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## AN ANALOGUE OF THE PALEY-WIENER THEOREM FOR THE EUCLIDEAN MOTION GROUP

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### 1. Introduction

The purpose of this paper is to give a detailed proof of an analogue of the Paley-Wiener theorem for the euclidean motion group which was announced in [3]. Restricting our attention to bi-invariant functions (with respect to the rotation group) we obtain an analogue of the Paley-Wiener theorem for the Fourier-Bessel transform.

### 2. Unitary representations

Let  $G$  be the group of all motions of the  $n$ -dimensional euclidean space  $\mathbf{R}^n$ . Then  $G$  is realized as the group of  $(n+1) \times (n+1)$ -matrices of the form  $\begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix}$ , ( $k \in SO(n)$ ,  $x \in \mathbf{R}^n$ ). Let  $K$  and  $H$  be the closed subgroups consisting of the elements  $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ , ( $k \in SO(n)$ ) and  $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ , ( $x \in \mathbf{R}^n$ ), respectively. Then  $G$  is the semi-direct product of  $H$  and  $K$ . We normalize the Haar measure  $dg$  on  $G$  such that  $dg = dx dk$ , where  $dx = (2\pi)^{-n/2} dx_1 \cdots dx_n$  and  $dk$  is the normalized Haar measure on  $K$ .

For any subgroup  $G_1$  of  $G$  we denote by  $\hat{G}_1$  the set of all equivalence classes of irreducible unitary representations of  $G_1$ . For an irreducible unitary representation  $\sigma$  of  $G_1$ , we denote by  $[\sigma]$  the equivalence class which contains  $\sigma$ . For simplicity we identify  $k \in SO(n)$  with  $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \in K$  and  $x \in \mathbf{R}^n$  with  $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in H$ . Denote by  $\langle, \rangle$  the euclidean inner product on  $\mathbf{R}^n$ . Then we can identify  $\hat{H}$  with  $\mathbf{R}^n$  so that the value of  $\xi \in \hat{H}$  at  $x \in H$  is  $e^{i\langle \xi, x \rangle}$ . Because  $H$  is normal,  $K$  acts on  $H$  and therefore on  $\hat{H}$  naturally:  $\langle k\xi, x \rangle = \langle \xi, k^{-1}x \rangle$ . Let  $K_\xi$  be the isotropy subgroup of  $K$  at  $\xi \in \hat{H}$ . If  $\xi \neq 0$ ,  $K_\xi$  is isomorphic to  $SO(n-1)$ .

The dual space  $\hat{G}$  of  $G$  was completely determined by G. W. Mackey [4] and S. Itô [2] as follows.

Let  $\mathfrak{H} = L_2(K)$  be the Hilbert space of all square integrable functions on  $K$ . We denote by  $U^\xi$  the unitary representation of  $G$  induced by  $\xi \in \hat{H}$ . Then for

$$g = \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \in G$$

$$(U_g^\xi F)(u) = e^{i\langle \xi, u^{-1}x \rangle} F(k^{-1}u), \quad (F \in \mathfrak{H}, u \in K).$$

Let  $\chi_\sigma$  and  $d_\sigma$  be the character and the degree of  $[\sigma] \in \hat{K}_\xi$ , respectively. Let  $L$  and  $R$  be the left and right regular representations of  $K$ , respectively. We also denote by  $L$  and  $R$  the corresponding representations of the universal enveloping algebra of the Lie algebra of  $K$  defined on  $C^\infty(K)$ , respectively. If  $\sigma(m) = (\sigma_{pq}(m)) (1 \leq p, q \leq d_\sigma)$ , we put

$$P^\sigma = d_\sigma \int_{K_\xi} \overline{\chi_\sigma(m)} R_m d_\xi m$$

and

$$P_q^\sigma = d_\sigma \int_{K_\xi} \overline{\sigma_{qq}(m)} R_m d_\xi m,$$

where  $d_\xi m$  is the normalized Haar measure on  $K_\xi$ . Then  $P^\sigma$  and  $P_q^\sigma$  are both orthogonal projections of  $\mathfrak{H}$ . Put  $\mathfrak{H}^\sigma = P^\sigma \mathfrak{H}$  and  $\mathfrak{H}_q^\sigma = P_q^\sigma \mathfrak{H}$ . The subspaces  $\mathfrak{H}_q^\sigma$  ( $1 \leq q \leq d_\sigma$ ) are stable under  $U^\xi$  and the representations of  $G$  induced on  $\mathfrak{H}_q^\sigma$  ( $1 \leq q \leq d_\sigma$ ) under  $U^\xi$  are equivalent for all  $q=1, \dots, d_\sigma$ . We fix one of them and denote by  $U^{\xi, \sigma}$ . It is easy to see that

$$U_g^{\xi, \sigma} = R_k U_\xi^\sigma R_k^{-1} (k \in K, \xi \in \hat{H}, g \in G). \quad (2.1)$$

Two representations  $U^{\xi, \sigma}$  and  $U^{\xi', \sigma'}$  are equivalent if and only if there exists an element  $k \in K$  such that  $\xi' = k\xi$  and  $[\sigma] = [\sigma'^k]$ , where

$$\sigma'^k(m) = \sigma'(kmk^{-1}), \quad (m \in K_\xi).$$

First we assume that  $\xi \neq 0$ . Then  $U^{\xi, \sigma}$  is irreducible and every infinite dimensional irreducible unitary representation is equivalent to one of  $U^{\xi, \sigma}$ ,

( $\xi \neq 0, [\sigma] \in \hat{K}_\xi$ ). Since  $\mathfrak{H} = \bigoplus_{[\sigma] \in \hat{K}_\xi} \mathfrak{H}^\sigma$  and  $\mathfrak{H}^\sigma = \bigoplus_{q=1}^{d_\sigma} \mathfrak{H}_q^\sigma$ , we have

$$U^\xi \cong \bigoplus_{[\sigma] \in \hat{K}_\xi} \underbrace{(U^{\xi, \sigma} \oplus \dots \oplus U^{\xi, \sigma})}_{d_\sigma \text{ times}}. \quad (2.2)$$

Next we assume that  $\xi = 0$ . Then  $U^{\xi, \sigma}$  is reducible and  $K_\xi = K$ . For any  $[\sigma] \in \hat{K}$  we define a finite dimensional unitary representation  $U^\sigma$  of  $G$  by  $U_g^\sigma = \sigma(k)$ , where  $g = \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \in G$ . Then we have  $U^{0, \sigma} \cong \underbrace{U^\sigma \oplus \dots \oplus U^\sigma}_{d_\sigma \text{ times}}$  and  $U^0 \cong \bigoplus_{[\sigma] \in \hat{K}} U^{0, \sigma}$ .

