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<th>The exchange property of quasi-continuous modules with the finite exchange property</th>
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Osaka University
Mohamed and Müller showed in [5] that continuous modules have the exchange property. And, recently, they also showed in [6] that for nonsingular quasi-continuous modules, the finite exchange property implies the exchange property. However, it is still open whether this is true or not for any quasi-continuous modules ([5, Problem 2]). The purpose of this paper is to answer this problem in the affirmative. This, then also provides another instance of modules for which the existence of the finite exchange property implies that of the exchange property in reference to the longstanding open question posed in Crawley-Jonsson [1].

Discrete (=semiperfect) modules and quasi-discrete (=quasi-semiperfect) modules are dual to continuous modules and quasi-continuous modules, respectively. We note that discrete modules have the exchange property and, for quasi-discrete modules, the finite exchange property implies the exchange property. These results follows by summarizing following results:

(1) (Oshiro [10]) Every quasi-discrete module $M$ has an indecomposable decomposition $M = \bigoplus_i M_i$ such that $M' = \bigoplus_i (M_i/M_i : \text{completely indecomposable})$ satisfies the finite exchange property. So, if $M$ is discrete then it satisfies the finite exchange property since $M = M'$.

(2) (Harada-Ishii [2], Yamagata [12], [13]) If a module has an indecomposable decomposition and satisfies the finite exchange property, then it satisfies the exchange property in direct sums of completely indecomposable modules.

(3) (Zimmermann-Huisgen and Zimmermann [11]) A module $M$ satisfies the exchange property if and only if for any $P = M \oplus X = \bigoplus_i M_i$ with each $M_i \cong M$, there exists $M_i < \bigoplus_i M_i$ for each $i \in I$ such that $P = M \oplus \bigoplus_i M_i$.

The reader is referred to Mohamed and Müller’s Book [5] for the background of these results.
1. Preliminaries. Throughout this paper $R$ will denote a ring with identity and all $R$-modules will be unital right $R$-modules. For two $R$-modules $X$ and $Y$, we use $X \subseteq_e Y$, $X \subseteq Y$, $X \subseteq_e Y$, $X \subseteq \oplus Y$ and $X \subseteq \oplus Y$ to mean that $X$ is an essential submodule of $Y$, $X$ is isomorphic to a submodule of $Y$, $X$ is isomorphic to an essential submodule of $Y$, $X$ is a direct summand and $X$ is isomorphic to a direct summand of $Y$, respectively. For a set $I$, $|I|$ stands for the cardinal of $I$.

An $R$-module $M$ is called an extending module (or a CS-module) if it satisfies

(C1): for any submodule $X$ of $M$, there exists a direct summand $X^*$ of $M$ such that $X \subseteq_e X^*$

$M$ is called continuous if it satisfies (C1) and

(C2): Every submodule of $M$ which is isomorphic to a direct summand of $M$ is a direct summand.

$M$ is called a quasi-continuous module if it satisfies (C1) and

(C3): If $X$ and $Y$ are direct summands of $M$ with $X \cap Y = 0$, then $X \oplus Y$ is a direct summand.

For an $R$-module $M$ with a decomposition $M = \sum_{i=1}^{n} M_i$, we use the following condition:

(A) For any choice of $x_i \in M_i (i \in \mathbb{N}, \alpha_i$ distinct) such that $(0 : x_1) \subseteq (0 : x_2) \subseteq \ldots$, the sequence becomes stationary, where $(0 : x)$ denotes the annihilator right ideal of $x$.

This condition appeared in [8] (cf.[5], [7]). One of interesting results on this condition is (2) of the following

**Proposition 1.1.** Let $M$ be a quasi-continuous module. Then the following hold:

1. for any decomposition $M = \sum_{i=1}^{n} M_i$ and any $J \subseteq I$, $\sum_{i \in J} M_i$ is $\sum_{i \in J} M_i$-injective
2. any decomposition $M = \sum_{i \in I} M_i$ satisfies the condition (A).
3. for any decomposition $M = \sum_{i \in I} M_i$ and any direct summand $X$ of $M$, there exists $N_i < \oplus M_i$ such that $M = X \oplus \sum_{i \in I} N_i$.

Proof. (1) follows from [9, Proposition 1.5]. (2) follows from [5, Proposition 2.13] or from (1) above and [5, Proposition 1.9]. And see [3] for (3).

A module $M$ is called a square module if $M \cong X \oplus X$ for some module $X$, and a module is called a square free if it does not contain non-zero square modules.

The following results are important and useful in the study of quasi-continuous modules...
Lemma 1.1 ([5, Theorem 2.37]). Any quasi-continuous module is a direct sum of a quasi-injective module and a square free module.

Proposition 1.2. For an $R$-module $X$, the following conditions are equivalent:

1. $X$ is quasi-continuous.
2. Any decomposition $E(X) = \sum_{\tau} M_i$ implies $X = \sum_{\tau} (M_i \cap X)$, where $E(X)$ is the injective hull of $X$.
3. For any $R$-module $Y$ with $X \subseteq Y$, any decomposition $Y = \sum_{\tau} Y_i$ implies $X = \sum_{\tau} (Y_i \cap X)$.

Proof. The equivalence of 1) and 2) is well known [5, Theorem 2.8]. We may show the implication 2) $\Rightarrow$ 3). Let $Y$ be an $R$-module with $X \subseteq Y$ and consider a decomposition $Y = \sum_{\tau} Y_i$. Let $x \in X$. Then there exists a finite subset $F = \{1, 2, ..., n\}$ of $I$ such that $x \in \sum_{\tau} Y_i$. Let $x = y_1 + ... + y_n$, where $y_i \in Y_i$. Consider $E(X) = \sum_{\tau} E(Y_i) \cup E(\sum_{\tau} Y_i)$. By 2), we have $X = \sum_{\tau} (E(\sum_{\tau} Y_i) \cap X) \cup E(\sum_{\tau} Y_i) \cap X$. Express the element $x$ as $x = p_1 + p_2 + ... + p_n + q$ where $p_i \in E(Y_i) \cap X$ and $q \in E(\sum_{\tau} Y_i) \cap X$. Since $x = y_1 + ... + y_n$ with $y_i \in E(Y_i)$, we see $p_i = y_i$, $i = 1, ..., n$. Hence $y_i \in Y_i \cap X$, $i = 1, ..., n$; so $x \in \sum_{\tau} (Y_i \cap X)$. Accordingly we see $X = \sum_{\tau} (Y_i \cap X)$.

