Title	The exchange property of quasi-continuous modules with the finite exchange property
Author(s)	Rizvi, S. Tariq; Oshiro, Kiyoichi
Citation	Osaka Journal of Mathematics. 1996, 33(1), p. 217-234
Version Type	VoR
URL	https://doi.org/10.18910/3617
rights	
Note	

Oshiro, K., Tariq Rizvi, S
Osaka J. Math.
33 (1996), 217-234

THE EXCHANGE PROPERTY OF QUASI-CONTINUOUS MODULES WITH THE FINITE EXCHANGE PROPERTY

KIYOICHI OSHIRO and S. TARIQ RIZVI

(Received October 4, 1994)

Mohamed and Müller showed in [5] that continuous modules have the exchange property. And, recently, they also showed in [6] that for nonsingular quasi-continuous modules, the finite exchange property implies the exchange property. However, it is still open whether this is true or not for any quasi-continuous modules ([5, Problem 2]). The purpose of this paper is to answer this problem in the affirmative. This, then also provides another instance of modules for which the existence of the finite exchange property implies that of the exchange property in reference to the longstanding open question posed in Crawley-Jonsson [1].

Discrete (=semiperfect) modules and quasi-discrete (=quasi-semiperfect) modules are dual to continuous modules and quasi-continuous modules, respectively. We note that discrete modules have the exchange property and, for quasi-discrete modules, the finite exchange property implies the exchange property. These results follows by summarizing following results:

(1) (Oshiro [10]) Every quasi-discrete module M has an indecomposable decomposition $M=\sum_I \oplus M_i$ such that $M'=\sum\{M_i | M_i$: completely indecomposable$\}$ satisfies the finite exchange property. So, if M is discrete then it satisfies the finite exchange property since $M=M'$.

(2) (Harada-Ishii [2], Yamagata [12], [13]) If a module has an indecomposable decomposition and satisfies the finite exchange property, then it satisfies the exchange property in direct sums of completely indecomposable modules.

(3) (Zimmermann-Huisgen and Zimmermann [11]) A module M satisfies the exchange property if and only if for any $P=M\oplus X=\sum_I \oplus M_i$ with each $M_i \simeq M$, there exists $M_i' < \oplus M_i$ for each $i \in I$ such that $P=M\oplus \sum_I \oplus M_i'$.

The reader is referred to Mohamed and Müller'Book [5] for the background of these results.

1. Preliminaries. Throughout this paper R will denote a ring with identity and all R-modules will be unital right R-modules. For two R-modules X and Y, we use $X \subseteq_e Y$, $X \underset{\sim}{\subset} Y$, $X \underset{\sim}{\subset}_e Y$, $X < \oplus Y$ and $X \lesssim \oplus Y$ to mean that X is an essential submodule of Y, X is isomorphic to a submodule of Y, X is isomorphic to an essential submodule of Y, X is a direct summand and X is isomorphic to a direct summand of Y, respectively. For a set I, $|I|$ stands for the cardinal of I.

An R-module M is called an extending module (or a CS-module) if it satisfies

(C_1): for any submodule X of M, there exists a direct summand X^* of M such that $X \subseteq_e X^*$

M is called continuous if it satisfies (C_1) and

(C_2): Every submodule of M which is isomorphic to a direct summand of M is a direct summand.

M is called a quasi-continuous module if it satisfies (C_1) and

(C_3): If X and Y are direct summands of M with $X \cap Y = 0$, then $X \oplus Y$ is a direct summand.

For an R-module M with a decomposition $M = \sum_I \oplus M_i$, we use the following condition:

(A) For any choice of $x_i \in M_{\alpha_i} (i \in \mathbb{N}, \alpha_i$ distinct) such that $(0: x_1) \subseteq (0: x_2) \subseteq \ldots$, the sequence becomes stationary, where $(0: x)$ denotes the annihilator right ideal of x.

This condition appeared in [8] (cf.[5], [7]). One of interesting results on this condition is (2) of the following

Proposition 1.1. *Let M be a quasi-continuous module. Then the following hold*:

(1) *for any decomposition $M = \sum_I \oplus M_i$ and any $J \subseteq I$, $\sum_I \oplus M_i$ is $\sum_{I-J} \oplus M_i$-injective*

(2) *any decomposition $M = \sum_I \oplus M_i$ satisfies the condition* (A).

(3) *for any decomposition $M = \sum_I \oplus M_i$ and any direct summand X of M, there exists $N_i < \oplus M_i$ such that $M = X \oplus \sum_I \oplus N_i$.*

Proof. (1) follows from [9, Proposition 1.5]. (2) follows from[5, Proposition2.13] or from (1) above and [5, Proposition 1.9]. And see[3] for (3).

A module M is called a square module if $M \simeq X \oplus X$ for some module X, and a module is called a square free if it does not contain non-zero square modules.

The following results are important and useful in the study of quasi-continuous modules

Lemma 1.1 ([5, Theorem 2.37]). *Any quasi-continuous module is a direct sum of a quasi-injective module and a square free module.*

Proposition 1.2. *For an R-module X, the following conditions are equivalent*:

(1) *X is quasi-continuous.*

(2) *Any decomposition* $\mathrm{E}(X)=\sum_I \oplus M_i$ *implies* $X=\sum_I \oplus(M_i \cap X)$, *where* $\mathrm{E}(X)$ *is the injective hull of* X.

(3) *for any R-module Y with $X \subseteq_e Y$, any decomposition $Y=\sum_I \oplus Y_i$ implies $X=\sum_I \oplus(Y_i \cap X)$.*

Proof. The equivalence of 1) and 2) is well known [5, Theorem 2.8]. We may show the implication 2) $\Rightarrow$ 3). Let Y be an R-module with $X \subseteq_e Y$ and consider a decomposition $Y=\sum_I \oplus Y_i$. Let $x \in X$. Then there exists a finite subset $F=\{1, 2, \ldots, n\}$ of I such that $x \in \sum_F \oplus Y_i$. Let $x=y_1+\ldots+y_n$, where $y_i \in Y_i$. Consider $\mathrm{E}(X)=\sum_F \oplus \mathrm{E}(Y_i) \oplus \mathrm{E}(\sum_{I-F} \oplus Y_i)$. By 2), we have $X=\sum_F \oplus(\mathrm{E}(Y_i) \cap X) \oplus \mathrm{E}(\sum_{I-F} \oplus Y_i) \cap X$. Express the element x as $x=p_1+p_2+\ldots+p_n+q$ where $p_i \in \mathrm{E}(Y_i) \cap X$ and $q \in \mathrm{E}(\sum_{I-F} \oplus Y_i) \cap X$. Since $x=y_1+\ldots+y_n$ with $y_i \in \mathrm{E}(Y_i)$, we see $p_i=y_i$, $i=1, \ldots, n$. Hence $y_i \in Y_i \cap X$, $i=1, \ldots, n$; so $x \in \sum_I \oplus(Y_i \cap X)$. Accordingly we see $X=\sum_I \oplus(Y_i \cap X)$.

