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AN ANALOGUE OF THE PALEY-WIENER THEOREM FOR THE EUCLIDEAN MOTION GROUP

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1. Introduction

The purpose of this paper is to give a detailed proof of an analogue of the Paley-Wiener theorem for the euclidean motion group which was announced in [3]. Restricting our attention to bi-invariant functions (with respect to the rotation group) we obtain an analogue of the Paley-Wiener theorem for the Fourier -Bessel transform.

2. Unitary representations

Let G be the group of all motions of the *n*-dimensional euclidean space \mathbb{R}^n . Then G is realized as the group of $(n+1) \times (n+1)$ -matrices of the form $\binom{k \ x}{0 \ 1}$, $(k \in SO(n), x \in \mathbb{R}^n)$. Let K and H be the closed subgroups consisting of the elements $\binom{k \ 0}{0 \ 1}$, $(k \in SO(n))$ and $\binom{1 \ x}{0 \ 1}$, $(x \in \mathbb{R}^n)$, respectively. Then G is the semi-direct product of H and K. We normalize the Haar measure dg on G such that dg = dxdk, where $dx = (2\pi)^{-n/2}dx_1 \cdots dx_n$ and dk is the normalized Haar measure on K.

For any subgroup G_1 of G we denote by \hat{G}_1 the set of all equivalence classses of irreducible unitary representations of G_1 . For an irreducible unitary representation σ of G_1 , we denote by $[\sigma]$ the equivalence class which contains σ . For simplicity we identify $k \in SO(n)$ with $\binom{k \ 0}{0} = K$ and $x \in \mathbb{R}^n$ with $\binom{1 \ x}{0} = H$. Denote by \langle , \rangle the euclidean inner product on \mathbb{R}^n . Then we can identify \hat{H} with \mathbb{R}^n so that the value of $\xi \in \hat{H}$ at $x \in H$ is $e^{i \langle \xi, x \rangle}$. Because H is normal, K acts on H and therefore on \hat{H} naturally: $\langle k\xi, x \rangle = \langle \xi, k^{-1}x \rangle$. Let K_{ξ} be the isotropy subgroup of K at $\xi \in \hat{H}$. If $\xi \neq 0$, K_{ξ} is isomorphic to SO(n-1).

The dual space \hat{G} of G was completely determined by G. W. Mackey [4] and S. Itô [2] as follows.

Let $\mathfrak{D}=L_2(K)$ be the Hilbert space of all square integrable functions on K. We denote by $U^{\mathfrak{k}}$ the unitary representation of G induced by $\mathfrak{k} \in \hat{H}$. Then for

$$g = \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \in G$$
$$(U_{\mathfrak{g}}^{\mathfrak{k}}F)(u) = e^{i < \mathfrak{k} \cdot u^{-1}x > F(k^{-1}u)}, (F \in \mathfrak{H}, u \in K).$$

Let χ_{σ} and d_{σ} be the character and the degree of $[\sigma] \in \hat{K}_{\xi}$, respectively. Let L and R be the left and right regular representations of K, respectively. We also denote by L and R the corresponding representations of the universal enveloping algebra of the Lie algebra of K defined on $C^{\infty}(K)$, respectively. If $\sigma(m) = (\sigma_{pq}(m))(1 \le p, q \le d_{\sigma})$, we put

$$P^{\sigma} = d_{\sigma} \int_{K_{\xi}} \overline{\chi_{\sigma}(m)} R_{m} d_{\xi} m$$

and

$$P_q^{\sigma} = d_{\sigma} \int_{K_{\xi}} \overline{\sigma_{qq}(m)} R_m d_{\xi} m \, ,$$

where $d_{\xi}m$ is the normalized Haar measure on K_{ξ} . Then P^{σ} and P_{q}^{σ} are both orthogonal projections of §. Put $\mathfrak{F}^{\sigma}=P^{\sigma}\mathfrak{F}$ and $\mathfrak{F}_{q}^{\sigma}=P_{q}^{\sigma}\mathfrak{F}$. The subspaces $\mathfrak{F}_{q}^{\sigma}$ $(1 \leq q \leq d_{\sigma})$ are stable under U^{ξ} and the representations of G induced on $\mathfrak{F}_{q}^{\sigma}$ $(1 \leq q \leq d_{\sigma})$ under U^{ξ} are equivalent for all $q=1, \dots, d_{\sigma}$. We fix one of them and denote by $U^{\xi,\sigma}$. It is easy to see that

$$U_{\mathfrak{g}}^{k\xi} = R_{\mathfrak{g}} U_{\mathfrak{g}}^{\xi} R_{\mathfrak{g}}^{-1} (k \in K, \, \xi \in \hat{H}, \, g \in G) \,.$$

$$(2.1)$$

Two representations $U^{\xi,\sigma}$ and $U^{\xi',\sigma'}$ are equivalent if and only if there exists an element $k \in K$ such that $\xi' = k\xi$ and $[\sigma] = [\sigma'^k]$, where

$$\sigma'^{k}(m) = \sigma'(kmk^{-1}), (m \in K_{\xi}).$$

First we assume that $\xi \neq 0$. Then $U^{\xi,\sigma}$ is irreducible and every infinite dimensional irreducible unitary representation is equivalent to one of $U^{\xi,\sigma}$, $(\xi \neq 0, [\sigma] \in \hat{K}_{\xi})$. Since $\mathfrak{H} = \bigoplus_{[\sigma] \in \hat{K}_{\xi}} \mathfrak{H}^{\sigma}$ and $\mathfrak{H}^{\sigma} = \bigoplus_{q=1}^{d\sigma} \mathfrak{H}^{\sigma}_{q}$, we have $U^{\xi} \simeq \bigoplus_{[\sigma] \in \hat{K}_{\xi}} (\underbrace{U^{\xi,\sigma} \oplus \cdots \oplus U^{\xi,\sigma}}_{d_{\sigma} \text{ times}})$. (2.2)

Next we assume that $\xi = 0$. Then $U^{\xi,\sigma}$ is reducible and $K_{\xi} = K$. For any $[\sigma] \in \hat{K}$ we define a finite dimensional unitary representation U^{σ} of G by $U_{g}^{\sigma} = \sigma(k)$, where $g = \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \in G$. Then we have $U^{0,\sigma} \cong \underbrace{U^{\sigma} \oplus \cdots \oplus U^{\sigma}}_{d_{\sigma} \text{ times}}$ and $U^{0} \cong \bigoplus_{[\sigma] \in \hat{K}}$

 $U^{0,\sigma}$. Moreover every finite dimensional irreducible unitary representation of G is equivalent to one of U^{σ} , $([\sigma] \in \hat{K})$.