Moreover every finite dimensional irreducible unitary representation of  $G$  is equivalent to one of  $U^\sigma$ , ( $[\sigma] \in \hat{K}$ ).

We denote by  $(\hat{G})_\infty$  and  $(\hat{G})_0$  the set of all equivalence classes of infinite and

finite dimensional irreducible unitary representations of  $G$ , respectively.

### 3. The Plancherel formula

Let  $\mathfrak{k}$  be the Lie algebra of  $K$ . We denote by  $\Delta$  the Casimir operator of  $K$  (In case  $n=2$ , we put  $\Delta = -X^2$  for a non-zero  $X \in \mathfrak{k}$ ). By the Peter-Weyl theorem we can choose a complete orthonormal basis  $\{\phi_j\}_{j \in J}$  of  $\mathfrak{H}$ , consisting of the matricial elements of irreducible unitary representations of  $K$ , that is,  $\phi_j = d_\tau^{1/2} \tau_{pq}$  for some  $[\tau] \in \hat{K}$  ( $\tau = (\tau_{pq})$ ) and  $p, q = 1, \dots, d_\tau$ . First, we prove the following

**Lemma 1.** *Let  $T$  be a bounded operator on  $\mathfrak{H} = L_2(K)$  which leaves the space  $C^\infty(K)$  stable. If for any non-negative integers  $l$  and  $m$ , there exists a constant  $C^{l,m}$  such that*

$$||\Delta^l T \Delta^m|| \leq C^{l,m},$$

*then the series  $\sum_{i,j \in J} |(T\phi_j, \phi_i)|$  converges.*

**Proof.** For the sake of brevity we assume that  $n \geq 3$ . In case  $n=2$  the same method is valid with a slight modification. Let  $\mathfrak{t}$  be a Cartan subalgebra of  $\mathfrak{k}$ . Denote by  $\mathfrak{k}^c$  and  $\mathfrak{t}^c$  the complexifications of  $\mathfrak{k}$  and  $\mathfrak{t}$ , respectively. Fix an order in the dual space of  $(-1)^{1/2}\mathfrak{t}$ . Let  $P$  be the positive root system of  $\mathfrak{k}^c$  with respect to  $\mathfrak{t}^c$ . Let  $\mathcal{F}$  be the set of all dominant integral forms. Then  $\Lambda \in \mathcal{F}$  is the highest weight of some irreducible unitary representation of  $K$  if and only if it is lifted to a unitary character of the Cartan subgroup corresponding to  $\mathfrak{t}$ . Let  $\mathcal{F}_0$  be the set of all such  $\Lambda$ 's. For any  $\Lambda \in \mathcal{F}_0$  we denote by  $\tau_\Lambda$  a representative of  $[\tau_\Lambda] \in \hat{K}$  which is a matricial representation of  $K$  with the highest weight  $\Lambda$ . Then the mapping  $\Lambda \mapsto [\tau_\Lambda]$  gives the bijection between  $\mathcal{F}_0$  and  $\hat{K}$ . Let  $d_\Lambda$  be the degree of  $\tau_\Lambda$ . Denote by  $J_\Lambda$  be the set of  $j \in J$  such that  $\phi_j = d_\Lambda^{1/2} (\tau_\Lambda)_{pq}$  for some  $p, q = 1, \dots, d_\Lambda$ . Let  $(,)$  be the inner product on the dual space of  $(-1)^{1/2}\mathfrak{t}$  induced by the Killing form and put  $|\Lambda| = (\Lambda, \Lambda)^{1/2}$ . As usual we put  $\rho = \frac{1}{2} \sum_{\alpha \in P} \alpha$ . We use the following known facts (i)~(iii):

- (i) For every  $\Lambda \in \mathcal{F}_0$  and  $j \in J_\Lambda$ , we have  $(\Delta + |\rho|^2)\phi_j = |\Delta + \rho|^2\phi_j$ .
- (ii) For every  $\Lambda \in \mathcal{F}_0$ ,  $d_\Lambda = \frac{\prod_{\alpha \in P} (\Lambda + \rho, \alpha)}{\prod_{\alpha \in P} (\rho, \alpha)}$ , (Weyl's dimension formula).
- (iii) The Dirichlet series  $\sum_{\Lambda \in \mathcal{F}_0} \frac{1}{|\Lambda + \rho|^s}$  converges if  $s > \left[\frac{n}{2}\right]$ .

(see [1(a)] and [9])

By (i)

$$\phi_j = \frac{(\Delta + |\rho|^2)^l}{|\Lambda + \rho|^{2l}} \phi_j \text{ for } j \in J_\Lambda \text{ and } l = 0, 1, 2, \dots,$$

Therefore

$$\begin{aligned} \sum_{j \in J_\Lambda} \sum_{i \in J_{\Lambda'}} |(T\phi_j, \phi_i)| &= \frac{1}{|\Lambda + \rho|^{2l} |\Lambda' + \rho|^{2m}} \sum_{j \in J_\Lambda} \sum_{i \in J_{\Lambda'}} |(T(\Delta + |\rho|^2)^l \phi_j, \\ &(\Delta + |\rho|^2)^m \phi_i)| = \frac{1}{|\Lambda + \rho|^{2l} |\Lambda' + \rho|^{2m}} \sum_{j \in J_\Lambda} \sum_{i \in J_{\Lambda'}} |((\Delta + |\rho|^2)^m T(\Delta + |\rho|^2)^l \phi_j, \phi_i)|. \end{aligned}$$