For a cardinal $\alpha$, an $R$-module $X$ has the $\alpha$-exchange property if for any $R$-module $M$ and any two decompositions $M = X \oplus N = \sum_{\tau} \oplus M_i$ with $|I| \leq \alpha$, there exists $M_i \subset \oplus M$ for each $i \in I$ such that $M = X \oplus (\sum_{\tau} M_i)$.

$X$ has the exchange property if this holds for any cardinal $\alpha$ and has the finite exchange property if this holds whenever the index set $I$ is finite. We note that, in these definitions, we may assume that $M_i \simeq X$ for all $i \in I$ (by Zimmermann-Huisgen and Zimmermann [11]).

2. A key lemma. In this section we show the following which is a key lemma of this paper. We note that this lemma is also used for the study of direct sums of relative continuous modules [3], [4].

Lemma 2.1. Let $P$ be an $R$-module with a decomposition $P = \sum_{\tau} \oplus M_i$ such that each $M_i$ is extending. We consider the index set $I$ as an well ordered set:
I = \{0, 1, \ldots, w, w+1, \ldots\}, and let X be a submodule of M. Then there are submodules \( T(i) \subseteq e T(i)^* \oplus \bigoplus M_i \), decompositions \( M_i = T(i)^* \oplus N_i \) and a submodule \( \sum_{i \in k} X(i) \subseteq e X \) for which the following properties hold:

1) \( X(0) = T(0) \subseteq e T(0)^* \).
2) \( X(k) \subseteq T(k) \oplus \bigoplus_{i \in k} N_i \) for all \( k \in I \).
3) \( \sigma(X(k)) = T(k) \subseteq e T(k)^* \), \( \sigma(X(k)) = \sigma(X(k)) \) (by \( \sigma|X(k) \)).
4) \( X \cong \sigma(X) \) (by \( \sigma|X \)).

For a proof of this result, we need two lemmas

**Lemma 2.2.** Let \( M \) be an \( R \)-module with a decomposition \( M = M_1 \oplus M_2 \) and let \( X \) be a submodule of \( M \). If there is a decomposition \( M_i = M_i^* \oplus M_i^{**} \) such that \( M_i \cap X = \varepsilon M_i^* \) for \( i = 1, 2, \ldots \), then \( X \cong (M_1^* 
 \cap X) \oplus (M_2^* \cap X) \cap X \). So, in particular if \( M_i \cap X = 0 \), then \( X \cong (M_1^* \cap X) \oplus (M_2^* \cap X) \cap X \).

**Proof.** Let \((0 \neq)x \in X\) and express \( x \) in \( M = M_1^* \oplus M_2^* \oplus M_1^{**} \oplus M_2^{**} \) as \( x = x_1^* + x_2^* + x_1^{**} + x_2^{**} \) where \( x_i^* \in M_i^* \) and \( x_i^{**} \in M_i^{**} \). If \( x_1^* + x_2^* \in (M_1^* \cap X) \oplus (M_2^* \cap X) \), then \( x_1^{**} + x_2^{**} \in (M_1^{**} \oplus M_2^{**}) \cap X \) and hence \( x \in (M_1^* \cap X) \oplus (M_2^* \cap X) \oplus (M_1^{**} \oplus M_2^{**}) \cap X \). In the case of \( x_1^* + x_2^* \notin (M_1^* \cap X) \oplus (M_2^* \cap X) \), we take \( r \in R \) such that \( 0 \neq (x_1^* + x_2^*) r \in (M_1^* \cap X) \oplus (M_2^* \cap X) \). Then \( 0 \neq x r \in (M_1^* \cap X) \oplus (M_2^* \cap X) \oplus (M_1^{**} \oplus M_2^{**}) \cap X \).

**Lemma 2.3.** Let \( M \) be an \( R \)-module with a decomposition \( M = A \oplus B \oplus C \oplus D \) and let \( X \) be a submodule of \( M \). If \( Y \) is a submodule of \( X \) such that \( Y \subseteq (A \oplus B) \cap X \) and \( Y \cong (B \oplus C) \cap X \), where \( \sigma \) is the projection: \( M = A \oplus B \oplus C \oplus D \rightarrow A \). Then \( Y \cong (B \oplus C) \cap X \).}

**Proof.** Let \((0 \neq)x = a + b + c \in (A \oplus B \oplus C) \cap X\), where \( a \in A \), \( b \in B \) and \( c \in C \). If \( a = 0 \), \( x = b + c \in (B \oplus C) \cap X \). If \( a \neq 0 \), then \( 0 \neq \sigma(xr) \in (Y) \) for some \( r \in R \); so there exists \( y \in Y \) such that \( \sigma(xr) = \sigma(y) \). Since \( xr - y \in \text{Ker} \ \sigma \cap (B \oplus C) \), we see \( xr \in Y \cap ((B \oplus C) \cap X) \). Hence we see \( Y \cong (B \oplus C) \cap X \).}

**Proof of Lemma 2.1.** We put \( X_i = M_i \cap Y \) for all \( i \in I \). Since \( M_i \) is extending, we have a decomposition
\[ M_i = X_i^* \oplus X_i^{**} \]
such that \( X_i \subseteq e X_i^* \) for all \( i \in I \). By Lemma 2.2,
\[ (M_0 \oplus M_1) \cap X_e \ni X_0 \oplus X_1 \oplus (X_0^{**} \oplus X_1^{**}) \cap X. \]

We put
\[ X(0) = X_0, \ X(1) = X_1 \oplus (X_0^{**} \oplus X_1^{**}) \cap X. \]