For a cardial α, an R-module X has the α-exchange property if for any R-module M and any two decompositions $M=X \oplus N=\sum_I \oplus M_i$ with $|I| \leq \alpha$, there exists $M_i' < \oplus M$ for each $i \in I$ such that

$$M=X \oplus(\sum_I \oplus M_i')$$

X has the exchange property if this holds for any cardinal α and has the finite exchange property if this holds whenever the index set I is finite. We note that, in these definitions, we may assume that $M_i \simeq X$ for all $i \in I$ (by Zimmermann-Huisgen and Zimmermann [11]).

2. A key lemma. In this section we show the following which is a key lemma of this paper. We note that this lemma is also used for the study of direct sums of relative continuous modules [3], [4].

Lemma 2.1. *Let P be an R-module with a decomposition $P=\sum_I \oplus M_i$ such that each M_i is extending. We consider the index set I as an well ordered set*:

$I=\{0, 1, \dots w, w+1, \dots\}$, and let X be a submodule oa M. Then there are submodules $T(i)\subseteq_e T(i)^* < \oplus M_i$, decompositions $M_i = T(i)^* \oplus N_i$ and a submodule $\sum_I \oplus X(i) \subseteq_e X$ for which the following properties hold:

1) $X(0)=T(0)\subseteq_e T(0)^*$.
2) $X(k)\subseteq T(k)\oplus \sum_{i<k}\oplus N_i$

for all $k\in I$.

3) $\sigma(X(k))=T(k)\subseteq_e T(k)^*$, $X(k)\simeq \sigma(X(k))$*(by* $\sigma|X(k)$*)*

for all $k\in I$, where σ is the projection: $P=\sum_I \oplus T(i)^* \oplus \sum_I \oplus N_i \rightarrow \sum_I \oplus T(i)^*$.

4) $X\simeq \sigma(X)$*(by* $\sigma|X$*).*

For a proof of this result, we need two lemmas

Lemma 2.2. *Let M be an R-module with a decomposition $M=M_1\oplus M_2$ and let X a submodule of M. If there is a decomposition $M_i=M_i^*\oplus M_i^{**}$ such that $M_i\cap X\subseteq_e M_i^*$ for $i=1, 2.$, then $X_e\supseteq(M_1^*\cap X)\oplus(M_2^*\cap X)\oplus(M_1^*\oplus M_2^{**})\cap X$. So, in particular if $M_1\cap X=0$, then $X_e\supseteq(M_2^*\cap X)\oplus(M_1\oplus M_2^{**})\cap X$.*

Proof. Let $(0\neq)x\in X$ and express x in $M=M_1^*\oplus M_1^{**}\oplus M_2^*\oplus M_2^{**}$ as $x=x_1^* + x_1^{**}+x_2^*+x_2^{**}$ where $x_i^*\in M_i^*$ and $x_i^{**}\in M_i^{**}$. If $x_1^*+x_2^*\in(M_1^*\cap X)\oplus(M_2^*\cap X)$, then $x_1^{**}+x_2^{**}\in(M_1^{**}\oplus M_2^{**})\cap X$ and hence $x\in(M_1^*\cap X)\oplus(M_2^*\cap X)\oplus(M_1^{**}\oplus M_2^{**})\cap X$. In the case of $x_1^*+x_2^*\notin(M_1^*\cap X)\oplus(M_2^*\cap X)$, we take $r\in R$ such that $0\neq(x_1^*+x_2^*)r\in(M_1^*\cap X)\oplus(M_2^*\cap X)$. Then $0\neq xr\in(M_1^*\cap X)\oplus(M_2^*\cap X)\oplus(M_1^{**}\oplus M_2^{**})\cap X$.

Lemma 2.3. *Let M be an R-module with a decomposition $M=A\oplus B\oplus C\oplus D$ and let X be a submodule of M. If Y is a submodule of X such that $Y\subseteq(A\oplus B)\cap X$ and $Y \overset{\sigma|Y}{\simeq} \sigma(Y)\subseteq_e A$, where σ is the projection*: $M=A\oplus B\oplus C\oplus D\rightarrow A$. *Then $Y\oplus((B\oplus C)\cap X)\subseteq_e(A\oplus B\oplus C)\cap X$.*

Proof. Let $(0\neq)x=a+b+c\in(A\oplus B\oplus C)\cap X$, where $a\in A$, $b\in B$ and $c\in C$. If $a=0$, $x=b+c\in(B\oplus C)\cap X$. If $a\neq 0$, then $0\neq\sigma(xr)\in\sigma(Y)$ for some $r\in R$; so there exists $y\in Y$ such that $\sigma(xr)=\sigma(y)$. Since $xr-y\in\text{Ker }\sigma\cap(B\oplus C)$, we see $xr\in Y\oplus((B\oplus C)\cap X)$. Hence we see $Y\oplus((B\oplus C)\cap X)\subseteq_e(A\oplus B\oplus C)\cap X$.

Proof of Lemma 2.1. We put $X_i=M_i\cap Y$ for all $i\in I$. Since M_i is extending, we have a decomposition

$$M_i = X_i^* \oplus X_i^{**}$$

such that $X_i \subseteq_e X_i^*$ for all $i \in I$. By Lemma 2.2,

$$(M_0 \oplus M_1) \cap X_e \supseteq X_0 \oplus X_1 \oplus (X_0^{**} \oplus X_1^{**}) \cap X.$$

We put

$$X(0) = X_0,\ X(1) = X_1 \oplus (X_0^{**} \oplus X_1^{**}) \cap X.$$

Let π_0, π_1 be the projections:

$$X_0^{**} \oplus X_1^{**} \to X_0^{**},\ X_0^{**} \oplus X_1^{**} \to X_1^{**}$$

respectively. Since $X_1^{**} \cap X = 0$, we see that

$$(X_0^{**} \oplus X_1^{**}) \cap X \simeq \pi_0((X_0^{**} \oplus X_1^{**}) \cap X) \simeq \pi_1((X_0^{**} \oplus X_1^{**}) \cap X). \dots (*)$$

canonically. Put

$$T(0) = X(0),\ T(0)^* = X(0)^* = X_0^*,\ T(1) = X_1 \oplus \pi_1((X_0^{**} \oplus X_1^{**}) \cap X).$$

Since M_1 is extending, we have a decomposition

$$M_1 = T(1)^* \oplus N_1$$

with $T(1) \subseteq_e T(1)^*$. Putting $N_0 = X_0^{**}$, we have

$$P = T(0)^* \oplus T(1)^* \oplus N_0 \oplus N_1 \oplus \sum_{2 \le i} \oplus M_i$$

such that

$$X(0) = T(0),\ X(1) \subseteq T(1) \oplus N_0,$$
$$\sigma_1(X(i)) = T = (i),\ X(i) \simeq T(i) \text{ by } (\sigma_1 | X(i)) \text{ (cf. (*) above)}$$

for $i = 1, 2$, where σ_1 is the projection:

$$P = T(0)^* \oplus T(1)^* \oplus N_0 \oplus N_1 \oplus \sum_{2 \le i} \oplus M_i \to T(0)^* \oplus T(1)^*$$