On the other hand by the assumption of the lemma we can prove that there exists a constant  $C_1^{l,m}$  such that

$$| |(\Delta + |\rho|^2)^m T(\Delta + |\rho|^2)^l| | \leq C_1^{l,m}$$

Then

$$\begin{aligned} \sum_{j \in J_\Lambda} \sum_{i \in J_{\Lambda'}} |(T\phi_j, \phi_i)| &\leq \frac{C_1^{l,m}}{|\Lambda + \rho|^{2l} |\Lambda' + \rho|^{2m}} (d_\Lambda)^2 (d_{\Lambda'})^2 \\ &= C_1^{l,m} \frac{1}{|\Lambda + \rho|^{2l} |\Lambda' + \rho|^{2m}} \frac{\prod_{\alpha \in P} (\Lambda + \rho, \alpha)^2 (\Lambda' + \rho, \alpha)^2}{\prod_{\alpha \in P} (\rho, \alpha)^4} \\ &\leq C_1^{l,m} \frac{\prod_{\alpha \in P} (\alpha, \alpha)^2}{\prod_{\alpha \in P} (\rho, \alpha)^4} \cdot \frac{1}{|\Lambda + \rho|^{2l - n(n-1)/2 + [n/2]} |\Lambda' + \rho|^{2m - n(n-1)/2 + [n/2]}}. \end{aligned} \quad (3.1)$$

Therefore if put  $l=m$ , we have

$$\sum_{i,j \in J} |(T\phi_j, \phi_i)| \leq C_1^{l,l} \frac{\prod_{\alpha \in P} (\alpha, \alpha)^2}{\prod_{\alpha \in P} (\rho, \alpha)^4} \left( \sum_{\Lambda \in \mathcal{G}_0} \frac{1}{|\Lambda + \rho|^{2l - n(n-1)/2 + [n/2]}} \right)^2. \quad (3.2)$$

If we take  $l=m > \frac{1}{2} \frac{n(n-1)}{2} = \frac{1}{2} \dim K$ , using the property (iii) we obtain

$$\sum_{i,j \in J} |(T\phi_j, \phi_i)| < +\infty.$$

q.e.d.

**Corollary.** *If  $T$  is an operator on  $\mathfrak{H}$  satisfying the conditions of Lemma 1,  $T$  is of the trace class.*

For the proof of this corollary, see Harish-Chandra [1(a), Lemma 1].

For any  $f \in C_c^\infty(G)$ . We put

$$T_f(\xi, \sigma) = \int_G f(g) U_g^{\xi, \sigma} dg \quad (\xi \neq 0, [\sigma] \in \hat{K}_\xi).$$

Then

$$(T_f(\xi, \sigma) F)(u) = \int_K K_f(\xi, \sigma; u, v) F(v) dv \quad (u \in K),$$

where

$$K_f(\xi, \sigma; u, v) = d_\sigma \int_{K_\xi} \overline{\sigma_{qq}(m)} d_\xi m \int_H f \begin{pmatrix} umv^{-1} & x \\ 0 & 1 \end{pmatrix} e^{i\langle \xi, u^{-1}x \rangle} dx.$$

It is easy to see that  $T_f(\xi, \sigma)F \in C^\infty(K)$  for any  $f \in C_c^\infty(G)$  and  $F \in C^\infty(K)$ .

We denote by  $\lambda$  and  $\mu$  the left and right regular representations of  $G$ , respectively. We also denote by  $\lambda$  and  $\mu$  the corresponding representations of the universal enveloping algebra of  $G$  defined on  $C^\infty(G)$ . We regard each element  $X \in \mathfrak{k}$  as a right invariant vector field on  $K$ . So that we have  $L(X) = -X$ . Since

$$(T_f(\xi, \sigma)F(\exp(-tX))u) = (T_{\lambda(\exp tX)f}(\xi, \sigma)F)(u) \quad (t \in \mathbf{R}),$$

we have

$$((-X)T_f(\xi, \sigma)F)(u) = (T_{\lambda(X)f}(\xi, \sigma)F)(u)$$

for  $F \in C^\infty(K)$ . Therefore for any non-negative integer  $l$

$$\Delta^l T_f(\xi, \sigma) = T_{\lambda(\Delta)^l f}(\xi, \sigma).$$

Also we have

$$T_f(\xi, \sigma)\Delta^m = T_{\mu(\Delta)^m f}(\xi, \sigma) \quad (m = 0, 1, 2, \dots)$$

by a similar way. On the other hand we notice that

$$||T_f(\xi, \sigma)|| \leq \int_G |f(g)| dg.$$

Hence

$$||\Delta^l T_f(\xi, \sigma)\Delta^m|| \leq \int_G |(\lambda(\Delta)^l \mu(\Delta)^m f)(g)| dg.$$

Thus the operator  $T_f(\xi, \sigma)$ ,  $f \in C_c^\infty(G)$ , satisfies the assumptions of Lemma 1. By the corollary to Lemma 1,  $T_f(\xi, \sigma)$  is of the trace class.

As it can be easily seen that  $K_f(\xi, \sigma; u, v) \in C^\infty(K \times K)$ , we have

$$\text{Tr}(T_f(\xi, \sigma)) = \int_K K_f(\xi, \sigma; u, u) du$$

(see [1(b), Lemma 5]). Making use of the relation

$$d_\sigma \int_{K_\xi} \sigma_{qq}(m_1 m m_1^{-1}) d_\xi m_1 = \chi_\sigma(m),$$

we have the following proposition.