Let \( \pi_0, \pi_1 \) be the projections:
\[ X_0^{**} \oplus X_1^{**} \to X_0^{**}, \ X_0^{**} \oplus X_1^{**} \to X_1^{**} \]
respectively. Since \( X_i^{**} \cap X = 0 \), we see that
\[ (X_0^{**} \oplus X_1^{**}) \cap X \simeq \pi_0((X_0^{**} \oplus X_1^{**}) \cap X) \simeq \pi_1((X_0^{**} \oplus X_1^{**}) \cap X) \ldots \]
\[ \text{(**) above) for } i = 1, 2, \text{ where } \pi_i \text{ is the projection :} \]
\[ P = T(0)^* \oplus T(1)^* \oplus N_0 \oplus N_1 \oplus \sum_{i \in I} M_i \]
such that
\[ X(0) = T(0), \ X(1) \subseteq T(1) \oplus N_0, \]
\[ \sigma_i(X(i)) = T = (i), \ X(i) \simeq T(i) \text{ by } (\sigma_i|X(i)) \text{ (cf. (**)) above) for } i = 1, 2. \]

Next consider \( (M_0 \oplus M_1 \oplus M_2) \cap X \). Let \( A = T(0)^* \oplus T(1)^*, \ B = N_0 \oplus N_1, \ C = M_2 \) and \( D = \sum_{i \in I} M_i \) and \( Y = X(0) \oplus X(1) \). Then
\[ X \subseteq P = A \oplus B \oplus C \oplus D, \]
\[ A \oplus B \oplus C = M_0 \oplus M_1 \oplus M_2 \text{ and } Y \simeq (Y) \subseteq e A. \] So we see from Lemma 2.3 that
\[ (M_0 \oplus M_1 \oplus M_2) \cap X_e \ni X(0) \oplus X(1) \oplus (N_0 \oplus N_1 \oplus N_2) \cap X. \]

Further, since \( (N_0 \oplus N_1) \cap X = 0 \), we see from Lemma 2.2 that
\[ (N_0 \oplus N_1 \oplus M_2) \cap X_e \ni X_2 \oplus (N_0 \oplus N_1 \oplus X_2^{**}) \cap X. \]

Let \( \pi_0, \pi_2 \) be the projections:
\[ N_0 \oplus N_1 \oplus X_2^{**} \to N_0 \oplus N_1, \ N_0 \oplus N_1 \oplus X_2^{**} \to X_2^{**} \]
respectively. Since \((N_0 \oplus N_1) \cap X = 0\) and \(X^* \cap X = 0\), we see
\[ (N_0 \oplus N_1 \oplus X^*) \cap X \approx \pi_0((N_0 \oplus N_1 \oplus X^*) \cap X) \approx \pi_0((N_0 \oplus N_1 \oplus X^*) \cap X) \]
canonically...(**)

Put
\[ X(2) = X_2 \oplus (N_0 \oplus N_1 \oplus X^*) \cap X, \]
\[ T(2) = X_2 \oplus \pi_2((N_0 \oplus N_1 \oplus X^*) \cap X). \]

Then
\[ (M_0 \oplus M_1 \oplus M_2) \cap X \equiv X(0) \oplus X(1) \oplus X(2). \]

Since \(M_2\) is extending, we have a decomposition
\[ M_2 = T(2)^* \oplus N_2 \]
with \(T(2) \subseteq e T(2)^*\). Here we see
\[ P = T(0)^* \oplus T(1)^* \oplus T(2)^* \oplus N_0 \oplus N_1 \oplus N_2 \oplus \sum_{3 \leq i} M_i, \]
\[ X(2) \subseteq T(2)^* \oplus N_0 \oplus N_1, \]
and for the projection :
\[ \sigma_2 : P = T(0)^* \oplus T(1)^* \oplus T(2)^* \oplus N_0 \oplus N_1 \oplus N_2 \oplus \sum_{3 \leq i} M_i \rightarrow \]
\[ T(0)^* \oplus T(1)^* \oplus T(2)^*. \]

We see that
\[ \sigma_2(X(i)) = T(i), \quad X(i) \approx T(i) \text{(by } \sigma_2|X(i)) \]
for \(i = 0, 1, 2\) (cf (**)).

Now we proceed our argument by transfinite induction on \(a \in I\). Let \(a \in I\) and put \(J = \{i \in I | i < a\}\)

Assume that there are submodules \(T(i) \subseteq e T(i)^* \oplus M_i\), decompositions \(M_i = T(i)^* \oplus N_i\), for which the following hold :

1) \(X(0) = T(0)\),
2) \(X(k) \subseteq T(k) \oplus \sum_{i \in k} N_i \forall k \in J\),
3) \((\sum_{i \in k} M_i) \cap X_e \supseteq \sum_{i \in k} X(i) \forall k \in J\); so \((\sum_{i \in k} M_i) \cap X_e \supseteq \sum_{i \in k} X(i)\),
4) \(X(k) \approx T(k)\) by \((\sigma_j|X(k)) \forall k \in J\)

where \(\sigma_j\) is the projection :
\[ P = \sum_{j} T(i)^* \oplus \sum_{j} N(i) \oplus \sum_{j} M(i) \rightarrow \sum_{j} T(i)^* \]

So
Consider $(\sum \oplus M_i \oplus M_a) \cap X$. We note that
\[ \sum \oplus X(i) \subseteq (\sum \oplus M_i) \cap X = (\sum \oplus T(i)^* \oplus \sum \oplus N_i) \cap X, \]
\[ \sum \oplus X(i) \approx \sum \oplus T(i) \text{(by } \sigma_i | \sum \oplus X(i)\text{)} \]

Considering
\[ \sum \oplus T(i)^* \oplus \sum \oplus N(i) \oplus M_a \oplus \sum \oplus M_i \to \sum \oplus T(i)^* \]
we infer from Lemma 2.3 that
\[ (\sum \oplus N_i \oplus M_a) \cap X_e \supseteq \sum \oplus X(i) \oplus (\sum \oplus N_i \oplus M_a) \cap X. \]