Next consider $(M_0 \oplus M_1 \oplus M_2) \cap X$. Put $A = T(0)^* \oplus T(1)^*$, $B = N_0 \oplus N_1$, $C = M_2$ and $D = \sum_{3 \le i} \oplus M_i$ and $Y = X(0) \oplus X(1)$. Then $X \subseteq P = A \oplus B \oplus C \oplus D$, $A \oplus B \oplus C = M_0 \oplus M_1 \oplus M_2$ and $Y \simeq \sigma_1(Y) \subseteq_e A$. So we see from Lemma 2.3 that

$$(M_0 \oplus M_1 \oplus M_2) \cap X_e \supseteq X(0) \oplus X(1) \oplus (N_0 \oplus N_1 \oplus N_2) \cap X$$

Furthere, since $(N_0 \oplus N_1) \cap X = 0$, we see from Lemma 2.2 that

$$(N_0 \oplus N_1 \oplus M_2) \cap X_e \supseteq X_2 \oplus (N_0 \oplus N_1 \oplus X_2^{**}) \cap X$$

Let π_{01}, π_2 be the projections:

$$N_0 \oplus N_1 \oplus X_2^{**} \to N_0 \oplus N_1,\ N_0 \oplus N_1 \oplus X_2^{**} \to X_2^{**}$$

respectively. Since $(N_0 \oplus N_1) \cap X = 0$ and $X_2^{**} \cap X = 0$, we see

$$(N_0 \oplus N_1 \oplus X_2^{**}) \cap X \simeq \pi_{01}((N_0 \oplus N_1 \oplus X_2^{**}) \cap X) \simeq \pi_2((N_0 \oplus N_1 \oplus X_2^{**}) \cap X)$$

canonically...(**)
Put

$$X(2) = X_2 \oplus (N_0 \oplus N_1 \oplus X_2^{**}) \cap X,$$
$$T(2) = X_2 \oplus \pi_2((N_0 \oplus N_1 \oplus X_2^{**}) \cap X).$$

Then

$$(M_0 \oplus M_1 \oplus M_2) \cap X_e \supseteq X(0) \oplus X(1) \oplus X(2).$$

Since M_2 is extending, we have a decomposition

$$M_2 = T(2)^* \oplus N_2$$

with $T(2) \subseteq_e T(2)^*$. Here we see

$$P = T(0)^* \oplus T(1)^* \oplus T(2)^* \oplus N_0 \oplus N_1 \oplus N_2 \oplus \sum_{3 \leq i} \oplus M_i,$$
$$X(2) \subseteq T(2) \oplus N_0 \oplus N_1,$$

and for the projection :

$$\sigma_2 : P = T(0)^* \oplus T(1)^* \oplus T(2)^* \oplus N_0 \oplus N_1 \oplus N_2 \oplus \sum_{3 \leq i} \oplus M_i \to$$
$$T(0)^* \oplus T(1)^* \oplus T(2)^*.$$

We see that

$$\sigma_2(X(i)) = T(i),\ X(i) \simeq T(i)(\text{by } \sigma_2|X(i))$$

for $i = 0, 1, 2$(cf (**)).

Now we proceed our argument by transfinite induction on $\alpha \in I$. Let $\alpha \in I$ and put $J = \{i \in I | i < \alpha\}$

Assume that there are submodules $T(i) \subseteq_e T(i)^* <_\oplus M_i$, decompositions $M_i = T(i)^* \oplus N_i$ for which the following hold :

1) $X(0) = T(0)$,
2) $X(k) \subseteq T(k) \oplus \sum_{i<k} \oplus N_i \forall k \in J$,
3) $(\sum_{i \leq k} \oplus M_i) \cap X_e \supseteq \sum_{i \leq k} \oplus X(i) \forall k \in J$; so$(\sum_J \oplus M_i) \cap X_e \supseteq \sum_J \oplus X(i)$,
4) $X(k) \simeq T(k)$ by $(\sigma_J | X(k)) \forall k \in J$

where σ_J is the projection :

$$P = \sum_J \oplus T(i)^* \oplus \sum_J \oplus N(i) \oplus \sum_{I-J} \oplus M(i) \to \sum_J \oplus T(i)^*$$

So

$$\sum_J \oplus X(i) \simeq \sum_J \oplus T(i) (\text{by } \sigma_J | \sum_J \oplus X(i))$$

Consider $(\sum_J \oplus M_i \oplus M_\alpha) \cap X$. We note that

$$\sum_J \oplus X(i) \subseteq_e (\sum_J \oplus M_i) \cap X = (\sum_J \oplus T(i)^* \oplus \sum_J \oplus N_i) \cap X,$$
$$\sum_J \oplus X(i) \simeq \sum_J \oplus T(i) (\text{by } \sigma_J | \sum_J \oplus X(i))$$

Considering

$$\sum_J \oplus T(i)^* \oplus \sum_J \oplus N(i) \oplus M_\alpha \oplus \sum_{I-J\cup\alpha} \oplus M_i \to \sum_J \oplus T(i)^*$$

we infer from Lemma 2.3 that

$$(\sum_J \oplus M_i \oplus M_\alpha) \cap X_e \supseteq \sum_J \oplus X(i) \oplus (\sum_J \oplus N_i \oplus M_\alpha) \cap X.$$

Since $\sum_J \oplus N_i \cap X = 0$, we see by Lemma 2.2 that

$$(\sum_J \oplus N_i \oplus M_\alpha) \cap X_e \supseteq X_\alpha \oplus (\sum_J \oplus N_i \oplus X_\alpha^{**}) \cap X$$

(where $X_\alpha = M_\alpha \cap X \subseteq_e X_\alpha^*$, $M_\alpha = X_\alpha^* \oplus X_\alpha^{**}$)

Let π_J and π_α be the projections:

$$\sum_J \oplus N_i \oplus X_\alpha^{**} \to \sum_J \oplus N_i, \ \sum_J \oplus N_i \oplus X_\alpha^{**} \to X_\alpha^{**}$$

respectively. We see that

$$(\sum_J \oplus N_i \oplus X_\alpha^{**}) \cap X \simeq \pi_J((\sum_J \oplus N_i \oplus X_\alpha^{**}) \cap X) \simeq \pi_\alpha((\sum_J \oplus N_i \oplus X_\alpha^{**} \cap X)$$

canonically. We put

$$X(\alpha) = X_\alpha \oplus (\sum_J \oplus N_i \oplus X_\alpha^{**}) \cap X,$$
$$T(\alpha) = X_\alpha \oplus \pi_\alpha((\sum_J \oplus N_i \oplus X_\alpha^{**}) \cap X).$$

Since M_α is extending, we have a decomposition

$$M_\alpha = T(\alpha)^* \oplus N_\alpha$$

with $T(\alpha) \subseteq_e T(\alpha)^*$. Now we see

$$X(\alpha) \subseteq T(\alpha) \oplus \sum_J \oplus N_i,$$
$$\sigma(X(\alpha)) = T(\alpha), \ X(\alpha) \simeq T(\alpha) (\text{by } \sigma | X(\alpha))$$

where σ is the projection:

$$P = \sum_J \oplus T(i)^* \oplus \sum_{J\cup\alpha} \oplus N(i) \oplus \sum_{I-J\cup\alpha} \oplus M(i) \to \sum_{J\cup\alpha} \oplus T(i)^*.$$

Furthermore we see

$$(\sum_{J\cup\alpha}\oplus M_i)\cap X_e \supseteq \sum_{J\cup\alpha}\oplus X(i)$$

Thus 1)-5) above hold for $J\cup\alpha$, and this completes the proof by transfinite induction.