**Proposition 1.** *For any  $f \in C_c^\infty(G)$ ,  $T_f(\xi, \sigma)$  ( $\xi \neq 0, [\sigma] \in \hat{K}_\xi$ ) is of the trace class and*

$$\text{Tr}(T_f(\xi, \sigma)) = \int_{K_\xi} \overline{\chi_\sigma(m)} d_\xi m \int_{H \times K} f \begin{pmatrix} u & m & u^{-1} & x \\ & & & 1 \end{pmatrix} e^{i \langle \xi, u^{-1}x \rangle} dx du.$$

Let  $\mathbf{R}_+$  be the set of all positive numbers and let  $M$  be the subgroup consisting of the elements  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{pmatrix}$ , ( $m \in SO(n-1)$ ). Then for any  $\xi \in \hat{H}$  of the form  $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$  ( $a \in \mathbf{R}_+$ ), we have  $K_\xi = M$ . It follows from the results of §2 that  $(\hat{G})_\infty$  can be identified with  $\mathbf{R}_+ \times \hat{M}$ . For  $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$  ( $a \in \mathbf{R}_+$ ), we write briefly  $T_f(\xi, \sigma) = T_f(a, \sigma)$ . Then we have the following Plancherel formula for  $G$ .

**Proposition 2.** For any  $f \in C_c^\infty(G)$

$$\int_G |f(g)|^2 dg = \frac{2}{2^{n/2} \Gamma(n/2)} \sum_{[\sigma] \in \hat{M}} d_\sigma \int_{\mathbf{R}_+} \|T_f(a, \sigma)\|_2^2 a^{n-1} da,$$

where  $\|\cdot\|_2$  denotes the Hilbert-Schmidt norm.

*Proof.* It is enough to prove that

$$f \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{2}{2^{n/2} \Gamma(n/2)} \sum_{[\sigma] \in \hat{M}} d_\sigma \int_{\mathbf{R}_+} \text{Tr}(T_f(a, \sigma)) a^{n-1} da.$$

For any  $f \in C_c^\infty(G)$ , we put

$$T_f(\xi) = \int_G f(g) U_\xi^g dg \quad (\xi \in \hat{H}).$$

As above we write  $T_f(\xi) = T_f(a)$  for  $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$  ( $a \in \mathbf{R}_+$ ). Then by (2.2)

$$T_f(\xi) = \bigoplus_{[\sigma] \in \hat{K}_\xi} (T_f(\xi, \sigma) \oplus \cdots \oplus_{d_\sigma \text{ times}} T_f(\xi, \sigma)) \quad (\xi \neq 0).$$

Therefore

$$\text{Tr}(T_f(\xi)) = \sum_{\sigma \in \hat{K}_\xi} d_\sigma \text{Tr}(T_f(\xi, \sigma)).$$

Hence it is enough to prove that

$$f \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{2}{2^{n/2} \Gamma(n/2)} \int_{\mathbf{R}_+} \text{Tr}(T_f(a)) a^{n-1} da. \quad (3.3)$$

Since

$$\phi(m) = \int_{H \times K} f \begin{pmatrix} u & m & u^{-1} & x \\ & & & 1 \end{pmatrix} e^{i \langle \xi, u^{-1}x \rangle} dx du$$

is a central function on  $K_\xi$ ,

$$\phi(m) = \sum_{[\sigma] \in \hat{K}_\xi} \left( \int_{K_\xi} \phi(m_1) \overline{\chi_\sigma(m_1)} d_\xi m_1 \right) \chi_\sigma(m)$$

(see [7], §24). Hence by Proposition 1 we have

$$\begin{aligned} \phi(1) &= \sum_{[\sigma] \in \hat{K}_\xi} d_\sigma \int_{K_\xi} \phi(m) \overline{\chi_\sigma(m)} d_\xi m \\ &= \sum_{[\sigma] \in \hat{K}_\xi} d_\sigma \text{Tr}(T_f(\xi, \sigma)). \end{aligned}$$

Thus we have

$$\begin{aligned} \text{Tr}(T_f(\xi)) &= \phi(1) = \int_{H \times K} f \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} e^{i \langle \xi, u^{-1}x \rangle} dx du \\ &= \int_H \left\{ \int_K f \begin{pmatrix} 1 & ux \\ 0 & 1 \end{pmatrix} du \right\} e^{i \langle \xi, x \rangle} dx. \end{aligned}$$

Hence

$$\int_K f \begin{pmatrix} 1 & ux \\ 0 & 1 \end{pmatrix} du = \int_H \text{Tr}(T_f(\xi)) e^{-i \langle \xi, x \rangle} d\xi,$$

where  $d\xi = \frac{1}{(2\pi)^{n/2}} d\xi_1 \cdots d\xi_n$ . When  $x=0$ ,

$$f \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \int_H \text{Tr}(T_f(\xi)) d\xi \quad (3.4)$$

By (2.1) we have  $\text{Tr}(T_f(k\xi)) = \text{Tr}(R_k T_f(\xi) R_k^{-1}) = \text{Tr}(T_f(\xi))$ .

Hence  $\text{Tr}(T_f(\xi)) = \text{Tr}(T_f(|\xi|))$ . So that we have (3.3) from (3.4).

q.e.d.

Let  $\mathbf{B}(\mathfrak{H})$  be the Banach space of all bounded linear operators on  $\mathfrak{H}$ . We define the **Fourier transform** of  $f \in C_c^\infty(G)$  by the  $\mathbf{B}(\mathfrak{H})$ -valued function  $T_f$  on  $\hat{H}$ . In terms of this transform Proposition 2 becomes the following

**Corollary.** For any  $f \in C_c^\infty(G)$

$$\int_G |f(g)|^2 dg = \frac{2}{2^{n/2} \Gamma(n/2)} \int_{\mathbb{R}_+} \|T_f(a)\|_2^2 a^{n-1} da.$$

#### 4. The Fourier-Laplace transform

For each  $\zeta \in \hat{H}^c (\cong \mathbb{C}^n)$  we define a bounded representation of  $G$  on  $\mathfrak{H}$  by

$$(U_\zeta^\zeta F)(u) = e^{i \langle \zeta, u^{-1}x \rangle} F(k^{-1}u), \quad (F \in \mathfrak{H}, u \in K),$$

where  $g = \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \in G$ . For  $f \in C_c^\infty(G)$ , put

$$T_f(\zeta) = \int_G f(g) U_g^\zeta dg.$$

Then  $T_f$  is a  $B(\mathfrak{H})$ -valued function on  $\hat{H}^c$ . We shall call  $T_f$  the **Fourier-Laplace transform** of  $f$ .