We denote by $(\hat{G})_{\infty}$ and $(\hat{G})_{0}$ the set of all equivalence classes of infinite and

finite dimensional irreducible unitary representations of G, respectively.

3. The Plancherel formula

Let \mathfrak{k} be the Lie algebra of K. We denote by Δ the Casimir operator of K (In case n=2, we put $\Delta = -X^2$ for a non-zero $X \in \mathfrak{k}$). By the Peter-Weyl theorem we can choose a complete orthonormal basis $\{\phi_j\}_{j\in J}$ of \mathfrak{H} , consisting of the matricial elements of irreducible unitary representations of K, that is, $\phi_j = d_{\tau}^{1/2} \tau_{pq}$ for some $[\tau] \in \hat{K} (\tau = (\tau_{pq}))$ and $p, q=1, \dots, d_{\tau}$. First, we prove the following

Lemma 1. Let T be a bounded operator on $\mathfrak{D}=L_2(K)$ which leaves the space $C^{\infty}(K)$ stable. If for any non-negative integers l and m, there exists a constant $C^{l,m}$ such that

$$||\Delta^{l}T\Delta^{m}|| \leq C^{l,m},$$

then the series $\sum_{i,j\in J} |(T\phi_j, \phi_i)|$ converges.

Proof. For the sake of brevity we assume that $n \ge 3$. In case n=2 the same method is valid with a slight modification. Let t be a Cartan subalgebra of t. Denote by t^c and t^c the complexifications of t and t, respectively. Fix an order in the dual space of $(-1)^{1/2}$ t. Let P be the positive root system of t^{e} with respect to t^c . Let \mathcal{F} be the set of all dominant integral forms. Then $\Lambda \in$ \mathcal{F} is the highest weight of some irreducible unitary representation of K if and only if it is lifted to a unitary character of the Cartan subgroup corresponding to t. Let \mathcal{F}_0 be the set of all such Λ 's. For any $\Lambda \in \mathcal{F}_0$ we doente by τ_{Λ} a representative of $[\tau_{\Lambda}] \in \hat{K}$ which is a matricial representation of K with the highest weight Λ . Then the mapping $\Lambda \mapsto [\tau_{\Lambda}]$ gives the bijection between \mathcal{F}_0 and \hat{K} . Let d_{Λ} be the degree of τ_{Λ} . Denote by J_{Λ} be the set of $j \in J$ such that $\phi_j = d_{\Lambda}^{1/2}$ $(\tau_{\Lambda})_{pq}$ for some $p, q=1, \dots, d_{\Lambda}$. Let (,) be the inner product on the dual space of $(-1)^{1/2}$ t induced by the Killing form and put $|\Lambda| = (\Lambda, \Lambda)^{1/2}$. As usual we put $\rho = \frac{1}{2} \sum_{n \in \mathbb{P}} \alpha$. We use the following known facts (i)~(iii): (i) For every $\Lambda \in \mathcal{F}_0$ and $j \in J_\Lambda$, we have $(\Delta + |\rho|^2)\phi_j = |\Delta + \rho|^2\phi_j$. (ii) For every $\Lambda \in \mathcal{F}_0$, $d_\Lambda = \frac{\prod_{\alpha \in P} (\Lambda + \rho, \alpha)}{\prod_{\alpha \in P} (\rho, \alpha)}$, (Weyl's dimension formula). The Dirichlet series $\sum_{\Lambda \in \mathcal{G}_n} \frac{1}{|\Lambda + \rho|^s}$ converges if $s > \left[\frac{n}{2}\right]$. (iii) (see [1(a)] and [9]) By (i) $\phi_j = \frac{(\Delta + |\rho|^2)^l}{|\Lambda + \rho|^{2l}} \phi_j \text{ for } j \in J_\Lambda \text{ and } l = 0, 1, 2, \cdots.$

Therefore

$$\sum_{j\in J_{\Lambda}i\in J_{\Lambda'}} \sum_{|(T\phi_j,\phi_i)| = \frac{1}{|\Lambda+\rho|^{2l}|\Lambda'+\rho|^{2m}}} \sum_{j\in J_{\Lambda}j\in J_{\Lambda'}} \sum_{|(T(\Delta+|\rho|^2)^l\phi_j, \phi_i)| = \frac{1}{|\Lambda+\rho|^{2l}|\Lambda'+\rho|^{2m}}} \sum_{j\in J_{\Lambda}i\in J_{\Lambda'}} \sum_{|((\Delta+|\rho|^2)^m T(\Delta+|\rho|^2)^l\phi_j, \phi_i)|.$$