3. The exchange property. Using Lemma 2.1 we shall give a proof of the exchange property of continuous modules from our point of view.

Proposition 3.1. *Let P be an R-module and X a submodule of P. If X is continuous and P has a decomposition $P=\sum_I\oplus M_i$ with each $M_i\simeq X$, then there exists direct summand $N_i<\oplus M_i$ for each $i\in I$ such that $P=X\oplus\sum_I\oplus N_i$. So, X is a direct summand of P.*

Proof. By Lemma 2.1, we have

$$P=\sum_I\oplus T(i)^*\oplus\sum_I\oplus N_i,\ X_e\supseteq\sum_I\oplus X(i)$$

such that, for each $i\in I$,

1) $T(i)\subseteq_e T(i)^*$,
2) $M_i=T(i)^*\oplus N_i$,
3) $\sigma(X(i))=T(i)$, $X(i)\simeq T(i)$(by $\sigma|X(i)$)

where σ is the projection :

$$P=\sum_I\oplus T(i)^*\oplus\sum_I\oplus N_i\longrightarrow\sum_I\oplus T(i)^*.$$

Since X is quasi-continuous and $X\simeq\sigma(X)\subseteq_e\sum_I\oplus T(i)^*$, we obtain, by Proposition 1.2,

$$\sigma(X)=\sum_I\oplus(T(i)^*\cap\sigma(X)).$$

Putting $X(i)^\#=\sigma^{-1}(T(i)^*\cap\sigma(X))$, we see

$$X=\sum_I\oplus X(i)^\#,$$
$$X(i)\subseteq_e X(i)^\#\ \forall i\in I,$$
$$T(i)\subseteq_e\sigma(X(i)^\#)\subseteq_e T(i)^*\ \forall i\in I.$$

Since $X\simeq M_i$ and $X(i)^\#<\oplus X$, we see from the condition (C_2) for X that $\sigma(X(i)^\#)<\oplus T(i)^*$; whence $\sigma(X(i)^\#)=T(i)^*$ for all $i\in I$.

As a result

$$X \simeq \sigma(X) = \sum_I \oplus T(i) \text{ (by } \sigma|X).$$

Hence it follows $P = X \oplus \sum_I \oplus N_i$.

As an immediate consequence we have

Theorem 3.1 ([5, Theorem 3.24]). *Continuous module have the exchange property.*

REMARK. We note that, in the proof above, the exchange property of quasi-injective modules is not used. (Compare our proof to the proof of [5, Theorem 3. 24])

Now, we are in a position to show our main result

Theorem 3.2. *Any quasi-continuous module with the finite exchange property has the exchange property.*

Proof. Let X be a quasi-continuous module with the finite exchange property. We may assume X to be a square free by Lemma 1.1. In order to show our result by transfinite induction, let α be an infinite cardinal and assume that X satisfies β-exchange property for any cardinal $\beta < \alpha$. To show that X satisfies the α-exchange property, consider the situation of R-modules:

$$P = \sum_I \oplus M_i = X \oplus Y$$

where $|I| = \alpha$ and $M_i \simeq X$ for all $i \in I$. We may consider I as a well ordered set; $I = \{0, 1, \dots, \omega, \dots\}$, whose ordinal is an initial ordinal; so, for any $\beta \in I$, the cardinal of $\{i \in I | i < \beta\} < \alpha$. By Lemma 2.1 and, as in the proof of Proposition 3. 1, we have decompositions:

$$\begin{aligned} &P = \sum_I \oplus T(i)^* \oplus \sum_I \oplus N_i, \\ &X = \sum_I \oplus X(i)^\# {}_e\supseteq \sum_I \oplus X_i \end{aligned}$$

such that, for all $k \in I$,

$$\begin{aligned} &M_k = T(k)^* \oplus N_k, \quad T(k) \subseteq_e T(k)^*, \quad X(k) \subseteq_e X(k)^\#, \\ &X(k) \subseteq T(k) \oplus \sum_{i<k} \oplus N_i, \\ &X(k)^\# \subseteq T(k)^* \oplus \sum_I \oplus N_k, \\ &\sigma(X(k)) = T(k) \subseteq_e \sigma(X(k)^\#) \subseteq_e T(k)^*, \\ &X(k) \simeq T(k) \quad \text{(by } \sigma|X(k)), \\ &X(k)^\# \simeq \sigma(X(k)^\#) \text{ (by } \sigma|X(k)^\#) \end{aligned}$$

where σ is the projection:

$$P=\sum_I\oplus T(i)^*\oplus\sum_I\oplus N_i\rightarrow\sum_I\oplus T(i)^*.$$

Since $N_k<\oplus M_k\simeq X=\sum_I\oplus X_i^\#$, by Proposition 1.1, we have a decomposition $N_k=\sum_{i\in I}\oplus N_k(i)$ for each $k\in I$ with $N_k(i)\underset{\sim}{<}\oplus X(i)^\#$.

We note that $\sum_I\oplus T(i)^*$ is square free, since $X\underset{\sim}{\subseteq}_e\sum_I\oplus T(i)^*$. Now, using the finite exchange property of $X(0)^\#$ for

$$X(0)^\#<\oplus T(0)^*\oplus\sum_I\oplus N_i$$

we have decompositions

$$T(0)^*=\overline{T(0)^*}\oplus\overline{T(0)^*},$$
$$\sum_I\oplus N_i=\overline{\sum_I\oplus N_i}\oplus\overline{\sum_I\oplus N_i}$$

such that

$$P=X(0)^\#\oplus\overline{T(0)^*}\oplus\sum_{I-0}\oplus T(i)^*\oplus\overline{\sum_I\oplus N_i}.$$

We denote, by π_0, the projection:

$$P=X(0)^\#\oplus\overline{T(0)^*}\oplus\sum_{I-0}\oplus T(i)^*\oplus\overline{\sum_I\oplus N_i}\longrightarrow X(0)^\#.$$

Then

$$\overline{T(0)^*}\oplus\overline{\sum_I\oplus N_i}\simeq X(0)^\#\ (\text{by } \pi_0|\overline{T(0)^*}\oplus\overline{\sum_I\oplus N_i}).$$

Assume $0\neq\overline{\sum_I\oplus N_i}$ and take $0\neq n''\in\overline{\sum_I\oplus N_i}$. We exprese n'' in $P=X(0)^\#\oplus\overline{T(0)^*}\oplus\sum_{I-0}\oplus T(i)^*\oplus\overline{\sum_I\oplus N_i}$ as $n''=a+b+n'$, where $a\in X(0)^\#$ $b\in\overline{T(0)^*}\oplus\sum_{I-0}\oplus T(i)^*$, $n'\in\overline{\sum_I\oplus N_i}$