Since $\sum \oplus N_i \cap X = 0$, we see by Lemma 2.2 that
\[ (\sum \oplus N_i \oplus M_a) \cap X_e \supseteq X_e \oplus (\sum \oplus N_i \oplus X_e^**) \cap X \]

(where $X_a = M_a \cap X \subseteq X^*_a$, $M_a = X^*_a \oplus X_a^{**}$)

Let $\pi_i$ and $\pi_a$ be the projections:
\[ \sum \oplus N_i \oplus X_a^{**} \to \sum \oplus N_i, \sum \oplus N_i \oplus X_a^{**} \to X_a^{**} \]
respectively. We see that
\[ (\sum \oplus N_i \oplus X_a^{**}) \cap X \approx \pi_i((\sum \oplus N_i \oplus X_a^{**}) \cap X) \approx \pi_a((\sum \oplus N_i \oplus X_a^{**} \cap X) \]
canonicaly. We put
\[ X(a) = X_a \oplus (\sum \oplus N_i \oplus X_a^{**}) \cap X, \]
\[ T(a) = X_a \oplus \pi_a((\sum \oplus N_i \oplus X_a^{**} \cap X). \]

Since $M_a$ is extending, we have a decomposition
\[ M_a = T(a)^* \oplus N_a \]
with $T(a) \subseteq e T(a)^*$. Now we see
\[ X(a) \subseteq T(a) \oplus \sum \oplus N_i, \]
\[ \sigma(X(a)) = T(a), \ X(a) \approx T(a) \text{(by } \sigma | X(a)) \]
where $\sigma$ is the projection:
\[ P = \sum \oplus T(i)^* \oplus \sum \oplus N(i) \oplus \sum \oplus M(i) \to \sum \oplus T(i)^*. \]
Furthermore we see

\[(\sum_{J \cup \alpha} \oplus M_i) \cap X_e \supseteq \sum_{J \cup \alpha} \oplus X(i)\]

Thus 1)-5) above hold for \(J \cup \alpha\), and this completes the proof by transfinite induction.

3. The exchange property. Using Lemma 2.1 we shall give a proof of the exchange property of continuous modules from our point of view.

Proposition 3.1. Let \(P\) be an \(R\)-module and \(X\) a submodule of \(P\). If \(X\) is continuous and \(P\) has a decomposition \(P = \sum \oplus M_i\) with each \(M_i \approx X\), then there exists direct summand \(N_i < \oplus M_i\) for each \(i \in I\) such that \(P = X \oplus \sum \oplus N_i\). So, \(X\) is a direct summand of \(P\).

Proof. By Lemma 2.1, we have

\[P = \sum \oplus T(i) \ast \oplus \sum \oplus N_i, \quad X_e \supseteq \sum \oplus X(i)\]

such that, for each \(i \in I\),

1) \(T(i) \subseteq \sigma T(i) \ast\),
2) \(M_i = T(i) \ast \oplus N_i\),
3) \(\sigma(X(i)) = T(i), \quad X(i) \approx T(i) (\text{by } \sigma[X(i)])\)

where \(\sigma\) is the projection:

\[P = \sum \oplus T(i) \ast \oplus \sum \oplus N_i \rightarrow \sum \oplus T(i) \ast.\]

Since \(X\) is quasi-continuous and \(X \approx \sigma(X) \subseteq \sigma \sum \oplus T(i) \ast\), we obtain, by Proposition 1.2,

\[\sigma(X) = \sum \oplus (T(i) \ast \cap \sigma(X)).\]

Putting \(X(i) \ast = \sigma^{-1}(T(i) \ast \cap \sigma(X))\), we see

\[X = \sum \oplus X(i) \ast,\]

\[X(i) \subseteq \sigma X(i) \ast \forall i \in I,\]

\[T(i) \subseteq \sigma(X(i) \ast) \subseteq \sigma T(i) \ast \forall i \in I.\]

Since \(X \approx M_i\) and \(X(i) \ast < \oplus X\), we see from the condition \((C_2)\) for \(X\) that \(\sigma(X(i) \ast) < \oplus T(i) \ast\); whence \(\sigma(X(i) \ast) = T(i) \ast\) for all \(i \in I\).

As a result
Hence it follows $P = X \oplus \sum T(i)$. (by $\sigma X$).

As an immediate consequence we have

**Theorem 3.1** ([5, Theorem 3.24]). Continuous module have the exchange property.

**Remark.** We note that, in the proof above, the exchange property of quasi-injective modules is not used. (Compare our proof to the proof of [5, Theorem 3.24])

Now, we are in a position to show our main result

**Theorem 3.2.** Any quasi-continuous module with the finite exchange property has the exchange property.

**Proof.** Let $X$ be a quasi-continuous module with the finite exchange property. We may assume $X$ to be a square free by Lemma 1.1. In order to show our result by transfinite induction, let $\alpha$ be an infinite cardinal and assume that $X$ satisfies $\beta$-exchange property for any cardinal $\beta < \alpha$. To show that $X$ satisfies the $\alpha$-exchange property, consider the situation of $R$-modules:

$$P = \sum T(i) \oplus \sum N_i,$$

$$X = \sum X(i) \oplus \sum X_i$$

such that, for all $k \in I$,

$$M_k = T(k) \ast \oplus N_k, \quad T(k) \subseteq T(k) \ast, \quad X(k) \subseteq X(k) \ast,$$

$$X(k) \subseteq T(k) \oplus \sum N_i,$$

$$X(k) \ast \subseteq T(k) \ast \oplus \sum N_k,$$

$$\sigma(X(k)) = T(k) \subseteq e \sigma(X(k)) \ast \subseteq e T(k) \ast,$$

$$X(k) \approx T(k) \ast (by \ \sigma X(k)),$$

$$X(k) \ast \approx \sigma(X(k) \ast) \ast (by \ \sigma X(k) \ast)$$
where \( \sigma \) is the projection:

\[
P = \sum T(i) \oplus \sum N_i \to \sum T(i)^*.
\]

