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AN ANALOGUE OF THE PALEY-WIENER THEOREM FOR THE EUCLIDEAN MOTION GROUP

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1. Introduction

The purpose of this paper is to give a detailed proof of an analogue of the Paley-Wiener theorem for the euclidean motion group which was announced in [3]. Restricting our attention to bi-invariant functions (with respect to the rotation group) we obtain an analogue of the Paley-Wiener theorem for the Fourier -Bessel transform.

2. Unitary representations

Let G be the group of all motions of the n-dimensional euclidean space $\boldsymbol{R}^n$. Then G is realized as the group of $(n+1)\times(n+1)$-matrices of the form $\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix}$, $(k\in SO(n),\ x\in\boldsymbol{R}^n)$. Let K and H be the closed subgroups consisting of the elements $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $(k\in SO(n))$ and $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $(x\in\boldsymbol{R}^n)$, respectively. Then G is the semi-direct product of H and K. We normalize the Haar measure dg on G such that $dg=dxdk$, where $dx=(2\pi)^{-n/2}dx_1\cdots dx_n$ and dk is the normalized Haar maesure on K.

For any subgroup G_1 of G we denote by $\hat{G}_1$ the set of all equivalence classses of irreducible unitary representations of G_1. For an irreducible unitary representation σ of G_1, we denote by $[\sigma]$ the equivalence class which contains σ. For simplicity we identify $k\in SO(n)$ with $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\in K$ and $x\in\boldsymbol{R}^n$ with $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\in H$. Denote by $\langle\ ,\ \rangle$ the euclidean inner product on $\boldsymbol{R}^n$. Then we can identify $\hat{H}$ with $\boldsymbol{R}^n$ so that the value of $\xi\in\hat{H}$ at $x\in H$ is $e^{i\langle\xi,x\rangle}$. Because H is normal, K acts on H and therefore on $\hat{H}$ naturally: $\langle k\xi, x\rangle=\langle\xi, k^{-1}x\rangle$. Let K_ξ be the isotropy subgroup of K at $\xi\in\hat{H}$. If $\xi\neq 0$, K_ξ is isomorphic to $SO(n-1)$.

The dual space $\hat{G}$ of G was completely determined by G. W. Mackey [4] and S. Itô [2] as follows.

Let $\mathfrak{H}=L_2(K)$ be the Hilbert space of all square integrable functions on K. We denote by U^ξ the unitary representation of G induced by $\xi\in\hat{H}$. Then for

$g=\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix}\in G$

$$(U_g^\xi F)(u) = e^{i<\xi,u^{-1}x>}F(k^{-1}u),\ (F\in\mathfrak{H},\ u\in K)\,.$$

Let χ_σ and d_σ be the character and the degree of $[\sigma]\in\hat{K}_\xi$, respectively. Let L and R be the left and right regular representations of K, respectively. We also denote by L and R the corresponding representations of the universal enveloping algebra of the Lie algebra of K defined on $C^\infty(K)$, respectively. If $\sigma(m)=(\sigma_{pq}(m))(1\leqq p,\ q\leqq d_\sigma)$, we put

$$P^\sigma = d_\sigma\int_{K_\xi}\overline{\chi_\sigma(m)}\,R_m d_\xi m$$

and

$$P_q^\sigma = d_\sigma\int_{K_\xi}\overline{\sigma_{qq}(m)}\,R_m d_\xi m\,,$$

where $d_\xi m$ is the normalized Haar measure on K_ξ. Then P^σ and P_q^σ are both orthogonal projections of $\mathfrak{H}$. Put $\mathfrak{H}^\sigma=P^\sigma\mathfrak{H}$ and $\mathfrak{H}_q^\sigma=P_q^\sigma\mathfrak{H}$. The subspaces $\mathfrak{H}_q^\sigma$ $(1\leqq q\leqq d_\sigma)$ are stable under U^ξ and the representations of G induced on $\mathfrak{H}_q^\sigma$ $(1\leqq q\leqq d_\sigma)$ under U^ξ are equivalent for all $q=1,\cdots,d_\sigma$. We fix one of them and denote by $U^{\xi,\sigma}$. It is easy to see that

$$U_g^{k\xi} = R_k U_g^\xi R_k^{-1}(k\in K,\ \xi\in\hat{H},\ g\in G)\,. \tag{2.1}$$

Two representations $U^{\xi,\sigma}$ and $U^{\xi',\sigma'}$ are equivalent if and only if there exists an element $k\in K$ such that $\xi'=k\xi$ and $[\sigma]=[\sigma'^k]$, where

$$\sigma'^k(m) = \sigma'(kmk^{-1}),\ (m\in K_\xi)\,.$$

First we assume that $\xi\neq 0$. Then $U^{\xi,\sigma}$ is irreducible and every infinite dimensional irreducible unitary representation is equivalent to one of $U^{\xi,\sigma}$, $(\xi\neq 0,\ [\sigma]\in\hat{K}_\xi)$. Since $\mathfrak{H}=\bigoplus_{[\sigma]\in\hat{K}_\xi}\mathfrak{H}^\sigma$ and $\mathfrak{H}^\sigma=\bigoplus_{q=1}^{d\sigma}\mathfrak{H}_q^\sigma$, we have

$$U^\xi\cong\bigoplus_{[\sigma]\in\hat{K}_\xi}(\underbrace{U^{\xi,\sigma}\oplus\cdots\oplus U^{\xi,\sigma}}_{d_\sigma\text{ times}})\,. \tag{2.2}$$

Next we assume that $\xi=0$. Then $U^{\xi,\sigma}$ is reducible and $K_\xi=K$. For any $[\sigma]\in\hat{K}$ we define a finite dimensional unitary representation U^σ of G by $U_g^\sigma=\sigma(k)$, where $g=\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix}\in G$. Then we have $U^{0,\sigma}\cong\underbrace{U^\sigma\oplus\cdots\oplus U^\sigma}_{d_\sigma\text{ times}}$ and $U^0\cong\bigoplus_{[\sigma]\in\hat{K}}U^{0,\sigma}$. Moreover every finite dimensional irreducible unitary representation of G is equivalent to one of U^σ, $([\sigma]\in\hat{K})$.