Since $0\neq\pi_0(n'')=a\in X(0)^\#{}_e\supseteq X(0)$, there exists $r\in R$ such that $0\neq ar\in X(0)$. Since $X(0)=T(0)$ and $n''r=ar+br+n'r$, we see from $n''r-n'r=ar+br\in\sum_I\oplus T(i)^*\cap\sum_I\oplus N_i=0$ that $n''r-n'r=0$; whence $n''r=0$, so $ar=0$, a contradiction. Accordingly, $\overline{\sum_I\oplus N_i}=0$ and hence

$$P=X(0)^\#\oplus\overline{T(0)^*}\oplus\sum_{I-0}\oplus T(i)^*\oplus\sum_I\oplus N_i,$$

so

$$X(0)^{\#}\oplus\overline{T(0)^{*}}\simeq T(0)^{*}.$$

Since $X(0)^{\#}\oplus\overline{T(0)^{*}}$ is square free and $X(0)^{\#}\subseteq_{e}T(0)^{*}$, we also see that $\overline{T(0)^{*}}=0$. Therefore

$$P=X(0)^{\#}\oplus\sum_{I-0}\oplus T(i)^{*}\oplus\sum_{I}\oplus N_{i},$$

Next using the finite exchange property of $X(1)^{\#}$ in

$$\begin{aligned}P&=X(0)^{\#}\oplus T(1)^{*}\oplus\sum_{I-\{0,1\}}\oplus T(i)^{*}\oplus N_{0}(1)\oplus\sum_{I-1}\oplus N_{0}(i)\oplus\sum_{I-0}\oplus N_{i}\\&=W\oplus T(1)^{*}\oplus N_{0}(1)\oplus\sum_{I-0}\oplus N_{i}\end{aligned}$$

where $W=X(0)^{\#}\oplus\sum_{i-\{0,1\}}\oplus T(i)^{*}\oplus\sum_{I-1}\oplus N_{0}(i)$, we have decompositions

$$\begin{aligned}&W=\overline{W}\oplus\overline{W}\\&T(1)^{*}=\overline{T(1)^{*}}\oplus\overline{T(1)^{*}},\\&N_{0}(1)=\overline{N_{0}(1)}\oplus\overline{N_{0}(1)},\\&\sum_{I-0}\oplus N_{i}=\overline{\sum_{I-0}\oplus N_{i}}\oplus\overline{\sum_{I-0}\oplus N_{i}}\end{aligned}$$

such that

$$P=\overline{W}\oplus\overline{T(1)^{*}}\oplus\overline{N_{0}(1)}\oplus\overline{\sum_{I-0}\oplus N_{i}}.$$

Since $\sum_{I}\oplus T(i)^{*}$ is square free, we see from $\overline{\overline{W}}\subsetsim X(1)^{1}\subsetsim T(1)^{*}$ that $\overline{\overline{W}}=0$. So

$$P=X(0)^{\#}\oplus X(1)^{\#}\oplus\overline{T(1)^{*}}\oplus\sum_{I-\{0,1\}}\oplus T(i)^{*}\oplus\overline{N_{0}(1)}\oplus\sum_{I-1}\oplus N_{0}(i)\oplus\overline{\sum_{I-0}\oplus N_{i}}.$$

In order to show $\overline{\overline{\sum_{I-0}\oplus N_{i}}}=0$, consider the projection $\pi_{1}:P=X(0)^{\#}\oplus X(1)^{\#}\oplus\overline{T(1)^{*}}\oplus\sum_{I-\{0,1\}}\oplus T(i)^{*}\oplus\overline{N_{0}(1)}\oplus\sum_{I-1}\oplus N_{0}(i)\oplus\overline{\sum_{I-0}\oplus N_{i}}\longrightarrow X(1)^{\#}$. Assuming $\overline{\overline{\sum_{I-0}\oplus N_{i}}}\neq0$ we take $0\neq n''\in\overline{\overline{\sum_{I-0}\oplus N_{i}}}$. We express n'' in $P=X(0)^{\#}\oplus X(1)^{\#}\oplus\overline{T(1)^{*}}\oplus\sum_{I-\{0,1\}}\oplus T(i)^{*}\oplus\overline{N_{0}(1)}\oplus\sum_{I-1}\oplus N_{0}(i)\oplus\overline{\sum_{I-0}\oplus N_{i}}$ as $n''=a+b+n'$, where $a\in X(0)^{\#}\oplus X(1)^{\#}$, $b\in\overline{T(1)^{*}}\oplus\sum_{I-\{0,1\}}\oplus T(i)^{*}\oplus\overline{N_{0}(1)}\oplus\sum_{I-1}\oplus N_{0}(i)$, $n'\in\overline{\sum_{I-0}\oplus N_{i}}$. Since $X(0)\oplus X(1)\subseteq_{e}X(0)^{\#}\oplus X(1)^{\#}$, we can take $r\in R$, such that $0\neq ar\in X(0)\oplus X(1)$. Note that $0\neq n''r=ar+br+n'r$. Since $X(0)\oplus X(1)\subseteq_{e}T(0)^{*}\oplus T(1)^{*}\oplus N_{0}$, this implies $ar+br=0$ and $n''r=n'r$; whence $n''r=0$, a contradiction.Thus we get

$$P=X(0)^{\#}\oplus X(1)^{\#}\oplus\overline{T(1)^{*}}\oplus\sum_{I-\{0,1\}}\oplus T(i)^{*}\oplus\overline{N_{0}(1)}\oplus\sum_{I-1}\oplus N_{0}(i)\oplus\sum_{I-1}\oplus N_{i},$$

We proceed with the same argument for $X(2)^{\#}$. We consider

$$\begin{aligned}P=&X(0)^{\#}\oplus X(1)^{\#}\oplus\overline{T(1)^*}\oplus T(2)^*\oplus\sum_{I-\{0,1,2\}}\oplus T(i)^*\\&\oplus\overline{N_0(1)}\oplus N_0(2)\oplus\sum_{I-\{0,1\}}\oplus N_0(i)\\&\oplus N_1(2)\oplus\sum_{I-2}\oplus N_1(i)\\&\oplus\sum_{I-\{0,1\}}\oplus N_i.\\=&W\oplus T(2)^*\oplus N_0(2)\oplus N_1(2)\oplus\sum_{I-\{0,1\}}\oplus N_i\end{aligned}$$

where

$$\begin{aligned}W=&X(0)^{\#}\oplus X(1)^{\#}\oplus\overline{T(1)^*}\oplus\sum_{I-\{0,1,2\}}\oplus T(i)^*\oplus\overline{N_0(1)}\\&\oplus\sum_{I-\{1,2\}}\oplus N_0(i)\oplus\sum_{I-2}\oplus N_1(i).\end{aligned}$$

And using the finite exchange property of $X(2)^{\#}$ in this decomposition, we have decompositions

$$\begin{aligned}&W=\overline{\overline{W}}\oplus\overline{W},\\&T(2)^*=\overline{\overline{T(2)^*}}\oplus\overline{T(2)^*},\\&N_0(2)=\overline{\overline{N_0(2)}}\oplus\overline{N_0(2)},\\&N_1(2)=\overline{\overline{N_1(2)}}\oplus\overline{N_1(2)},\\&\sum_{I-\{0,1\}}\oplus N_i=\overline{\overline{\sum_{I-\{0,1\}}\oplus N_i}}\oplus\overline{\sum_{I-\{0,1\}}\oplus N_i}\end{aligned}$$

such that

$$P=X(2)^{\#}\oplus\overline{W}\oplus\overline{T(2)^*}\oplus\overline{N_0(2)}\oplus\overline{N_1(2)}\oplus\overline{\sum_{I-\{0,1\}}\oplus N_i}.$$