Since \( N_k < \oplus M_k \simeq X = \sum X^* \), by Proposition 1.1, we have a decomposition

\[
N_k = \sum \oplus N_k(i) \text{ for each } k \in I \text{ with } N_k(i) < \oplus X(i)^*.
\]

We note that \( \sum T(i)^* \) is square free, since \( X \subseteq \sum T(i)^* \). Now, using the finite exchange property of \( X(0)^* \) for

\[
X(0)^* < \oplus T(0)^* \oplus \sum T(i)^* \oplus \sum N_i
\]

we have decompositions

\[
T(0)^* = T(0)^* \oplus T(0)^*^*,
\]

\[
\sum T(i)^* = \sum N_i \oplus \sum N_i
\]

such that

\[
P = X(0)^* \oplus T(0)^* \oplus \sum T(i)^* \oplus \sum N_i.
\]

We denote, by \( \pi_0 \), the projection:

\[
P = X(0)^* \oplus T(0)^* \oplus \sum T(i)^* \oplus \sum N_i \to X(0)^*.
\]

Then

\[
\overline{T(0)^*} \oplus \sum N_i \simeq X(0)^* \text{ (by } \pi_0 [T(0)^* \oplus \sum N_i]).
\]

Assume \( 0 \neq \sum N_i \) and take \( 0 \neq n'' \in \sum N_i \). We express \( n'' \) in

\[
P = X(0)^* \oplus T(0)^* \oplus \sum T(i)^* \oplus \sum N_i \text{ as } n'' = a + b + n', \text{ where } a \in X(0)^*, b \in \overline{T(0)^*} \oplus \sum N_i, n' \in \sum N_i.
\]

Since \( 0 \neq \pi_0(n'') = a \in X(0)^* \), \( X(0) \) exists \( r \in R \) such that \( 0 \neq ar \in X(0) \). Since \( X(0) = T(0) \) and \( n''r = ar + br + n'r \), we see from \( n''r - n'r = ar + br \in \sum T(i)^* \cap \sum N_i = 0 \) that \( n''r - n'r = 0 \); whence \( n''r = 0 \), so \( ar = 0 \), a contradiction. Accordingly, \( \sum N_i = 0 \) and hence

\[
P = X(0)^* \oplus T(0)^* \oplus \sum T(i)^* \oplus \sum N_i,
\]

so
Since \( X(0)^* \oplus T(0)^* \) is square free and \( X(0)^* \subseteq \varepsilon T(0)^* \), we also see that \( T(0)^* = 0 \). Therefore

\[
P = X(0)^* \oplus \bigoplus_{i=0}^{\infty} T(i)^* \oplus \bigoplus_{i=0}^{\infty} \mathcal{N}_i.
\]

Next using the finite exchange property of \( X(1)^* \) in

\[
P = X(0)^* \oplus T(1)^* \oplus \bigoplus_{i=0}^{\infty} T(i)^* \oplus \mathcal{N}_0(1) \oplus \bigoplus_{i=0}^{\infty} \mathcal{N}_i
\]

where \( W = X(0)^* \oplus \bigoplus_{i=0}^{\infty} T(i)^* \oplus \bigoplus_{i=0}^{\infty} \mathcal{N}_i \), we have decompositions

\[
W = \overline{W} \oplus \overline{W},
\]

\[
T(1)^* = \overline{T(1)^*} \oplus \overline{T(1)^*},
\]

\[
\mathcal{N}_0(1) = \overline{\mathcal{N}_0(1)} \oplus \overline{\mathcal{N}_0(1)},
\]

\[
\bigoplus_{i=0}^{\infty} \mathcal{N}_i = \bigoplus_{i=0}^{\infty} \mathcal{N}_i \oplus \bigoplus_{i=0}^{\infty} \mathcal{N}_i
\]

such that

\[
P = \overline{W} \oplus T(1)^* \oplus \mathcal{N}_0(1) \oplus \bigoplus_{i=0}^{\infty} \mathcal{N}_i.
\]

Since \( \bigoplus T(i)^* \) is square free, we see from \( \overline{W} \subseteq X(1)^* \subseteq T(1)^* \) that \( \overline{W} = 0 \). So

\[
P = X(0)^* \oplus X(1)^* \oplus \overline{T(1)^*} \oplus \bigoplus_{i=0}^{\infty} T(i)^* \oplus \mathcal{N}_0(1) \oplus \bigoplus_{i=0}^{\infty} \mathcal{N}_i.
\]

In order to show \( \bigoplus_{i=0}^{\infty} \mathcal{N}_i = 0 \), consider the projection \( \pi_1 : P = X(0)^* \oplus C(1)^* \oplus \overline{T(1)^*} \oplus \bigoplus_{i=0}^{\infty} T(i)^* \oplus \mathcal{N}_0(1) \oplus \bigoplus_{i=0}^{\infty} \mathcal{N}_i \rightarrow X(1)^* \).

Assuming \( \bigoplus_{i=0}^{\infty} \mathcal{N}_i \neq 0 \) we take \( 0 \neq n'' \in \bigoplus_{i=0}^{\infty} \mathcal{N}_i \). We express \( n'' \) in \( P = X(0)^* \oplus X(1)^* \oplus \overline{T(1)^*} \oplus \bigoplus_{i=0}^{\infty} T(i)^* \oplus \mathcal{N}_0(1) \oplus \bigoplus_{i=0}^{\infty} \mathcal{N}_i \) as \( n'' = a + b + n' \), where \( a \in X(0)^* \oplus X(1)^* \), \( b \in \overline{T(1)^*} \oplus \bigoplus_{i=0}^{\infty} T(i)^* \oplus \mathcal{N}_0(1) \) \( \oplus \bigoplus_{i=0}^{\infty} \mathcal{N}_i \), \( n' \in \bigoplus_{i=0}^{\infty} \mathcal{N}_i \). Since \( X(0)^* \oplus X(1) \subseteq \varepsilon X(0)^* \oplus X(1)^* \), we can take \( r \in R \) such that \( 0 \neq ar \in X(0)^* \oplus X(1) \). Note that \( 0 \neq n'' r = ar + br + n'r \). Since \( X(0)^* \oplus X(1) \subseteq \varepsilon T(0)^* \oplus T(1)^* \), this implies \( ar + br = 0 \) and \( n'' r = n'r \); whence \( n'' r = 0 \), a contradiction. Thus we get

\[
P = X(0)^* \oplus X(1)^* \oplus \overline{T(1)^*} \oplus \bigoplus_{i=0}^{\infty} T(i)^* \oplus \mathcal{N}_0(1) \oplus \bigoplus_{i=0}^{\infty} \mathcal{N}_i.
\]