We denote by $(\hat{G})_\infty$ and $(\hat{G})_0$ the set of all eqiuvalence classes of infinite and finite dimensional irreducible unitary representations of G, respectively.

3. The Plancherel formula

Let $\mathfrak{k}$ be the Lie algebra of K. We denote by Δ the Casimir operator of K (In case $n=2$, we put $\Delta=-X^2$ for a non-zero $X\in\mathfrak{k}$). By the Peter-Weyl theorem we can choose a complete orthonormal basis $\{\phi_j\}_{j\in J}$ of $\mathfrak{H}$, consisting of the matricial elements of irreducible unitary representations of K, that is, $\phi_j=d_\tau^{1/2}\tau_{pq}$ for some $[\tau]\in\hat{K}$ $(\tau=(\tau_{pq}))$ and $p, q=1, \cdots, d_\tau$. First, we prove the following

Lemma 1. *Let T be a bounded operator on $\mathfrak{H}=L_2(K)$ which leaves the space $C^\infty(K)$ stable. If for any non-negative integers l and m, there exists a constant $C^{l,m}$ such that*

$$||\Delta^l T\Delta^m||\leqq C^{l,m},$$

then the series $\sum_{i,j\in J}|(T\phi_j, \phi_i)|$ converges.

Proof. For the sake of brevity we assume that $n\geqq 3$. In case $n=2$ the same method is valid with a slight modification. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$. Denote by $\mathfrak{k}^c$ and $\mathfrak{t}^c$ the complexifications of $\mathfrak{k}$ and $\mathfrak{t}$, respectively. Fix an order in the dual space of $(-1)^{1/2}\mathfrak{t}$. Let P be the positive root system of $\mathfrak{k}^c$ with respect to $\mathfrak{t}^c$. Let $\mathcal{F}$ be the set of all dominant integral forms. Then $\Lambda\in\mathcal{F}$ is the highest weight of some irreducible unitary representation of K if and only if it is lilfted to a unitary character of the Cartan subgroup corresponding to $\mathfrak{t}$. Let $\mathcal{F}_0$ be the set of all such Λ's. For any $\Lambda\in\mathcal{F}_0$ we doente by τ_Λ a representative of $[\tau_\Lambda]\in\hat{K}$ which is a matricial representation of K with the highest weight Λ. Then the mapping $\Lambda\mapsto[\tau_\Lambda]$ gives the bijection between $\mathcal{F}_0$ and $\hat{K}$. Let d_Λ be the degree of τ_Λ. Denote by J_Λ be the set of $j\in J$ such that $\phi_j=\mathrm{d}_\Lambda{}^{1/2}$ $(\tau_\Lambda)_{pq}$ for some $p, q=1, \cdots, d_\Lambda$. Let (,) be the inner product on the dual space of $(-1)^{1/2}\mathfrak{t}$ induced by the Killing form and put $|\Lambda|=(\Lambda, \Lambda)^{1/2}$. As usual we put $\rho=\frac{1}{2}\sum_{\alpha\in P}\alpha$. We use the following known facts (i)~(iii):

(i) For every $\Lambda\in\mathcal{F}_0$ and $j\in J_\Lambda$, we have $(\Delta+|\rho|^2)\phi_j=|\Delta+\rho|^2\phi_j$.

(ii) For every $\Lambda\in\mathcal{F}_0$, $d_\Lambda=\dfrac{\prod_{\alpha\in P}(\Lambda+\rho, \alpha)}{\prod_{\alpha\in P}(\rho, \alpha)}$, (Weyl's dimension formula).

(iii) The Dirichlet series $\sum_{\Lambda\in\mathcal{F}_0}\dfrac{1}{|\Lambda+\rho|^s}$ converges if $s>\left[\dfrac{n}{2}\right]$.

(see [1(a)] and [9])

By (i)

$$\phi_j=\frac{(\Delta+|\rho|^2)^l}{|\Lambda+\rho|^{2l}}\phi_j \text{ for } j\in J_\Lambda \text{ and } l=0, 1, 2, \cdots.$$

Therefore

$$\sum_{j\in J_\Lambda}\sum_{i\in J_{\Lambda'}}|(T\phi_j,\phi_i)|=\frac{1}{|\Lambda+\rho|^{2l}|\Lambda'+\rho|^{2m}}\sum_{j\in J_\Lambda}\sum_{j\in J_{\Lambda'}}|(T(\Delta+|\rho|^2)^l\phi_j,$$
$$(\Delta+|\rho|^2)^m\phi_i)|=\frac{1}{|\Lambda+\rho|^{2l}|\Lambda'+\rho|^{2m}}\sum_{j\in J_\Lambda}\sum_{i\in J_{\Lambda'}}|((\Delta+|\rho|^2)^mT(\Delta+|\rho|^2)^l\phi_j,\phi_i)|.$$