But, as $\overline{\overline{W}}\underset{\sim}{\subset}X(2)^{|}\underset{\sim}{\subset}T(2)^*$ and as $\sum_{I}\oplus T(i)^*$ is square free, we obtain $\overline{\overline{W}}=0$. So

$$\begin{aligned}P=&W\oplus T(2)^*\oplus N_0(2)\oplus N_1(2)\oplus\sum_{I-\{0,1\}}\oplus N_i\\=&X(0)^{\#}\oplus X(1)^{\#}\oplus X(2)^{\#}\oplus\overline{T(1)^*}\oplus\overline{T(2)^*}\oplus\sum_{I-\{0,1,2\}}\oplus T(i)^*\\&\oplus\overline{N_0(1)}\oplus\overline{N_0(2)}\oplus\sum_{I-\{1,2\}}\oplus N_0(i)\\&\oplus\overline{N_1(2)}\oplus\sum_{I-2}\oplus N_1(i)\\&\oplus\overline{\sum_{I-\{0,1\}}\oplus N_i}.\end{aligned}$$

We denote by π_2 the projection :

$$\begin{aligned}P=&X(0)^{\#}\oplus X(1)^{\#}\oplus X(2)^{\#}\oplus\overline{T(1)^*}\oplus\overline{T(2)^*}\oplus\sum_{I-\{0,1,2\}}\oplus T(i)^*\\&\oplus\overline{N_0(1)}\oplus\overline{N_0(2)}\oplus\sum_{I-\{1,2\}}\oplus N_0(i)\\&\oplus\overline{N_1(2)}\oplus\sum_{I-2}\oplus N_1(i)\oplus\overline{\sum_{I-\{0,1\}}\oplus N_i}\longrightarrow X(2)^{\#}.\end{aligned}$$

Then note that

$$\overline{\overline{T(2)^*}}\oplus\overline{\overline{N_0(2)}}\oplus\overline{\overline{N_1(2)}}\oplus\overline{\overline{\sum_{I-\{0,1\}}\oplus N_i}}\simeq X(2)^{\#}\ldots(*)$$

(by $\pi_2|\overline{\overline{T(2)^*}}\oplus\overline{\overline{N_0(2)}}\oplus\overline{\overline{N_1(2)}}\oplus\overline{\overline{\sum_{I-\{0,1\}}\oplus N_i}}$).

We shall show $\overline{\overline{\sum_{I-\{0,1\}}\oplus N_i}}=0$. Assume not, and take $0\neq n''\in\overline{\overline{\sum_{I-\{0,1\}}\oplus N_i}}$. We express n'' as $n''=a+b+n'$
where $a\in X(0)^{\#}\oplus X(1)^{\#}\oplus X(2)^{\#}$, $b\in\overline{T(1)^*}\oplus\overline{T(2)^*}\oplus\sum_{I-\{0,1,2\}}\oplus T(i)^*\oplus\overline{N_0(1)}\oplus\overline{N_0(2)}\oplus\sum_{I-\{1,2\}}\oplus N_0(i)\oplus\sum_{I-2}\oplus N_1(i)\oplus\overline{N_1(2)}$, $n'\in\overline{\sum_{I-\{0,1\}}\oplus N_i}$. Note that $a\neq 0$ by (*). Since $X(0)\oplus X(1)\oplus X(2)\subseteq_e X(0)^{\#}\oplus X(1)^{\#}\oplus X(2)^{\#}$, we can take $r\in R$ such that $0\neq ar\in X(0)\oplus X(1)\oplus X(2)$; so $0\neq n''r$. As $X(0)\oplus X(1)\oplus X(2)\subseteq T(0)^*\oplus T(1)^*\oplus T(2)^*\oplus N_0\oplus N_1$, we see that $ar+br=0$ and $n''r-n'r=0$. But $n''r-n'r=0$ implies $n''r=0$, a contradiction. Hence $\overline{\overline{\sum_{I-\{0,1\}}\oplus N_i}}=0$. As a result, we have

$$\begin{aligned}P=&X(0)^{\#}\oplus X(1)^{\#}\oplus X(2)^{\#}\oplus\overline{T(1)^*}\oplus\overline{T(2)^*}\oplus\sum_{I-\{0,1,2\}}\oplus T(i)^*\\&\oplus\overline{N_0(1)}\oplus\overline{N_0(2)}\oplus\sum_{I-\{1,2\}}\oplus N_0(i)\\&\oplus\overline{N_1(2)}\oplus\sum_{I-2}\oplus N_1(i)\\&\oplus\sum_{I-\{0,1\}}\oplus N_i.\end{aligned}$$

We transfinitely proceed with this argument. For the sake of convenience, for any k in I, we put

$$I(k)=\{i\in I|i<k\}.$$

Now, let $\beta\in I$ and assume that we have obtained decompositions:

$$\begin{aligned}&T(i)^*=\overline{\overline{T(i)^*}}\oplus\overline{T(i)^*}\,\forall\, i\in I(\beta),\\&N_0(i)=\overline{\overline{N_0(i)}}\oplus\overline{N_0(i)}\,\forall\, i\in I(\beta)-0,\\&N_1(i)=\overline{\overline{N_1(i)}}\oplus\overline{N_1(i)}\,\forall\, i:\ 1<i<\beta,\\&\ldots\ldots\ldots\ldots\ldots,\\&N_k(i)=\overline{\overline{N_k(i)}}\oplus\overline{N_k(i)}\,\forall\, i:\ k<i<\beta,\\&\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

such that, fo any $k\in I(\beta)$,

$$\begin{aligned}P=&\sum_{0\leq i\leq k}\oplus X(i)^{\#}\sum_{0<i\leq k}\oplus\overline{T(i)^*}\oplus\sum_{k<i}\oplus T(i)^*\\&\oplus N_0(0)\oplus\sum_{0<i\leq k}\oplus\overline{N_0(i)}\oplus\sum_{k<i}\oplus N_0(i)\\&\oplus N_1(0)\oplus N_1(1)\oplus\sum_{1<i\leq k}\oplus\overline{N_1(i)}\oplus\sum_{k<i}\oplus N_1(i)\\&\oplus N_2(0)\oplus N_2(1)\oplus N_2(2)\oplus\sum_{2<i\leq k}\oplus\overline{N_2(i)}\oplus\sum_{k<i}\oplus N_2(i)\end{aligned}$$

$$\cdots\cdots\cdots\cdots$$
$$\oplus\sum_{k\leq i}\oplus N_i$$

For $k\in I(\beta)$, we put

$$Q(k)=\overline{T(k)^*}\oplus\sum_{0\leq i<k}\oplus\overline{N_i(k)}$$
$$Q=\sum_{I(\beta)-0}\oplus Q(k).$$

Since $Q(k)\simeq X(k)^{\#}\underset{\sim}{\subset} T(k)^*$ for all $k\in I(\beta)-0$ and $\sum_I\oplus T(k)^*$ satisfies the condition A, $Q=\sum_{I(\beta)-0}\oplus Q(k)$ satisfies the condition A. We note that, for any $q_k\in Q_k$ and $q_l\in Q_l$, $(0:\ q_k)\neq(0:\ q_l)$, since Q is square free.