We proceed with the same argument for \( X(2)^* \). We consider
\[ P = X(0)^* \oplus X(1)^* \oplus T(1)^* \oplus T(2)^* \oplus \sum_{i \in \{0,1,2\}} T(i)^* \\
\oplus N_0(1) \oplus N_0(2) \oplus \sum_{i \in \{0,1\}} N_0(i) \\
\oplus N_1(2) \oplus \sum_{i \in \{0,1\}} N_1(i) \\
\oplus \sum_{i \in \{0,1\}} N_i. \\
= W \oplus T(2)^* \oplus N_0(2) \oplus \sum_{i \in \{0,1\}} N_i. \\
\]

where
\[ W = X(0)^* \oplus X(1)^* \oplus T(1)^* \oplus \sum_{i \in \{0,1,2\}} T(i)^* \oplus N_0(1) \oplus \sum_{i \in \{0,1,2\}} N_0(i) \oplus \sum_{i \in \{0,1\}} N_1(i). \]

And using the finite exchange property of \( X(2)^* \) in this decomposition, we have decompositions
\[ W = W \oplus \overline{W}, \]
\[ T(2)^* = T(2)^* \oplus T(2)^*, \]
\[ N_0(2) = N_0(2) \oplus \overline{N_0(2)}, \]
\[ N_1(2) = N_1(2) \oplus \overline{N_1(2)}, \]
\[ \sum_{i \in \{0,1\}} N_i = \sum_{i \in \{0,1\}} N_i \oplus \sum_{i \in \{0,1\}} \overline{N_i}. \]

such that
\[ P = X(2)^* \oplus \overline{W} \oplus T(2)^* \oplus N_0(2) \oplus \overline{N_0(2)} \oplus \sum_{i \in \{0,1\}} N_i. \]

But, as \( \overline{W} \subseteq X(2)^* \subseteq T(2)^* \) and as \( \sum_{i} T(i)^* \) is square free, we obtain \( \overline{W} = 0 \). So
\[ P = W \oplus T(2)^* \oplus N_0(2) \oplus \sum_{i \in \{0,1,2\}} N_0(i) \oplus \sum_{i \in \{0,1,2\}} T(i)^* \\
\oplus N_1(2) \oplus \sum_{i \in \{0,1,2\}} N_1(i) \\
\oplus \sum_{i \in \{0,1,2\}} N_i. \\
\]

We denote by \( \pi_2 \) the projection:
\[ P = X(0)^* \oplus X(1)^* \oplus X(2)^* \oplus T(1)^* \oplus T(2)^* \oplus \sum_{i \in \{0,1,2\}} T(i)^* \\
\oplus N_0(1) \oplus N_0(2) \oplus \sum_{i \in \{0,1,2\}} N_0(i) \\
\oplus N_1(2) \oplus \sum_{i \in \{0,1,2\}} N_1(i) \\
\oplus \sum_{i \in \{0,1,2\}} N_i \rightarrow X(2)^*. \]

Then note that
By π2, \( T(2)^* \oplus N_0(2) \oplus N_1(2) \oplus \sum_{i \in \{0,1\}} N_i \simeq X(2)^* \ldots (*) \)

We shall show \( \sum_{i \in \{0,1\}} N_i = 0 \). Assume not, and take \( 0 \neq n'' \in \sum_{i \in \{0,1\}} N_i \). We express \( n'' \) as \( n'' = a + b + n' \)

where \( a \in X(0)^* \oplus X(1)^* \oplus X(2)^* \), \( b \in T(1)^* \oplus T(2)^* \oplus \sum_{i \in \{0,1,2\}} T(i)^* \oplus N_0(1) \oplus N_0(2) \oplus \sum_{i \in \{0,1,2\}} N_0(i) \oplus \sum_{i \in \{0,1\}} N_i(1) \oplus N_1(2) \oplus \sum_{i \in \{0,1\}} N_i(i) \oplus \sum_{i \in \{0,1\}} N_i \).

Note that \( a \neq 0 \) by (*). Since \( X(0)^* \oplus X(1)^* \oplus X(2)^* \subseteq X(0)^* \oplus X(1)^* \oplus X(2)^* \), we can take \( r \in R \) such that \( 0 \neq ar \in X(0)^* \oplus X(1)^* \oplus X(2)^* \); so \( 0 \neq n''r \). As \( X(0)^* \oplus X(1)^* \oplus X(2)^* \subseteq T(0)^* \oplus T(1)^* \oplus T(2)^* \oplus N_0 + N_1 \), we see that \( ar + br = 0 \) and \( n''r - n'r = 0 \). But \( n''r - n'r = 0 \) implies \( n''r = 0 \), a contradiction. Hence \( \sum_{i \in \{0,1\}} N_i = 0 \). As a result, we have

\[
P = X(0)^* \oplus X(1)^* \oplus X(2)^* \oplus T(1)^* \oplus T(2)^* \oplus \sum_{i \in \{0,1,2\}} T(i)^* \oplus N_0(1) \oplus N_0(2) \oplus \sum_{i \in \{0,1,2\}} N_0(i) \oplus N_1(2) \oplus \sum_{i \in \{0,1\}} N_1(i) \oplus \sum_{i \in \{0,1\}} N_i \]

We transfinitely proceed with this argument. For the sake of convenience, for any \( k \) in \( I \), we put

\[
I(k) = \{ i \in I \mid i < k \}.
\]