On the other hand by the assumption of the lemma we can prove that there exists a constant $C_1^{l,m}$ such that

$$||(\Delta+|\rho|^2)^mT(\Delta+|\rho|^2)^l||\leqq C_1^{l,m}$$

Then

$$\sum_{j\in J_\Lambda}\sum_{i\in J_{\Lambda'}}|(T\phi_j,\phi_i)|\leqq\frac{C_1^{l,m}}{|\Lambda+\rho|^{2l}|\Lambda'+\rho|^{2m}}(d_\Lambda)^2(d_{\Lambda'})^2$$
$$=C_1^{l,m}\frac{1}{|\Lambda+\rho|^{2l}|\Lambda'+\rho|^{2m}}\frac{\prod_{\alpha\in P}(\Lambda+\rho,\alpha)^2(\Lambda'+\rho,\alpha)^2}{\prod_{\alpha\in P}(\rho,\alpha)^4}$$
$$\leqq C_1^{l,m}\frac{\prod_{\alpha\in P}(\alpha,\alpha)^2}{\prod_{\alpha\in P}(\rho,\alpha)^4}\cdot\frac{1}{|\Lambda+\rho|^{2l-n(n-1)/2+[n/2]}|\Lambda'+\rho|^{2m-n(n-1)/2+[n/2]}}. \qquad (3.1)$$

Therefore if put $l=m$, we have

$$\sum_{i,j\in J}|(T\phi_j,\phi_i)|\leqq C_1^{l,l}\frac{\prod_{\alpha\in P}(\alpha,\alpha)^2}{\prod_{\alpha\in P}(\rho,\alpha)^4}\Big(\sum_{\Lambda\in\mathcal{F}_0}\frac{1}{|\Lambda+\rho|^{2l-n(n-1)/2+[n/2]}}\Big)^2. \qquad (3.2)$$

If we take $l=m>\frac{1}{2}\frac{n(n-1)}{2}=\frac{1}{2}\dim K$, using the property (iii) we obtain

$$\sum_{i,j\in J}|(T\phi_j,\phi_i)|<+\infty\,.$$

q.e.d.

Corollary. *If T is an operator on $\mathfrak{H}$ satisfying the conditions of Lemma* 1, *T is of the trace class.*

For the proof of this corollary, see Harish-Chandra [1(a), Lemma 1].

For any $f\in C_c^\infty(G)$. We put

$$T_f(\xi,\sigma)=\int_G f(g)U_g^{\xi,\sigma}dg \qquad (\xi\neq 0,\ [\sigma]\in\hat{K}_\xi)\,.$$

Then

$$(T_f(\xi,\sigma)F)(u)=\int_K K_f(\xi,\sigma;u,v)F(v)dv \qquad (u\in K)\,,$$

where

$$K_f(\xi, \sigma; u,v) = d_\sigma \int_{K_\xi} \overline{\sigma_{qq}(m)}\, d_\xi m \int_H f\begin{pmatrix} umv^{-1} & x \\ 0 & 1 \end{pmatrix} e^{i<\xi, u^{-1}x>} dx\,.$$

It is easy to see that $T_f(\xi, \sigma)F \in C^\infty(K)$ for any $f \in C_c^\infty(G)$ and $F \in C^\infty(K)$.

We denote by λ and μ the left and right regular representations of G, respectively. We also denote by λ and μ the corresponding representations of the universal enveloping algebra of G defined on $C^\infty(G)$. We regard each element $X \in \mathfrak{k}$ as a right invariant vector field on K. So that we have $L(X) = -X$. Since

$$(T_f(\xi, \sigma) F(\exp(-tX))u) = (T_{\lambda(\exp tX)f}(\xi, \sigma) F)(u) \qquad (t \in \boldsymbol{R})\,,$$

we have

$$((-X)T_f(\xi, \sigma)F)(u) = (T_{\lambda(X)f}(\xi, \sigma)F)(u)$$

for $F \in C^\infty(K)$. Therefore for any non-negative integer l

$$\Delta^l T_f(\xi, \sigma) = T_{\lambda(\Delta)^l f}(\xi, \sigma)\,.$$

Also we have

$$T_f(\xi, \sigma)\Delta^m = T_{\mu(\Delta)^m f}(\xi, \sigma) \qquad (m = 0, 1, 2, \cdots)$$

by a similar way. On the other hand we notice that

$$\|T_f(\xi, \sigma)\| \leqq \int_G |f(g)|\, dg.$$

Hence

$$\|\Delta^l T_f(\xi, \sigma)\Delta^m\| \leqq \int_G |(\lambda(\Delta)^l \mu(\Delta)^m f)(g)|\, dg\,.$$

Thus the opreator $T_f(\xi, \sigma)$, $f \in C_c^\infty(G)$, satisfies the assumptions of Lemma 1. By the corollary to Lemma 1, $T_f(\xi, \sigma)$ is of the trace class.

As it can be easily seen that $K_f(\xi, \sigma; u, v) \in C^\infty(K \times K)$, we have

$$Tr(T_f(\xi, \sigma)) = \int_K K_f(\xi, \sigma; u, u)\, du$$

(see [1(b), Lemma 5]). Making use of the relation

$$d_\sigma \int_{K_\xi} \sigma_{qq}(m_1 m m_1^{-1}) d_\xi m_1 = \chi_\sigma(m)\,,$$

we have the following proposition.

Proposition 1. *For any* $f \in C_c^\infty(G)$, $T_f(\xi, \sigma)$ $(\xi \neq 0, [\sigma] \in \hat{K}_\xi)$ *is of the trace class and*

$$Tr(T_f(\xi,\sigma)) = \int_{K_\xi} \overline{\chi_\sigma(m)}\, d_\xi m \int_{H\times K} f\begin{pmatrix} u\,m\,u^{-1} & x \\ 0 & 1\end{pmatrix} e^{i<\xi, u^{-1}x>}\, dx\, du\,.$$

Let $\boldsymbol{R}_+$ be the set of all positive numbers and let M be the subgroup consisting of the elements $\begin{pmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1\end{pmatrix}$, $(m \in SO(n-1))$. Then for any $\xi \in \hat{H}$ of the form $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0\end{pmatrix}$ $(a \in \boldsymbol{R}_+)$, we have $K_\xi = M$. It follows from the results of §2 that $(\hat{G})_\infty$ can be indetified with $\boldsymbol{R}_+ \times \hat{M}$. For $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0\end{pmatrix}$ $(a \in \boldsymbol{R}_+)$, we write briefly $T_f(\xi,\sigma) = T_f(a,\sigma)$. Then we have the following Plancherel formula for G.