Since  $K$  is compact, for each  $f \in C_c^\infty(G)$  there exists a positive number  $a$  such that  $\text{Supp}(f) \subset \left\{ \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \in G; |x| \leq a, k \in K \right\}$ , where  $\text{Supp}(f)$  denotes the support of  $f$ . We denote by  $r_f$  the greatest lower bound of such  $a$ 's. Throughout this section we assume that  $r_f \leq a$  for a fixed  $a \in \mathbf{R}_+$ .

**Lemma 2.** *There exists a constant  $C \geq 0$  depending only on  $f$  such that  $||T_f(\zeta)|| \leq C \exp a |\text{Im } \zeta|$ .*

*Proof.* Making use of the Schwarz's inequality we have

$$\begin{aligned} ||T_f(\zeta) F||^2 &\leq \int_K \left\{ \int_{H \times K} \left| f \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \right| e^{-\langle \text{Im } \zeta, u^{-1}x \rangle} |F(k^{-1}u)| dx dk \right\}^2 du \\ &\leq e^{2a |\text{Im } \zeta|} \int_K \left\{ \int_K \left( \int_H \left| f \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \right| dx \right) |F(k^{-1}u)| dk \right\}^2 du \\ &\leq e^{2a |\text{Im } \zeta|} \int_K \left\{ \int_K \left( \int_H \left| f \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \right| dx \right)^2 dk \int_K |F(k^{-1}u)|^2 dk \right\} du \\ &= e^{2a |\text{Im } \zeta|} \int_K \left( \int_H \left| f \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \right| dx \right)^2 dk ||F||^2 \end{aligned}$$

for any  $F \in \mathfrak{H}$ . Therefore it is enough to put

$$C = \left\{ \int_K \left( \int_H \left| f \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \right| dx \right)^2 dk \right\}^{1/2}.$$

q.e.d.

**Lemma 3.** *The  $B(\mathfrak{H})$ -valued function  $T_f$  on  $\hat{H}^c$  is entire analytic.*

*Proof.* For any  $n$ -tuple  $(m_1, \dots, m_n)$  of non-negative integers we define a bounded operator  $T_f^{m_1 \dots m_n}$  by

$$(T_f^{m_1 \dots m_n} F)(u) = \int_{H \times K} f \begin{pmatrix} k & ux \\ 0 & 1 \end{pmatrix} x_1^{m_1} \dots x_n^{m_n} F(k^{-1}u) dx dk,$$

where  $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ . Then we have

$$||T_f^{m_1 \dots m_n}|| \leq a^{m_1 + \dots + m_n} \left\{ \int_K \left( \int_H \left| f \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \right| dx \right)^2 dk \right\}^{1/2}.$$

Hence for any fixed  $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n$  the series

$$\sum_{m=0}^{\infty} i^m \sum_{m_1 + \dots + m_n = m} \frac{m!}{m_1! \dots m_n!} T_f^{m_1 \dots m_n} \zeta_1^{m_1} \dots \zeta_n^{m_n}$$

converges in  $B(\mathfrak{H})$ -norm. It is easy to see that this series is equal to  $T_f(\zeta)$ .

q.e.d.

For any polynomial function  $p$  on  $\hat{H}^c$ , we define a differential operator  $p(D)$  on  $H$  by  $p(D) = p\left(\frac{1}{i} \frac{\partial}{\partial x_1}, \dots, \frac{1}{i} \frac{\partial}{\partial x_n}\right)$ . A polynomial function  $p$  on  $\hat{H}^c$  is called  $K$ -invariant if  $p(k\zeta) = p(\zeta)$  for any  $k \in K$  and  $\zeta \in \hat{H}^c$ . As is easily seen,  $T_f(\zeta)$  leaves the space  $C^\infty(K)$  stable.

**Lemma 4.** 1) For any non-negative integers  $l$  and  $m$  we have  $\Delta^l T_f(\zeta) \Delta^m = T_{\lambda(\Delta^l) \mu(\Delta^m) f}(\zeta)$ , ( $\zeta \in \hat{H}^c$ ).  
2) For any  $K$ -invariant polynomial function  $p$  on  $\hat{H}^c$ , we have  $p(\zeta) T_f(\zeta) = T_{p^*(D) f}(\zeta)$ , ( $\zeta \in \hat{H}^c$ ), where  $p^*(\zeta) = p(-\zeta)$ .

The statement 1) can be proved by a similar way mentioned in §3. The statement 2) is easily proved, using the fact  $\frac{\partial}{\partial x_j} e^{i\langle \zeta, x \rangle} = i\zeta_j e^{i\langle \zeta, x \rangle}$  and the integration by parts. From Lemma 2 and Lemma 4 we have the following

**Proposition 3.** For any  $K$ -invariant polynomial function  $p$  on  $\hat{H}^c$  and for any non-negative integers  $l$  and  $m$ , there exists a constant  $C_p^{l,m}$  such that

$$||p(\zeta) \Delta^l T_f(\zeta) \Delta^m|| \leq C_p^{l,m} \exp a |Im \zeta|.$$

Finally from the definition of  $T_f$  we have the following functional equations for  $T_f$ .

**Proposition 4.**  $T_f(k\zeta) = R_k T_f(\zeta) R_k^{-1}$  ( $\zeta \in \hat{H}^c$ ,  $k \in K$ ).

## 5. The analogue of the Paley-Wiener theorem

**Theorem 1.** A  $B(\mathfrak{H})$ -valued function  $T$  on  $\hat{H}$  is the Fourier transform of  $f \in C_c^\infty(G)$  such that  $r_f \leq a$  ( $a > 0$ ) if and only if it satisfies the following conditions:

- (I)  $T$  can be extended to an entire analytic function on  $\hat{H}^c$ .
- (II) For any  $\zeta \in \hat{H}^c$ ,  $T(\zeta)$  leaves the space  $C^\infty(K)$  stable. Moreover for any  $K$ -invariant polynomial function  $p$  on  $\hat{H}^c$  and for any non-negative integers  $l$  and  $m$ , there exists a constant  $C_p^{l,m}$  such that

$$||p(\zeta) \Delta^l T(\zeta) \Delta^m|| \leq C_p^{l,m} \exp a |Im \zeta|.$$

- (III) For any  $k \in K$

$$T(k\zeta) = R_k T(\zeta) R_k^{-1} \quad (\zeta \in \hat{H}^c).$$

**Proof.** We have already proved the necessity of the theorem in §4. In the following we shall prove the sufficiency of the theorem.