On the other hand by the assumption of the lemma we can prove that there exists a constant $C_1^{i,m}$ such that

$$||(\Delta + |\rho|^2)^m T(\Delta + |\rho|^2)^l|| \leq C_1^{l,m}$$

Then

$$\sum_{j\in J_{\Lambda}i\in J_{\Lambda'}} |(T\phi_{j},\phi_{i})| \leq \frac{C_{1}^{i,m}}{|\Lambda+\rho|^{2l}|\Lambda'+\rho|^{2m}} (d_{\Lambda})^{2} (d_{\Lambda'})^{2}$$

$$= C_{1}^{i,m} \frac{1}{|\Lambda+\rho|^{2l}|\Lambda'+\rho|^{2m}} \frac{\prod_{\alpha\in P}(\Lambda+\rho,\alpha)^{2} (\Lambda'+\rho,\alpha)^{2}}{\prod_{\alpha\in P}(\rho,\alpha)^{4}}$$

$$\leq C_{1}^{i,m} \frac{\prod_{\alpha\in P}(\alpha,\alpha)^{2}}{\prod_{\alpha\in P}(\rho,\alpha)^{4}} \cdot \frac{1}{|\Lambda+\rho|^{2l-n(n-1)/2+[n/2]}|\Lambda'+\rho|^{2m-n(n-1)/2+[n/2]}}.$$
(3.1)

Therefore if put l=m, we have

$$\sum_{i,j\in J} |(T\phi_j,\phi_i)| \leq C_1^{l,l} \frac{\prod_{\alpha \in P}(\alpha,\alpha)^2}{\prod_{\alpha \in P}(\rho,\alpha)^4} \left(\sum_{\Lambda \in \mathcal{F}_0} \frac{1}{|\Lambda+\rho|^{2l-n(n-1)/2+[n/2]}} \right)^2.$$
(3.2)

If we take $l=m>\frac{1}{2}\frac{n(n-1)}{2}=\frac{1}{2}$ dim K, using the property (iii) we obtain

$$\sum_{i,j\in J} |(T\phi_j,\phi_i)| < +\infty$$
 .

q.e.d.

Corollary. If T is an operator on \mathfrak{S} satisfying the conditions of Lemma 1, T is of the trace class.

For the proof of this corollary, see Harish-Chandra [1(a), Lemma 1]. For any $f \in C^{\infty}_{c}(G)$. We put

$$T_f(\xi, \sigma) = \int_G f(g) U_g^{\xi, \sigma} dg \qquad (\xi \neq 0, \, [\sigma] \in \hat{K}_{\xi}) \,.$$

Then

$$(T_f(\xi,\sigma)F)(u) = \int_K K_f(\xi,\sigma;u,v)F(v)dv \qquad (u \in K),$$

where

Analogue of the Paley-Wiener Theorem

$$K_f(\xi, \sigma; u, v) = d_{\sigma} \int_{K_{\xi}} \overline{\sigma_{qq}(m)} \ d_{\xi} m \int_H f \begin{pmatrix} umv^{-1} \ x \\ 0 \ 1 \end{pmatrix} e^{i < \xi, u^{-1}x >} dx$$

It is easy to see that $T_f(\xi, \sigma)F \in C^{\infty}(K)$ for any $f \in C^{\infty}_{\mathfrak{c}}(G)$ and $F \in C^{\infty}(K)$.

We denote by λ and μ the left and right regular representations of G, respectively. We also denote by λ and μ the corresponding representations of the universal enveloping algebra of G defined on $C^{\infty}(G)$. We regard each element $X \in \mathfrak{k}$ as a right invariant vector field on K. So that we have L(X) = -X. Since

$$(T_f(\xi,\sigma)F(\exp(-tX))u) = (T_{\lambda(\exp tX)f}(\xi,\sigma)F)(u) \qquad (t \in \mathbf{R}),$$

we have

$$((-X)T_f(\xi,\sigma)F)(u) = (T_{\lambda(X)f}(\xi,\sigma)F)(u)$$

for $F \in C^{\infty}(K)$. Therefore for any non-negative integer l

$$\Delta^{\prime}T_{f}(\xi,\sigma) = T_{\lambda(\Delta)}\iota_{f}(\xi,\sigma)$$
.

Also we have

$$T_f(\xi,\sigma)\Delta^m = T_{\mu(\Delta)}m_f(\xi,\sigma) \qquad (m=0,1,2,\cdots)$$

by a similar way. On the other hand we notice that

$$||T_f(\xi,\sigma)|| \leq \int_G |f(g)| dg.$$

Hence

$$||\Delta^{i}T_{f}(\xi,\sigma)\Delta^{m}|| \leq \int_{G} |(\lambda(\Delta)^{i}\mu(\Delta)^{m}f)(g)| dg$$

Thus the opreator $T_f(\xi, \sigma)$, $f \in C^{\infty}_{c}(G)$, satisfies the assumptions of Lemma 1. By the corollary to Lemma 1, $T_f(\xi, \sigma)$ is of the trace class.

As it can be easily seen that $K_f(\xi, \sigma; u, v) \in C^{\infty}(K \times K)$, we have

$$Tr(T_f(\xi,\sigma)) = \int_K K_f(\xi,\sigma;u,u) du$$

(see [1(b), Lemma 5]). Making use of the relation

$$d_{\sigma}\int_{K_{\xi}}\sigma_{qq}(m_1mm_1^{-1})d_{\xi}m_1=\chi_{\sigma}(m)$$
,

we have the following proposition.

Proposition 1. For any $f \in C^{\infty}_{c}(G)$, $T_{f}(\xi, \sigma)(\xi \neq 0, [\sigma] \in \hat{K}_{\xi})$ is of the trace class and

$$Tr(T_f(\xi,\sigma)) = \int_{K_{\xi}} \overline{\mathcal{X}_{\sigma}(m)} \ d_{\xi}m \int_{H \times K} f\left(\begin{matrix} u \ m \ u^{-1} \ x \\ 0 \ 1 \end{matrix} \right) e^{i < \xi, u^{-1}x >} \ dx \ du$$

Let \mathbf{R}_+ be the set of all positive numbers and let M be the subgroup consisting of the elements $\begin{pmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(m \in SO(n-1))$. Then for any $\xi \in \hat{H}$ of the form $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $(a \in \mathbf{R}_+)$, we have $K_{\xi} = M$. It follows from the results of §2 that $(\hat{G})_{\infty}$ can be indetified with $\mathbf{R}_+ \times \hat{M}$. For $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $(a \in \mathbf{R}_+)$, we write briefly $T_f(\xi, \sigma)$

 $=T_f(a, \sigma)$. Then we have the following Plancherel formula for G.