Proposition 2. *For any* $f \in C_c^\infty(G)$

$$\int_G |f(g)|^2 dg = \frac{2}{2^{n/2}\Gamma(n/2)} \sum_{[\sigma]\in\hat{M}} d_\sigma \int_{\boldsymbol{R}_+} ||T_f(a,\sigma)||_2^2 a^{n-1} da\,,$$

where $||\quad||_2$ *denotes the Hilbert-Schmidt norm.*

Proof. It is enough to prove that

$$f\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} = \frac{2}{2^{n/2}\Gamma(n/2)} \sum_{[\sigma]\in\hat{M}} d_\sigma \int_{\boldsymbol{R}_+} Tr(T_f(a,\sigma)) a^{n-1} da\,.$$

For any $f \in C_c^\infty(G)$, we put

$$T_f(\xi) = \int_G f(g) U_g^\xi dg \quad (\xi \in \hat{H})\,.$$

As above we write $T_f(\xi) = T_f(a)$ for $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0\end{pmatrix}$ $(a \in \boldsymbol{R}_+)$. Then by (2.2)

$$T_f(\xi) = \bigoplus_{[\sigma]\in\hat{K}_\xi} (\underbrace{T_f(\xi,\sigma)\oplus\cdots\oplus T_f(\xi,\sigma)}_{d_\sigma \text{ times}}) \quad (\xi \neq 0)\,.$$

Therefore

$$Tr(T_f(\xi)) = \sum_{\sigma\in\hat{K}_\xi} d_\sigma Tr(T_f(\xi,\sigma))\,.$$

Hence it is enough to prove that

$$f\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} = \frac{2}{2^{n/2}(n/2)} \int_{\boldsymbol{R}_+} Tr(T_f(a)) a^{n-1} da\,. \tag{3.3}$$

Since

$$\phi(m) = \int_{H\times K} f\begin{pmatrix} u\,m\,u^{-1} & x \\ 0 & 1\end{pmatrix} e^{i<\xi, u^{-1}x>} dx du$$

is a central function on K_ξ,

$$\phi(m) = \sum_{[\sigma]\in\hat{K}_\xi} \left(\int_{K_\xi} \phi(m_1) \overline{\chi_\sigma(m_1)}\, d_\xi m_1 \right) \chi_\sigma(m)$$

(see [7], §24). Hence by Proposition 1 we have

$$\begin{aligned}\phi(1) &= \sum_{[\sigma]\in\hat{K}_\xi} d_\sigma \int_{K_\xi} \phi(m) \overline{\chi_\sigma(m)}\, d_\xi m \\ &= \sum_{[\sigma]\in\hat{K}_\xi} d_\sigma Tr(T_f(\xi, \sigma)) .\end{aligned}$$

Thus we have

$$\begin{aligned}Tr(T_f(\xi)) = \phi(1) &= \int_{H\times K} f\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} e^{i<\xi, u^{-1}x>} dx du \\ &= \int_H \left\{ \int_K f\begin{pmatrix}1 & ux\\ 0 & 1\end{pmatrix} du \right\} e^{i<\xi, x>} dx .\end{aligned}$$

Hence

$$\int_K f\begin{pmatrix}1 & ux\\ 0 & 1\end{pmatrix} du = \int_H Tr(T_f(\xi)) e^{-i<\xi, x>} d\xi ,$$

where $d\xi = \frac{1}{(2\pi)^{n/2}} d\xi_1 \cdots d\xi_n$. When $x=0$,

$$f\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \int_H Tr(T_f(\xi)) d\xi \tag{3.4}$$

By (2.1) we have $Tr(T_f(k\xi)) = Tr(R_k T_f(\xi) R_k^{-1}) = Tr(T_f(\xi))$.
Hence $Tr(T_f(\xi)) = Tr(T_f(|\xi|))$. So that we have (3.3) from (3.4).

q.e.d.

Let $\boldsymbol{B}(\mathfrak{H})$ by the Banach space of all bounded linear operators on $\mathfrak{H}$. We define the **Fourier transform** of $f \in C_c^\infty(G)$ by the $\boldsymbol{B}(\mathfrak{H})$-valued function T_f on $\hat{H}$. In terms of this transform Proposition 2 becomes the following

Corollary. *For any* $f \in C_c^\infty(G)$

$$\int_G |f(g)|^2 dg = \frac{2}{2^{n/2}\Gamma(n/2)} \int_{R_+} \| T_f(a) \|_2^2\, a^{n-1} da.$$

4. The Fourier-Laplace transform

For each $\zeta \in \hat{H}^c (\simeq \boldsymbol{C}^n)$ we define a bounded representation of G on $\mathfrak{H}$ by

$$(U_g^\zeta F)(u) = e^{i<\zeta, u^{-1}x>} F(k^{-1}u), \; (F\in\mathfrak{H},\ u\in K) ,$$

where $g = \begin{pmatrix}k & x\\ 0 & 1\end{pmatrix} \in G$. For $f \in C_c^\infty(G)$, put

$$T_f(\zeta) = \int_G f(g)\, U_g^\zeta dg\,.$$

Then T_f is a $\boldsymbol{B}(\mathfrak{H})$-valued function on $\hat{H}^c$. We shall call T_f the **Fourier-Laplace transform** of f.

Since K is compact, for each $f \in C_c^\infty(G)$ there exists a positive number a such that $\operatorname{Supp}(f) \subset \left\{ \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \in G;\ |x| \leqq a,\ k \in K \right\}$, where $\operatorname{Supp}(f)$ denotes the support of f. We denote by r_f the greatest lower bound of such a's. Throughout this section we assume that $r_f \leqq a$ for a fixed $a \in \boldsymbol{R}_+$.