Now, let \( \beta \in I \) and assume that we have obtained decompositions:

\[
T(i)^* = T(i)^* \oplus T(i)^* \forall i \in I(\beta),
\]

\[
N_0(i) = N_0(i) \oplus N_0(i) \forall i \in I(\beta) - 0,
\]

\[
N_1(i) = N_1(i) \oplus N_1(i) \forall i : 1 < i < \beta,
\]

\[
N_k(i) = N_k(i) \oplus N_k(i) \forall i : k < i < \beta,
\]

such that, for any \( k \in I(\beta) \),

\[
P = \sum_{0 \leq i \leq k} \oplus X(i)^* \oplus \sum_{0 < i \leq k} \oplus T(i)^* \oplus \sum_{k < i} \oplus T(i)^* \oplus \sum_{0 < i \leq k} \oplus N_0(i) \oplus \sum_{k < i} \oplus N_0(i) \\
\oplus \sum_{1 < i \leq k} \oplus N_1(i) \oplus \sum_{k < i} \oplus N_1(i) \\
\oplus \sum_{2 < i \leq k} \oplus N_2(i) \oplus \sum_{k < i} \oplus N_2(i)
\]
For \( k \in I(\beta) \), we put

\[
Q(k) = \frac{T(k)^*}{\Theta M} \oplus \sum_{0 \leq i < k} \Theta N_i(k)
\]

\[
Q = \sum_{l(\beta) = 0} \Theta^* Q(k).
\]

Since \( Q(k) \subset X(k)^* \subset T(k)^* \) for all \( k \in I(\beta) - 0 \) and \( \sum_t \Theta T(k)^* \) satisfies the condition \( A \), \( Q = \sum_t \Theta^* Q(k) \) satisfies the condition \( A \). We note that, for any \( q_k \in Q_k \) and \( q_i \in Q_i \), \( (0 : q_k) \neq (0 : q_i) \), since \( Q \) is square free.

Putting

\[
\tilde{N}_0 = N_0(0) \oplus \sum_{0 < i < \beta} \Theta N_0(i) \oplus \sum_{\beta < i} \Theta N_0(i) \subseteq N_0,
\]

\[
\tilde{N}_k = N_0(0) \oplus N_k(1) \oplus \ldots \oplus N_k(k) \oplus \sum_{k < i < \beta} \Theta N_k(i) \oplus \sum_{\beta < i} \Theta N_k(i) \subseteq N_k
\]

for \( 0 \neq k \in I(\beta) \), we claim that

\[
P = \sum_{i < \beta} \Theta X(i)^* \oplus \sum_{0 < i < \beta} \Theta T(i)^* \oplus \sum_{\beta < i} \Theta T(i)^* \oplus \sum_{\beta < i} \Theta \tilde{N}(i) \oplus \sum_{\beta < i} \Theta N_i
\]

To show this we may show that \( Q \) is contained in

\[
Z = \sum_{i < \beta} \Theta X(i)^* \oplus \sum_{0 < i < \beta} \Theta T(i)^* \oplus \sum_{\beta < i} \Theta T(i)^* \oplus \sum_{\beta < i} \Theta \tilde{N}(i) \oplus \sum_{\beta < i} \Theta N_i
\]

Assume \( Q \notin Z \). Since \( Q = \sum_t \Theta Q(k) \) satisfies the condition \( A \), we can take \( q_k \in Q_k \) and \( q_i \in Q_i \) such that \( q_k \notin Z \) and, for any \( k < l \) and \( q_i \in Q_i \),

\[
(0 : q_k) \subset (0 : q_i) \implies q_i \in Z \ldots (*)
\]

We express \( q_k \) in

\[
P = \sum_{0 \leq i \leq k} \Theta X(i)^* \oplus \sum_{0 < i < k} \Theta T(i)^* \oplus \sum_{k < i} \Theta T(i)^*
\]

\[
\oplus N_0(0) \oplus \sum_{0 < i \leq k} \Theta N_0(i) \oplus \sum_{k < i} \Theta N_0(i)
\]

\[
\oplus N_0(0) \oplus N_0(1) \oplus \sum_{0 < i < k} \Theta N_0(i) \oplus \sum_{k < i} \Theta N_0(i)
\]

\[
\oplus N_0(0) \oplus N_0(1) \oplus N_2(1) \oplus \sum_{k < i} \Theta N_2(i) \oplus \sum_{k < i} \Theta N_2(i)
\]

\[
\sum_{\beta < i} \Theta N_k(i) \oplus \sum_{\beta < i} \Theta N_k(i)
\]

\[
\oplus \sum_{k < i} \Theta N(i)
\]

as \( q_k = a + b \), where

\[
a = \sum_{0 \leq i < k} \Theta X(i)^* \oplus \sum_{0 < i < k} \Theta T(i)^* \oplus \sum_{k < i} \Theta T(i)^* \oplus \sum_{\beta < i} \Theta T(i)^*
\]

\[
\oplus \sum_{k < i} \Theta N(i)
\]

as \( q_k = a + b \), where

\[
a = \sum_{0 \leq i < k} \Theta X(i)^* \oplus \sum_{0 < i < k} \Theta T(i)^* \oplus \sum_{k < i} \Theta T(i)^* \oplus \sum_{\beta < i} \Theta T(i)^*
\]

\[
\oplus \sum_{k < i} \Theta N(i)
\]
\[\bigoplus N_0(0) \oplus \sum_{0 < i < k} N_0(i) \oplus \sum_{k < i < \beta} N_0(i) \oplus \sum_{\beta < i} N_0(i)\]
\[\bigoplus N_1(0) \oplus N_1(1) \oplus \sum_{1 < i < k} N_1(i) \oplus \sum_{k < i < \beta} N_1(i) \oplus \sum_{\beta < i} N_1(i)\]
\[\bigoplus N_2(0) \oplus N_2(1) \oplus N_2(2) \oplus \sum_{2 < i < k} N_2(i) \oplus \sum_{k < i < \beta} N_2(i) \oplus \sum_{\beta < i} N_2(i)\]

\[\sum_{\beta < i} N_i\]

and

\[b \in \sum_{k < i < \beta} \bigoplus T(i)\oplus \sum_{k < i < \beta} \bigoplus N_3(i) \oplus \sum_{k < i < \beta} \bigoplus N_3(i)\]

\[= \sum_{k < i < \beta} Q(i)\]