Proposition 2. For any $f \in C_c^{\infty}(G)$

$$\int_{G} |f(g)|^{2} dg = \frac{2}{2^{n/2} \Gamma(n/2)} \sum_{[\sigma] \in \hat{M}} d_{\sigma} \int_{R_{+}} ||T_{f}(a,\sigma)||^{2}_{2} a^{n-1} da ,$$

where || ||₂ denotes the Hilbert-Schmidt norm.

Proof. It is enough to prove that

$$f\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{2}{2^{n/2}\Gamma(n/2)} \sum_{[\sigma] \in \hat{\mathfrak{A}}} d_{\sigma} \int_{\mathbf{R}_{+}} Tr(T_{f}(a, \sigma)) a^{n-1} da .$$

For any $f \in C^{\infty}_{c}(G)$, we put

$$T_f(\xi) = \int_G f(g) U_s^{\xi} dg \quad (\xi \in \hat{H}) \,.$$

As above we write $T_f(\xi) = T_f(a)$ for $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ c \end{pmatrix}$ $(a \in \mathbf{R}_+)$. Then by (2.2)

$$T_{f}(\xi) = \bigoplus_{[\sigma] \in \hat{\mathcal{K}}_{\xi}} (T_{f}(\xi, \underline{\sigma}) \oplus \cdots \oplus T_{f}(\xi, \sigma)) \quad (\xi \neq 0) .$$

Therefore

$$Tr(T_f(\xi)) = \sum_{\sigma \in \hat{K}_{\xi}} d_{\sigma} Tr(T_f(\xi, \sigma))$$

Hence it is enough to prove that

$$f\begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix} = \frac{2}{2^{n/2}(n/2)} \int_{\mathbf{R}_{+}} Tr(T_{f}(a)) a^{n-1} da .$$
(3.3)

Since

$$\phi(m) = \int_{H \times K} f \begin{pmatrix} u \ m \ u^{-1} \ x \\ 0 \ 1 \end{pmatrix} e^{i < \xi, u^{-1} x >} dx du$$

is a central function on K_{ξ} ,

$$\phi(m) = \sum_{[\sigma] \in \hat{K}_{\xi}} (\int_{K_{\xi}} \phi(m_1) \,\overline{\chi_{\sigma}(m_1)} \, d_{\xi}m_1) \,\chi_{\sigma}(m)$$

(see [7], §24). Hence by Proposition 1 we have

$$\phi(1) = \sum_{[\sigma] \in \hat{\mathcal{K}}_{\xi}} d_{\sigma} \int_{K_{\xi}} \phi(m) \,\overline{\chi_{\sigma}(m)} \, d_{\xi}m$$
$$= \sum_{[\sigma] \in \hat{\mathcal{K}}_{\xi}} d_{\sigma} Tr(T_{f}(\xi, \sigma)) \,.$$

Thus we have

$$Tr(T_{f}(\xi)) = \phi(1) = \int_{H \times K} f\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} e^{i \langle \xi, u^{-1}x \rangle} dx du$$
$$= \int_{H} \left\{ \int_{K} f\begin{pmatrix} 1 & ux \\ 0 & 1 \end{pmatrix} du \right\} e^{i \langle \xi, x \rangle} dx .$$

Hence

$$\int_{K} f\begin{pmatrix} 1 & ux \\ 0 & 1 \end{pmatrix} du = \int_{H} Tr(T_{f}(\xi)) e^{-i \langle \xi, x \rangle} d\xi ,$$

where $d\xi = \frac{1}{(2\pi)^{n/2}} d\xi_1 \cdots d\xi_n$. When x=0,

$$f\begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix} = \int_{H} Tr(T_{f}(\xi)) d\xi$$
(3.4)

By (2.1) we have $Tr(T_f(k\xi)) = Tr(R_k T_f(\xi) R_k^{-1}) = Tr(T_f(\xi))$. Hence $Tr(T_f(\xi)) = Tr(T_f(|\xi|))$. So that we have (3.3) from (3.4).

q.e.d.

Let $B(\mathfrak{H})$ by the Banach space of all bounded linear operators on \mathfrak{H} . We define the **Fourier transform** of $f \in C^{\infty}_{e}(G)$ by the $B(\mathfrak{H})$ -valued function T_{f} on \hat{H} . In terms of this transform Proposition 2 becomes the following

Corollary. For any $f \in C^{\infty}_{c}(G)$

$$\int_{G} |f(g)|^{2} dg = \frac{2}{2^{n/2} \Gamma(n/2)} \int_{R_{+}} ||T_{f}(a)||^{2}_{2} a^{n-1} da.$$

4. The Fourier-Laplace transform

For each $\zeta \in \hat{H}^{c}(\simeq C^{n})$ we define a bounded representation of G on \mathfrak{H} by

$$(U_{\mathfrak{s}}^{\zeta}F)(u)=e^{i<\zeta,u^{-1}x>}F(k^{-1}u),\,(F\in\mathfrak{Y},\,u\in K)\,,$$

where $g = \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \in G$. For $f \in C^{\infty}_{\sigma}(G)$, put

$$T_f(\zeta) = \int_G f(g) \, U_g^{\zeta} \, dg$$

Then T_f is a $B(\mathfrak{Y})$ -valued function on \hat{H}^c . We shall call T_f the Fourier-Laplace transform of f.

Since K is compact, for each $f \in C_{c}^{\infty}(G)$ there exists a positive number a such that $\operatorname{Supp}(f) \subset \left\{ \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \in G; |x| \leq a, k \in K \right\}$, where $\operatorname{Supp}(f)$ denotes the support of f. We denote by r_{f} the greatest lower bound of such a's. Throughout this section we assume that $r_{f} \leq a$ for a fixed $a \in \mathbf{R}_{+}$.