Lemma 2. *There exists a constant $C \geqq 0$ depending only on f such that $\|T_f(\zeta)\| \leqq C \exp a|\operatorname{Im} \zeta|$.*

Proof. Making use of the Schwarz's inequality we have

$$\begin{aligned} \|T_f(\zeta)F\|^2 &\leqq \int_K \left\{ \int_{H\times K} \left| f\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \right| e^{-<Im\zeta, u^{-1}x>} |F(k^{-1}u)|\, dx dk \right\}^2 du \\ &\leqq e^{2a|Im\zeta|} \int_K \left\{ \int_K \left(\int_H \left| f\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \right| dx \right) |F(k^{-1}u)|\, dk \right\}^2 du \\ &\leqq e^{2a|Im\zeta|} \int_K \left\{ \int_K \left(\int_H \left| f\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \right| dx \right)^2 dk \int_K |F(k^{-1}u)|^2 dk \right\} du \\ &= e^{2a|Im\zeta|} \int_K \left(\int_H \left| f\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \right| dx \right)^2 dk\, \|F\|^2 \end{aligned}$$

for any $F \in \mathfrak{H}$. Therefore it is enough to put

$$C = \left\{ \int_K \left(\int_H \left| f\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \right| dx \right)^2 dk \right\}^{1/2}.$$

q.e.d.

Lemma 3. *The $\boldsymbol{B}(\mathfrak{H})$-valued function T_f on $\hat{H}^c$ is entire analytic.*

Proof. For any n-tuple $(m_1, \cdots, m_n)$ of non-negative integers we define a bounded operator $T_f^{m_1\cdots m_n}$ by

$$(T_f^{m_1\cdots m_n}F)(u) = \int_{H\times K} f\begin{pmatrix} k & ux \\ 0 & 1 \end{pmatrix} x_1^{m_1} \cdots x_n^{m_n} F(k^{-1}u)\, dx\, dk\,,$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Then we have

$$\|T_f^{m_1\cdots m_n}\| \leqq a^{m_1+\cdots+m_n} \left\{ \int_K \left(\int_H \left| f\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \right| dx \right)^2 dk \right\}^{1/2}.$$

Hence for any fixed $\zeta = (\zeta_1 \cdots, \zeta_n) \in \boldsymbol{C}^n$ the series

$$\sum_{m=0}^\infty i^m \sum_{m_1+\cdots+m_n=m} \frac{m!}{m_1! \cdots m_n!}\, T_f^{m_1\cdots m_n} \zeta_1^{m_1} \cdots \zeta_n^{m_n}$$

converges in $\boldsymbol{B}(\mathfrak{H})$-norm. It is easy to see that this series is equal to $T_f(\zeta)$.

q.e.d.

For any polynomial function p on $\hat{H}^c$, we define a differential operator $p(D)$ on H by $p(D)=p\left(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_n}\right)$. A polynomial funciton p on $\hat{H}^c$ is called K-invariant if $p(k\zeta)=p(\zeta)$ for any $k\in K$ and $\zeta\in\hat{H}^c$. As is easily seen, $T_f(\zeta)$ leaves the space $C^\infty(K)$ stable.

Lemma 4. 1) *For any non-negative integers l and m we have* $\Delta^l T_f(\zeta)\Delta^m = T_{\lambda(\Delta^l)\mu(\Delta^m)f}(\zeta)$, $(\zeta\in\hat{H}^c)$.
2) *For any K-invariant polynomial function p on $\hat{H}^c$, we have* $p(\zeta)T_f(\zeta)= T_{p^*(D)f}(\zeta)$, $(\zeta\in\hat{H}^c)$, *where* $p^*(\zeta)=p(-\zeta)$.

The statement 1) can be proved by a similar way mentioned in §3. The statement 2) is easily proved, using the fact $\frac{\partial}{\partial x_j} e^{i<\zeta,x>}=i\zeta_j e^{i<\zeta,x>}$ and the integration by parts. From Lemma 2 and Lemma 4 we have the following

Proposition 3. *For any K-invariant polynomial function p on $\hat{H}^c$ and for any non-negative integers l and m, there exists a constant $C_p^{l,m}$ such that*

$$||p(\zeta)\Delta^l T_f(\zeta)\Delta^m|| \leqq C_p^{l,m} \exp a|Im\zeta|.$$

Finally from the definition of T_f we have the following functional equations for T_f.

Proposition 4. $T_f(k\zeta)=R_k T_f(\zeta)R_k^{-1}$ $(\zeta\in\hat{H}^c, k\in K)$.

5. The analogue of the Paley-Wiener theorem

Theorem 1. *A $\boldsymbol{B}(\mathfrak{H})$-valued function T on $\hat{H}$ is the Fourier transform of $f \in C_c^\infty(G)$ such that $r_f\leqq a$ $(a>0)$ if and only if it satisfies the following conditions:*

(I) *T can be extended to an entire analytic function on $\hat{H}^c$.*

(II) *For any $\zeta\in\hat{H}^c$, $T(\zeta)$ leaves the space $C^\infty(K)$ stable. Moreover for any K-invariant polynomial function p on $\hat{H}^c$ and for any non-negative integers l and m, there exists a constant $C_p^{l,m}$ such that*

$$||p(\zeta)\Delta^l T(\zeta)\Delta^m|| \leqq C_p^{l,m} \exp a|Im\zeta|.$$

(III) *For any $k\in K$*

$$T(k\zeta) = R_k T(\zeta) R_k^{-1} \quad (\zeta\in\hat{H}^c).$$

Proof. We have already proved the necessity of the theorem in §4. In the following we shall prove the sufficiency of the theorem.