Then \(a \in Z\) and (*) shows \(b \in Z\), so \(Q_k = a + b \in Z\), a contradiction. Thus we get

\[P = \sum_{i < k} \bigoplus X(i) \oplus \sum_{0 < i < \beta} \bigoplus T(i) \oplus \sum_{\beta < i} T(i) \oplus \sum_{\beta < i} T(i)\]

\[\bigoplus N_0(0) \oplus \sum_{0 < i < \beta} N_0(i) \oplus \sum_{\beta < i} N_0(i)\]
\[\bigoplus N_1(0) \oplus N_1(1) \oplus \sum_{1 < i < k} N_1(i) \oplus \sum_{\beta < i} N_1(i)\]
\[\bigoplus N_2(0) \oplus N_2(1) \oplus N_2(2) \oplus \sum_{2 < i < k} N_2(i) \oplus \sum_{\beta < i} N_2(i)\]

\[\sum_{\beta < i} N_i\]

We put

\[W = \sum_{i < k} \bigoplus X(i) \oplus \sum_{0 < i < \beta} \bigoplus T(i) \oplus \sum_{\beta < i} T(i)\]

\[\bigoplus N_0(0) \oplus \sum_{0 < i < \beta} N_0(i) \oplus \sum_{\beta < i} N_0(i)\]
\[\bigoplus N_1(0) \oplus N_1(1) \oplus \sum_{1 < i < k} N_1(i) \oplus \sum_{\beta < i} N_1(i)\]
\[\bigoplus N_2(0) \oplus N_2(1) \oplus N_2(2) \oplus \sum_{2 < i < k} N_2(i) \oplus \sum_{\beta < i} N_2(i)\]

\[\sum_{\beta < i} N_i\]

And we consider the decomposition : \(P = W \oplus T(\beta) \oplus \sum_{\beta < i} N_i\). Here using the \(|I(\beta)|\)-exchange property of \(X(\beta)\) in this decomposition, we get decompositions :

\[W = \overline{W} \oplus \overline{W}\]
\[T(\beta) = T(\beta) \oplus \overline{T(\beta)}\]
\[N_i(\beta) = N_i(\beta) \oplus \overline{N_i(\beta)}\]
for \( i < \beta \)
\[
\sum_{\beta \leq i} \oplus N_i = \sum_{\beta \leq i} \oplus N_i \oplus \sum_{\beta \leq i} \oplus N_i
\]
such that
\[
P = X(\beta)^* \oplus W \oplus \sum_{0 < i \leq \beta} \oplus T(i)^* \oplus \sum_{\beta < i} \oplus N_i(\beta) \oplus \sum_{\beta \leq i} \oplus N_i
\]
But, by the same argument above, we can that
\[
W = 0, \sum_{\beta \leq i} \oplus N_i = 0
\]
so, we have
\[
P = \sum_{0 \leq i \leq \beta} \oplus X(i)^* \\
\oplus \sum_{0 < i \leq \beta} \oplus T(i)^* \\
\oplus \sum_{\beta < i} \oplus T(i)^* \\
\oplus N_0(0) \oplus \sum_{0 < i \leq \beta} \oplus N_0(i) \oplus \sum_{\beta < i} \oplus N_i(i) \\
\oplus N_1(0) \oplus N_1(1) \oplus \sum_{0 < i \leq \beta} \oplus N_1(i) \oplus \sum_{\beta < i} \oplus N_i(i) \\
\oplus N_2(0) \oplus N_2(1) \oplus N_2(2) \oplus \sum_{2 < i \leq \beta} \oplus N_2(i) \oplus \sum_{\beta < i} \oplus N_i(i)
\]
Thus, by transfinite induction, we have decompositions
\[
T(i)^* = T(i)^* \oplus T(i)^* \forall i \in I \\
N_0(i) = N_0(i) \oplus N_0(i) \forall i \in I - 0, \\
N_1(i) = N_1(i) \oplus N_1(i) \forall i \in I - I(2),
\]
\[
N_k(i) = N_k(i) \oplus N_k(i) \forall i \in I - I(k),
\]
such that, for any \( k \in I, \)
\[
P = \sum_{0 \leq i \leq k} \oplus X(i)^* \\
\oplus \sum_{0 < i \leq k} \oplus T(i)^* \oplus \sum_{k < i} \oplus T(i)^* \\
\oplus N_0(0) \oplus \sum_{0 < i \leq k} \oplus N_0(i) \oplus \sum_{k < i} \oplus N_i(i) \\
\oplus N_1(0) \oplus N_1(1) \oplus \sum_{0 < i \leq k} \oplus N_1(i) \oplus \sum_{k < i} \oplus N_i(i) \\
\oplus N_2(0) \oplus N_2(1) \oplus N_2(2) \oplus \sum_{2 < i \leq k} \oplus N_2(i) \oplus \sum_{k < i} \oplus N_i(i)
\]
So, by the quite similar argument above for \( X(1)^* \) or \( X(2)^* \), we have

\[
P = \sum \bigoplus X(i)^* + \sum \bigoplus T(i)^*
\]

\[
\oplus N_0(0) \oplus \sum_{i=0} \bigoplus N_0(i)
\]

\[
\oplus N_1(0) \oplus N_1(1) \oplus \sum_{i=0} \bigoplus N_1(i)
\]

\[
\oplus N_2(0) \oplus N_2(1) \oplus N_2(2) \oplus \sum_{i=0} \bigoplus N_2(i)
\]

\[
\oplus N_k(0) \oplus N_k(1) \oplus N_k(2) \oplus \cdots \oplus N_k(k) \oplus \sum_{i=0} \bigoplus N_k(i)
\]

This completes the proof, as \( X = \sum \bigoplus X(i)^* \).

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