Lemma 2. There exists a constant $C \ge 0$ depending only on f such that $||T_f(\zeta)|| \le C \exp a |\operatorname{Im} \zeta|$.

Proof. Making use of the Schwarz's inequality we have

$$||T_{f}(\zeta)F||^{2} \leq \int_{K} \left\{ \int_{H\times K} |f\binom{k}{0} \frac{x}{1}| e^{-\langle Im\zeta, u^{-1}x \rangle} |F(k^{-1}u)| dx dk \right\}^{2} du$$

$$\leq e^{2a|Im\zeta|} \int_{K} \left\{ \int_{K} \left(\int_{H} |f\binom{k}{0} \frac{x}{1}| dx \rangle| F(k^{-1}u)| dk \right\}^{2} du$$

$$\leq e^{2a|Im\zeta|} \int_{K} \left\{ \int_{K} \left(\int_{H} |f\binom{k}{0} \frac{x}{1}| dx \rangle^{2} dk \int_{K} |F(k^{-1}u)|^{2} dk \right\} du$$

$$= e^{2a|Im\zeta|} \int_{K} \left(\int_{H} |f\binom{k}{0} \frac{x}{1}| dx \rangle^{2} dk ||F||^{2} dx$$

for any $F \in \mathfrak{D}$. Therefore it is enough to put

$$C = \left\{ \int_{K} \left(\int_{H} |f \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix}| dx \right)^{2} dk \right\}^{1/2}.$$

Lemma 3. The $B(\mathfrak{H})$ -valued function T_f on \hat{H}^c is entire analytic.

Proof. For any *n*-tuple (m_1, \dots, m_n) of non-negative integers we define a bounded operator $T_f^{m_1 \dots m_n}$ by

$$(T_f^{m_1\cdots m_n}F)(u) = \int_{H\times K} f\binom{k \ ux}{0 \ 1} x_1^{m_1\cdots x_n}F(k^{-1}u)dx \ dk ,$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Then we have
 $||T_f^{m_1\cdots m_n}|| \leq a^{m_1+\cdots+m_n} \left\{ \int_K \left(\int_H |f\binom{k \ x}{0 \ 1}| dx \right)^2 dk \right\}^{1/2}.$

Hence for any fixed $\zeta = (\zeta_1 \cdots, \zeta_n) \in C^n$ the series

$$\sum_{m=0}^{\infty} i^m \sum_{m_1+\cdots+m_n=m} \frac{m!}{m_1!\cdots m_n!} T_f^{m_1\cdots m_n} \zeta_1^{m_1\cdots} \zeta_n^{m_n}$$

converges in $B(\mathfrak{Y})$ -norm. It is easy to see that this series is equal to $T_f(\zeta)$. q.e.d.

For any polynomial function p on \hat{H}^c , we define a differential operator p(D)on H by $p(D)=p\left(\frac{1}{i}\frac{\partial}{\partial x_1}, \dots, \frac{1}{i}\frac{\partial}{\partial x_n}\right)$. A polynomial function p on \hat{H}^c is called K-invariant if $p(k\zeta)=p(\zeta)$ for any $k \in K$ and $\zeta \in \hat{H}^c$. As is easily seen, $T_f(\zeta)$ leaves the space $C^{\infty}(K)$ stable.

Lemma 4. 1) For any non-negative integers l and m we have $\Delta^{l}T_{f}(\zeta)\Delta^{m} = T_{\lambda(\Delta^{l})\mu(\Delta^{m})f}(\zeta), \ (\zeta \in \hat{H}^{c}).$

2) For any K-invariant polynomial function p on \hat{H}^c , we have $p(\zeta)T_f(\zeta) = T_{p^*(D)f}(\zeta), (\zeta \in \hat{H}^c)$, where $p^*(\zeta) = p(-\zeta)$.

The statement 1) can be proved by a similar way mentioned in §3. The statement 2) is easily proved, using the fact $\frac{\partial}{\partial x_j}e^{i\langle\zeta,x\rangle}=i\zeta_je^{i\langle\zeta,x\rangle}$ and the integration by parts. From Lemma 2 and Lemma 4 we have the following

Proposition 3. For any K-invariant polynomial function p on \hat{H}^c and for any non-negative integers l and m, there exists a constant $C_p^{l,m}$ such that

$$||p(\zeta)\Delta^{l}T_{f}(\zeta)\Delta^{m}|| \leq C_{p}^{l,m} exp \ a |Im\zeta|.$$

Finally from the definition of T_f we have the following functional equations for T_f .

Proposition 4. $T_f(k\zeta) = R_k T_f(\zeta) R_k^{-1}$ ($\zeta \in \hat{H}^c, k \in K$).

5. The analogue of the Paley-Wiener theorem

Theorem 1. A $B(\mathfrak{D})$ -valued function T on \hat{H} is the Fourier transform of $f \in C^{\infty}_{c}(G)$ such that $r_{f} \leq a$ (a > 0) if and only if it satisfies the following conditions:

(I) T can be extended to an entire analytic function on \hat{H}^c .

(II) For any $\zeta \in \hat{H}^c$, $T(\zeta)$ leaves the space $C^{\infty}(K)$ stable. Moreover for any K-invariant polynomial function p on \hat{H}^c and for any non-negative integers l and m, there exists a constant $C_p^{l,m}$ such that

 $||p(\zeta)\Delta^{l}T(\zeta)\Delta^{m}|| \leq C_{p}^{l,m} \exp a|Im\zeta|.$

(III) For any $k \in K$

$$T(k\zeta) = R_k T(\zeta) R_k^{-1} \quad (\zeta \in \hat{H}^c) .$$

Proof. We have already proved the necessity of the theorem in §4. In the following we shall prove the sufficiency of the theorem.