Let T be an arbitrary $\boldsymbol{B}(\mathfrak{H})$-valued function on $\hat{H}$ statisfying the conditions (I)~(III) in the theorem. Let $\{\phi_j\}_{j\in J}$ be the complete orthonomal basis of $\mathfrak{H}$

which we have chosen in §3. If $|Im\zeta|\leqq b (b>0)$, by the condition (II) for any non-negateive integers l and m there exists a constant $C^{l}{}_{1}^{m}$ such that

$$||\Delta^l T(\zeta)\Delta^m||\leqq C^{l}{}_{1}^{m}\exp ab\,.$$

Therefore by Lemma 1 the series

$$\sum_{i,j\in J}|(T(\zeta)\phi_j,\ \phi_i)|$$

converges and $T(\zeta)$ is of the trace class. We assume that $n\geqq 3$. If $\phi_j=d_\Lambda^{1/2}(\tau_\Lambda)_{pq}$, we have $|\phi_j(u)|\leqq d_\Lambda^{1/2}$ because $|\tau_\Lambda(u)_{pq}|\leqq 1$. So we have

$$\sum_{j\in J_\Lambda}\sum_{i\in J_{\Lambda'}}|(T(\zeta)\phi_j,\ \phi_i)\phi_i(u)\overline{\phi_j(v)}|\leqq\frac{C_1^{l,m}e^{ab}}{|\Lambda+\rho|^{2l}|\Lambda'+\rho|^{2m}}(d_\Lambda)^{3/2}(d_{\Lambda'})^{3/2}\,.$$

Hence

$$\begin{aligned}&\sum_{i,j\in J}|(T(\zeta)\phi_j,\ \phi_i)\phi_i(u)\overline{\phi_j(v)}|\\&\leqq C_1^{l,l}e^{ab}\frac{\prod_{\alpha\in P}(\alpha,\ \alpha)^3}{\prod_{\alpha\in P}(\rho,\ \alpha)^5}\Big(\sum_{\Lambda\in\mathcal{F}_0}\frac{1}{|\Lambda+\rho|^{2l-3(n(n-1)/2-[n/2])/2}}\Big)^2<+\infty\end{aligned}\tag{5.1}$$

for $2\,l>\frac{3}{2}\frac{n(n-1)}{2}-\frac{1}{2}\Big[\frac{n}{2}\Big]$. In case $n=2$, $|\phi_j|=1$ for all $j\in J$. Therefore $\sum_{i,j\in J}|(T(\zeta)\phi_j,\ \phi_i)\phi_i(u)\ \overline{\phi_j(v)}|=\sum_{i,j\in J}|(T(\zeta)\phi_j,\ \phi_i)|<+\infty$.

Now let us define the kernel function of $T(\zeta)$ $(\zeta\in\hat{H}^c)$ by

$$K(\zeta;\ u,\ v)=\sum_{i,j\in J}(T(\zeta)\phi_j,\ \phi_i)\Phi_i(u)\overline{\phi_j(v)}\,.\tag{5.2}$$

By the facts stated above and the property (I) it is easy to see that for any $\zeta\in\hat{H}^c$ the right hand side of (5.2) is absolutely convergent and that it is uniformly covergent on every compact subset of $\hat{H}^c\times K\times K$. Thus we have the following

Lemma 5. *The function $\hat{H}^c\times K\times K\ni(\zeta,u,v)\rightarrow K(\zeta;u,v)$ is of class C^∞ and entire analytic with repsect to ζ.*

If we adopt the formula (5.1) to $p(\zeta)T(\zeta)$ instead of $T(\zeta)$, we have the following lemma by making use of (II).

Lemma 6. *For any K-invariant polynomial function p on $\hat{H}^c$, there exists a constant C_p such that*

$$|p(\zeta)K(\zeta;\ u,\ v)|\leqq C_p\exp a|Im\zeta|,\ (\zeta\in\hat{H}^c,\ u,\ v\in K)\,.$$

REMARK. $K(\zeta;\ u,\ v)$ is rapidly decreasing on the real axis $\hat{H}$.

Let us define a function f on G by the inversion formula corresponding to the Fourier transform, i.e.

$$f(g)=\frac{2}{2^{n/2}\Gamma(n/2)}\int_{R_+} Tr(T(a)U^{a}_{g^{-1}})a^{n-1}da\,.$$

By the property (III) we have

$$\begin{aligned}&(T(k\zeta)\phi_j,\ \phi_i)\phi_i(u)\,\overline{\phi_j(v)}\\&=(R_kT(\zeta)R_k^{-1}\phi_j,\ \phi_i)\phi_i(u)\,\overline{\phi_j(v)}=(T(\zeta)R_k^{-1}\phi_j,\ R_k^{-1}\phi_i)\phi_i(u)\,\overline{\phi_j(v)}\,.\end{aligned}$$

Let $\phi_j=d_\tau^{1/2}\tau_{pq}$ and $\phi_i=d_\sigma^{1/2}\sigma_{rs}([\tau],\ [\sigma]\in\hat{K})$. Then

$$R_k^{-1}\phi_j(w)=d_\tau^{1/2}\tau_{pq}(wk^{-1})=d_\tau^{1/2}\sum_{l=1}^{d_\tau}\tau_{pl}(w)\,\overline{\tau_{ql}(k)}$$

and

$$R_k^{-1}\phi_i(w)=d_\sigma^{1/2}\sum_{m=1}^{d_\sigma}\sigma_{rm}(w)\,\overline{\sigma_{sm}(k)}\,.$$