Putting

$$\hat{N}_0=N_0(0)\oplus\sum_{0<i<\beta}\oplus\overline{N_0(i)}\oplus\sum_{\beta\leq i}\oplus N_0(i)\subseteq N_0,$$
$$\hat{N}_k=N_k(0)\oplus N_k(1)\oplus\ldots\oplus N_k(k)\oplus\sum_{k<i<\beta}\oplus\overline{N_k(i)}\oplus\sum_{\beta\leq i}\oplus N_k(i)\subseteq N_k$$

for $0\neq k\in I(\beta)$, we claim that

$$P=\sum_{i<\beta}\oplus X(i)^{\#}\oplus\sum_{0<i<\beta}\oplus\overline{T(i)^*}\oplus\sum_{\beta\leq i}\oplus T(i)^*\oplus\sum_{i<\beta}\oplus\hat{N}(i)\oplus\sum_{\beta\leq i}\oplus N_i$$

To show this we may show that Q is contained in

$$Z=\sum_{i<\beta}\oplus X(i)^{\#}\oplus\sum_{0<i<\beta}\oplus\overline{T(i)^*}\oplus\sum_{\beta\leq i}\oplus T(i)^*\oplus\sum_{i<\beta}\oplus\hat{N}(i)\oplus\sum_{\beta\leq i}\oplus N_i.$$

Assume $Q\not\subseteq Z$. Since $Q=\sum_{I-0}\oplus Q(k)$ satisfies the condition A, we can take Q_k and $q_k\in Q_k$ such that $q_k\notin Z$ and, for any $k<l$ and $q_l\in Q_l$,

$$(0:\ q_k)\subset(0:\ q_l)\Longrightarrow q_l\in Z.\ \ldots(*)$$

We express q_k in

$$\begin{aligned}P=&\sum_{0\leq i\leq k}\oplus X(i)^{\#}\oplus\sum_{0<i\leq k}\oplus\overline{T(i)^*}\oplus\sum_{k<i}\oplus T(i)^*\\&\oplus N_0(0)\oplus\sum_{0<i\leq k}\oplus\overline{N_0(i)}\oplus\sum_{k<i}\oplus N_0(i)\\&\oplus N_1(0)\oplus N_1(1)\oplus\sum_{1<i\leq k}\oplus\overline{N_1(i)}\oplus\sum_{k<i}\oplus N_1(i)\\&\oplus N_2(0)\oplus N_2(1)\oplus N_2(2)\oplus\sum_{2<i\leq k}\oplus N_2(i)\oplus\sum_{k<i}\oplus N_2(i)\\&\cdots\cdots\cdots\cdots\\&\oplus\sum_{k\leq i}\oplus N_i\end{aligned}$$

as $q_k=a+b$, where

$$a\in\sum_{0\leq i\leq k}\oplus X(i)^{\#}\oplus\sum_{0<i\leq k}\oplus\overline{T(i)^*}\oplus\sum_{k<i<\beta}\oplus\overline{T(i)^*}\oplus\sum_{\beta\leq i}\oplus T(i)^*$$

$$\oplus N_0(0)\oplus\sum_{0<i\le k}\oplus\overline{N_0(i)}\oplus\sum_{k<i<\beta}\oplus\overline{N_0(i)}\oplus\sum_{\beta\le i}\oplus N_0(i)$$
$$\oplus N_1(0)\oplus N_1(1)\oplus\sum_{1<i\le k}\oplus\overline{N_1(i)}\oplus\sum_{k<i<\beta}\oplus\overline{N_1(i)}\oplus\sum_{\beta\le i}\oplus N_1(i)$$
$$\oplus N_2(0)\oplus N_2(1)\oplus N_2(2)\oplus\sum_{2<i\le k}\oplus\overline{N_2(i)}\oplus\sum_{k<i<\beta}\oplus\overline{N_2(i)}\oplus\sum_{\beta\le i}\oplus N_2(i)$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\oplus\sum_{\beta\le i}\oplus N_i$$

and

$$b\in\sum_{k<i<\beta}\oplus\overline{\overline{T(i)^*}}\oplus\sum_{k<i<\beta}\oplus\overline{\overline{N_0(i)}}\oplus\sum_{k<i<\beta}\oplus\overline{\overline{N_1(i)}}\oplus\sum_{k<i<\beta}\oplus\overline{\overline{N_2(i)}})\oplus$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$=\sum_{k<i<\beta}\oplus Q(i).$$

Then $a\in Z$ and (*) shows $b\in Z$, so $q_k=a+b\in Z$, a contradiction. Thus we get

$$P=\sum_{i<\beta}\oplus X(i)^\#\oplus\sum_{0<i<\beta}\oplus\overline{T(i)^*}\oplus\sum_{\beta\le i}\oplus T(i)^*\oplus\sum_{i<\beta}\oplus\hat{N}_i\oplus\sum_{\beta\le i}\oplus N_i$$
$$=\sum_{0<i<\beta}\oplus X(i)^\#\oplus\sum_{0<i<\beta}\oplus\overline{T(i)^*}\sum_{\beta\le i}T(i)^*$$
$$\oplus N_0(0)\oplus\sum_{0<i<\beta}\oplus\overline{N_0(i)}\oplus\sum_{\beta\le i}\oplus N_0(i)$$
$$\oplus N_1(0)\oplus N_1(1)\oplus\sum_{1<i<\beta}\oplus\overline{N_1(i)}\oplus\sum_{\beta\le i}\oplus N_1(i)$$
$$\oplus N_2(0)\oplus N_2(1)\oplus N_2(2)\oplus\sum_{2<i<\beta}\oplus\overline{N_2(i)}\oplus\sum_{\beta\le i}\oplus N_2(i)$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\oplus\sum_{\beta\le i}\oplus N_i$$

We put

$$W=\sum_{i<\beta}\oplus X(i)^\#\oplus\sum_{0<i<\beta}\oplus\overline{T(i)^*}\sum_{\beta<i}\oplus T(i)^*$$
$$\oplus N_0(0)\oplus\sum_{0<i<\beta}\oplus\overline{N_0(i)}\oplus\sum_{\beta<i}\oplus N_0(i)$$
$$\oplus N_1(0)\oplus N_1(1)\oplus\sum_{1<i<\beta}\oplus\overline{N_1(i)}\oplus\sum_{\beta<i}\oplus N_1(i)$$
$$\oplus N_2(0)\oplus N_2(1)\oplus N_2(2)\oplus\sum_{2<i<\beta}\oplus\overline{N_2(i)}\oplus\sum_{\beta<i}\oplus N_2(i)$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\oplus\sum_{i<\beta}\oplus N_\beta(i)$$