Let T be an arbitrary $B(\mathfrak{H})$ -valued function on \hat{H} statisfying the conditions (I)~(III) in the theorem. Let $\{\phi_j\}_{j\in J}$ be the complete orthonomal basis of \mathfrak{H}

which we have chosen in §3. If $|Im\zeta| \leq b(b>0)$, by the condition (II) for any non-negateive integers l and m there exists a constant C'_{1}^{m} such that

$$||\Delta^{\iota}T(\zeta)\Delta^{\boldsymbol{m}}|| \leq C^{\iota,\boldsymbol{m}}_{1} \exp ab$$
.

Therefore by Lemma 1 the series

$$\sum_{i,j\in J} |(T(\zeta)\phi_j,\phi_i)|$$

converges and $T(\zeta)$ is of the trace class. We assume that $n \ge 3$. If $\phi_j = d_{\Delta}^{1/2}(\tau_{\Delta})_{pq}$, we have $|\phi_j(u)| \le d_{\Delta}^{1/2}$ because $|\tau_{\Delta}(u)_{pq}| \le 1$. So we have

$$\sum_{j\in J_{\Lambda}}\sum_{i\in J_{\Lambda'}}|(T(\zeta)\phi_{j},\phi_{i})\phi_{i}(u)\overline{\phi_{j}(v)}| \leq \frac{C_{1}^{\prime,m}e^{av}}{|\Lambda+\rho|^{2l}|\Lambda'+\rho|^{2m}}(d_{\Lambda})^{3/2}(d_{\Lambda'})^{3/2}.$$

Hence

$$\sum_{i,j\in J} |(T(\zeta)\phi_j,\phi_i)\phi_i(u)\overline{\phi_j(v)}|$$

$$\leq C_1^{l,l}e^{a_b} \frac{\prod_{\alpha\in P}(\alpha,\alpha)^3}{\prod_{\alpha\in P}(\rho,\alpha)^5} \left(\sum_{\Lambda\in\mathcal{F}_0} \frac{1}{|\Lambda+\rho|^{2l-3(\pi(n-1)/2-[\pi/2])/2}}\right)^2 < +\infty$$
(5.1)

for $2l > \frac{3}{2} \frac{n(n-1)}{2} - \frac{1}{2} \left[\frac{n}{2} \right]$. In case n=2, $|\phi_j|=1$ for all $j \in J$. Therefore $\sum_{i,j \in J} |(T(\zeta)\phi_j, \phi_i)\phi_i(u) \ \overline{\phi_j(v)}| = \sum_{i,j \in J} |(T(\zeta)\phi_j, \phi_i)| < +\infty.$

Now let us define the kernel function of $T(\zeta)$ ($\zeta \in \hat{H}^c$) by

$$K(\zeta; u, v) = \sum_{i, j \in J} (T(\zeta)\phi_j, \phi_i) \Phi_i(u) \overline{\phi_j(v)}.$$
(5.2)

By the facts stated above and the property (I) it is easy to see that for any $\zeta \in \hat{H}^c$ the right hand side of (5.2) is absolutely convergent and that it is uniformly covergent on every compact subset of $\hat{H}^c \times K \times K$. Thus we have the following

Lemma 5. The function $\hat{H}^c \times K \times K \ni (\zeta, u, v) \rightarrow K(\zeta; u, v)$ is of class C^{∞} and entire analytic with repsect to ζ .

If we adopt the formula (5.1) to $p(\zeta)T(\zeta)$ instead of $T(\zeta)$, we have the following lemma by making use of (II).

Lemma 6. For any K-invariant polynomial function p on \hat{H}^c , there exists a constant C_p such that

$$|p(\zeta)K(\zeta; u, v)| \leq C_p \exp a |Im\zeta|, (\zeta \in \hat{H}^c, u, v \in K).$$

REMARK. $K(\zeta; u, v)$ is rapidly decreasing on the real axis \hat{H} .

Let us define a function f on G by the inversion formula corresponding to

the Fourier transform, i.e.

$$f(g) = \frac{2}{2^{n/2} \Gamma(n/2)} \int_{R_+} Tr(T(a) U_g^{a-1}) a^{n-1} da$$

By the property (III) we have

$$(T(k\zeta)\phi_j, \phi_i)\phi_i(u)\overline{\phi_j(v)} = (R_kT(\zeta)R_k^{-1}\phi_j, \phi_i)\phi_i(u)\overline{\phi_j(v)} = (T(\zeta)R_k^{-1}\phi_j, R_k^{-1}\phi_i)\phi_i(u)\overline{\phi_j(v)}.$$

Let $\phi_j = d_{\tau}^{1/2} \tau_{pq}$ and $\phi_i = d_{\sigma}^{1/2} \sigma_{rs}([\tau], [\sigma] \in \hat{K})$. Then

$$R_{k}^{-1}\phi_{j}(w) = d_{\tau}^{1/2}\tau_{pq}(wk^{-1}) = d_{\tau}^{1/2}\sum_{l=1}^{d_{\tau}}\tau_{pl}(w) \overline{\tau_{ql}(k)}$$

and

$$R_{k}^{-1}\phi_{i}(w) = d_{\sigma}^{1/2}\sum_{m=1}^{d_{\sigma}}\sigma_{rm}(w) \overline{\sigma_{sm}(k)}$$

Therefore

$$(T(\zeta)R_{k}^{-1}\phi_{j}, R_{k}^{-1}\phi_{i})\phi_{i}(u)\overline{\phi_{j}(v)} = \sum_{l=1}^{d_{\tau}}\sum_{m=1}^{d_{\sigma}} (T(\zeta)d^{1/2}\tau_{pl}, d^{1/2}\sigma_{rm})d^{1/2}\sigma_{rs}(u)\sigma_{sm}(k)d^{1/2}\overline{\tau_{pq}(v)\tau_{ql}(k)}.$$