Therefore

$$\begin{aligned}&(T(\zeta)R_k^{-1}\phi_j,\ R_k^{-1}\phi_i)\phi_i(u)\,\overline{\phi_j(v)}\\&=\sum_{l=1}^{d_\tau}\sum_{m=1}^{d_\sigma}(T(\zeta)d^{1/2}\tau_{pl},\ d_\sigma^{1/2}\sigma_{rm})d_\sigma^{1/2}\sigma_{rs}(u)\sigma_{sm}(k)d_\tau^{1/2}\,\overline{\tau_{pq}(v)\tau_{ql}(k)}\,.\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{p,q=1}^{d_\tau}\sum_{r,s=1}^{d_\sigma}(T(k\zeta)d_\tau^{1/2}\tau_{pq},\ d_\sigma^{1/2}\sigma_{rs})d_\sigma^{1/2}\sigma_{rs}(u)d_\tau^{1/2}\,\overline{\tau_{pq}(v)}\\&=\sum_{p,l=1}^{d_\tau}\sum_{r,m=1}^{d_\sigma}(T(\zeta)d_\tau^{1/2}\tau_{pl},\ d_\sigma^{1/2}\sigma_{rm})\sum_{s=1}^{d_\sigma}d_\sigma^{1/2}\sigma_{rs}(u)\sigma_{sm}(k)\times\sum_{q=1}^{d_\tau}d_\tau^{1/2}\,\overline{\tau_{pq}(v)\tau_{ql}(k)}\\&=\sum_{p,l=1}^{d_\tau}\sum_{r,m=1}^{d_\sigma}(T(\zeta)d_\tau^{1/2}\tau_{pl},\ d_\sigma^{1/2}\sigma_{rm})d_\sigma^{1/2}\sigma_{rm}(uk)d_\tau^{1/2}\,\tau_{pl}(vk)\,.\end{aligned}$$

Since $K(\zeta;\ u,\ v)=\sum_{[\sigma],[\tau]\in\hat{K}}\sum_{p,q=1}^{d_\tau}\sum_{r,s=1}^{d_\sigma}(T(\zeta)d_\tau^{1/2}\tau_{pq},\ d_\sigma^{1/2}\sigma_{rs})d_\sigma^{1/2}\sigma_{rs}(u)\times d_\tau^{1/2}\,\overline{\tau_{pq}(v)}$, we have the following functional equation for $K(\zeta;\ u,\ v)$:

$$K(k\zeta;\ u,\ v)=K(\zeta;\ uk,\ vk)\,. \tag{5.3}$$

On the other hand

$$Tr(T(k\xi)U^{k\xi}_{g^{-1}})=Tr(R_kT(\xi)R_k^{-1}U^{\xi}_{g^{-1}})=Tr(T(\xi)U^{\xi}_{g^{-1}}),\ (\xi\in\hat{H})\,.$$

Hence

$$\frac{2}{2^{n/2}\Gamma(2/n)}\int_{R_+}Tr(T(a)U^{a}_{g^{-1}})a^{n-1}da=\int_H Tr(T(\xi)U^{\xi}_{g^{-1}})d\xi\,,$$

where $d\xi=\frac{1}{(2\pi)^{n/2}}d\xi_1\cdots d\xi_n$. As $T(\xi)F(u)=\int_K K(\xi;\ u,\ v)F(v)dv\quad(F\in\mathfrak{H})$

and $g^{-1}=\begin{pmatrix} k^{-1} & -k^{-1}x \\ 0 & 1 \end{pmatrix}$ for $g=\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix}\in G$, we have

$$U^{\xi}_{g^{-1}}T(\xi)F(u)=\int_K e^{-i<\xi,u^{-1}k^{-1}x>}K(\xi;\,ku,\,v)F(v)dv\,.$$

Since $T(\xi)$ is of the trace class, so is $U^{\xi}_g T(\xi)$. Moreover the function $K\times K\ni (u,\,v)\mapsto e^{-i<\xi,u^{-1}k^{-1}x>}K(\xi;\,ku,\,v)$ is clearly of class C^{∞}. Hence

$$Tr(T(\xi)^{\xi}_{g^{-1}})=\mathrm{Tr}(U^{\xi}_{g^{-1}}T(\xi))=\int_K e^{-i<\xi,u^{-1}k^{-1}x>}K(\xi,\,ku,\,u)du\,.$$

Therefore the equation (5.3) and the remark to Lemma 6 imply that

$$\begin{aligned}\int_H Tr(T(\xi)U^{\xi}_{g^{-1}})d\xi&=\int_H\int_K e^{-i<ku\xi,x>}K(\xi;\,ku,\,u)dud\xi\\&=\int_K\int_H e^{-i<\xi,x>}K(u^{-1}k^{-1}\xi;\,ku,\,u)d\xi\,du\\&=\int_K\int_H e^{-i<\xi,x>}K(\xi;\,1,\,k^{-1})d\xi\,du\\&=\int_H e^{-i<\xi,x>}K(\xi;\,1,\,k^{-1})d\xi\,.\end{aligned}$$

Thus we have

$$f\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix}=\int_H e^{-i<\xi,x>}K(\xi;\,1,\,k^{-1})d\xi\,, \tag{5.4}$$

($k\in K$, $x\in H$). It follows from Lemma 5 and the remark to Lemma 6 that f is of class C^{∞}. Making use of Lemma 6, it follows from the classical Paley-Wiener theorem that if $|x|>a$, $f\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix}=0$ for any $k\in K$.

Finally we have to check that $T_f=T$. Since

$$T_f(\xi)F(u)=\int_K K_f(\xi:\,u,\,v)F(v)dv$$

where

$$K_f(\xi;\,u,\,v)=\int_H f\begin{pmatrix} uv^{-1} & x \\ 0 & 1 \end{pmatrix} e^{i<\xi,u^{-1}x>}dx\,,$$

so it is enough to prove that

$$K(\xi;\,u,\,v)=\int_H f\begin{pmatrix} uv^{-1} & x \\ 0 & 1 \end{pmatrix} e^{i<\xi,u^{-1}x>}dx\,.$$

By the relation (5.4),

$$f\begin{pmatrix} uv^{-1} & x \\ 0 & 1 \end{pmatrix}=\int_H e^{-i<\xi,x>}K(\xi;\,1,\,vu^{-1})d\xi$$

$$= \int_H e^{-i<\xi,x>} K(u^{-1}\xi; u, v) d\xi .$$

Hence

$$K(u^{-1}\xi; u, v) = \int_H f \begin{pmatrix} uv^{-1} & x \\ 0 & 1 \end{pmatrix} e^{i<\xi,x>} dx .$$

If we replace $u\xi$ for ξ,

$$K(\xi; u, v) = \int_H f \begin{pmatrix} uv^{-1} & x \\ 1 & 0 \end{pmatrix} e^{i<u\xi,x>} dx$$

$$= \int_H f \begin{pmatrix} uv^{-1} & x \\ 0 & 1 \end{pmatrix} e^{i<\xi,u^{-1}x>} dx .$$

This completes the proof of the theorem.