Let  $T$  be an arbitrary  $B(\mathfrak{H})$ -valued function on  $\hat{H}$  satisfying the conditions (I)~(III) in the theorem. Let  $\{\phi_j\}_{j \in J}$  be the complete orthonormal basis of  $\mathfrak{H}$

which we have chosen in §3. If  $|Im\zeta| \leq b(b > 0)$ , by the condition (II) for any non-negative integers  $l$  and  $m$  there exists a constant  $C_1^{l,m}$  such that

$$||\Delta^l T(\zeta) \Delta^m|| \leq C_1^{l,m} \exp ab.$$

Therefore by Lemma 1 the series

$$\sum_{i,j \in J} |(T(\zeta) \phi_j, \phi_i)|$$

converges and  $T(\zeta)$  is of the trace class. We assume that  $n \geq 3$ . If  $\phi_j = d_\Lambda^{1/2}(\tau_\Lambda)_{pq}$ , we have  $|\phi_j(u)| \leq d_\Lambda^{1/2}$  because  $|\tau_\Lambda(u)_{pq}| \leq 1$ . So we have

$$\sum_{j \in J_\Lambda} \sum_{i \in J_{\Lambda'}} |(T(\zeta) \phi_j, \phi_i) \phi_i(u) \overline{\phi_j(v)}| \leq \frac{C_1^{l,m} e^{ab}}{|\Lambda + \rho|^{2l} |\Lambda' + \rho|^{2m}} (d_\Lambda)^{3/2} (d_{\Lambda'})^{3/2}.$$

Hence

$$\begin{aligned} & \sum_{i,j \in J} |(T(\zeta) \phi_j, \phi_i) \phi_i(u) \overline{\phi_j(v)}| \\ & \leq C_1^{l,m} e^{ab} \frac{\prod_{\alpha \in P} (\alpha, \alpha)^3}{\prod_{\alpha \in P} (\rho, \alpha)^5} \left( \sum_{\Lambda \in \mathcal{F}_0} \frac{1}{|\Lambda + \rho|^{2l - 3(n-1)/2 - [n/2]/2}} \right)^2 < +\infty \end{aligned} \quad (5.1)$$

for  $2l > \frac{3}{2} \frac{n(n-1)}{2} - \frac{1}{2} \left[ \frac{n}{2} \right]$ . In case  $n=2$ ,  $|\phi_j| = 1$  for all  $j \in J$ . Therefore

$$\sum_{i,j \in J} |(T(\zeta) \phi_j, \phi_i) \phi_i(u) \overline{\phi_j(v)}| = \sum_{i,j \in J} |(T(\zeta) \phi_j, \phi_i)| < +\infty.$$

Now let us define the kernel function of  $T(\zeta)$  ( $\zeta \in \hat{H}^c$ ) by

$$K(\zeta; u, v) = \sum_{i,j \in J} (T(\zeta) \phi_j, \phi_i) \Phi_i(u) \overline{\phi_j(v)}. \quad (5.2)$$

By the facts stated above and the property (I) it is easy to see that for any  $\zeta \in \hat{H}^c$  the right hand side of (5.2) is absolutely convergent and that it is uniformly convergent on every compact subset of  $\hat{H}^c \times K \times K$ . Thus we have the following

**Lemma 5.** *The function  $\hat{H}^c \times K \times K \ni (\zeta, u, v) \rightarrow K(\zeta; u, v)$  is of class  $C^\infty$  and entire analytic with respect to  $\zeta$ .*

If we adopt the formula (5.1) to  $p(\zeta)T(\zeta)$  instead of  $T(\zeta)$ , we have the following lemma by making use of (II).

**Lemma 6.** *For any  $K$ -invariant polynomial function  $p$  on  $\hat{H}^c$ , there exists a constant  $C_p$  such that*

$$|p(\zeta)K(\zeta; u, v)| \leq C_p \exp a |Im\zeta|, (\zeta \in \hat{H}^c, u, v \in K).$$

REMARK.  $K(\zeta; u, v)$  is rapidly decreasing on the real axis  $\hat{H}$ .

Let us define a function  $f$  on  $G$  by the inversion formula corresponding to

the Fourier transform, i.e.

$$f(g) = \frac{2}{2^{n/2}\Gamma(n/2)} \int_{R_+} \text{Tr}(T(a)U_g^{a_{-1}})a^{n-1}da.$$

By the property (III) we have

$$\begin{aligned} & (T(k\xi)\phi_j, \phi_i)\phi_i(u)\overline{\phi_j(v)} \\ &= (R_k T(\xi)R_k^{-1}\phi_j, \phi_i)\phi_i(u)\overline{\phi_j(v)} = (T(\xi)R_k^{-1}\phi_j, R_k^{-1}\phi_i)\phi_i(u)\overline{\phi_j(v)}. \end{aligned}$$

Let  $\phi_j = d_\tau^{1/2}\tau_{pq}$  and  $\phi_i = d_\sigma^{1/2}\sigma_{rs}([\tau], [\sigma] \in \hat{K})$ . Then

$$R_k^{-1}\phi_j(w) = d_\tau^{1/2}\tau_{pq}(wk^{-1}) = d_\tau^{1/2} \sum_{l=1}^{d_\tau} \tau_{pl}(w) \overline{\tau_{ql}(k)}$$

and

$$R_k^{-1}\phi_i(w) = d_\sigma^{1/2} \sum_{m=1}^{d_\sigma} \sigma_{rm}(w) \overline{\sigma_{sm}(k)}.$$