Hence

$$\sum_{p,q=1}^{d_{\tau}} \sum_{r,s=1}^{d_{\sigma}} (T(k\zeta) d_{\tau}^{1/2} \tau_{pq}, d_{\sigma}^{1/2} \sigma_{rs}) d_{\sigma}^{1/2} \sigma_{rs}(u) d_{\tau}^{1/2} \overline{\tau_{pq}(v)}$$

$$= \sum_{p,l=1}^{d_{\tau}} \sum_{r,m=1}^{d_{\sigma}} (T(\zeta) d_{\tau}^{1/2} \tau_{pl}, d_{\sigma}^{1/2} \sigma_{rm}) \sum_{s=1}^{d_{\sigma}} d_{\sigma}^{1/2} \sigma_{rs}(u) \sigma_{sm}(k) \times \sum_{q=1}^{d_{\tau}} d_{\tau}^{1/2} \overline{\tau_{pq}(v) \tau_{ql}(k)}$$

$$= \sum_{p,l=1}^{d_{\tau}} \sum_{r,m=1}^{d_{\sigma}} (T(\zeta) d_{\tau}^{1/2} \tau_{pl}, d_{\sigma}^{1/2} \sigma_{rm}) d_{\sigma}^{1/2} \sigma_{rm}(uk) d_{\tau}^{1/2} \tau_{pl}(vk) .$$

Since $K(\zeta; u, v) = \sum_{[\sigma], [\tau] \in \hat{K}} \sum_{p,q=1}^{d_{\tau}} (T(\zeta) d_{\tau}^{1/2} \tau_{pq}, d_{\sigma}^{1/2} \sigma_{rs}) d_{\sigma}^{1/2} \sigma_{rs}(u) \times d_{\tau}^{1/2} \overline{\tau_{pq}(v)},$ we have the following functional equation for $K(\zeta; u, v)$:

$$K(k\zeta; u, v) = K(\zeta; uk, vk).$$
(5.3)

On the other hand

$$Tr(T(k\xi)U_{\mathfrak{s}^{\pm_{1}}})=Tr(R_{\mathfrak{k}}T(\xi)R_{\mathfrak{s}^{-1}}U_{\mathfrak{s}^{\pm_{1}}})=Tr(T(\xi)U_{\mathfrak{s}^{\pm_{1}}}), (\xi \in \hat{H}).$$

Hence

$$\frac{2}{2^{n/2}\Gamma(2/n)} \int_{R_+} Tr(T(a)U_{g^{-1}}^{a})a^{n-1}da = \int_{H} Tr(T(\xi)U_{g^{-1}}^{\epsilon})d\xi ,$$

where $d\xi = \frac{1}{(2\pi)^{n/2}}d\xi_1 \cdots d\xi_n$. As $T(\xi)F(u) = \int_{K} K(\xi; u, v)F(v)dv \quad (F \in \mathfrak{H})$

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and
$$g^{-1} = {\binom{k^{-1} - k^{-1}x}{0}}$$
 for $g = {\binom{k}{0}} \frac{x}{1} \in G$, we have
 $U_{g^{-1}}^{\xi} T(\xi) F(u) = \int_{K} e^{-i \langle \xi, u^{-1}k^{-1}x \rangle} K(\xi; ku, v) F(v) dv$

Since $T(\xi)$ is of the trace class, so is $U_{\varepsilon}^{\xi}T(\xi)$. Moreover the function $K \times K \ni$ $(u, v) \mapsto e^{-i < \xi \cdot u^{-1}k^{-1}x >} K(\xi; ku, v)$ is clearly of class C^{∞} . Hence

$$Tr(T(\xi)_{g}^{\xi-1}) = Tr(U_{g}^{\xi-1}T(\xi)) = \int_{K} e^{-i < \xi \cdot u^{-1}k^{-1}x > K(\xi, ku, u) du}$$

Therefore the equation (5.3) and the remark to Lemma 6 imply that

$$\begin{split} &\int_{H} Tr(T(\xi)U_{\xi}^{\xi_{-1}})d\xi = \int_{H} \int_{K} e^{-i\langle ku\xi,x\rangle} K(\xi;\,ku,\,u) dud\xi \\ &= \int_{K} \int_{H} e^{-i\langle \xi,x\rangle} K(u^{-1}k^{-1}\xi;\,ku,\,u) d\xi \,du \\ &= \int_{K} \int_{H} e^{-i\langle \xi,x\rangle} K(\xi;\,1,\,k^{-1}) d\xi \,du \\ &= \int_{H} e^{-i\langle \xi,x\rangle} K(\xi;\,1,\,k^{-1}) d\xi \,. \end{split}$$

Thus we have

$$f\begin{pmatrix}k&x\\0&1\end{pmatrix} = \int_{H} e^{-i\langle\xi,x\rangle} K(\xi;\,1,\,k^{-1})d\xi\,,\qquad(5.4)$$

 $(k \in K, x \in H)$. It follows from Lemma 5 and the remark to Lemma 6 that f is of class C^{∞} . Making use of Lemma 6, it follows from the classical Paley-Wiener theorem that if |x| > a, $f \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} = 0$ for any $k \in K$.

Finally we have to check that $T_f = T$. Since

$$T_f(\xi)F(u) = \int_K K_f(\xi: u, v)F(v)dv$$

where

$$K_f(\xi; u, v) = \int_H f\begin{pmatrix} uv^{-1} & x \\ 0 & 1 \end{pmatrix} e^{i \langle \xi, u^{-1}x \rangle} dx ,$$

so it is enough to prove that

$$K(\xi; u, v) = \int_{H} f\begin{pmatrix} uv^{-1} & x \\ 0 & 1 \end{pmatrix} e^{i \langle \xi, u^{-1}x \rangle} dx \, .$$

By the relation (5.4),

$$f\binom{uv^{-1} \ x}{0 \ 1} = \int_{H} e^{-i < \xi, x > K(\xi; 1, vu^{-1}) d\xi}$$

Analogue of the Paley-Wiener Theorem

$$=\int_{H}e^{-i\langle\xi,x\rangle}K(u^{-1}\xi;u,v)d\xi.$$

Hence

$$K(u^{-1}\xi; u, v) = \int_{H} f\begin{pmatrix} uv^{-1} & x \\ 0 & 1 \end{pmatrix} e^{i \langle \xi, x \rangle} dx$$

If we replace $u\xi$ for ξ ,

$$K(\xi; u, v) = \int_{H} f \begin{pmatrix} uv^{-1} & x \\ 1 & 0 \end{pmatrix} e^{i < u\xi \cdot x > } dx$$
$$= \int_{H} f \begin{pmatrix} uv^{-1} & x \\ 0 & 1 \end{pmatrix} e^{i < \xi \cdot u^{-1} x > } dx .$$

This completes the proof of the theorem.