6. The Fourier-Bessel transform

Let $C_c^\infty(K\backslash G/K)$ be the set of all complex valued K-bi-invariant functions on G which are infinitely differentiable and with compact support. For $f \in C_c^\infty(G)$, put

$$(\mathcal{F}_\xi f)(g) = \int_H f\left(g \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) e^{-i<\xi,y>} dy$$

and

$$(\mathcal{P}f)(g) = \int_K f(gu) du .$$

For $f \in C_c^\infty(K\backslash G/K)$ it is easy to to see that

$$(\mathcal{P}\mathcal{F}_\xi f)\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \left(\int_H f\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \phi_\xi(y) dy\right) \phi_{-\xi}(x) , \qquad (6.1)$$

where

$$\phi_\xi(x) = \int_K e^{i<\xi,ux>} du .$$

REMARK. The formula (6.1) is regarded as an analogue of the Poisson integral for semisimple Lie groups (see [5]). And the function ϕ_ξ is the zonal spherical function.

Let us define the **Fourier-Bessel transform** $\mathcal{B}\mathcal{F}f$ of $f \in C_c^\infty(K\backslash G/K)$ by

$$(\mathcal{B}\mathcal{F}f)(\xi) = \int_H f\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \phi_\xi(y) dy .$$

If $x = \begin{pmatrix} r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $(r>0)$ and $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $(a>0)$, we can prove that

$$\phi_\xi(x) = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{n-1}{2}\right)}\int_0^\pi e^{i\, ar\cos\theta}\sin^{n-2}\theta\, d\theta$$
$$= \Gamma\left(\frac{n}{2}\right)\frac{J_{(n-2)/2}(ar)}{\left(\frac{ar}{2}\right)^{(n-2)/2}}$$

(see [8] for the the notation of the Bessel function $J_n(r)$).

If $g_r = \begin{pmatrix} 1 & 0 & r \\ \ddots & & 0 \\ & \ddots & \vdots \\ 0 & 1 & 0 \\ 0 \cdots 0 & & 1 \end{pmatrix}$, $(r \geqq 0)$, we write briefly $f(r) = f(g_r)$. Then for any $f \in C_c^\infty(K\backslash G/K)$ f is uniquely determined by $f(r)$, $(r \geqq 0)$. Let $C^\infty(K\backslash \hat{H})$ be the set of all complex valued K-invariant functions on $\hat{H}$ which are infinitely differentiable. If $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, we write $F(\xi) = F(a)$ for $F \in C^\infty(K\backslash \hat{H})$. It is obvious that $\mathcal{BF} f \in C^\infty(K\backslash \hat{H})$ for $f \in C_c^\infty(K\backslash G/K)$. Moreover we have

$$(\mathcal{BF} f)(a) = \int_{R_+} f(r)\frac{(ar)^{(n-2)/2}}{J_{(n-2)/2}(ar)} r^{n-1} dr \quad (a>0)\,.$$

Since for $f \in C_c^\infty(K\backslash G/K)$

$$(\mathcal{BF} f)(\xi) = \int_{H\times K} f\begin{pmatrix} 1 & u^{-1}y \\ 0 & 1 \end{pmatrix} e^{i<\xi,y>} dy du$$
$$= \int_{H\times K} f\left(\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right) e^{i<\xi,y>} dy du$$
$$= \int_H f\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} e^{i<\xi,y>} dy\,,$$

we have

$$f\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \int_{\hat{H}} (\mathcal{BF} f)(\xi) e^{-i<\xi,y>} d\xi$$
$$= \int_{\hat{H}} (\mathcal{BF} f)(\xi)\phi_{-\xi}(y) d\xi\,.$$

On the other hand we remark that $\phi_{-\xi}(x) = \phi_\xi(x)$ for any $\xi \in \hat{H}$ and $x \in H$. Hence we have the following inversion formula

$$f(r) = \int_{R_+} (\mathcal{BF} f)(a)\frac{J_{(n-2)/2}(ar)}{(ar)^{(n-2)/2}} a^{n-1} da$$

Then we can easily prove the following analogue of the Paley-Wiener theorem for the Fourier-Bessel transform.

Theorem 2. *A function F on $\hat{H}$ is the Fourier-Bessel transform of $f \in C_c^\infty(K\backslash G/K)$ such that $r_f \leqq a$ $(a>0)$ if and only if it satisfies the following conditions:*

(I) *F can be extended to an entire analytic function on $\hat{H}^c$.*

(II) *For any K-invariant polynomial function p of $\hat{H}^c$ there exists a constant C_p such that*

$$|p(\zeta)F(\zeta)| \leqq C_p \exp a|Im\zeta| \quad (\zeta \in \hat{H}^c).$$

(III) *For any $k \in K$*

$$F(k\zeta) = F(\zeta) \qquad (\zeta \in \hat{H}).$$

REMARK. In case $n=2$, we have

$$(\mathscr{B}\mathscr{F}f)(a) = \int_0^\infty f(r)J_0(ar)rdr.$$

This is the classical Fourier-Bessel transform [8].

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