Therefore

$$\begin{aligned} & (T(\xi)R_k^{-1}\phi_j, R_k^{-1}\phi_i)\phi_i(u)\overline{\phi_j(v)} \\ &= \sum_{l=1}^{d_\tau} \sum_{m=1}^{d_\sigma} (T(\xi)d_\tau^{1/2}\tau_{pl}, d_\sigma^{1/2}\sigma_{rm})d_\sigma^{1/2}\sigma_{rs}(u)\sigma_{sm}(k)d_\tau^{1/2} \overline{\tau_{pq}(v)\tau_{ql}(k)}. \end{aligned}$$

Hence

$$\begin{aligned} & \sum_{p,q=1}^{d_\tau} \sum_{r,s=1}^{d_\sigma} (T(k\xi)d_\tau^{1/2}\tau_{pq}, d_\sigma^{1/2}\sigma_{rs})d_\sigma^{1/2}\sigma_{rs}(u)d_\tau^{1/2} \overline{\tau_{pq}(v)} \\ &= \sum_{p,l=1}^{d_\tau} \sum_{r,m=1}^{d_\sigma} (T(\xi)d_\tau^{1/2}\tau_{pl}, d_\sigma^{1/2}\sigma_{rm}) \sum_{s=1}^{d_\sigma} d_\sigma^{1/2}\sigma_{rs}(u)\sigma_{sm}(k) \times \sum_{q=1}^{d_\tau} d_\tau^{1/2} \overline{\tau_{pq}(v)\tau_{ql}(k)} \\ &= \sum_{p,l=1}^{d_\tau} \sum_{r,m=1}^{d_\sigma} (T(\xi)d_\tau^{1/2}\tau_{pl}, d_\sigma^{1/2}\sigma_{rm})d_\sigma^{1/2}\sigma_{rm}(uk)d_\tau^{1/2} \tau_{pl}(vk). \end{aligned}$$

Since  $K(\xi; u, v) = \sum_{[\sigma], [\tau] \in \hat{K}} \sum_{p,q=1}^{d_\tau} \sum_{r,s=1}^{d_\sigma} (T(\xi)d_\tau^{1/2}\tau_{pq}, d_\sigma^{1/2}\sigma_{rs})d_\sigma^{1/2}\sigma_{rs}(u) \times d_\tau^{1/2} \overline{\tau_{pq}(v)}$ ,

we have the following functional equation for  $K(\xi; u, v)$ :

$$K(k\xi; u, v) = K(\xi; uk, vk). \quad (5.3)$$

On the other hand

$$\text{Tr}(T(k\xi)U_g^{k\xi_{-1}}) = \text{Tr}(R_k T(\xi)R_k^{-1}U_g^{\xi_{-1}}) = \text{Tr}(T(\xi)U_g^{\xi_{-1}}), (\xi \in \hat{H}).$$

Hence

$$\frac{2}{2^{n/2}\Gamma(2/n)} \int_{R_+} \text{Tr}(T(a)U_g^{a_{-1}})a^{n-1}da = \int_H \text{Tr}(T(\xi)U_g^{\xi_{-1}})d\xi,$$

where  $d\xi = \frac{1}{(2\pi)^{n/2}} d\xi_1 \cdots d\xi_n$ . As  $T(\xi)F(u) = \int_K K(\xi; u, v)F(v)dv$  ( $F \in \mathfrak{F}$ )

and  $g^{-1} = \begin{pmatrix} k^{-1} & -k^{-1}x \\ 0 & 1 \end{pmatrix}$  for  $g = \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \in G$ , we have

$$U_g^{\xi^{-1}} T(\xi) F(u) = \int_K e^{-i\langle \xi, u^{-1}k^{-1}x \rangle} K(\xi; ku, v) F(v) dv.$$

Since  $T(\xi)$  is of the trace class, so is  $U_g^{\xi} T(\xi)$ . Moreover the function  $K \times K \ni (u, v) \mapsto e^{-i\langle \xi, u^{-1}k^{-1}x \rangle} K(\xi; ku, v)$  is clearly of class  $C^\infty$ . Hence

$$\text{Tr}(T(\xi) U_g^{\xi^{-1}}) = \text{Tr}(U_g^{\xi} T(\xi)) = \int_K e^{-i\langle \xi, u^{-1}k^{-1}x \rangle} K(\xi, ku, u) du.$$

Therefore the equation (5.3) and the remark to Lemma 6 imply that

$$\begin{aligned} \int_H \text{Tr}(T(\xi) U_g^{\xi^{-1}}) d\xi &= \int_H \int_K e^{-i\langle \xi, u^{-1}k^{-1}x \rangle} K(\xi; ku, u) du d\xi \\ &= \int_K \int_H e^{-i\langle \xi, x \rangle} K(u^{-1}k^{-1}\xi; ku, u) d\xi du \\ &= \int_K \int_H e^{-i\langle \xi, x \rangle} K(\xi; 1, k^{-1}) d\xi du \\ &= \int_H e^{-i\langle \xi, x \rangle} K(\xi; 1, k^{-1}) d\xi. \end{aligned}$$

Thus we have

$$f \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} = \int_H e^{-i\langle \xi, x \rangle} K(\xi; 1, k^{-1}) d\xi, \quad (5.4)$$

( $k \in K, x \in H$ ). It follows from Lemma 5 and the remark to Lemma 6 that  $f$  is of class  $C^\infty$ . Making use of Lemma 6, it follows from the classical Paley-Wiener theorem that if  $|x| > a$ ,  $f \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} = 0$  for any  $k \in K$ .

Finally we have to check that  $T_f = T$ . Since

$$T_f(\xi) F(u) = \int_K K_f(\xi; u, v) F(v) dv$$

where

$$K_f(\xi; u, v) = \int_H f \begin{pmatrix} uv^{-1} & x \\ 0 & 1 \end{pmatrix} e^{i\langle \xi, u^{-1}x \rangle} dx,$$

so it is enough to prove that

$$K(\xi; u, v) = \int_H f \begin{pmatrix} uv^{-1} & x \\ 0 & 1 \end{pmatrix} e^{i\langle \xi, u^{-1}x \rangle} dx.$$

By the relation (5.4),

$$f \begin{pmatrix} uv^{-1} & x \\ 0 & 1 \end{pmatrix} = \int_H e^{-i\langle \xi, x \rangle} K(\xi; 1, vu^{-1}) d\xi$$

$$= \int_H e^{-i\langle \xi, x \rangle} K(u^{-1}\xi; u, v) d\xi.$$

Hence

$$K(u^{-1}\xi; u, v) = \int_H f \begin{pmatrix} uv^{-1} & x \\ 0 & 1 \end{pmatrix} e^{i\langle \xi, x \rangle} dx.$$

If we replace  $u\xi$  for  $\xi$ ,

$$\begin{aligned} K(\xi; u, v) &= \int_H f \begin{pmatrix} uv^{-1} & x \\ 1 & 0 \end{pmatrix} e^{i\langle u\xi, x \rangle} dx \\ &= \int_H f \begin{pmatrix} uv^{-1} & x \\ 0 & 1 \end{pmatrix} e^{i\langle \xi, u^{-1}x \rangle} dx. \end{aligned}$$

This completes the proof of the theorem.

