/2} + 1)$$

for a positive constant C . Hence, $E_{+, \phi_+}(t, s)$ is a Fourier integral operator with infinite order. For a conic set V in T^*R^1 we set $\Gamma(t, s; V) = \cup_\pm \{(x, \xi); (x, \xi) \text{ is a point at } t \text{ of the bicharacteristic strip of } \pm g(x)^j \xi \text{ emanating from } (y, \eta) \text{ in } V\}$. Then, using the fundamental solution (3.7) we have

Theorem 3.2 ([20]). *Let $u(t)$ be the solution of the Cauchy problem (3.2)*

of the operator (3.6) for u_0 in $\mathcal{S}\{\kappa\}'$ with compact support. Then, when μ satisfies $\kappa < \mu < 2$ we have

$$UWF^{(\mu)}(u(t)) = \Gamma(t, s; UWF^{(\mu)}(u_0))$$

and when $\mu \geq 2$ we have

$$UWF^{(\mu)}(u(t)) \subset \Gamma(t, s; UWF^{(\mu)}(u_0)) \cup T_0^*R,$$

especially, we have

$$UWF^{(\mu)}(u(t)) \setminus T_0^*R = \Gamma(t, s; UWF^{(\mu)}(u_0) \setminus T_0^*R),$$

where $T_0^*R = \{(0, \xi); \xi \in R \setminus \{0\}\}$. In particular, when $u_0 = \delta(x)$ (Dirac function) we have

$$(0, \pm 1) \in UWF^{(2)}(u(t)).$$

For the construction of the fundamental solution (3.7) we use finite order Fourier integral operators with complex phase functions $\phi_{\pm}(t, s; x, \xi) - if_{\pm}(t, s; x, \xi)$ as in [7] instead of Fourier integral operators of exponential order. Then, we can give the estimate (3.10) from below.

REMARK. In the above we assumed $a > 0$. But, if we assume $a < 0$ we can also construct the fundamental solution $E(t, s)$ for (3.6) in the same form (3.7) with (3.9)–(3.10) replaced by

$$\begin{cases} \operatorname{Re} f_{-}(t, s; x, \xi) \geq C(t-s)\langle \xi \rangle^{1/2} / (|x|^j \langle \xi \rangle^{1/2} + 1), \\ \operatorname{Re} f_{+}(t, s; x, \xi) \leq -C(t-s)\langle \xi \rangle^{1/2} / (|x|^j \langle \xi \rangle^{1/2} + 1). \end{cases}$$

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