6. The Fourier-Bessel transform

Let $C^{\infty}_{\mathfrak{c}}(K \setminus G/K)$ be the set of all complex valued K-bi-invariant functions on G which are infinitely differentiable and with compact support. For $f \in C^{\infty}_{\mathfrak{c}}(G)$, put

$$(\mathcal{F}_{\xi}f)(g) = \int_{H} f\left(g\begin{pmatrix}1 & y\\0 & 1\end{pmatrix}\right) e^{-i\langle\xi, y\rangle} dy$$

and

$$(\mathscr{P}f)(g) = \int_{K} f(gu) du$$

For $f \in C_c^{\infty}(K \setminus G/K)$ it is easy to to see that

$$(\mathscr{PF}_{\xi}f)\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \left(\int_{H} f\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \phi_{\xi}(y) dy \right) \phi_{-\xi}(x) , \qquad (6.1)$$

where

$$\phi_{\xi}(x) = \int_{K} e^{i < \xi, ux >} du \; .$$

REMARK. The formula (6.1) is regarded as an analogue of the Poisson integral for semisimple Lie groups (see [5]). And the function ϕ_{ξ} is the zonal spherical function.

Let us define the Fourier-Bessel transform $\mathscr{B}\mathcal{F}f$ of $f \in C^{\infty}_{\mathfrak{c}}(K \setminus G/K)$ by

$$(\mathscr{BF}f)(\xi) = \int_{H} f\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \phi_{\xi}(y) dy.$$

If $x = \begin{pmatrix} r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $(r > 0)$ and $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $(a > 0)$, we can prove that

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$$\phi_{\xi}(x) = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{n-1}{2}\right)} \int_{0}^{\pi} e^{i \operatorname{ar} \cos\theta} \sin^{n-2\theta} d\theta$$
$$= \Gamma\left(\frac{n}{2}\right) \frac{J_{(n-2)/2}(ar)}{\left(\frac{ar}{2}\right)^{(n-2)/2}}$$

(see [8] for the notation of the Bessel function $J_n(r)$).

If
$$g_r = \begin{pmatrix} 1 & 0 & r \\ \ddots & 0 \\ 0 & 1 & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$
, $(r \ge 0)$, we write briefly $f(r) = f(g_r)$. Then for any

 $f \in C^{\infty}_{c}(K \setminus G/K) f$ is uniquely determined by f(r), $(r \ge 0)$. Let $C^{\infty}(K \setminus \hat{H})$ be the set of all complex valued K-invariant functions on \hat{H} which are infinitely differ-

entiable. If $\xi = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, we write $F(\xi) = F(a)$ for $F \in C^{\infty}(K \setminus \hat{H})$. It is obvious that $\mathscr{BF} f \in C^{\infty}(K \setminus \hat{H})$ for $f \in C^{\infty}_{e}(K \setminus G/K)$. Moreover we have

$$(\mathscr{BF}f)(a) = \int_{\mathbf{R}_{+}} f(r) \frac{(ar)^{(n-2)/2}}{\int_{(n-2)/2} (ar)} r^{n-1} dr \quad (a > 0) \,.$$

Since for $f \in C^{\infty}_{c}(K \setminus G/K)$

$$(\mathscr{BF}f)(\xi) = \int_{H \times K} f\left(\begin{matrix} 1 & u^{-1}y \\ 0 & 1 \end{matrix} \right) e^{i \langle \xi, y \rangle} dy du$$
$$= \int_{H \times K} f\left(\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{matrix} \right) \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \right) e^{i \langle \xi, y \rangle} dy du$$
$$= \int_{H} f\left(\begin{matrix} 1 & y \\ 0 & 1 \end{matrix} \right) e^{i \langle \xi, y \rangle} dy,$$

we have

$$\begin{split} f\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} &= \int_{\hat{H}} (\mathscr{BF}f)(\xi) e^{-i \langle \xi \cdot y \rangle} d\xi \\ &= \int_{\hat{H}} (\mathscr{BF}f)(\xi) \phi_{-\xi}(y) d\xi \,. \end{split}$$

On the other hand we remark that $\phi_{-\xi}(x) = \phi_{\xi}(x)$ for any $\xi \in \hat{H}$ and $x \in H$. Hence we have the following inversion formula

$$f(r) = \int_{\mathbf{R}_{+}} (\mathscr{BF}f)(a) \frac{\int_{(n-2)/2} (ar)}{(ar)^{(n-2)/2}} a^{n-1} da$$

Then we can easily prove the following analogue of the Paley-Wiener theorem

for the Fourier-Bessel transform.

Theorem 2. A function F on \hat{H} is the Fourier-Bessel transform of $f \in C_{\bullet}^{\infty}$ $(K \setminus G/K)$ such that $r_f \leq a$ (a > 0) if and only if it satisfies the following conditions:

(I) F can be extended to an entire analytic function on \hat{H}^c .

(II) For any K-invariant polynomial function p of \hat{H}^c there exists a constant C_p such that

$$|p(\zeta)F(\zeta)| \leq C_p \exp a |Im\zeta| \quad (\zeta \in \hat{H}^c).$$

(III) For any $k \in K$

$$F(k\zeta) = F(\zeta)$$
 $(\zeta \in \hat{H}).$

REMARK. In case n = 2, we have

$$(\mathscr{BF}f)(a) = \int_0^\infty f(r) J_0(ar) r dr$$

This is the classical Fourier-Bessel transform [8].

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