## 6. The Fourier-Bessel transform

Let  $C_c^\infty(K \backslash G / K)$  be the set of all complex valued  $K$ -bi-invariant functions on  $G$  which are infinitely differentiable and with compact support. For  $f \in C_c^\infty(G)$ , put

$$(\mathcal{F}_\xi f)(g) = \int_H f \left( g \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right) e^{-i\langle \xi, y \rangle} dy$$

and

$$(\mathcal{P}f)(g) = \int_K f(gu) du.$$

For  $f \in C_c^\infty(K \backslash G / K)$  it is easy to see that

$$(\mathcal{P}\mathcal{F}_\xi f) \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \left( \int_H f \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \phi_\xi(y) dy \right) \phi_{-\xi}(x), \quad (6.1)$$

where

$$\phi_\xi(x) = \int_K e^{i\langle \xi, ux \rangle} du.$$

REMARK. The formula (6.1) is regarded as an analogue of the Poisson integral for semisimple Lie groups (see [5]). And the function  $\phi_\xi$  is the zonal spherical function.

Let us define the **Fourier-Bessel transform**  $\mathcal{BF}f$  of  $f \in C_c^\infty(K \backslash G / K)$  by

$$(\mathcal{BF}f)(\xi) = \int_H f \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \phi_\xi(y) dy.$$

If  $x = \begin{pmatrix} r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ , ( $r > 0$ ) and  $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ , ( $a > 0$ ), we can prove that

$$\begin{aligned}\phi_\xi(x) &= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi e^{i ar \cos \theta} \sin^{n-2} \theta d\theta \\ &= \Gamma\left(\frac{n}{2}\right) \frac{J_{(n-2)/2}(ar)}{\left(\frac{ar}{2}\right)^{(n-2)/2}}\end{aligned}$$

(see [8] for the notation of the Bessel function  $J_n(r)$ ).

If  $g_r = \begin{pmatrix} 1 & 0 & r \\ & \ddots & 0 \\ 0 & 1 & \vdots \\ & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$ , ( $r \geq 0$ ), we write briefly  $f(r) = f(g_r)$ . Then for any

$f \in C_c^\infty(K \backslash G / K)$   $f$  is uniquely determined by  $f(r)$ , ( $r \geq 0$ ). Let  $C^\infty(K \backslash \hat{H})$  be the set of all complex valued  $K$ -invariant functions on  $\hat{H}$  which are infinitely differentiable.

If  $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ , we write  $F(\xi) = F(a)$  for  $F \in C^\infty(K \backslash \hat{H})$ . It is obvious that  $\mathcal{BF}f \in C^\infty(K \backslash \hat{H})$  for  $f \in C_c^\infty(K \backslash G / K)$ . Moreover we have

$$(\mathcal{BF}f)(a) = \int_{R_+} f(r) \frac{(ar)^{(n-2)/2}}{J_{(n-2)/2}(ar)} r^{n-1} dr \quad (a > 0).$$

Since for  $f \in C_c^\infty(K \backslash G / K)$

$$\begin{aligned}(\mathcal{BF}f)(\xi) &= \int_{H \times K} f \begin{pmatrix} 1 & u^{-1}y \\ 0 & 1 \end{pmatrix} e^{i \langle \xi, y \rangle} dy du \\ &= \int_{H \times K} f \left( \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \right) e^{i \langle \xi, y \rangle} dy du \\ &= \int_H f \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} e^{i \langle \xi, y \rangle} dy,\end{aligned}$$

we have

$$\begin{aligned}f \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} &= \int_{\hat{H}} (\mathcal{BF}f)(\xi) e^{-i \langle \xi, y \rangle} d\xi \\ &= \int_{\hat{H}} (\mathcal{BF}f)(\xi) \phi_{-\xi}(y) d\xi.\end{aligned}$$

On the other hand we remark that  $\phi_{-\xi}(x) = \phi_\xi(x)$  for any  $\xi \in \hat{H}$  and  $x \in H$ . Hence we have the following inversion formula

$$f(r) = \int_{R_+} (\mathcal{BF}f)(a) \frac{J_{(n-2)/2}(ar)}{(ar)^{(n-2)/2}} a^{n-1} da$$

Then we can easily prove the following analogue of the Paley-Wiener theorem

for the Fourier-Bessel transform.

**Theorem 2.** *A function  $F$  on  $\hat{H}$  is the Fourier-Bessel transform of  $f \in C_c^\infty(K \backslash G/K)$  such that  $r_f \leq a$  ( $a > 0$ ) if and only if it satisfies the following conditions:*

(I)  *$F$  can be extended to an entire analytic function on  $\hat{H}^c$ .*

(II) *For any  $K$ -invariant polynomial function  $p$  of  $\hat{H}^c$  there exists a constant  $C_p$  such that*

$$|p(\zeta)F(\zeta)| \leq C_p \exp a |Im \zeta| \quad (\zeta \in \hat{H}^c).$$

(III) *For any  $k \in K$*

$$F(k\zeta) = F(\zeta) \quad (\zeta \in \hat{H}).$$

REMARK. In case  $n = 2$ , we have

$$(\mathcal{BF}f)(a) = \int_0^\infty f(r)J_0(ar)rdr.$$

This is the classical Fourier-Bessel transform [8].

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