And we consider the decomposition : $P=W\oplus T(\beta)^*\oplus\sum_{i<\beta}\oplus N_i(\beta)\oplus\sum_{\beta\le i}\oplus N_i$ Here using the $|I(\beta)|$-exchange property of $X(\beta)$ in this decomposition, we get decompositions :

$$W=\overline{\overline{W}}\oplus\overline{W}$$
$$T(\beta)^*=\overline{\overline{T(\beta)^*}}\oplus\overline{T(\beta)^*}$$
$$N_i(\beta)=\overline{\overline{N_i(\beta)}}\oplus\overline{N_i(\beta)}$$

for $i<\beta$

$$\sum_{\beta\leq i}\oplus N_i=\overline{\sum_{\beta\leq i}\oplus N_i}\oplus\overline{\overline{\sum_{\beta\leq i}\oplus N_i}}$$

such that

$$P=X(\beta)^{\#}\oplus\overline{W}\oplus\sum_{0<i\leq\beta}\oplus\overline{T(i)^{*}}\oplus\sum_{\beta<i}\oplus\overline{N_i(\beta)}\oplus\overline{\sum_{\beta\leq i}\oplus N_i}$$

But, by the same argument above, we can that

$$\overline{\overline{W}}=0,\ \overline{\overline{\sum_{\beta\leq i}\oplus N_i}}=0$$

so, we have

$$\begin{aligned}P=&\sum_{0\leq i\leq\beta}\oplus X(i)^{\#}\\&\oplus\sum_{0<i\leq\beta}\oplus\overline{T(i)^{*}}\\&\oplus\sum_{\beta<i}\oplus T(i)^{*}\\&\oplus N_0(0)\oplus\sum_{0<i\leq\beta}\oplus\overline{N_0(i)}\oplus\sum_{\beta<i}\oplus N_0(i)\\&\oplus N_1(0)\oplus N_1(1)\oplus\sum_{0<i\leq\beta}\oplus\overline{N_1(i)}\oplus\sum_{\beta<i}\oplus N_1(i)\\&\oplus N_2(0)\oplus N_2(1)\oplus N_2(2)\oplus\sum_{2<i\leq\beta}\oplus\overline{N_2(i)}\oplus\sum_{\beta<i}\oplus N_2(i)\\&\cdots\cdots\cdots\cdots\cdots\\&\oplus\sum_{\beta\leq i}\oplus N_i\end{aligned}$$

Thus, by transfinite induction, we have decompositions

$$\begin{aligned}&T(i)^{*}=\overline{\overline{T(i)^{*}}}\oplus\overline{T(i)^{*}}\ \forall\, i\in I\\&N_0(i)=\overline{\overline{N_0(i)}}\oplus\overline{N_0(i)}\ \forall\, i\in I-0,\\&N_1(i)=\overline{\overline{N_1(i)}}\oplus\overline{N_1(i)}\ \forall\, i\in I-I(2),\\&\cdots\cdots\cdots\cdots\cdots\\&N_k(i)=\overline{\overline{N_1(i)}}\oplus\overline{N_1(i)}\ \forall\,_i\in I-I(k),\\&\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

such that, for any $k\in I$,

$$\begin{aligned}P=&\sum_{0\leq i\leq k}\oplus X(i)^{\#}\\&\oplus\sum_{0<i\leq k}\oplus\overline{T(i)^{*}}\oplus\sum_{k<i}\oplus T(i)^{*}\\&\oplus N_0(0)\oplus\sum_{0<i\leq k}\oplus\overline{N_0(i)}\oplus\sum_{k<i}\oplus N_0(i)\\&\oplus N_1(0)\oplus N_1(1)\oplus\sum_{0<i\leq k}\oplus\overline{N_1(i)}\oplus\sum_{k<i}\oplus N_1(i)\\&\oplus N_2(0)\oplus N_2(1)\oplus N_2(2)\oplus\sum_{2<i\leq k}\oplus\overline{N_2(i)}\oplus\sum_{k<i}\oplus N_2(i)\end{aligned}$$

....

$\oplus\sum_{k\leq i}\oplus N_i$

So, by the quite similar argument above for $X(1)^{\#}$ or $X(2)^{\#}$, we have

$$
\begin{aligned}
P=&\sum_I\oplus X(i)^{\#}\oplus\sum_{I-0}\oplus\overline{T(i)^{*}}\\
&\oplus N_0(0)\oplus\sum_{I-0}\oplus\overline{N_0(i)}\\
&\oplus N_1(0)\oplus N_1(1)\oplus\sum_{I-I(2)}\oplus\overline{N_1(i)}\\
&\oplus N_2(0)\oplus N_2(1)\oplus N_2(2)\oplus\sum_{I-I(3)}\oplus\overline{N_2(i)}\\
&\dots\dots\dots\dots\dots\\
&\oplus N_k(0)\oplus N_k(1)\oplus N_k(2)\cdots\oplus N_k(k)\oplus\sum_{I-I(k)}\oplus\overline{N_{k(i)}}\\
&\dots\dots\dots\dots\dots
\end{aligned}
$$

This completes the proof, as $X=\sum_I\oplus X(i)^{\#}$.

ACKNOWLEDGEMENT. The authors would like to thank Prof. B. Müller for his kind comments for a preliminary version of this paper.

References

[1] P. Crawley and B. Jónsson: *Refinements for infinite direct decompositions of algebraic systems*, Pac. J. Math. **14** (1964), 797-855

[2] M. Harada and T. Ishii: *On perfect rings and the exchange property*, Osaka J. Math. **12**, (1975), 483-491.

[3] S.K. Jain, K. Oshiro and S.T. Rizvi: *Direct sum of relative quasi-continiuous modules*, preprint.

[4] S.R. Lopez-Permouth, K. Oshiro and S.T. Rizvi: *On retative continuous modules*, preprint.

[5] S.H. Mohamed and B.J. Müller: *Continuous and Discrete Modules*, London Math. Soc. Lecture Notes Series **147**, Cambridge Univ. Press 1990.

[6] S.H. Mohamed and B.J. Müller: *The exchange property for quasi-continuous modules*, Ring theory (Ohio State-Denison, 1992), 242-247, World Scientific (1993).

[7] B.J. Müller and S.T. Rizvi: *direct sums of indecomposable modules*, Osaka J. Math. **21**, (1984), 365-374.

[8] K. Okado: *On the decomposition of extending modules*, Math. Japonica **29**, (1984) 939-941.

[9] K. Oshiro: *Continuous modules and quasi-continuous modules*, Osaka J. Math.**20**, (1983), 681-694.

[10] K. Oshiro: *Semiperfect modules and quasi-semiperfect modules*, Osaka J. Math. **20**, (1983), 337-372.

[11] B. Zimmermann-Huisgen and W. Zimmermann: *Classes of modules with the exchange property*, J. Algebra **88**, (1984), 416-434.

[12] K. Yamagata: *The exchange property and direct sums of indecomposable injective modules*, Pacific J. Math. **55**, (1974), 301-317.

[13] K. Yamagata: *On rings of finite representation type and modules with the finite exchange proprty*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **13**, (1975), 1-6.

Kiyoichi Oshiro
Department of Mthematics
Yamaguchi University
Yamaguchi, 753, Japan

S. Tariq Rizvi
Department of Mathematics
Ohio State University
Lima, Ohio 45804, U.S.A.